Testing Weak Factors in Asset Pricing

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Outline

Empirical Asset Pricing

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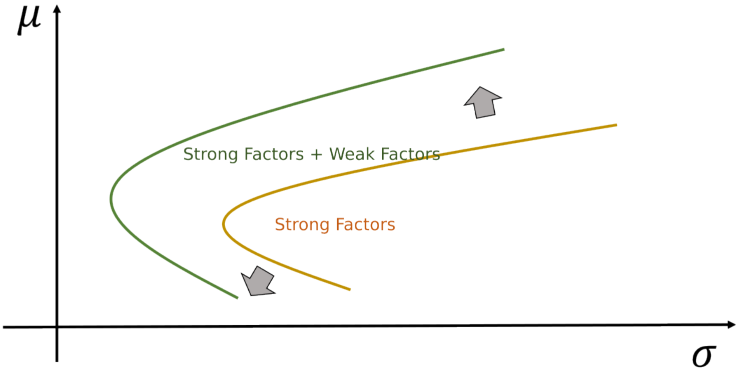
- Seemingly benign but captivating
  - Standard empirical approach is two-pass CSR method
  - Once you decide to take it seriously, lots of complexity arise in empirical application
- This paper considers the issue of weak factors
  - When some factor loadings are close to zeros for most assets

Weak Factors and Investment

- APT is fine with the following Mean-Variance analysis:



- Hence, as an investor, s/he will have a strong incentive to search for the weak factors!

Weak Factors and Asset Pricing Test

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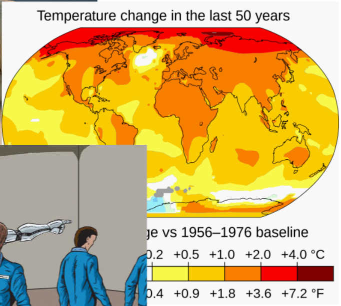
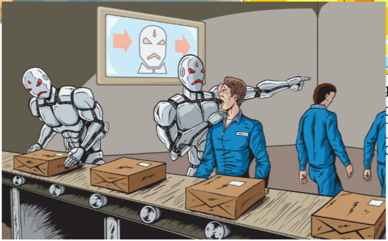
  - weak factors without premium may appear to be important
  - strong factors with significant premium may apprear to be insignificant
- Especially, when the literature proposes hundreds of factors, we need some criteria

Furthermore, rapidly changing economic landscape

- We need to discern which factors are strong weak
  - in a rapidly changing economic environment
- For example, paradigm shifts such as climate changes or job destruction due to AI beg for a short-T method





Key insight of this paper

- Back to the famous equation,

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- Taxonomy of asset pricing econometrics

	Small T	Large T
Strong Factors	$B_{strong} \sim me$	$B_{strong} \gg me$
Weak Factors	$B_{weak} \ll me$	$B_{weak} \sim me$

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1. Traditionally, estimation errors in estimated beta are the cause of trouble
2. We flip it as a blessing to reveal whether a given factor is weak or not

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  - How to test whether a factor of interest is Weak/Semi-strong
    - Pesaran (2012), Pesaran and Smith (2021), Connor and Korajczyk (2022)
- We propose a novel test for weak factors under *small T* setup
  - builds on the two-pass methodology
  - detect whether observed risk factors are *(locally)* weak or not

Outline

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RGP

- Conditional Factor Structure for asset $i = 1, \cdots, N$ at $t = 1, \cdots, T$:

$$R_{it} = \alpha_{it-1} + \underbrace{\boldsymbol{\beta}'_{fit-1}\boldsymbol{f}_t}_{\text{strong}} + \underbrace{\boldsymbol{\beta}'_{git-1}\boldsymbol{g}_t}_{\text{weak}} + e_{it},$$

where

$$\boldsymbol{\beta}_{fit-1} = \left(\beta_{f_1 it-1}, \cdots, \beta_{f_K it-1}\right)', \ \boldsymbol{f}_t = (f_{1t}, \cdots, f_{Kt})$$
$$\boldsymbol{\beta}_{git-1} = \left(\beta_{g_1 it-1}, \cdots, \beta_{g_L it-1}\right)', \ \boldsymbol{g}_t = (g_{1t}, \cdots, g_{Lt})$$

RGP+APT+Local Smoothness

- We can treat (smoothness assumption) the conditional model as a locally unconditional model:

$$\mathbf{R}_t = \gamma_{zt-1}\mathbf{1}_N + \mathbf{B}_f\boldsymbol{\delta}_{ft} + \mathbf{B}_g\boldsymbol{\delta}_{gt} + \boldsymbol{\epsilon}_t,$$

where $\boldsymbol{\delta}_{ft}$ and $\boldsymbol{\delta}_{gt}$ are *expost* risk premia:

$$\boldsymbol{\delta}_{ft} = \gamma_{ft-1} + \boldsymbol{f}_t - E\left[\boldsymbol{f}_t|\mathcal{I}_{t-1}\right], \boldsymbol{\delta}_{gt} = \gamma_{gt-1} + \boldsymbol{g}_t - E\left[\boldsymbol{g}_t|\mathcal{I}_{t-1}\right]$$

Local Factor Strength

- For some $0 \leq \rho \leq 1$, the matrix $\mathbf{B}_g$ satisfies

$$\|\mathbf{B}_g\|^2 \asymp O\left(N^{\rho}\right), \ \|\mathbf{B}_g' 1_N\| \asymp o\left(N^{\frac{\rho+1}{2}}\right)$$

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  - When $\rho = 1$, $\frac{\mathbf{B}_g' \mathbf{B}_g}{N} \asymp O\left(1\right)$, or $\boldsymbol{g}_t$ is strong
  - The difference in the convergence speed plays a key role to learn ρ
    - Analogy to well-spread portfolio $\boldsymbol{w}$, $\boldsymbol{w}'\mathbf{1}_N = 1$ and $\boldsymbol{w}'\boldsymbol{w} \to 0$

Outline

Target Equation

- First, we consider the case that there is no strong factor $\boldsymbol{f}$:
    - RGP

$$\mathbf{R}_t = \boldsymbol{\alpha}_{t-1} + \mathbf{B}_g \boldsymbol{g}_t + \boldsymbol{\epsilon}_t$$

Target Equation

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  - RGP

$$\mathbf{R}_t = \boldsymbol{\alpha}_{t-1} + \mathbf{B}_g \boldsymbol{g}_t + \boldsymbol{\epsilon}_t$$

  - Along with the pricing, $\mu = B \times \gamma$

$$\mathbf{R}_t = \gamma_{zt-1} \mathbf{1}_N + \mathbf{B}_g \boldsymbol{\delta}_{gt} + \boldsymbol{\epsilon}_t,$$

    which gives the target equation:

$$\overline{\mathbf{R}} = \overline{\gamma}_z \mathbf{1}_N + \mathbf{B}_g \overline{\boldsymbol{\delta}}_g + \overline{\boldsymbol{\epsilon}}$$

  - Note that we are interested in whether $\boldsymbol{g}$ is weak or not

FMB two-pass

- First-pass time-series OLS gives

$$\widehat{\mathbf{B}}_{g0} = \mathbf{B}_g + \epsilon \mathcal{P}_g,$$

where $\mathbf{R} = (\mathbf{R}_1, \cdots, \mathbf{R}_T)'$, $G = (\boldsymbol{g}_1, \cdots, \boldsymbol{g}_T)'$, $\mathcal{J}_T = I_T - \frac{1}{T}\mathbf{1}_T\mathbf{1}_T', \mathcal{P}_g = \mathcal{J}_T G \left(G'\mathcal{J}_T G\right)^{-1}$

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- Second-pass cross-sectional OLS gives:

$$\widehat{\boldsymbol{\Gamma}}_{g0} = \begin{bmatrix} \widehat{\gamma}_{0g0} \\ \widehat{\boldsymbol{\delta}}_{g0} \end{bmatrix} = \left(\widehat{\mathbf{X}}_{g0}'\widehat{\mathbf{X}}_{g0}\right)^{-1} \widehat{\mathbf{X}}_{g0}'\overline{\mathbf{R}}$$

$$\asymp \begin{bmatrix} \overline{\gamma}_z \\ 0_L \end{bmatrix} + \begin{bmatrix} O\left(\frac{\mathbf{B}_g' 1_N}{N}\right) \\ O\left(\frac{\mathbf{B}_g'\mathbf{B}_g}{N}\right) \end{bmatrix} + O_p\left(\frac{1}{\sqrt{N}}\right),$$

where

$$\widehat{\mathbf{X}}_{g0} = \left[1_N \ \hat{\mathbf{B}}_{g0}\right]$$

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where

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Properties of FMB 1

Theorem 1. *Under some Assumptions, the two-pass estimator* $\widehat{\boldsymbol{\delta}}_{g0}$ *in* $\widehat{\boldsymbol{\Gamma}}_{g0} = \left[\widehat{\gamma}_{z0}\ \widehat{\boldsymbol{\delta}}'_{g0}\right]'$ *behaves as follows:*

	$\widehat{\boldsymbol{\delta}}_{g0} \rightarrow_p$	$\sqrt{N}\widehat{\boldsymbol{\delta}}_{g0} \rightarrow_d$
$\rho < \frac{1}{2}$		$\mathcal{N}\left(\mathbf{0}_L, \frac{\kappa_4 + Ts_4}{T^2 s_2^2} G' \mathcal{J}_T G\right)$
$\rho = \frac{1}{2}$	$\mathbf{0}_L$	$\mathcal{N}\left(\mathbf{0}_L, \frac{\kappa_4 + Ts_4}{T^2 s_2^2} G' \mathcal{J}_T G\right) + O_p(1)$
$\frac{1}{2} < \rho < 1$		$\pm\infty$
$\rho = 1$	$\widehat{\boldsymbol{\delta}}_{g0} \nrightarrow_p \overline{\boldsymbol{\delta}}_g$	

where $s_2 = \lim_N \frac{1}{N}\sum_i \epsilon_{it}^2$, $\kappa_4 = \left(\lim_N \frac{1}{N}\sum_i \epsilon_{it}^4 - 3s_4\right)$ *and* $s_4 = \lim_N \frac{1}{N}\sum_i E\left[\epsilon_{it}^2\right]^2$

Relation to Standard OLS

Theorem 2. *Under the assumption that residuals are normal i.i.d, the OLS statistics R^2_{g0} and t-stats and F-stat on $\widehat{\boldsymbol{\delta}}_{g0}$ behaves as follows:*

<table>
<tr><th></th><th>$R^2_{\text{g0}} \to_p$</th><th>$t_{g0,k} \to_p$</th><th>$F_{g0} \to_p$</th></tr>
<tr><td>$\rho < \frac{1}{2}$</td><td rowspan="3">0</td><td>$\mathcal{N}(0,1)$</td><td>$\frac{\chi^2_L}{L}$</td></tr>
<tr><td>$\rho = \frac{1}{2}$</td><td>$\mathcal{N}(0,1) + O_p(1)$</td><td>$\frac{\chi^2_L}{L} + O_p(1)$</td></tr>
<tr><td>$\frac{1}{2} < \rho < 1$</td><td rowspan="2">$\pm\infty$</td><td rowspan="2">∞</td></tr>
<tr><td>$\rho = 1$</td><td>$(0,1)$</td></tr>
</table>

Properties of FMB 2

Theorem 3. *Under some Assumptions, the two-pass estimator* $\widehat{\gamma}_{z0}$ *in* $\widehat{\boldsymbol{\Gamma}}_{g0} = \left[\widehat{\gamma}_{z0}\ \widehat{\boldsymbol{\delta}}'_{g0}\right]'$ *behaves as follows:*

	$\widehat{\gamma}_{z0} \to_p$	$\sqrt{N}\left(\widehat{\gamma}_{0g0} - \overline{\gamma}_0\right) \to_d$
$\rho = 0$	$\overline{\gamma}_z$	$\mathcal{N}\left(0, \frac{s_2}{T}\right)$
$0 < \rho < 1$	$\overline{\gamma}_z$	$\pm\infty$
$\rho = 1$	$\widehat{\gamma}_{z0} \nrightarrow_p \overline{\gamma}_z$	$\pm\infty$

where $s_2 = \lim_N \frac{1}{N} \sum_i \epsilon_{it}^2$

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where $s_2 = \lim_N \frac{1}{N}\sum_i \epsilon_{it}^2$

- *Given that we do not observe* $\overline{\gamma}_z$ *(except R is an excess return), the asymptotic distribution is not directly useful*
  - *This property originates from that* $\widehat{\gamma}_{z0}$ *contains* $\frac{\mathbf{1}'_N \mathbf{B}_g}{N}$
- *Hence, we propose a new test using* $\sqrt{N}\frac{\mathbf{1}'_N \widehat{\mathbf{B}}_{g0}}{N}\overline{\boldsymbol{\delta}}_g$

Weakness Test

Theorem 4. *Under some Assumptions,* $\sqrt{N}\frac{\mathbf{1}_N'\widehat{\mathbf{B}}_{g0}}{N}\overline{\boldsymbol{\delta}}_g$ *behaves as follows:*

	$\sqrt{N}\frac{\mathbf{1}_N'\widehat{\mathbf{B}}_{g0}}{N}\overline{\boldsymbol{\delta}}_g \to_d$
$\rho = 0$	$\mathcal{N}\left(0, s_2\overline{\boldsymbol{\delta}}_g'\left(G'\mathcal{J}_T G\right)^{-1}\overline{\boldsymbol{\delta}}_g\right)$
$0 < \rho < 1$	$\pm\infty$
$\rho = 1$	

- *Furthermore, we observe all the elements for the asymptotic variance except* $s_2 = \lim_N \frac{1}{N}\sum_i \epsilon_{it}^2$*!*

Estimation of Asymptotic Variance

- Recall that when $\rho < \frac{1}{2}$
  - $\sqrt{N}\widehat{\boldsymbol{\delta}}_{\mathrm{g}0} \to_d \mathcal{N}\left(\mathbf{0}_L, \frac{\kappa_4 + T s_4}{T^2 s_2^2} G' \mathcal{J}_T G\right)$

Estimation of Asymptotic Variance

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- We need to estimate $s_2 = \lim_N \frac{1}{N} \sum_i \epsilon_{it}^2$, $\kappa_4 = \left(\lim_N \frac{1}{N} \sum_i \epsilon_{it}^4 - 3s_4\right)$ and $s_4 = \lim_N \frac{1}{N} \sum_i E\left[\epsilon_{it}^2\right]^2$!
  - We can do that by exploiting estimated residuals from first-pass as well as those from second-pass

Summary of tests

- We utilize two tests: (i) coefficients on the noisy betas from FMB and (ii) average of the noisy betas

<table>
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<tr><td>$\rho = 0$</td><td rowspan="2">Null</td><td>Null</td></tr>
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<tr><td>$\rho \geq \frac{1}{2}$</td><td>Alternative</td></tr>
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Modified FMB two-pass

- First-pass time-series OLS gives

$$\widehat{\mathbf{B}}_f = \mathbf{B}_f + \epsilon \mathcal{P}_f, \widehat{\mathbf{B}}_g = \mathbf{B}_g + \epsilon \mathcal{P}_{g_\perp},$$

where $\mathcal{P}_f = \mathcal{J}_T F \left(F' \mathcal{J}_T F\right)^{-1}$, $\mathcal{P}_{g\perp} = \mathcal{J}_T G_\perp \left(G'_\perp \mathcal{J}_T G_\perp\right)^{-1}$

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- Second-pass cross-sectional OLS gives:

$$\widehat{\Gamma}_g = \left[\begin{array}{c} \widehat{\gamma}_z \\ \widehat{\boldsymbol{\delta}}_g \end{array} \right] = \left(\widehat{\mathbf{X}}_g' \widehat{\mathbf{X}}_g\right)^{-1} \widehat{\mathbf{X}}_g' \left(\overline{\mathbf{R}} - \widehat{\mathbf{B}}_f \overline{\boldsymbol{\delta}}_f\right),$$

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where

$$\widehat{\mathbf{X}}_g = \left[1_N \ \widehat{\mathbf{B}}_g\right]$$

- If we include $\widehat{\mathbf{B}}_f$ in the second pass regressor
  - It is well known that the estimator is biased due to estimation error
  - The bias-correction such as Shaken (1992) does not work (See Pesaran and Smith (2021))

Slight modification of tests

- Two tests have similar properties

	$\sqrt{N}\widehat{\boldsymbol{\delta}}_g \to_d$	$\frac{\mathbf{1}_N' \widehat{\mathbf{B}}_g}{\sqrt{N}} \overline{\boldsymbol{\delta}}_g \to_d$
$\rho = 0$	$\mathcal{N}\left(\mathbf{0}_L, V_1\right)$	$\mathcal{N}\left(0, V_2\right)$
$\rho < \frac{1}{2}$	$\mathcal{N}\left(\mathbf{0}_L, V_1\right)$	
$\rho = \frac{1}{2}$	$\mathcal{N}\left(\mathbf{0}_L, V_1\right) + O_p\left(1\right)$	$\pm\infty$
$\frac{1}{2} < \rho \leq 1$	$\pm\infty$	

where

$$V_1 = \frac{s_4}{s_2^2}\mathbf{l}'\mathbf{l}G_\perp' G_\perp + \frac{\kappa_4}{s_2^2} G_\perp' \mathrm{diag}\left(\mathbf{l} \odot \mathbf{l}\right) G_\perp$$

$$\mathbf{l} = \frac{1_T}{T} - \mathcal{P}_f \overline{\boldsymbol{\delta}}_f$$

$$V_2 = s_2 \overline{\boldsymbol{\delta}}_g' \left(G_\perp' G_\perp\right)^{-1} \overline{\boldsymbol{\delta}}_g$$

- Furthermore, we can operationalize the tests using consistent estimators for components in the asymptotic variance

Outline

PCA

- Following Zaffaroni (2023), we obtain the systematic factors up to rotation

$$F_* - F\tilde{H} \rightarrow_p 0_{T \times K}$$

Modified FMB two-pass with PCA factors

- First-pass time-series OLS gives

$$\widehat{\mathbf{B}}_{f_*} = \mathbf{B}_{f_*} + \epsilon_* \mathcal{P}_{f_*},$$
$$\widehat{\mathbf{B}}_{g_*} = \mathbf{B}_{g_*} + \epsilon_* \mathcal{P}_{g_{*\perp}},$$

where $\mathcal{P}_{f_*} = \mathcal{J}_T F_* \left(F_*' \mathcal{J}_T F_*\right)^{-1}, \ \mathcal{P}_{g_{*\perp}} = \mathcal{J}_T G_{*\perp} \left(G_{*\perp}' \mathcal{J}_T G_{*\perp}\right)^{-1}$

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$$\widehat{\Gamma}_{g_*} = \begin{bmatrix} \widehat{\gamma}_{z_*} \\ \widehat{\boldsymbol{\delta}}_{g_*} \end{bmatrix} = \left(\widehat{\mathbf{X}}_{g_*}' \widehat{\mathbf{X}}_{g_*}\right)^{-1} \widehat{\mathbf{X}}_{g_*}' \left(\overline{\mathbf{R}} - \widehat{\mathbf{B}}_{f_*} \overline{\boldsymbol{\delta}}_{f_*}\right),$$

where

$$\widehat{\mathbf{X}}_{g_*} = \left[1_N \ \widehat{\mathbf{B}}_{g_*}\right]$$

Slight modification of tests

- Two tests have similar properties

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$\rho = 0$	$\mathcal{N}(\mathbf{0}_L, V_{1*})$	$\mathcal{N}(0, V_{2*})$
$\rho < \frac{1}{2}$	$\mathcal{N}(\mathbf{0}_L, V_{1*})$	$\pm\infty$
$\rho = \frac{1}{2}$	$\mathcal{N}(\mathbf{0}_L, V_{1*}) + O_p(1)$	$\pm\infty$
$\frac{1}{2} < \rho \leq 1$	$\pm\infty$	$\pm\infty$

where

$$V_{1*} = \frac{s_4}{s_2^2}\mathbf{l}'\mathbf{l}G_\perp' G_\perp + \frac{\kappa_4}{s_2^2}G_\perp' \mathrm{diag}(\mathbf{l} \odot \mathbf{l}) G_\perp$$

$$V_{2*} = c_* + s_2\overline{\boldsymbol{\delta}}_g' (G_\perp' G_\perp)^{-1} \overline{\boldsymbol{\delta}}_g$$

- Furthermore, we can operationalize the tests using consistent estimator of the asymptotic variance

Outline

Simulation Design

1. Calibration: MacKinlay and Pastor (2000)

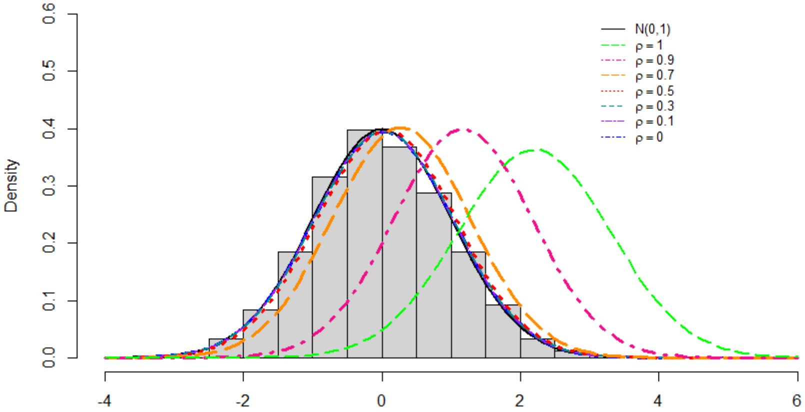
$$R_{it} = 0 + \beta_{fi}\boldsymbol{f}_t + \beta_{gi}\boldsymbol{g}_t + e_{it}$$

2. We consider a single strong factor and a single weak factor, $N = 3000,\ T = 24$
3. We focus on the distribution of the following two tests

	test 1: $\frac{1}{\sqrt{\widehat{AsyVar}}}\left(\sqrt{N}\widehat{\boldsymbol{\delta}}_g\right)$	test 2: $\frac{1}{\sqrt{\widehat{AsyVar}}}\left(\frac{\mathbf{1}_N'\widehat{\mathbf{B}}_g}{\sqrt{N}}\overline{\boldsymbol{\delta}}_g\right)$
$\rho = 0$	$\mathcal{N}(0,1)$	$\mathcal{N}(0,1)$
$\rho < \frac{1}{2}$		
$\rho = \frac{1}{2}$	$\mathcal{N}(0,1) + O_p(1)$	$\pm\infty$
$\frac{1}{2} < \rho \leq 1$	$\pm\infty$	

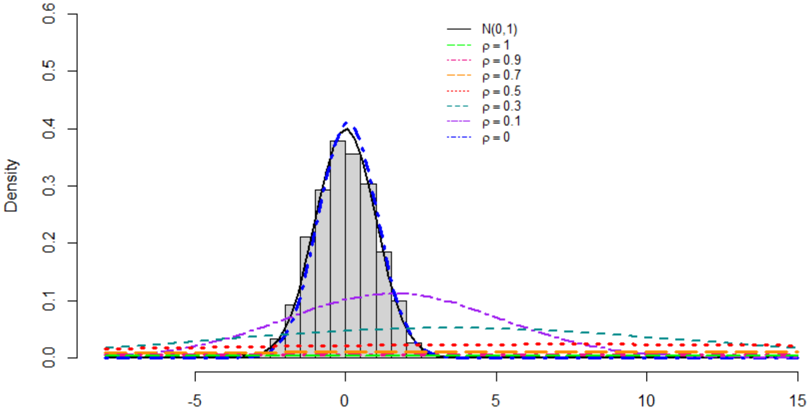
test 1: under the null $\rho < \frac{1}{2}$ + DGP with $\rho \in [0, 1]$

- 3000 repetitions



test 2: under the null $\rho = 0$ + DGP with $\rho \in [0, 1]$

- 3000 repetitions



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- We focus on the test 2 on the null $\rho = 0$ over 1968-2022
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    - We test whether a factor in FF5 is weak or not
    - Strong/weak depends on industry

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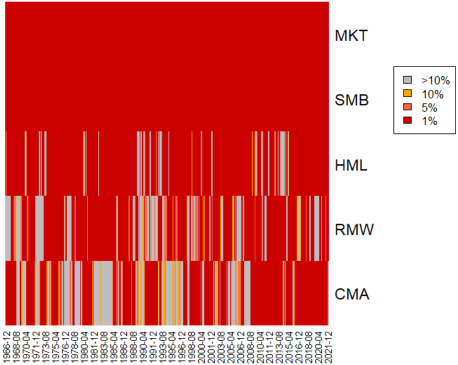
    - Strong/weak depends on industry
- Given a set of strong factors, we perform weak factor test on
    - Factor zoo
        - 150 factors from Feng, Giglio and Xiu (2020)
    - Likelihood of being weak on recession/post-publication

Are there any strong factors in FF5?

- HeatMap (Strong Red - ... - Weak Gray)

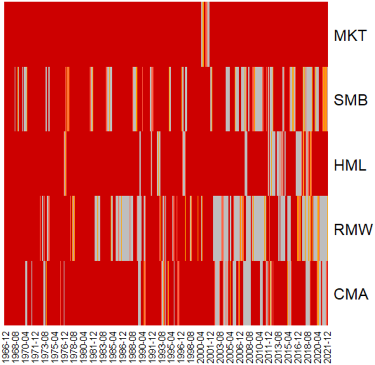


- Null on F and G

F	G
No Strong Factor	MKT
CAPM	SMB, HML
FF3	RMW, CMA

Stong/Weak of FF5 in Utility Industry

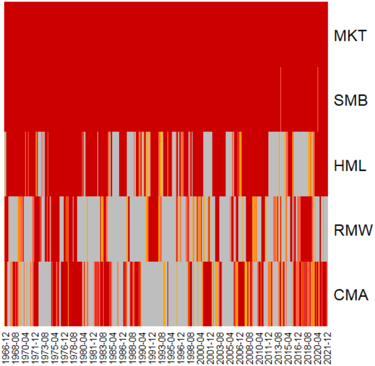
- HeatMap (Strong Red - ... - Weak Gray)



- SMB tend to be weaker in Utility industry
    - 20% of tests in Uitlity vs 0% of tests in CRSP

Stong/Weak of FF5 in Consumer Nondurables

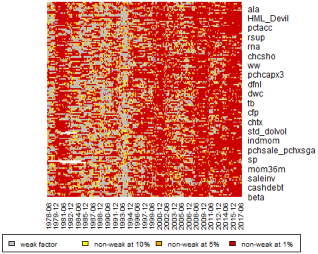
- HeatMap (Strong Red - ... - Weak Gray)



- HML tend to be weaker in Consumer Nondurables industry
  - 27% of tests in Consumer Non-durables vs 10% of tests in CRSP

Factor Zoo with strong subset of FF5

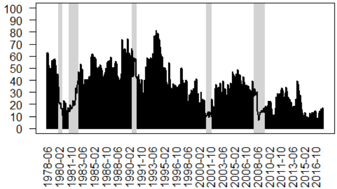
- We select strong factors from FF5 at each local time



Business Cycle and Weakness of Factors

- Business Cycle and % of Weak factors in factor zoo

$$\%\text{ of weak factors} = a - \underbrace{10.6}_{t=19.69} *\text{NBER recession dummy} + e$$



Post-publication effect

- What happens to the weakness of a given factor post publication
  - We regress [the dummy on $|t| > 1.96$ from our test] on [the post-publication dummy]

    $$\text{Strong Dummy using our test} = a + \underbrace{0.19}_{t=45.92} * \text{Post Publication dummy} + e$$

  - Nice contrast with the results that the average returns tend to be lower post publication (McLean and Pontiff, 2016)
    - Public information => Pervasive & Fair price

Outline

Conclusion

- We provide a framework to test local weakness of factors using a short panel
  - Asymptotic theory
  - Simulation evidence
- Empirical findings
  - anomaly factors: tend to be stronger during recession and post publication