Learning in a Network of Cournot Markets

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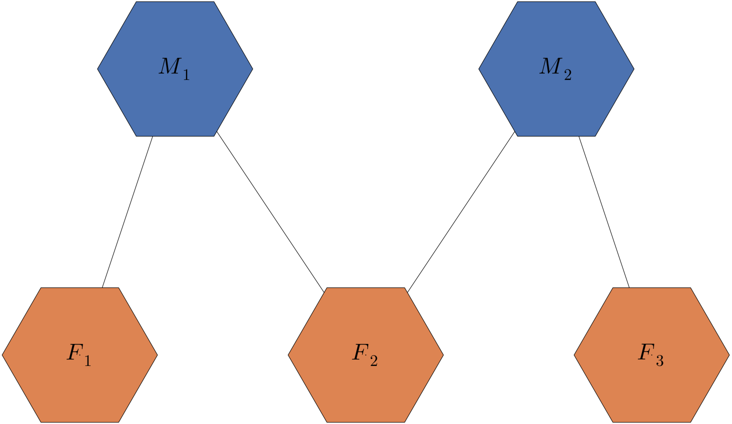
Quick Setup

- Single good, multiple markets Cournot economy with linear inverse demand
- Firms and Consumers connected by bipartite graph
- Assumed Information: known slope of demand and price-quantity history
- Market size (intercepts) unknown $\rightarrow$ least-squares learning

Quick Setup



Motivation

- Rational expectations require:
  - a lot of information
  - high computational ability
- How do agents learn equilibrium?
  - eductive
  - **evolutive**
- If there are shocks or structural changes, agents need to learn fast!
- Not all firms and consumers interact with each other

Research Question

- Is there convergence to an equilibrium? Which?
- How fast?
- How does network affect convergence speeds?

Summary of findings

- Convergence to full information Cournot-Nash equilibrium
- Network has no effect on stability but affects convergence speed
- Individual quantities converge polynomially at a constant rate
  - independent of network structure
- Aggregate (market and firm) quantities converge faster than individual quantities
  - speed dependent on network structure

Model - Notation

- markets: $\mathcal{M} = \{1, \ldots, M\}$, indexed by m
- firms: $\mathcal{J} = \{1, \ldots, J\}$, indexed by j
- price in market m in period t: p_t^m
- quantity produced by firm j for market m in period t: $q_t^{m,j}$
- True parameters: α and β
- Estimates by firm j for market m at time t: $a_t^{m,j}$
- $\mathcal{J}_m$: firms connected to market m
- $\mathcal{M}_j$: markets that firm j is connected to
- demand shock: ε_t^m

Model – Network

- Network: bipartite graph $\mathcal{G} = (\mathcal{M}, \mathcal{J}, E)$
- biadjacency matrix G with elements g_{ij}
- Example graph:

$$G = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Model – Firm

Inverse demand function:

$$p_t^m = \alpha - \beta \left(\sum_{j \in \mathcal{J}_m} q_t^{m,j} \right) + \varepsilon_t^m$$

Perceived inverse demand:

$$p^{m,j} = a^{m,j} - \beta q^{m,j} + \upsilon^{m,j}$$

Parameters are unknown $\rightarrow$ firm estimates:

$$\hat{p}_t^{m,j}(q_t^{m,j}) = a_{t-1}^{m,j} - \beta q^{m,j}$$

Model – Firm

Objective:

$$\mathbf{q}_t^j = \underset{\{q^{m,j}\}_{m\in\mathcal{M}_j}}{\arg\max} \left[\left(\sum_{m\in\mathcal{M}_j} \hat{p}_t^{m,j}(q^{m,j})q^{m,j} \right) - \frac{c}{2}\left(Q^j\right)^2 \right]$$

Yields:

$$q_t^{m,j} = \frac{2}{\beta}\left(\frac{1}{2} a_{t-1}^{m,j} - \frac{1}{2\left(M_j + \beta\right)} \sum_{i\in\mathcal{M}} g_{ij} a_{t-1}^{i,j} \right)$$

Model – Firm

Vectorize:

$$q_t^{m,j} = \frac{2}{\beta} g_{mj} \left(\operatorname{diag} G_j \left(\frac{1}{2} \mathbf{e}_m + t_j \mathbf{1} \right) \right)^\top a_{t-1}^j ,$$

where $t_j = \frac{1}{2\left(M_j + \beta\right)}$, $\mathbf{e}_m$ is the m-th unit vector, $\mathbf{1}$ a vector of ones, and $\operatorname{diag} G_j$ is the diagonal matrix with the j-th column of G as its diagonal.

Model – Firm

Vectorize more:

$$q_t^j = \frac{2}{\beta} L_j a_{t-1}^j \,,$$

where

$$L_j = \operatorname{diag} G_j \left(t_j \mathbb{1} + \frac{1}{2} I \right) \operatorname{diag} G_j \,,$$

and $\mathbb{1}$ is a matrix of ones.

Model – Learning

Recursive updating:

$$a_t^{m,j} = a_{t-1}^{m,j} + \frac{1}{t}\left(\underbrace{p_t^m + \frac{\beta}{2}q_t^{m,j}}_{\text{Inferred } \alpha} - \underbrace{a_{t-1}^{m,j}}_{\text{Current belief about } \alpha}\right)$$

$$= a_{t-1}^{m,j} + \frac{1}{t}\left(\alpha - \frac{\beta}{2}\left(\sum_{i\in\mathcal{J}_m\setminus j} q_t^{m,i}\right) - a_{t-1}^{m,j} + \varepsilon_t^m\right) .$$

Model – Learning

By stacking the difference equations for all firms we can write the learning process in matrix form as

$$a_t = a_{t-1} + \frac{1}{t}\left(\alpha \operatorname{vec} G - A a_{t-1} + \mathcal{E}_t\right) .$$

Proposition 1

Steady-state beliefs $\bar{a}$ induce the Cournot-Nash equilibrium quantities.

▸ Matrix form

Stochastic approximation

In deviations from the steady state, $\hat{a}_t = a_t - \bar{a}$:

$$\hat{a}_t = \hat{a}_{t-1} - \frac{1}{t}\left(A\hat{a}_{t-1} - \mathcal{E}_t\right) .$$

Approximation:

$$\frac{\hat{a}_t - \hat{a}_{t-1}}{\frac{1}{t}} \approx \dot{a} = -A\hat{a} + \mathcal{E}_t .$$

Stochastic approximation

Proposition 2

Discrete learning dynamics are approximated by ODE

$$\dot{a} = -Aa\,.$$

In particular, if $a(\tau)$ is a solution to the ODE, then

$$a_t \approx a(\tau)\ ,$$

with $\tau \approx \log t$.

Dynamics of discrete system can be analyzed using the eigenvalues and eigenvectors of A.

Stochastic approximation

ODE solution:

$$a(\tau) = \sum_{i=1}^{JM} c_i e^{-\lambda_i \tau} v_i \,,$$

where λ_i are the eigenvalues of A and v_i the corresponding eigenvectors.

Results

Theorem 1 (Individual Learning)

For any strongly connected network, quantities converge polynomially at a rate of $-\frac{1}{2}$ to the steady state values.

Theorem 2 (Informational Efficiency)

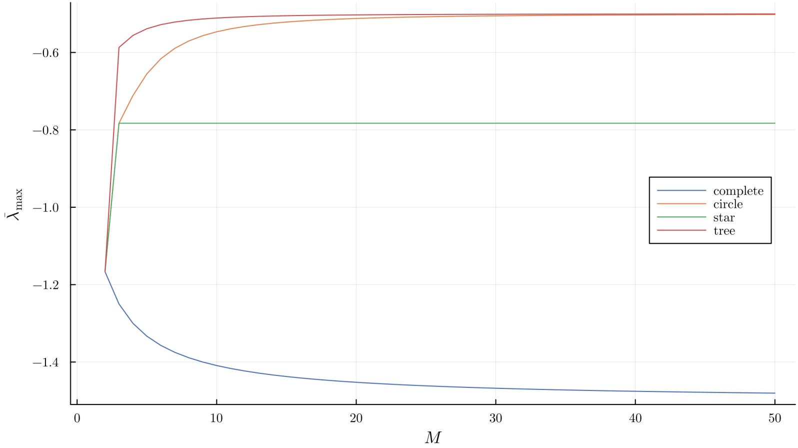
Aggregate production converges at a faster rate than individual production both within markets and within firms. Prices are determined by aggregate production and are thus also learned at the faster rate.

▸ Connectivity

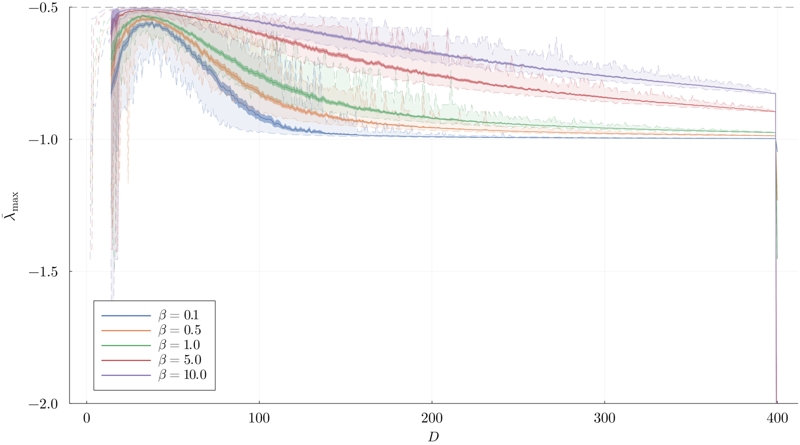
Proof sketch

1. Show that smallest eigenvalue of A is $\lambda_{\min} = \frac{1}{2}$
2. Characterize the eigenspace of $\lambda_{\min}$, $E_{\lambda_{\min}}(A)$
3. Construct a mapping u^m that aggregates individual beliefs to aggregate (market) quantities
4. Show that $u^m \in \ker E_{\lambda_{\min}}(A)$

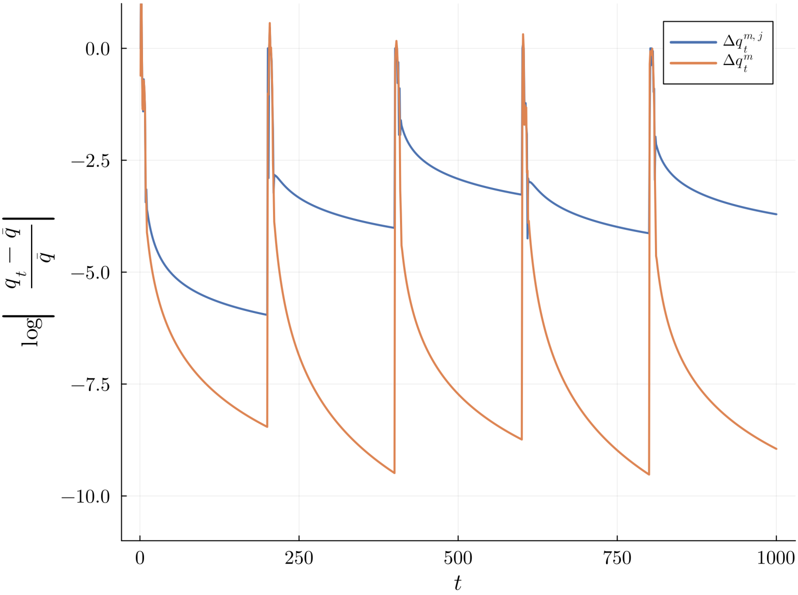
Network comparison



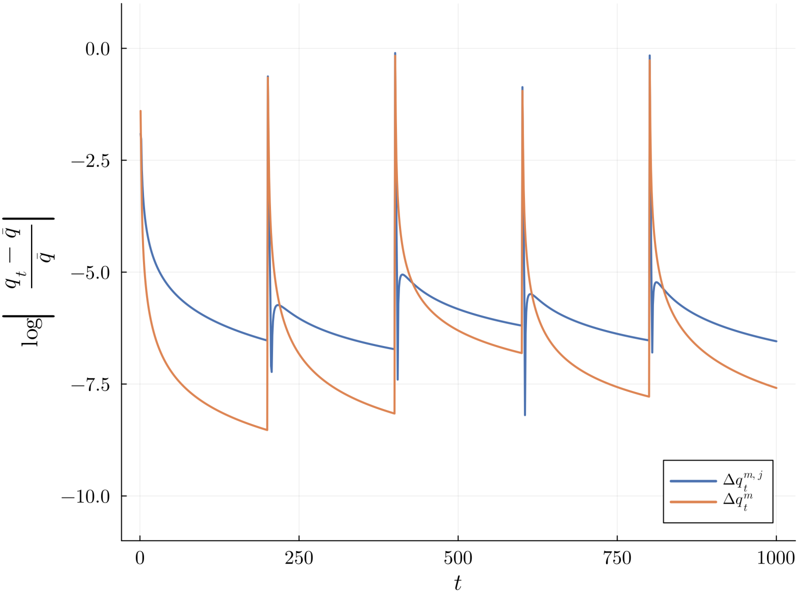
Erdős-Rényi random graph



Learning time series – Complete network



Learning time series – Tree network



Conclusion

- Firms are able to learn the Cournot-Nash equilibrium
- Individual quantities converge polynomially at a constant rate independent of network structure
- Aggregate (market and firm) quantities converge faster than individual quantities
- The convergence speed depends on the network structure

Thank you!

Model – Learning in matrix form

Where

$$A = \operatorname{diag} G \, (L + I) \in \mathbb{R}^{JM \times JM} \, , \tag{1}$$

$$L = \begin{pmatrix} 0 & L_2 & \dots & L_J \\ L_1 & 0 & \dots & L_J \\ \vdots & \vdots & \ddots & \vdots \\ L_1 & L_2 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{J \times J} \, , \tag{2}$$

and,

$$\mathcal{E}_t = (\mathbf{1} \otimes \varepsilon_t) \circ \operatorname{vec} G \, . \tag{3}$$

▸ Back

Definitions

Definition 1 (Weak connectivity)

A network is weakly connected if the number of connections $D = |E|$ satisfies

$$D > M + J - 1\,.$$

Definition 2 (Strong connectivity)

A network is strongly connected if

$$M_j \geq 2 \quad \forall j \in \mathcal{J} \text{ and } J_m \geq 2 \quad \forall m \in \mathcal{M}\,.$$

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