Oligopoly, Complementarities, and Transformed Potentials

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Introduction

- Multiproduct firms selling horizontally differentiated products are ubiquitous in real-world markets
- They feature prominently in
  - the structural IO literature (Berry, Levinsohn, and Pakes, 1995; Nevo, 2000; 2001; Miller and Weinberg, 2017)
  - the international trade literature (Bernard, Redding, and Schott, 2010; 2011; Mayer, Melitz, and Ottaviano, 2014; Hottman, Redding, and Weinstein, 2016)

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- We use a potential games approach to establish equilibrium existence in a large class of multiproduct-firm pricing games
- Also characterize class of demand systems such that multiproduct-firm pricing games admit a potential
- Class of demand system allows for (price-dependent) patterns of complementarities, which have recently received much attention in
  - the structural IO literature (Gentzkow, 2007)
  - the theoretical IO literature (Rey and Tirole, 2019)

Multiproduct Oligopoly Pricing with IIA Demand

Demand side:

- Finite set of differentiated products $\mathcal{N}$.
- Representative consumer has quasi-linear indirect utility

$$y + V(p) = y + \Psi\Big(\sum_{j \in \mathcal{N}} h_j(p_j)\Big)$$

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- Roy's identity gives us the demand for product i:

$$D_i(p) = -h_i'(p_i)\Psi'\Big(\sum_j h_j(p_j)\Big)$$

- The demand system has the IIA property as $D_i(p)/D_j(p) = h_i'(p_i)/h_j'(p_j)$ is independent of p_k for $k \neq i, j$

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- Examples:
  - If $\Psi(H) = \log(1 + H)$ and $h_j(p_j) = e^{\beta_j - \alpha p_j}$, then demand system is logit
  - If $\Psi(H) = \log H$ and $h_j(p_j) = a_j p_j^{1-\sigma}$, then demand system is CES

Multiproduct Oligopoly Pricing with IIA Demand

In Nocke and Schutz (2018), we show that the demand system can be derived from multistage discrete/continuous choice if and only if:

(i) Each h_i is $\mathcal{C}^1$, strictly positive, strictly decreasing, and log-convex.

(ii) Ψ is $\mathcal{C}^1$, Ψ' is non-negative, and $H \mapsto H\Psi'(H)$ is non-decreasing.

We assume throughout that conditions (i) and (ii) hold.

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Complements vs. substitutes:

$$\frac{\partial D_i}{\partial p_k} = \overbrace{(-h_i')}^{>0}\overbrace{(-h_k')}^{>0}\Psi'' \qquad (k \neq i)$$

- Products are (locally) substitutes if Ψ' is (locally) decreasing, and (locally) complements if Ψ' is (locally) increasing.
- In particular, products may be, e.g., complements when prices are low, but substitutes when prices are high.

Multiproduct Oligopoly Pricing

Supply side:

- $\mathcal{F}$, the set of firms, is a partition of $\mathcal{N}$ (the set of products).
- $|\mathcal{F}| \geq 2$.
- $c_i > 0$: Constant unit cost of product i.

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- $\mathcal{F}$, the set of firms, is a partition of $\mathcal{N}$ (the set of products).
- $|\mathcal{F}| \geq 2$.
- $c_i > 0$: Constant unit cost of product i.
- Firm f's payoff function:

$$\pi^f(p) = \sum_{k \in f} (p_k - c_k)(-h_k'(p_k)) \Psi' \left(\sum_{j \in \mathcal{N}} h_j(p_j) \right) \ \forall p \in (0, \infty]^{\mathcal{N}}$$

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- Infinite prices are allowed: $p_k = \infty \implies$ the firm does not make any profit on product k.
- Firms compete by setting prices simultaneously.

Existence of an Ordinal Potential

The resulting multiproduct-firm pricing game has an ordinal potential (Monderer and Shapley, 1996):

$$W(p) = \Psi' \left(\sum_{j \in \mathcal{N}} h_j(p_j) \right) \prod_{g \in \mathcal{F}} \sum_{k \in g} (p_k - c_k)\,(-h_k'(p_k))$$

$$= \pi^f(p) \times \underbrace{\prod_{g \neq f} \sum_{k \in g} (p_k - c_k)\,(-h_k'(p_k))}_{>0, \text{ independent of } p^f}, \quad \forall f \in \mathcal{F}$$

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- $W(\cdot)$ is indeed an ordinal potential, as

$$\pi^f(p^{f\prime}, p^{-f}) - \pi^f(p^f, p^{-f}) > 0 \iff W(p^{f\prime}, p^{-f}) - W(p^f, p^{-f}) > 0$$

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- The ordinal potential is useful because it captures deviation incentives: Firm f has a strict incentive to deviate from p^f to $p^{f\prime}$ (holding p^{-f} fixed) if and only if the deviation strictly raises the value of the potential function.

Using the Potential Function

- Suppose that p^* solves maximization problem $\max_p W(p)$.
- Then, if firm f unilaterally deviates from that price vector, the value of W decreases, and thus π^f decreases
- This implies that p^* is a pure-strategy Nash equilibrium.

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- We show that W has a global maximizer under very weak restrictions. This implies:

Proposition

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- New method to compute equilibria: Instead of solving
  - multidimensional fixed-point problem (best-response analysis)
  - or a unidimensional fixed-point problem (aggregative-games approach),

  simply maximize W.

Transformed Potentials

Consider the pricing game with logged payoffs:

$$\widetilde{\pi}^f \equiv \log \pi^f = \log \left(\sum_{k \in f} (p_k - c_k)(-h'_k(p_k)) \right) + \log \Psi' \left(\sum_{j \in \mathcal{N}} h_j(p_j) \right).$$

Take the log of the ordinal potential W, and note that, for every f,

$$\widetilde{W} \equiv \log W = \widetilde{\pi}^f + \sum_{g \neq f} \log \left(\sum_{k \in g} (p_k - c_k)(-h'_k(p_k)) \right).$$

Therefore,

$$\widetilde{W}(p^{f\prime}, p^{-f}) - \widetilde{W}(p^f, p^{-f}) = \widetilde{\pi}^f(p^{f\prime}, p^{-f}) - \widetilde{\pi}^f(p^f, p^{-f}), \ \forall f, p^f, p^{f\prime}, p^{-f}.$$

So $\widetilde{W}$ is a potential for the logged pricing game.

- Compared to the weaker concept of ordinal potential, the difference is that the potential function captures deviation incentives in a cardinal way.

Transformed Potentials

The demand system D thus has the following property:

- There exists a transformation function G (here: $G = \log$) such that, for every $\mathcal{F}$ and $(c_j)_{j \in \mathcal{N}}$, the pricing game with payoffs $G \circ \pi^f$ has a potential.

We say that D has a transformed potential, or a G-potential.

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Why not look for demand systems admitting an ordinal potential?

- Monderer and Shapley (1996): A smooth normal-form game $(\mathcal{N}, A, u)$ admits a potential if and only if

$$\frac{\partial^2 u_i}{\partial a_i \partial a_j} = \frac{\partial^2 u_j}{\partial a_i \partial a_j}, \quad \forall i, j. \tag{1}$$

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- There is no known analogue of condition (1) for the weaker concept of ordinal potential (e.g., Ewerhart, 2020).
- By contrast, for a given transformation function G, we can apply condition (1) to $G \circ u$ to determine whether D has a G-potential.

Transformed Potentials

Theorem

Let D be a demand system. The following assertions are equivalent:

(a) *D admits a transformed potential.*

(b) *At least one of the following assertions holds true:*

(i) *The demand system D takes the IIA form*

$$D_i(p) = -h_i'(p_i)\Psi'\big(\sum_{j\in\mathcal{N}} h_j(p_j)\big).$$

(ii) *The demand system D takes the generalized linear form*

$$D_i(p) = -h_i'(p_i) + \sum_{j\neq i} \alpha_{ij} p_j.$$

If assertion (i) (resp., assertion (ii)) holds, then the logarithm (resp., the identity function) is an admissible transformation function for demand system D.

Key steps of the proof

Transformed Potentials

Recall the definition of D having a transformed potential:

- There exists a transformation function G such that, for every $\mathcal{F}$ and $(c_j)_{j \in \mathcal{N}}$, the associated pricing game with payoffs $G \circ \pi^f$ has a potential.

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- There exists a transformation function G such that, for every $\mathcal{F}$ and $(c_j)_{j\in\mathcal{N}}$, the associated pricing game with payoffs $G \circ \pi^f$ has a potential.

We now fix the ownership partition $\mathcal{F}$ and study the weaker concept of D having an $\mathcal{F}$-specific transformed potential:

- There exists a transformation function G such that, for every $c = (c_j)_{j\in\mathcal{N}}$, the pricing game with marginal cost vector c, ownership structure $\mathcal{F}$, and payoffs $G \circ \pi^f$ has a potential.

$\mathcal{F}$-Specific Transformed Potentials

Proposition

Let D be a demand system and $\mathcal{F}$ a firm partition. The following assertions are equivalent:

(a) $(D, \mathcal{F})$ *admits a transformed potential.*

(b) *At least one of the following assertions holds true:*

  (i) *The demand system D satisfies the following properties: For every $f, g \in \mathcal{F}$ with $f \neq g$, $i, j \in f$, and $k \in g$, $\partial_k D_i / D_j = 0$ and $\partial^2_{ik} \log D_i / D_k = 0$.*

  (ii) *The demand system D takes the following form: For any $i \in f \in \mathcal{F}$,*

$$D_i(p) = -\partial_i \psi^f(p^f) + \sum_{\substack{\mathcal{F}' \subseteq \mathcal{F}: \\ f \in \mathcal{F}'}} \sum_{\substack{\iota \in \prod_{g \in \mathcal{F}'} g: \\ \iota(f) = i}} \alpha(\iota) \prod_{\substack{g \in \mathcal{F}': \\ g \neq f}} p_{\iota(g)},$$

where p^f is the vector of prices set by firm f.

If assertion (i) (resp., assertion (ii)) holds, then the logarithm (resp., the identity function) is an admissible transformation function for demand system D.

$\mathcal{F}$-Specific Transformed Potentials

Remarks:

- Case (ii): This is a generalization of the generalized linear demand system
- Case (i): PDEs are hard to integrate; in the special case where $|\mathcal{F}| = 2$, we obtain:

$$D_i(p) = -\partial_i \psi^f(p^f)\Psi'\Big(\sum_{g \in \mathcal{F}} \psi^g(p^g)\Big) \quad \forall i \in f \in \mathcal{F},$$

where ψ^f and Ψ are arbitrary functions

$\mathcal{F}$-Specific Transformed Potentials

What about case (i) when $|\mathcal{F}| \geq 3$?

- The following class of indirect utility functions still give rise to an $\mathcal{F}$-specific log-potential:

$$V(p) = \Psi\Big(\sum_{f \in \mathcal{F}} \psi^f(p^f)\Big)$$

- This can be micro-founded by three-stage discrete / continuous choice:
  1. Consumer decides whether to take up the outside option
  2. If not, consumer chooses from which firm to purchase
  3. Consumer chooses which products to purchase (and how much) from selected firm

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- Comments:
  - Substitution patterns go beyond IIA
  - Can also obtain more flexible patterns of complementarity / substitutability: e.g., products can be complements within firms and substitutes across firms
  - Such demand patterns resemble those implied by one-stop shopping behavior
  - Nested logit or nested CES is a special case

$\mathcal{F}$-Specific Transformed Potentials

- A richer class, which also gives rise to an $\mathcal{F}$-specific log-potential:

$$V(p) = \Psi\Big(\sum_{B \in 2^{\mathcal{F}} \setminus \emptyset} a(B) \prod_{f \in B} \psi^f(p^f) \Big)$$

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- This can again be be micro-founded by discrete / continuous choice:
  1. Consumer decides whether to take up outside option
  2. If not, consumer chooses among all consumption baskets $B \in 2^{\mathcal{F}} \setminus \emptyset$. If consumer goes for basket B, he/she receives utility

  $$\log a(B) + \sum_{f \in B} \log \psi^f(p^f) + \varepsilon_B$$

     where the ε_B terms are i.i.d. Gumbel
  3. Conditional on having chosen bundle B, consumer consumes $-\partial_i \psi^f / \psi^f$ units of every product $i \in f \in B$
- This basket structure gives rise to even richer patterns of complementarity / substitutability

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- This basket structure gives rise to even richer patterns of complementarity / substitutability
- In the empirical IO literature, this approach has been used to model joint consumption and complementarities within the discrete choice framework (Ershov et al. 2023, Wang 2024)

$\mathcal{F}$-Specific Transformed Potentials

- Another class allowing for more flexible substitution patterns:
- Let $\mathcal{L}$ (set of nests) be a partition of $\mathcal{F}$ (set of firms)
- The following indirect utility function gives rise to an $\mathcal{F}$-specific log-potential:

$$V(p) = \Psi\Big(\sum_{\ell \in \mathcal{L}} \Phi^{\ell}\Big(\sum_{f \in \mathcal{F}} \psi^{f}(p^{f})\Big)\Big)$$

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- Again, this can be micro-founded by discrete/continuous choice.
- The nested structure and the basket structure can be combined together to obtain flexible substitution patterns and complex patterns of complementarities

Conclusion

- A multiproduct-firm pricing game based on an IIA demand system has a log potential.
- Therefore, any such pricing game has a Nash equilibrium.
- The potential games approach allows us to handle price-dependent patterns of complementarity / substitutability.
- Conversely, if a demand system D has a transformed potential, then D takes:
  - Either the generalized linear form,
  - or the IIA form.
- The weaker concept of an $\mathcal{F}$-specific transformed potential permits richer patterns of complementarity / substitutability by allowing for a nest structure and a basket structure

Transformed Potentials

Key steps of the proof:

- Let $\epsilon(\pi) = -\pi G''(\pi)/G'(\pi)$ be the curvature of G.
- Condition (1):

$$\forall f \neq g,\ \forall (i,j) \in f \times g, \quad \frac{\partial^2 G(\pi^f)}{\partial p_i \partial p_j} = \frac{\partial^2 G(\pi^g)}{\partial p_i \partial p_j}.$$

- Therefore,

$$\frac{\partial^3 G(\pi^f)}{\partial c_i \partial p_i \partial p_j} = 0.$$

- This gives us a parameterized ordinary differential equation:
  For every profit level π, products $i \neq j$, and price vector p such that $p_i D_i(p) > \pi$,

$$\partial_j D_i \left(D_i + \pi \frac{\partial_i D_i}{D_i} \right) \left(\epsilon'(\pi) + \frac{\epsilon(\pi)(1-\epsilon(\pi))}{\pi} \right) = \partial^2_{ij} D_i \left(1 - \epsilon(\pi)\right)$$

  where D_i and its derivatives are evaluated at p.
- Solving the parameterized ODE (and doing a few other things), we find that $G(\pi)$ must be affine in $(\pi, \log \pi)$.

Transformed Potentials

Key steps of the proof (Cont'd):

- Suppose $G(\pi) = A + C\pi$:
- Then, condition (1) boils down to

$$\forall (i,j,k) \in \mathcal{N}^3 \text{s.t. } k \neq i,j, \quad \partial^2_{jk} D_i = 0.$$

- Thus, D_i is additively separable:

$$D_i(p) = \sum_{j \in \mathcal{N}} h_{ij}(p_j)$$

- Slutsky symmetry: For $i \neq j$,

$$h'_{ij}(p_j) = \partial_j D_i = \partial_i D_j = h'_{ji}(p_i)$$

- Hence, $h'_{ij}(\cdot)$ is constant, and substitution effects are thus linear.
- Therefore, D takes the generalized linear form.

Transformed Potentials

Key steps of the proof (Cont'd):

- Suppose $G(\pi) = A + B \log \pi$:
- Then, condition (1) boils down to

$$\forall (i,j,k) \in \mathcal{N}^3 \text{ s.t. } k \neq i,j, \quad \partial_k (D_i / D_j) = 0 \tag{2}$$

$$\forall (i,j) \in \mathcal{N}^2, \ \forall (k,l) \in (\mathcal{N} \setminus \{i,j\})^2 \quad \partial^2_{ik} \log(D_j / D_l) = 0.$$

- Suppose first $|\mathcal{N}| \geq 3$.
- Letting V be an indirect utility function for D, (2) can be rewritten as:

$$\forall (i,j,k) \in \mathcal{N}^3 \text{ s.t. } k \neq i,j, \quad \partial_k (\partial_i V / \partial_j V) = 0$$

- Thus, V is strongly separable with respect to the partition $\{\{i\}\}_{i \in \mathcal{N}}$.
- By Theorem 1 in Goldman and Uzawa (1964) / Proposition 1 in Anderson, Erkal, and Piccinin (2019), V must take the form

$$V(p) = \Psi \left(\sum_{j \in \mathcal{N}} h_j(p_j) \right)$$

Transformed Potentials

Key steps of the proof (Cont'd):

- If $|\mathcal{N}| = 2$, then condition (2) no longer has bite.
- We are then left with a single PDE:

$$\partial^2_{12} \log \frac{D_1}{D_2} = 0.$$

- Hence, $\log(D_1/D_2)$ is additively separable in (p_1, p_2):

$$\log \frac{D_1(p)}{D_2(p)} = \phi_1(p_1) - \phi_2(p_2) \quad \Rightarrow \quad \frac{D_1(p)}{\exp \phi_1(p_1)} = \frac{D_2(p)}{\exp \phi_2(p_2)}$$

- Letting h_i be an antiderivative of $\exp \phi_i$, we obtain:

$$\frac{\partial_1 V}{\partial_1 (h_1(p_1) + h_2(p_2))} = \frac{\partial_2 V}{\partial_2 (h_1(p_1) + h_2(p_2))}$$

- By Lemma 1 in Goldman and Uzawa (1964), V is thus a transformation of $h_1(p_1) + h_2(p_2)$.
- This means that V takes the form

$$V(p) = \Psi \left(h_1(p_1) + h_2(p_2) \right)$$

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