Two-Sided Matching with Common Priority

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Motivation

- How to match students to colleges/schools?
  - College admissions model:
    - colleges have **preferences** (P_C) over students.
  - School choice model:
    - schools have rankings of **priorities** ($\succ_C$) over students.

    $\Rightarrow$ schools **DO NOT** have preferences over students.
- Key Difference Colleges/schools act as either agents or objects.
- There are situations in which colleges/schools play a **dual role**!

1. Colleges/schools as agents AND objects (1)

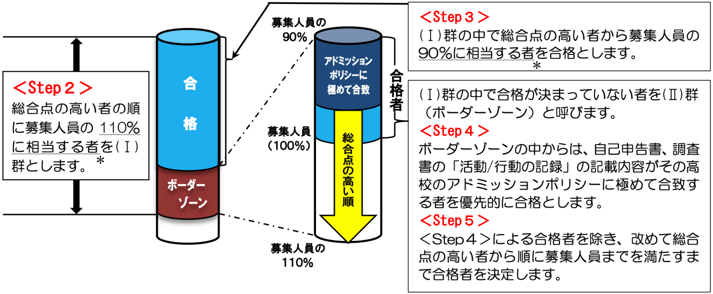
- In some student assignment problems, there is **ambiguity**:
  - Colleges/schools should be treated as agents or objects?
  - Or both? – public high-school choice in **Osaka** (Japan), national college admissions in China,[1] etc...

- We consider an **extended** matching model: $G = (P_S, P_C, \succ)$
  - We define a new stability concept: **Double Stability (DS)**.
  - Its properties are closely related to existing mechanisms.
    - Serial Dictatorship (SD) & Deferred Acceptance (DA)

- We derive a **characterization** of double stable matching.

[1] Related paper: Fang, Y. and Yasuda, Y. (2023) "Improving Matching under Information Constraint: Chinese College Admission Reconsidered," mimeo.

1. Colleges/schools as agents AND objects (2)



- In public school choice in Osaka, high schools play a dual role.[2]
  - They act as objects in Step 3 and agents in Step 4.
  - As a result, priorities and preferences are **BOTH** used.

[2]Source: Osaka's public high school choice in 2021.

2. The extended matching model（1）

- We consider an extended one-to-one matching model as follows.

$P_{s_1}: c_1, c_3, c_2$	$P_{c_1}: s_3, s_2, s_1$	$\succ: s_1, s_2, s_3$
$P_{s_2}: c_1, c_2, c_3$	$P_{c_2}: s_3, s_2, s_1$	
$P_{s_3}: c_2, c_1, c_3$	$P_{c_3}: s_3, s_2, s_1$	

- First, we define several notions that are needed for our analysis.
  - □ **Individual Rationality**: for each $i \in S \cup C$, $\mu_i R_i \emptyset$.
  - □ **Preference Blocking Pair**: (s, c) satisfies the condition

$$cP_s\mu_s \quad \text{and} \quad sP_c\mu_c \tag{1}$$

  - □ **Priority Blocking Pair**: (s, c) satisfies the condition

$$cP_s\mu_s \quad \text{and} \quad s \succ \mu_c \tag{2}$$

2. The extended matching model（2）

- These notions lead us to consider a new stability concept.

Double Stability (DS) $= \mu^*$

A matching μ is **double stable** if it is (i) individually rational, and (ii) neither preference blocked nor priority blocked by any pair.

- Although DS looks attractive, its existence is **NOT** guaranteed.[3]

$P_{s_1}: c_1, c_3, c_2$	$P_{c_1}: s_3, s_2, s_1$	$\succ: s_1, s_2, s_3$
$P_{s_2}: c_1, c_2, c_3$	$P_{c_2}: s_3, s_2, s_1$	
$P_{s_3}: c_2, c_1, c_3$	$P_{c_3}: s_3, s_2, s_1$	

[3]The unique priority stable matching $\binom{s_1\ s_2\ s_3}{c_1\ c_2\ c_3}$ is preference blocked by (s_2, c_1).

2. The extended matching model (3)

- Can we find a DS matching μ^* whenever it exists?

- Mechanism We classify mechanisms $\phi(\cdot)$ into two categories:

  - Extreme: **ONLY** preferences are used, i.e.,

$$\phi(P_S, P_C, \succ) = \phi(P_S, P_C, \succ') \quad \text{for any } \succ \text{ and } \succ' . \qquad (3)$$

  or **ONLY** priorities are used.

$$\phi(P_S, P_C, \succ) = \phi(P_S, P'_C, \succ) \quad \text{for any } P_C \text{ and } P'_C. \qquad (4)$$

    - e.g., deferred acceptance (DA), serial dictatorship (SD), ...

  - Moderate: Preferences and priorities **BOTH** matter. (in Osaka)

- Preview We focus on those **extreme** mechanisms.

3. Extreme mechanisms（1）

- SD Rule Let students choose according to the priority order.

$P_{s_1}: c_1, c_2, c_3$	$P_{c_1}: s_1, s_2, s_3$
$P_{s_2}: c_1, c_2, c_3$	$P_{c_2}: s_1, s_3, s_2$
$P_{s_3}: c_1, c_3, c_2$	$P_{c_3}: s_1, s_2, s_3$
	$\succ: s_1, s_2, s_3$

SD	c_1	c_2	c_3
Step 1	s_1		
Step 2		s_2	
Step 3			s_3

- Result 1 SD **implements** a **DS** matching μ^* whenever it exists.
  - SD finds a **unique** priority stable (PS) matching, μ^{PS}.
  - If a DS matching exists, then $\mu^{PS} = \mu^*$ **must** hold.[4]

[4]This implies there exists at most one double stable matching.

3. Extreme mechanisms (2)

- DA Rule Wait until the end to see who is matched to whom.

$P_{s_1}: c_1, c_2, c_3$	$P_{c_1}: s_1, s_2, s_3$
$P_{s_2}: c_1, c_2, c_3$	$P_{c_2}: s_1, s_3, s_2$
$P_{s_3}: c_1, c_3, c_2$	$P_{c_3}: s_1, s_2, s_3$
	$\succ: s_1, s_2, s_3$

DA	c_1	c_2	c_3
Step 1	s_1, s_2, s_3		
Step 2		s_2	s_3

- Result 2 There is a difference depending on which side proposes.
  - □ Student-proposing DA **implements** a DS matching if it exists.
  - □ However, college-proposing DA **fails** to do so.[5]

[5] In the example, $\mu^{COSM} = \binom{s_1\ s_2\ s_3}{c_1\ c_3\ c_2}$ is different from $\mu^* = \binom{s_1\ s_2\ s_3}{c_1\ c_2\ c_3}$.

3. Extreme mechanisms（3）

- Question Why does the student-proposing DA succeed?

- To explain this, we use the following two properties.

  - **P1**: $\mu^{PS} = \mu^*$ is Pareto efficient for students.
  - **P2**: μ^{SOSM} is a student optimal stable matching.[6]

- Proof Suppose that the student-proposing DA fails, then

$$\mu^{SOSM} \neq \mu^* \tag{5}$$

  - **P2** implies that μ^* is Pareto dominated by μ^{SOSM}.
  - μ^* is **NOT** Pareto efficient for students. This contradicts **P1**.

[6] That is, μ^{SOSM} is the best preference stable matching for students.

3. Extreme mechanisms（4）

- Armed with these findings, we obtain the following result.

Main Result

A DS matching exists if and only if the outcomes of the SD and the student-proposing DA mechanism **coincide**.

- This result provides a **necessary and sufficient condition** for DS.[7]
- Proof The "if" part is trivial. To prove the "only if" part,
  - Suppose that a DS matching exists but $\mu^{PS} \neq \mu^{SOSM}$.
  - Then, at least one of Result 1 and 2 must be **violated**.

[7] It is computationally **EASY** to figure out whether a DS matching exists or not.

4. Extensions（1）

- Minimal Stability there is no pair (s, c) such that

$$cP_s\mu_s \quad \text{and} \quad sP_c\mu_c \quad \text{and} \quad s \succ \mu_c \tag{6}$$

  - MS only requires the elimination of a **double blocking pair**.[8]

$P_{s_1}: c_1, c_3, c_2$	$P_{c_1}: s_3, s_2, s_1$	$\succ: s_1, s_2, s_3$
$P_{s_2}: c_1, c_2, c_3$	$P_{c_2}: s_3, s_2, s_1$	
$P_{s_3}: c_2, c_1, c_3$	$P_{c_3}: s_3, s_2, s_1$	

  - Thus, MS is weak enough that its existence is guaranteed.
  - Any other stability notion should lie between MS and DS.

[8] In the example, $\mu^{MS} = \binom{s_1\ s_2\ s_3}{c_3\ c_2\ c_1}$ is neither priority stable nor preference stable.

4. Extensions（2）

- College-specific Priorities What if the priority order is **not** common?

$P_{s_1}: c_2, c_1$	$P_{c_1}: s_1, s_2, s_3$	$\succ_{c_1}: s_1, s_2, s_3$
$P_{s_2}: c_1, c_2$	$P_{c_2}: s_2, s_1, s_3$	$\succ_{c_2}: s_2, s_3, s_1$
$P_{s_3}: c_2, c_1$		

- In the above example, we have the following relation.

$$\mu^* = \begin{pmatrix} s_1 & s_2 & s_3 \\ c_1 & c_2 & \emptyset \end{pmatrix} \neq \begin{pmatrix} s_1 & s_2 & s_3 \\ c_2 & c_1 & \emptyset \end{pmatrix} = \mu^{SOSM} \quad (7)$$

- This means that **Result 2** and **Main Result** no longer hold.
- As noted, the common priority is the **KEY** for our paper.

Summary

- We study a new concept, **DS** in an extended matching model.
- While a DS matching does not always exist, we show:
  - Both SD and student-proposing DA mechanisms **implement** a DS matching whenever it exists.
  - However, college-proposing DA mechanism **fails** to do so.
  - A DS matching exists **if and only if** the outcomes of the SD and student-proposing DA mechanism **coincides**.
- Two **extensions**: minimal stability and college-specific priorities.