Linear Models with Interval Censored Variables

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Motivation: Interval Censored Data

Measuring continuous variables by intervals

- reduce nonresponse / reflect measurement error

Illustration: Wealth and Assets Survey in the UK

- asset holdings by intervals

Why is it relevant?

- widespread[1]

- distort genuine dependence between variables

[1] Survey of Consumer Finances, Health&Retirement Survey, PSID, ...

This Project: Linear Moments

How to estimate single coefficient when

- outcome and covariate interval censored
- linear specification with instruments

Illustration: UK Wealth and Assets Survey

$$risky_{it} = \beta wealth_{it} + \gamma controls_{it} + \alpha_i + \epsilon_{it} \text{ with } \mathbb{E}[instru_{it}\epsilon_{it}] = 0$$

– $risky_{it}$ and $wealth_{it}$ by intervals

– compare estimates β across pop. (const. rel. risk aversion?)

Preview of Results: Features

Illustration: compare UK wealth elasticity

Feature:

interval both

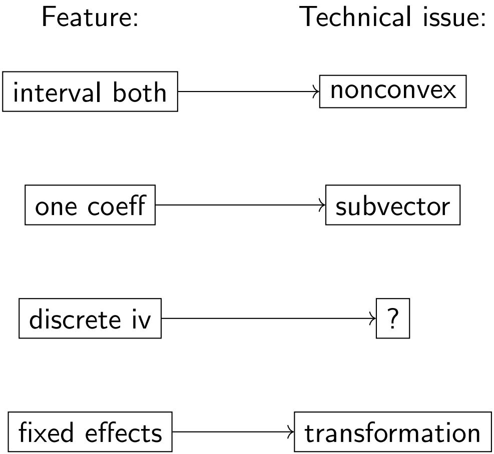
one coeff

discrete iv

fixed effects

Preview of Results: Issues

Illustration: compare UK wealth elasticity



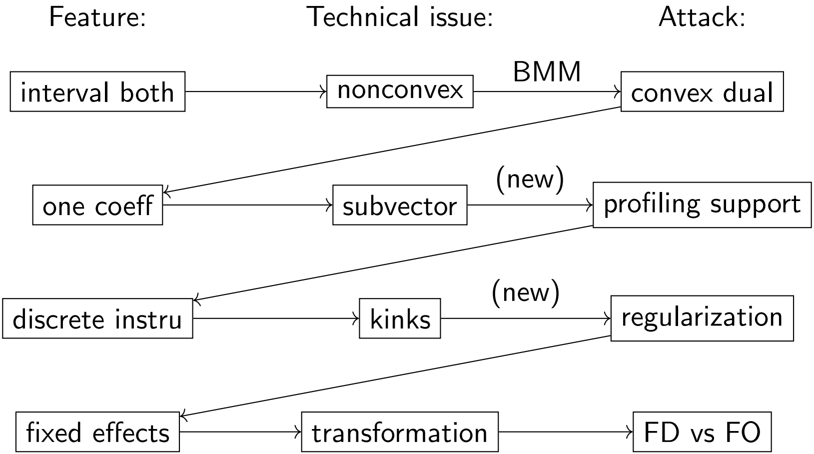
Preview of Results: Solutions

Illustration: compare UK wealth elasticity

Feature:		Technical issue:		Attack:
interval both	→	nonconvex	BMM →	convex dual
one coeff	→	subvector	(new) →	profiling support
discrete instru	→	?=kinks	(new) →	regularization
fixed effects	→	transformation	→	FD vs FO

Preview of Results: Roadmap

Illustration: UK wealth elasticity 1, like Italy unlike Sweden



Related Literature

Interval Covariate:

...

Interval Outcome:

...

Interval Both:

BMM: Beresteanu, Molchanov, Molinari ('11) convex dual set

my paper: closed-form support function + profiling support
+ regularization + work out panel data application

Moment Inequalities:

...

Related Literature

Interval Covariate:

Hsiao ('83); Manksi and Tamer ('02)

Interval Outcome:

Stewart('83); Beresteanu and Molinari ('08); Bontemps, Magnac, Maurin ('11)

Interval Both:

Beresteanu, Molchanov, Molinari ('11); my paper

Moment Inequalities:

Andrews and co. ('10,'13); Cho and Russell ('18); Gafarov ('19); reviews by Canay and co. ('17, '24)

Outline

PART I: IDENTIFICATION

(1) interval both: from nonconvex to convex (dual)

(2) one coeff: from all to subvector (profiling support)

PART II: ESTIMATION

(3) discrete instru: from kinks=nonpivotal to pivotal (regu'tion)

PART III: ILLUSTRATION

(4) fixed-effects: from cross section to panel (FD vs. FO)

PART I: IDENTIFICATION

- Setup	
- Identified set:	nonconvex
- Auxiliary set:	convex
- Support function:	all coeff
- Subvector profiling:	one coeff

Setup

Linear specification:

$$y^\star = x^\star \beta + z^\top \gamma + u$$

- interval censored outcome: $y^\star \in \left[\underline{y}, \overline{y}\right]$
- interval censored covariate: $x^\star \in \left[\underline{x}, \overline{x}\right]$
- control covariates: $\underset{L\times 1}{z}$
- instrumental variables: $\mathbb{E}(wu) = \underset{M\times 1}{0}$
- data: $\{\underline{y}_i, \overline{y}_i, \underline{x}_i, \overline{x}_i, z_i, w_i\}_{i=1}^n$ iid

Midpoint is a nonstarter

Challenge #1: characterize observationally equivalent β
Attack: profiling convex dual set

Identified Set: Θ_I

Observationally equivalent $\theta = (\beta, \gamma)$

$$\Theta_I := \{\theta \in \mathbb{R}^{1+L} : \mathbb{E}[w(y^\star - x^\star\beta - z\gamma)] = \underset{M\times 1}{0}$$
$$, y^\star \in [\underline{y}, \overline{y}], x^\star \in [\underline{x}, \overline{x}]\}$$

Identified set may be nonconvex

Why is nonconvex inconvenient?

Auxiliary Dual Set: $S_{\theta_\star}$

Fix coefficients at $\theta_\star$

$$S_{\theta_\star} := \{s \in \mathbb{R}^M : \underset{M\times 1}{s} = \mathbb{E}[w(y^\star - x^\star\beta_\star - z\gamma_\star)] \\ , y^\star \in [\underline{y}, \overline{y}], x^\star \in [\underline{x}, \overline{x}]\}$$

Auxiliary dual set is convex (verify)

$$\theta_\star \in \Theta_I \text{ iff } 0 \in S_{\theta_\star}$$

Compare: from nonconvex Θ_I to convex $S_{\theta_\star}$

$$\Theta_I := \{\theta \in \mathbb{R}^{1+L} : \underset{M\times 1}{0} = \mathbb{E}[w(y^\star - x^\star\beta - z\gamma)] \\ , y^\star \in [\underline{y}, \overline{y}], x^\star \in [\underline{x}, \overline{x}]\}$$

Auxiliary Dual Set as a Support Function: $q \mapsto \delta_{\theta_\star}(q)$

Write convex set as a function

$$\delta_{\theta_\star}(q) := \sup_{s \in S_{\theta_\star}} q^\top s, \text{ for } q \in \mathbb{S}\text{=sphere in } \mathbb{R}^M$$

draw support function

Characterization as infinite moment inequalities (BMM):

$$\theta_\star \in \Theta_I \text{ iff } 0 \leq \delta_{\theta_\star}(q) \text{ for each } q \in \mathbb{S}$$

Characterization using midpoints and half-lenghts (new)

$$\delta_{\theta_\star}(q) = \mathbb{E}[q^\top w(y_c - x_c\beta_\star - z\gamma_\star)] + \mathbb{E}[|q^\top w\Delta_y|] + \mathbb{E}[|q^\top w\Delta_x\beta_\star|]$$

where y_c, x_c midpoints; Δ_y, Δ_x half-lengths

Subvector Profiling

Only β is relevant

$$\beta_\star \in \Theta_I \text{ iff } 0 \leq \delta_{\beta_\star,0}(q) \text{ for each } q \in \mathbb{S} \text{ such that } \mathbb{E}(zw^\top q) = \underset{L\times 1}{0}$$

Characterization as constrained optimization (new):

$$\boxed{\beta_\star \in \Theta_I \text{ iff } 0 \leq \min_{q\in\mathbb{S}} \delta_{\beta_\star,0}(q) \quad \text{s.t. } \mathbb{E}(zw^\top q) = \underset{L\times 1}{0}}$$

Compare BMM infinite moment inequalities:

$$(\beta_\star, \gamma_\star) \in \Theta_I \text{ iff } 0 \leq \delta_{\beta_\star,\gamma_\star}(q) = \sup_{s\in S_{\theta_\star}} q^\top s \text{ for each } q \in \mathbb{S}$$

- Setup
- Identified set: nonconvex
- Auxiliary set: convex
- Support function: all coeff
- Subvector profiling: one coeff

PART II: ESTIMATION

- Estimation problem
- Asymptotic distribution: nonpivotal
- Discrete controls and regularization
- Regularized asymptotic distribution: pivotal
- Estimation method
- Monte Carlo

Estimation Problem

Interval estimator from inverting test statistic

Alternative 1: θ in grid $\mathbb{R}^{1+L}$ such that $\hat{\delta}_{\theta}(\hat{q}_n) \geq cv_{\theta}$

Alternative 2: β in grid $\mathbb{R}^{1}$ such that $\hat{\delta}_{\beta,0}(\hat{q}_n) \geq cv_{\beta}$

Both *cv*s computationally expensive, even second grid line

Challenge #2: construct test statistic with pivotal distribution

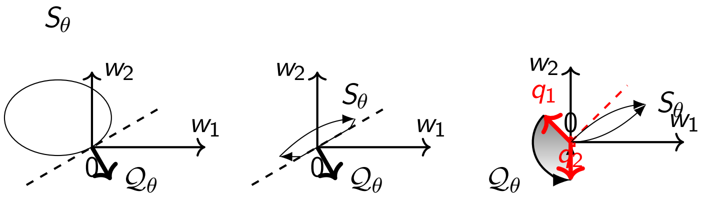
Attack: minimizing directions are binding moments

Asymptotic Distribution

Sample analog

$$\sqrt{n} \min_{q \in \mathbb{S}} \hat{\delta}_\theta(q) \rightsquigarrow \mathbb{L}_\theta := \min_{q \in \mathcal{Q}_\theta} \mathbb{G}(q) \text{ when } 0 \in \partial S_\theta$$

where $\mathcal{Q}_\theta = \arg\min_{q \in \mathbb{S}} \delta_\theta(q)$= set pop. minimizing directions



(a) $\mathcal{Q}_\theta$ when 0 is smooth

(b) $\mathcal{Q}_\theta$ when 0 is exposed

(c) $\mathcal{Q}_\theta$ when 0 is kink

Takeaway: indexing binding moments =nonpivotal

Discrete Controls and Regularization

Why nonpivotal?
kink at zero = multiple minimizing directions
= infinitely many binding moments

Discrete instruments responsible for kinks

Fix unique minimizing direction $q_\star$ by regularization

$$q_\star = \lim_{\kappa \to 0} \arg\min_{q \in \mathbb{S}} \delta_\theta(q) + \kappa pen(q) \text{ such that } q_\star \in \mathcal{Q}_\star$$

$q \to pen(k)$ makes seq. strictily convex programms

Regularized Asymptotic Distribution

Let $\hat{q}_{n,\kappa}$ denote a penalized sample minimizing direction

$$\hat{q}_{n,\kappa} := \arg\min_{q \in \mathbb{S}} \hat{\delta}_\theta(q) + \kappa q^\top \hat{s}_c, \text{ where } \hat{s}_c = \mathbb{E}_n[w(y_c - x_c\beta - z\gamma)]$$

If rate condition on κ hold, then (new)

$$\sqrt{n}\hat{\delta}_\theta(\hat{q}_{n,\kappa}) \rightsquigarrow \mathcal{N}\Big(0, avar(q_\star)\Big) \text{ when } 0 \in \partial S_\theta$$

Takeaway: pivotal after standardization

Estimation One Coeff

Penalized sample direction

$$\hat{q}_{n,\kappa} := \arg\min_{q} \hat{\delta}_{\beta,0}(q) + \kappa q^\top \hat{s}_c \text{ s.t. } \mathbb{E}_n(z_i w_i^\top q) = 0$$

Test statistic:

$$T_{n,\kappa}(\beta) = \sqrt{n}\frac{\hat{\delta}_n(\hat{q}_{n,\kappa})}{\widehat{avar}(\hat{q}_{n,\kappa})}$$

Test inversion:

$$\beta \text{ in grid } \mathbb{R}^1 \text{ such that } T_{n,\kappa}(\beta) \geq cv_\beta = \text{normal quantile}$$

- computationally "cheaper": cv same for every β
- penalized vs. unpenalized minimand
- ("uniform") asymptotic validity

Monte Carlo: Races and Horses

Races: three DGPs, only outcome is censored

- Exogenous: binary exogenous covariate (kinks and flats)
- Endogenous: binary instrumental variable (kinks and flats)
- Mixture: discrete-continuous exo. covariate (kinks and smooth)

Horses: four estimators

- penalized sample direction (our test)
- naive sample analog
- subsampling
- generalized moment selection (gms)

Monte Carlo: Selected Results

Percentage of Rejections ($n = 1,000$) - full parameter inference

Experiment	Test	Interior		Frontier	Exterior		Time
		.9	.99	1	1.01	1.1	(100th of s)
	Our Test $\kappa = 10$	0.00	3.80	5.80	7.30	62.50	11.39
	Our Test $\kappa = 1$	0.00	4.50	6.50	8.90	65.40	11.39
	Our Test $\kappa = 0.1$	0.10	6.70	9.60	12.20	67.70	11.39
Exogeneous	Naive	0.10	6.80	9.60	12.30	67.80	11.34
	Subs	0.00	0.00	0.00	0.00	0.30	92.53
	GMS	0.00	1.50	2.10	3.50	39.70	203.44
	Our Test $\kappa = 10$	0.00	3.70	5.40	7.30	59.80	11.35
	Our Test $\kappa = 1$	0.10	4.00	5.50	7.90	61.20	11.35
	Our Test $\kappa = 0.1$	0.10	5.50	7.60	10.30	63.20	11.35
Endogeneous	Naive	0.10	5.50	7.60	10.40	63.20	11.20
	Subs	0.00	0.10	0.00	0.00	0.60	115.67
	GMS	0.00	1.40	2.20	3.10	39.00	202.25
	Our Test $\kappa = 10$	0.10	6.20	8.90	10.70	67.80	76.19
	Our Test $\kappa = 1$	0.80	9.70	12.10	15.70	71.60	76.19
	Our Test $\kappa = 0.1$	0.80	9.40	12.00	14.70	71.10	76.19
Mixture	Naive	0.80	9.40	12.00	14.70	71.10	79.95
	Subs	0.00	0.10	0.10	0.10	2.90	264.26
	GMS	0.00	1.80	2.50	3.50	39.80	247.75

The frontier point is a kink point

- Setup
- Identified set: nonconvex
- Auxiliary set: convex
- Support function: all coeff
- Subvector profiling: one coeff

- Estimation problem
- Asympotitic distribution: nonpivotal
- Discrete controls (kinks) and regularization
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PART III: ILLUSTRATION

- Panel Linear Moment Model
- Wealth and Assets Survey
- Empirical Results

Panel Linear Moment Model

$$\mathbb{E}[W_i^\top H(Y_i^\star - X_i^\star \beta - Z_i^\top \gamma - 1_T \alpha_i)] = 0$$

Coeff of interest is β (if $=1$ constant relative risk aversion)

- $\underset{T\times 1}{Y_i^\star}$ log risky asset holdings household i
- $\underset{T\times 1}{X_i^\star}$ log financial wealth
- $\underset{(T-1)\times T}{H}$ such that $H 1_T = 0$ and upper-triangular
- $\underset{M\times(T-1)}{W_i}$ lower, upper wealth, age, age-sq, hou'ld size, time dummies

Same across populations? Italy=constant, Sweden=decreasing, UK=?

Wealth and Assets Survey

Biennial panel survey on GB households: 2006/07 to 2014/15

Financial asset holdings (26 cat.) measured by intervals (some exact)

Aggregate into

- risky asset holdings (add 5 cat.)
- financial wealth (+ 21 other cat.)

Table 1:Sample Descriptive Statistics

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Table 1: Sample Descriptive Statistics

Wave	average $risky_{Lit}$	$risky_{Uit}$	% Censored	average $wealth_{Lit}$	$wealth_{Uit}$	% Censored
2006/07	9.90	10.02	16.18%	11.65	11.76	32.36%
2008/09	9.68	9.88	22.88%	11.63	11.75	31.92%
2010/11	9.84	9.94	12.09%	11.76	11.81	19.82%
2012/13	9.95	10.03	10.03%	11.98	11.96	17.63%
2014/15	10.06	10.17	10.05%	12.01	12.09	18.80%

Source: UK Wealth and Assets Survey Waves 1 to 5. The sample consists only of households reporting positive risky assets holdings during the five waves. Risky asset holdings and financial wealth are in log sterling pound scale. Number of households $n = 686$.

Results: First Exercise

Table 2: Alternative Estimates for the Financial Wealth Elastiticity of Household Risky Assets Demand

	(1) POLS	(2) POLS	(3) FD imp.	(4) FD uncen.	(6) FD	(7) FO
wealth					[0.78,1.24]	[0.73,1.11]
age	✗	✓	✓	✓	✓	✓
$age^2/100$	✗	✓	✓	✓	✓	✓
household size	✗	✓	✓	✓	✓	✓
year 07/08	✗	✓	✓	✓	✓	✓
year 09/10	✗	✓	✓	✓	✓	✓
year 11/12	✗	✓	✓	✓	✓	✓
year 13/14	✗	✓	✓	✓	✓	✓
Constant	✓	✓	✗	✗	✗	✗

Note: Same specification as in Chiappori and Paiella (2011, Relative Risk Aversion is Constant: Evidence from Panel Data, JEEA)

Results: Fragility Checks

Table 2: Alternative Estimates for the Financial Wealth Elastiticity of Household Risky Assets Demand

	(1) POLS	(2) POLS	(3) FD imp.	(4) FD uncen.	(6) FD	(7) FO
wealth	1.14	1.15	.99	.86		
	[1.10,1.18]	[1.11,1.20]	[.90,1.07]	[.75,.97]	[0.78,1.24]	[0.73,1.11]
age	✗	✓	✓	✓	✓	✓
$age^2/100$	✗	✓	✓	✓	✓	✓
ho'hold size	✗	✓	✓	✓	✓	✓
year 07/08	✗	✓	✓	✓	✓	✓
year 09/10	✗	✓	✓	✓	✓	✓
year 11/12	✗	✓	✓	✓	✓	✓
year 13/14	✗	✓	✓	✓	✓	✓
Constant	✓	✓	✗	✗	✗	✗

CONCLUSIONS

Motivation: interval censored outcome and covariate

Question (how to): estimate one coefficient in linear moment

- point estimators are invalid
- valid interval estimator by inverting m-type test statistic
- tractable computation by leveraging linearity

Illustration: UK Wealth and Assets Panel Data