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On the Pettis Integral Approach to Large Population Games

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A large Bayesian game with linear best responses

A Bayesian game with linear best responses (linear in opponents' actions and payoff states) has numerous applications

A continuum of agents is a useful assumption to study the role of information in a large economy

E.g., Bergemann and Morris (2013) build a framework for information design based on Angeltos and Pavan (2007), henceforth AP&BM

A large Bayesian game with linear best responses

The aggregated action is the integration of a continuum of random variables (a stochastic process), often assumed to be i.i.d.

It is implicitly assumed (without mathematical justification)

- an i.i.d. non-degenerate stochastic process is measurable
- the law of large numbers (LLN) holds

but mathematically incompatible (Judd, 1985; Uhlig, 1996)

Rigorous mathematical foundations are required for further analysis of more general models

Summary

This paper introduces a general class of large Bayesian games with linear best responses that have a solid mathematical foundation

- The model is formulated based on the Pettis integral framework (Pettis, 1938; Uhlig, 1996).
- Conditions for the uniqueness of equilibrium are derived
- Information design is addressed based on the condition.
- An optimal information structure is characterized in AP&BM

A large Bayesian game with linear best responses

$I = [0, 1]$: the set of agents (with the Lebesgue measure ν)

$\theta(i)$: agent i's payoff state, a random variable with a finite variance

$x(i)$: i's private information, an arbitrary random variable

$\{x(i)\}_{i \in I}$: an information structure

$f(i)$: i's strategy, an $x(i)$-measurable random variable with a finite variance

f: a strategy profile

A large Bayesian game with linear best responses

Agent i's best response is linear in the conditional expectation of

$$\{f(j)\}_{j \neq i} \text{ and } \theta(i)$$

Let $R(i, j) \in \mathbb{R}$ be the weight on $f(j)$

The best response strategy is

$$f(i) = \mathbf{E}\left[\mathbf{R}f(i) + \theta(i) \mid x(i)\right]$$

$\mathbf{R}$ is the integral operator

$$f(i) \mapsto \mathbf{R}f(i) = \int R(i, j) f(j) \mathrm{d}\nu(j)$$

$\langle \{\theta(i), x(i)\}_{i \in I}, R \rangle$: a Bayesian game

We call $R : I \times I \to \mathbb{R}$ a payoff structure

What assumption do we need to ensure equilibria are well-defined?

The model of AP&BM

$R(i, j) = r < 1$ for all $i, j \in I$ (uniform payoff structure)

$\theta(i) = \theta$ for all $i \in I$ (common payoff state), θ is normally distributed

$$x(i) = \begin{pmatrix} \theta + \varepsilon_i \\ \theta + \varepsilon_0 \end{pmatrix} \begin{matrix} \text{a private signal} \\ \text{a public signal} \end{matrix}$$

The idiosyncratic errors $\{\varepsilon_i\}_{i \in I}$ constitute an i.i.d. Gaussian process

The unique equilibrium strategy $f(i)$ is linear in signals

LLN

AP&BM rely on

$$\int \varepsilon_i \mathrm{d}\nu(i) = 0$$

- an i.i.d. non-degenerate stochastic process is measurable
- the law of large numbers (LLN) holds

mathematically incompatible (Judd, 1985; Uhlig, 1996)

This problem is resolved by the Pettis integral (Uhlig, 1996)

We formulate $\langle \{\theta(i), x(i)\}_{i \in I}, R \rangle$ adopting the Pettis integral

A stochastic process

We regard a strategy profile f as a process

$$f : I \to X$$

taking values in the set of all random variables with finite second moments over a fixed probability space

The set of random variables X is endowed with

$$\langle x, y \rangle = \mathrm{E}\left[xy\right] \text{ for } x, y \in X$$

The Pettis integral

A process $f : I \to X$ is Pettis integrable if there exists $\bar{f} \in X$ with

$$\langle x, \bar{f} \rangle = \int \langle x, f(i) \rangle \, \mathrm{d}\nu(i) \quad \forall x \in X$$

$\bar{f} \in X$ is called the Pettis integral of f

We denote the Pettis integral of f (with abuse of notation) by

$$\int f(i) \mathrm{d}\nu(i)$$

The Pettis integral

A process $f : I \to X$ is regular if

(Q1) The map $(i, j) \mapsto \langle f(i), f(j) \rangle$ is measurable

(Q2) The map $i \mapsto ||f(i)||^2 = \langle f(i), f(i) \rangle$ is integrable

Lemma

A regular process $f : I \to X$ is Pettis integrable.

A weaker condition suffices for Pettis integrability

But we use (Q1) and (Q2) to discuss the uniqueness of equilibrium

The equilibrium strategy in AP&BM satisfies (Q1) and (Q2)

The Pettis integral

Proposition

If f is regular,

$$\mathbf{Var}\left[\int f(i)\mathrm{d}\nu(i)\right] = \int\int \mathbf{Cov}\left[f(i), f(j)\right]\mathrm{d}\nu(i)\mathrm{d}\nu(j).$$

If $f(i)$ and $f(j)$ are uncorrelated for all $i \neq j$,

$$\mathbf{Var}\left[\int f(i)\mathrm{d}\nu(i)\right] = 0, \text{ i.e., } \int f(i)\mathrm{d}\nu(i) = \text{const. a.s.}$$

The latter claim is the LLN of Uhlig (1996)

The Pettis integral

The variance of the average action is referred to as volatility in AP

$$\mathbf{Var}\left[\int f(i)\mathrm{d}\nu(i)\right]$$

Assuming the LLM, BM show that

$$\mathbf{Var}\left[\int f(i)\mathrm{d}\nu(i)\right] = \mathbf{Cov}\left[f(i), f(j)\right] \text{ for } i \neq j$$

which is consistent with our result

$$\mathbf{Var}\left[\int f(i)\mathrm{d}\nu(i)\right] = \int\int \mathbf{Cov}\left[f(i), f(j)\right]\mathrm{d}\nu(i)\mathrm{d}\nu(j)$$

The Pettis integral framework for $\langle\{\theta(t), x(t)\}_{t\in T}, R\rangle$

$\langle\{\theta(i), x(i)\}_{i\in I}, R\rangle$ is assumed to satisfy

- $\theta : I \to X$ is regular
- $R : I \times I \to \mathbb{R}$ is measurable and bounded

A strategy profile $f : I \to X$ is a regular process such that $f(i) \in X$ is $x(i)$-measurable for every $i \in I$

A strategy profile f is an equilibrium if

$$f(i) = \mathrm{E}\left[\mathrm{R}f(i) + \theta(i) \mid x(i)\right] \quad \forall i \in I$$

where $\mathrm{R}f(i)$ is the Pettis integral

$$\mathrm{R}f(i) = \int R(i, j)f(j)\mathrm{d}\nu(j)$$

The Pettis integral framework for $\langle\{\theta(t), x(t)\}_{t\in T}, R\rangle$

Lemma

AP&BM can be reformulated in accordance with the Pettis integral framework, where every result remains the same.

Uniqueness of equilibrium

$\mathcal{L}_2$: the Hilbert space of all square-integrable functions on I

$$\langle \phi, \psi \rangle_{\mathcal{L}_2} \equiv \int \phi(i)\psi(i)\mathrm{d}\nu(i) \quad \forall \phi, \psi \in \mathcal{L}_2$$

Define the integral operator $\mathbf{R} : \mathcal{L}_2 \to \mathcal{L}_2$ using $R : I \times I \to \mathbb{R}$

$$\phi(i) \mapsto \mathbf{R}\phi(i) = \int R(i, j)\phi(j)\mathrm{d}\nu(j)$$

The restriction of $\mathbf{R}$ on the set of the Pettis integrable processes to $\mathcal{L}_2$

$$f(i) \mapsto \mathbf{R}f(i) = \int R(i, j)f(j)\mathrm{d}\nu(j)$$

We use the same notation for simplicity

Uniqueness of equilibrium

The numerical range of $\mathbf{R}$

$$W(\mathbf{R}) = \{\langle \mathbf{R}\phi, \phi \rangle_{\mathcal{L}_2} \mid \phi \in H, \ \|\phi\|_{\mathcal{L}_2} = 1\} \subset (-\infty, 1)$$

is the set of all possible values that $\langle \mathbf{R}\phi, \phi \rangle_{\mathcal{L}_2}$ can take when ϕ is any unit vector in $\mathcal{L}_2$

(R1) The maximum of the numerical range of $\mathbf{R}$ is less than 1

(R2) The maximum real eigenvalues of $\mathbf{R}$ is less than 1

Lemma

(R1) implies (R2).

(R1) and (R2) are equivalent if $R(i,j) = R(j,i)$ for all $i, j \in I$.

Uniqueness of equilibrium

(R1) is sufficient, while (R2) is necessary for the uniqueness

If $R(i, j) = R(j, i)$ for all $i, j \in I$, (R1) is both necessary and sufficient since (R1) and (R2) are equivalent

Proposition

If R satisfies (R1) and the game has an equilibrium, this is a unique equilibrium in the following sense:

> If f and g are two equilibria such that $(i, j) \mapsto \mathbf{E}\left[f(i)g(j)\right]$ is measurable, then $f(i) = g(i)$ a.s. for all $i \in I$.

If R does not satisfy (R2), there exists $\{x(i)\}_{i \in I}$ such that the game has either no equilibria or a continuum of equilibria.

Uniqueness of equilibrium

Suppose that $R(i, j) = r$. The uniqueness holds if and only if $r < 1$

Corollary

Suppose that $R(i, j) = r$ for all $i, j \in I$. If $r < 1$ and the game has an equilbrium, this is a unique equilibrium.

If $r \geq 1$, there exists $\{x(i)\}_{i \in I}$ such that the game has either no equilibria or a continuum of equilibria.

AP&BM show the uniqueness of symmetric equilibrium (invariant under permutations of agents)

By the above corollary, there are no asymmetric equilibria

Information design

We address information design, where an information structure is endogenously determined by an information designer

Assume AP&BM, but θ can follow an arbitrary distribution

The objective function is quadratic in strategies and a state, where agents' strategies are treated equally (invariant under permutations)

An information structure is optimal if it maximizes the objective when the agents follow the unique equilibrium

We have characterized optimal information structures

An application to large Cournot games

Firm $i \in I$ produces a_i units of a homogeneous product

The inverse demand function

$$\theta - \gamma \int_0^1 a_j \mathrm{d}\nu(j)$$

θ is a random intercept; $\gamma > 0$ is a slope

Firm i's cost

$$\frac{{a_i}^2}{2}$$

The objective is the weighted average of PS and CS

$$\lambda \cdot \mathrm{PS} + (1 - \lambda) \cdot \mathrm{CS}$$

Optimal information structures

Proposition

It is optimal to inform all firms of θ (full disclosure) if and only if

$$\text{the slope } \gamma \leq \frac{4}{\text{the weight of PS } \lambda} - 3.$$

Otherwise, it is optimal to inform a certain fraction of the firms of θ and to inform the other firms of nothing at all (partial disclosure).

The fraction is $\dfrac{\lambda}{\lambda\gamma - 4(1-\lambda)} > 0$, which is decreasing in λ and γ

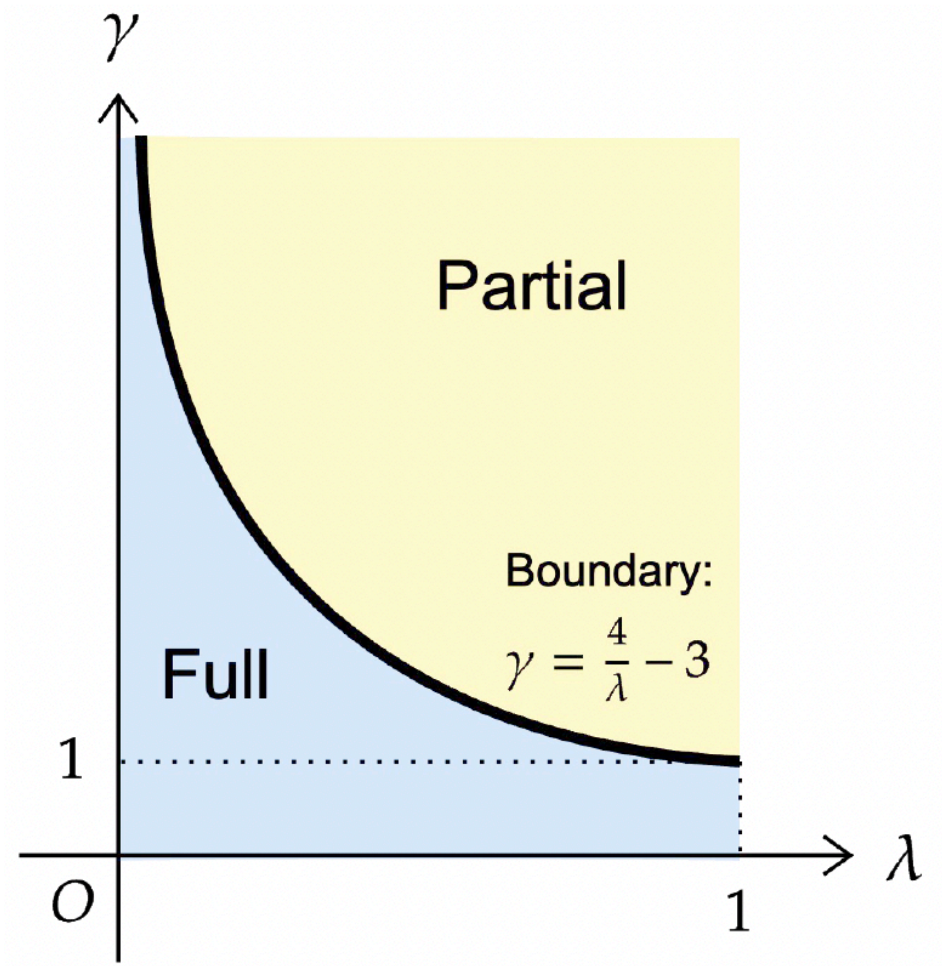
Such an information structure is referred to as targeted disclosure

Optimal information structures

BM (2013) identify an optimal information structure that maximizes PS, assuming θ is normally distributed (can take negative values)

Our result allows for any weighted sum of PS and CS and any random variable θ (can be restricted to take positive values)

Optimal information structures



Takeaway

More generally, there exists targeted disclosure that is optimal among all information structures in our model

This is based on

- the Pettis integral framework
- the condition for uniqueness

Thank you very much!