The Endogenous Growth and Asset Prices Nexus Revisited with Closed-Form Solution*

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Abstract

Endogenous growth models exhibit long-run risks, which are considered a potential explanation of the equity premium puzzle. Unlike previous literature, we use a closed-form solution of a simplified model to make the following contributions. First, we derive a set of conditions for a positive and large equity premium. Second, we match a key driver of endogenous growth, the R&D spending-to-GDP ratio. Third, we include a novel discussion on the role of patent obsolescence. Given that the literature concerns the accuracy of loglinear-lognormal solutions, we solve our model numerically with third-order perturbation. We find additional risk correction due to higher-order terms.

JEL: E13, E31, E43, E44, E62.

Keywords: endogenous growth, loglinear-lognormal solution, asset pricing, equity premium, third-order perturbation

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1 Introduction

Endogenous growth models have been shown to explain the hysteresis effects of supply shocks (e.g. Covid-19, Ukraine War) on output and inflation (see, e.g. Fornaro and Wolf, 2023, FW for short). These models can also replicate financial data well. For instance, Kung and Schmid (2015, KS for short) show that these models produce a high mean of the equity premium, and also capture the variation in market returns. This is because endogenous technology movements create significant, low-frequency variation in consumption (long-run risks), which are reflected in asset prices (see, e.g. KS for a closed-economy and Grüning (2017, 2018) for open-economy setups).

In this paper we use a simplified endogenous growth model to deliver approximate closed-form solutions for asset prices. The complexity of the cited models and that of many other endogenous growth models precludes even approximate closed-form solutions and, thus, they are solved numerically. Unlike the literature using exogenous growth models with long-run risk (see e.g. Bansal et al., 2010), there is no paper, which shows using at least an approximate closed-form solution how the endogenous growth model is successful in resolving the equity premium puzzle. For the closed-form solution, we take the standard loglinear-lognormal approach. In particular, we loglinarise the macroeconomic model, and calculate asset prices by assuming that the pricing kernel and the return on the consumption claim are jointly conditionally lognormally distributed.

What is the advantage of having an approximate closed-form solution? Closed-form solutions of simplified models are well-suited to illuminate how model assumptions and mechanisms shape the predictions of a model at the expense of matching the data less well than complex models. In particular, Bansal et al. (2010) calculate approximate closed-form solutions based on the loglinear-lognormal approach from an exogenous growth model with long-run risk. Their model is driven by both temporary, business cycle (mean-reverting) and permanent shocks. They show that positive business cycle shocks lead to falling asset prices due to their mean-reversion property. In the absence of at least one approximate closed-form solution, it is difficult to see clearly how business cycle shocks in endogenous growth models produce rising asset prices in response to positive mean-reverting shocks. We have the following contributions to make to the literature.

First, we use the closed-form solution to show that the salient asset pricing implications of the model are driven by the trend component of consumption growth. To do so, we decompose expected consumption growth into cyclical and trend components, which are endogenous in our set up, while they are driven by cyclical and permanent shocks, respectively, in Bansal et al. (2010). With the closed-form solution, we can analytically show the conditions under which the negative cyclical component is sufficiently muted, so that the positive trend component engineers a procyclical price-consumption ratio in line with empirical evidence. The latter happens, for instance, when the shock process is sufficiently persistent with an AR(1) parameter larger than 0.95. The closed-form solution also shows how the interaction of Epstein-Zin preferences and endogenous growth leads to a larger precautionary savings effect, and keeps the risk-free rate low.

Second, we show that the R&D spending-to-GDP ratio is crucial for a significant trend component, and for the asset pricing implications of the model. Unlike the previous macro-finance literature, which calibrates growth models using estimated input shares, our model matches R&D spending-to-GDP ratio in US data. Figure 1 exhibits R&D spending-to-GDP ratio varies around two percent in annual US data from 1953-2023. For the benchmark value of the R&D spending-to-GDP ratio, the model captures a small fraction of the mean and standard deviation of the equity premium (31 basis points and 5.5 percent, respectively). However, a small increase of 0.13 percentage points in the R&D spending-to-GDP ratio relative to the empirical benchmark of 2.46 percent raises annual productivity and consumption growth by 0.71 percentage points, and quadruples the equity premium (from 31 basis points to 130 basis points) roughly in line with empirical estimates of Kogan et al. (2017).

Insert Figure 1 here.

Third, we point to the importance of patent obsolescence in explaining asset prices. A rise in the obsolescence parameter magnifies the effect of supply shocks on macroeconomic variables as they make the hysteresis effects coming from the endogenous trend component more pronounced similarly to the arguments in FW. With patent obsolescence, the precautionary savings effect is powerful in helping to match the low risk free rate in the data.

Fourth, we find that a third-order perturbation solution of the model as well as stochastic volatility imply additional risk-correction relative to the loglinear-lognormal solution. Since the previous literature finds that loglinear-lognormal solutions might be inaccurate (see e.g. Pohl et al., 2018), we solve the model with third-order perturbation, which the computational macro-finance literature considers to be highly accurate (see e.g. Caldara et al., 2012). We find that macroeconomic and financial moments increase when the full model is solved with third-order perturbation solution due to the positive risk-correction coming from higher-order terms. In addition, we show that stochastic volatility helps capture the full variation in the price-consumption ratio.

Finally, the simplicity of our model allows us to gauge the importance of intangible capital accumulation on asset prices in isolation. The previous literature employs models with both tangible and intangible capital accumulation. It is difficult to judge the contribution of intangible capital since the previous literature (e.g. Kaltenbrunner and Lochstoer, 2010) already shows that even models equipped with tangible capital only and permanent shocks can yield reasonable asset pricing implications.

We use the endogenous growth model of FW but unlike FW we focus on macro-financial issues. We confirm that uncertainty shocks further elevate risk-premia in keeping with the findings of Bandi et al. (2023) and Bonciani and Oh (2022). Recently, Dou et al. (2024) have considered a version of the KS model with heterogeneous firms facing financial friction. Their model features a new endogenous state variable capturing misallocation, which creates low-frequency uncertainty about growth. Donadelli and Grüning (2016) find that the risk-premium is higher in the KS model with endogenous labour decisions and wage rigidity. Donadelli and Grüning (2021) explore how fiscal policies affect innovation dynamics and welfare in the KS model.

2 Model

2.1 Short summary of the non-linear model

In this section, we birefly summarise the four key non-linear equations of the endogenous growth model of FW. In the Online Appendix, we include full derivation of the FW model, and explain how it differs from the seminal asset pricing model of KS.

The first equation captures the positive relation between the growth rate and R&D spending:

$$G_{t+1} \equiv \frac{A_{t+1}}{A_t} = 1 - \phi + \chi s_t^{\zeta}. \tag{1}$$

where G_{t+1} is the growth rate of technology, A_t, and $s_t \equiv S_t/A_t$ is detrended spending on R&D. ϕ measures the obsolescence of technology (patent obsolescence in KS), which is zero in FW but positive in KS. Parameter $0 < \zeta < 1$ captures decreasing returns in innovation. χ is a constant, which helps to calibrate productivity growth. We will show that the introduction of patent obsolescence significantly improves the asset pricing implications of the model.

The second equation is the usual Euler equation, which takes the constant relative risk aversion (CRRA) form:

$$1 = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\frac{1}{\psi}} G_{t+1}^{-\frac{1}{\psi}} r_t, \tag{2}$$

where r_t is the risk-free real interest rate, and $c_t \equiv C_t/A_t$ is detrended consumption, and ψ is the elasticity of intertemporal substitution (EIS). In contrast to FW, we have relaxed their $\psi = 1$ choice to $\psi > 1$, which is the usual choice in long-run risk literature.

The third equilibrium condition describes the optimal path of investment spending:

$$s_t^{1-\zeta} = \left(\frac{1}{1 + r_t + \eta} \right) \left(\zeta\chi\bar{\omega} Z_{t+1} L + (1 - \phi) s_{t+1}^{1-\zeta} \right) \tag{3}$$

Note that the discount rate, $r_t + \eta$ is higher than the risk-free rate, r_t since $\eta > 0$ is chosen such that the steady-state of $r_t + \eta$ matches corporate discount rates in post-war US data[1]. $\zeta\chi\bar{\omega} Z_{t+1} L$ is the profits of the monopolist, and we assume that labour is fixed ($L = 1$) (see online appendix for the derivation of the profit function).[2] The value of a claim to profits is equal to v_t. Perfect competition and free entry to the innovation sector imply that the marginal benefit of innovation, v_t is equal to the marginal cost, $s_t^{1-\zeta}$. Hence, the Euler equation (3) is associated with pricing the profit claim. When $\phi = 0$, we are back to the expression of FW in their Appendix E.2.

The fourth is the usual market clearing:

$$\Psi Z_t L = c_t + s_t, \tag{4}$$

[1] Cash-flows from innovative projects are highly uncertain and the discount rates used to evaluate them are also significantly higher than the risk-free rate—see FW for corporate discount rates of about fourteen percent in real terms.

[2] We deviate from FW and consider an oligopolistic setup in the sense of Benigno and Fornaro (2017), whose setup slightly changes the $\bar{\omega}$ of FW to $\bar{\omega} \equiv (\xi - 1)(\alpha/\xi)^{\frac{1}{1-\alpha}}$, where α is the share of intermediate inputs in production and ξ is the gross markup. In this way the input share and the markup can be calibrated separately, so that $\xi \neq 1/\alpha$.

where $\Psi Z_t L$ is the value added (GDP).[3]

Finally, the exogenous component of technology is given by an AR(1) process in logs:

$$log(Z_{t+1}/Z) \equiv \widehat{z}_{t+1} = \rho \widehat{z}_t + \sigma_t \varepsilon_{t+1}, \quad (5)$$

where σ_t captures stochastic volatility, and the innovation comes from a standard Normal distribution: $\varepsilon_t \sim N(0,1)$. σ_t^2 follows the process in Bansal et al. (2010):

$$\sigma_{t+1}^2 = \sigma^2 + v_1(\sigma_t^2 - \sigma^2) + \sigma_w \varepsilon_{t+1}^w$$

where $\varepsilon_t^w \sim N(0,1)$. Since we choose $Z = 1$ it is true that $\widehat{z}_t = log(Z_t) - log(Z) = log(Z_t) = z_t$. For given stochastic processes $\{z_t, \sigma_t^2\}_{t=0}^{\infty}$, the model can be reduced to two equations in two variables, $\{c_t\}_{t=0}^{\infty}$ and $\{G_{t+1}\}_{t=0}^{\infty}$ as in FW.

2.2 Model solution strategy

To present an approximate closed-form solution of the model, we make the following simplification that is frequently used in macro-finance literature. First, we derive loglinear solution of the macroeconomic model of FW with CRRA preferences. Second, we calculate asset prices assuming lognormality of the Euler equations containing Epstein-Zin curvature. This way, we can present the main mechanisms of the model with a closed-form solution that is comparable to those in the exogenous growth literature.

2.3 Loglinear macroeconomic model

The combination of equations (2) and (3), as well as equation (4) can be approximated to the first-order as

$$\widehat{c}_t = \gamma_c z_t \text{ and } \widehat{g}_{t+1} = \gamma_g z_t \quad (6)$$

where

$$\gamma_c \equiv \frac{1}{s_c} \frac{\frac{1}{\psi} G^{\frac{1}{\psi}} + \frac{1-\zeta}{\zeta} \frac{G}{G-1+\phi} \left[(G^{\frac{1}{\psi}} - \beta(\rho(1-\phi) - \eta)) - (1-s_c) \frac{1}{\zeta} (G^{\frac{1}{\psi}} - \beta(1-\phi-\eta))\rho \right]}{\frac{1}{\psi} G^{\frac{1}{\psi}} + \frac{G}{G-1+\phi} \left[\frac{1-\zeta}{\zeta} (G^{\frac{1}{\psi}} - \beta(\rho(1-\phi) - \eta)) + \frac{1}{\psi}(1-\rho) \frac{1-s_c}{s_c} \frac{1}{\zeta} \right]},$$

[3] In particular, the value added is given by $Y_t - \int_0^1 x_{j,t} dj = \Psi A_t Z_t L$ where Y_t is output, $\int_0^1 x_{j,t} dj$ is the sum of intermediate inputs, and $\Psi \equiv (\alpha/\xi)^{\alpha/(1-\alpha)}(1-\alpha/\xi)$ is a positive constant.

$$\gamma_g \equiv \frac{(G^{\frac{1}{\psi}} - \beta(1-\phi-\eta))\rho + \frac{1-\rho}{\psi s_c}}{\frac{G^{\frac{1}{\psi}}}{\psi} + \frac{G}{G-1+\phi}\left[\frac{1-\zeta}{\zeta}(G^{\frac{1}{\psi}} - \beta(\rho(1-\phi)-\eta)) + \frac{(1-s_c)}{\psi s_c}\frac{1-\rho}{\zeta}\right]},$$

where s_c is the share of consumption in GDP. Hence, $1 - s_c$ is the share of R&D spending in GDP. Variables are written in log-deviation from steady-state: $\widehat{c}_t \equiv log(c_t/c)$ and $\widehat{g}_{t+1} = log(G_{t+1}/G)$. The omission of the time subscript refers to deterministic steady-states. With $\psi = 1$ and $\phi = 0$ we obtain the formulas in FW (see section E.2 of their Appendix).

Expected consumption growth, $\Delta\widehat{c}^{total}_{t+1}$ can be decomposed into the growth rate of a cyclical term, $\Delta\widehat{c}_{t+1}$, and the trend growth rate, $\widehat{g}_{t+1}$:

$$\begin{aligned} \Delta\widehat{c}^{total}_{t+1} &= \Delta\widehat{c}_{t+1} + \widehat{g}_{t+1} \\ &= \gamma_c \Delta\widehat{z}_{t+1} + \gamma_g \widehat{z}_t \\ &= [-\gamma_c(1-\rho) + \gamma_g]\widehat{z}_t. \end{aligned} \tag{7}$$

For reasonable calibrations of the model $\gamma_c > 0$, $\gamma_g > 0$, and $\gamma_c > \gamma_g$. Since $0 < \rho < 1$, the cyclical term, $-\gamma_c(1-\rho)$ always has negative reaction to positive shocks, z_t. The trend growth term, γ_g, is always positive. As long as the shock is sufficiently persistent, $|\gamma_c(\rho - 1)| < \gamma_g$ i.e. the negative cyclical component, $\gamma_c(\rho - 1)$ is smaller in absolute value than the trend component, consumption growth, $\Delta\widehat{c}^{total}_{t+1}$, will be procyclical in the endogenous growth model.[4]

The bottom panel of Figure 2 shows the sensitivity of $-(1-\rho)\gamma_c + \gamma_g$ (and its components, $-(1-\rho)\gamma_c$ and γ_g in the top panel) to the persistence of the shock, ρ, the curvature of the investment function, ζ, patent obsolescence, ϕ, growth rate, g, IES, ψ, gross markup, ξ, and R&D spending-to-GDP ratio, $1 - s_c$. A higher ζ and ϕ increase hysteresis effects, and make the trend component, γ_g react more keenly to supply shocks. Notably, $-(1-\rho)\gamma_c + \gamma_g$ turns to positive for highly persistent shocks of $\rho > 0.95$: this is mainly because the negative cyclical component, $-(1-\rho)\gamma_c$ is shrinking for larger ρ. The trend component, γ_g is positive and slightly decreasing with ρ. For each parameter, the total effect, $-(1-\rho)\gamma_c + \gamma_g$ is governed by the trend component, γ_g. When IES is increasing, households view fluctuations in current total consumption less negatively, so the reaction of total consumption to

[4]The exogenous growth endowment model of Bansal et al. (2010) predicts rising consumption only for permanent technology shocks. In the endogenous growth model permanent shocks are not needed as the endogenous trend component picks up the role of permanent shocks.

shocks, $-(1-\rho)\gamma_c+\gamma_g$ also decreases. The total effect also increases with more investment spending, which enhances the trend component.

Insert Figure 2 here.

3 Calibration

We mainly follow the calibration strategy of FW and explain when we diverge from it (see calibration in Table 1). Time is in quarters. χ, α, and β are chosen to target three moments: i) consumption growth rate of 1.82 percent per annum (based on annual US data 1929-2017), ii) an R&D spending-to-GDP ratio of 2.46 (based on annual US data 1953-2017), and iii) a risk-free rate of 2.40 percent per annum, which is higher than 0.32 estimated from US data 1929-2017.[5] Hence, the resulting values for χ, α, and β are 1.4181, 0.1627, and 0.9955, respectively.

Insert Table 1 here.

Instead of setting a wedge η to increase the discount rate for profits, we use patent obsolescence, ϕ, which is another tool to induce a corporate discount rate of about 14 percent in line with post-war US data (see FW). Hence, we apply a quarterly obsolescence rate of 3.75 percent, which the BLS uses to calculate the stock of intangible capital in the US. Figure 3 shows that the price-consumption ratio, the risk-free rate and the risk-premia are more sensitive to ϕ than η.

Regarding the curvature of the innovation spending function ζ, FW argues that a value close to one might be the relevant choice. However, we stick to the estimate 0.83, used in KS. To be clear, a value of ζ closer to one would further raise risk-premia as it would make hysteresis effects stronger.

A risk-aversion of ten and an EIS of 1.85 mimics the choices of KS and are standard in the long-run risk literature (both are one in FW since they do not focus on the macro-financial implications of their model). The linearisation coefficient, κ_1 is close to one as in Campbell and Shiller (1988); see asset pricing section below.

[5] We could choose a higher β and achieve somewhat lower risk-free rate at the cost of having a growth rate below one percent per annum (half of the value in US data), which would also reduce the risk-premia below one percent (around six percent in US data)—see also the mid-panel of Figure 3 for the positive correspondence between the growth rate and the risk-free rate.

Unlike the crises-experiment-like calibration of FW, our macro-finance perspective requires more conservative calibration of the technology shock. In particular, we set $\rho = 0.9925$ and $\sigma = 0.0064$ to match the standard deviation of the business cycle frequency of consumption growth (cycles between 6 and 32 quarters isolated through the band-pass filter of Christiano and Fitzgerald, 2003), which are in the ballpark of the business cycle literature as well as KS. In our model the steady-state of the technology shock is fixed to one for simplicity, $Z = 1$. The AR(1) parameter and the size of the innovation in the stochastic volatility shock process—v_1 and σ_w, respectively—are reasonably close to the values in Bansal et al. (2010).

The leverage, ϕ_{lev} on equity premium is 1.67 as in KS. The net markup $\xi - 1$ is 20 percent, which is lower than the choice of KS.

4 Loglinear-lognormal asset pricing

Using the Campbell-Shiller (1988) log-linear approximation for the return on the consumption claim we write:

$$r_{c,t+1} = \kappa_0 + \kappa_1 p_{c,t+1} - p_{c,t} + \Delta c_{t+1}^{total} \quad (8)$$

where $r_{c,t+1} = log(R_{c,t+1})$ is the log return on the consumption claim and $p_{c,t+1} = log(P_{c,t+1})$ is the log of the price-consumption ratio. $\kappa_0 \equiv log(1 + exp(p_c)) - \kappa_1 p_c$ and $\kappa_1 \equiv exp(p_c)/(1+exp(p_c))$ are constants. Note that $\Delta c_{t+1}^{total} = \Delta c_{t+1} + g + g_{t+1}$, where $g_{t+1} = log(G_{t+1})$.

We guess that the solution for the price-consumption ratio takes the form of:

$$p_{c,t} = \eta_0 + \eta_1 z_t + \eta_2 \sigma_t^2 \quad (9)$$

where η_0, η_1 and η_2 are constants, which are determined below. Below we show that positive technology shocks raise asset prices, $\eta_1 > 0$, while a rise in uncertainty reduce asset valuations, $\eta_2 < 0$.

We assume that the pricing kernel and the return on the consumption claim are jointly conditionally log-normally distributed:

$$
\begin{aligned}
0 &= log(E_t[exp(m_{t+1} + r_{c,t+1})]) \qquad (10)\\
&= E_t[m_{t+1} + r_{c,t+1}] + \frac{1}{2} Var_t[m_{t+1} + r_{c,t+1}]\\
&= log(E_t[exp(\theta log(\beta) + \theta(1 - 1/\psi)\Delta c^{total}_{t+1} + \theta(\kappa_0 + \kappa_1 p_{c,t+1} - p_{c,t}))])\\
&= log(E_t[exp(const. + \theta(1 - 1/\psi)[\gamma_c(z_{t+1} - z_t) + \gamma_g z_t]\\
&\quad +\theta\kappa_1\eta_1 z_{t+1} - \theta\eta_1 z_t + \theta\kappa_1\eta_2\sigma^2_{t+1} - \theta\eta_2\sigma^2_t)])\\
&= const. + \theta(1 - 1/\psi)[\gamma_c(\rho - 1) + \gamma_g]z_t + \frac{1}{2}\theta^2(1 - 1/\psi)^2\gamma_c^2\sigma_t^2\\
&\quad +\theta\eta_1(\kappa_1\rho - 1)z_t + \frac{1}{2}\theta^2\kappa_1^2\eta_1^2\sigma_t^2 + \theta\eta_2(\kappa_1 v_1 - 1)\sigma_t^2 + \frac{1}{2}\theta^2\kappa_1^2\eta_2^2\sigma_w^2
\end{aligned}
$$

where $const. \equiv \theta[log(\beta) + \kappa_0 + \eta_0(\kappa_1 - 1) + (1 - 1/\psi)g]$ are constants, and $\theta \equiv (1-\gamma)/(1-1/\psi)$ appears due to Epstein-Zin preferences, which separates risk-aversion, γ from the inverse of the elasticity of intertemporal substitution (EIS) $1/\psi$. When $\theta = 1$ we obtain CRRA preferences with $\gamma = 1/\psi$ (assuming that $1/\psi \neq 1$).

The third row uses the return representation of the pricing kernel in the case of Epstein-Zin preferences ($M_{t+1} = \beta^\theta \left(\frac{c_{t+1}}{c_t} G_{t+1}\right)^{-\frac{\theta}{\psi}} r^{\theta-1}_{c,t+1}$). Further, we use the Campbell-Shiller log-linear approximation for the return on the consumption claim from equation (8), the expression for total consumption growth, Δc^{total}_{t+1}, and the price-consumption ratio from equations (7) and (9) respectively. In the last two rows, we have applied the log-normality assumption.

The last two rows in equation (10) imply the following exclusion restrictions (for constants, z_t-terms, and σ_t^2-terms, respectively):

$$
\eta_0 = \frac{log(\beta) + (1 - 1/\psi)g + \kappa_0 + \frac{1}{2}\theta\kappa_1^2\eta_2^2\sigma_w^2}{1 - \kappa_1}, \qquad (11)
$$

$$
\eta_1 = \frac{(1 - 1/\psi)[\gamma_c(\rho - 1) + \gamma_g]}{1 - \kappa_1\rho}, \qquad (12)
$$

$$
\eta_2 = \frac{1}{2}\frac{[(\theta - \theta/\psi)^2\gamma_c^2 + \theta^2\kappa_1^2\eta_1^2]}{\theta(1 - \kappa_1 v_1)}. \qquad (13)
$$

As long as the shock is sufficiently persistent, $|\gamma_c(\rho - 1)| < \gamma_g$ i.e. the positive trend component, γ_g outweighs the negative reaction of the cyclical component $|\gamma_c(\rho-1)|$. On the mid-left panel of Figure 3, one can see that η_1 is positive for highly persistent shocks only ($\rho > 0.95$), which are consistent with the estimates in the business cycle literature. Our calibration of $\psi > 1$ ensures that the price-consumption ratio responds positively to cyclical technology shocks $\eta_1 > 0$, which is a core assumption in the long-run risk literature. $\eta_2 < 0$ captures the negative effects of volatility on asset prices.

Insert Figure 3 here.

Note that the η coefficients are directly comparable to those in the exogenous growth literature with long-run risk. Notably, Bansal et al. (2010) reports a negative coefficient, $\eta_1 = \frac{(1-1/\psi)(\rho-1)}{1-\kappa_1\rho} < 0$ for $\psi > 1$ (see the second component of equation (6) in their paper). Hence in endowment (Bansal et al., 2010) or production models of exogenous growth where positive transitory shocks lead to a fall in the price-consumption ratio $\eta_1 < 0$ if $\psi > 1$. The sign and size of η coefficients are important for the risk-free rate and the equity premium (see equations (14) and (15) below, respectively).

Risk-free rate. To derive an expression for the risk-free rate recall log version of the return representation of the pricing kernel. In line with the strategy of de Groot et al. (2022) we substitute in for $r_{c,t+1}$ from equation (8) and impose the restrictions from equations (11) and (12) on the resulting equation. Applying a lognormality assumption for the pricing kernel we derive the risk-free rate as:

$$r_{f,t} = -log(\beta) + \frac{g}{\psi} - \frac{1}{\psi}[-\gamma_c(1-\rho) + \gamma_g]z_t \quad (14)$$
$$-\tfrac{1}{2}\gamma^2\gamma_c^2\sigma_t^2 + \frac{1}{2}(1/\psi - \gamma)(1-\gamma)\gamma_c^2\sigma_t^2 + \frac{1}{2}(\theta-1)\kappa_1^2\eta_1^2\sigma_t^2 + \frac{1}{2}\theta\kappa_1^2\eta_2^2\sigma_w^2$$

In the absence of uncertainty, $\sigma_t^2 = \sigma_w^2 = z_t = 0$, the risk free is the log of the growth-adjusted discount factor, $g/\psi - log(\beta)$. In the case of CRRA preferences, $\theta = 1 \Leftrightarrow 1/\psi = \gamma$, precautionary savings effect is $-\frac{1}{2}\gamma^2\gamma_c^2\sigma_t^2$, which is growing with risk-aversion γ, the sensitivity of cyclical consumption to technology γ_c, and the variance of the shock σ_t^2. Hence, the precautionary savings effect keeps the unconditional risk free rate below the steady-state risk-free rate. This is because the lognormal approximation allows for the variance of the shock—a measure of uncertainty. Higher uncertainty leads to precautionary savings, which drives down the risk-free rates.

The expression, $-\frac{1}{\psi}[-\gamma_c(1-\rho) + \gamma_g]z_t$, in the first row shows that the risk-free rate rises in response to negative temporary technology shocks as long as the shock is sufficiently persistent: $\rho > 0.95$. In particular, supply disruptions lead to persistently lower spending on innovation causing a shortage in demand which is greater than the shock itself (see also FW). The upshot is a rise in the risk-free rate, which is in contrast to the logic from exogenous growth models (see e.g. Bansal et al. 2010), where the risk-free rate rises in response to negative technological innovations since there is no trend component when only temporary shocks are considered.

With Epstein-Zin preferences, $\theta \neq 1 \Leftrightarrow 1/\psi \neq \gamma$ and $\psi > 1$ (see the second, third, and fourth terms in row two, which are negative in total), precautionary savings are stronger. The endogenous growth mechanism affects precautionary savings through η_1, which appear joint with Epstein-Zin preferences. Persistent low growth episodes in the endogenous growth model magnify precautionary savings effect as long as $\eta_1 > 0$. Stochastic volatility captured by the last term, $\frac{1}{2}\theta\kappa_1^2\eta_2^2\sigma_w^2$ is another source of uncertainty and, thus, has a negative influence on the risk-free rate.

Equity premium. The levered[6] excess return on the consumption claim is given by

$$\begin{aligned} \phi_{lev} logE_t[exp(r_{c,t+1} - r_{f,t})] &= \phi_{lev}[\ E_t r_{c,t+1} - r_{f,t} + \frac{1}{2}Var_t[r_{c,t+1}]] \\ &= -\phi_{lev}Cov_t(m_{t+1} - E_t m_{t+1}, r_{c,t+1} - E_t r_{c,t+1}) \\ &= \phi_{lev}[\gamma\gamma_c\kappa_1\eta_1\sigma_t^2 + (1-\theta)(\kappa_1^2\eta_1^2\sigma_t^2 + \kappa_1^2\eta_2^2\sigma_w^2)] \end{aligned} \tag{15}$$

In the first row $\frac{1}{2}Var_t[r_{c,t+1}]$ is the so-called Jensen-term that appears in lognormal approximations. The conditional covariance in the second row contains the innovation component of the pricing kernel $m_{t+1} - E_t m_{t+1}$ and of the return on the consumption claim $r_{c,t+1} - E_t r_{c,t+1}$ respectively (see the Online Appendix). In row three, the first term $\phi_{lev}\gamma\gamma_c\kappa_1\eta_1\sigma_t^2$ is due to CRRA preferences. The second term $\phi_{lev}(1-\theta)\kappa_1^2\eta_1^2\sigma_t^2$ arises due to Epstein-Zin preferences ($\theta \neq 1$), and raises the equity premium (even in the absence of stochastic volatility with constant variances). The last term, $\phi_{lev}(1-\theta)\kappa_1^2\eta_2^2\sigma_w^2$ appears due to stochastic volatility.

To understand the key role of patent obsolescence ϕ in macro-finance we contrast it with the discount rate wedge η. The macro literature considers ϕ and η to be very similar tools keeping the discount rate high and the

[6] We could have alternatively introduced a multiplicative constant leverage term for expected consumption growth in the log-linear approximation of the return in equation (8), as in Bansal et al. (2010).

growth rate low. Next, we show that patent obsolescence is more effective in matching a low growth rate and risk-free rate.

On the top panel of Figure 3, we assess the sensitivity of the price-consumption ratio η_1, the unconditional risk-free rate and the equity premium to the choice of either the patent obsolescence rate, ϕ or the discount rate wedge η. Figure 3 suggests that all three measures are much more sensitive to the choice of ϕ relative to η. Technically, ϕ reduces the growth rate and the positively-related risk-free rate directly through equation (1)—see also the mid-panel of Figure 3. With η, the risk-free rate and the growth rate are only indirectly-related by discounting the profits in equation (3).

In general, a positive choice of ϕ leads to higher precautionary savings through increased η_1, and helps to fit a low risk-free rate and a high equity premium. Positive patent obsolescence makes low-growth episodes more persistent, i.e. it increases the trend component of consumption γ_g (see the top panel of Figure 2) and magnifies long-run risks. In sum, we find that the patent obsolescence channel is essential for jointly matching risk-free rate and risk-premia.[7] The bottom panel shows the sensitivity of the price-consumption ratio, the risk-free rate and the equity premium to the R&D spending-to-GDP ratio $1 - s_c$. Higher investment spending raises the growth rate, the risk-free rate and the price-consumption ratio through the trend component (see also Figure 2). Whereas higher investment spending lowers the equity premium as the cyclical component of the equity premium (see part on Figure 3) shrinks more than the rise in the trend component (see Epstein-Zin part on Figure 3). Below we show that this prediction of the loglinear-lognormal model is reversed in the non-linearly solved model (see section 5), where investment spending entails risk correction so economies with higher R&D spending-to-GDP ratio exhibit higher equity premia at the stochastic steady-state.

5 Results

In this section, we calculate macroeconomic and financial moments based on the loglinear-lognormal model solution. For robustness, we also include

[7] The Online Appendix further shows that risk-premia increase with the IES and the markup. This tells us that the direct positive influence of a higher IES on η_1 (see IES$=1/\psi$ in the nominator of equation (12)) is stronger than the negative indirect influence that the IES has on consumption growth (see Figure 2).

a numerical solution of the model with third-order perturbation, which the computational literature (see e.g. Caldara et al. (2012)) finds to be highly accurate. We consider third-order perturbation for two reasons. First, the loglinearly-solved macroeconomic model did not include Epstein-Zin curvature to facilitate the closed-form solution (except the asset pricing part, which is solved with the assumption of lognormality). Second, Pohl et al. (2018) has shown inaccuracies of the solution when the Campbell-Shiller loglinear approximation is used for the return on the consumption claim.

In Table 2 we report moments of the R&D spending-to-GDP ratio, consumption growth, risk-free rate, equity premia, and price-consumption ratio. We report US empirical moments for the so-called long-sample, including pre-WWII data (1929-2017), as well as the short-sample, excluding pre-WWII data (in our case 1953-2017). The starting date of the short sample follows from the availability of the R&D investment spending data for the US. The long-sample implies a higher standard deviation of consumption growth and lower standard deviation of the risk-free rate. In the absence of physical capital our model better matches moments of the long-sample similar to Bansal et al. (2010).

Insert Table 2 here.

We also report simulated moments from the benchmark models solved in two ways: loglinear-lognormal (see columns LL) and fully non-linear with third-order perturbation (see columns NL). In addition, we consider models including stochastic volatility (see columns titled LL+SV and NL+SV). In columns 1-4, the deterministic steady state is the same across models and solution methods.

In line with KS we assess how the model matches business cycle and low frequencies of the US data. For the standard deviations of consumption growth, risk-free rate, equity return and price-consumption ratio, we report both business cycle (cycles between 6 and 32 quarters, denoted as (bc)) and growth cycle (cycles between 100 and 200 quarters as defined in KS, denoted as (gc)) frequencies. We have chosen the standard deviation of the technology shock such that it closely matches the standard deviation of consumption growth at the business cycle frequency over 1929-2017.

The third-order perturbation solution NL includes higher-order terms implying positive risk-correction. Hence, the NL solution better matches the standard deviation of the return on equity. Due to the positive risk-correction, the unconditional mean of the R&D spending-to-GDP ratio as well as the unconditional mean of consumption growth in the stochastic steady-state is somewhat in excess of the empirical moments. The discrepancy between the moments at the stochastic (also called the ergodic mean of the distribution) and the deterministic steady-state can be traced back to the non-linearities of the model, which are reflected by the third-order perturbation solution.

Adding time-varying and stochastic volatility to the models (see column titles including SV) improves the fit of the model to financial data moments. In particular, stochastic volatility raises standard deviation of the return on equity as it implies more uncertainty in growth prospects. Higher uncertainty due to stochastic volatility implies a rise in precautionary savings, which pushes the risk-free rate below the value of the benchmark model in the case of the LL solution[8]). The model with stochastic volatility offers more reasonable prediction for the standard deviation of equity in the setting with LL solution (about sixteen in the model versus nineteen in the data) relative to the NL solution, which overshoots the empirical value (thirty-five versus nineteen). Notably, the model with SV can capture the full variation in the price-consumption ratio (including all frequencies).

In the last column entitled NL1, we also consider a robustness check where we set the discount factor such that the model exactly matches $E[s/y]$ at the stochastic steady-state. In this scenario the mean and standard deviation of the equity premium calculated from simulated data significantly falls short of the empirical moments. The comparision of column 2 and 5 reveals that a 0.13 percentage point increase in the R&D spending-to-GDP ratio raises consumption growth by 0.71 percentage points and quadruples the mean of the equity premia (from 31 to 130 basis points). Hence, economies with more R&D activity require higher excess returns for the risks associated with investment projects.

[8]This is not true for the NL and NL+SV cases due to the positive risk-correction induced by the third-order solution. In the absence of further frictions such production capital with adjustment costs, the tight positive comovement between the growth rate and the risk-free rate in the stochastic steady-state dominates the negative influence of the precautionary savings effect on the risk-free rate.

6 Conclusion

We show approximate closed-form solution for asset prices in a simplified endogenous growth model. Our closed-form solution method makes it clear that asset prices rise when the positive trend component dominates the negative cyclical component. We point to the importance of patent obsolescence in keeping the growth rate and risk-free rate low. A solution of the model with third-order perturbation, and the extension with stochastic volatility helps better explain some key financial moments such as the volatility of the price-consumption ratio as well as the mean and variance of the equity premium.

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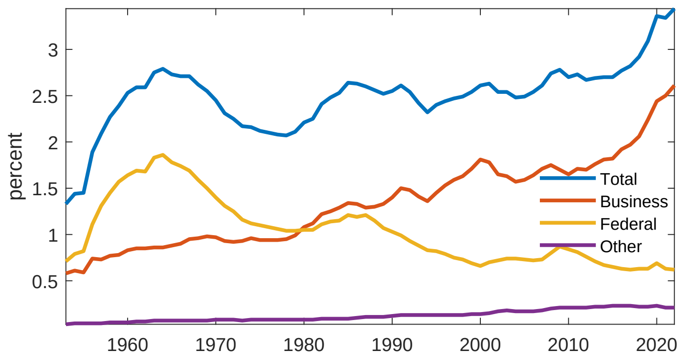
7 Figures and Tables

Table 1: Benchmark Calibration

γ	10	χ	1.4181	ζ	0.83	α	0.1627	ρ	0.992	ϕ_{lev}	1.67
ψ	1.85	β	0.9955	κ_1	0.995	ϕ	0.0375	σ	0.0064	ξ	1.20

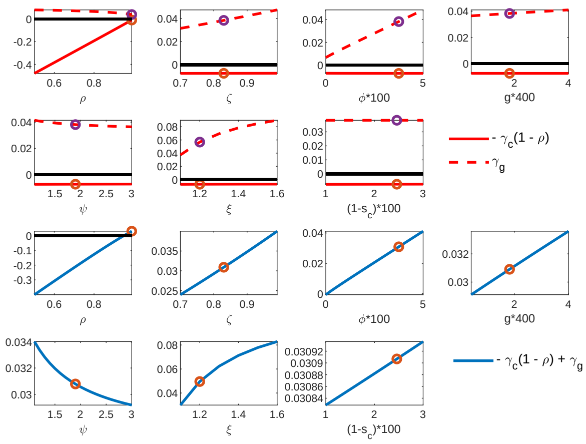
Notes: we parameterise the persistence of the shock ρ, the curvature of innovation technology ζ, the patent obsolescence ϕ, the risk-aversion γ, the elasticity of intertemporal substitution ψ, the gross markup ξ, and the linearisation constant κ_1 in the Campbell-Shiller equation, and the leveraging factor ϕ_{lev} based on previous estimates. The discount factor β, the share of intermediate inputs in production, α the level parameter of the innovation technology χ, and the size of the shock σ are chosen to target the following four moments: the mean and standard deviation of consumption growth rate, 1.82 and 1.34 percent per annum respectively, at the business cycle frequency of US data 1929-2017; the R&D spending-to-GDP ratio of 2.46 percent (data is only available from 1953); and the risk-free rate of 2.40 percent per annum. We also consider an extension of the benchmark model with stochastic volatility. The AR(1) and the shock-size parameters of the time-varying volatility process are given by $v_1 = 0.995$, and $\sigma_w = 0.0008\%$, respectively.

Figure 1: Research and Development (R&D) spending (total and main components) as a share of US Gross Domestic Product (GDP).



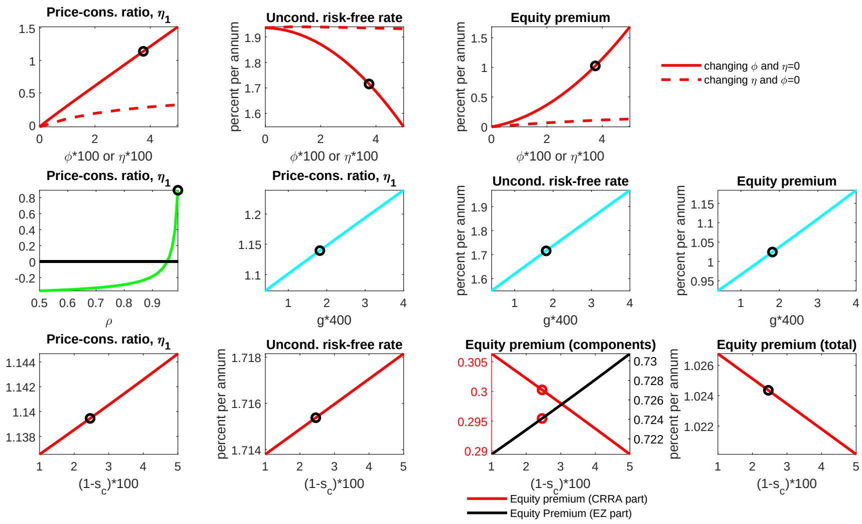
Source: NCSES. Publication ID NSF 24-318.

Figure 2: the sensitivity of the cyclical and trend components of consumption growth to the parameters of the model, the growth rate, and the R&D spending–to–GDP ratio.



Notes: circles denote the benchmark parametrisation in Table 1. The horizontal black line marks the zero on the vertical axis. The cyclical and trend components of expected consumption growth as a function of the technology shock are given by $-(1-\rho)\gamma_c$ and γ_g respectively. Parameters include the persistence of the shock ρ, the curvature of innovation technology ζ, the patent obsolescence ϕ, the growth rate g, the elasticity of intertemporal substitution ψ, the gross markup ξ, and the R&D spending-to-GDP ratio $1-s_c$.

Figure 3: the sensitivity of the price-consumption ratio, the risk-free rate, and the equity premium to the parameters, the calibrated growth rate, and the R&D spending-to-GDP ratio (based on the loglinear-lognormal solution in the absence of stochastic volatility)



Notes: circles denote the benchmark parametrisation in Table 1. The horizontal black line (middle left) marks the zero on the vertical axis. Parameters include the patent obsolescence ϕ, the discount rate wedge η, the persistence of the shock ρ, the growth rate g, and the R&D spending-to-GDP ratio $1 - s_c$. In the bottom panel the component of the equity premium is measured on the left-axis (red) while the component linked to Epstein-Zin curvature (EZ part) is displayed on the right-axis (black).

Table 2: Empirical and Simulated Moments

	US data 1929-2017	US data 1953-2017	LL (1)	LL+SV (2)	NL (3)	NL+SV (4)	NL1 (5)
$E[s/y]$	2.46*	2.46	2.46	2.46	2.59	2.66	2.46
$E[\Delta c^{total}]$	1.82	1.92	1.82	1.82	2.48	3.05	1.77
$AC1(\Delta c^{total})$	0.48	0.46	–	–	0.08	0.37	0.05
$\sigma(\Delta c^{total})$	2.10	1.21	0.65	0.65	2.65	4.09	2.63
$\sigma(\Delta c^{total})(bc)$	1.34	0.85	–	–	1.34	1.67	1.35
$\sigma(\Delta c^{total})(gc)$	0.81	0.25	–	–	0.32	0.66	0.30
$E[r_f]$	0.32	0.86	1.72	1.38	2.45	2.57	2.40
$\sigma(r_f)$	2.78	1.62	0.35	0.41	1.02	1.52	0.98
$\sigma(r_f)(bc)$	1.54	0.69	–	–	0.13	0.19	0.12
$\sigma(r_f)(gc)$	1.30	0.53	–	–	0.23	0.32	0.23
$\phi_{lev}E[r_c - r_f]$	5.81	5.93	1.02	2.22	1.30	1.43	0.31
$\sigma(r_c)$	19.49	17.36	5.37	6.55	15.16	35.19	5.54
$\sigma(r_c)(bc)$	17.03	15.86	–	–	5.19	11.26	2.75
$\sigma(r_c)(gc)$	4.74	4.86	–	–	4.18	5.16	4.10
$E[pc]$	3.41	3.57	–	–	6.30	6.43	6.21
$\sigma(pc)$	0.46	0.39	0.06	0.10	0.14	0.53	0.06
$\sigma(pc)(bc)$	0.14	0.11	–	–	0.02	0.06	0.01
$\sigma(pc)(gc)$	0.20	0.21	–	–	0.04	0.11	0.02
$AC1(pc)$	0.87	0.87	–	–	0.99	0.99	0.99

Notes: $E[.]$, $\sigma(.)$, AC1(.) means unconditional mean, standard deviation, and first-order autocorrelation. Variables reported are the following: the R&D spending-to-GDP ratio s/y, the consumption growth including both cyclical and trend components Δc^{total}, the risk-free rate r_f, the return on the consumption claim r_c, and the price-consumption ratio p_c. *US data for s/y is only available from 1953. In the title of the columns, LL refers to the model solved with loglinear-lognormal (LL) assumption. LL+SV is the model with stochastic volatility (SV) solved in the LL-way. NL is the model solved non-linearly solved with third-order perturbation simulated 100 times for 10000 periods. The column entitled NL+SV refers to the model with stochastic volatility solved in the NL-way. In each of the NL solutions, we assume that the stochastic discount factor in the macroeconomic part of the model contains Epstein-Zin curvature (see equation (3)) . The deterministic steady-state is the same across columns (1)-(4). For NL1 in column (5), we set $\beta = 0.99615$ to achieve $E[s/y] = 2.46$. For the standard deviations, we also report the statistics for the business cycle frequency, (bc) for cycles between 6-32 quarters and, for the growth cycle window, (gc), the cycles between 100-200 quarters. We isolate these frequencies using the band-pass filter of Christiano and Fitzgerald (2003). In the data, the levered return on the consumption claim $\phi_{lev}E[r_c - r_f]$ corresponds to the excess return on dividends. We use CRSP data to calculate the annualised return on dividend. Risk-free rate refers to the estimated annualised real return on 3-month US Treasury Bills (see more in the Online Appendix). Dividends are calculated as the difference between the cum-dividend and ex-dividend returns. The price-consumption ratio is captured by the price-dividend ratio in the data.