Nash Bargaining with Coalitional Threats

Rajiv Vohra ⓡ Debraj Ray

Outline

- Nash meets vN-M: games with coalitional threats
- Endogenizing coalitional threats
- TU games and constrained egalitarianism

Nash (1950)

- 2-player "game" (F, d)
  - $F \subseteq \mathbb{R}^2$ feasible payoffs
  - $d \in F$ disagreement point: payoffs if no agreement
- **Domain restriction:**
  - F nonempty compact convex with $x \gg d$ for some $x \in F$

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- **Solution**:
  - $\Psi(F, d) \in F$ for every (F, d)

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- **Axioms on Solution:**
  - Pareto Optimality
  - Symmetry: $[d_1 = d_2]$ + [symmetric F] $\Rightarrow$ $[\Psi_1(F,d) = \Psi_2(F,d)]$
  - Invariance: Affine payoff transforms generate the same solution transform
  - IIA: $[F' \subseteq F]$ + $[\Psi(F,d) \in F']$ $\Rightarrow$ $[\Psi(F',d) = \Psi(F,d)]$

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Theorem (Nash)

Ψ *satisfies these four axioms if and only if for all* (F, d)*:*

$$\Psi(F, d) = \arg\max_{x \in F, x \geq d} (x_1 - d_1)(x_2 - d_2)$$

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- $F \subseteq \mathbb{R}^n$ feasible payoffs;
- $d \in \mathbb{R}^n$ disagreement point: payoffs if no agreement at all;
- $\boldsymbol{\Theta} = \{(\Theta(S))_{S \subset N}\}$, where:

  $\Theta(S) \subset \mathbb{R}^S$ are sets of threats for each subcoalition S

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- **Remarks:**

1. Domain allows for nonconvex feasible sets.
2. S is ineffective if $\theta(S) = \{d_S\}$, and effective otherwise.
3. It's possible that $\zeta_i \equiv \max \Theta(\{i\}) > d_i$.

Solution

- $x \in F$ is blocked by threat (S, y) if $y \in \Theta(S)$ and $y \gg x_S$.

- Unblocked set:

$$U(G) \equiv \{x \in F | x \text{ is not blocked by any threat } (S, y)\}.$$

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- Remark:
- Allow solution to be multi-valued:

Coalitional threats $\Rightarrow$ nonconvexities.

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If, for some permutation π of N, $y(\pi) \equiv (y_{\pi(1)}, \ldots, y_{\pi(n)}) \in F$ for any $y \in F$,

then $x(\pi) \equiv (x_{\pi(1)}, \ldots, x_{\pi(n)}) \in \sigma(G)$ for any $x \in \sigma(G)$.

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[**UHC**] If $G^k = (F^k, \boldsymbol{\Theta}^k, d^k)$ converges in the (product) Hausdorff metric to $G = (F, \boldsymbol{\Theta}, d)$, and $x^k \in \sigma(G^k)$ for all k with $x^k \to x$, then $x \in \sigma(G)$.

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- Consider G such that $F = \{x, y\}$, $d = 0$ and all coalitions ineffective.
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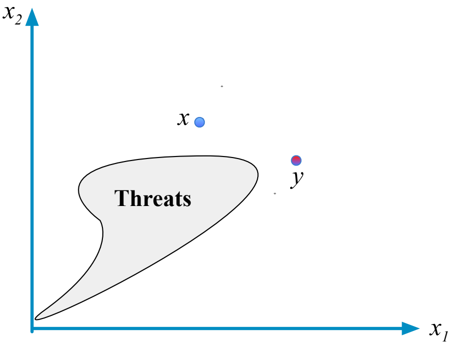

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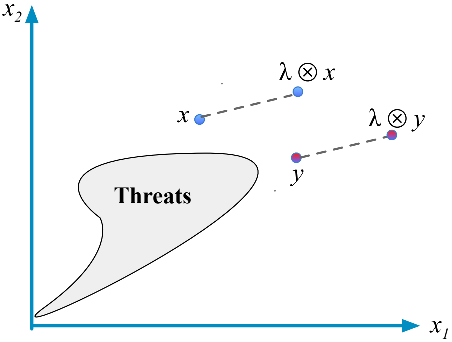

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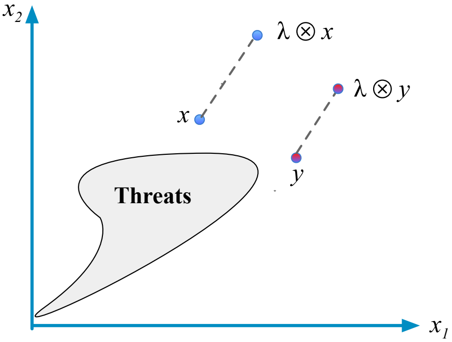

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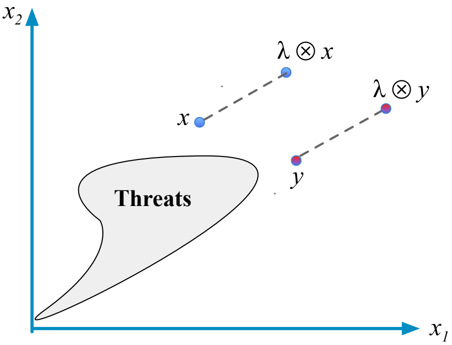

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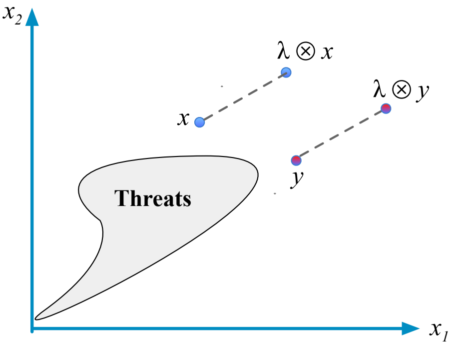

$[x \in \sigma(\{x, y\}, \mathbf{\Theta}, 0)] \Rightarrow [\lambda \otimes x \in \sigma(\lambda \otimes \{x, y\}, \mathbf{\Theta}, 0)$ for some $\lambda \gg 1]$.

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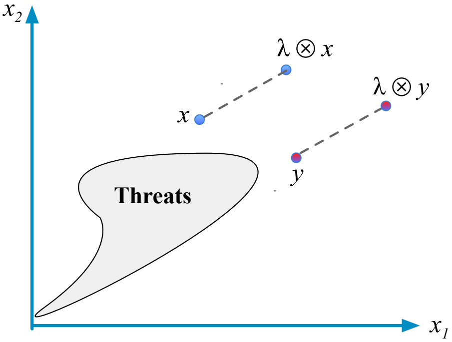

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  - It's not needed when all coalitions are ineffective or if x is the only solution.

Characterization

Theorem 1 (Coalitional Nash Solution)

A solution $\sigma(G)$ satisfies axioms [Par], [Inv], [Sym], [IIA], [UHC] *and* [Exp] *for every game G if and only if*

$$\sigma(G) = \arg\max_{x \in U(G)} \prod_{j \in N} [x_j - d_j].$$

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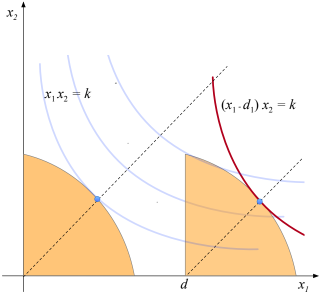
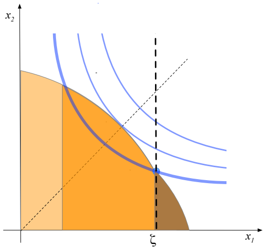
- An asymmetry:

  - The solution subtracts disagreement points, but no coalitional threat.
  - The latter appear as “conventional” constraints.

Asymmetry

- The asymmetry is particularly stark across d_i and $\zeta_i \equiv \max \Theta(\{i\})$:





$\arg\max_{\mathbf{x} \in F} \{(x_1 - d_1)(x_2 - d_2) | x \geq d\}$ if $d = \zeta$

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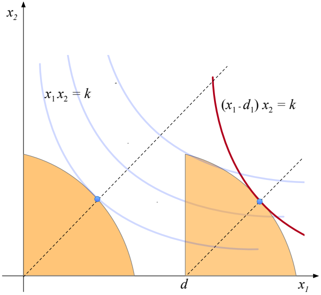

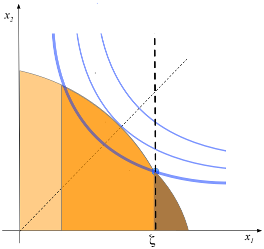

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- Do you think that $d_i = \zeta_i$ by definition? If so, subtract ζ_i, but no other coalitional worth is treated the same way.

Noncooperative vs Axiomatic Bargaining

- This schizophrenia often comes up in noncooperative bargaining models

  E.g. Binmore-Shaked-Sutton (1989), Chatterjee-Dutta-Ray-Sengupta (1993), Compte-Jehiel (2010).

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  "The attraction of split-the-difference lies in the fact that a larger outside option seems to confer greater bargaining power. But how can a bargainer use his outside option to gain leverage? By threatening to play the deal-me-out card. When is such a threat credible? Only when dealing himself out gives the bargainer a bigger payoff than dealing himself in. It follows that the agreement that would be reached without outside options is immune to deal-me-out threats, unless the deal assigns one of the bargainers less than he can get elsewhere" [emphasis ours].

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- We obtain the same solution axiomatically.

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- Define the solution

$$\phi(G) = \arg\max_{x \in U(G)} \prod_{j \in N} [x_j - a_j].$$

  - ϕ satisfies [Par], [Inv], [Sym], [IIA] and [UHC], but not [Exp].

Details for Example 1

- $N = \{1, 2\}$ with $d = 0$:
  - $\zeta_1 = 1$ and $\zeta_2 = 0$, so $a = (1, 0)$.
  - $F = \{y, z\}$ where $y = (2, 2)$ and $z = (3, 1)$. Both unblocked.

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- No expansion $\lambda \gg 1$ can maintain this indifference:
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- By Theorem 1, [Exp] eliminates not just this objective function but everything else except our coalitional Nash solution.
  - Can construct more complex violations even if we insist on $d = \zeta$.

Internally Consistent Threats

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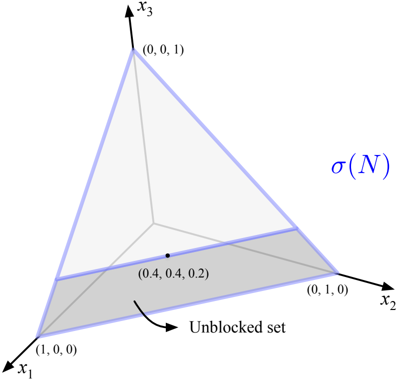

$\sigma(N) = (0.4, 0.4, 0.2)$

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  - One approach: impose the same coalitional Nash procedure on subcoalitions.
  - Then only "credible block" is $(0.4, 0.4)$, and so:
  - $U(N) = \{x \in F \mid x \geq 0$ and $\max\{x_1, x_2\} \geq 0.4\}$.

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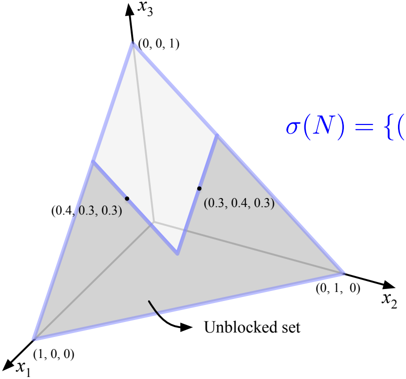

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Internally Consistent Coalitional Nash Solution

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3. If $U^*(S)$ is empty, set $\Theta^*(S) = \{d_S\} = \{0_S\}$ and $\sigma^*(S) = \emptyset$.

- If $U^*(S)$ is nonempty, it is compact: Set $\sigma^*(S) = \Theta^*(S) = \arg\max_{x \in U^*(S)} \prod_{j \in N} x_j$.

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$$U^*(S) \equiv \{x \in F(S) | x \text{ is unblocked by any } (T, y) \text{ with } T \subset S \text{ and } y \in \Theta^*(T)\}.$$

3. If $U^*(S)$ is empty, set $\Theta^*(S) = \{d_S\} = \{0_S\}$ and $\sigma^*(S) = \emptyset$.

- If $U^*(S)$ is nonempty, it is compact: Set $\sigma^*(S) = \Theta^*(S) = \arg\max_{x \in U^*(S)} \prod_{j \in N} x_j$.

4. Keep going: up to N.

Internally Consistent Coalitional Nash Solution

- **Solution:** $$\sigma^*(N) = \Theta^*(N) = \underset{x \in U^*(N)}{\arg\max} \prod_{j \in N} x_j.$$
- Unwieldy, because of the recursion.

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   - Super simple (especially if Ψ is a singleton), but also inconsistent.

Naïve Blocking

Example 3: 3-player game.

- $d = 0$, and $F(S) = \{0_S\}$ for all S except:
- $F(\{i\}) = \{\zeta_1\} = \{1.1\}$
- $F(\{1, 2\}) = \{(1, 1), (1.2, 0.8)\}$ – not convex
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In this example naive blocking does not give the same solution as the recursive solution.

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Naïve Blocking

- It's not surprising that the naive solution doesn't work.
  - But if $F(S)$ is convex for every S, it does!

Theorem 2

Assume $F(S)$ nonempty compact and convex for every coalition S. Then $U^*(S) = U^{\text{naïve}}(S)$, and so the internally consistent Nash solution need only guard against the threats posed by the unconstrained Nash solutions of its subcoalitions:

$$\sigma^*(S) = \arg\max_{x \in U^*(S)} \prod_{j \in S} x_j = \arg\max_{x \in U^{\text{naïve}}(S)} \prod_{j \in S} x_j.$$

TU Games

- TU game: There is an affine transform of payoffs such that $d = 0$ and such that for each S, there is $v(S)$ with

$$F(S) = \{x \in \mathbb{R}^S \mid \sum_{i \in S} v(\{i\}) \leq \sum_{i \in S} x_i \leq v(S)\}.$$

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- Consider x, y in $\mathbb{R}^k_+$ with $x_i \leq x_{i+1}$ and $y_i \leq y_{i+1}$ for all $i = 1, \ldots, k-1$.
- x majorizes y if $x \neq y$ and $\sum_{i=1}^j x_i \geq \sum_{i=1}^j y_i$ for every $j = 1, \ldots, k$.
- For $A \subset \mathbb{R}^k$, let $L(A)$ be its set of unmajorized or Lorenz-maximal elements.

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  - For $A \subset \mathbb{R}^k$, let $L(A)$ be its set of unmajorized or Lorenz-maximal elements.
- Dutta and Ray (1989, 1991) use these ideas to define a solution that respects egalitarianism as well as coalitional threats.

Constrained Egalitarianism in TU Games

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3. Set $E(S) = L(U^e(S))$.

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In a TU game, the internally consistent Nash solution is a subset of the constrained egalitarian solution for every coalition S*:*

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Suppose additionally that a TU game is superadditive. Then for all S, $\sigma^*(S)$ is nonempty, and is found by maximizing the Nash product over the set of allocations that are unblocked by any subcoalition using equal division.

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- Axiomatization of Nash bargaining with coalitional threats.
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- Axiomatization of Nash bargaining with coalitional threats.
  - Solution maxes Nash product net of disagreement payoffs
  - But treats all coalitional threats as “conventional” constraints.
- We endogenize coalitional threats:
  - Recursively applying the coalitional Nash solution to every subcoalition.
  - Simple characterization for games with convex payoff sets:

    max Nash product over allocations unblocked by unconstrained Nash product.