Dynamic Contracting with Many Agents

Bruno Biais (HEC), Hans Gersbach (ETH),
Elu von Thadden (Mannheim), Jean-Charles Rochet (TSE),
Stéphane Villeneuve (TSE)

Econometric Society European Meeting,
Rotterdam, August 2024

Rochet acknowledges support by the European Research Council
(ERC Grant 101055239, DIPAMUTA).

Research Question

- How to allocate capital and consumption among several agents subject to Incentive Compatibility Constraints?
- Example: an investment bank with several trading desks.
- How to compensate traders and reallocate their capital under management as a function of their performance?
- The question is also relevant in macro-finance.
- To solve this problem, we extend Sannikov (2008)'s dynamic contract theory to several agents.

The model

- Single perishable good, can be consumed or invested.
- Continuous time $t \in (0, \infty)$, continuum of agents $i \in (0, 1)$.
- At each date t, principal allocates capital k_t^i and consumption c_t^i to agent i.
- Individual output subject to idiosyncratic Brownian risks Z_t^i :

$$dY_t^i = k_t^i \left[\mu dt + \sigma dZ_t^i \right]$$

- Incentive Compatibility Constraints needed to ensure that agents do no secretly divert output : Bolton-Scharfstein (1990).

Resource Constraints

- Constraint 1: aggregate capital allocated among agents

$$K_t = \int_0^1 k_t^i di.$$

- Constraint 2: Investment= output minus consumption:

$$dK_t = (\mu K_t - c_t^p - \int_0^1 c_t^i di)dt,$$

where c_t^P: principal's consumption.

Diversification implies that K_t is not random. More complex case with aggregate risk solved in the paper.

Sannikov's Martingale Approach

- Principal and agents: discount rate ρ, concave utility $u(c)$.
- Continuation utility of agent i from t onwards:

$$\omega_t^i = \mathbb{E}_t \int_t^\infty e^{-\rho(s-t)} u(c_s^i) ds.$$

- Martingale Representation Theorem implies that

$$d\omega_t^i = \left(\rho\omega_t^i - u(c_t^i)\right) dt + \sigma y_t^i dZ_t^i$$

where y_t^i = sensitivity of ω_t^i to agent i's performance.

Incentive compatibility

- Markov Contracts: $(c_t^i, k_t^i, y_t^i)(\omega_t^i)$.
- Anonymity and stationarity: we drop indices i and t.
- Incentive Compatibility Condition

$$y(\omega) \geq k(\omega)u'(c(\omega)) \text{ for all } \omega$$

  Minimum sensitivity of continuation utility to performance.
- The dynamics of ω_t cannot be deterministic: endogenous heterogeneity.
- To apply the dynamic programming principle: must compute the value function of the principal $V(K, \mathbb{P})$ for any probability distribution $\mathbb{P}$ of continuation utilities.

The Principal's problem

Infinite dimensional control problem

$$V(K, \mathbb{P}) = \sup_{c_t^P, k(.), c(.), y(.)} \int_0^\infty e^{-\rho t} u(c_t^P) dt$$

under two state equations: the dynamics of capital:

$$\dot{K}_t = \mu K_t - \int c(\omega) d\mathbb{P}(\omega) - c_t^P, K_0 = K,$$

and the Fokker Planck equation for the density $p(t, \omega)$ of $\mathbb{P}_t$:

$$\frac{\partial p(t,\omega)}{\partial t} = -\frac{\partial}{\partial \omega}[(\rho \omega_t - u(c(\omega_t)))\, p(t,\omega)] + \frac{\partial^2}{\partial \omega^2}[\sigma^2 y^2(\omega_t) p(t,\omega)],$$

with $\mathbb{P}_0 = \mathbb{P}$ plus feasibility and incentive constraints:

$$\int k(\omega_t) d\mathbb{P}(\omega_t) = K_t, y(\omega_t) \geq k(\omega_t) u'(c(\omega_t)).$$

The Hamilton Jacobi Bellman equation

- IC constraint always binding: $k(\omega) = \frac{y(\omega)}{u'(c(\omega)}$.
- Generalized Hamilton Jacobi Bellman equation:

$$\rho V(K, \mathbb{P}) = \sup_{c, c^P, y} \left\{ u(c^P) + \frac{\partial V}{\partial K} \left(\mu K - c^P - \int c(\omega) d\mathbb{P}(\omega) \right) \right.$$

$$+ \int [g'(\omega)(\rho\omega - u(c(\omega)))) + \frac{\sigma^2}{2} g''(\omega) y^2(\omega)] d\mathbb{P}(\omega),$$

under the constraint: $K = \int \frac{y(\omega)}{u'(c(\omega)))} d\mathbb{P}(\omega)$,

- $g(\omega)$ denotes the gradient of V with respect to $\mathbb{P}$.

Explicit solution when utility is log (or CRRA)

- Consumption is proportional to capital, both for principal:

$$c_t^P = \gamma^P K_t,$$

and agents:

$$c(\omega) = \frac{k(\omega)}{y},$$

- constant sensitivity y of continuation utility to performance.
- Capital under management reallocated according to performance:

$$\frac{dk_t}{k_t} = gdt + \rho\sigma y dZ_t,$$

where $g = \mu - \gamma^P - \frac{1}{y}$.

Dynamics of optimal allocations

- Cross sectional distribution of capital at date t is log normal:

$$\log k_t = \log k_0 + (g - \frac{1}{2}\rho^2\sigma^2 y^2)t + \rho\sigma y Z_t.$$

- Aggregate capital grows over time at rate $g = \mu - \gamma^p - \frac{1}{y}$
- Variance of $\log k_t$ grows over time at rate $v = \rho^2\sigma^2 y^2$.
- Growth of aggregate capital and growth of inequality vary in the same direction: if y increases, both g and v increase.
- Optimal y trades off between growth and inequality.

Decentralized implementation of optimal allocations

- In the log utility case, optimal allocations can be implemented by a decentralized market for capital coupled with appropriate monetary policy.
- Principal distributes to the agents a safe asset (bond or money) that can be traded for capital.
- Principal commits to risk-free return r s.t. agents keep fraction ρy of their wealth invested in risky capital:

$$\rho y = \frac{\mu - r}{\sigma^2}$$

- Principal's budget is balanced by continuous wealth transfer.

Conclusion

- Given the time constraint I could not explain in detail the Mean Field Control techniques underlying our results.
- These techniques are complex, but much simpler than Mean Field Games techniques used for equilibrium analyses with heterogeneous agents.
- When utilities are log or CRRA, the solution is quasi explicit and easy to implement.
- We are currently working on applications of the same techniques in several extensions of our model.