Full Surplus Extraction from Colluding Bidders

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August 25, 2024

Introduction

- **Can Seller effectively fight collusion among Buyers?**
- Infinitely repeated first-price auctions.
  - Seller sets dynamic reserve prices without long-term commitment.
  - Buyers are patient.
  - Buyers are privately informed about their willingness-to-pay.

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- Infinitely repeated first-price auctions.
  - Seller sets dynamic reserve prices without long-term commitment.
  - Buyers are patient.
  - Buyers are privately informed about their willingness-to-pay.
- **Yes, Seller can get as much revenue as without collusion!**

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- **Why can Buyers collude?**
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  - Public disclosure of bids facilitates collusion.
- **Why is collusion hard to fight?**
  - Seller faces uncertainty both about Buyers' willingness-to-pay and the details of Buyers' collusive scheme $\Rightarrow$ collusion is hard to detect.

What do I do?

- **Introduce collusive equilibria**.
  - Equilibrium is collusive if, given Seller's equilibrium strategy, Buyers play the best equilibrium in the corresponding reduced game among themselves.

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- **Construct a collusive equilibrium that allows Seller to extract (almost) full surplus from patient Buyers.**

$\Rightarrow$ There is an effective strategy for fighting collusion.

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  - Even though Seller has to publicly disclose bids.
- **Full surplus extraction is not implied by existing folk theorems!**

Model: setup

Seller (player 0) and $n \geq 2$ Buyers, interact over $T = \infty$ periods.

- Seller offers one unit of a good in every period.
- Seller's valuation is 0.
- Seller's discount factor is δ_0.

- Buyers demand a new unit in every period.
- Buyers' valuations: binary ($\overline{\theta} > \underline{\theta}$); *iid* across time and Buyers.
  - $\mathbb{P}[\ \underline{\theta}\ ] = q$.
  - Buyers are privately informed about their valuations.
- Buyers' common discount factor is $\delta \geq \delta_0$.

Model: timing

In every period:

1. Seller announces reserve price r.
2. Buyers privately learn their valuations.
3. Buyers bid/abstain in the first-price auction with reserve price r.
4. Bids and/or abstentions are publicly disclosed.

Roadmap

1. Collusive public perfect equilibrium
2. Full surplus extraction
3. Concluding remarks

Collusive public perfect equilibrium: motivation

- **One-shot equilibrium** (q is high, i.e. many low types).

$$r^*_{os} = \underline{\theta}, \quad \underline{b}^*_{os} = \underline{\theta}, \quad \overline{b}^*_{os} \text{ mixed on } (\underline{\theta}, \cdot]$$

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- **Repetition of the one-shot equilibrium is "non-collusive"**.
  - Buyers can collude by playing $\underline{b} = \emptyset$, $\overline{b} = \underline{\theta}$.
  - Outcome can be supported by a grim-trigger strategy for high δ's.

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- **Rule out this and other non-collusive equilibria.**

Collusive public perfect equilibrium: definition

1. **Strongly symmetric public perfect equilibrium (SSPPE)**.
   - PPE $\approx$ Analog of SPE in games with imperfect public monitoring.
   - Strongly symmetric $=$ symmetric on and off equilibrium path.

SSPPE formal

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1. **Strongly symmetric public perfect equilibrium (SSPPE).**
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   - Strongly symmetric = symmetric on and off equilibrium path.

   SSPPE formal

2. **Given Seller's equilibrium strategy, Buyers collude.**
   - Seller's strategy induces a buyer-game.
   - Buyer-game is a *stochastic* game between Buyers in which reserve prices are set according to Seller's strategy.
   - Buyers cannot gain by choosing another strongly symmetric public perfect equilibrium in the induced buyer-game.

   Collusive formal

Full surplus extraction

Construct a collusive public perfect equilibrium, in which Seller extracts full surplus as $\delta \to 1$.

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- 3 (on path) $\times$ 2 (off path) $=$ 6 cases depending on parameter values.
  - On-path: stationary and separating ($\overline{b}^* > \underline{b}^*$) in all 3 cases.
  - Off-path: (i) **pooling** and (ii) separating.

Full surplus extraction

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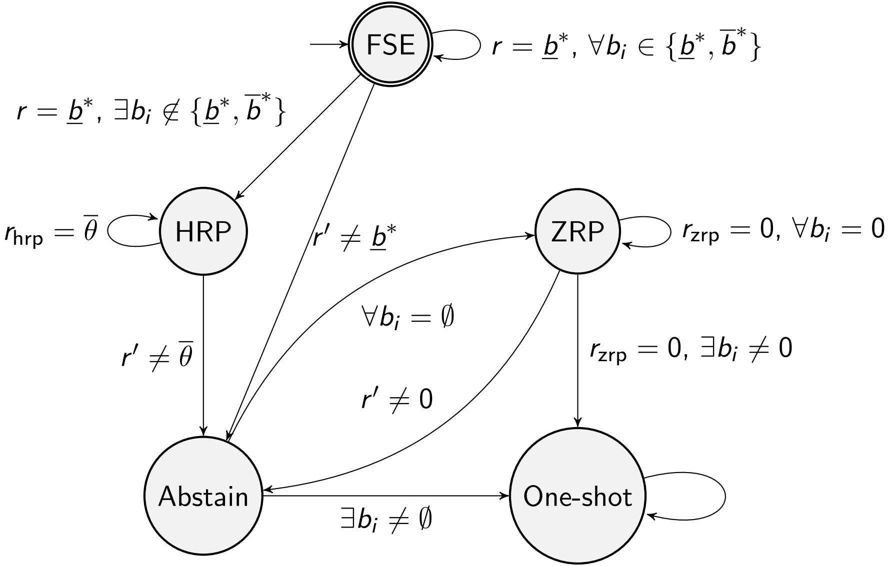
- 3 (on path) $\times$ 2 (off path) $=$ 6 cases depending on parameter values.
  - On-path: stationary and separating ($\overline{b}^* > \underline{b}^*$) in all 3 cases.
  - Off-path: (i) **pooling** and (ii) separating.
- Off-path collusive public perfect equilibria (**pooling case**):
  - ZRP: *zero-revenue pooling* (pool at $\underline{b} = 0$) (*to punish Seller*).
  - HRP: *high-reserve-price* ($r = \overline{\theta}, v = 0$) (*to punish Buyers*).

Full Surplus Extraction (FSE) equilibrium, illustration



Buyers' incentive compatibility constraints

$$\text{(LowIC)} \quad \underbrace{(1-\delta)\frac{q^{n-1}}{n}(\underline{\theta} - \underline{b}^*) + \delta v^*_{\text{fse}} \geq 0}_{\text{Low type eq. } \geq \text{ Low type deviates to } \emptyset},$$

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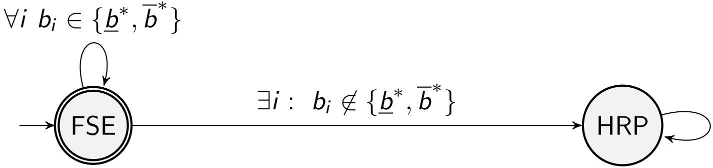
$$\text{(HighIC-on-sch)} \quad \underbrace{(1-\delta)\frac{1-q^n}{n(1-q)}(\overline{\theta} - \overline{b}^*) + \delta v^*_{\text{fse}} \geq (1-\delta)\frac{q^{n-1}}{n}(\overline{\theta} - \underline{b}^*) + \delta v^*_{\text{fse}}}_{\text{High type eq.} \geq \text{Mimic low type}}.$$

Buyer-game and Collusiveness constraints



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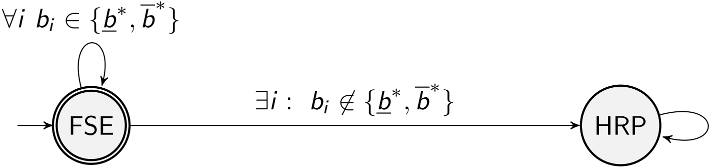

Collusiveness: $v^*_{fse} \geq \underbrace{\sup {\color{magenta}v'} | {\color{magenta}v'} \text{ is a SSPPE payoff in the Buyer-game}}_{\text{Optimal buyer-equilibrium problem}}.$

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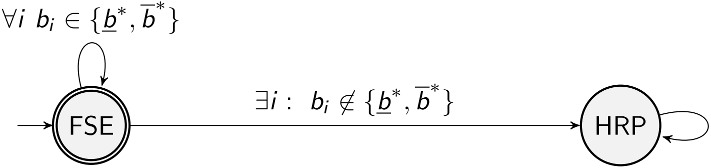

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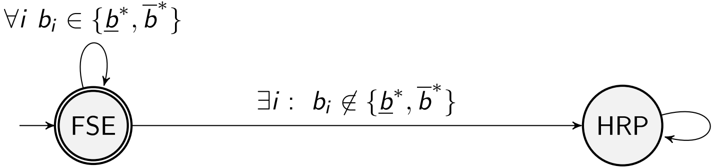
Optimal buyer-equilibrium problem: solution approach.

- Lemma: any buyer-equilibrium is monotonic $(\overline{b}' \geq \underline{b}'$ or $\underline{b}' = \emptyset)$.
- Relaxed problem: maximize v' over monotonic bidding profiles.
  - MDP $\Rightarrow$ stationary bidding profiles are w.l.o.g.

Buyer-game and Collusiveness constraints



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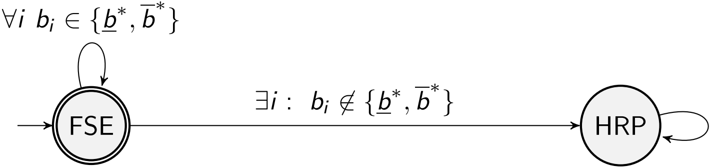

$$\text{(Col-sep-1)} \quad v^*_{\text{fse}} \geq v'_1 = \underbrace{(1-\delta)\frac{1}{n}\Big[(1-q^n)(\overline{\theta} - \underline{b}^*) + q^n 0\Big] + \delta(1-q)^n v'_1}_{\text{High types bid } \underline{b}^*\text{, Low types abstain } \emptyset},$$

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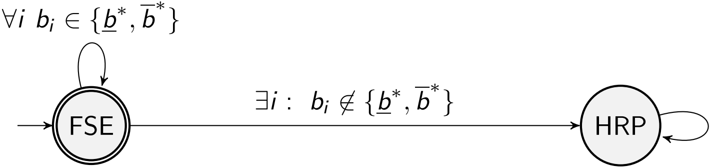

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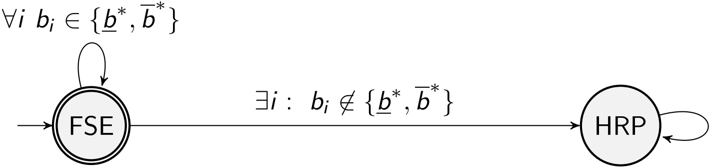

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$$\text{(Col-pool)} \quad v^*_{\text{fse}} \geq v'_{\text{p}} = \underbrace{(1-\delta)\frac{1}{n}\left[(1-q)(\overline{\theta}-\underline{b}^*)+q(\underline{\theta}-\underline{b}^*)\right]+\delta v'_{\text{p}}}_{\text{Both types pool at } \underline{b}^*}.$$

Revenue maximization problem

$$\mathcal{RM}: \quad (\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}) \in \arg\max_{\overline{b}, \underline{b}, v} \text{ Revenue}, \quad \text{s.t.}$$

(i) Incentive compatibility,

(ii) Collusiveness.

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Lemma

$\left(\overline{b}^*, \underline{b}^*, v^*_{\text{fse}}\right)$ *defines a collusive public perfect equilibrium of the repeated auction game for high enough values of* δ*.*

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Lemma

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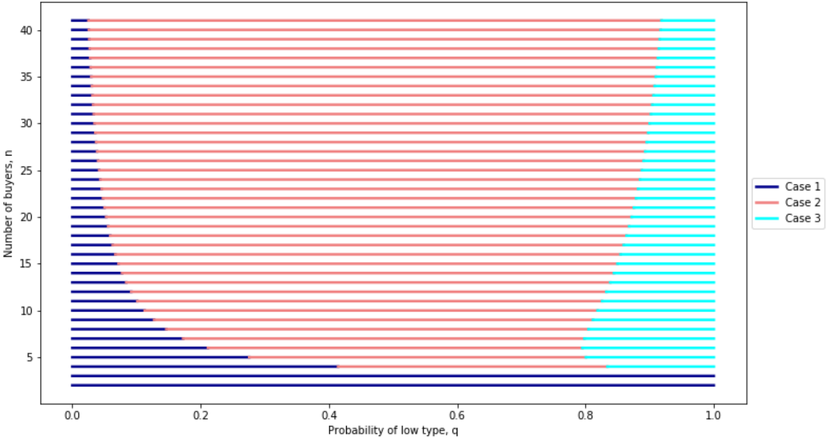
- **Solve $\mathcal{RM}$ by identifying binding constraints (3 on-path cases).**
  - Relax $\mathcal{RM}$ $\rightarrow$ show relaxed dual is feasible $\rightarrow$ check remaining con's.
- **Show $v^*_{\text{fse}}(\delta) \xrightarrow[\delta \to 1]{} 0$, which $\Rightarrow$ full surplus extraction as $\delta \to 1$.**

Solution to $\mathcal{RM}$: 3 parameter regions



Solution to $\mathcal{RM}$: 3 cases

- **Case 1: High expected valuation (low q).**

$$\text{(LowIC)} \quad \text{Low type eq.} = \text{Low type deviates to } \emptyset = 0,$$
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- **Case 2: Medium expected valuation (medium q)**.

$$\text{(HighIC-up)} \quad \text{High type eq.} = \text{High type deviates to } \overline{b}^* + \epsilon,$$
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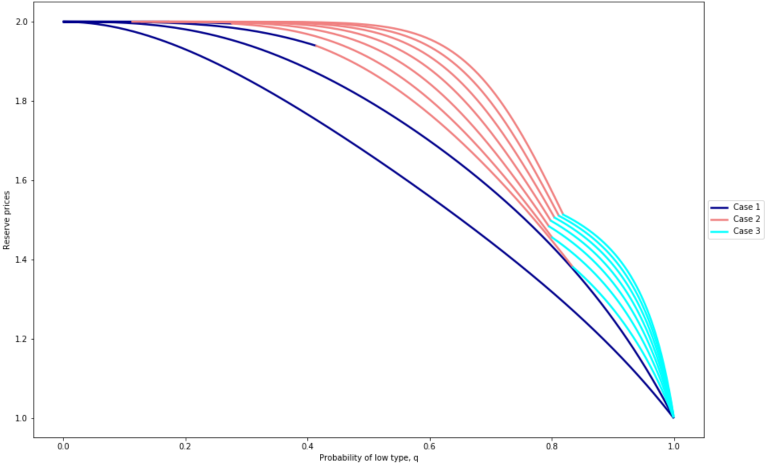
- **Case 3: Low expected valuation (high q).**

$$\begin{aligned} &\text{(HighIC-up)} && \text{High type eq.} \;=\; \text{High type deviates to } \overline{b}^* + \epsilon, \\ &\text{(HighIC-down)} && \text{High type eq.} \;=\; \text{High type deviates to } \underline{b}^* + \epsilon. \end{aligned}$$

Solution to $\mathcal{RM}$: $\lim_{\delta\to 1} \underline{b}^*(q,n,\delta)$

Parameters: $\underline{\theta} = 1$, $\overline{\theta} = 2$, $n \in \{2, 3, \ldots, 10\}$.



Pattern of binding constraints

Concluding remarks

- Repeated first-price auction game with a strategic Seller.
- Seller uses reserve prices to counteract Buyers' collusion.
- *Collusive public perfect equilibrium*.
- A collusive *PPE* that allows Seller to extract full surplus as $\delta \to 1$.

$\Rightarrow$ Seller can successfully fight collusion using dynamic reserve prices.

Appendix

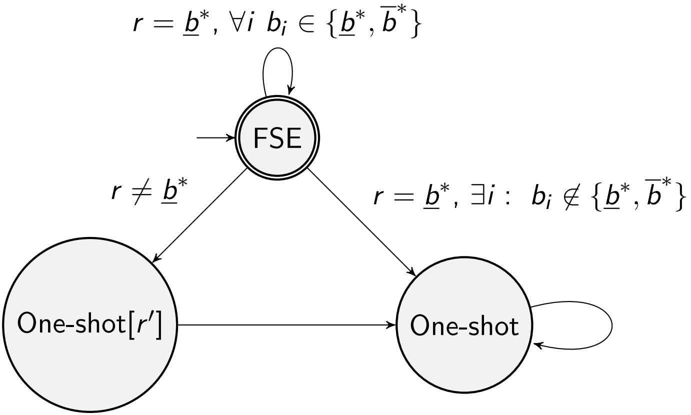
Roadmap

Literature

- Reserve price as anti-collusion device: Thomas (2005); Zhang (2021); Iossa, Loertscher, Marx and Rey (2022).
- Stage game design: Abdulkadiroglu and Chung (2004).
- Collusion detection in auctions with adaptive bidders: Chassang, Kawai, Nakabayashi and Ortner (2022a, 2022b, 2022c).
- Repeated games/oligopolies/auctions: Abreu, Pearce and Stachetti (1990); Fudenberg, Levine and Maskin (1994); Athey, Bagwell and Sanchirico (2004); Skrzypacz and Hopenhayn (2004); ...
- Dynamic Mechanism Design: Pavan, Segal, and Toikka (2014); ...

Appendix

Full surplus extraction (FSE) equilibrium, High-reserve-price region, illustration



- One-shot: $r^*_{os} = \overline{\theta}$.
- One-shot[r']: low types abstain, high types mix on $[r', \cdot]$.

Appendix

Zero-revenue pooling equilibrium, proof sketch

- Seller has no profitable deviation.
- Buyers' off-schedule deviations:

$$\underbrace{\delta\frac{1}{n}\left[(1-q)\overline{\theta}+q\underline{\theta}\right]}_{\text{High type abstains}} \geq \underbrace{(1-\delta)\overline{\theta}+\delta(1-q)q^{n-1}(\overline{\theta}-\underline{\theta})}_{\text{High type deviates to } \epsilon},$$

unprofitable for:

$$\delta \geq \frac{n\overline{\theta}}{n\overline{\theta}+q\underline{\theta}+(1-q)\overline{\theta}-n(1-q)q^{n-1}(\overline{\theta}-\underline{\theta})}.$$

Appendix

1. Strongly symmetric public perfect equilibrium

- **Public histories**:

  Seller: $\mathcal{H}_0 \ni h_0^{t+1} = \big(\varnothing, (r^0, b_1^0, ..., b_n^0), \ldots, (r^t, b_1^t, ..., b_n^t)\big).$

  Buyers: $\mathcal{H} \ni h^{t+1} = \big(\varnothing,$ —"— $, \ldots ,$ —"— $, r^{t+1}\big).$

- **Public strategies**:

  Seller: $\sigma_0 : \mathcal{H}_0 \to \mathbb{R}_+.$

  Buyers: $\sigma_i : \mathcal{H} \times \Theta \to \{\emptyset\} \cup \mathbb{R}_+.$

Definition

- A public strategy profile is a **public perfect equilibrium** if it induces a Nash equilibrium after any public history.
- A public perfect equilibrium is **strongly symmetric** if Buyers adopt symmetric bidding profiles on and off equilibrium path.

- Focus on pure strategies.

Back to Informal

Appendix

2. Buyer-game: preliminaries

Definition

- A public history in $\mathcal{H}_0$ is called σ_0-**consistent** if it is consistent with Seller's play of public strategy σ_0. Formal
- Two σ_0-consistent histories are called σ_0-**equivalent** if they prescribe the same Seller's continuation play according to σ_0.

- **Set of equivalence classes** $\equiv$ **set of states of the Buyer-game**.
- r : **maps states into reserve prices**.

  For any history h_0 from state ω, we have $r(\omega) = \sigma_0(h_0)$.
- τ : **defines state transitions**. For a bid profile b, $\tau(\omega, b) = \omega'$ if

  For any history h_0 from state ω, history $(h_0, r(\omega), b)$ is in state ω'.

2. Buyer-game: definition

Definition

The **buyer-game induced by** σ_0 is a stochastic game where:

- Players: Buyers.
- Actions: *same as in full repeated auction game.*
- States: classes of σ_0-equivalent histories.
- State transitions occur according to τ.
- Set of valuations: *same as in full repeated auction game*.
- Utility functions:

$$\tilde{u}_i(\omega, b, \theta_i) = \begin{cases} \frac{1}{\#(\text{win})}(\theta_i - b_i), & \text{if } b_i \geq r(\omega) \ \& \ \big(b_i = \max\{b\} \text{ or } b_{-i} = \emptyset\big) \\ 0, & \text{otherwise.} \end{cases}$$

2. Buyer-game: equilibria

Public hist.: $\mathbf{H}(\sigma_0) \ni \mathbf{h}^{t+1} = \left(\omega^0, (b_1^0, ..., b_n^0), \ldots, \omega^t, (b_1^t, ..., b_n^t),\ \omega^{t+1}\right)$

Public Strat.: $\rho_i : \mathbf{H}(\sigma_0) \times \Theta \to \{\emptyset\} \cup \mathbb{R}_+$.

Definition

A public strategy profile $(\rho_1^*, \ldots, \rho_n^*)$ is a **strongly symmetric public perfect equilibrium of the buyer-game** induced by σ_0 if

1. It induces a Nash equilibrium after every public history in $\mathbf{H}(\sigma_0)$.
2. Buyers use strongly symmetric strategies, i.e. $\rho_i^*(\mathbf{h}, \cdot) = \rho_j^*(\mathbf{h}, \cdot)$ after every public history $\mathbf{h} \in \mathbf{H}(\sigma_0)$ for any two buyers i, j.

Back to Informal

Appendix

Collusive public perfect equilibrium

Definition

A public strategy profile $(\sigma_0^*, \sigma^*, ..., \sigma^*)$ is a **collusive (on-path) public perfect equilibrium** of the repeated auction game if

1. It is a strongly symmetric public perfect equilibrium.
2. There is no strongly symmetric public perfect equilibrium in the buyer-game induced by σ_0^*, whose equilibrium payoff exceeds the buyer payoff from $(\sigma_0^*, \sigma^*, ..., \sigma^*)$ in the repeated auction game.

Back to Informal

Appendix

σ_0-consistent histories

- A typical time-t history consistent with the seller's play according to σ_0 is:

$$h_0^t = \Big(\varnothing,\ (\sigma_0(\varnothing),\ b^0),\ (\sigma_0(h_0^0),\ b^1),\ \ldots,\ (\sigma_0(h_0^{t-2}),\ b^{t-1})\Big),$$

where

$$h_0^0 = (\sigma_0(\varnothing),\ b^0),$$

$$h_0^1 = \Big((\sigma_0(\varnothing),\ b^0),\ (\sigma_0(h_0^0),\ b^1)\Big),$$

$$\ldots,$$

$$h_0^{t-1} = \Big((\sigma_0(\varnothing),\ b^0),\ (\sigma_0(h_0^0),\ b^1),\ \ldots,\ (\sigma_0(h_0^{t-3}),\ b^{t-2})\Big).$$

Go back

Appendix

Pattern of binding constraints

LowIC, HighIC-up, HighIC-down, Col-sep-1 determine the eq. bids.

- For every n: LowIC binds for low q, HighIC-up binds for high q,
- For every n: Col-sep-1 binds for low q, HighIC-down binds for high q.

Pattern of binding constraints

LowIC, HighIC-up, HighIC-down, Col-sep-1 determine the eq. bids.

- For every n: LowIC binds for low q, HighIC-up binds for high q,
- For every n: Col-sep-1 binds for low q, HighIC-down binds for high q.

Col-sep-2, Col-pool, HighIC-on-sch don't bind for any q, n.

LowIC vs. HighIC-up

LowIC & HighIC-up are con's on ex post reward ratios:

(LowIC) $$\frac{\underline{\theta} - \underline{b}}{\overline{\theta} - \overline{b}} \geq \frac{0 - \frac{\delta}{1-\delta}(1-q)\,\mathbb{P}\left(\text{win}\middle|\overline{\theta}\right)}{\mathbb{P}\left(\text{win}\middle|\underline{\theta}\right) + \frac{\delta}{1-\delta} q\,\mathbb{P}\left(\text{win}\middle|\underline{\theta}\right)} = \underline{R}_{\mathrm{L}}(\delta, q, n),$$

(HighIC-up) $$\frac{\underline{\theta} - \underline{b}}{\overline{\theta} - \overline{b}} \geq \frac{1 - \left[\mathbb{P}\left(\text{win}\middle|\overline{\theta}\right) + \frac{\delta}{1-\delta}(1-q)\,\mathbb{P}\left(\text{win}\middle|\overline{\theta}\right)\right]}{\frac{\delta}{1-\delta} q\,\mathbb{P}\left(\text{win}\middle|\underline{\theta}\right)} = \underline{R}_{\mathrm{H}}(\delta, q, n).$$

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- $\underline{R}_{\text{L}}(\delta,q,n)$ & $\underline{R}_{\text{H}}(\delta,q,n)$ both go to $-\frac{1-q^n}{q^n}$ as $\delta \to 1$, but for every n

  $\underline{R}_{\text{L}}(\delta,q,n) < \underline{R}_{\text{H}}(\delta,q,n)$ for any high q,δ; and vice versa.

  $\Rightarrow$ **HighIC-up must be binding for high q,δ.**

Illustration of $\underline{R}_{\mathrm{L}}(\delta, \frac{1}{5}, 4)$ and $\underline{R}_{\mathrm{H}}(\delta, \frac{1}{5}, 4)$

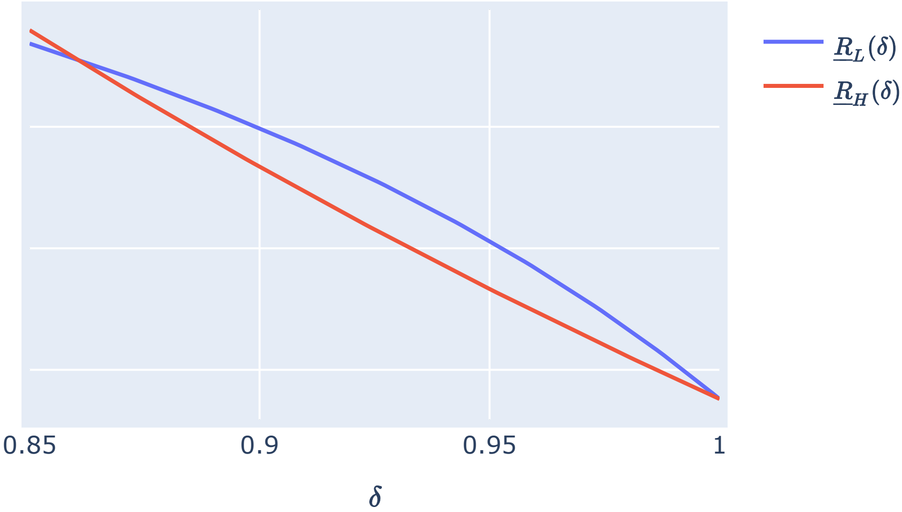


Illustration of $\underline{R}_{\mathsf{L}}(\delta, \frac{1}{4}, 4)$ and $\underline{R}_{\mathsf{H}}(\delta, \frac{1}{4}, 4)$

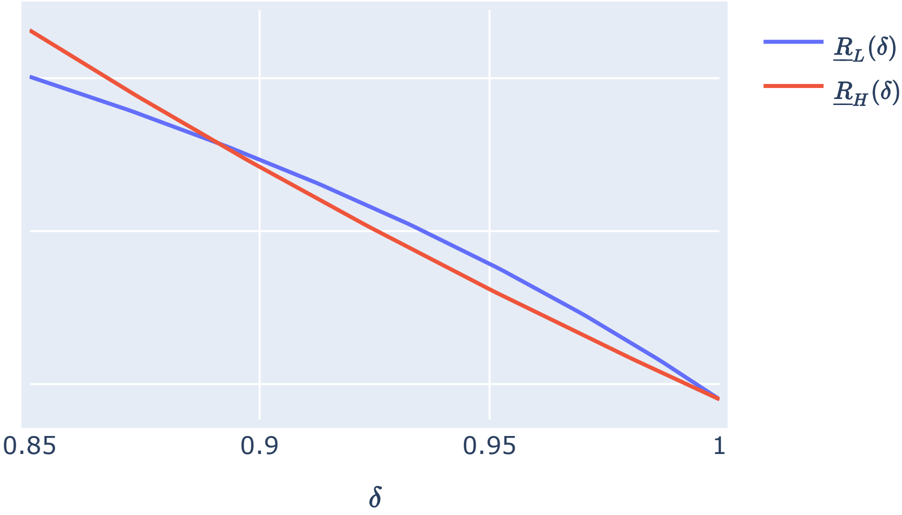
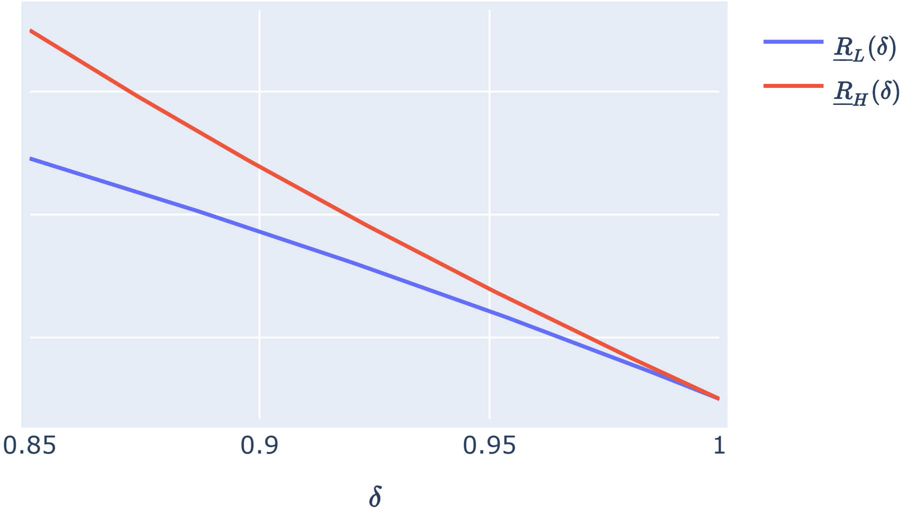


Illustration of $\underline{R}_{\mathsf{L}}(\delta, \frac{1}{2}, 4)$ and $\underline{R}_{\mathsf{H}}(\delta, \frac{1}{2}, 4)$



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$\Rightarrow$ **For high q (many low types) collusion is not a concern.**

Non-binding collusiveness constraints

(Col-sep-2) $v^*_{\text{fse}} \geq v_2' = \underbrace{(1-\delta)\frac{1}{n}\left[(1-q^n)(\overline{\theta} - \underline{b}^*) + q^n(\underline{\theta} - \underline{b}^*)\right] + \delta q^n v_2'}_{\text{High types bid } \underline{b}^* + \epsilon\text{, Low types bid } \underline{b}^*}.$

$\underline{b}^* > \underline{\theta} \Rightarrow$ **all positive-reward types punished with probability 1**.

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(Col-pool) $v^*_{\text{fse}} \geq v'_{\text{p}} = \underbrace{(1-\delta)\frac{1}{n}\left[(1-q)(\overline{\theta}-\underline{b}^*) + q(\underline{\theta}-\underline{b}^*)\right] + \delta v'_{\text{p}}}_{\text{Both types pool at } \underline{b}^*}.$

Gain from lower bidding, but allocative efficiency loss from pooling.

- Turns out, $|\text{Gain}| < |\text{Loss}|$, moreover $\lim_{\delta \to 1} v'_p(\delta) < 0$ in all 3 cases.

Non-binding IC constraint

Consider HighIC-on-sch and compare to HighIC-down:

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Given $v^*_{\text{fse}} \approx 0$ for high δ, deviating off-schedule is more tempting.

Conclusion