Interim Information and Seller's Revenue in Standard Auctions

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Introduction

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How does the Interim Revenue depend jointly on a and v?

Motivation

Statistical interpretation: Exploration of the properties of ex-ante equivalent formats

- Bidders play the symmetric efficient equilibrium in a, a'
- Econometrician learns one bidder's value v and forecasts revenues in a, a'
  - What makes a better than a' when a bidder is v (and worse when v')?
  - Understand differences that average out ex-ante

Application: Informed seller chooses format based on Interim Revenue

- *Endogenous*: Repeated auctions (e.g. procurement, online-ad auctions)
- *Exogenous*: Rating based on purchasing history
- Manipulation *of* auction format vs. *within* format (credible auctions, shill bidding...)
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Preview of Results

FPA vs. SPA

- Single crossing
- FPA better for v low, worse for v high

Standard auctions: Who-pays-what specification (in the space of order statistics)

- $a \succ_v a' \Leftrightarrow$ A bidder's transfer is higher in a than a' when ***a competitor*** is v
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Outline

- FPA vs SPA
  - Example: 2 bidders, uniform distribution
  - Single crossing
- Standard Auctions
  - Winner Pay Auctions
  - Pay-as-bid Auctions
  - FPA best at $v = 0$ and worst at $v \approx 1$

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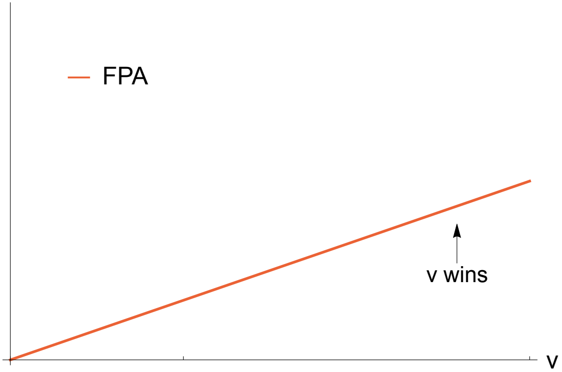

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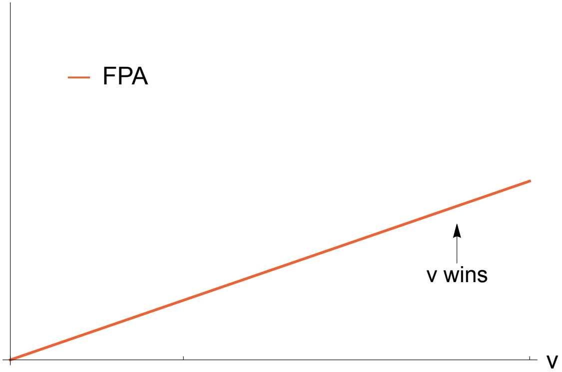

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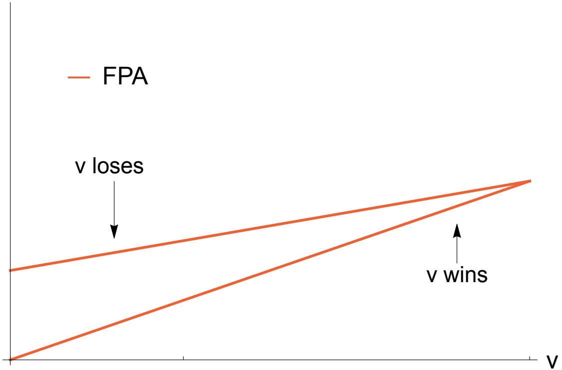

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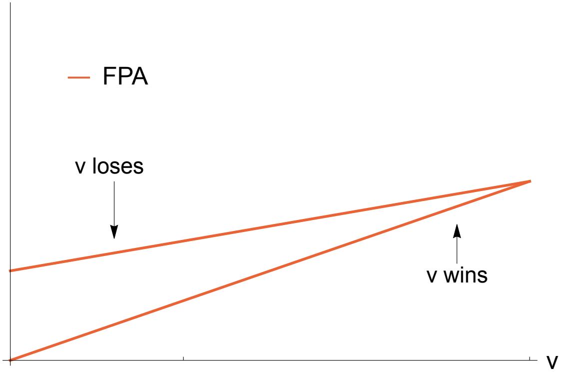

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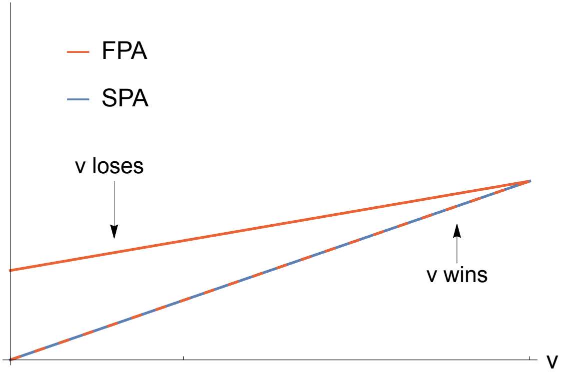

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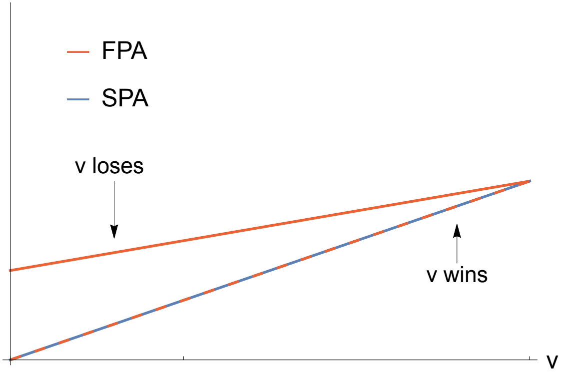

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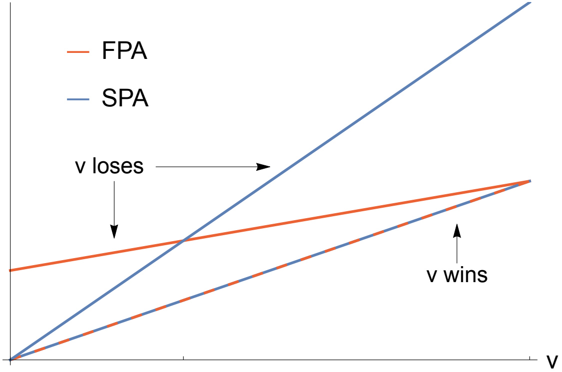

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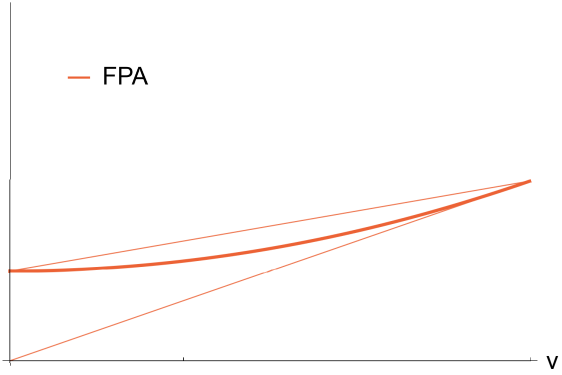

- IRFs: Weight events by their likelihood

$$\begin{aligned} \Pi^a(v) &= \mathbb{P}(v \text{ wins}) \cdot \mathbb{E}[\Pi^a | v \text{ wins}] \\ &+ \mathbb{P}(v \text{ loses}) \cdot \mathbb{E}[\Pi^a | v \text{ loses}] \end{aligned}$$

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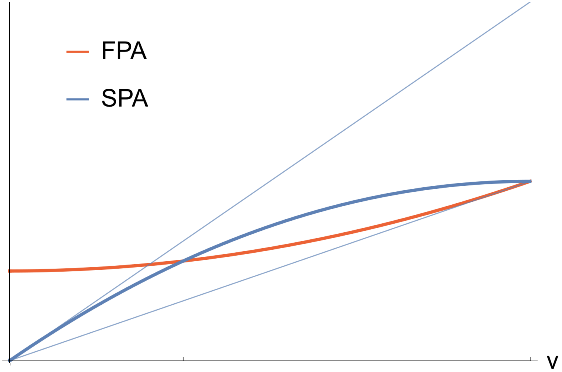

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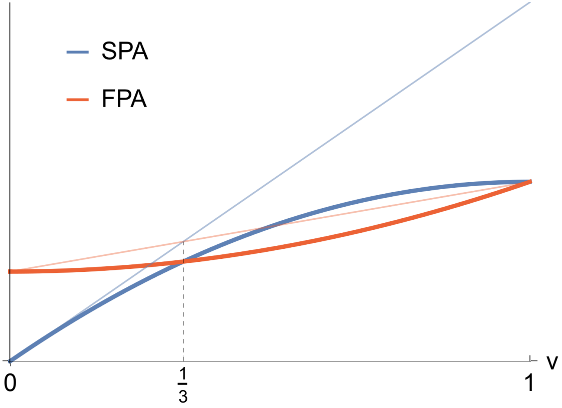
  - v loses for sure...
  - and sets the price in SPA



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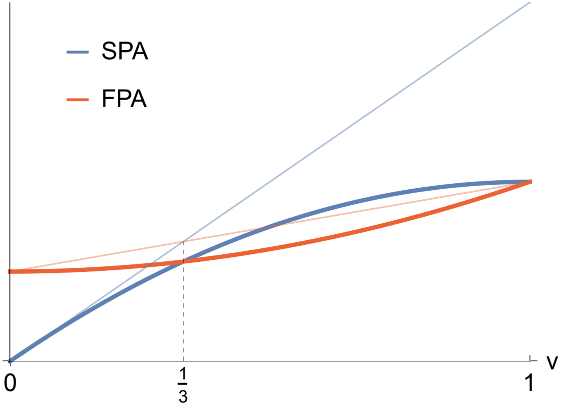

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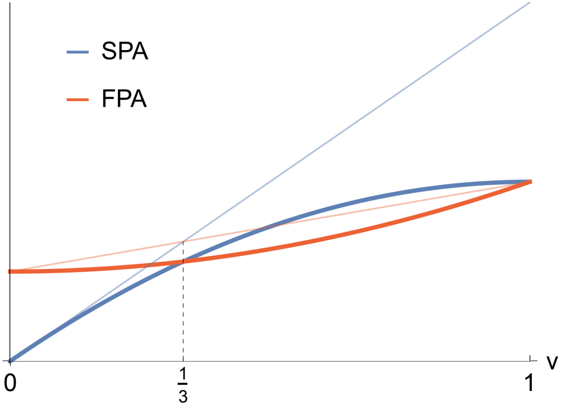
  - $v = 1$ (v never loses)
  - $v = \frac{1}{3}$ (Same rev. even if v loses)

$$\Pi^S(v) > \Pi^F(v) \iff v > \frac{1}{3}$$



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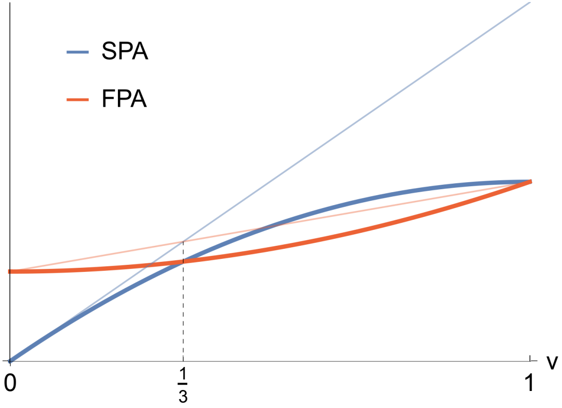
- $\mathbb{E}_v\left[\Pi^S(v)\right] = \mathbb{E}_v\left[\Pi^F(v)\right]$ (RET+LIE)

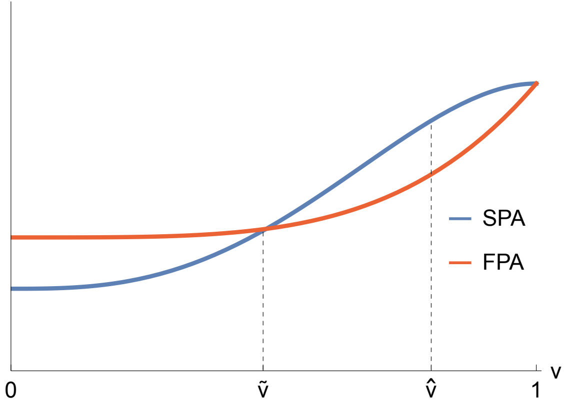


FPA vs. SPA: Single Crossing

- n bidders, valuation $\sim F$
- Virtual value ψ

- **Proposition**: *If $\psi(v) = b^F(v)$ has unique solution, then there is a unique $\widetilde{v}$ s.t.*
  - $\Pi^F(v) > \Pi^S(v)$ *if* $v < \widetilde{v}$
  - $\Pi^S(v) > \Pi^F(v)$ *if* $v > \widetilde{v}$

- $\Pi^F(v) - \Pi^S(v)$ *is*
  - *maximized at* $v = 0$
  - *minimized at* $b^F(\hat{v}) = \psi(\hat{v})$



▸ Sketch of Proof

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- **Standard Auctions**
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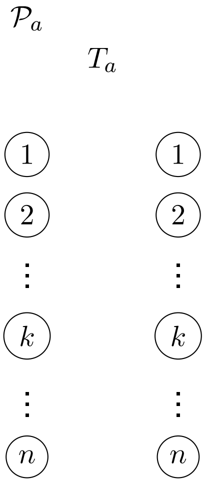

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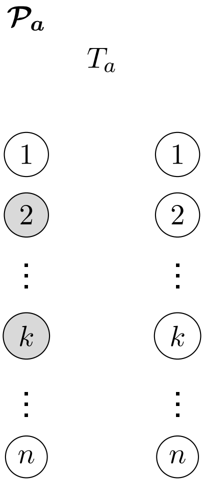

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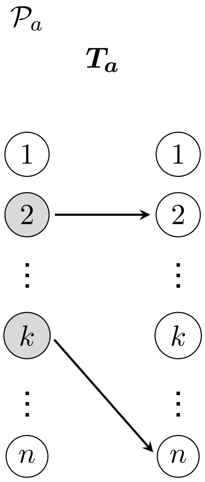
- Who pays? $\mathcal{P}_a$ specifies the order statistics that pay
- **What do they pay?** T_a associates to each payer the the order statistic of the bid that he pays
  - *Constraint*: A bidder cannot pay a bid higher than his own



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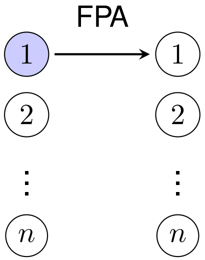

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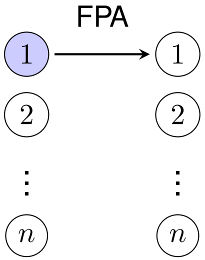

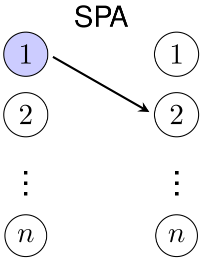

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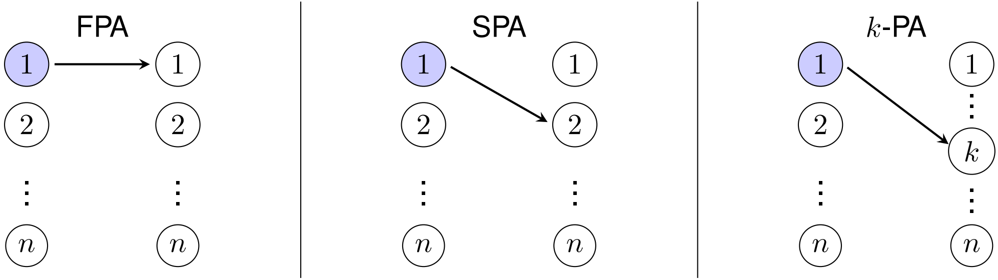

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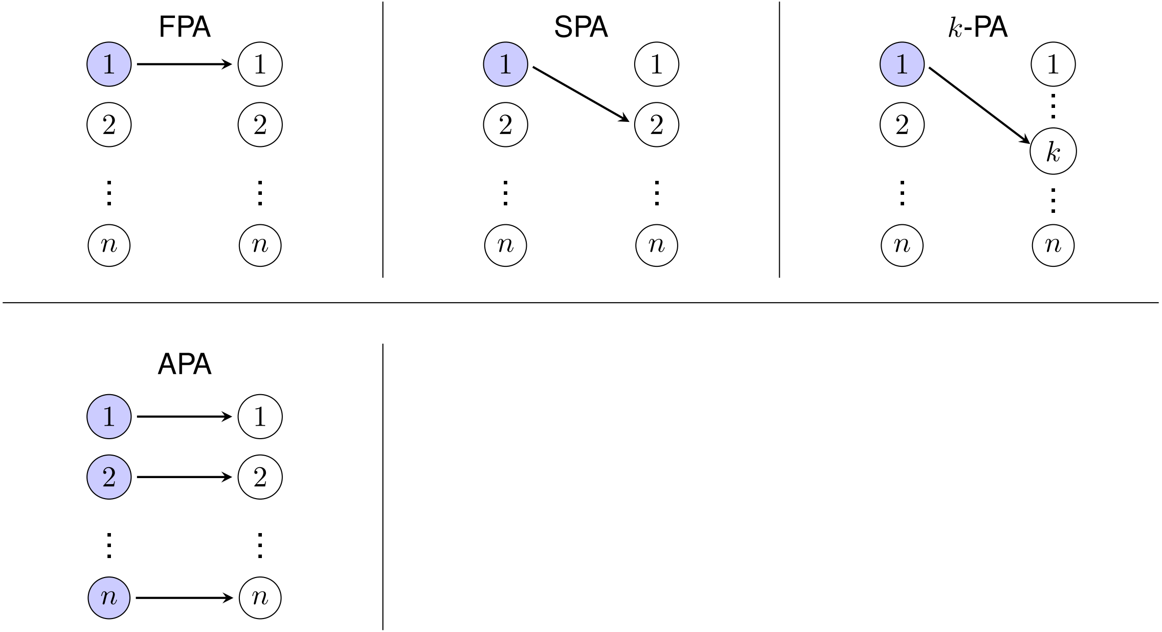

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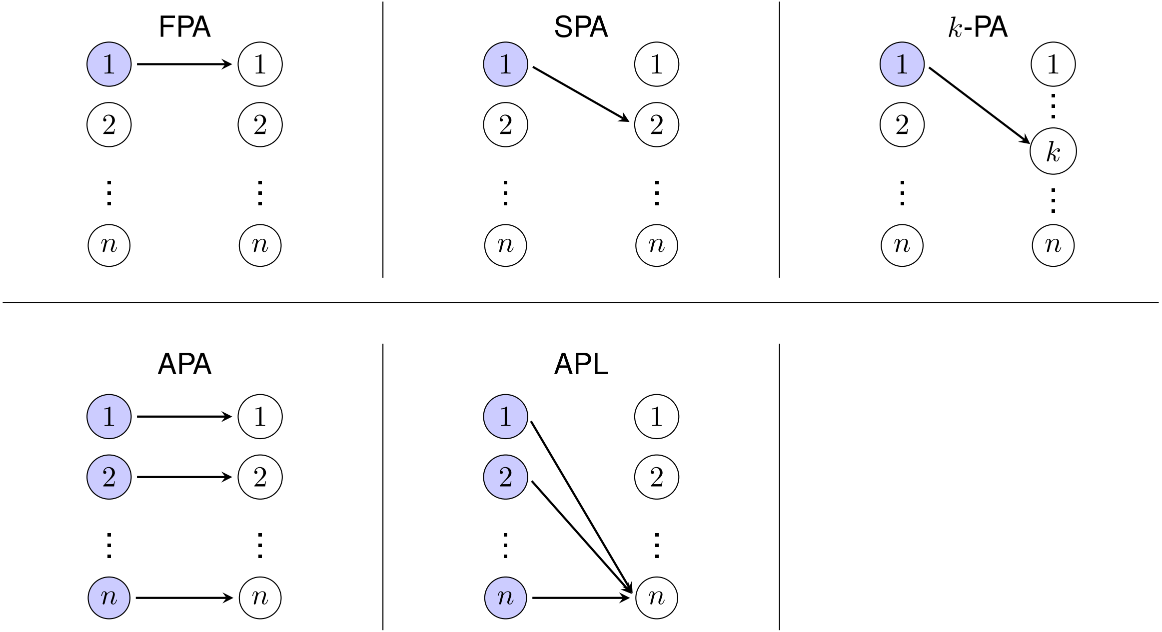

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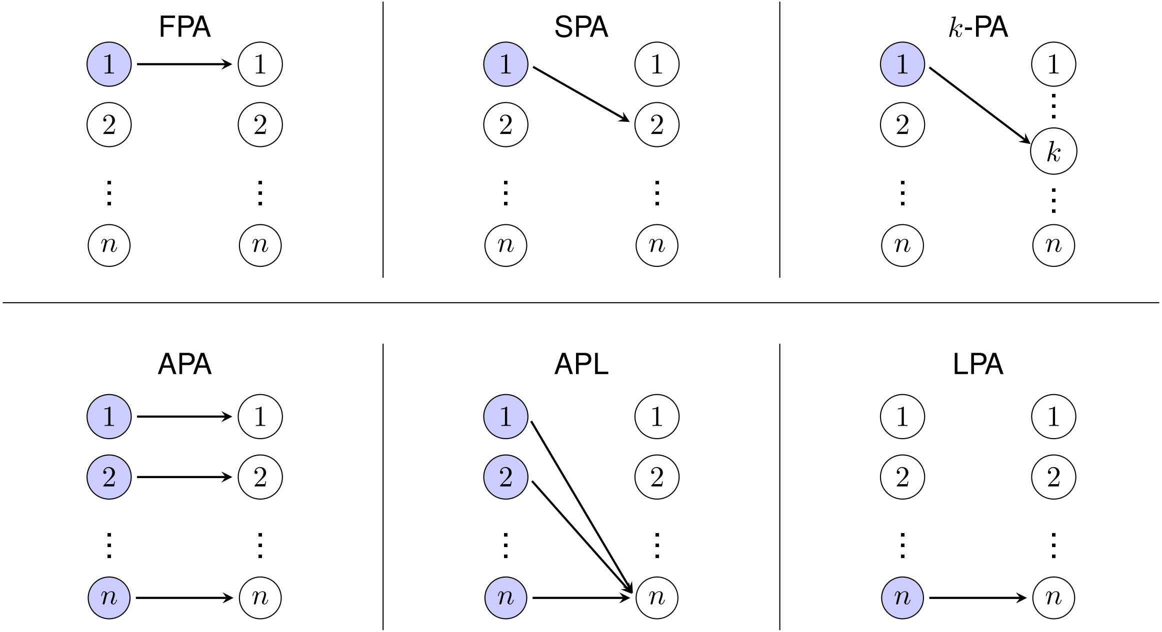

Decomposition

- **Equilibrium transfer vector** $\tilde{t}^a(\boldsymbol{v}) : [0,1]^n \to \mathbb{R}^n$
  - Associates a valuation vector $\boldsymbol{v}$ to a vector of transfers made by each bidder
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$$\tilde{t}^{SPA}\left(\begin{bmatrix} 0.2 \\ \\ 0.6 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \\ b^S(0.2) \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0.2 \end{bmatrix}$$

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$$\tilde{t}^{APA}\left(\begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix}\right) = \begin{bmatrix} b^A(0.2) \\ b^A(0.6) \end{bmatrix} = \begin{bmatrix} 0.02 \\ 0.18 \end{bmatrix}$$

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  - Associates a valuation vector $\boldsymbol{v}$ to a vector of transfers made by each bidder
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- By construction,

$$\Pi^a(\boldsymbol{v}) = \sum_{i=1}^{n} \tilde{t}_i^a(\boldsymbol{v})$$

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  - Depends on the auction format and equilibrium bidding ▸ Bid fct
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  - Denote with $t^a(v)$, determines differences across IRFs

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$$\mathbb{E}\left[t^a(v)\right] = \mathbb{E}\left[t(v)\right]$$

▸ Details

Study IRF = study $t^a(v)$

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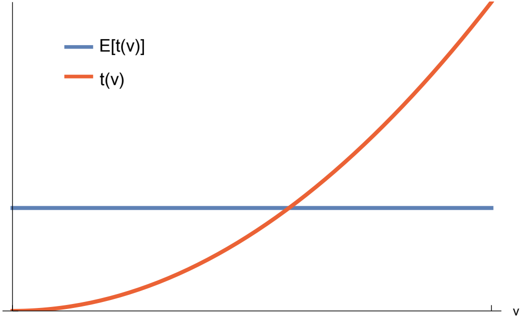

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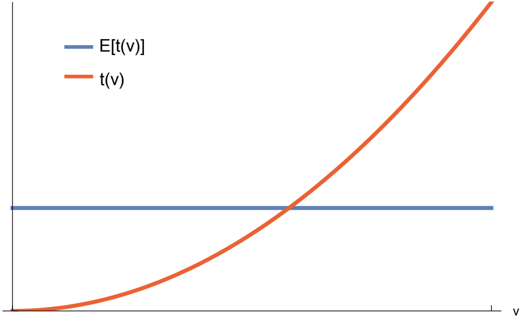

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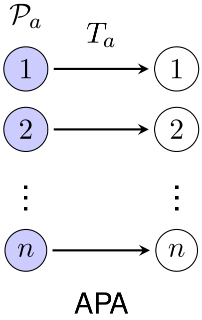

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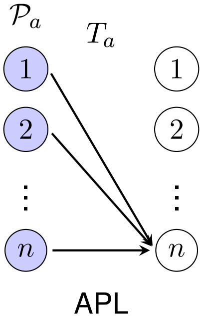

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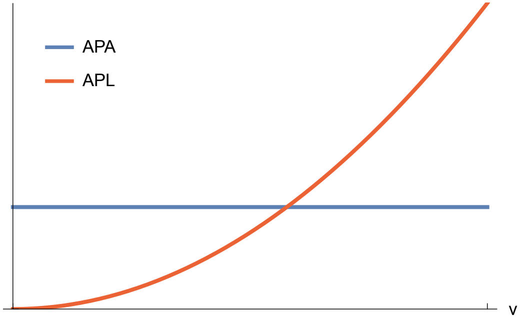

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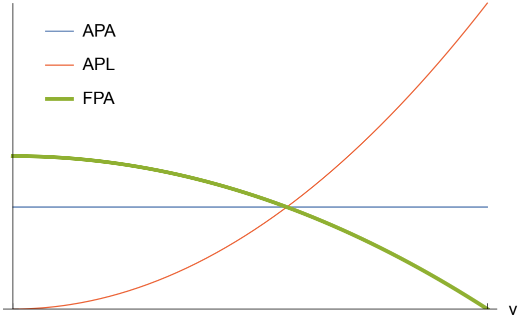

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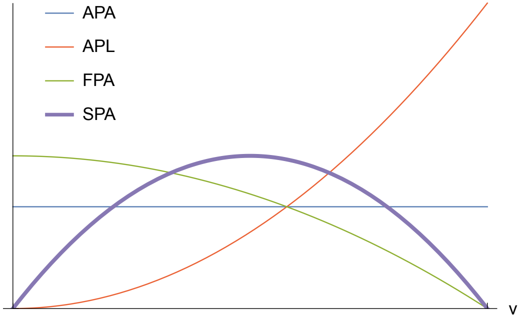

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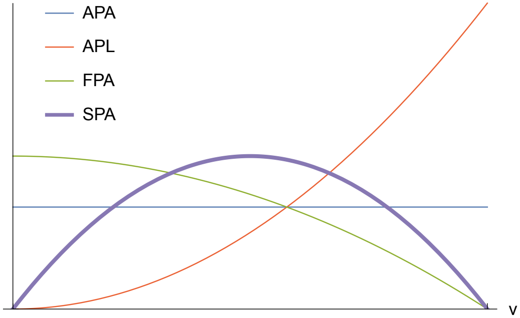

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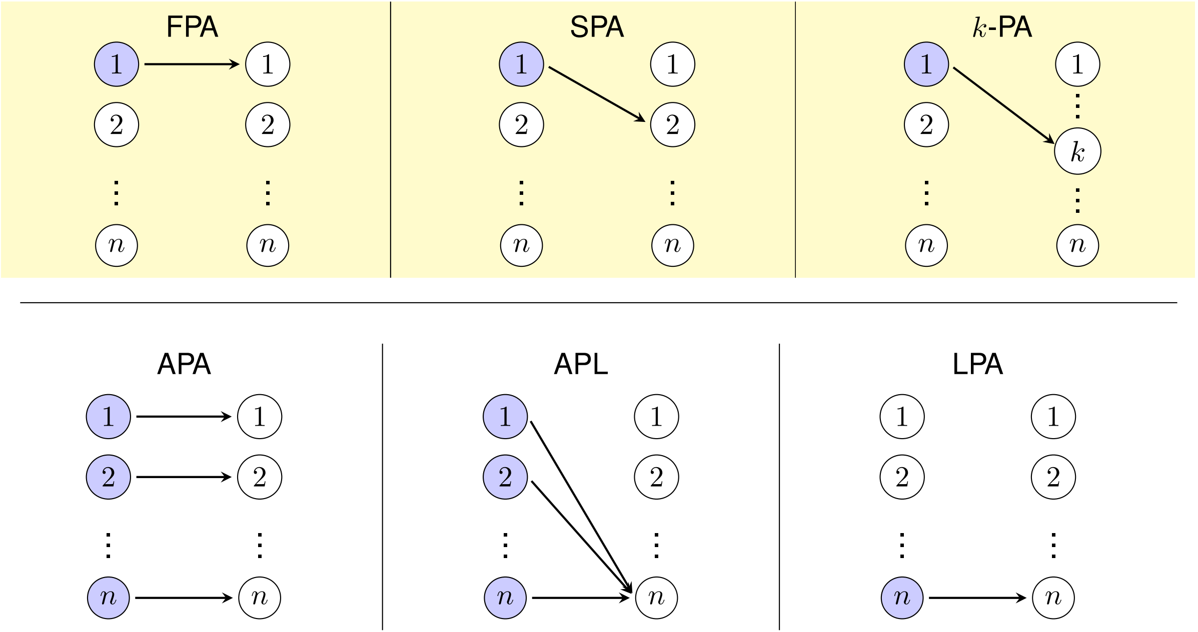
$\Rightarrow$ Single (multiple) crossings among some formats

Outline

- FPA vs SPA
  - Example: 2 bidders, uniform distribution
  - Single crossing
- Standard Auctions
  - **Winner Pay Auctions**
  - Pay-as-bid Auctions
  - FPA best at $v = 0$ and worst at $v \approx 1$

Winner-Pay Auction (WPA): $\mathcal{P}_a = \{1\}$



WPA: Properties

At $v = 0$, the seller's interim revenue in the FPA is higher than in any other WPA
For $v \approx 1$, the seller's interim revenue in the FPA is lower than in any other WPA

Key Intuition

- v affects transfer conditional on paying in all kPA except in FPA (if pay, pay own bid)
  $\Rightarrow$ bad if $v = 0$, good if $v \approx 1$

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[RET] $\quad ||$

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- A bidder at $v = 0$ depresses expectation ($\boldsymbol{v} > \boldsymbol{v}|0$)

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$$t^k(v) - t^j(v) \propto b^k(v) - b^j(1)$$

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▸ All WPA

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- $b^j(v)$ is increasing in j: $b^F < b^S < ... < b^n$ ▸ All WPA
- Similar result for APA vs APL
  - Pay your bid $\Rightarrow$ Hedge the risk conditional on payer
    Good at $v = 0$ $(FPA \succ kPA$ & $APA \succ APL)$
    Bad at $v \approx 1$ $(kPA \succ FPA$ & $APL \succ APA)$

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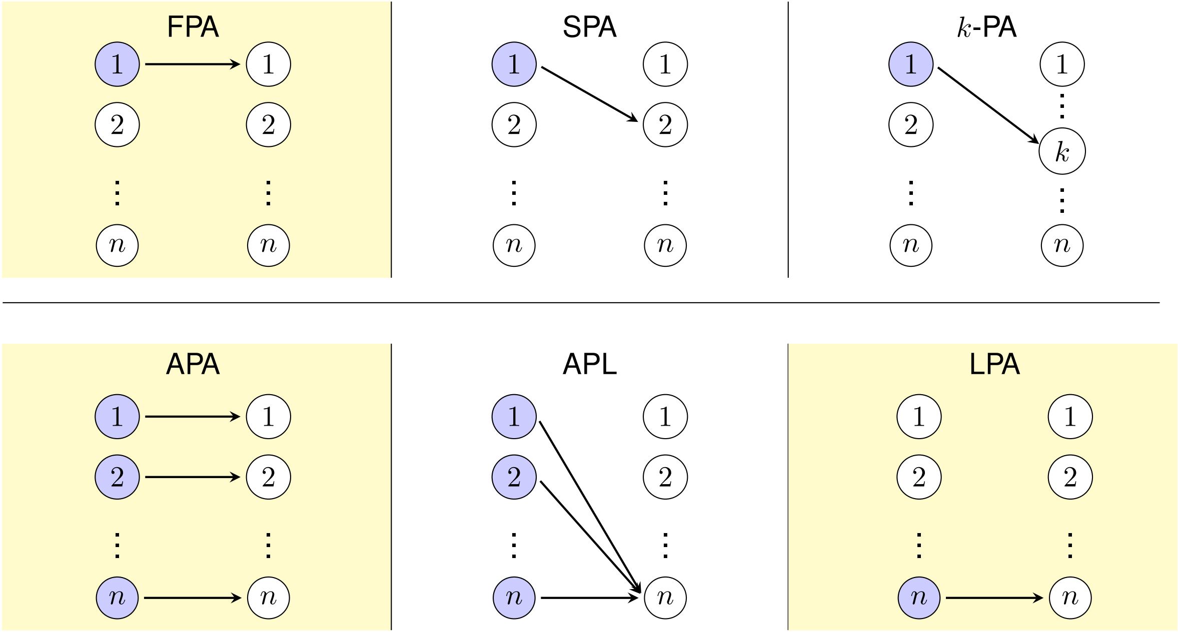
Pay-as-Bid-Auction (PBA): $T_a(i) = i, \forall i \in \mathcal{P}_a$



PBA

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- Bid function (by RET)

$$b^{\text{PB-}\mathcal{P}}(x) = \frac{t(x)}{\mathbb{P}(x \in \mathcal{P})}$$

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$$t^{\text{PB-}\mathcal{P}}(x, v) = b^{\text{PB-}\mathcal{P}}(x)\,\mathbb{P}_v(x \in \mathcal{P}) = t(x)\,\frac{\mathbb{P}_v(x \in \mathcal{P})}{\mathbb{P}(x \in \mathcal{P})}$$

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- Likelihood ratio: How the probability that x is a payer changes with the information that a competitor has value v

PB-$\{\mathcal{P}\}$: Examples

$$t^{\text{PB-}\mathcal{P}}(v) = \mathbb{E}\left[t(x)\frac{\mathbb{P}_v(x \in \mathcal{P})}{\mathbb{P}(x \in \mathcal{P})}\right]$$

All-Pay Auction: PB-$[n]$

$$\frac{\mathbb{P}_v(x \in [n])}{\mathbb{P}(x \in [n])} = \frac{1}{1}$$

- Then,

$$t^{APA}(v) = \mathbb{E}[t(x)]$$

- Realized transfer independent of competitors' values

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First-Price Auction: PB-$\{1\}$

$$\frac{\mathbb{P}_v(x \in \{1\})}{\mathbb{P}(x \in \{1\})} = \begin{cases} 0 & \text{if } x < v \\ \frac{F^{n-2}(x)}{F^{n-1}(x)} & \text{if } x > v \end{cases} = \frac{1}{F(x)}\mathbf{1}\{x > v\}$$

- Then,

$$t^{FPA}(v) = \int_v^1 \frac{t(x)}{F(x)}\text{d}F(x)$$

PB-$\{\mathcal{P}\}$: Examples

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Last Pay Auction: PB-$\{n\}$

$$\frac{\mathbb{P}_v(x \in \{n\})}{\mathbb{P}(x \in \{n\})} = \begin{cases} \frac{(1-F(x))^{n-2}}{(1-F(x))^{n-1}} & \text{if } x < v \\ 0 & \text{if } x > v \end{cases} = \frac{1}{1-F(x)}\mathbf{1}\{x < v\}$$

- Then,

$$t^{LPA}(v) = \int_0^v \frac{t(x)}{1-F(x)}\text{d}F(x)$$

- Increasing and unbounded
  - Unbounded bid (necessary whenever $1 \notin \mathcal{P}$)

$$b^{LPA}(x) = \frac{t(x)}{(1-F(x))^{n-1}}$$

Ranking of IRF among PBA

Given v, finding the interim optimal PBA $=$ Solving:

$$\overline{\mathsf{PB}}\left(v\right) = \max_{\mathcal{P}\subseteq[n]} \mathbb{E}\left[t\left(x\right)\frac{\mathbb{P}_{v}\left(x\in\mathcal{P}\right)}{\mathbb{P}\left(x\in\mathcal{P}\right)}\right]$$

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Prop: For any $\mathcal{P}\subseteq[n]$

- $\mathit{\Pi}^{\mathsf{PB}\text{-}\{1\}}\left(0\right) > \mathit{\Pi}^{\mathsf{PB}\text{-}\mathcal{P}}\left(0\right) > \mathit{\Pi}^{\mathsf{PB}\text{-}\{n\}}\left(0\right)$
- $\mathit{\Pi}^{\mathsf{PB}\text{-}\{n\}}\left(v\right) > \mathit{\Pi}^{\mathsf{PB}\text{-}\mathcal{P}}\left(v\right) > \mathit{\Pi}^{\mathsf{PB}\text{-}\{1\}}\left(v\right)$ for $v\approx 1$

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Prop: For any $\mathcal{P} \subseteq [n]$

- $\Pi^{\text{PB-}\{1\}}(0) > \Pi^{\text{PB-}\mathcal{P}}(0) > \Pi^{\text{PB-}\{n\}}(0)$ *[FPA best among PBAs at 0]*
- $\Pi^{\text{PB-}\{n\}}(v) > \Pi^{\text{PB-}\mathcal{P}}(v) > \Pi^{\text{PB-}\{1\}}(v)$ for $v \approx 1$ *[LPA best among PBAs at 1]*

- At $v = 0$ special bidder is the minimum (n^{th} order stat)

$$\frac{\mathbb{P}_0(x\in\{1\})}{\mathbb{P}(x\in\{1\})} > \frac{\mathbb{P}_0(x\in\mathcal{P})}{\mathbb{P}(x\in\mathcal{P})} > \frac{\mathbb{P}_0(x\in\{n\})}{\mathbb{P}(x\in\{n\})} \qquad \forall x,\mathcal{P}$$

- Likelihood that a generic bidder is any other order statistics increases
- Most significant increase for likelihood of being the maximum

$\Rightarrow$ Seller prefers to receive payments only from the first-order statistic

- At $v \approx 1$ argument is reversed

Main Result

Prop:

- At $v = 0$, the FPA interim *dominates* all standard auctions
- At $v \to 1$, the FPA is interim *dominated* by all standard auctions
- Moreover, $1 \notin \{\mathcal{P}_a\} \iff \lim_{v \to 1} \Pi^a (v) = \infty$

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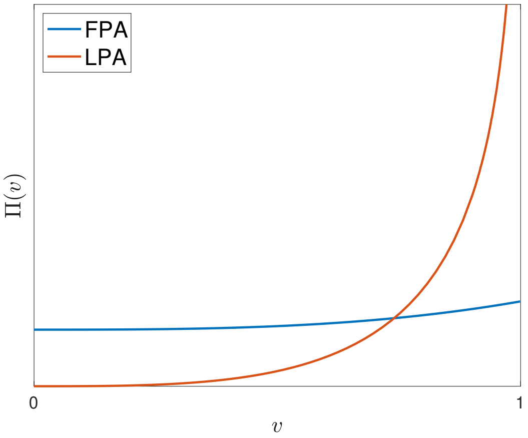

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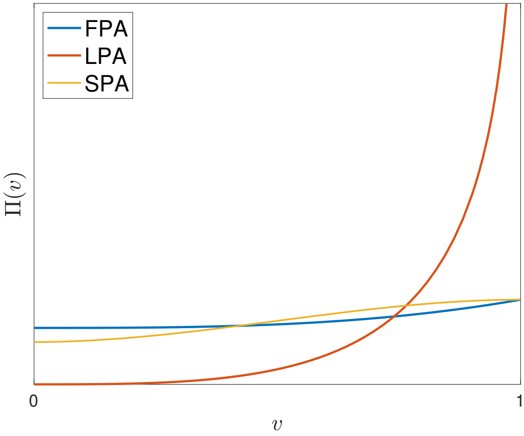

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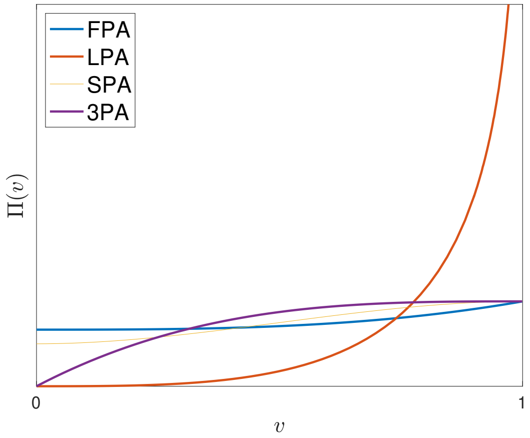

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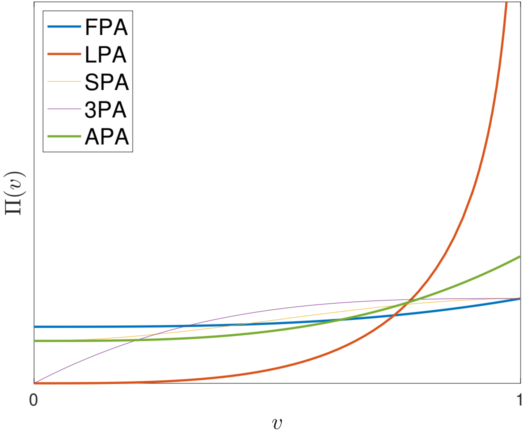

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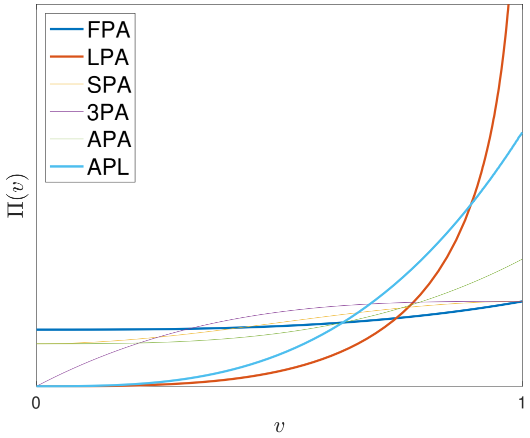

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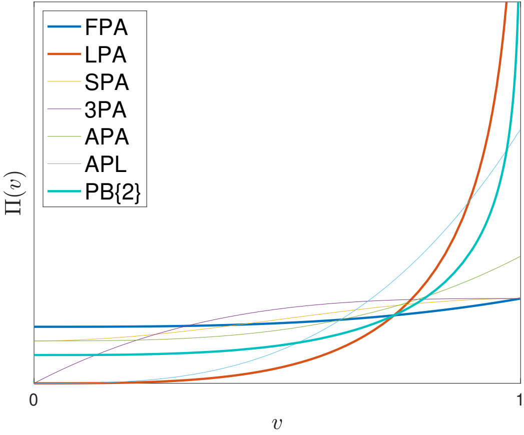

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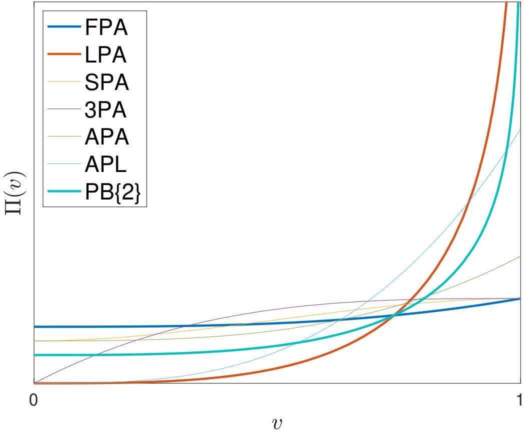


- Then,

$$\text{Im}\left(\mathit{\Pi}^{FPA}\right) \subset \text{Im}\left(\mathit{\Pi}^{a}\right) \subseteq \text{Im}\left(\mathit{\Pi}^{LPA}\right)$$

$\Rightarrow$ FPA less risky

Conclusion

- We analyze how the *marginal contribution of a single bidder* varies across formats
  - This contribution **is not** equal to $t(v)$, but...
  - Depends on how presence of v impacts expected transfer from *other* bidders
    - Driver of interim difference across formats
- Bidders play the efficient equilibrium of the format with symmetric competitors
  - Preliminary analysis: bidders' sophistication limits ability to exploit information

Savvy Bidders

- Bidders are aware that the seller knows v before choosing the auction format
  - The identity, but not the valuation, of the special bidder is known
- Let $\mathcal{A}$ be the set of possible auction formats
- Seller chooses $\mathcal{E} : [0,1] \to \mathcal{A}$, $\mathcal{E}(v)$ is format chosen when special bidder is v
  - $\mathcal{E}(a)^{-1}$ is the set of values that induce $a \Rightarrow$ information about a competitor
- Bidders: Observe $a \Rightarrow$ play equm of asym. auction $\mathcal{E}(a)^{-1} \times [0,1]^{n-1}$
  - v best responds to deviations (which he detects!)
- Seller: Observe $v \Rightarrow$ play $\mathcal{E}(v)$

Equilibrium Algorithm

1. Compute equilibrium of auction a with asymmetric bidders $V \times [0,1]^{n-1}$
   - Bids $b^a_{V,S} : V \to \mathbb{R}$ and $b^a_{V,N} : [0,1] \to \mathbb{R}$ that are mutual best responses
2. Extend the equilibrium to $[0,1]^n$
   - Compute for each $v \notin V$, the best response to $n-1$ bidders playing $b_V^{a,N}$
   - $\tilde{b}^a_{V,S} : [0,1] \to \mathbb{R}$ extends $b^a_{V,S}$ on $[0,1] \setminus V$
     * Types of the special bidders for which the seller should not choose format a, play a best response to the equilibrium in auction a
3. Define interim revenue $\mathit{\Pi}^a_V(v) := \mathbb{E}\left[\Pi^a_V | v_1 = v\right]$ (also defined for $v \notin V$)

Equilibrium Definition

Def: The function $\mathcal{E}(v)$ is a *savvy-bidder equilibrium* if:

1. $\Pi^{a}_{\Omega_a}$ is well-definite $\forall a \in \mathcal{A}$ (There exist bid functions as defined in Step 1 and 2)

2. For all $v \in [0,1]$ and $a \in \mathcal{A}$, $\mathit{\Pi}^{\mathcal{E}(v)}_{\mathcal{E}^{-1}(\mathcal{E}(v))}(v) \geq \mathit{\Pi}^{a}_{\mathcal{E}^{-1}(a)}(v)$

A Savvy Bidders Equilibrium

Prop: Suppose F is the uniform CDF and $\mathcal{A} = \{FPA, SPA\}$.Then, for each n

$$\mathcal{E}(v) = \begin{cases} FPA & v = 0 \\ SPA & v > 0 \end{cases}$$

constitutes a savvy-bidder equilibrium where

$$b^F_{0,N}(x) = \frac{n-2}{n-1}x, \quad \tilde{b}^F_{0,S}(x) = \max\left\{\frac{n-1}{n}x, \frac{n-2}{n-1}\right\}, \quad b^{SPA}_{(0,1],N}(x) = \tilde{b}^S_{(0,1],S}(x) = x$$

- With savvy-bidder the seller cannot exploit his information
- Others will adjust their bids leading to an unraveling process

  $\Rightarrow$ Choice of format where bids are unaffected by information about competitors

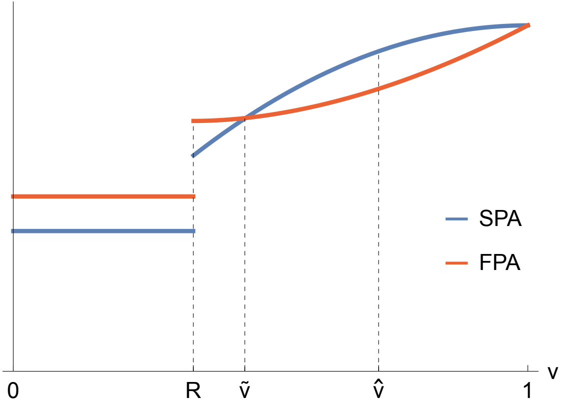
  $\Rightarrow$ Only the SPA is immune to manipulations

Reserve Price

- Seller sets reserve price R in both FPA and SPA

- **Proposition**: *There is a unique* $\widetilde{v} > R$ *such that*
  - $\Pi^F(v) > \Pi^S(v)$ *if* $v < \widetilde{v}$
  - $\Pi^S(v) > \Pi^F(v)$ *if* $v > \widetilde{v}$

- $\Pi^F(v) - \Pi^S(v)$ *is*
  - *maximized at any* $v \leq R$
  - *minimized at* $b^F(\hat{v}, R) = \psi(\tilde{v})$



FPA is best at 0

- Using $t^{FPA}(x,0) = \frac{t(x)}{F(x)}$ we obtain

$$t^{FPA}(x,0) > t^{a}(x,0) \Leftrightarrow$$

$$\sum_{j \in \mathcal{P}_a} \mathbb{P}_{\boldsymbol{v}|x}\left[v_{(j)} = x\right] \mathbb{E}_{\boldsymbol{v}}\left[b^{a}\left(v_{(T_a(j))}(\boldsymbol{v})\right) | v_{(j)} = x\right] >$$

$$\sum_{j \in \mathcal{P}_a} \frac{n-j}{n-1} \mathbb{P}_{\boldsymbol{v}|x}\left[v_{(j)} = x\right] \mathbb{E}_{\boldsymbol{v}}\left[b^{a}\left(v_{(T_a(j))}(\boldsymbol{v})\right) | v_{(j)} = x,\ v_{(n)} = 0\right]$$

  that holds as

- $\frac{n-j}{n-1} < 1$ for all $j > 1$ ($\approx$ want highest bidder to pay), and

$$\mathbb{E}_{\boldsymbol{v}}\left[b^{a}\left(v_{(k)}(\boldsymbol{v})\right) | v_{(j)} = x\right] \geq \mathbb{E}_{\boldsymbol{v}}\left[b^{a}\left(v_{(k)}(\boldsymbol{v})\right) | v_{(j)} = x,\ v_{(n)} = 0\right]$$

want payers to pay their bids

Bidding functions: 3 uniform bidders

- LPA:

$$b^{\text{PB-}\{3\}} = \frac{t(v)}{\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(3)} = v\right)} = \frac{2}{3}\frac{v^3}{(1-v)^2}$$

- APL:

$$\begin{aligned} b^{APA}(v) = \frac{2}{3}v^3 &= \mathbb{E}_{\boldsymbol{v}|v}\left[b^{APL}\left(v_{(3)}(\boldsymbol{v})\right)\right] \\ &= b^{APL}(v)(1-v)^2 + \int_0^v b^{APL}(w)\, 2(1-w)\,\mathrm{d}w \end{aligned}$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}v} b^{APL}(v) = \frac{2v^2}{(1-v)^2}$$

$$\implies b^{APL}(v) = \frac{2v(2-v)}{1-v} + 4\log(1-v)$$

- $2 \to 3$ auction

$$b^{\text{PB-}\{2\}}(v)\, v = \int_0^v b^{2,3}(w)\,\mathrm{d}w$$

$$\implies b^{2,3}(v) = \frac{v^2(3-2v)}{3(1-v)^2}$$

$\mathbb{E}\left[t^a(v)\right]$

- Let

$$t^a(x,v) := \mathbb{E}_{\boldsymbol{v}|x,v}\left[\tilde{t}_1^a(\boldsymbol{v})\right]$$

be the expected transfer of a bidder with value x given a competitor has value v.

- By construction,

$$\mathbb{E}_v\left[t^a(x,v)\right] = t(x), \quad \mathbb{E}_x\left[t^a(x,v)\right] = t^a(v)$$

- Then

$$\mathbb{E}_x\left[t(x)\right] = \mathbb{E}_{x,v}\left[t^a(x,v)\right] = \mathbb{E}_v\left[t^a(v)\right]$$

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Equilibrium Bidding

- Denote $F^{v}_{(j,m)} : [0,1] \to [0,1]$ the CDF of the j^{th} order statistic of m draws from F truncated at v
- Using the structure of the standard auction,

$$t(v) = \sum_{j \in \mathcal{P}_a} \mathbb{P}_{\boldsymbol{v}|v}\left[v_{(j)} = v\right] \mathbb{E}_{\boldsymbol{v}}\left[b^a\left(v_{(T_a(j))}(\boldsymbol{v})\right) | v_{(j)} = v\right]$$

$$= \sum_{j \in \mathcal{P}_a} \mathbb{P}_{\boldsymbol{v}|v}\left[v_{(j)} = v\right] \int_0^v b^a(x)\, \mathsf{d}F^{v}_{(T_a(j)-j,n-j)}(x)$$

  where the unknown is the bidding function $b^a : [0,1] \to \mathbb{R}$
  - v pays only if he is in the set of payers $\mathcal{P}_a$, and
  - conditional on being the j^{th}-order statistic he pays the $T_a(j)^{th}$-highest bid
- If the above admits a monotone solution (with initial condition $b^a(0) = 0$), then such solution constitutes an equilibrium of the standard auction a

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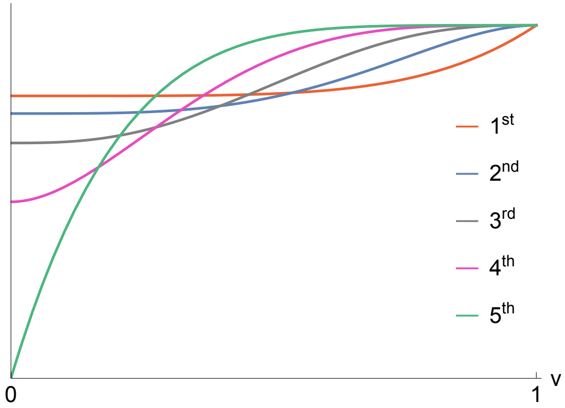
WPA: Ranking at the extrema

- Interim ranking between kPA and $(k+1)$PA is a race between:

1. Collect bids of higher types (kPA better)
2. Higher bid functions ($(k+1)$PA better)



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Single Crossing: Sketch of Proof

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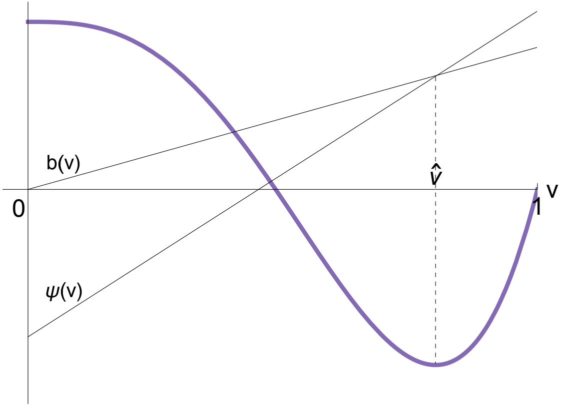
$$\Delta(v) := \Pi^F(v) - \Pi^S(v)$$
$$\propto \int_v^1 \left[b^F(x,n) - \psi(x)\right] \mathrm{d}F^{n-1}(x)$$



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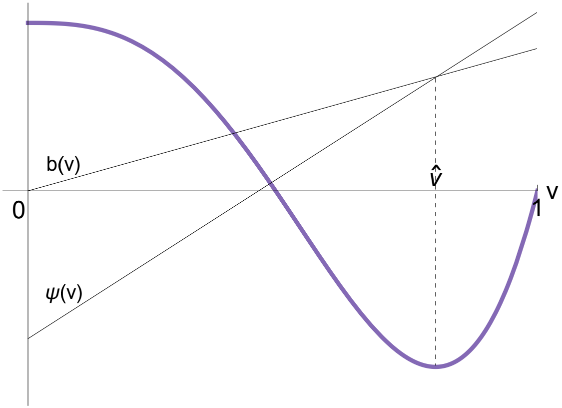
1. $\mathbb{E}_v[\Delta(v)] = 0$
2. $\Delta(1) = 0, \qquad \Delta(0) > 0$
3. Δ has a single minimum



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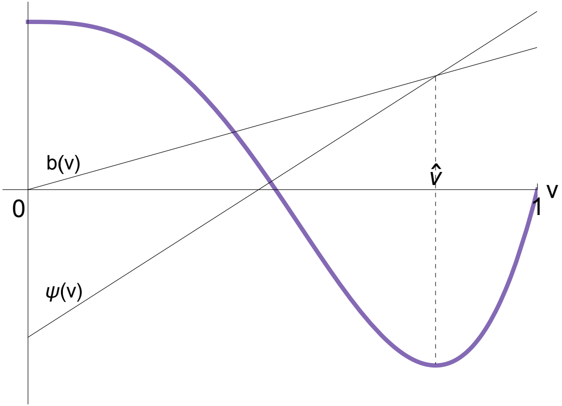

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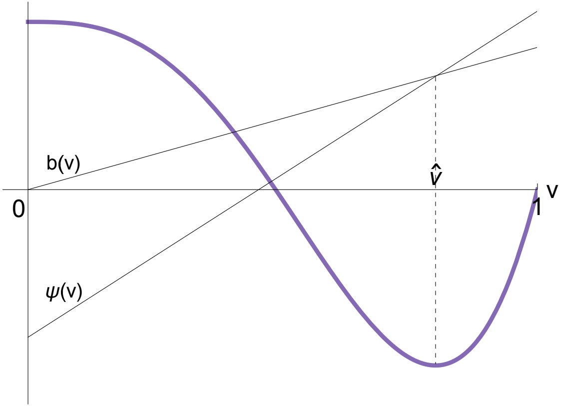

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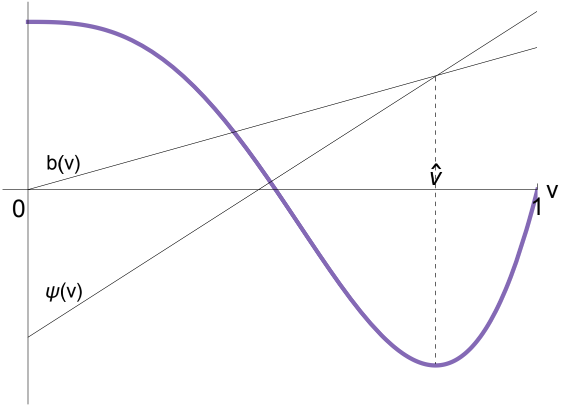
   $\Delta'(v) = 0$ when
   - $v = 0$: maximum
   - $\psi(\hat{v}) = b^F(\hat{v})$

   If unique solution, then unique minimum



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