The Trilemma for Low Interest Rate Macroeconomics *

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Abstract

Three desirable goals of macroeconomic policy are: full employment, low inflation, and a low debt level with no Ponzi scheme. This paper shows that, when the natural real interest rate is persistently depressed, at most two of these three goals can be simultaneously achieved. Depending on the parameters of the economy, each of these three possibilities can be the preferred option, resulting in a non-trivial policy trilemma.

Keywords: Liquidity trap, Ponzi scheme, Secular stagnation

JEL Classification: E12, E31, E63, H63

1 Introduction

Over the past decades, the natural real interest rate has progressively and persistently declined across the industrialized world to such an extent that Japan, the Eurozone, and even the United States have spent long stretches of time with a binding zero lower bound on the nominal interest rate. It took massive fiscal stimuli following the Covid pandemic and large contractionary supply shocks to raise inflation above target. However, population aging, declining productivity growth, high inequality, and persistently low demand for investment suggest that the natural real interest rate is likely to remain depressed over the coming decades. This paper argues that this entails a major challenge to macroeconomic policy.

The traditional response to this challenge has been to advocate for a rise in the inflation target such as to prevent the zero lower bound from binding (Krugman, 1998).

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However, central banks are reluctant to raise their target above 2%. An alternative response to persistently depressed demand consists in implementing a fiscal expansion financed by an increase in the debt-to-GDP ratio, unbacked by future tax increases, which can be sustainable provided that the real interest rate remains below the growth rate of the economy (Blanchard, 2019; Krugman, 2021). This paper formalizes this idea within a secular stagnation framework, while also highlighting the risk this strategy entails for price stability.

Secular stagnation is characterized by low inflation, determined by binding downward nominal wage rigidities, and by a binding zero lower bound. This induces the real interest rate to be above its natural counterpart, resulting in underemployment. This suggests a policy dilemma: either raise the inflation target sufficiently to depart from the zero lower bound or keep the economy depressed. But, under secular stagnation, the natural real interest rate is so low that a Ponzi scheme of public debt is likely to be sustainable. Government transfers to households financed by rolling over debt, rather than by raising future taxes, generate a wealth effect that stimulates aggregate demand, which can restore full employment. However, a Ponzi scheme is inherently unstable and can collapse, either because households coordinate on running away from it following a sunspot shock or because a positive shock to the natural real interest rate brings stagnation to an end. Such a run restores the fiscal theory of the price level, which results in an upward jump in the price level that shrinks the Ponzi scheme to zero. While the Ponzi scheme can restore full employment without raising the inflation target, it entails a small probability of a sudden debasement of the currency.[1]

The government must therefore choose between a depressed economy, a higher inflation target, or a Ponzi scheme. If either the welfare cost of changes to the price level or the likelihood of collapse of the Ponzi scheme is sufficiently low, then the Ponzi debt scheme is optimal... until it collapses. Otherwise, the optimal policy consists in permanently higher inflation and full employment, unless the welfare cost of inflation is so high that a persistently depressed economy is preferable.

Related literature. In a highly influential AEA Presidential Lecture, Blanchard (2019) has argued that public debt sustainability need not be a concern in a low interest rate environment. Building on this insight, Krugman (2021) has argued that, under secular stagnation, public debt is an attractive alternative to higher inflation to achieve full employment. Mankiw (2022) has warned about the possibility that, even when a Ponzi

[1] A Ponzi scheme can also crowd out capital. However, for this to be an adverse effect, the marginal product of capital must be larger than the growth rate of the economy, which must itself be larger than the real interest rate for the Ponzi scheme to exist. But, for the marginal product of capital to exceed the real interest rate, some financial frictions must exist, such as imperfect competition or liquidity constraints. My analysis abstracts from capital and therefore abstracts from these effects.

scheme can be sustainable, a run on public debt can occur and is likely to be painful. This paper formalizes these insights.

Billi, Galí, and Nakov (2024) have characterized the optimal trade-off between higher inflation and insufficient economic activity within a New Keynesian economy with a persistently depressed natural real interest rate, but without bubbles. When a Ponzi scheme is sustainable, Kocherlakota (2022), Blanchard (2022), Miao and Su (2023), and Aguiar, Amador, and Arellano (2023a) have shown that fiscal policy can be essential to stabilize economic activity. Also, Campos, Fernández-Villaverde, Nuño, and Paz (2024) have shown that a debt-financed fiscal expansion may be necessary to stimulate aggregate demand such as to prevent the zero lower bound from binding. My analysis emphasizes that expansionary fiscal policy can reduce the inflation rate that is necessary to achieve full employment, but it entails the risk of a price level jump when the Ponzi scheme collapses.

Corsetti and Maćkowiak (2023) have argued that delaying a fiscal adjustment is a gamble that may be worth taking. It reduces inflation in the short-run, but it may eventually result in a large jump in the price level. However, their analysis abstracts from the stimulating wealth effect from running a Ponzi scheme.

Bassetto and Cui (2018) have shown that the fiscal theory of the price level does not uniquely pin down the price level when the interest rate is below the growth rate of the economy, since a Ponzi scheme may or may not arise. Brunnermeier, Merkel, and Sannikov (2023) have argued that the steady state with a Ponzi scheme can be made the unique equilibrium provided that the government makes an off-the-equilibrium commitment to run primary surpluses forever if the Ponzi scheme collapses (which raises the value of public debt following the run). I instead consider that the government is not able to make such a strong commitment and therefore cannot prevent the possibility of a run on Ponzi debt.

My analysis assumes a preference for wealth, which makes it possible to have secular stagnation (Michau, 2018) and rational bubbles (Michau, Ono, and Schlegl, 2023) within a representative household model of the economy.[2] Relying on this framework, Michau (2024a) has shown that, under secular stagnation, helicopter drops of money can be both stimulative and non-inflationary due to the sustainability of a Ponzi scheme. To explore the policy trilemma, the present analysis adds two features to this framework: i) price instability has a negative impact on welfare and ii) the Ponzi scheme can collapse through a stochastic jump in the price level, which introduces the stochastic discount factor into the valuation of future budget surpluses.[3]

[2]The possibility of obtaining rational bubbles under a preference for wealth had previously been shown by Ono (1994), chapter 11, and by Zhou (2016). However, they did not investigate Ponzi schemes.

[3]The Ono (1994, 2001) model of secular stagnation assumes a constant marginal utility of wealth (or

Relying on a tractable model of bubbles with financial frictions and downward nominal wage rigidity, Hanson and Phan (2017) and Biswas, Hanson, and Phan (2020) have shown that a bubble can boost economic activity, and its collapse can result in secular stagnation. While they take the bubble as exogenous, I focus on a Ponzi scheme that is partly determined by fiscal policy. Also, in their work, the bubble stimulates economic activity by relaxing financial constraints, whereas in my work, the Ponzi scheme stimulates aggregate demand through a wealth effect.[4]

Mian, Straub, and Sufi (2024) have characterized the maximum budget deficit that can be sustained forever when the natural real interest rate is depressed, with and without a binding zero lower bound. While their analysis focuses on fiscal space, I instead characterize the trade-off between inflation, the output gap, and the Ponzi scheme. They assume that households have a preference for wealth, that they interpret as a convenience benefit of liquidity, which implies that public debt stimulates aggregate demand. By contrast, to have the Ricardian equivalence, I assume a preference for net wealth, i.e. wealth net of the present value of taxes, which implies that only a Ponzi scheme can be stimulative.[5]

Section 2 presents the setup of the economy, section 3 defines the equilibrium, and section 4 characterizes the steady state equilibria. The policy trilemma is investigated in section 5. Section 6 discusses the nature of the shock inducing the price level jump. Possible ways to break through the trilemma are discussed in section 7. The paper ends with a conclusion.

2 Economy

The economy consists of identical firms, identical households, and a government. The only friction is a downward nominal wage rigidity.

Time is continuous. There is a unit mass of infinitely lived households. Population within each household grows at rate n. At time t, the total population of the economy is equal to $N_t = e^{nt}$.

of real money balances), which annihilates the wealth effect from the Ponzi scheme. This results in a dilemma: higher inflation or underemployment.

[4] This line of research belongs to a growing literature on the interactions between monetary policy and bubbles (Galí, 2014, 2021; Dong, Miao, and Wang, 2020; Asriyan, Fornaro, Martin, and Ventura, 2021; Ikeda, 2022; Plantin, 2023).

[5] Recently, Barro (2020), Mehrotra and Sergeyev (2021), Reis (2021), Cochrane (2021), Abel and Panageas (2022), Amol and Luttmer (2022), Brumm, Feng, Kotlikoff, and Kubler (forthcoming), Kocherlakota (2023), Aguiar, Amador, and Arellano (2023b) have also carefully investigated the sustainability of public debt in low interest rate environments, but in real economies without the possibility of depressed demand.

2.1 Government

Nominal indebtedness at time t amounts to B_t. Real lump-sum taxes per capita are set equal to τ_t. Public indebtedness therefore evolves according to

$$\dot{B}_t = i_t B_t - \tau_t P_t N_t, \tag{1}$$

where i_t denotes the nominal interest rate and P_t the aggregate price level at t.

Real indebtedness per capita is given by $b_t = B_t/(P_t N_t)$. I denote by Φ_t the expected present value of the real primary surpluses per capita τ_t from time t onward. In the absence of Ponzi scheme, we would have $b_t = \Phi_t$. It is therefore natural to define the magnitude of the government Ponzi debt scheme as

$$\Delta_t = b_t - \Phi_t. \tag{2}$$

Whether a Ponzi scheme is sustainable will be determined endogenously in equilibrium.

An important drawback from running a Ponzi scheme is that it can collapse at any moment. I therefore assume that a sunspot shock occurs at exogenous Poisson rate ε. It induces households to run away from the Ponzi scheme, resulting in its collapse. By assumption, the government remains committed to the same present value of surpluses Φ_t, which is therefore not affected by the sunspot shock. The shock induces an upward jump in the price level that is multiplied by $1 + \Delta_t/\Phi_t$.[6] The evolution of the price level is therefore given by the following stochastic process

$$dP_t = \pi_t P_t dt + \frac{\Delta_t}{\Phi_t} P_t dJ_t, \tag{3}$$

where π_t denotes the inflation rate at time t in the absence of price level jump, while dJ_t denotes the Poisson jump, which is equal to 1 with probability εdt and to 0 with probability $1 - \varepsilon dt$. Throughout my analysis, I consider that lump-sum taxes are set such that the present value of surpluses Φ_t is strictly positive; otherwise, an arbitrarily large price level jump would not be sufficient to eliminate the Ponzi scheme.

Using Itô's lemma with jumps, we can compute $d(1/P_t)$ and, hence, $d(B_t/(P_t N_t))$, which gives

$$db_t = \left[(i_t - \pi_t - n)b_t - \tau_t\right] dt - \Delta_t dJ_t. \tag{4}$$

The derivation is provided in appendix A. When the shock occurs, public debt falls

[6] Indeed, as the price level increases from P_t to $P_t(1 + \Delta_t/\Phi_t)$, public indebtedness falls from $B_t/(N_t P_t) = \Delta_t + \Phi_t$ to $B_t/(N_t P_t(1 + \Delta_t/\Phi_t)) = \Phi_t$, which reduces the magnitude of the Ponzi scheme to zero.

from b_t to $b_t - \Delta_t = \Phi_t$.

Finally, monetary policy follows a Taylor rule, unless the zero lower bound is binding, which implies

$$i_t = \max\{r^n + \pi^* + \phi[\pi_t - \pi^*], 0\}, \tag{5}$$

where π^* is the inflation target, r^n is the natural real interest rate to be subsequently defined, and ϕ determines the responsiveness of the nominal interest rate to inflation.

2.2 Firms

For simplicity, I assume that labor is the only factor of production. The representative firm employs L_t units of labor per capita to produce output Y_t using a constant returns to scale production function

$$Y_t = N_t L_t. \tag{6}$$

Employment therefore amounts to $N_t L_t$. The real wage w_t is equal to the marginal product of labor, which gives

$$w_t = 1. \tag{7}$$

2.3 Households

The representative household discounts the future at rate ρ, where $\rho > n$. It inelastically supplies one unit of labor per capita, resulting in aggregate labor supply being equal to N_t. The household derives utility $u(c_t)$ from consuming c_t per capita at time t, where $u'(\cdot) > 0$, $u''(\cdot) < 0$, and $\lim_{c \to 0} u'(c) = \infty$.

The household also derives utility from holding wealth, which is equal to a_t per capita. However, government debt b_t must eventually be repaid through taxes; unless the government is running a Ponzi scheme. The expected present value of taxes is therefore equal to $b_t - \Delta_t$, where Δ_t denotes the magnitude of the government's Ponzi scheme. The household perceives its *net wealth* to be equal to $a_t - b_t + \Delta_t$ at time t, and derives utility $\gamma(a_t - b_t + \Delta_t)$ from holding it. This specification of net wealth implies that households are Ricardian. A lump-sum transfer eventually repaid through a lump-sum tax temporarily raises both a_t and b_t by the same amount, while leaving $a_t - b_t + \Delta_t$ unchanged. Thus, the marginal utility of wealth $\gamma'(a_t - b_t + \Delta_t)$ is unaffected by the transfer, consistently with the Ricardian equivalence proposition. The preference for wealth satisfies $\gamma'(\cdot) > 0$, $\gamma''(\cdot) < 0$, $\gamma'(0) < \infty$, $\lim_{k \to \infty} \gamma'(k) = 0$, and $\int_0^\infty \gamma'(e^{\lambda t})dt < \infty$ for any $\lambda > 0$.[7]

[7] This last technical condition, which makes it possible to rule out explosive Ponzi schemes, is very mild. It is satisfied for any CRRA specification $\gamma(k) = [(k - \underline{k})^{(1-\sigma)} - 1]/(1 - \sigma)$ with reference wealth level $\underline{k} < 0$.

Finally, the household gets disutility $\psi c(dP_t/P_t)$ from changes to the price level, where ψ determines the strength of this disutility, while the function $c(\cdot)$ is given by

$$
\begin{aligned}
c\left(\frac{dP_t}{P_t}\right) &= \begin{cases} \frac{1}{dt}\left|\frac{dP_t}{P_t}\right| & \text{if } dJ_t = 0 \\ \frac{1}{dt}C\left(\left|\frac{dP_t}{P_t}\right|\right) & \text{if } dJ_t = 1 \end{cases}, \\
&= \begin{cases} |\pi_t| & \text{if } dJ_t = 0 \\ \frac{1}{dt}C\left(\frac{\Delta_t}{\Phi_t}\right) & \text{if } dJ_t = 1 \end{cases},
\end{aligned} \tag{8}
$$

where the function $C(\cdot)$ satisfies $C(0) = 0$, $C'(0) \geq 1$, and $C''(\cdot) \geq 0$.[8] This specification nests a linear flow cost of inflation under normal circumstances, together with a discrete cost when the price level jumps. The discrete cost is weakly convex in the magnitude of the jump. This utility cost of inflation has no impact on the behavior of the representative household, since inflation is beyond its control, but it will be relevant for our subsequent welfare analysis. The adverse effect of inflation can be interpreted as the mental cost of optimizing purchases when the price level changes, together with an additional convex cost from the financial disruption and the loss of monetary policy credibility entailed by a sudden debasement of the currency.

The household's expected intertemporal utility is given by

$$
\mathbb{E}_0\left[\int_0^\infty e^{-(\rho-n)t}\left[u(c_t) + \gamma(a_t - b_t + \Delta_t) - \psi c\left(\frac{dP_t}{P_t}\right)\right] dt\right]. \tag{9}
$$

Let r_t denote the real return that is risk-free in *real* terms (whereas $i_t - \pi_t$ is the real return that is risk-free in *nominal* terms, but risky in real terms due to the inflation risk). The portfolio of the representative household h is composed of two assets: government bonds b_t^h and bonds that are risk-free in real terms d_t^h. Thus, at any point in time $a_t = b_t^h + d_t^h$. The risk-free bonds yield a return $r_t - n$ per capita. Government bonds yield $i_t - \pi_t - n$ and their value drops by Δ_t/b_t when the price level jumps, which occurs with probability εdt at time t.[9] The household receives labor income $w_t L_t$, pays lump-sum taxes τ_t, and consumes c_t per capita.[10] Hence, household wealth per capita

[8] If we impose $C'(0) = 1$, then the two parts of (8) can be nested into the single expression $c\left(\frac{dP_t}{P_t}\right) = \frac{C(|dP_t/P_t|)}{dt}$. Indeed, by (3), if $dJ_t = 1$, we have $|dP_t/P_t| = \Delta_t/\Phi_t$; while, if $dJ_t = 0$, we have $\frac{C(|dP_t/P_t|)}{dt} = \frac{C(|\pi_t|dt)}{dt} = \frac{C(0)+|\pi_t|dtC'(0)}{dt} = |\pi_t|$.

[9] As the price level is multiplied by $1 + \Delta_t/\Phi_t$, the value of bonds b_t^h held by the household decreases by $b_t^h - b_t^h/(1 + \Delta_t/\Phi_t) = b_t^h(b_t - \Phi_t)/b_t = b_t^h\Delta_t/b_t$.

[10] If there is less than full employment, labor demand L_t is below 1 and labor income is equal to $w_t L_t$.

follows

$$
\begin{aligned}
da_t &= \left[(r_t - n)d_t^h + (i_t - \pi_t - n)b_t^h + w_t L_t - \tau_t - c_t\right] dt - b_t^h \frac{\Delta_t}{b_t} dJ_t, \\
&= \left[(r_t - n)a_t + w_t L_t - \tau_t - c_t\right] dt + b_t^h \left[(i_t - \pi_t - r_t)dt - \frac{\Delta_t}{b_t} dJ_t\right].
\end{aligned} \tag{10}
$$

Finally, the household is subject to a no-borrowing constraint

$$
a_t \geq 0. \tag{11}
$$

In equilibrium, this constraint is never binding since households are identical and the supply of assets is always positive.

The representative household maximizes its expected utility (9) subject to its flow of funds constraint (10) with initial wealth a_0 and to the no-borrowing constraint (11).

Before the jump in the price level has occurred, the intertemporal allocation of consumption satisfies the Euler equation

$$
\frac{\dot{c}_t}{c_t} = \left(\frac{u'(c_t)}{-c_t u''(c_t)}\right) \left[r_t - \rho + \frac{\gamma'(a_t - b_t + \Delta_t)}{u'(c_t)} + \varepsilon \left(\frac{u'(\bar{c}_t)}{u'(c_t)} - 1\right)\right], \tag{12}
$$

where $\bar{c}_t$ denotes consumption at time t immediately after the price level jump. Once the price level jump has occurred, consumption follows a similar Euler equation with both Δ_t and ε equal to zero. The optimal portfolio allocation between risky government debt b_t^h and risk-free bonds d_t^h results in

$$
r_t = i_t - \pi_t - \varepsilon \frac{\Delta_t}{b_t} \frac{u'(\bar{c}_t)}{u'(c_t)}. \tag{13}
$$

The real return on government bonds $i_t - \pi_t$ is above the risk-free real interest rate r_t due to the price level risk. Finally, the optimizing behavior of the household implies that the following transversality condition must be satisfied

$$
\lim_{t \to \infty} \mathbb{E}_0 \left[e^{-(\rho - n)t} u'(c_t) a_t\right] = 0. \tag{14}
$$

The Euler equation (12), the risk premium relationship (13), and the transversality condition (14) are sufficient conditions to characterize a solution to the household's problem. This is formally established in appendix B.

The role of the preference for wealth can be seen from the Euler equation (12): in addition to raising the propensity to save, it ensures that in steady state, i.e. when $\dot{c}_t = 0$, consumption is a decreasing function of the real interest rate. This is essential to allow for the possibility of a sustainable Ponzi scheme or of secular stagnation.

2.4 Downward nominal wage rigidity

Following Schmitt-Grohé and Uribe (2016), I impose a downward nominal wage rigidity. Workers never accept a rate of nominal wage growth that falls below a reference rate of inflation π^R. But, the profit maximizing behavior of firms implies that, under our linear production function, the real wage must always be equal to 1, as given by (7). Hence, the price level P_t must be equal to the nominal wage rate. So, the downward nominal wage rigidity prevents inflation from ever falling below π^R. This results in two possibilities: if inflation is above π^R, the downward nominal wage rigidity is not binding, ensuring full employment with $L_t = 1$; conversely, if there is less than full employment with $L_t < 1$, the downward nominal wage rigidity must be binding, resulting in inflation being equal to π^R. We must therefore have[11]

$$\pi_t \geq \pi^R \text{ and } L_t \leq 1 \text{ with complementary slackness.} \tag{15}$$

Throughout my analysis, I assume that the inflation target π^* from the Taylor rule (5) is greater or equal to the reference rate of inflation π^R.

2.5 Market clearing

For the economy to be in equilibrium, markets must clear. Goods market clearing requires aggregate demand $N_t c_t$ to be equal to aggregate supply $Y_t = N_t L_t$, which gives

$$c_t = L_t. \tag{16}$$

Financial market clearing requires households' demand for government bonds b_t^h and for risk-free bonds d_t^h to be equal to their respective supply, equal to b_t and 0. As $a_t = b_t^h + d_t^h$, this implies

$$a_t = b_t. \tag{17}$$

Hence, net household wealth $a_t - b_t + \Delta_t$ must always be equal to Δ_t. Finally, the labor market clearing condition is replaced by the downward nominal wage rigidity (15).

[11] Michau (2018) offers a slightly more general specification, where under-employment induces workers to accept a rate of nominal wage growth below π^R. However, empirically, the Phillips curve is very flat at low rates of inflation (Forbes, Gagnon, and Collins, 2021), suggesting that downward nominal wage flexibility is very limited.

3 Equilibrium

Given the structure of the economy, and the preference for wealth of the representative household, the stochastic discount factor is given by

$$\Lambda_t = e^{-\int_0^t \left(\rho - n - \frac{\gamma'(\Delta_u)}{u'(c_u)}\right) du} u'(c_t). \tag{18}$$

Let us now rely on this stochastic discount factor to provide a precise definition of the present value of primary surpluses Φ_t, from which the magnitude of the Ponzi debt scheme $\Delta_t = b_t - \Phi_t$ can be deduced.

Assuming that the present value of real primary surpluses Φ_t does not jump when the sunspot shock occurs, Φ_t is akin to a safe asset whose dividends must be discounted by the risk-free real interest rate. Thus, Φ_t can be defined by

$$d\Phi_t = \left[(r_t - n)\Phi_t - \tau_t\right] dt, \tag{19}$$

together with the boundary condition $\lim_{T\to\infty} \mathbb{E}_t \left[\Lambda_T \Phi_T\right] = 0$. As required, this definition implies

$$\Phi_t = \mathbb{E}_t \left[\int_t^{\infty} \frac{\Lambda_s}{\Lambda_t} \tau_s ds\right]. \tag{20}$$

This is formally established in appendix C.

For simplicity and clarity, I am assuming that, when the price level jumps at time t, the path of lump-sum taxes $(\tau_s)_{s=t}^{\infty}$ adjusts such as to leave the present value of primary surpluses Φ_t unchanged. Alternatively, we could be tempted to assume that, when the shock occurs, the government raises the present value of surpluses Φ_t to the level of public debt b_t such as to prevent any jump in the price level. However, as shown in appendix D, when $\varepsilon > 0$, this generically results in $\Delta_t = 0$. Intuitively, a Ponzi cannot exist if the government commits to raising the present value of surpluses conditional on the realization of the sunspot shock, since public liabilities would then always be backed by fiscal surpluses.

The evolution of public debt b_t and of the present value of surpluses Φ_t, respectively given by (4) and (19), together with expression for the risk premium (13), imply that the Ponzi scheme $\Delta_t = b_t - \Phi_t$ follows

$$d\Delta_t = \left[r_t - n + \varepsilon \frac{u'(\bar{c}_t)}{u'(c_t)}\right] \Delta_t dt - \Delta_t dJ_t, \tag{21}$$

which entails

$$\Delta_t = \lim_{T\to\infty} \mathbb{E}_t \left[\frac{\Lambda_T}{\Lambda_t} \Delta_T\right]. \tag{22}$$

This is also shown in appendix C. Hence, a Ponzi scheme is only valuable if households expect it to be valuable in the future.

Throughout my analysis, I do not impose the government's no-Ponzi condition $\Delta_t \leq 0$. Instead, the limit to public indebtedness is endogenously determined by households' willingness to lend to their government, which is itself determined by their transversality condition (14).[12]

The no-borrowing constraint (11) prevents households from running Ponzi schemes. Hence, by Walras' law, the government's no-Ponzi condition must either be binding $\Delta_t = 0$ or violated $\Delta_t > 0$, but cannot be slack. I henceforth consider that $\Delta_t \geq 0$.[13]

Recall that Φ_t was assumed to be strictly positive; otherwise, by (3), an arbitrarily large price level jump could not eliminate the Ponzi scheme. The supply of assets $b_t = \Phi_t + \Delta_t$ must therefore always be strictly positive. This implies that the household's no-borrowing constraint (11) cannot be binding in equilibrium. Also, in equilibrium, the household's transversality condition (14) can be written as $\lim_{t\to\infty} e^{-(\rho-n+\varepsilon)t} u'(c_t)\Delta_t = 0$ conditional on the absence of a price level jump. This is shown in appendix E.

This allows us to define the equilibrium of the economy.

Definition 1 *An equilibrium of the economy* before *the price level jump* $(c_t, \pi_t, r_t, i_t, \Delta_t)_{t=0}^{\infty}$ *is characterized by the Euler equation (12) with asset market clearing (17):*

$$\frac{\dot{c}_t}{c_t} = \left(\frac{u'(c_t)}{-c_t u''(c_t)}\right)\left[r_t - \rho + \frac{\gamma'(\Delta_t)}{u'(c_t)} + \varepsilon\left(\frac{u'(\bar{c}_t)}{u'(c_t)} - 1\right)\right]; \tag{23}$$

the downward nominal wage rigidity (15) with goods market clearing (16):

$$\pi_t \geq \pi^R \text{ and } c_t \leq 1 \text{ with complementary slackness}; \tag{24}$$

the risk-premium equation (13) with $b_t = \Phi_t + \Delta_t$:

$$r_t = i_t - \pi_t - \varepsilon \frac{\Delta_t}{\Phi_t + \Delta_t} \frac{u'(\bar{c}_t)}{u'(c_t)}; \tag{25}$$

the Taylor rule (5):

$$i_t = \max\{r^n + \pi^* + \phi[\pi_t - \pi^*], 0\}; \tag{26}$$

[12]Note that $\Lambda_t \Delta_t = \lim_{T\to\infty} \mathbb{E}_t[\Lambda_T \Delta_T] = \lim_{T\to\infty} \mathbb{E}_t[\Lambda_T(b_T - \Phi_T)] = \lim_{T\to\infty} \mathbb{E}_t[\Lambda_T b_T]$, where the last equality follows from $\lim_{T\to\infty} \mathbb{E}_t[\Lambda_T \Phi_T] = 0$. Using the asset market clearing condition (17) and the stochastic discount factor (18), we obtain $\Lambda_t \Delta_t = \lim_{T\to\infty} \mathbb{E}_t\left[e^{-\int_0^T\left(\rho - n - \frac{\gamma'(\Delta_u)}{u'(c_u)}\right)du} u'(c_T) a_T\right]$. Hence, without the preference for wealth, the households' transversality condition (14) implies $\Delta_t = 0$. But, as we shall see, with a preference for wealth, we can have $\Delta_t > 0$.

[13]Formally, by the previous footnote, we have $\Lambda_t \Delta_t = \lim_{T\to\infty} \mathbb{E}_t[\Lambda_T b_T]$. Hence, the no-borrowing constraint $a_t \geq 0$ together with the asset market clearing condition $a_t = b_t$ implies $\Delta_t \geq 0$.

the dynamics of the Ponzi scheme (21) with $dJ_t = 0$:

$$\dot{\Delta}_t = \left[r_t - n + \varepsilon \frac{u'(\bar{c}_t)}{u'(c_t)} \right] \Delta_t; \tag{27}$$

the transversality condition conditional on the absence of a price level jump:

$$\lim_{t \to \infty} e^{-(\rho - n + \varepsilon)t} u'(c_t) \Delta_t = 0; \tag{28}$$

and the initial magnitude of the Ponzi scheme Δ_0 chosen by the government. The equilibrium after *the price level jump is also characterized by (23)-(28), but with $\Delta_t = 0$ and $\varepsilon = 0$. This determines the consumption level immediately after the price level jump $\bar{c}_t$, which affects the economy before the jump through (23), (25), and (27).*

In equilibrium, net household wealth is equal to Δ_t. Hence, assuming a preference for *net wealth* implies that public debt can only stimulate aggregate demand if it is unbacked, i.e. if it corresponds to a Ponzi scheme. An alternative would be to assume a preference for wealth (i.e. $\gamma(a_t)$ rather than $\gamma(a_t - b_t + \Delta_t)$). This would imply that a high level of public debt would be stimulating, even if backed by future surpluses. A Ponzi scheme would nonetheless still be possible and, as the Ricardian equivalence would not hold, the timing of tax collection would affect the equilibrium of the economy. This would substantially complicate the analysis. For a careful comparison of the preference for net wealth and for wealth, see Michau (2024b).

I henceforth assume that the fiscal policy of the government is characterized by a constant present value of surpluses equal to Φ until right after the Ponzi scheme has collapsed. Thus, when $\Delta_t > 0$, the primary surplus τ_t is determined residually such as to keep $\Phi_t = \Phi$. From equation (19), this implies that $\tau_t = (r_t - n)\Phi$, which entails $\tau_t < 0$ whenever $r_t < n$. Once the Ponzi scheme has collapsed, the present value of surpluses Φ_t disappears from the definition of equilibrium.[14]

4 Steady state equilibria

Let us now characterize the steady state equilibria of the economy (c, π, r, i, Δ) before the occurrence of the price level jump. From the downward nominal wage rigidity (24), we must either have full employment with $c = 1$ or low inflation with $\pi = \pi^R$. From the dynamics of the Ponzi scheme (27), we must either have no Ponzi scheme

[14] Once the Ponzi scheme has collapsed and uncertainty has dissolved, equations (18) and (20), together with the Euler equation (23) with $\Delta_t = 0$ and $\varepsilon = 0$, imply that $\Phi_t = \int_t^\infty e^{-\int_t^s (r_u - n) du} \tau_s ds$. Hence, if $\Phi_t > 0$ and the economy is in a steady state with $r < n$, then surpluses τ_t must be converging to zero. With $\tau_t \geq 0$ for all t, the present value of surpluses Φ_t must then be shrinking over time.

with $\Delta = 0$ or a Ponzi scheme of constant magnitude with $r = n - \varepsilon u'(\bar{c})/u'(c)$.[15] This gives the following four steady state equilibrium possibilities:

- A *neoclassical steady state* with full employment $c = 1$ and no Ponzi scheme $\Delta = 0$;
- A *secular stagnation steady state* with low inflation $\pi = \pi^R$, no Ponzi scheme $\Delta = 0$, and under-employment $c < 1$;
- A *Ponzi steady state* with full employment $c = 1$, interest rate $r = n - \varepsilon u'(\bar{c})/u'(c)$, and a Ponzi scheme $\Delta > 0$;
- A *Ponzi-stagnation steady state* with low inflation $\pi = \pi^R$, interest rate $r = n - \varepsilon u'(\bar{c})/u'(c)$, under-employment $c < 1$, and a Ponzi scheme $\Delta > 0$.

Once the price level jump has occurred, only the first two steady state survive. Of course, the latter two steady state are only "steady" conditional on the absence of a price level jump. For simplicity, I nonetheless refer to them as steady states. I now characterize each of these four steady state equilibria.

4.1 Neoclassical steady state

A neoclassical steady state $(c^n, \pi^n, r^n, i^n, \Delta^n)$ is characterized by full employment $c^n = 1$ and no Ponzi scheme $\Delta^n = 0$. Recall from (3) that, in the absence of Ponzi scheme, the price level cannot jump. We can therefore consider that, once in the neoclassical steady state, the economy remains there. Hence, $\bar{c} = c^n = 1$. From the consumption Euler equation (23), the real interest rate is therefore given by

$$r^n = \rho - \frac{\gamma'(0)}{u'(1)}. \tag{29}$$

This is the *natural real interest rate*, which enters the Taylor rule (26). A persistent lack of demand corresponds to a low natural real interest rate r^n. In this framework, this results from a strong marginal utility of wealth $\gamma'(0)$. This can be seen as a proxy for other factors depressing aggregate demand, such as population aging, which would not change the nature of the underlying policy trilemma.

In the absence of price level risk, the risk premium in (25) is trivially equal to zero, which gives $i^n = r^n + \pi^n$. The Taylor rule (26) therefore entails $r^n = \max\{r^n + (\phi - 1)[\pi^n - \pi^*], -\pi^n\}$. Hence, we must either have $\pi^n = \pi^*$ or $\pi^n = -r^n$; and both possibilities require

$$\pi^* \geq -r^n. \tag{30}$$

[15]Note that an explosive Ponzi scheme, with $\lim_{t\to\infty} \Delta_t = \infty$, cannot be an equilibrium outcome. This is shown in appendix F.

This shows that, when the natural real interest rate r^n is depressed due to a lack of demand, the existence of the neoclassical steady state requires the inflation target π^* to be sufficiently high to overcome the zero lower bound on the nominal interest rate. This is an important element of the policy trilemma.

Finally, the downward nominal wage rigidity (24) requires $\pi^n \geq \pi^R$. But, we must either have $\pi^n = \pi^*$ and $\pi^* \geq -r^n$ or $\pi^n = -r^n$, both of which imply $\pi^n \geq -r^n$. And, as we are about to see, $-r^n > \pi^R$ is a necessary condition for the secular stagnation steady state to exist and, hence, for the trilemma to arise. It follows that $\pi^n > \pi^R$, implying that the downward nominal wage rigidity is slack.

4.2 Secular stagnation steady state

A secular stagnation steady state $(c^{ss}, \pi^{ss}, r^{ss}, i^{ss}, \Delta^{ss})$ is characterized by low inflation $\pi^{ss} = \pi^R$, no Ponzi scheme $\Delta^{ss} = 0$, and underemployment $c^{ss} < 1$. Again, in the absence of Ponzi scheme, the price level cannot jump and we can consider that, once in the secular stagnation steady state, the economy remains there. This implies that $\bar{c} = c^{ss}$ and $r^{ss} = i^{ss} - \pi^{ss}$. The Euler equation (23) in steady state, given by $1/u'(c^{ss}) = (\rho - r^{ss})/\gamma'(0)$, implies that c^{ss} is a decreasing function of r^{ss}. Hence, to have underemployment with $c^{ss} < 1 = c^n$, the stagnation real interest rate r^{ss} must be above the natural real interest rate r^n.

The Taylor rule $r^{ss} = \max\{r^n + (\phi - 1)[\pi^R - \pi^*], -\pi^R\}$ with $r^{ss} > r^n$ and $\pi^R \leq \pi^*$ implies that $r^{ss} = -\pi^R$ and, hence, $i^{ss} = 0$. Thus, for the secular stagnation steady state to exist, and for the trilemma to arise, aggregate demand must be so depressed that $r^n < r^{ss} = -\pi^R$. I henceforth assume that the condition $r^n < -\pi^R$ is satisfied.

Finally, by the Euler equation (23), output is demand determined with

$$\frac{1}{u'(c^{ss})} = \frac{\rho + \pi^R}{\gamma'(0)}, \tag{31}$$

where I consider that $\pi^R > -\rho$. Note that a relaxation of the downward nominal wage rigidity, through a reduction in π^R, raises the real interest rate $-\pi^R$, which further depresses the economy. This is the paradox of flexibility, which shows that the fundamental cause of stagnation is not the downward nominal wage rigidity, but the existence of money that prevents the nominal, and hence the real, interest rate from being sufficiently low. Underemployment is a general equilibrium phenomenon: the interest rate is excessively high in the financial market, which depresses the demand for goods and, hence, firms' demand for labor. The downward nominal wage rigidity is only necessary to put a break on the deflationary spiral, which would otherwise be so strong as to prevent the existence of the secular stagnation steady state.

4.3 Ponzi steady state

A Ponzi steady state $(c^p, \pi^p, r^p, i^p, \Delta^p)$ is characterized by full employment $c^p = 1$ and $r^p = n - \varepsilon \frac{u'(\bar{c})}{u'(1)}$. Hence, by the Euler equation (23), the magnitude of the Ponzi scheme Δ^p must be given by

$$\gamma'(\Delta^p) = (\rho - n + \varepsilon)u'(1). \tag{32}$$

This is the size of the Ponzi scheme that is necessary to induce aggregate demand to be sufficiently high to have full employment under interest rate $r^p = n - \varepsilon \frac{u'(\bar{c})}{u'(1)}$.

The existence of this Ponzi steady state requires $\Delta^p > 0$ or, equivalently, $\gamma'(\Delta^p) < \gamma'(0)$. Hence, by (29) and (32), we must have $r^n < n - \varepsilon$. The larger the likelihood ε of collapse of the Ponzi scheme, the more stringent the existence condition for this Ponzi steady state. This insight was originally obtained by Weil (1987) in his seminal analysis of stochastic bubbles.

From the risk-premium equation (25), together with $r^p = n - \varepsilon \frac{u'(\bar{c})}{u'(1)}$, we have $i^p - \pi^p = n - \varepsilon \frac{\Phi}{\Phi + \Delta^p} \frac{u'(\bar{c})}{u'(1)}$. The Taylor rule (26) can be written as $i^p - \pi^p = \max\{r^n + (\phi - 1)[\pi^p - \pi^*], -\pi^p\}$. Hence, from these two equations, we must either have $\pi^p = \varepsilon \frac{\Phi}{\Phi + \Delta^p} \frac{u'(\bar{c})}{u'(1)} - n$ or $\pi^p = \pi^* + \frac{1}{\phi - 1}\left[n - r^n - \varepsilon \frac{\Phi}{\Phi + \Delta^p} \frac{u'(\bar{c})}{u'(1)}\right]$; and both possibilities require

$$\pi^* \geq -r^n - \frac{\phi}{\phi - 1}\left[(n - \varepsilon - r^n) + \varepsilon\left(1 - \frac{\Phi}{\Phi + \Delta^p} \frac{u'(\bar{c})}{u'(1)}\right)\right]. \tag{33}$$

If the collapse of the Ponzi scheme does not entail an output risk, i.e. if $\bar{c} = 1$, then the lower bound for the inflation target (33) is lower than in the neoclassical steady state (30). This is due to the fact that the Ponzi scheme Δ^p generates a wealth effect, which stimulates aggregate demand. Hence, the corresponding real interest rate $r^p = n - \varepsilon$ is higher than in the neoclassical steady state r^n, which relaxes the zero lower bound constraint. This is the essence of the policy trilemma: offsetting a depressed level of aggregate demand either requires high inflation or a Ponzi scheme.

Even with an output risk, i.e. with $\bar{c} < 1$, which depresses demand and reduces the real interest rate $r^p = n - \varepsilon \frac{u'(\bar{c})}{u'(1)}$, the threshold for the inflation target is likely to be lower in the Ponzi steady state (33) than in the neoclassical steady state (30). This is always the case when either ε or Φ is sufficiently close to zero.

Finally, to have full employment, the downward nominal wage rigidity must be non-binding. If $\varepsilon \frac{\Phi}{\Phi + \Delta^p} \frac{u'(\bar{c})}{u'(1)} - n \geq \pi^R$, then the constraint is always trivially satisfied. Otherwise, the threshold (33) for the inflation target must be raised to $\pi^* \geq \pi^R - \frac{1}{\phi - 1}\left[n - r^n - \varepsilon \frac{\Phi}{\Phi + \Delta^p} \frac{u'(\bar{c})}{u'(1)}\right]$ such as to have $\pi^p = \pi^* + \frac{1}{\phi - 1}\left[n - r^n - \varepsilon \frac{\Phi}{\Phi + \Delta^p} \frac{u'(\bar{c})}{u'(1)}\right] \geq \pi^R$ as the unique possibility for the inflation rate in the Ponzi steady state.

4.4 Ponzi-stagnation steady state

A Ponzi-stagnation steady state only exists under stringent conditions. It is derived in appendix G.

5 Policy options

Our first three steady state possibilities capture the essence of the policy trilemma, illustrated in Figure 1. The neoclassical steady state has full employment and no Ponzi scheme, but fairly high inflation equal to at least $-r^n$. The secular stagnation steady state has low inflation equal to π^R and no Ponzi scheme, but under-employment with $c^{ss} < 1$. The Ponzi steady state has full employment and typically fairly low inflation, but with a Ponzi scheme $\Delta^p > 0$ that can collapse at any moment. For simplicity, I henceforth consider that $\pi^R \geq 0$, which implies that the downward nominal wage rigidity prevents the occurrence of deflation. Hence, from a pure welfare perspective, inflation cannot be excessively low.

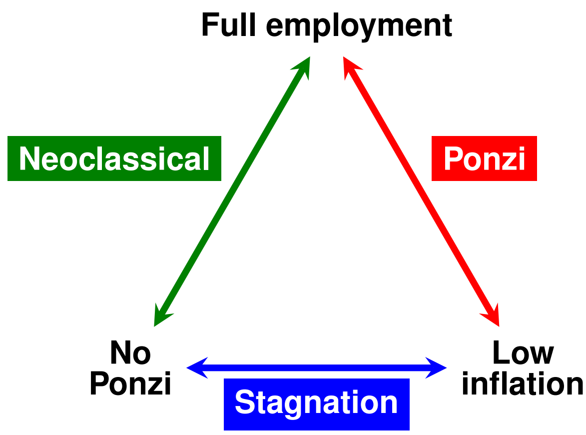

Figure 1: Trilemma for low interest rate macroeconomics

However, the trilemma does not always arise. First, to have a policy trade-off, the secular stagnation steady state must exist, which requires aggregate demand to be sufficiently depressed to have $r^n < -\pi^R$. Otherwise, the neoclassical steady state can combine full employment, no Ponzi scheme, and inflation as low as π^R. Second, when $r^n \geq n - \varepsilon$, a Ponzi scheme is not sustainable. In that case, the policy options amount to a dilemma: either full employment or low inflation.

The government has two policy instruments: the inflation target π^* and the initial magnitude of the Ponzi scheme Δ_0, which can be implemented through a lump-sum transfer of magnitude Δ_0 financed by the issuance of nominal bonds (which can be interpreted as a helicopter drop of nominal bonds). I assume that the inflation target is constant over time, and can only be changed when the Ponzi scheme collapses. The government sets these instruments such as to maximize welfare.

Throughout my analysis, I focus on the best case scenario where households spontaneously coordinate on the best equilibrium consistent with the policy (Δ_0, π^*) chosen by the government. I only impose one restriction. It concerns the equilibrium path following the realization of the sunspot shock. I assume that, if the economy is depressed with $c_t < 1$ and the Ponzi scheme collapses with $\Delta_t < \bar{\Delta}$ where $\bar{\Delta}$ is fairly small, the economy subsequently remains in the secular stagnation steady state. In other words, the sunspot shock can only trigger a persistent increase in inflation if the Ponzi scheme that is collapsing is fairly large. Otherwise, even with an arbitrarily small Ponzi scheme, the sunspot shock can trigger a persistent upward jump in the rate of inflation and in employment, the expectation of which can stimulate economic activity. This appears to be implausible.[16]

5.1 The dilemma

Before investigating the trilemma, we need to solve the dilemma facing the government once the Ponzi scheme has collapsed. The definition of equilibrium (given by (23)-(28) with $\Delta_t = 0$ and $\varepsilon = 0$) implies that, once the Ponzi scheme has collapsed, the economy must either reach the neoclassical or the secular stagnation steady state. I assume that if the two steady states can be reached, which requires both a sufficiently high inflation target and a situation before the collapse such that either $c_t = 1$ or $\Delta_t \geq \bar{\Delta}$, then households spontaneously coordinate on the neoclassical steady state. In other words, households rationally expect inflation to be on target if that is possible. Let $(\bar{c}_t, \bar{\pi}_t)$ denote the steady state equilibrium that solves this dilemma at time t.

When the economy is depressed with $c_t < 1$ and the Ponzi scheme is small with $\Delta_t < \bar{\Delta}$, the economy remains depressed, which implies $(\bar{c}_t, \bar{\pi}_t) = (c^{ss}, \pi^R)$. Let us now solve the dilemma when either $c_t = 1$ or $\Delta_t \geq \bar{\Delta}$. As the government sets the lowest inflation target π^* consistent with the desired steady state, inflation in the neoclassical steady state must be equal to $-r^n$. Inflation is always equal to π^R in the secular stagnation steady state. The welfare of the representative household is equal to $[u(1) + \gamma(0) + \psi r^n]/(\rho - n)$ in the neoclassical steady state and to $[u(c^{ss}) + \gamma(0) - \psi\pi^R]/(\rho - n)$ under secular stagnation. It follows that full employment is chosen if and only if the welfare cost of higher inflation $\psi(-r^n - \pi^R) > 0$ is lower than the welfare cost of

[16] My results on the nature of the trilemma are not fundamentally modified if $\bar{\Delta} = 0$, except that for some parameters the optimal equilibrium trajectory seems implausible.

under-employment $u(1) - u(c^{ss}) > 0$. Hence, when either $c_t = 1$ or $\Delta_t \geq \bar{\Delta}$, we have

$$(\bar{c}_t, \bar{\pi}_t) = \begin{cases} (1, -r^n) & \text{if } \psi \leq \frac{u(1)-u(c^{ss})}{-r^n - \pi^R} \\ (c^{ss}, \pi^R) & \text{if } \psi > \frac{u(1)-u(c^{ss})}{-r^n - \pi^R} \end{cases}. \tag{34}$$

This also characterizes the solution to the dilemma at time 0.

Note that, in the presence of a Ponzi scheme, $\bar{c}$ has an impact on welfare, as reflected by the risk-premium (25). I am assuming that the government chooses $\bar{c}$ once the Ponzi scheme has collapsed and cannot commit *ex-ante* to a different value of $\bar{c}$.

5.2 The trilemma

When $r^n < \min\{-\pi^R, n - \varepsilon\}$, the secular stagnation and the Ponzi steady state both exist, resulting in a policy trilemma. If we denote by $(c_t, \pi_t, \Delta_t)_{t=0}^{\infty}$ the equilibrium of the economy conditional on the absence of a price level jump and by $(\bar{c}_s, \bar{\pi}_s)_{s=t}^{\infty}$ the steady state equilibrium afterwards, then welfare along this path is given by

$$\int_0^{\infty} e^{-(\rho-n+\varepsilon)t} \left[u(c_t) + \gamma(\Delta_t) - \psi\pi_t - \psi\varepsilon C\left(\frac{\Delta_t}{\Phi}\right) \right.$$
$$\left. + \varepsilon \left(\int_t^{\infty} e^{-(\rho-n)(s-t)} \left[u(\bar{c}_s) + \gamma(0) - \psi\bar{\pi}_s \right] ds \right) \right] dt. \tag{35}$$

This expression is derived in appendix H. At each point in time, a price level jump occurs with probability εdt and momentarily raises the cost of inflation to $\psi C(\Delta_t/\Phi)/dt$. This inflation risk is the welfare cost of running a Ponzi scheme of public debt. Note that, if households never run away from the Ponzi scheme, i.e. if $\varepsilon = 0$, then the Ponzi steady state is always superior to the neoclassical steady state, thanks to the welfare gain from higher wealth and to lower inflation equal to $\max\{-n, \pi^R\}$ instead of $-r^n$.

To solve for the optimal policy, for any value of $\Delta_0 \in [0, \infty)$, we need to characterize the equilibrium paths that exist, before determining the lowest inflation target π^* consistent with each path, and then evaluating the corresponding level of welfare using (35). The solution to the trilemma is given by the policy (Δ_0, π^*) that maximizes welfare.

For simplicity and clarity of exposition, I first consider the case where the government chooses among steady state equilibria, i.e. $\Delta_0 \in \{0, \Delta^p\}$, which captures the essence of the trilemma. I then allow for other values of $\Delta_0 \in [0, \infty)$. As my analysis relies on numerical simulations, I now calibrate the model.

5.2.1 Calibration

Households have constant relative risk aversion for consumption

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \tag{36}$$

and, following Kumhof, Rancière, and Winant (2015) and Michau (2024a), constant relative risk aversion for wealth, relative to a reference wealth level $\underline{a} < 0$,

$$\gamma(a) = k \frac{(a - \underline{a})^{1-\sigma} - 1}{1-\sigma}. \tag{37}$$

The convex welfare cost of a price level jump is given by

$$C(x) = \alpha \frac{(x+1)^{\beta} - 1}{\beta}, \tag{38}$$

where $\alpha \geq 1$ determines the cost of a price level jump relative to the cost of a continuous increase in the price level[17] and $\beta \geq 1$ determines the convexity of this cost.

I assume a 5% discount rate and constant population, which gives $\rho = 5\%$ and $n = 0\%$. Core inflation in Japan has been close to 0% on average over the past three decades, while it was close to 1% in the Eurozone from 2013 to 2021 when the zero lower bound was binding. I therefore set $\pi^R = 0.5\%$.

Eggertsson, Mehrotra, and Robbins (2019) and Rachel and Summers (2019) have estimated the U.S. natural real interest rate to be equal to -2.2% and 0.4%, respectively. However, throughout my analysis, the natural real interest rate r^n is defined by (29) as the real interest rate consistent with full employment *in the absence of Ponzi scheme*. But, Eggertsson, Mehrotra, and Robbins (2019) and Rachel and Summers (2019) have found that the rise in public indebtedness in the U.S. over the past four decades has raised the natural real interest rate by about 2%. Hence, their estimation implies that the U.S. natural real interest rate as defined by (29) is between -4.2% and -1.6%. The natural real interest rate is probably even lower in the Eurozone and in Japan.[18] I therefore set the coefficient of relative risk aversion for consumption θ such that the natural real interest rate r^n is equal to -3%, which gives $\theta = 3.56$. Note that with $r^n = -3\%$, in the absence of Ponzi scheme, a 2% inflation target is inconsistent with the economy being at full employment.

I set the scale parameter k of the preference for wealth such that, under secular

[17] An infinitesimally small price level *jump* $|dP_t/P_t|$ entails a welfare cost $\frac{C(|dP_t/P_t|)}{dt} = \frac{C(0)+|dP_t/P_t|C'(0)}{dt} = C'(0)|\pi_t| = \alpha|\pi_t|$. By (8), a *continuous* increase in price of the same magnitude entails a welfare cost $|\pi_t|$.

[18] Holston, Laubach, and Williams (2017) have estimated the natural real interest rate in the Eurozone to be about 0.7% below the U.S..

stagnation, the output gap amounts to 10% of GDP, i.e. $c^{ss} = 0.9c^n$ with c^n normalized to 1, which gives $k = 0.08$. According to Hausman and Wieland (2014), the output gap in Japan was about 10% in 2013, before the monetary and fiscal expansion of Abenomics, while Hall (2017) reported a 15% output gap for the U.S. in 2015. The reference wealth level $\underline{a}$, which gives the theoretical upper bound to household indebtedness, is set equal to one year of output at full employment, which gives $\underline{a} = -1$. Neither the calibration of other parameters nor the simulation results are very sensitive to $\underline{a}$. The present value of surpluses is set equal to 100% of GDP at full employment, which gives $\Phi = 1$. Public debt in excess of that threshold must correspond to a Ponzi scheme. The maximal magnitude of a Ponzi scheme, reached when $\varepsilon = 0$, is set equal to 150% of GDP, implying that public debt could potentially rise to 250% of GDP, but not higher. This gives $\sigma = 0.51$. Finally, the threshold magnitude of the Ponzi scheme $\bar{\Delta}$ below which a depressed economy must remain stuck in secular stagnation following the occurrence of the sunspot shock is set equal to 20% of GDP, which gives $\bar{\Delta} = 0.2$.

The calibration is summarized in Table 1. While this calibration is plausible, the model remains stylized. Importantly, the qualitative insights from my simulations are robust to plausible changes to this calibration.

Parameter	Calibrated value	Moment
Discount rate	$\rho = 5\%$	.
Population growth	$n = 0\%$	.
Reference rate of inflation for wage bargaining	$\pi^R = 0.5\%$	.
CRRA for consumption	$\theta = 3.56$	$r^n = -3\%$
CRRA for wealth (relative to reference level)	$\sigma = 0.51$	$\Delta^p = 1.5c^n$ when $\varepsilon = 0$
Scale parameter of preference for wealth	$k = 0.08$	$c^{ss} = (1 - 0.1)c^n$
Reference wealth level	$\underline{a} = -1$	$\underline{a} = -c^n$
Present value of primary surpluses	$\Phi = 1$	$\Phi = c^n$
Threshold to be stuck in depression	$\bar{\Delta} = 0.2$	$\bar{\Delta} = 0.2c^n$

Table 1: Calibration of the model

5.2.2 Choosing among steady states

Let us now solve the policy trilemma assuming that the government chooses among the three steady state equilibria. In the Ponzi steady state, when inflation is set as low as possible, we have $\pi^p = \max\left\{\varepsilon \frac{\Phi}{\Phi+\Delta^p} \frac{u'(\bar{c})}{u'(1)} - n, \pi^R\right\}$. Thus, from the welfare function (35) and after simplifications, the Ponzi steady state is the preferred option if and only if

$$u(1) + \gamma(\Delta^p) - \psi\pi^p - \psi\varepsilon C\left(\frac{\Delta^p}{\Phi}\right) \geq u(\bar{c}) + \gamma(0) - \psi\bar{\pi}, \tag{39}$$

where $\bar{c}$ and $\bar{\pi}$ are given by (34) and Δ^p by (32).[19] If this inequality is not satisfied, then the neoclassical steady state is the solution to the trilemma when ψ is below the threshold from the dilemma (34) and the secular stagnation steady is the solution when ψ is above this threshold.

The critical parameters for the policy trilemma are the the welfare cost of changes to the price level ψ and the likelihood of collapse of the Ponzi scheme ε. The parameters α and β of the cost of a price level jump (38) are also important. I therefore characterize numerically the optimal steady state equilibrium as a function of ψ and ε for different values of α and β.

I first consider the possibility that $\alpha = 5$ and $\beta = 1$. This implies that an increase in the price level is 5 times more costly when it is due to the collapse of a Ponzi scheme than when it results from a high inflation target (see footnote 17). Intuitively, when $\alpha = 5$ and $\beta = 1$, a 1% chance per year of having a 50% upward jump in the price level, which raises expected inflation by 0.5%, is as costly as a $0.005\alpha = 2.5\%$ anticipated increase in inflation. The idea is that sudden and uncontrolled fiscal inflation generates financial disruption and a loss of monetary policy credibility that makes it markedly more costly than a smooth and anticipated increase in the price level induced by monetary policy. Similarly, the sovereign debt literature typically assumes a sizeable deadweight cost of default (Aguiar and Amador, 2021).

The optimal steady state, as a function of ψ and ε, is displayed in Figure 2. Recall that, when $\varepsilon = 0$, the Ponzi scheme is of maximal magnitude (calibrated to be equal to 150% of GDP) and never collapses; while, when $\varepsilon = n - r^n = 3\%$, the Ponzi scheme is of zero magnitude. Figure 2 shows that when either ε or ψ is close to zero, a Ponzi scheme is either so unlikely to collapse or the cost of a collapse is so low that it is the preferred option, thanks to the welfare gain from higher wealth. The vertical dashed line corresponds to the threshold from the dilemma, given by (34). After the collapse of the Ponzi scheme, the economy must be in the neoclassical steady state to the left of this line and in secular stagnation to the right.[20]

As ε increases, a Ponzi scheme is more likely to collapse and to induce an upward jump in the price level, making it less attractive. At some point, a Ponzi scheme is no longer desirable. If the cost of inflation ψ is below the threshold from the dilemma given by (34), then the neoclassical steady state is optimal even though inflation permanently rises to $-r^n = 3\%$; while, if the cost of inflation is above the threshold, then a permanently depressed economy with consumption equal to $c^{ss} = 0.9 < c^n = 1$ is preferable.

[19] Note that, in the Ponzi steady state, we have $c = 1$. Thus, even if $\Delta^p < \bar{\Delta}$, $\bar{c}$ and $\bar{\pi}$ are given by (34) following the collapse of the Ponzi scheme.

[20] This implies that, before the price level jump, the Ponzi steady state yields slightly higher welfare to the left of that line, where $\bar{c} = 1$, than to the right, where $\bar{c} = c^{ss}$.

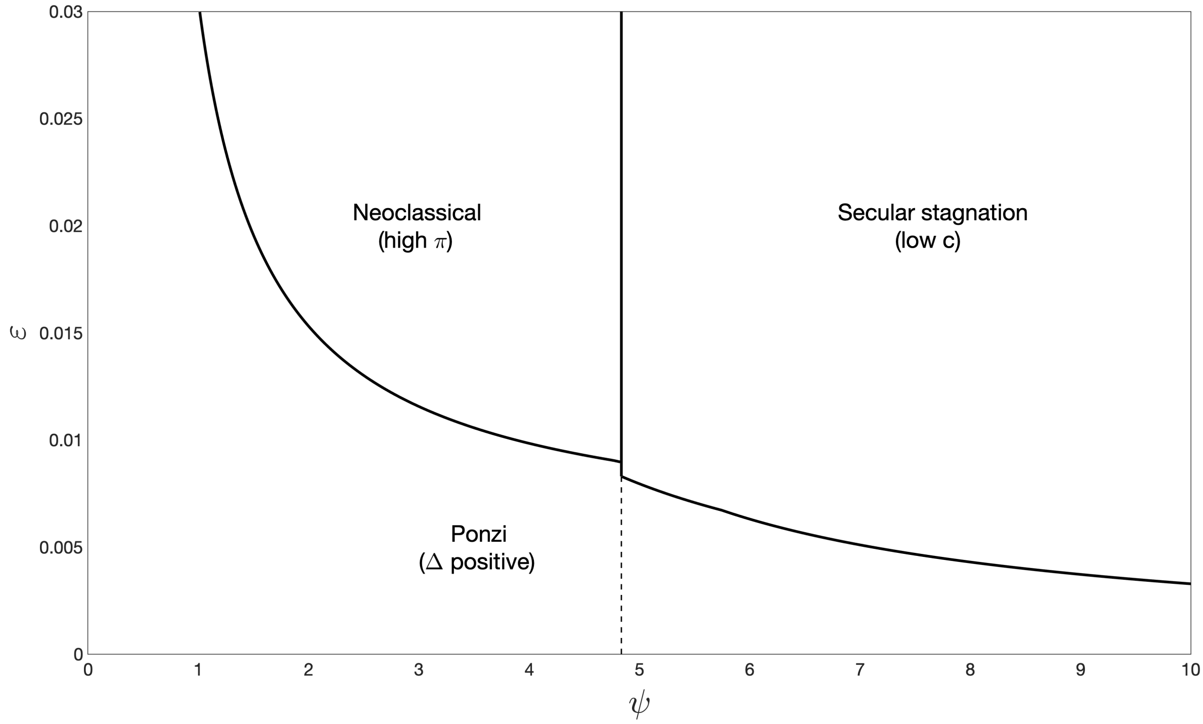

Figure 2: Trilemma for $\alpha = 5$ and $\beta = 1$

What is the solution to the trilemma when the cost of the price level jump is a convex function of the magnitude of the jump? To emphasize the effect of convexity, I set $\alpha = 1$ and $\beta = 6$. With $\alpha = 1$, there is no discontinuity between a continuous rise in the price level and an infinitesimal jump. With $\beta = 6$, a 100% increase in the price level is about 6 times more costly than a 50% increase. The corresponding trilemma is displayed in Figure 3. A small likelihood of collapse ε entails a large magnitude of the Ponzi scheme and, hence, a sizeable welfare cost when the price level jump does occur. So the Ponzi steady state is optimal when ε is either so large that the Ponzi scheme is small or so close to zero that the Ponzi scheme is unlikely to collapse. The neoclassical steady state with permanently higher inflation becomes optimal for intermediate values of ε. The secular stagnation steady state remains optimal for a high welfare cost of inflation ψ, unless ε is very close to zero.

Another situation of interest arises when $\alpha = 1$ and $\beta = 1$, implying that changes to the price level are equally costly whether they occur through jumps or through continuous changes.[21] As shown in Figure 4, under our calibration, the neoclassical steady state is never preferred to the Ponzi steady state: the welfare benefit from the Ponzi scheme $\gamma(\Delta^p)$ outweighs the expected cost of a higher price level $\psi\varepsilon\Delta^p/\Phi$.

[21] Formally, when $\alpha = 1$ and $\beta = 1$, we have $C(x) = x$ and, hence, equation (8) simplifies to $c(dP_t/P_t) = |dP_t/P_t|/dt$ regardless of whether a price level jump occurs.

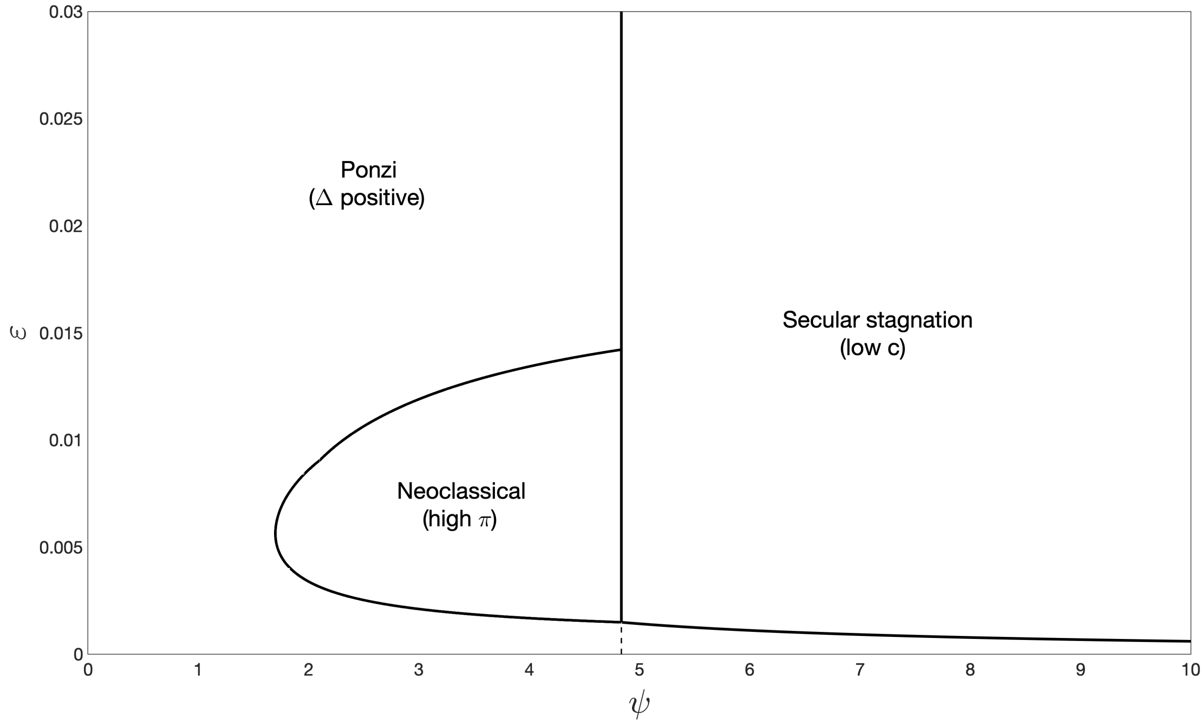

Figure 3: Trilemma for $\alpha = 1$ and $\beta = 6$

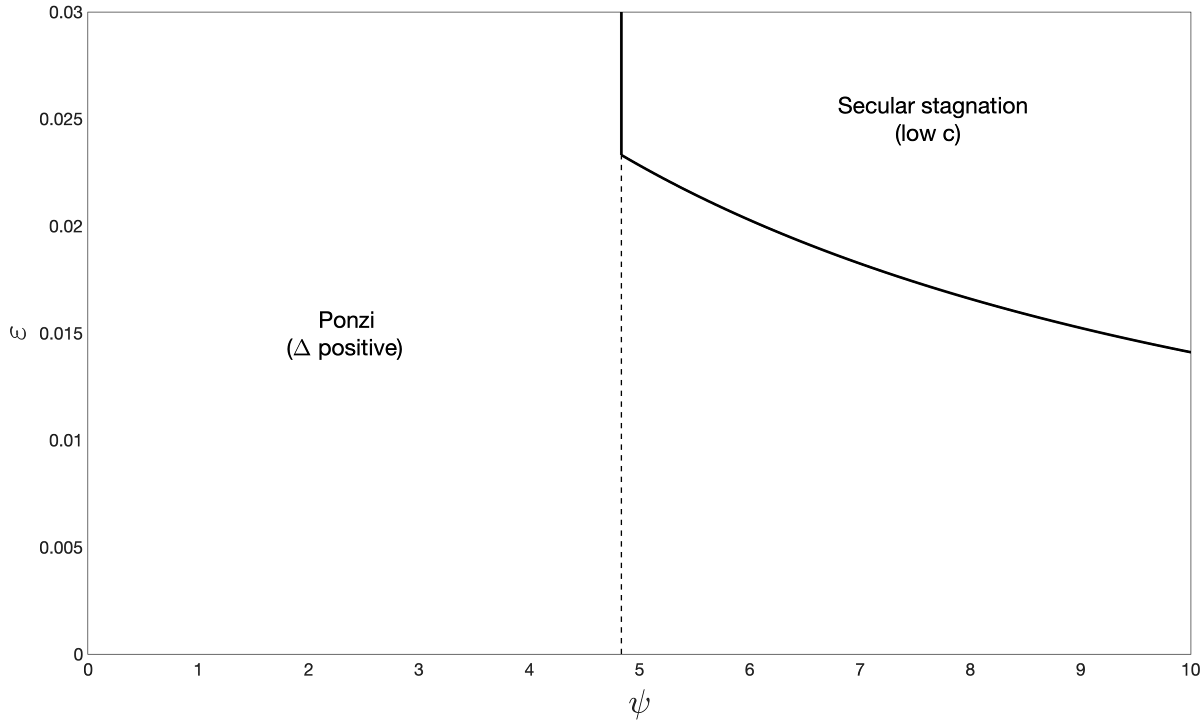

Figure 4: Trilemma (or dilemma) for $\alpha = 1$ and $\beta = 1$

5.2.3 Starting away from steady state

Let us now allow for the possibility that the government sets an initial magnitude of the Ponzi scheme Δ_0 that is inconsistent with being in steady state. As shown in appendix I, there are three equilibrium possibilities: i) a depressed path with $c_t < 1$ converging to the Ponzi steady state, ii) a depressed path with $c_t < 1$ converging to the secular stagnation steady state, and iii) and a full employment path converging to the neoclassical steady state. The existence conditions for each possibility is given in appendix I.

For each Δ_0, I compute the equilibrium paths that exist, determine the lowest inflation target π^* consistent with each path, and then compute the corresponding level of welfare using (35). The solution to the trilemma is given by the policy (Δ_0, π^*) that yields the highest welfare. Figure 5 displays the unrestricted solution to the policy trilemma when $\alpha = 5$ and $\beta = 1$. The white surface corresponds to situation where the optimal policy consists in going straight to a steady state, as in the previous subsection. The light grey area corresponds to an optimal path with a depressed economy converging to the Ponzi steady state, while the dark grey area corresponds to a depressed economy converging to the secular stagnation steady state.

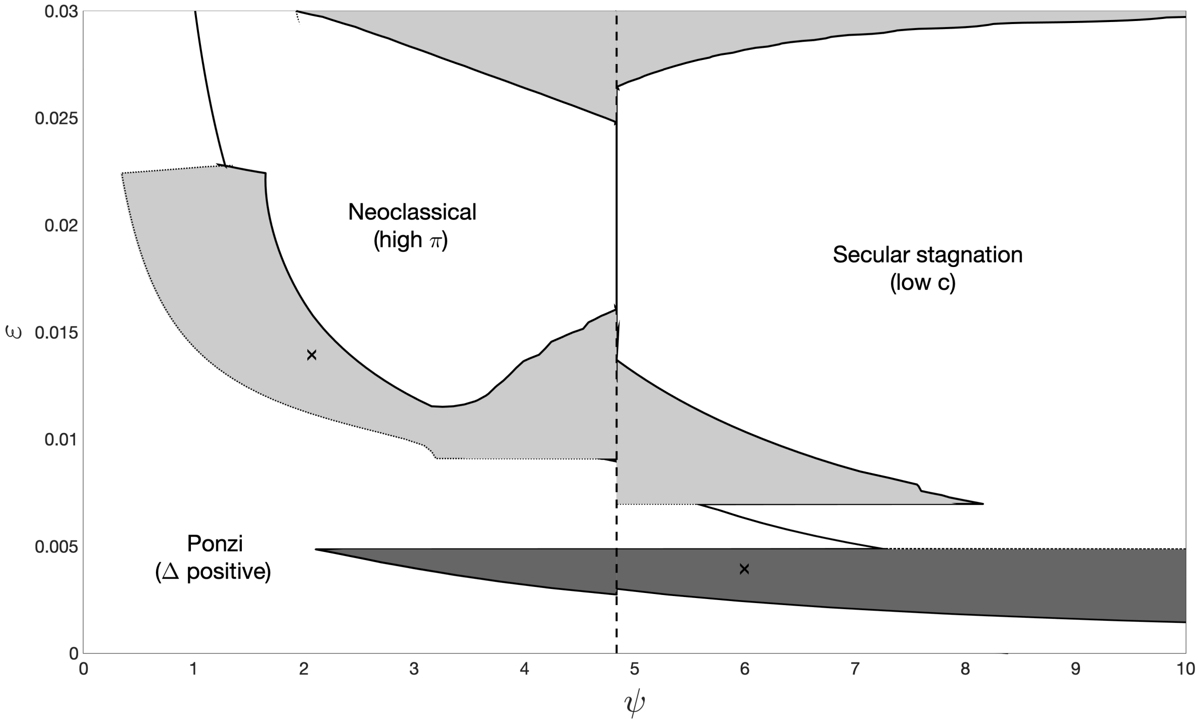

Figure 5: Trilemma for $\alpha = 5$ and $\beta = 1$ with $\Delta_0 \in [0, \infty)$

To understand the rationale for setting the economy on a depressed path leading to the Ponzi steady state, let us consider the situation where $\varepsilon = 0.014$ and $\psi = 2.1$ (shown in Figure 5 by a cross within the light grey area). The corresponding equilibrium path, which is optimal, is displayed in Figure 6. While the economy fails to produce at full capacity for 8.3 years, it is nonetheless close to full employment thanks to the stimulating wealth effect of the Ponzi scheme. The main benefit from being away from the Ponzi steady state is that the downward nominal wage rigidity is binding, resulting in low inflation. This can be seen from the right panel of Figure 6, which displays both the inflation rate π_t and the effective rate of inflation from a welfare perspective given by $\pi_t + \varepsilon C(\Delta_t/\Phi_t)$.

Figure 6: Optimal equilibrium path when $\alpha = 5$, $\beta = 1$, $\varepsilon = 0.014$, and $\psi = 2.1$.

This theoretical possibility is reminiscent of the situation of Japan over the previous decades. The economy has remained depressed with inflation persistently well below target. But, to reduce the shortfall in the output gap, Japan has implemented a highly expansionary fiscal policy eventually resulting in a debt-to-GDP ratio over 250%. This suggests that the Japanese economy has navigated between the Ponzi and the secular stagnation steady state.

Figure 7 displays the optimal equilibrium path for $\varepsilon = 0.004$ and $\psi = 6.0$ (shown in Figure 5 by a cross within the dark grey area), which converges to the secular stagnation steady state. The initial magnitude of the Ponzi scheme Δ_0 is such that, at time 0, the economy produces at full capacity.[22] The Ponzi scheme shrinks at a very low rate, below 1% per year. Convergence to the secular stagnation steady state therefore takes centuries. The main benefit from this equilibrium path is to be nearly at full employment with utility from Ponzi wealth, while having very low inflation thanks to the binding downward nominal wage rigidity.

Multiplicity of equilibrium paths. Throughout this analysis, I have been assuming that households spontaneously coordinate on the best equilibrium path consistent with the policy chosen by the government. While this assumption seems strong, it should be emphasized that for a given policy (Δ_0, π^*), the number of possible equilibrium paths is limited. If the optimal path converges to either the secular stagnation

[22]The initial magnitude of the Ponzi scheme Δ_0 is equal to 104% of GDP, which is smaller than in the Ponzi steady state for $\varepsilon = 0.004$ where $\Delta^p = 116\%$ of GDP.

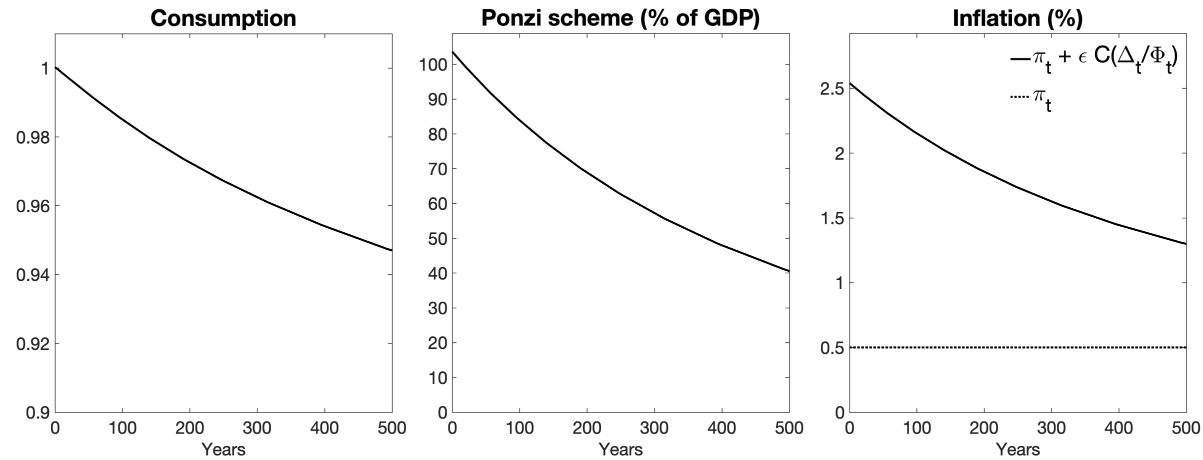

Figure 7: Optimal equilibrium path when $\alpha = 5$, $\beta = 1$, $\varepsilon = 0.004$, and $\psi = 6.0$.

or the Ponzi steady state, the inflation target is too low to be consistent with the existence of the neoclassical steady state. Moreover, convergence to the secular stagnation steady state requires $-\pi^R - n + \varepsilon < 0$, while convergence to the Ponzi steady state requires $-\pi^R - n + \varepsilon \frac{\Phi}{\Phi+\Delta^p} \frac{u'(\bar{c}_t)}{u'(1)} > 0$ (see appendix I). Under my calibration, for any value of ε, these two conditions are mutually inconsistent. Hence, if the optimal policy (Δ_0, π^*) induces convergence to either the secular stagnation or the Ponzi steady state, then the corresponding equilibrium path is unique.

If the optimal path converges to the neoclassical steady state, the corresponding policy (Δ_0, π^*) is typically also consistent with another equilibrium path converging to one of the other two steady states. Our assumption that households spontaneously coordinate on the best equilibrium path is equivalent to assuming that inflation will be on target provided that this is possible. Hence, if the inflation target is set sufficiently high to have full employment, inflation will be on target and the economy will produce at full capacity. Michau (2023) provides a detailed investigation of the difficulty of moving from the secular stagnation to the neoclassical steady state when inflation is persistent. In theory, a state-contingent fiscal policy can eliminate the low inflation equilibrium by committing to engage in massive government spending whenever inflation falls below target. This is an off-the-equilibrium threat that entails a zero welfare cost. But, when fiscal policy must be non-contingent, the transition from stagnation to full employment can be quite costly, reducing the attractiveness of the neoclassical steady state. These considerations are beyond the scope of this analysis, which emphasizes that, even in the best case scenario where this equilibrium selection problem can easily be overcome, a fundamental trilemma remains.

Finally, note that, in the absence of Ponzi scheme, a private sector bubble may arise. On the one hand, it can boost aggregate demand without necessarily being a threat to price stability. But, on the other hand, the bursting of a bubble can be a major source of financial disruption, especially in the presence of credit constraints. A careful analysis of bubbles is beyond the scope of this analysis.

6 Persistent lack of demand

So far, we have assumed a *permanent* lack of demand, with a permanently depressed natural real interest rate. But the analysis remains unchanged for a sufficiently *persistent* lack of demand. In particular, we can assume that ε is the likelihood of a decline in the marginal utility of wealth, which raises aggregate demand sufficiently to eliminate the secular stagnation and the Ponzi steady state. Hence, the upward jump in the price level can be driven by a fundamental shock, rather than by a sunspot shock. Appendix J shows that the analysis remains unchanged, with the equilibrium before the shock still given by (23)-(28), but with $\bar{c}_t = 1$.

The steady states also remain unchanged, except for the secular stagnation steady state that is now characterized by a higher output level, i.e. by less underemployment, since the prospect of an economic recovery raises aggregate demand. The existence condition for the secular stagnation steady state remains unchanged, and given by $r^n < -\pi^R$. The existence of a Ponzi scheme still requires $\varepsilon < n - r^n$ with r^n given by (29). Hence, the existence of the trilemma now requires a very persistent, rather than a permanent, depression in aggregate demand.

7 Breaking through the trilemma

Is there a way to break through the policy trilemma? First, the country can switch to electronic currency, i.e. abolish cash, such as to remove the zero lower bound on the nominal interest rate. A negative nominal interest rate can then be implemented by taxing bank deposits, which, in practice, would make such a reform politically difficult to implement. Inflation could always be set equal to π^R and, hence, secular stagnation would never be optimal. The Ponzi steady state would be preferred to the neoclassical steady state if and only if $\gamma(\Delta^p) - \psi\varepsilon C\left(\Delta^p/\Phi\right) \geq \gamma(0)$.[23]

Another way to circumvent the zero lower bound is through tax policy. As shown in Michau (2018), a negative nominal interest rate can be replicated through either a wealth tax or an increasing rate of consumption tax, together with offsetting adjustments to the taxation of labor and investment. However, in practice, wealth is neither easily observable nor very liquid, while the rate of consumption tax can hardly keep increasing for a prolonged period of time.

[23] Note that if the planner, unlike households, does not value Ponzi wealth, then a Ponzi scheme would never be desirable.

Government spending financed by lump-sum taxes can boost aggregate demand such as to overcome the trilemma. However, if households do not value the goods and services that are publicly produced, it is preferable to have a negative output gap such as to minimize the disutility from supplying labor.

In a heterogeneous household economy, redistribution policy can be used to break through the trilemma. The government can stimulate aggregate demand by redistributing income from households with a low marginal propensity to consume to those with a higher one (Rachel and Summers, 2019). However, the scope for such policies seems limited in Europe and Japan, where aggregate demand is particularly weak despite an already extensive welfare state. Moreover, a one-off redistribution of wealth from rich-thrifty households to poor-spendthrift ones can only temporarily boost aggregate demand, until wealth inequality regains its original level (Illing, Ono, and Schlegl, 2018; Mian, Straub, and Sufi, 2021). Such policies therefore require a constant flow of income redistribution across households, and therefore a permanent distortion to the allocation of resources.

These considerations suggest that the trilemma cannot easily be overcome.

8 Conclusion

This paper has shown that, when aggregate demand is permanently or persistently depressed, we cannot simultaneously have full employment, low inflation, and no Ponzi scheme.[24]

While I have assumed that inflation should ideally be as low as possible, alternatively the central bank can be strongly committed to its inflation target π^* with $\pi^* \in (\pi^R, -r^n)$, which rules out the neoclassical steady state. In that case, the government must choose between secular stagnation, where the liquidity trap makes monetary policy unable to raise inflation from π^R to π^*, and a Ponzi scheme, where monetary policy cannot prevent a price level jump that temporarily raises inflation much above π^*. In both cases, when inflation is too low and when it is too high, monetary policy is powerless. A persistently low real interest rate therefore entails a major challenge to the inflation targeting framework of central banks whenever the inflation target is below $-r^n$.

My analysis has abstracted from capital accumulation. When $r^n < n$, the capital stock is below the golden rule level, implying that the crowding out of capital by the Ponzi scheme is welfare enhancing (as in a dynamically inefficient OLG economy). But, in the presence of financial frictions, we can simultaneously have $r^n < n$ and the

[24]Under a preference for wealth, rather than a preference for *net* wealth, high public debt would replace the Ponzi scheme within the trilemma.

capital stock below the golden rule level, implying that the crowding out of capital is a detrimental consequence of the Ponzi scheme. A careful quantitative analysis of the trilemma under financial frictions is left to future research.

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A Government's flow of funds

By Itô's lemma with jumps, if X_t follows a continuous time stochastic process with jumps given by $dX_t = \mu(X_t)dt + \gamma(X_t)dJ_t$, then for any differentiable function $f(\cdot)$ we have

$$df(X_t) = \frac{\partial f(X_t)}{\partial X_t}\mu(X_t)dt + \left[f(X_t + \gamma(X_t)) - f(X_t)\right]dJ_t.$$

Applying this lemma to $dP_t = \pi_t P_t dt + (\Delta_t/\Phi_t)P_t dJ_t$ yields

$$\begin{aligned} d\left(\frac{1}{P_t}\right) &= \frac{-1}{P_t^2}\pi_t P_t dt + \left(\frac{1}{P_t + \frac{\Delta_t}{\Phi_t}P_t} - \frac{1}{P_t}\right)dJ_t, \\ &= \frac{-\pi_t}{P_t}dt - \frac{\Delta_t}{b_t}\frac{1}{P_t}dJ_t, \end{aligned} \tag{A1}$$

where $b_t = \Phi_t + \Delta_t$.

We therefore have

$$\begin{aligned} d\left(\frac{B_t}{P_t N_t}\right) &= \frac{dB_t}{P_t N_t} + \frac{B_t}{N_t}d\left(\frac{1}{P_t}\right) + \frac{B_t}{P_t}d\left(\frac{1}{N_t}\right), \\ &= \frac{i_t B_t - \tau_t P_t N_t}{P_t N_t}dt + \frac{B_t}{N_t}\left(\frac{-\pi_t}{P_t}dt - \frac{\Delta_t}{b_t}\frac{1}{P_t}dJ_t\right) - \frac{B_t}{P_t}\frac{nN_t}{N_t^2}dt, \\ &= \left[(i_t - \pi_t - n)b_t - \tau_t\right]dt - \Delta_t dJ_t, \end{aligned}$$

where the second line was obtained using equations (1), (A1), and the fact that $N_t = e^{nt}$.

B Solving the household's problem

To provide sufficient conditions that characterize a solution to the household's problem, let us introduce slightly more specific notations than in the text. The only source of uncertainty is the time T when the price level jump occurs, which follows an exponential distribution with parameter ε. A state-contingent allocation chosen by a household can therefore be denoted by $(\tilde{c}_t, \tilde{a}_t, \tilde{b}_t^h, (\bar{c}_s(t), \bar{a}_s(t))_{s=t}^{\infty})_{t=0}^{\infty}$, where $\tilde{c}_t$, $\tilde{a}_t$, and $\tilde{b}_t^h$ denote consumption, wealth, and government bond holdings at time t conditional on the absence of a jump, while $\bar{c}_t(T)$ and $\bar{a}_t(T)$ denote consumption and wealth at time t conditional on the price level having jumped at time T.[25]

[25] After time T, the economy is risk-free and the household no longer needs to make a portfolio decision.

Using these notations, the Euler equation before the jump in the price level is

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \left(\frac{u'(\tilde{c}_t)}{-\tilde{c}_t u''(\tilde{c}_t)}\right)\left[r_t - \rho + \frac{\gamma'(\tilde{a}_t - b_t + \Delta_t)}{u'(\tilde{c}_t)} + \varepsilon\left(\frac{u'(\bar{c}_t(t))}{u'(\tilde{c}_t)} - 1\right)\right], \tag{A2}$$

where $\bar{c}_t(t)$ is the consumption level immediately after a price level jump occurring at time t. Once the price level jump has occurred at time T, the Euler equation simplifies to

$$\frac{\dot{\bar{c}}_t(T)}{\bar{c}_t(T)} = \left(\frac{u'(\bar{c}_t(T))}{-\bar{c}_t(T)u''(\bar{c}_t(T))}\right)\left[r_t - \rho + \frac{\gamma'(\bar{a}_t(T) - b_t)}{u'(\bar{c}_t(T))}\right]. \tag{A3}$$

The expression for the risk premium (13) can be expressed as

$$r_t = i_t - \pi_t - \varepsilon\frac{\Delta_t}{b_t}\frac{u'(\bar{c}_t(t))}{u'(\tilde{c}_t)}. \tag{A4}$$

As T is exponentially distributed, we have

$$\begin{aligned}
\mathbb{E}_0\left[e^{-(\rho-n)t}u'(c_t)a_t\right] &= \int_0^\infty \varepsilon e^{-\varepsilon T}\left[e^{-(\rho-n)t}u'(c_t)a_t\right]dT, \\
&= \int_0^t \varepsilon e^{-\varepsilon T}\left[e^{-(\rho-n)t}u'(\bar{c}_t(T))\bar{a}_t(T)\right]dT \\
&\qquad + \int_t^\infty \varepsilon e^{-\varepsilon T}\left[e^{-(\rho-n)t}u'(\tilde{c}_t)\tilde{a}_t\right]dT, \\
&= \int_0^t \varepsilon e^{-\varepsilon T}\left[e^{-(\rho-n)t}u'(\bar{c}_t(T))\bar{a}_t(T)\right]dT, \\
&\qquad + e^{-(\rho-n+\varepsilon)t}u'(\tilde{c}_t)\tilde{a}_t.
\end{aligned} \tag{A5}$$

Hence, the transversality condition (14) can be written as

$$\lim_{t\to\infty}\left[e^{-(\rho-n+\varepsilon)t}u'(\tilde{c}_t)\tilde{a}_t + \int_0^t \varepsilon e^{-\varepsilon T}\left[e^{-(\rho-n)t}u'(\bar{c}_t(T))\bar{a}_t(T)\right]dT\right] = 0. \tag{A6}$$

I shall now prove that (A2), (A3), (A4), and (A6) are sufficient to characterize a solution to the household's problem.[26]

Any feasible allocation satisfies the household's flow of funds equation (10) with a_0 given and the no-borrowing constraint (11). Let $(\tilde{c}_t^*, \tilde{a}_t^*, \tilde{b}_t^{h*}, (\bar{c}_s^*(t), \bar{a}_s^*(t))_{s=t}^\infty)_{t=0}^\infty$ denote any feasible allocation that satisfies (A2), (A3), (A4), and (A6). Throughout the proof, I am assuming that such allocations satisfy the no-borrowing constraint (11), which is therefore non-binding.

The objective of the household is to maximize the following objective, where the cost of changes to the price level is omitted since it is exogenous to the household's

[26]Note that the Euler equations (A2) and (A3) as well as the equation for the risk-premium (A4) can be derived from the Hamilton-Jacobi-Bellman equation for the household's problem.

behavior,

$$
\begin{aligned}
\mathbb{E}_0 & \left[\int_0^{\infty} e^{-(\rho-n)t} \left[u(c_t) + \gamma(a_t - b_t + \Delta_t)\right] dt\right] \\
&= \int_0^{\infty} \varepsilon e^{-\varepsilon T} \left[\int_0^{T} e^{-(\rho-n)t} \left[u(\tilde{c}_t) + \gamma(\tilde{a}_t - b_t + \Delta_t)\right] dt \right. \\
&\qquad\qquad \left. + \int_T^{\infty} e^{-(\rho-n)t} \left[u(\bar{c}_t(T)) + \gamma(\bar{a}_t(T) - b_t)\right] dt\right] dT, \\
&= \int_0^{\infty} \left[\int_t^{\infty} \varepsilon e^{-\varepsilon T} dT\right] e^{-(\rho-n)t} \left[u(\tilde{c}_t) + \gamma(\tilde{a}_t - b_t + \Delta_t)\right] dt \\
&\qquad + \int_0^{\infty} \varepsilon e^{-\varepsilon T} \left[\int_T^{\infty} e^{-(\rho-n)t} \left[u(\bar{c}_t(T)) + \gamma(\bar{a}_t(T) - b_t)\right] dt\right] dT, \\
&= \int_0^{\infty} e^{-(\rho-n+\varepsilon)t} \left[u(\tilde{c}_t) + \gamma(\tilde{a}_t - b_t + \Delta_t)\right] dt \\
&\qquad + \int_0^{\infty} \varepsilon e^{-\varepsilon T} \left[\int_T^{\infty} e^{-(\rho-n)t} \left[u(\bar{c}_t(T)) + \gamma(\bar{a}_t(T) - b_t)\right] dt\right] dT.
\end{aligned}
$$

Let D be defined by

$$
D = \mathbb{E}_0 \left[\int_0^{\infty} e^{-(\rho-n)t} \left[u(c_t^*) + \gamma(a_t^* - b_t + \Delta_t) - u(c_t) - \gamma(a_t - b_t + \Delta_t)\right] dt\right]. \qquad \text{(A7)}
$$

I shall now show that, under equations (A2), (A3), (A6), and (A4), we must have $D \geq 0$.

To establish this result, the following lemma will be used.

Lemma A1 *For any differentiable and concave functions $u(\cdot)$ and $\gamma(\cdot)$, we have*

$$
u(c_t^*)+\gamma(a_t^*)-u(c_t)-\gamma(a_t) \geq \left(\dot{a}_t - \dot{a}_t^*\right) u'(c_t^*) - \left(a_t - a_t^*\right)\left[r_t u'(c_t^*) + \gamma'(a_t^*)\right] - i_t(b_t - b_t^*) u'(c_t^*),
$$

where c_t is given by $c_t = r_t a_t + i_t b_t + x_t - \dot{a}_t$ for some scalars r_t, i_t, and x_t at time t.

Proof. Let c_t be defined by

$$
c_t = r_t a_t + i_t b_t + x_t - \frac{a_{t+\delta t} - a_t}{\delta t},
$$

with $\delta > 0$. We have

$$
\begin{aligned}
u(c_t) + \gamma(a_t) &= u\left(r_t a_t + i_t b_t + x_t - \frac{a_{t+\delta t} - a_t}{\delta t}\right) + \gamma(a_t), \\
&= u\left(\left(\frac{1}{\delta t} + r_t\right) a_t + i_t b_t + x_t - \frac{a_{t+\delta t}}{\delta t}\right) + \gamma(a_t).
\end{aligned}
$$

For δ sufficiently small, $1/(\delta t) + r_t$ must be positive. Hence, the above expression is concave in a_t, $-a_{t+\delta t}$, and $i_t b_t$. It follows that

$$\begin{aligned} u(c_t)+\gamma(a_t) &\leq u(c_t^*)+\gamma(a_t^*)+(a_t-a_t^*)\left[\left(\frac{1}{\delta t}+r_t\right)u'(c_t^*)+\gamma'(a_t^*)\right] \\ &\qquad -(a_{t+\delta t}-a_{t+\delta t}^*)\frac{u'(c_t^*)}{\delta t}+i_t(b_t-b_t^*)u'(c_t^*), \\ &\leq u(c_t^*)+\gamma(a_t^*)+(a_t-a_t^*)\left[r_t u'(c_t^*)+\gamma'(a_t^*)\right] \\ &\qquad -\left(\frac{a_{t+\delta t}-a_t}{\delta t}-\frac{a_{t+\delta t}^*-a_t^*}{\delta t}\right)u'(c_t^*)+i_t(b_t-b_t^*)u'(c_t^*), \end{aligned}$$

where $c_t^* = r_t a_t^* + i_t b_t^* + x_t - \frac{a_{t+\delta t}^* - a_t^*}{\delta t}$. Taking the limit as δ tends to zero gives the desired result. ■

For any $t \neq T$, by the household's flow of funds constraint (10), we have

$$\dot{a}_t = (r_t-n)a_t + w_t L_t - \tau_t - c_t + b_t^h(i_t - \pi_t - r_t),$$

with $r_t = i_t - \pi_t$ for $t \geq T$. Hence, from Lemma A1, for any $t \neq T$ we have

$$\begin{aligned} u(c_t^*)+\gamma(a_t^*-b_t+\Delta_t)-u(c_t)-\gamma(a_t-b_t+\Delta_t) &\geq (\dot{a}_t-\dot{a}_t^*)\,u'(c_t^*) \\ -(a_t-a_t^*)\left[(r_t-n)\,u'(c_t^*)+\gamma'(a_t^*-b_t+\Delta_t)\right] &-(i_t-\pi_t-r_t)(b_t^h-b_t^{h*})u'(c_t^*). \end{aligned}$$

Substituting this inequality into the expression for D yields

$$\begin{aligned} D \geq & \int_0^\infty e^{-(\rho-n+\varepsilon)t}\Big[\left(\dot{\tilde{a}}_t-\dot{\tilde{a}}_t^*\right)u'(\tilde{c}_t^*)-\left(\tilde{a}_t-\tilde{a}_t^*\right)\Big[(r_t-n)\,u'(\tilde{c}_t^*)+\gamma'(\tilde{a}_t^*-b_t+\Delta_t)\Big] \\ & \qquad -(i_t-\pi_t-r_t)(\tilde{b}_t^h-\tilde{b}_t^{h*})u'(\tilde{c}_t^*)\Big]\,dt \\ & +\int_0^\infty \varepsilon e^{-\varepsilon T}\Big[\int_T^\infty e^{-(\rho-n)t}\left[\left[\left(\dot{\bar{a}}_t(T)-\dot{\bar{a}}_t^*(T)\right)u'(\bar{c}_t^*(T))\right]\right. \\ & \qquad \left.-\left(\bar{a}_t(T)-\bar{a}_t^*(T)\right)\left[(r_t-n)\,u'(\bar{c}_t^*(T))+\gamma'(\bar{a}_t^*(T)-b_t)\right]\right]dt\Big]\,dT. \end{aligned} \tag{A8}$$

Integrating by parts, we have

$$\begin{aligned} & \int_0^\infty e^{-(\rho-n+\varepsilon)t}\left(\dot{\tilde{a}}_t-\dot{\tilde{a}}_t^*\right)u'(\tilde{c}_t^*)dt \\ &= \left[\lim_{t\to\infty} e^{-(\rho-n+\varepsilon)t}\left(\tilde{a}_t-\tilde{a}_t^*\right)u'(\tilde{c}_t^*)\right]-\left(\tilde{a}_0-\tilde{a}_0^*\right)u'(\tilde{c}_0^*) \\ &\qquad -\int_0^\infty \left(\tilde{a}_t-\tilde{a}_t^*\right)d\left[e^{-(\rho-n+\varepsilon)t}u'(\tilde{c}_t^*)\right], \\ &= \left[\lim_{t\to\infty} e^{-(\rho-n+\varepsilon)t}\left(\tilde{a}_t-\tilde{a}_t^*\right)u'(\tilde{c}_t^*)\right] \\ &\qquad -\int_0^\infty e^{-(\rho-n+\varepsilon)t}\left(\tilde{a}_t-\tilde{a}_t^*\right)\left[-(\rho-n+\varepsilon)u'(\tilde{c}_t^*)+u''(\tilde{c}_t^*)\dot{\tilde{c}}_t^*\right]dt, \end{aligned}$$

where, to obtain the last line, I have used the fact that initial wealth is exogenously given, implying that $\tilde{a}_0 = \tilde{a}_0^*$. Similarly, we have

$$\begin{aligned}
&\int_T^\infty e^{-(\rho-n)t} \left(\dot{\bar{a}}_t(T) - \dot{\bar{a}}_t^*(T)\right) u'(\bar{c}_t^*(T)) dt \\
&\quad = \left[\lim_{t\to\infty} e^{-(\rho-n)t} \left(\bar{a}_t(T) - \bar{a}_t^*(T)\right) u'(\bar{c}_t^*(T))\right] - e^{-(\rho-n)T} \left(\bar{a}_T(T) - \bar{a}_T^*(T)\right) u'(\bar{c}_T^*(T)) \\
&\qquad\qquad - \int_T^\infty e^{-(\rho-n)t} \left(\bar{a}_t(T) - \bar{a}_t^*(T)\right) \left[-(\rho-n)u'(\bar{c}_t^*(T)) + u''(\bar{c}_t^*(T))\dot{\bar{c}}_t^*(T)\right] dt.
\end{aligned}$$

Substituting these two equations into (A8) yields

$$\begin{aligned}
D \;\geq\; & \int_0^\infty e^{-(\rho-n+\varepsilon)t}(\tilde{a}_t - \tilde{a}_t^*) \left[(\rho - n + \varepsilon)u'(\tilde{c}_t^*) - u''(\tilde{c}_t^*)\dot{\tilde{c}}_t^*\right. \\
& \qquad\qquad \left. - (r_t - n)\, u'(\tilde{c}_t^*) - \gamma'(\tilde{a}_t^* - b_t + \Delta_t)\right] dt \\
& - \int_0^\infty e^{-(\rho-n+\varepsilon)t}(i_t - \pi_t - r_t)(\tilde{b}_t^h - \tilde{b}_t^{h*})u'(\tilde{c}_t^*)dt \\
& - \int_0^\infty \varepsilon e^{-\varepsilon T} \left[e^{-(\rho-n)T} \left(\bar{a}_T(T) - \bar{a}_T^*(T)\right) u'(\bar{c}_T^*(T))\right] dT \\
& + \int_0^\infty \varepsilon e^{-\varepsilon T} \left[\int_T^\infty e^{-(\rho-n)t}(\bar{a}_t(T) - \bar{a}_t^*(T)) \left[(\rho-n)u'(\bar{c}_t^*(T)) - u''(\bar{c}_t^*(T))\dot{\bar{c}}_t^*(T)\right.\right. \\
& \qquad\qquad \left.\left. - (r_t - n)\, u'(\bar{c}_t^*(T)) - \gamma'(\bar{a}_t^*(T) - b_t)\right] dt\right] dT \\
& + \lim_{t\to\infty} \left[e^{-(\rho-n+\varepsilon)t} \left(\tilde{a}_t - \tilde{a}_t^*\right) u'(\tilde{c}_t^*)\right. \\
& \qquad\qquad \left. + \int_0^\infty \varepsilon e^{-\varepsilon T} \left[e^{-(\rho-n)t} \left(\bar{a}_t(T) - \bar{a}_t^*(T)\right) u'(\bar{c}_t^*(T))\right] dT\right]. \qquad \text{(A9)}
\end{aligned}$$

By the households' flow of funds constraint (10), at time T, when the price level jumps, we have

$$\bar{a}_T(T) = \tilde{a}_T - \tilde{b}_T^h \frac{\Delta_T}{b_T}.$$

This implies

$$\begin{aligned}
&\int_0^\infty \varepsilon e^{-\varepsilon T} \left[e^{-(\rho-n)T} \left(\bar{a}_T(T) - \bar{a}_T^*(T)\right) u'(\bar{c}_T^*(T))\right] dT \\
&\qquad = \int_0^\infty e^{-(\rho-n+\varepsilon)T} \left[(\tilde{a}_T - \tilde{a}_T^*) - (\tilde{b}_T^h - \tilde{b}_T^{h*})\frac{\Delta_T}{b_T}\right] \varepsilon u'(\bar{c}_T^*(T))dT.
\end{aligned}$$

Substituting this expression into the third term of the right-hand side of the inequality

(A9) and rearranging terms yields

$$
\begin{aligned}
D \geq & \int_0^\infty e^{-(\rho-n+\varepsilon)t}(\tilde{a}_t - \tilde{a}_t^*)u'(\tilde{c}_t^*) \\
& \qquad \left[\rho + \varepsilon - \frac{u''(\tilde{c}_t^*)}{u'(\tilde{c}_t^*)}\dot{\tilde{c}}_t^* - r_t - \frac{\gamma'(\tilde{a}_t^* - b_t + \Delta_t)}{u'(\tilde{c}_t^*)} - \varepsilon\frac{u'(\bar{c}_t^*(t))}{u'(\tilde{c}_t^*)}\right] dt \\
& + \int_0^\infty e^{-(\rho-n+\varepsilon)t}(\tilde{b}_t^h - \tilde{b}_t^{h*})u'(\tilde{c}_t^*)\left[\varepsilon\frac{\Delta_t}{b_t}\frac{u'(\bar{c}_t^*(t))}{u'(\tilde{c}_t^*)} - i_t + \pi_t + r_t\right] dt \\
& + \int_0^\infty \varepsilon e^{-\varepsilon T}\left[\int_T^\infty e^{-(\rho-n)t}(\bar{a}_t(T) - \bar{a}_t^*(T))u'(\bar{c}_t^*(T))\right. \\
& \qquad \left.\left[\rho - \frac{u''(\bar{c}_t^*(T))}{u'(\bar{c}_t^*(T))}\dot{\bar{c}}_t^*(T) - r_t - \frac{\gamma'(\bar{a}_t^*(T) - b_t)}{u'(\bar{c}_t^*(T))}\right] dt\right] dT \\
& + \lim_{t\to\infty}\left[e^{-(\rho-n+\varepsilon)t}\left(\tilde{a}_t - \tilde{a}_t^*\right)u'(\tilde{c}_t^*)\right. \\
& \qquad \left. + \int_0^\infty \varepsilon e^{-\varepsilon T}\left[e^{-(\rho-n)t}\left(\bar{a}_t(T) - \bar{a}_t^*(T)\right)u'(\bar{c}_t^*(T))\right] dT\right].
\end{aligned}
$$

By the Euler equation before the price level jump (A2), the expression for the risk premium (A4), the Euler equation after the price level jump (A3), respectively, the first three terms must be equal to zero, which yields

$$D \geq \lim_{t\to\infty} \mathbb{E}_0\left[e^{-(\rho-n)t}u'(c_t^*)(a_t - a_t^*)\right].$$

By the transversality condition (A6), this simplifies to

$$D \geq \lim_{t\to\infty} \mathbb{E}_0\left[e^{-(\rho-n)t}u'(c_t^*)a_t\right].$$

But the borrowing constraint (11) implies that a_t cannot be negative. This establishes that, for any feasible allocation $(\tilde{c}_t, \tilde{a}_t, \tilde{b}_t^h, (\bar{c}_s(t), \bar{a}_s(t))_{s=t}^\infty)_{t=0}^\infty$, we have $D \geq 0$, with D defined by (A7). Hence, the feasible allocation $(\tilde{c}_t^*, \tilde{a}_t^*, \tilde{b}_t^{h*}, (\bar{c}_s^*(t), \bar{a}_s^*(t))_{s=t}^\infty)_{t=0}^\infty$ that satisfies (A2), (A3), (A4), and (A6) must be welfare maximizing.

C Intertemporal government budget constraint

With a preference for wealth, the stochastic discount factor is given by

$$\Lambda_t = e^{-\int_0^t\left(\rho - n - \frac{\gamma'(\Delta_u)}{u'(c_u)}\right)du}u'(c_t).$$

Applying Itô's lemma with jumps (from appendix A) yields

$$\begin{aligned} d\Lambda_t &= -\left(\rho - n - \frac{\gamma'(\Delta_t)}{u'(c_t)}\right)\Lambda_t dt + e^{-\int_0^t\left(\rho - n - \frac{\gamma'(\Delta_u)}{u'(c_u)}\right)du} u''(c_t)\dot{c}_t dt \\ &\qquad + e^{-\int_0^t\left(\rho - n - \frac{\gamma'(\Delta_u)}{u'(c_u)}\right)du}\left[u'(\bar{c}_t) - u'(c_t)\right] dJ_t, \\ &= -\left(\rho - n - \frac{\gamma'(\Delta_t)}{u'(c_t)}\right)\Lambda_t dt + \frac{u''(c_t)}{u'(c_t)}\dot{c}_t\Lambda_t dt + \left[\frac{u'(\bar{c}_t)}{u'(c_t)} - 1\right]\Lambda_t dJ_t. \end{aligned}$$

where $\bar{c}_t$ denote consumption immediately after the price level jump (which was denoted more preciesly by $\bar{c}_t(t)$ in appendix B). Using the Euler equation (12), together with the fact that $\mathbb{E}_t[dJ_t] = \varepsilon dt$, yields $\mathbb{E}_t[d\Lambda_t] = -(r_t - n)\Lambda_t dt$, as expected for the stochastic discount factor.

Using the definition of the present value of surpluses Φ_t given by (19), which is not directly affected by the price level jump, we have

$$\begin{aligned} d(\Lambda_s\Phi_s) &= \Lambda_s d\Phi_s + \Phi_s d\Lambda_s, \\ &= \left[(r_s - n)\Phi_s - \tau_s\right]\Lambda_s ds + \Phi_s d\Lambda_s. \end{aligned}$$

Taking expectations and using the fact that $\mathbb{E}_s[d\Lambda_s] = -(r_s - n)\Lambda_s ds$ yields

$$\begin{aligned} \mathbb{E}_s\left[d(\Lambda_s\Phi_s)\right] &= \left[(r_s - n)\Phi_s - \tau_s\right]\Lambda_s ds - (r_s - n)\Phi_s\Lambda_s ds, \\ &= -\tau_s\Lambda_s ds. \end{aligned}$$

Taking expectation at time t with $t \leq s$, using the law of iterated expectations, integrating with respect to s from time t to infinity, and using the limit condition $\lim_{T\to\infty}\mathbb{E}_t\left[\Lambda_T\Phi_T\right] = 0$ yields

$$\Lambda_t\Phi_t = \mathbb{E}_t\left[\int_t^\infty \Lambda_s\tau_s ds\right].$$

This gives expression (20) for the present value of real primary surpluses.

Recall from equation (2) that the Ponzi debt scheme is defined by $\Delta_t = b_t - \Phi_t$. From the evolution of public debt b_t and of the present value of surpluses Φ_t, respectively given by (4) and (19), and then using the expression for the risk premium (13), we obtain

$$\begin{aligned} d\Delta_t &= \left[(i_t - \pi_t - n)b_t - \tau_t\right]dt - \Delta_t dJ_t - \left[(r_t - n)\Phi_t - \tau_t\right]dt, \\ &= \left[\left(r_t - n + \varepsilon\frac{\Delta_t}{b_t}\frac{u'(\bar{c}_t)}{u'(c_t)}\right)b_t - (r_t - n)\Phi_t\right]dt - \Delta_t dJ_t, \\ &= \left[r_t - n + \varepsilon\frac{u'(\bar{c}_t)}{u'(c_t)}\right]\Delta_t dt - \Delta_t dJ_t, \end{aligned}$$

which corresponds to equation (21).

Itô's lemma with jumps implies that $d(\Lambda_s\Delta_s) = \Lambda_s d\Delta_s + \Delta_s d\Lambda_s + d\Lambda_s d\Delta_s$. From the above expressions for $d\Lambda_s$ and $d\Delta_s$, we therefore have

$$\begin{aligned} d(\Lambda_s\Delta_s) &= \left[r_s - n + \varepsilon\frac{u'(\bar{c}_s)}{u'(c_s)}\right]\Delta_s\Lambda_s ds - \Delta_s\Lambda_s dJ_s + \Delta_s d\Lambda_s - \left[\frac{u'(\bar{c}_s)}{u'(c_s)} - 1\right]\Delta_s\Lambda_s dJ_s, \\ &= \left[r_s - n + \varepsilon\frac{u'(\bar{c}_s)}{u'(c_s)}\right]\Delta_s\Lambda_s ds + \Delta_s d\Lambda_s - \frac{u'(\bar{c}_s)}{u'(c_s)}\Delta_s\Lambda_s dJ_s. \end{aligned}$$

Taking expectations and using the fact that $\mathbb{E}_s[d\Lambda_s] = -(r_s - n)\Lambda_s ds$ and $\mathbb{E}_s[dJ_s] = \varepsilon ds$ yields

$$\begin{aligned} \mathbb{E}_s\left[d(\Lambda_s\Delta_s)\right] &= \left[r_s - n + \varepsilon\frac{u'(\bar{c}_s)}{u'(c_s)}\right]\Delta_s\Lambda_s ds - (r_s - n)\Delta_s\Lambda_s ds - \varepsilon\frac{u'(\bar{c}_s)}{u'(c_s)}\Delta_s\Lambda_s ds, \\ &= 0. \end{aligned}$$

Using the law of iterated expectations and integrating from time t to infinity yields

$$\Lambda_t\Delta_t = \lim_{T\to\infty}\mathbb{E}_t\left[\Lambda_T\Delta_T\right],$$

which corresponds to equation (22).

Note that these results imply that

$$\begin{aligned} b_t &= \Phi_t + \Delta_t, \\ &= \mathbb{E}_t\left[\int_t^\infty \frac{\Lambda_s}{\Lambda_t}\tau_s ds\right] + \lim_{T\to\infty}\mathbb{E}_t\left[\frac{\Lambda_T}{\Lambda_t}\Delta_T\right], \\ &= \mathbb{E}_t\left[\int_t^\infty \frac{\Lambda_s}{\Lambda_t}\tau_s ds\right] + \lim_{T\to\infty}\mathbb{E}_t\left[\frac{\Lambda_T}{\Lambda_t}b_T\right], \end{aligned}$$

where, to obtain the last line, I have used the fact that $\lim_{T\to\infty}\mathbb{E}_t\left[\Lambda_T\Phi_T\right] = 0$. This is the government's intertemporal budget constraint at time t.

D Jump in the present value of surpluses

Let us now allow for the possibility that, when the sunspot shock occurs, the present value of surpluses Φ_t jumps.

Using a state-contingent notation as in appendix B, let us denote by $\tilde{c}_t$, $\tilde{r}_t$, $\tilde{\tau}_t$, $\tilde{b}_t$, $\tilde{\Phi}_t$, $\tilde{\Delta}_t$ the variables at time t conditional on the absence of a sunspot shock at or before time t and by $\bar{c}_t(T)$, $\bar{r}_t(T)$, $\bar{\tau}_t(T)$, $\bar{b}_t(T)$, $\bar{\Phi}_t(T)$, $\bar{\Delta}_t(T)$ the same variables at t conditional on a sunspot shock having occurred at T, where $T \leq t$. Note that we trivially have $\bar{\Delta}_t(T) = 0$ since, by construction, the shock reduces the magnitude of the Ponzi scheme to zero.

The present value of surpluses before the sunspot shock is still defined by equation (20).[27] Assuming a finite horizon of length H, we have

$$\begin{aligned}
\tilde{\Phi}_t &= \mathbb{E}_t\left[\int_t^H \frac{\Lambda_s}{\Lambda_t}\tau_s ds\right], \\
&= \int_t^H \varepsilon e^{-\varepsilon(T-t)}\left[\int_t^T \frac{\tilde{\Lambda}_s}{\tilde{\Lambda}_t}\tilde{\tau}_s ds + \int_T^H \frac{\bar{\Lambda}_s(T)}{\tilde{\Lambda}_t}\bar{\tau}_s(T)ds\right]dT + e^{-\varepsilon(H-t)}\int_t^H \frac{\tilde{\Lambda}_s}{\tilde{\Lambda}_t}\tilde{\tau}_s ds,
\end{aligned}$$

where $e^{-\varepsilon(H-t)}$ is the probability that the sunspot shock does not happen by time H, while $\tilde{\Lambda}_s$ and $\bar{\Lambda}_s(T)$ are given by

$$\begin{aligned}
\tilde{\Lambda}_s &= e^{-\int_0^s\left(\rho-n-\frac{\gamma'(\tilde{\Delta}_u)}{u'(\tilde{c}_u)}\right)du}u'(\tilde{c}_s), \\
\bar{\Lambda}_s(T) &= e^{-\int_0^T\left(\rho-n-\frac{\gamma'(\tilde{\Delta}_u)}{u'(\tilde{c}_u)}\right)du-\int_T^s\left(\rho-n-\frac{\gamma'(0)}{u'(\bar{c}_u(T))}\right)du}u'(\bar{c}_s(T)).
\end{aligned}$$

The present value of surpluses at time t after the sunspot shock has occurred at T is defined by

$$\bar{\Phi}_t(T) = \int_t^H \frac{\bar{\Lambda}_s(T)}{\bar{\Lambda}_t(T)}\bar{\tau}_s(T)ds.$$

We therefore have

$$\begin{aligned}
\tilde{\Phi}_t &= \int_t^H \varepsilon e^{-\varepsilon(T-t)}\left[\int_t^T \frac{\tilde{\Lambda}_s}{\tilde{\Lambda}_t}\tilde{\tau}_s ds + \frac{\tilde{\Lambda}_T}{\tilde{\Lambda}_t}\frac{\bar{\Lambda}_T(T)}{\tilde{\Lambda}_T}\bar{\Phi}_T(T)\right]dT + e^{-\varepsilon(H-t)}\int_t^H \frac{\tilde{\Lambda}_s}{\tilde{\Lambda}_t}\tilde{\tau}_s ds, \\
&= \int_t^H\left[\int_s^H \varepsilon e^{-\varepsilon(T-t)}dT\right]\frac{\tilde{\Lambda}_s}{\tilde{\Lambda}_t}\tilde{\tau}_s ds + \int_t^H \varepsilon e^{-\varepsilon(T-t)}\frac{\tilde{\Lambda}_T}{\tilde{\Lambda}_t}\frac{\bar{\Lambda}_T(T)}{\tilde{\Lambda}_T}\bar{\Phi}_T(T)dT \\
&\qquad\qquad + e^{-\varepsilon(H-t)}\int_t^H \frac{\tilde{\Lambda}_s}{\tilde{\Lambda}_t}\tilde{\tau}_s ds, \\
&= \int_t^H e^{-\varepsilon(T-t)}\frac{\tilde{\Lambda}_T}{\tilde{\Lambda}_t}\left[\tilde{\tau}_T + \varepsilon\frac{\bar{\Lambda}_T(T)}{\tilde{\Lambda}_T}\bar{\Phi}_T(T)\right]dT, \\
&= \int_t^H e^{-\int_t^T\left(\rho-n-\frac{\gamma'(\tilde{\Delta}_u)}{u'(\tilde{c}_u)}+\varepsilon\right)du}\frac{u'(\tilde{c}_T)}{u'(\tilde{c}_t)}\left[\tilde{\tau}_T + \varepsilon\frac{u'(\bar{c}_T(T))}{u'(\tilde{c}_T)}\bar{\Phi}_T(T)\right]dT.
\end{aligned}$$

Finally, using the Euler equation before the realization of the shock, given by (12) with (17), and integrating it from time t to T (as in the first two equations of appendix F),

[27] To derive this equation, Φ_t should now be defined by

$$d\Phi_t = \left[\left(r_t - n + \varepsilon\frac{u'(\bar{c}_t(t))}{u'(c_t)}\frac{\Phi_t - \bar{\Phi}_t(t)}{\Phi_t}\right)\Phi_t - \tau_t\right]dt + [\bar{\Phi}_t(t) - \Phi_t]dJ_t,$$

where the state-contingent notation is not used within this stochastic differential equation. This expression generalizes (19). Using exactly the same steps as in appendix C yields equation (20).

this simplifies to

$$\tilde{\Phi}_t = \int_t^H e^{-\int_t^T \left(\tilde{r}_u - n + \varepsilon \frac{u'(\bar{c}_u(u))}{u'(\tilde{c}_u)}\right) du} \left[\tilde{\tau}_T + \varepsilon \frac{u'(\bar{c}_T(T))}{u'(\tilde{c}_T)} \bar{\Phi}_T(T)\right] dT.$$

Let us now assume that, when the sunspot shock occurs, the present value of surpluses $\bar{\Phi}_t(t)$ is set equal to the level of government debt $\tilde{b}_t$ such as to avoid a jump in the price level. This policy ensures that government bonds are safe in real terms, which implies that the risk-premium equation (13) is replaced by $r_t = i_t - \pi_t$. Hence, it follows from (4) that, before the realization of the sunspot shock, government debt evolves according to $d\tilde{b}_t = [(\tilde{r}_t - n)\tilde{b}_t - \tilde{\tau}_t]dt$. This implies that, when the shock occurs at time T, with $T \geq t$, we have

$$\tilde{b}_T = e^{\int_t^T (\tilde{r}_u - n) du} \tilde{b}_t - \int_t^T e^{\int_s^T (\tilde{r}_u - n) du} \tilde{\tau}_s ds.$$

Setting $\bar{\Phi}_T(T) = \tilde{b}_T$ into the above expression for $\tilde{\Phi}_t$ yields

$$\begin{aligned}
\tilde{\Phi}_t &= \int_t^H e^{-\int_t^T \left(\tilde{r}_u - n + \varepsilon \frac{u'(\bar{c}_u(u))}{u'(\tilde{c}_u)}\right) du} \tilde{\tau}_T dT + \tilde{b}_t \int_t^H e^{-\int_t^T \varepsilon \frac{u'(\bar{c}_u(u))}{u'(\tilde{c}_u)} du} \varepsilon \frac{u'(\bar{c}_T(T))}{u'(\tilde{c}_T)} dT \\
&\quad - \int_t^H e^{-\int_t^T \left(\tilde{r}_u - n + \varepsilon \frac{u'(\bar{c}_u(u))}{u'(\tilde{c}_u)}\right) du} \varepsilon \frac{u'(\bar{c}_T(T))}{u'(\tilde{c}_T)} \int_t^T e^{\int_s^T (\tilde{r}_u - n) du} \tilde{\tau}_s ds dT, \\
&= \int_t^H e^{-\int_t^T \left(\tilde{r}_u - n + \varepsilon \frac{u'(\bar{c}_u(u))}{u'(\tilde{c}_u)}\right) du} \tilde{\tau}_T dT + \tilde{b}_t \int_t^H e^{-\int_t^T \varepsilon \frac{u'(\bar{c}_u(u))}{u'(\tilde{c}_u)} du} \varepsilon \frac{u'(\bar{c}_T(T))}{u'(\tilde{c}_T)} dT \\
&\quad - \int_t^H e^{-\int_t^s \left(\tilde{r}_u - n + \varepsilon \frac{u'(\bar{c}_u(u))}{u'(\tilde{c}_u)}\right) du} \left[\int_s^H e^{-\int_s^T \varepsilon \frac{u'(\bar{c}_u(u))}{u'(\tilde{c}_u)} du} \varepsilon \frac{u'(\bar{c}_T(T))}{u'(\tilde{c}_T)} dT\right] \tilde{\tau}_s ds, \\
&= \tilde{b}_t \left[1 - e^{-\int_t^H \varepsilon \frac{u'(\bar{c}_u(u))}{u'(\tilde{c}_u)} du}\right] + \int_t^H e^{-\int_t^s \left(\tilde{r}_u - n + \varepsilon \frac{u'(\bar{c}_u(u))}{u'(\tilde{c}_u)}\right) du} e^{-\int_s^H \varepsilon \frac{u'(\bar{c}_u(u))}{u'(\tilde{c}_u)} du} \tilde{\tau}_s ds, \\
&= \tilde{b}_t - e^{-\int_t^H \varepsilon \frac{u'(\bar{c}_u(u))}{u'(\tilde{c}_u)} du} \left[\tilde{b}_t - \int_t^H e^{-\int_t^s (\tilde{r}_u - n) du} \tilde{\tau}_s ds\right].
\end{aligned}$$

Hence

$$\begin{aligned}
\tilde{\Delta}_t &= \tilde{b}_t - \tilde{\Phi}_t, \\
&= e^{-\int_t^H \varepsilon \frac{u'(\bar{c}_u(u))}{u'(\tilde{c}_u)} du} \left[\tilde{b}_t - \int_t^H e^{-\int_t^s (\tilde{r}_u - n) du} \tilde{\tau}_s ds\right].
\end{aligned}$$

Taking the limit as H tends to infinity shows that, if $\varepsilon > 0$ and $\int_t^\infty e^{-\int_t^s (\tilde{r}_u - n) du} \tilde{\tau}_s ds$ is finite, then $\tilde{\Delta}_t = 0$.

While the limit is typically indeterminate when $\int_t^\infty e^{-\int_t^s (\tilde{r}_u - n) du} \tilde{\tau}_s ds$ is not finite, progress can be made in the special case where the economy is in steady state and primary surpluses are constant until the realization of the sunspot shock. I henceforth consider that $\tilde{\tau}_s = \tilde{\tau}$, $\tilde{c}_s = \tilde{c}$, $\bar{c}_s(s) = \bar{c}$, and $\tilde{r}_s = \tilde{r}$ for all $s \geq t$. Note that, as $\int_t^\infty e^{-(\tilde{r}-n)(t-s)}\tilde{\tau}ds$ is not finite, we must have $\tilde{r} < n$. The magnitude of the Ponzi scheme is therefore given by

$$\begin{aligned}\tilde{\Delta}_t &= e^{-\varepsilon\frac{u'(\bar{c})}{u'(\tilde{c})}(H-t)}\left[\tilde{b}_t - \int_t^H e^{-(\tilde{r}-n)(t-s)}\tilde{\tau}ds\right], \\ &= e^{-\varepsilon\frac{u'(\bar{c})}{u'(\tilde{c})}(H-t)}\left[\tilde{b}_t - \frac{\tilde{\tau}}{\tilde{r}-n}\right] + e^{-\left(\tilde{r}-n+\varepsilon\frac{u'(\bar{c})}{u'(\tilde{c})}\right)(H-t)}\frac{\tilde{\tau}}{\tilde{r}-n}.\end{aligned}$$

In the limit as H tends to infinity, we have

$$\tilde{\Delta}_t = \begin{cases} \pm\infty & \text{if } \tilde{r} \in \left(-\infty, n - \varepsilon\frac{u'(\bar{c})}{u'(\tilde{c})}\right) \\ \frac{\tilde{\tau}}{\tilde{r}-n} & \text{if } \tilde{r} = n - \varepsilon\frac{u'(\bar{c})}{u'(\tilde{c})} \\ 0 & \text{if } \tilde{r} \in \left(n - \varepsilon\frac{u'(\bar{c})}{u'(\tilde{c})}, n\right) \end{cases}.$$

In equilibrium, we cannot have $\tilde{\Delta}_t < 0$ or $\tilde{\Delta}_t = \infty$, thereby ruling out the first possibility. If $\tilde{r} = n - \varepsilon\frac{u'(\bar{c})}{u'(\tilde{c})}$, then $\tilde{\Delta}_t = \frac{\tilde{\tau}}{\tilde{r}-n}$ where, to ensure $\tilde{\Delta}_t > 0$, we can consider that $\tilde{\tau} < 0$, i.e. the government is running permanent deficits until the sunspot shock occurs. However, for the economy to be in steady state, the magnitude of the Ponzi scheme must also satisfy $\gamma'(\tilde{\Delta}) = (\rho - n + \varepsilon)u'(1)$ (which formally follows from (32)). Hence, the possibility to have an equilibrium with $\tilde{\Delta} > 0$ and no jump in the price level only exists in a knife-edge case. We can therefore consider that, if an equilibrium exists with $\bar{\Phi}_t(t) = \tilde{b}_t$ such as to avoid a jump in the price level, then it must generically satisfy $\tilde{\Delta}_t = 0$.

E Transversality condition in equilibrium

Using expression (18) for the stochastic discount factor, we have

$$\mathbb{E}_0\left[\Lambda_t\Phi_t\right] = \mathbb{E}_0\left[e^{-\int_0^t\left(\rho-n-\frac{\gamma'(\Delta_u)}{u'(c_u)}\right)du}u'(c_t)\Phi_t\right] \geq \mathbb{E}_0\left[e^{-(\rho-n)t}u'(c_t)\Phi_t\right],$$

where the inequality follows from the fact that $\int_0^t(\gamma'(\Delta_u)/u'(c_u))du$ is always non-negative and, hence, $e^{\int_0^t\frac{\gamma'(\Delta_u)}{u'(c_u)}du}$ must always be greater or equal to one, while Φ_t was assumed to be non-negative. But, by definition of Φ_t, we know that $\lim_{t\to\infty}\mathbb{E}_0\left[\Lambda_t\Phi_t\right] = 0$. Hence, $\lim_{t\to\infty}\mathbb{E}_0\left[e^{-(\rho-n)t}u'(c_t)\Phi_t\right] = 0$.

Using the asset market clearing condition (17), the household's transversality condition (14) can be written as $\lim_{t\to\infty}\mathbb{E}_0\left[e^{-(\rho-n)t}u'(c_t)\left(\Phi_t + \Delta_t\right)\right] = 0$. But, we always have $\lim_{t\to\infty}\mathbb{E}_0\left[e^{-(\rho-n)t}u'(c_t)\Phi_t\right] = 0$. Hence, in equilibrium, the transversality condition (14) can be written as $\lim_{t\to\infty}\mathbb{E}_0\left[e^{-(\rho-n)t}u'(c_t)\Delta_t\right]=0$.

Using a state-contingent notation as in appendix B or D, let us denote by $\tilde{c}_t$ and $\tilde{\Delta}_t$ the variables at time t conditional on the absence of a sunspot shock at or before time t and by $\bar{c}_t(T)$and $\bar{\Delta}_t(T)$ the same variables at t conditional on a sunspot shock having occurred at T, where $T\leq t$. Using equation (A5) from appendix B, we have

$$\mathbb{E}_0\left[e^{-(\rho-n)t}u'(c_t)\Delta_t\right]=\int_0^t \varepsilon e^{-\varepsilon T}\left[e^{-(\rho-n)t}u'(\bar{c}_t(T))\bar{\Delta}_t(T)\right]dT+e^{-(\rho-n+\varepsilon)t}u'(\tilde{c}_t)\tilde{\Delta}_t.$$

But, after a price level jump, the magnitude of the Ponzi scheme must be equal to zero; which is formally implied by (21). Hence, $\bar{\Delta}_t(T)=0$. It follows that

$$\lim_{t\to\infty}\mathbb{E}_0\left[e^{-(\rho-n)t}u'(c_t)\Delta_t\right]=\lim_{t\to\infty}e^{-(\rho-n+\varepsilon)t}u'(\tilde{c}_t)\tilde{\Delta}_t.$$

F Ruling out explosive Ponzi schemes

The consumption Euler equation before the price level jump (23) can be written as

$$\frac{d\ln\left[u'(c_t)\right]}{dt}=\rho-r_t-\frac{\gamma'(\Delta_t)}{u'(c_t)}-\varepsilon\left(\frac{u'(\bar{c}_t)}{u'(c_t)}-1\right).$$

Integrating this expression from time t to T yields

$$u'(c_T)=u'(c_t)e^{\int_t^T\left(\rho-r_u-\frac{\gamma'(\Delta_u)}{u'(c_u)}-\varepsilon\left(\frac{u'(\bar{c}_u)}{u'(c_u)}-1\right)\right)du}.$$

Integrating the dynamics of the Ponzi scheme given by (27) from time t to T yields

$$\Delta_T=\Delta_t e^{\int_t^T\left(r_u-n+\varepsilon\frac{u'(\bar{c}_u)}{u'(c_u)}\right)du}.$$

From these two equations, conditional on the absence of a price level jump, we have

$$\lim_{T\to\infty}e^{-(\rho-n+\varepsilon)(T-t)}u'(c_T)\Delta_T=u'(c_t)\Delta_t\lim_{T\to\infty}e^{-\int_t^T\frac{\gamma'(\Delta_u)}{u'(c_u)}du}.$$

Hence, for the transversality condition (28) to be satisfied, we must have

$$\lim_{T\to\infty}\int_t^T\frac{\gamma'(\Delta_u)}{u'(c_u)}du=\infty. \tag{A10}$$

If the Ponzi scheme is explosive, there must eventually be a strictly positive lower bound x to its growth rate, i.e. $r_u-n+\varepsilon u'(\bar{c}_u)/u'(c_u)\geq x>0$ for all $u\geq t$ for some arbitrarily large t. We therefore have

$$\Delta_T = \Delta_t e^{\int_t^T \left(r_u - n + \varepsilon \frac{u'(\bar{c}_u)}{u'(c_u)} \right) du} \geq \Delta_t e^{x(T-t)}.$$

Note that, as labor supply is equal to 1, we must have $c_u \leq 1$ for all u. Hence, if the Ponzi scheme is explosive, we have

$$\lim_{T \to \infty} \int_t^T \frac{\gamma'(\Delta_u)}{u'(c_u)} du \leq \frac{1}{u'(1)} \lim_{T \to \infty} \int_t^T \gamma' \left(\Delta_t e^{x(u-t)} \right) du < \infty,$$

where the last inequality follows from our assumption that $\int_0^\infty \gamma'(e^{\lambda t}) dt < \infty$ for any $\lambda > 0$. This establishes that the limit (A10), and hence the transversality condition (28), cannot be satisfied for a Ponzi scheme that is explosive conditional on the absence of a price level jump.[28]

G Ponzi-stagnation steady state

A Ponzi-stagnation steady state $(c^{ps}, \pi^{ps}, r^{ps}, i^{ps}, \Delta^{ps})$ is characterized by low inflation $\pi^{ps} = \pi^R$ and $r^{ps} = n - \varepsilon \frac{u'(\bar{c})}{u'(c^{ps})}$. Hence, by the Euler equation, the magnitude of the Ponzi scheme Δ^{ps} is given by

$$\gamma' \left(\Delta^{ps} \right) = (\rho - n + \varepsilon) u'(c^{ps}). \tag{A11}$$

Also, by the Taylor rule $i^{ps} = \max\{r^n + \pi^R + (\phi - 1)[\pi^R - \pi^*], 0\}$ with $\pi^R \leq \pi^*$ and $r^n < -\pi^R$, we must have $i^{ps} = 0$. Finally, from the risk-premium equation (25), we have

$$\varepsilon \frac{\Phi}{\Phi + \Delta^{ps}} \frac{u'(\bar{c})}{u'(c^{ps})} = n + \pi^R, \tag{A12}$$

where we assume that Φ is constant in the Ponzi-stagnation steady state. The steady state values of c^{ps} and Δ^{ps} are jointly determined by (A11) and (A12). The Ponzi-stagnation steady state exists if and only if there exists a solution to these two equations that satisfy $c^{ps} < 1$ and $\Delta^{ps} > 0$.[29]

From (A12), this steady state cannot exist when $n + \pi^R \leq 0$. Even when $n + \pi^R > 0$, it does not exist when either ε or Φ is sufficiently close to zero.

If it exists, we can characterize a number of its properties. First, under-employment $c^{ps} < 1$ implies by (A11) that the Ponzi scheme is of a smaller magnitude than in the

[28]A similar proof of the impossibility of explosive bubbles under a preference for wealth was provided by Michau, Ono, and Schlegl (2023) for the case of deterministic bubbles, i.e. with $\varepsilon = 0$.

[29]If $c^{ps} = 1$, then $\Delta^{ps} = \Delta^p$ and this steady state coincides with the Ponzi steady state with a binding zero lower bound.

Ponzi steady state $\Delta^{ps} < \Delta^{p}$. Second, by (29), (A11), $\Delta^{ps} > 0$, and $c^{ps} < 1$, the Ponzi-stagnation steady state can only exist if $r^n < n - \varepsilon$, i.e. if the Ponzi steady state also exists.

Finally, with $\bar{c} \in \{c^{ss}, 1\}$, we must have $c^{ps} > c^{ss}$. Substituting (A12) into (A11) yields $\gamma'(\Delta^{ps}) = (\rho + \pi^R + \varepsilon)u'(c^{ps}) - \varepsilon \frac{\Phi}{\Phi + \Delta^{ps}} u'(\bar{c})$, which defines c^{ps} as an increasing function of Δ^{ps}. Moreover, this relationship together with (31) implies that, if $\Delta^{ps} = 0$, then $(\rho + \pi^R)u'(c^{ss}) + \varepsilon u'(\bar{c}) = (\rho + \pi^R + \varepsilon)u'(c^{ps})$. Thus, if $\Delta^{ps} = 0$, then $c^{ps} \in [c^{ss}, \bar{c}]$. We therefore have a relationship that defines c^{ps} as an increasing function of Δ^{ps} with $c^{ps} \geq c^{ss}$ when $\Delta^{ps} = 0$. This establishes that $c^{ps} > c^{ss}$.

H Welfare function

From the welfare function of the representative household (9), together with the asset market clearing condition (17), the objective of the government is to maximize

$$W = \mathbb{E}_0 \left[\int_0^\infty e^{-(\rho - n)t} \left[u(c_t) + \gamma(\Delta_t) - \psi c \left(\frac{dP_t}{P_t} \right) \right] dt \right].$$

The only source of uncertainty is the time T when the price level jumps, which is exponentially distributed with parameter ε. Let us denote by $(c_t, \pi_t, \Delta_t, \Phi_t)_{t=0}^\infty$ the equilibrium of the economy conditional on the absence of a price level jump and by $(\bar{c}, \bar{\pi})$ the steady state equilibrium afterwards. Using the specification for the cost of inflation (8), we have

$$\begin{aligned}
W &= \int_0^\infty \varepsilon e^{-\varepsilon T} \left[\int_0^T e^{-(\rho - n)t} \left[u(c_t) + \gamma(\Delta_t) - \psi |\pi_t| \right] dt - e^{-(\rho - n)T} \psi C \left(\frac{\Delta_T}{\Phi_T} \right) \right. \\
&\qquad \left. + \int_T^\infty e^{-(\rho - n)t} \left[u(\bar{c}_t) + \gamma(0) - \psi |\bar{\pi}_t| \right] dt \right] dT, \\
&= \int_0^\infty \left[\int_t^\infty \varepsilon e^{-\varepsilon T} dT \right] e^{-(\rho - n)t} \left[u(c_t) + \gamma(\Delta_t) - \psi |\pi_t| \right] dt \\
&\qquad - \int_0^\infty e^{-(\rho - n + \varepsilon)T} \varepsilon \psi C \left(\frac{\Delta_T}{\Phi_T} \right) dT \\
&\qquad + \int_0^\infty \varepsilon e^{-(\rho - n + \varepsilon)T} \left[\int_T^\infty e^{-(\rho - n)(t - T)} \left[u(\bar{c}_t) + \gamma(0) - \psi |\bar{\pi}_t| \right] dt \right] dT, \\
&= \int_0^\infty e^{-(\rho - n + \varepsilon)t} \left[u(c_t) + \gamma(\Delta_t) - \psi |\pi_t| - \psi \varepsilon C \left(\frac{\Delta_t}{\Phi_t} \right) \right. \\
&\qquad \left. + \varepsilon \int_t^\infty e^{-(\rho - n)(s - t)} \left[u(\bar{c}_s) + \gamma(0) - \psi |\bar{\pi}_s| \right] ds \right] dt.
\end{aligned}$$

Using $\pi_t \geq \pi^R \geq 0$ and $\Phi_t = \Phi$, this gives equation (35). When there is a Ponzi scheme, at each instant, there is a probability εdt of a price level jump that raises the cost of inflation to $\psi C(\Delta_t/\Phi_t)/dt$.

If the economy is in steady state before the price level jump, then the expected welfare of the representative household is equal to

$$W = \frac{1}{\rho - n + \varepsilon}\left[u(c) + \gamma(\Delta) - \psi|\pi| - \psi\varepsilon C\left(\frac{\Delta}{\Phi}\right)\right] + \frac{1}{\rho - n}\frac{\varepsilon}{\rho - n + \varepsilon}\left[u(\bar{c}) + \gamma(0) - \psi|\bar{\pi}|\right].$$

I Possible equilibrium paths

Let us now characterize the possible equilibrium paths for $\Delta_0 \notin \{0, \Delta^p\}$. Before reaching steady state, the economy can either be producing at full capacity with $c_t = 1$ or it can be depressed with $c_t < 1$. Let us consider each possibility in turn. Throughout this appendix, I assume that the present value of surpluses is constant over time with $\Phi_t = \Phi$.

I.1 Full employment paths

When the economy produces at full capacity, from the Euler equation (23), its real interest rate must be given by

$$r_t = \rho - \frac{\gamma'(\Delta_t)}{u'(1)} + \varepsilon\left(1 - \frac{u'(\bar{c})}{u'(1)}\right),$$

where $\bar{c}$ is the solution to the dilemma given by equation (34). Substituting this expression into the dynamics of the Ponzi scheme given by (27) yields

$$\dot{\Delta}_t = \left[\rho - n + \varepsilon - \frac{\gamma'(\Delta_t)}{u'(1)}\right]\Delta_t.$$

This implies that the Ponzi steady state, with $\Delta = \Delta^p$, is locally unstable.

Full employment paths leading to the neoclassical steady state. The only equilibrium possibility with full employment is to have $\Delta_0 \in (0, \Delta^p)$ followed by convergence to the neoclassical steady state. By the risk-premium equation (25), we must have

$$i_t - \pi_t = \rho - \frac{\gamma'(\Delta_t)}{u'(1)} + \varepsilon\left(1 - \frac{u'(\bar{c})}{u'(1)}\right) + \varepsilon\frac{\Delta_t}{\Phi + \Delta_t}\frac{u'(\bar{c})}{u'(c_t)}.$$

As Δ_t is decreasing over time, $i_t - \pi_t$ must also be decreasing. It converges to $r^n + \varepsilon(1-$

$u'(\bar{c})/u'(1))$. By the Taylor rule (26), we have

$$i_t - \pi_t = \max\{r^n + (\phi - 1)[\pi_t - \pi^*], -\pi_t\}.$$

Hence, we must either have $i_t - \pi_t = r^n + (\phi - 1)[\pi_t - \pi^*]$ with $i_t - \pi_t \geq -\pi_t$ or $i_t - \pi_t = -\pi_t$ with $i_t - \pi_t \geq r^n + (\phi - 1)[\pi_t - \pi^*]$. Both possibilities require

$$\pi^* \geq \frac{-\phi}{\phi - 1}(i_t - \pi_t) + \frac{r^n}{\phi - 1}.$$

As $i_t - \pi_t$ is decreasing over time, existence of an equilibrium path with full employment converging to the neoclassical steady state requires this inequality to be satisfied for $\Delta_\infty = 0$, which yields

$$\pi^* \geq -r^n + \frac{\phi}{\phi - 1}\varepsilon\left(\frac{u'(\bar{c})}{u'(1)} - 1\right).$$

Whenever $\bar{c} = 1$, this coincides with the existence condition for the neoclassical steady state.[30]

I.2 Underemployment paths

Let us now consider equilibrium paths that are characterized by underemployment before reaching steady state. By the downward wage rigidity (24), we must have $\pi_t = \pi^R$. The Taylor rule (26) can be written as $i_t = \max\{r^n + \pi^R + (\phi - 1)[\pi^R - \pi^*], 0\}$. Since $r^n < -\pi^R$ (otherwise the secular stagnation steady state does not exist and there is no trilemma), we must have $i_t = 0$. Using the resulting risk-premium equation (25), the equilibrium of the economy with under-employment is fully characterized by the Euler equation (23) and the Ponzi dynamics (27), which gives

$$\dot{c}_t = \left(\frac{u'(c_t)}{-c_t u''(c_t)}\right)\left[-\pi^R - \rho - \varepsilon + \frac{\gamma'(\Delta_t)}{u'(c_t)} + \varepsilon\frac{\Phi}{\Phi + \Delta_t}\frac{u'(\bar{c}_t)}{u'(c_t)}\right]c_t, \tag{A13}$$

$$\dot{\Delta}_t = \left[-\pi^R - n + \varepsilon\frac{\Phi}{\Phi + \Delta_t}\frac{u'(\bar{c}_t)}{u'(c_t)}\right]\Delta_t. \tag{A14}$$

As explained in the text, I am assuming that, if the Ponzi scheme collapses with $\Delta_t < \bar{\Delta}$ and $c_t < 1$, the economy subsequently remains in the secular stagnation steady state. Hence, $\bar{c}_t = c^{ss}$ when $\Delta_t < \bar{\Delta}$ and $\bar{c}_t$ is determined by equation (34) when $\Delta_t \geq \bar{\Delta}$.

By the two differential equations, convergence to the neoclassical steady state can

[30] Clearly, when ψ is so high that $\bar{c} = c^{ss}$, this equilibrium path is very unlikely to be the solution to the trilemma.

be ruled out since c_t cannot *asymptotically* converge to 1 as Δ_t converges to zero.[31] The remaining possibilities are that underemployment paths either lead to the Ponzi or to the secular stagnation steady state.

Underemployment paths leading to the Ponzi steady state. An equilibrium path with underemployment can only reach the Ponzi steady state in finite time. Let T denote the point in time when $c_T = 1$ and $\Delta_T = \Delta^p$. When $\Delta_0 \in (0, \Delta^p)$, the existence of such an equilibrium path requires both $\dot{c}_T > 0$ and $\dot{\Delta}_T > 0$. From (A13) and (A14), with the magnitude of Δ^p given by equation (32), both conditions are satisfied if and only if

$$-\pi^R - n + \varepsilon \frac{\Phi}{\Phi + \Delta^p} \frac{u'(\bar{c}_t)}{u'(1)} > 0.$$

When this condition is satisfied, the equilibrium path from time 0 to T is fully characterized by the two differential equations (A13) and (A14) where Δ_0 is exogenously given and T is determined such that $c_T = 1$ and $\Delta_T = \Delta^p$.

When $\Delta_0 \in (\Delta^p, \infty)$, the existence of an equilibrium path with underemployment leading to the Ponzi steady state requires both $\dot{c}_T > 0$ and $\dot{\Delta}_T < 0$. However, these two conditions, with Δ^p given by equation (32), are mutually inconsistent.

Underemployment paths leading to the secular stagnation steady state. When $\Delta_0 > 0$, convergence to the secular stagnation steady state requires $\dot{\Delta}_t < 0$ when Δ_t is close to zero. As $\bar{c}_t = c^{ss}$ when $\Delta_t < \bar{\Delta}$, by the differential equation (A14), we must have

$$-\pi^R - n + \varepsilon < 0.$$

If it exists, the corresponding equilibrium path is fully characterized by the two differential equations (A13) and (A14) with Δ_0 exogenously given and $c_\infty = c^{ss}$ and $\Delta_\infty = 0$. The highest feasible value of Δ_0 is such that $c_0 = 1$.

J Fundamental shock

Recall that J_t is initially equal to zero and can jump to one at any point in time at Poisson rate ε. Rather than being a pure sunspot shock that raises the price level, let us now assume that it is a shock to the marginal utility of wealth. More specifically,

[31] It is possible for the economy to be depressed until $c_t = 1$, followed by full-employment and convergence to the neoclassical steady state as described in the previous subsection. However, for any Δ_0, there is a continuum of such paths, which makes them fundamentally indeterminate. I therefore ignore these equilibrium possibilities.

the representative household's intertemporal utility is now given by

$$\mathbb{E}_0 \left[\int_0^{\infty} e^{-(\rho-n)t} \left[u(c_t) + (1-\lambda J_t)\gamma(a_t - b_t + \Delta_t) - \psi c\left(\frac{dP_t}{P_t}\right)\right] dt \right].$$

The parameter $\lambda \in [0,1]$ is assumed to be sufficiently high to satisfy

$$\rho - (1-\lambda)\frac{\gamma'(0)}{u'(1)} \geq \max\{-\pi^R, n-\varepsilon\}.$$

This implies that, once $J_t = 1$, the natural real interest rate, equal to the left-hand side of this inequality, is too high to allow for the possibility of either a secular stagnation or a Ponzi steady state. Hence, after the realization of the shock, the economy must be in the neoclassical steady state.[32] The equilibrium before the shock remains characterized by equations (23)-(28), but with $\bar{c}_t = 1$.

This establishes that the upward jump in the price level can be driven by a fundamental shock to the economy that reduces the marginal utility of wealth, rather than by a sunspot shock inducing households to run away from the Ponzi scheme of public debt.

The steady states remain unchanged, except for the secular stagnation steady state $(c^{ss}, \pi^{ss}, r^{ss}, i^{ss}, \Delta^{ss})$, which is no longer an absorbing state. This steady state is still characterized by the absence of Ponzi scheme $\Delta^{ss} = 0$ and by a binding downward nominal wage rigidity $\pi^{ss} = \pi^R$. From (23) with $\bar{c}_t = 1$, the Euler equation in steady state is now given by

$$\frac{1}{u'(c^{ss})} = \frac{1}{\gamma'(0)}\left[\rho - r^{ss} - \varepsilon\left(\frac{u'(1)}{u'(c^{ss})} - 1\right)\right]. \tag{A15}$$

As $c^{ss} < 1$, we have

$$\frac{\rho - r^n}{\gamma'(0)} = \frac{1}{u'(1)} > \frac{1}{u'(c^{ss})} = \frac{1}{\gamma'(0)}\left[\rho - r^{ss} - \varepsilon\left(\frac{u'(1)}{u'(c^{ss})} - 1\right)\right] \geq \frac{\rho - r^{ss}}{\gamma'(0)},$$

which establishes that $r^{ss} > r^n$, where r^n is the natural real interest rate before the occurrence of the fundamental shock. From the risk premium relationship (25) with $\Delta^{ss} = 0$, we have $r^{ss} = i^{ss} - \pi^{ss}$. Hence, from the Taylor rule (26), we have $r^{ss} = \max\{r^n + (\phi - 1)[\pi^R - \pi^*], -\pi^R\}$. As $r^{ss} > r^n$ and $\pi^R \leq \pi^*$, we must have $r^{ss} = -\pi^R$ and, hence, $i^{ss} = 0$. From the steady state Euler equation (A15), the output level is

[32] Naturally, I consider that the natural real interest rate r^n of the Taylor rule (26) is equal to $\rho - \frac{\gamma'(0)}{u'(1)}$ before the shock and to $\rho - (1-\lambda)\frac{\gamma'(0)}{u'(1)}$ afterwards.

given by

$$\frac{1}{u'(c^{ss})} = \frac{\rho + \pi^R + \varepsilon}{\gamma'(0) + \varepsilon u'(1)}.$$

Existence of the secular stagnation steady state requires $c^{ss} < 1$ or, equivalently, $r^n < -\pi^R$, which is the same condition as under a permanent lack of demand.

Output c^{ss} is an increasing function of likelihood ε of occurrence of the fundamental shock, with $c^{ss} = 1$ in the limit as ε tends to infinity. When $\varepsilon > 0$, the secular stagnation steady state is the same as under a permanent lack of demand, except that output is higher, i.e. the economy is less depressed. The prospect of an economic recovery raises households' demand for consumption and, hence, output under secular stagnation.