Inference in Auctions with Many Bidders Based on Transaction Prices

Federico Bugni
Northwestern

Yulong Wang
Syracuse

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Motivation

- Classic econometric analysis of auction data, **very large** literature
  - number of bidders K is small and known (e.g., Athey and Haile, 2002)
  - number of auctions n is large (e.g., Guerre, Perrigne, and Vuong, 2000)
  - multiple (sometimes all) bids are observed

- Example: homogeneous timber auction

Second-Price Auction

- For illustration, consider the classic second-price auction with IPVs
  - equilibruim strategy: bidder i submits her value $V_i \sim F_V$
    - K is the number of (**potential**) bidders
    - order statistics $V_{(1)} \geq V_{(2)} \geq \cdots \geq V_{(K)}$
  - transaction price $P = V_{(2)}$, the **second largest order stat**

$$F_P(\cdot) = F_{V_{(2)}}(\cdot) = F_V(\cdot)^K + K F_V(\cdot)^{K-1}(1 - F_V(\cdot))$$

  - number of auctions n is large $\Rightarrow$ nonparametrically estimate F_P
  - K is small and known $\Rightarrow$ estimate F_V by inverting the above

Motivation cont'd

- We consider the different situation
  - number of bidders K is large in each auction
  - number of auctions n is small/fixed
  - only the winning bid/transaction price is observed

- Example: art painting and Hong Kong vehicle license plate

$n < 21$ $\qquad$ $n = 4$

The New Framework

- We develop a new framework
  - number of auctions n is small/fixed
  - number of **potential** bidders K is large in each auction
  - only P is observed, but not $K \Rightarrow$

$$\text{only require observing } (P_1, ..., P_n) \text{ for a fixed } n \geq 3$$

- Our asymptotic framework:

$$n \text{ is fixed (small) while } K \to \infty \text{ (large)}$$

  - present second-price auctions with IPVs
  - extend to first-price auctions and to conditional IPV

Asymptotic Frameworks

Existing	Bidder 1	Bidder 2	$\cdots$	Bidder K
Auction 1	$V_{(1),1}$	$V_{(2),1}$		$V_{(K),1}$
Auction 2	$V_{(1),2}$	$V_{(2),2}$		$V_{(K),2}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
Auction n	$V_{(1),n}$	$V_{(2),n}$		$V_{(K),n}$
$n \to \infty$	$\vdots$	$\vdots$		$\vdots$

New	Bidder 1	Bidder 2	$\cdots$	$K \to \infty$
Auction 1	$V_{(1),1}$	$V_{(2),1}$		$V_{(\infty),1}$
Auction 2	$V_{(1),2}$	$V_{(2),2}$		$V_{(\infty),2}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
Auction n	$V_{(1),n}$	$V_{(2),n}$		$V_{(\infty),n}$

Plate	Price
D	\$26m
R	\$33m
W	\$33m
V	\$17m

EV theory ...

Review of Extreme Value Theory

- Consider one auction first. We assume F_V is within the domain of attraction (DoA) of **Extreme Value** (EV) distribution

- **Extreme Value Theory:** There exist constants a_K and b_K such that

$$\frac{V_{(1)} - b_K}{a_K} \xrightarrow{d} \tilde{Z}_1$$

  where the CDF of $\tilde{Z}_1$ must be the generalized EV dist.

$$G_\xi(x) = \begin{cases} \exp(-(1+\xi x)^{-1/\xi}) & \xi \neq 0 \\ \exp(-\exp(-x)) & \xi = 0 \end{cases}$$

  – EV theory to the sample maximum is similar as CLT to sample mean

  – ξ is the **tail index** that characterizes the tail heaviness

is this condition strong? ...

Review of EV Theory, cont'd

- The DoA assumption is mild and satisfied by many distributions

Dist.	Cauchy	Pareto(α)	t(v)	Gaussian	Uniform	Poisson
$\xi =$	1	$1/\alpha$	$1/v$	0	-1	X

  - essentially requires f_V is smooth (von Mises condition)

- Joint convergence of first d order statistics:

$$\frac{(V_{(1)}, \overbrace{V_{(2)}}^{=P}, \ldots, V_{(d)}) - b_K}{a_K} \xrightarrow{d} (\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_d)$$

  where joint PDF is given by

$$G_\xi(z_k) \prod_{i=1}^{d} g_\xi(z_i)/G_\xi(z_i) \text{ with } g_\xi(z) = \frac{\partial G_\xi(z)}{\partial z}$$

Coming Back to Auction

- EV theory implies that

$$\frac{P - b_K}{a_K} \xrightarrow{d} \tilde{Z}_2 \equiv Z$$

  with density

$$f_{Z|\xi}(x) = \begin{cases} (1+\xi x)^{-\frac{2+\xi}{\xi}} \exp(-(1+\xi x)^{-1/\xi}) & \xi \neq 0 \\ \exp(-2x)\exp(-\exp(-x)) & \xi = 0 \end{cases}$$

- If a_{K_j} and b_{K_j} for $j = 1, ..., n$ are known, the problem is straightforward:
  - let K_j be the numbers of bidder in the jth auction
  - $(P_j - b_{K_j})/a_{K_j} \xrightarrow{d} Z_j$ for $j = 1, ..., n$
  - inference about ξ and other features using n i.i.d. draws from $f_{Z|\xi}(x)$

Asymptotic Framework

- Unfortunately a_{K_j} and b_{K_j} are unknown and difficult to estimate
  - they depend on details of F_V beyond ξ

- Let $K = \min_{1 \leq j \leq n}\{K_j\}$ and assume $K_j/K \to 1$ for all j

- **Lemma 1:** there exist constants a_K and b_K such that for any auction j,

$$\frac{P_j - b_K}{a_K} = \frac{V_{(2),j} - b_K}{a_K} \xrightarrow{d} Z$$

$\Rightarrow P_1, ..., P_n$ share the same constants a_K and b_K, which are still unknown...

Self-normalization

- Sort the transaction prices as $P_{(1)} \geq P_{(2)} \geq \cdots \geq P_{(n)}$
  - consider the following self-normalized statistics

$$\begin{aligned} \mathbf{P}^* &= \left(1, \frac{P_{(2)} - P_{(n)}}{P_{(1)} - P_{(n)}}, \ldots, \frac{P_{(n-1)} - P_{(n)}}{P_{(1)} - P_{(n)}}, 0\right) \\ \text{Data} &= \left(1, \frac{33-17}{33-17}, \frac{26-17}{33-17}, 0\right) \end{aligned}$$

  - EV theory and continuous mapping theorem imply

$$\mathbf{P}^* \xrightarrow{d} \mathbf{Z}^* = \left(1, \frac{Z_{(2)} - Z_{(n)}}{Z_{(1)} - Z_{(n)}}, \ldots, 0\right),$$

whose PDF is $f_{\mathbf{Z}^*|\xi}$ is derived by change of variables, that is ...

The Density

- In particular

$$
\begin{aligned}
& f_{\mathbf{Z}^*|\xi}(\mathbf{z}^*) \\
= \; & n!\Gamma(2n) \int_0^{b(\xi)} s^{n-2} \exp\left(\begin{array}{c} -2n \log\left(\sum_{j=1}^{n}\left(1+\xi z_j^* s\right)^{-1/\xi}\right) \\ -\left(1+\frac{2}{\xi}\right)\sum_{j=1}^{n}\log\left(1+\xi z_j^* s\right) \end{array} \right) ds
\end{aligned}
$$

  - $\mathbf{z}^* = (1, z_2^*, ..., z_{n-1}^*, 0)$
  - $\Gamma(\cdot)$ is the gamma function
  - $b(\xi) = \infty$ if $\xi \geq 0$ and $-1/\xi$ otherwise

- We can compute this density via Gaussian quadrature

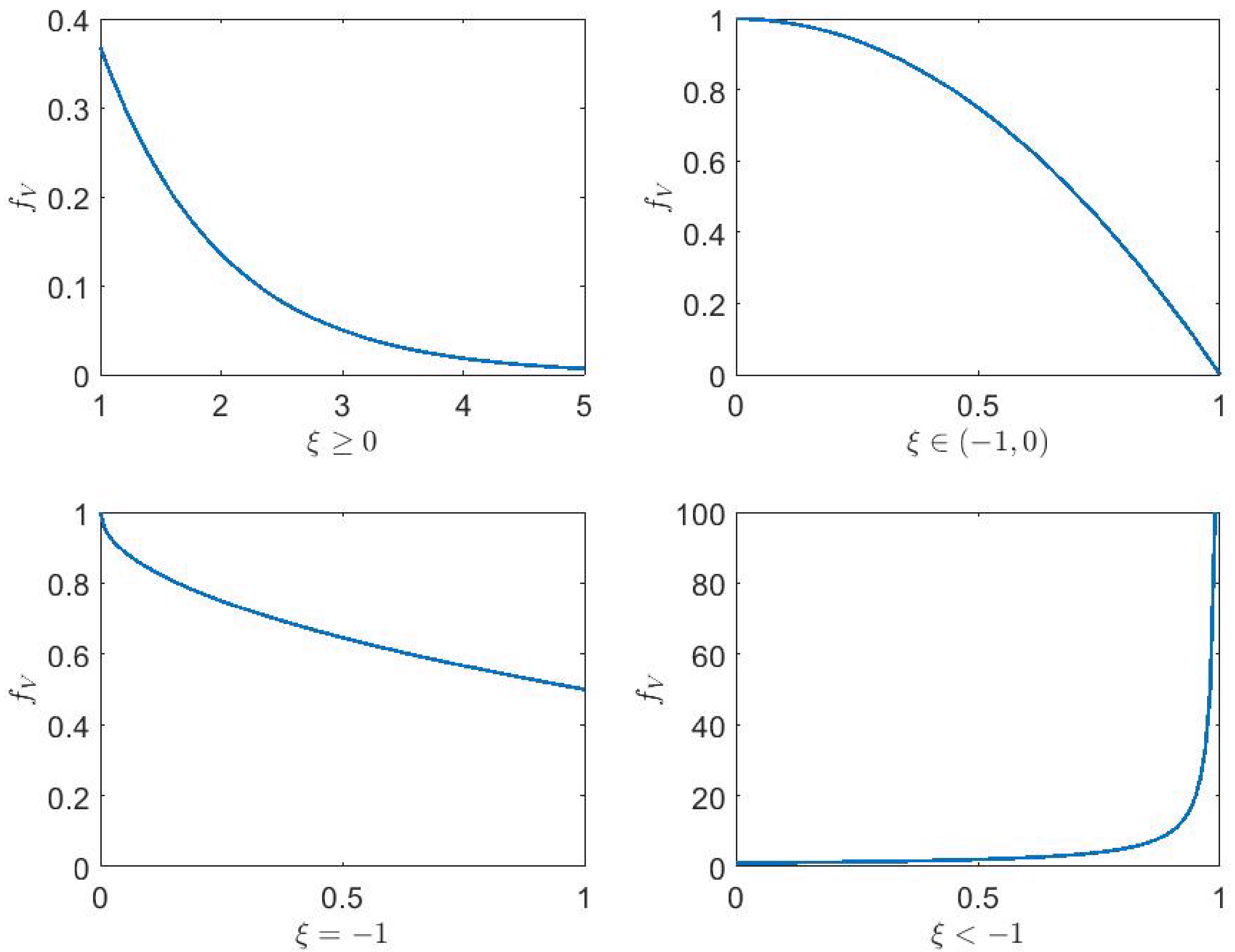
- Now using $f_{\mathbf{Z}^*|\xi}(\mathbf{z}^*)$, we illustrate the inference about ξ and other features

Why Care about ξ

- Commonly adopted assumption (e.g., Guerre, Perrigne, and Vuong, 2000)
  - $v^* = \sup\{v : F_V(v) < 1\} < \infty$
  - the density f_V is bounded in $(0, \infty)$

- Four cases given a continuous f_V
  - $\xi \geq 0 : v^* \leq \infty$ and $f_V(v) \to 0$ as $v \to v^*$
  - $\xi \in (-1, 0) : v^* < \infty$ and $f_V(v) \to 0$ as $v \to v^*$
  - $\xi = -1 : v^* < \infty$ and $f_V(\cdot) \in [\underline{C}, \overline{C}] \subset (0, \infty)$
  - $\xi < -1 : v^* < \infty$ and $f_V(v) \to \infty$ as $v \to v^*$

Figure Illustration



Necessary Condition Stated in ξ

- **Lemma 2:** Suppose (i) $v^* < \infty$, (ii) $f_V \in [\underline{C}, \overline{C}] \subset (0, \infty)$, and (iii) f_V is continuous, then

$$\xi = -1.$$

  - if we exclude $f_V(v) \to \infty$, we have $\xi \geq -1$
  - if we assume $\mathbb{E}\left[V_i^2\right] < \infty$, we have $\xi < 1/2$

- Then the hypothesis testing problem becomes

$$\mathsf{H}_0 : \xi = -1 \text{ against } \mathsf{H}_1 : (-1, 0.5)$$

Likelihood Ratio Test

- We have a composite alternative

$$\text{H}_0 : \xi = -1 \text{ against } \text{H}_1 : \xi \in (-1, 0.5)$$

$\Rightarrow$ the asym. **optimal** test is (Müller, 2011)

$$\varphi(\mathbf{P}^*) = \mathbf{1}\left[\frac{\int_{[-1,0.5]} f_{\mathbf{Z}^*|\xi}\left(\mathbf{P}^*\right) w\left(\xi\right) d\xi}{f_{\mathbf{Z}^*|\xi=-1}\left(\mathbf{P}^*\right)} > \text{cv}\right]$$

$\Rightarrow$ it maximizes the w-averaged average power

$\Rightarrow$ among all equivariant tests relying on $\mathbf{P}^* \xrightarrow{d} \mathbf{Z}^*$

$\Rightarrow$ $w\left(\cdot\right)$ some weight, say uniform

inference about other features ...

Other Features of the Auction

- **Lemma 3:** often-studied objects of interests are **all** functions of ξ within our framework

1. winner's expected utility $\mu_K \equiv \mathbb{E}\left[V_{(1)} - V_{(2)}\right]$

$$\frac{\mu_K}{a_K} \to \Gamma(1-\xi),$$

2. **seller's expected revenue** $\pi_K \equiv \mathbb{E}\left[V_{(2)}\right]$

$$\frac{\pi_K - b_K}{a_K} \to \frac{\Gamma(2-\xi) - 1}{\xi}$$

3. optimal reserve price $\gamma_K = \arg\max_{\gamma} \pi_K(\gamma)$

$$\frac{\gamma_K - b_K}{a_K} \to \frac{1}{1-\xi}$$

Seller's Expected Revenue

- We construct confidence intervals $U(P_1, ..., P_n)$ for $\pi_K \equiv \mathbb{E}\left[V_{(2)}\right]$ that
  - satisfy the asymptotic coverage

$$\mathbb{P}(\pi_K \in U) \geq 1 - \alpha \text{ for all } \xi \in [-1, 0.5]$$

  - (nearly) minimize the weighted average length

$$\int \mathbb{E}[\text{lgth}(U)]\, w(\xi)\, d\xi$$

  - satisfy the equivariance

$$U(a\mathbf{P} + b) = aU(\mathbf{P}) + b$$

CI about Seller's Expected Revenue

- Recall that our data and object of interest satisfy

$$\left(\frac{\mathbf{P}-b_K}{a_K}, \frac{\pi_K - b_K}{a_K}\right) \xrightarrow{d} (\mathbf{Z}, \pi^*)$$

where $\mathbf{P} = (P_1, ..., P_n)$, $\mathbf{Z} = (Z_1, ..., Z_n)$

- Challenge: ξ is unknown and (a_K, b_K) are unknown

- Aim to construct an asymptotically valid confidence interval $U(\cdot)$

$$\min \int_{\xi \in [-1, 0.5]} \mathbb{E}\left[\text{lgth}\left(U(\mathbf{P})\right)\right] w(\xi)\, d\xi$$

$$\text{s.t. } \mathbb{P}\left(\pi_K \in U(\mathbf{P})\right) \geq 95\% \text{ for all } \xi \in [-1, 0.5]$$

- Impose equivariance $U(a\mathbf{P} + b) = aU(\mathbf{P}) + b$ to eliminate (a_K, b_K)...

Extensions

- First-price auctions

$$P_j = V_{(1),j} - \frac{\int_{-\infty}^{V_{(1),j}} F_V(u)^{K_j-1}\,du}{F_V\left(V_{(1),j}\right)^{K_j-1}}$$

- **Lemma 4:** there exist constants a_K and b_K with $K = \min_{1\leq j\leq n}\{K_j\}$,

$$\frac{(P_1,...,P_n) - b_K}{a_K} \xrightarrow{d} (X_1,...,X_n)$$

  where $f_{X|\xi}$ depends only on ξ

- Binding reserve price is allowed

- Conditional IPV is allow if we observe ≥ 3 bids in a single auction

Conclusion

- Better approximation to empirical settings when K is large
  - K might not be observed because of
    - binding reserve price
    - selective entry (Gentry and Li, 2014)
  - multiple bids might not be observed
    - ascending-price auction (Athey and Haile, 2002)

$\Rightarrow$ we cannot identify F_V but can do inference about its tail feature

asymptotically, it's all is about ξ


Thanks!