# Information matrix tests for Gaussian mixtures

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# Objective

- We are interested in studying the correct specification of the following model for an observable, $M$-variate time series process $\mathbf{y}_t$:

$$
\begin{aligned}
\mathbf{y}_t|\boldsymbol{\xi}_t, \mathbf{y}_{t-1}, \boldsymbol{\xi}_{t-1}\dots &= \mathbf{A}(\boldsymbol{\xi}_t)\mathbf{y}_{t-1} + \boldsymbol{\varepsilon}_t, \\
\boldsymbol{\varepsilon}_t|\boldsymbol{\xi}_t, \mathbf{y}_{t-1}, \boldsymbol{\xi}_{t-1}\dots &\sim N[\boldsymbol{\nu}(\boldsymbol{\xi}_t), \boldsymbol{\Omega}(\boldsymbol{\xi}_t)], \\
\boldsymbol{\xi}_t|\mathbf{y}_{t-1}, \boldsymbol{\xi}_{t-1}\dots &\sim MC[\mathbf{P}(\mathbf{y}_{t-1})],
\end{aligned}
$$

  where $\mathbf{P}(\mathbf{y}_{t-1})$ is the transition matrix of the latent Markov chain (MC) process with $K$ states $\boldsymbol{\xi}_t$.

- This model nests a broad class of models of interest in empirical work:

1. $\mathbf{y}_t|\boldsymbol{\xi}_t, \mathbf{y}_{t-1}, \boldsymbol{\xi}_{t-1}\dots \sim N(\boldsymbol{\nu}, \boldsymbol{\Omega})$
   "Multivariate Hermite polynomials and information matrix tests", forthcoming in *Econometrics and Statistics*

# Objective

2 $\mathbf{y}_t|\boldsymbol{\xi}_t, \mathbf{y}_{t-1}, \boldsymbol{\xi}_{t-1} \ldots \sim N(\boldsymbol{\nu} + \mathbf{A}\mathbf{y}_{t-1}, \boldsymbol{\Omega})$
"Tests for random coefficient variation in vector autoregressive models", in J.J. Dolado, L. Gambetti and C. Matthes (eds.) Essays in honour of Fabio Canova, *Advances in Econometrics* 44B, 1-35, 2022.

3 $\mathbf{y}_t|\boldsymbol{\xi}_t, \mathbf{y}_{t-1}, \boldsymbol{\xi}_{t-1} \ldots \sim N[\boldsymbol{\nu}(\boldsymbol{\xi}_t), \boldsymbol{\Omega}(\boldsymbol{\xi}_t)]$,
$\boldsymbol{\xi}_t|\mathbf{y}_{t-1}, \boldsymbol{\xi}_{t-1} \ldots \sim MN(\boldsymbol{\pi})$
"IM tests for Gaussian mixtures", CEMFI WP 2401.

4 $\boldsymbol{\xi}_t|\mathbf{y}_{t-1}, \boldsymbol{\xi}_{t-1} \ldots \sim MN[\boldsymbol{\pi}(\mathbf{y}_{t-1})]$
"IM tests for multinomial logit models", CEMFI WP 2406.

5 $\mathbf{y}_t|\boldsymbol{\xi}_t, \mathbf{y}_{t-1}, \boldsymbol{\xi}_{t-1} \ldots \sim N[\boldsymbol{\nu}(\boldsymbol{\xi}_t) + \mathbf{A}(\boldsymbol{\xi}_t)\mathbf{y}_{t-1}, \boldsymbol{\Omega}(\boldsymbol{\xi}_t)]$,
$\boldsymbol{\xi}_t|\mathbf{y}_{t-1}, \boldsymbol{\xi}_{t-1} \ldots \sim MN[\boldsymbol{\pi}(\mathbf{y}_{t-1})]$
"IM tests for switching regression models".

6 $\boldsymbol{\xi}_t|\mathbf{y}_{t-1}, \boldsymbol{\xi}_{t-1} \ldots \sim MC[\mathbf{P}(\mathbf{y}_{t-1})]$
"IM tests for Markov chains".

# The information matrix test

- Consider a parametric model for $\mathbf{y}$ characterised by its (unconditional) probability distribution/density function $f(\mathbf{y};\boldsymbol{\phi})$, with $dim(\boldsymbol{\phi}) < \infty$.
- The information matrix (IM) test directly assesses the IM equality, which states that the sum of the Hessian matrix and the outer product of the score (OPS) vector should be zero in expected value when the estimated model is correctly specified.
- As Newey (1985) and Tauchen (1985) showed, the IM test can be regarded as a moment test based on the influence functions:

$$vech[\mathbf{h}_i(\boldsymbol{\phi}) + \mathbf{s}_i(\boldsymbol{\phi})\mathbf{s}_i'(\boldsymbol{\phi})].$$

- In practice, we evaluate these influence functions at the maximum likelihood estimator (MLE), $\hat{\boldsymbol{\phi}}_N$, so we need the asymptotic covariance matrix of

$$\frac{\sqrt{N}}{N}\sum_{i=1}^{N} vech[\mathbf{h}_i(\hat{\boldsymbol{\phi}}_N) + \mathbf{s}_i(\hat{\boldsymbol{\phi}}_N)\mathbf{s}_i'(\hat{\boldsymbol{\phi}}_N)].$$

# The information matrix test

- Chesher (1983) and Lancaster (1984) realised that one can use the generalised information matrix equality to obtain the expected value of the Jacobian of the influence functions with respect to $\boldsymbol{\phi}$ from the asymptotic covariance matrix between them and the score evaluated at the true values of the parameters, $\boldsymbol{\phi}_0$.

- Thus, we simply need the residual covariance matrix from their least squares projection onto the linear span of $\mathbf{s}_i(\boldsymbol{\phi}_0)$:

$$\mathcal{R}(\boldsymbol{\phi}_0) - \mathcal{U}(\boldsymbol{\phi}_0)\mathcal{I}^{-1}(\boldsymbol{\phi}_0)\mathcal{U}'(\boldsymbol{\phi}_0),$$

$$\begin{bmatrix} \mathcal{R}(\boldsymbol{\phi}_0) & \mathcal{U}(\boldsymbol{\phi}_0) \\ \mathcal{U}'(\boldsymbol{\phi}_0) & \mathcal{I}(\boldsymbol{\phi}_0) \end{bmatrix} = \left\{ \begin{array}{c} vech[\mathbf{h}_i(\boldsymbol{\phi}_0) + \mathbf{s}_i(\boldsymbol{\phi}_0)\mathbf{s}_i'(\boldsymbol{\phi}_0)] \\ \sum_{i=1}^N \mathbf{s}_i(\boldsymbol{\phi}_0) \end{array} \right\}.$$

# The information matrix test

- Therefore, the infeasible IM test statistic is the quadratic form

$$N\left\{\frac{1}{N}\sum_{i=1}^{N} vech'[\mathbf{h}_i(\hat{\boldsymbol{\phi}}_N)+\mathbf{s}_i(\hat{\boldsymbol{\phi}}_N)\mathbf{s}_i'(\hat{\boldsymbol{\phi}}_N)]\right\}$$
$$\times[\mathcal{R}(\boldsymbol{\phi}_0)-\mathcal{U}(\boldsymbol{\phi}_0)\mathcal{I}^{-1}(\boldsymbol{\phi}_0)\mathcal{U}(\boldsymbol{\phi}_0)]^{-}$$
$$\times\left\{\frac{1}{N}\sum_{i=1}^{N} vech[\mathbf{h}_i(\hat{\boldsymbol{\phi}}_N)+\mathbf{s}_i(\hat{\boldsymbol{\phi}}_N)\mathbf{s}_i'(\hat{\boldsymbol{\phi}}_N)]\right\}.$$

- A generalised inverse is often necessary because some of the influence functions underlying the IM test may be an exact linear combination of $\mathbf{s}_i(\boldsymbol{\phi}_0)$ or appear multiple times.

- As a result, the number of degrees of freedom of the asymptotic $\chi^2$ distribution under the null of correct specification is $rank[\mathcal{R}(\boldsymbol{\phi}_0)-\mathcal{U}(\boldsymbol{\phi}_0)\mathcal{I}^{-1}(\boldsymbol{\phi}_0)\mathcal{U}(\boldsymbol{\phi}_0)]$, which requires careful derivation.

# The information matrix test

- Chesher (1983) and Lancaster (1984) suggested a feasible version as $N$ times the $R^2$ in the regression of a vector of $N$ ones onto $\mathbf{s}_i(\hat{\boldsymbol{\phi}}_N)$ and $vech[\mathbf{h}_i(\hat{\boldsymbol{\phi}}_N) + \mathbf{s}_i(\hat{\boldsymbol{\phi}}_N)\mathbf{s}_i'(\hat{\boldsymbol{\phi}}_N)]$ using an OLS routine robust to multicollinearity.
- The inclusion of $\mathbf{s}_i(\hat{\boldsymbol{\phi}}_N)$ as additional regressors makes the statistic robust to the fact that the influence functions are evaluated at $\hat{\boldsymbol{\phi}}_N$.
- Nevertheless, this OPS regression has very poor finite sample properties, as stressed by Horowitz (1994) among many others.
- We apply the parametric bootstrap to an alternative feasible version that evaluates the theoretical expressions for the asymptotic covariance matrix at the MLE $\hat{\boldsymbol{\phi}}_N$.

# The case of incomplete data

- We follow Dempster, Laird and Rubin (1977) in using ‘incomplete data” to denote situations in which the observed data $\mathbf{y}$ is the output of a mapping $\mathbf{g}(.)$ from the complete sample space $\mathbf{Z}$ to the observed sample space $\mathbf{Y}$, so that the complete data $\zeta$ is only known to lie in $R$, the subset of $\mathbf{Z}$ implicitly defined by the equation $\mathbf{y} = \mathbf{g}(\zeta)$.
- Let $f(\zeta; \boldsymbol{\phi})$ denote the joint density of $\zeta$ given the parameters $\boldsymbol{\phi}$.
- Basic probability theory implies that

$$f(\mathbf{y}; \boldsymbol{\phi}) = \int_R f(\zeta; \boldsymbol{\phi}) d\zeta.$$

- We maintain the following regularity condition:

**Assumption:** *The boundary of $R$ does not depend on $\boldsymbol{\phi}$.*

# The case of incomplete data

- We can then prove the following result:

  **Proposition:** *The influence functions of the IM test of the model for observed variables are*

  $$E\left\{ vech\left[\frac{\partial^2 \ln f(\zeta;\boldsymbol{\phi})}{\partial\boldsymbol{\phi}\partial\boldsymbol{\phi}'} + \frac{\partial \ln f(\zeta;\boldsymbol{\phi})}{\partial\boldsymbol{\phi}}\frac{\partial \ln f(\zeta;\boldsymbol{\phi})}{\partial\boldsymbol{\phi}'}\right] \middle| \mathbf{y} \right\},$$

  *where the expectation is taken with respect to the conditional distribution of* $\zeta$ *given* $\mathbf{y}$ *over* $R$.

- Thus, we simply need to compute the expected value conditional on the observed variables of the influence functions underlying the IM test of the complete log-likelihood.

# The case of incomplete data

- This relationship is very convenient in those set ups in which the complete log-likelihood function adopts a particularly simple form, such as in the limited dependent variable models considered by Gouriéroux, Monfort, Renault and Trognon (1987), who proved a special case of the previous result when $f(\zeta;\boldsymbol{\phi})$ belongs to what they called a "bilinear" exponential family, and $\mathbf{y} = \mathbf{g}(\zeta)$.

- These include univariate probit and Tobit models among others, as well as their simultaneous equation versions studied by Smith (1987).

- Gaussian mixtures and their various generalisations mentioned at the beginning of the talk provide another case in point.

# The case of incomplete data

- To compute the IM test, we also need expressions for the different elements that appear in the asymptotic covariance matrices.
- Let $\mathbf{n}(\boldsymbol{\zeta};\boldsymbol{\phi})$ denote a vector influence functions of the complete data $\boldsymbol{\zeta}$ such that
$$E_{\boldsymbol{\zeta}}[\mathbf{n}(\boldsymbol{\zeta};\boldsymbol{\phi})] = \mathbf{0}$$
when both the expectation and the influence function are evaluated at the same value of the model parameters, $\boldsymbol{\phi}$.
- In addition, let
$$\mathbf{m}(\mathbf{y};\boldsymbol{\phi}) = E_{\boldsymbol{\zeta}|\mathbf{y}}[\mathbf{n}(\boldsymbol{\zeta};\boldsymbol{\phi})].$$

# The case of incomplete data

- We can prove the following result, which generalises Lemma 4 in Gouriéroux et al. (1987), who focused on the case in which the latent influence functions $\mathbf{n}(\zeta;\boldsymbol{\phi})$ coincide with $\partial \ln f(\zeta;\boldsymbol{\phi})/\partial\boldsymbol{\phi}$ when $f(\zeta;\boldsymbol{\phi})$ belongs to an exponential family:

**Proposition:**

$$V_{\mathbf{y}}[\mathbf{m}(\mathbf{y};\boldsymbol{\phi})] = V_{\zeta}[\mathbf{n}(\zeta;\boldsymbol{\phi})] - E_{\mathbf{y}}\{V_{\zeta|\mathbf{y}}[\mathbf{n}(\zeta;\boldsymbol{\phi})]\}$$

*and*

$$E_{\mathbf{y}}\left[\mathbf{m}(\mathbf{y};\boldsymbol{\phi})\frac{\partial \ln f(\mathbf{y};\boldsymbol{\phi})}{\partial\boldsymbol{\phi}'}\right] = -E_{\mathbf{y}}\left[\frac{\partial\mathbf{m}(\mathbf{y};\boldsymbol{\phi})}{\partial\boldsymbol{\phi}'}\right]$$

$$= E_{\zeta}\left[\mathbf{n}(\zeta;\boldsymbol{\phi})\frac{\partial \ln f(\zeta;\boldsymbol{\phi})}{\partial\boldsymbol{\phi}}\right] - E_{\mathbf{y}}\left\{cov_{\zeta|\mathbf{y}}\left[\mathbf{n}(\zeta;\boldsymbol{\phi}),\frac{\partial \ln f(\zeta;\boldsymbol{\phi})}{\partial\boldsymbol{\phi}}\right]\right\}.$$

# The case of incomplete data

- Thus, we can compute the different elements as:

$$\mathcal{I}(\boldsymbol{\phi}) = V_{\zeta}\left[\frac{\partial \ln f(\zeta;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}}\right] - E_{\mathbf{y}}\left\{V_{\zeta|\mathbf{y}}\left[\frac{\partial \ln f(\zeta;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}}\right]\right\},$$

$$\mathcal{U}(\boldsymbol{\phi}) = cov_{\zeta}\left\{vech\left[\frac{\partial^2 \ln f(\zeta;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}\partial \boldsymbol{\phi}'} + \frac{\partial \ln f(\zeta;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}}\frac{\partial \ln f(\zeta;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}'}\right], \frac{\partial \ln f(\zeta;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}}\right\}$$
$$-E_{\mathbf{y}}\left[cov_{\zeta|\mathbf{y}}\left\{vech\left[\frac{\partial^2 \ln f(\zeta;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}\partial \boldsymbol{\phi}'} + \frac{\partial \ln f(\zeta;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}}\frac{\partial \ln f(\zeta;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}'}\right], \frac{\partial \ln f(\zeta;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}}\right\}\right],$$

$$\mathcal{R}(\boldsymbol{\phi}) = V_{\zeta}\left\{vech\left[\frac{\partial^2 \ln f(\zeta;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}\partial \boldsymbol{\phi}'} + \frac{\partial \ln f(\zeta;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}}\frac{\partial \ln f(\zeta;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}'}\right]\right\}$$
$$-E_{\mathbf{y}}\left[V_{\zeta|\mathbf{y}}\left\{vech\left[\frac{\partial^2 \ln f(\zeta;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}\partial \boldsymbol{\phi}'} + \frac{\partial \ln f(\zeta;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}}\frac{\partial \ln f(\zeta;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}'}\right]\right\}\right].$$

- Once again, the advantage of this procedure arises when the complete model is much simpler to work with than the observed one.

# Multivariate Gaussian mixtures

- If $\epsilon|\boldsymbol{\xi} \sim N(\mathbf{0}, \mathbf{I}_M)$, $\boldsymbol{\nu}_k$ is an $M \times 1$ vector and $\boldsymbol{\Gamma}_k$ an $M \times M$ positive definite matrix with $\gamma_k = vech(\boldsymbol{\Gamma}_k)$, then $\mathbf{y} = \sum_{k=1}^{K} \xi_k(\boldsymbol{\nu}_k + \boldsymbol{\Gamma}_k^{1/2}\boldsymbol{\epsilon})$ is an $M$-variate, $K$-component mixture of normals.
- The natural model parameters are the mean vectors and covariance matrices of the components $\boldsymbol{\nu} = (\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_k, \ldots, \boldsymbol{\nu}_K)'$ and $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_k, \ldots, \boldsymbol{\gamma}_K)'$, respectively, and their probabilities $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k, \ldots, \lambda_K)$, which are subject to the unit simplex restrictions $\lambda_k \geq 0 \ \forall k$ and $\sum_{k=1}^{K} \lambda_k = 1$.
- Observations belong to the components with posterior probabilities

$$P(\xi_k = 1|y; \boldsymbol{\nu}, \boldsymbol{\gamma}, \boldsymbol{\lambda}) = \frac{\frac{\lambda_k}{\gamma_k}\phi[\varepsilon_k(\boldsymbol{\nu}, \boldsymbol{\gamma})]}{\sum_{l=1}^{K} \frac{\lambda_l}{\gamma_l}\phi[\varepsilon_l(\boldsymbol{\nu}, \boldsymbol{\gamma})]} = w_k(\boldsymbol{\nu}, \boldsymbol{\gamma}, \boldsymbol{\lambda}).$$

- Boldea and Magnus (2009) obtained analytical expressions for the score vector and Hessian matrix, and studied the OPS version of the IM test, but their number of degrees of freedom is incorrect in the multivariate case.

# Multivariate Gaussian mixtures

**Proposition:**

1) *The IM matrix test of a multivariate Gaussian mixture numerically coincides with a moment test based on the influence functions:*

$$w_k(\boldsymbol{\phi}) \left\{ \begin{array}{c} \mathbf{H}_3[\boldsymbol{\varepsilon}^*(\boldsymbol{\theta}_k)] \\ \mathbf{H}_4[\boldsymbol{\varepsilon}^*(\boldsymbol{\theta}_k)] \end{array} \right\}, \quad k = 1, \ldots, K$$

*evaluated at the MLE, where*

$$\mathbf{H}_j(\boldsymbol{\varepsilon}^*) = \left[ \begin{array}{c} H_{j,0,\cdots,0}(\boldsymbol{\varepsilon}^*) \\ H_{j-1,1,\cdots,0}(\boldsymbol{\varepsilon}^*) \\ \vdots \\ H_{0,\cdots,0,j}(\boldsymbol{\varepsilon}^*) \end{array} \right] = \left[ \begin{array}{c} H_j(\varepsilon_1^*) \\ H_{j-1}(\varepsilon_1^*) H_1(\varepsilon_2^*) \\ \vdots \\ H_j(\varepsilon_M^*) \end{array} \right]$$

*is the $\binom{M+j-1}{j}$ vector containing the distinct multivariate Hermite polynomials of order $j$ of a standardised random vector $\boldsymbol{\varepsilon}^*$, which can be expressed as products of the corresponding univariate Hermite polynomials of its elements.*

**Proposition (cont):**

2) *The asymptotic covariance matrix of these influence functions corrected for the sampling uncertainty in estimating the model parameters under the null is the residual covariance matrix in the multivariate theoretical regression of them on*

$$w_k(\boldsymbol{\phi}) \left\{ \begin{array}{c} 1 \\ \mathbf{H}_1[\boldsymbol{\varepsilon}^*(\boldsymbol{\theta}_k)] \\ \mathbf{H}_2[\boldsymbol{\varepsilon}^*(\boldsymbol{\theta}_k)] \end{array} \right\}, \quad k = 1, \ldots, K.$$

3) *If the effective number of components is $K$, then the asymptotic distribution of the IM test will be a $\chi^2$ random variable with degrees of freedom equal to*

$$\frac{KM(M+1)(M+2)(M+7)}{24}.$$

# Simulation evidence

- The asymptotic $\chi^2$ approximation of the IM test might not be very reliable in finite samples.
- For that reason, we assess test sizes by simulating 10,000 samples of length $N = 100$, $N = 400$ and $N = 1600$ for univariate and bivariate models.
- We also consider a parametric bootstrap procedure in which we simulate $B = 99$ samples from the mixture model estimated under the null.
- It is also of interest to investigate the power properties of our test.
- To do so, we simulate 2500 samples from three types of alternatives:
    1. mixtures with the same number of non-Gaussian components,
    2. mixtures with a larger number of Gaussian components, and
    3. non-mixture distributions.

# Monte Carlo simulations: conclusions

- Our Monte Carlo experiments confirm that using the theoretical expressions for the covariance matrices of the influence functions involved leads to substantial reductions in the size distortions of our testing procedures in finite samples relative to the OPS versions.
- In addition, the parametric bootstrap practically eliminates those size distortions.
- We also show that the IM test has power against various misspecification alternatives.

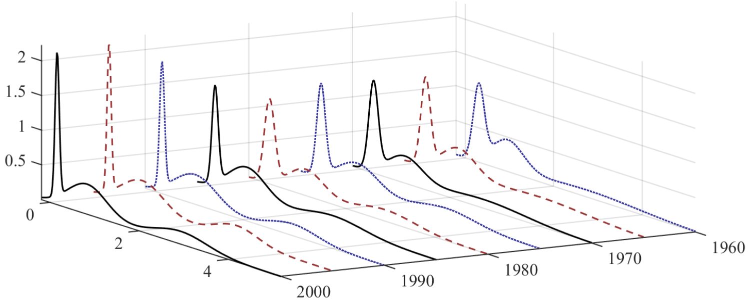
# Empirical illustration

- Gaussian mixtures feature pre-eminently in the empirical literature on "convergence clubs" in cross-country GDP per capita comparisons.
- We revisit the empirical application in Pittau, Zelli and Johnson (2010), who found that a Gaussian mixture with three components provides a very good fit for the distributions of per capita income in the Penn World Tables for 1960, 65, 70, 75, etc. all the way to the year 2000.
- In addition, they found that the within-group variances of both the rich and poor groups of countries decreased over time, while the distance between their means increased.
- Finally, they found that the sizes of the different groups fluctuated somewhat.

# Empirical illustration: parameter estimates

| Sample | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
|---|---|---|---|---|---|---|---|---|---|
| 1960 | 2.74 | 0.95 | 0.31 | 1.14 | 0.36 | 0.12 | 0.29 | 0.39 | 0.32 |
| 1965 | 2.84 | 1.01 | 0.28 | 1.05 | 0.39 | 0.11 | 0.27 | 0.40 | 0.33 |
| 1970 | 2.74 | 0.96 | 0.27 | 0.96 | 0.40 | 0.10 | 0.31 | 0.37 | 0.33 |
| 1975 | 3.08 | 1.07 | 0.26 | 0.65 | 0.47 | 0.10 | 0.24 | 0.45 | 0.31 |
| 1980 | 2.87 | 1.08 | 0.26 | 0.68 | 0.40 | 0.12 | 0.28 | 0.38 | 0.34 |
| 1985 | 2.86 | 0.92 | 0.20 | 0.69 | 0.43 | 0.07 | 0.27 | 0.49 | 0.24 |
| 1990 | 3.12 | 0.93 | 0.18 | 0.56 | 0.48 | 0.05 | 0.24 | 0.52 | 0.24 |
| 1995 | 3.02 | 0.89 | 0.15 | 0.49 | 0.48 | 0.05 | 0.25 | 0.50 | 0.25 |
| 2000 | 2.93 | 0.82 | 0.15 | 0.59 | 0.44 | 0.05 | 0.28 | 0.48 | 0.24 |

# Convergence clubs in cross-country GDP



# Empirical illustration: IM test

| | (p-values) | |
|---|---|---|
| Sample | Asym. | Boot. |
| 1960 | 0.68 | 0.44 |
| 1965 | 0.63 | 0.37 |
| 1970 | 0.34 | 0.13 |
| 1975 | 0.47 | 0.24 |
| 1980 | 0.74 | 0.49 |
| 1985 | 0.40 | 0.20 |
| 1990 | 0.55 | 0.39 |
| 1995 | 0.70 | 0.56 |
| 2000 | 0.51 | 0.35 |

# Extensions to Bartlett identities

- It would also interesting to extend the Bartlett identities test proposed by Chesher, Dhaene, Gouriéroux and Scaillet (1999) to incomplete data situations.
- In the context of univariate finite Gaussian mixtures, in particular, we would expect the influence functions to coincide with the fifth- and higher-order Hermite polynomials of the observed variable $y$ standardised as if it belonged to the $k^{th}$ component of the mixture weighted by the appropriate posterior probability.