Inference in Auctions with Many Bidders Using Transaction Prices*

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Abstract

This paper considers inference in first-price and second-price sealed-bid auctions in empirical settings where we observe auctions with a large number of bidders. Relevant applications include online auctions, treasury auctions, spectrum auctions, art auctions, and IPO auctions, among others. Given the abundance of bidders in each auction, we propose an asymptotic framework in which the number of bidders diverges while the number of auctions remains fixed. This framework allows us to perform asymptotically exact inference on key model features using only transaction price data. Specifically, we examine inference on the expected utility of the auction winner, the expected revenue of the seller, and the tail properties of the valuation distribution. Simulations confirm the accuracy of our inference methods in finite samples. Finally, we also apply them to Hong Kong car license auction data.

Keywords: auctions, hypothesis testing, confidence intervals, extreme value theory, tail index.

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1 Introduction

This paper considers inference in first-price and second-price sealed-bid auctions in empirical settings where we observe a possibly small number of auctions, each with a large number of bidders. The abundance of bidders in each auction prompts us to consider a novel asymptotic framework in which the number of bidders diverges, while allowing the number of auctions to be small and remain fixed. This framework differs substantially from the more conventional approach in which the researcher observes multiple bids from a large number of independent and identically distributed (i.i.d.) auctions. See Athey and Haile (2002); Haile and Tamer (2003); Athey and Haile (2007); Guerre, Perrigne, and Vuong (2000), among others. Our analysis can deliver an accurate approximation in empirical settings in which the number of bidders is large relative to the number of auctions. Relevant examples include online auctions, treasury auctions, spectrum auctions, art auctions, and IPO auctions, among many others. Since our asymptotic framework does not require the number of auctions to diverge, our analysis is suitable for applications with substantial heterogeneity across auctions, implying a limited number of truly homogeneous auctions.

Within our novel asymptotic framework, we introduce new inference methods for the winner's expected utility, the seller's expected revenue, and the valuation distribution's tail behavior. We show that the latter can be used to test the regularity conditions commonly assumed in auction literature. Our data requirements are minimal; our methods rely on observing transaction prices from a finite number of auctions. We do not need to observe multiple bids or the number of participating bidders in these auctions.

Our methodology characterizes the limiting behavior of transaction prices as the number of bidders increases. Unlike the traditional asymptotic framework with a growing number of auctions, the transaction price data does not allow us to fully identify the valuation distribution. However, it can reveal the distribution's tail properties, which are sufficient for conducting inference on practical objects of interest, such as the winner's expected utility and the seller's expected revenue. Moreover, the tail behavior allows us to evaluate whether the valuation distribution has bounded support and positive density at its highest value, which is a common regularity in the auction literature. (e.g., Maskin and Riley, 1984; Guerre et al., 2000; Guerre and Luo, 2022). Within our asymptotic framework, our inference methods are shown to control size and exhibit desirable power properties. Specifically, our confidence intervals are shown to minimize expected length and our hypothesis tests are shown to maximize weighted average power. Finally, Monte Carlo simulations confirm the accuracy of our methodology in finite samples, and a study of Hong Kong car license plate auctions illustrates its empirical relevance.

Our assumptions are arguably mild. The baseline version of the model assumes that the bidders are symmetric and have independent private values (IPV). We require a large number of bidders and that the valuation distribution falls within the domain of attraction of the extreme value (EV) distribution, a condition satisfied by commonly used distributions like Pareto, Student-t, Gaussian, and uniform distributions (see Section 2 for details). In this setup, we can conduct inference using transaction prices from just three or more auctions. Furthermore, the auctions do not need to have the same number of bidders (which may remain unknown), but should diverge at the same rate. Beyond the IPV setup, we discuss how to extend our analysis to allow for conditional IPV and reserve price.

The asymptotic framework with many bidders has been extensively employed in the economic theory literature (e.g., Hong and Shum, 2004; Virág, 2013; Di Tillio, Ottaviani, and Sørensen, 2021) but less so in econometrics. Hong, Paarsch, and Xu (2014) studies the asymptotic distribution of the transaction price in a clock model of a multi-unit, oral, ascending-price auction as the numbers of bidders and units increase. Krasnokutskaya, Song, and Tang (2022) considers a latent group structure on the set of agents and allows both the number of agents and the number of markets to grow. Menzel and Morganti (2013) shows that the nonparametric estimator of the valuation distribution may become irregular and perform poorly when the number of bidders increases. This issue does not affect our method, which relies on the EV approximation. In comparison, much of the existing econometric literature has focused on identifying and estimating the valuation distribution. In cases with a small number of bidders, like timber auctions, identifying the valuation distribution is the key to understanding the relevant auction features. The seminal work by Athey and Haile (2002) derives general results about identifying the valuation distribution from the distribution of bids. Haile and Tamer (2003) investigates English auctions and establishes bounds on the valuation distribution and other objects of interest with minimal structural assumptions. Chesher and Rosen (2017) extends these bounds to the non-IPV setup. Aradillas-Lopez, Gandhi, and Quint (2013) nonparametrically identifies bounds on seller profit and bidder surplus while accounting for variations in the number of bidders across auctions. Brendstrup and Paarsch (2006) and Komarova (2013) derive nonparametric identification of the valuation distribution with the transaction price and the winner's identity. Brendstrup and Paarsch (2007) investigates multi-object English auctions and establishes semiparametric identification using winning bids under the Archimedean copula assumption.

There is a vast literature that considers inference in the more traditional asymptotic framework in which the number of auctions diverges to infinity. We now briefly highlight some of its recent contributions. For example, Li (2005) studies first-price auctions with entry and binding reservation prices and estimates the valuation distribution with the observed bids and the number of actual bidders. Without knowing the number of bidders, An, Hu, and Shum (2010) proposes using a proxy of the number of bidders and an instrument variable. Kim and Lee (2014), Song (2015), Mbakop (2017), and Freyberger and Larsen (2022) construct identification of the valuation distribution with two or more order statistics of bids. Shneyerov and Wong (2011) derive nonparametric identification of model primitives based on finitely many groups of bidders. Recently, Luo and Xiao (2023) derives identification results with two consecutive order statistics and an instrument or three consecutive ones. All these methods of identification, estimation, and inference are based on the traditional asymptotic framework with many auctions and multiple bids from each auction. We refer to Hickman, Hubbard, and Sağlam (2012) and Gentry, Hubbard, Nekipelov, and Paarsch (2018) for recent surveys.

As already mentioned, our paper provides inference based only on transaction prices. This aspect of our paper resembles the recent work by Guerre and Luo (2022). However, there are considerable differences between this paper and our contribution. In particular, Guerre and Luo (2022) considers first-price auctions in which the number of bidders in each auction is random and has finite support. In this context, they establish that the winning bid is increasing in the number of bidders and, hence, the density of the winning bids exhibits discontinuities as the number of bidders changes. Our framework has many differences with Guerre and Luo (2022). First, their identification strategy relies on the assumption that the number of bidders is finite, while we specialize in the case when this amount diverges. Thus, the contributions are designed for different empirical environments. Second, their argument does not apply to second-price auctions, while ours does. Third, their method relies on observing a large number of auctions, while ours can deal with applications with a finite number of them. Finally, their paper delivers identification analysis, while ours focuses on inference.

Finally, our paper is also connected to the literature on testing in auction models. Donald and Paarsch (1996) introduces parametric tests within the context of IPV setups. Haile, Hong, and Shum (2003) devises nonparametric tests for common values in first-price auctions. Jun, Pinkse, and Wan (2010) develops a nonparametric test for affiliation. Hortaçsu and Kastl (2012) proposes a test of common values when some bidders have information about rivals' bids. Hill and Shneyerov (2013) develops a test for common values in first-price auctions utilizing tail indices. Liu and Luo (2017) puts forward a nonparametric test for comparing valuation distributions in first-price auctions. Relative to these studies, our work stands out as we address testing within a comprehensive framework accommodating numerous bidders in both first and second-price auctions. Furthermore, our methods remain applicable when multiple bids from each auction are available, operating effectively after conditioning on unobserved heterogeneity and requiring only a small number of auctions.

The rest of the paper is organized as follows. Section 2 establishes the new asymptotic framework with many bidders and discusses its relationship with extreme value (EV) theory. Section 3 focuses on second-price auctions and introduces our new inference methods. We begin by outlining the auction format and deriving the asymptotic distribution of transaction prices within the new framework. Subsequently, Section 3.1 presents confidence intervals for the winner's expected utility, Section 3.2 introduces confidence intervals for the seller's expected revenue, and Section 3.3 outlines hypothesis tests for the tail index. Section 4 offers analogous results for first-price auctions. Section

5 describes how to extend our analysis to allow for conditional IPV and the presence of reserve prices. Section 6 provides the Monte Carlo simulation results. Section 7 presents an empirical illustration of our methodology. Finally, Section 8 concludes. The paper's appendix provides all proofs, auxiliary results, and computational details.

2 Asymptotic framework with many bidders

We consider inference in sealed-bid auctions for a single object, where the data consist of transaction prices from $n \geq 3$ independent auction realizations, denoted as $\{P_j : j = 1, \ldots, n\}$. Importantly, our framework does not require n to diverge to infinity.

For each auction $j = 1, \ldots, n$, the setup is as follows. There is a single object for sale, and K_j potential buyers are bidding for it. These bidders have independent private values (IPV) $\{V_{i,j} : i = 1, \ldots, K_j\}$ distributed according to a common cumulative distribution function (CDF) F_V with support on $[v_L, v_H]$, with $0 \leq v_L < v_H$, where $v_H = \infty$ is allowed. We assume that F_V strictly increases on its support and admits a continuous probability density function (PDF) $f_V = F_V'$. Bidders are assumed to be risk-neutral and maximize expected profits without facing any liquidity or budget constraints. F_V and K_j are common knowledge to all bidders, but unknown to the researcher. Our inference methods rely on the asymptotics with diverging numbers of bidders K_j and a finite number of auctions n. To this end, we assume that $K \equiv \min\{K_1, \ldots, K_n\} \to \infty$ and $K_i/K \to 1$ for each $i = 1, \ldots, n$. That is, we assume all n auctions have approximately the same large number of bidders.[1]

We make an additional assumption about the valuation distribution F_V. We assume that it is in the domain of attraction of the EV distribution G_ξ, where ξ denotes the tail index. Formally, this means that there is a sequence of normalizing constants $\{(a_K, b_K) \in \mathbb{R}_{++} \times \mathbb{R} : K \in \mathbb{N}\}$ such that, for all x that is a continuity point of G_ξ,

$$\lim_{K\to\infty} (F_V(a_K x + b_K))^K = G_\xi(x). \tag{2.1}$$

See de Haan and Ferreira (2006, Chapter 1) or David and Nagaraja (2004, Chapter 10) for recent expositions on this topic, including sufficient conditions for (2.1). Under the condition in (2.1), standard asymptotic results imply that G_ξ belongs to one of three types: Weibull (if $\xi < 0$), Gumbel (if $\xi = 0$), or Fréchet (if $\xi > 0$). We can unify these distributions into the generalized EV

[1] Allowing for an unknown number of bidders and $K_j/K \not\to 1$ for some $j = 1, \ldots, n$ significantly complicates our inference problem. In particular, this would require observing more than the transaction price in each auction, contradicting the core premise of this paper.

distribution, with the following CDF:

$$G_\xi(x) = \begin{cases} \exp(-(1+\xi x)^{-1/\xi}) I(1+\xi x > 0) & \text{if } \xi > 0, \\ \exp(-\exp(-x)) & \text{if } \xi = 0, \\ \exp(-(1+\xi x)^{-1/\xi}) I(1+\xi x > 0) + I(1+\xi x \leq 0) & \text{if } \xi < 0, \end{cases} \tag{2.2}$$

Condition (2.1) is an arguably mild restriction, as is satisfied by most commonly used valuation distributions F_V. The case with $\xi > 0$ covers distributions with unbounded support (i.e., $v_H = \infty$) and polynomial decaying (i.e., "heavy") right tail. In this case, moments of order less than $1/\xi$ exist, and moments of order greater than $1/\xi$ do not (see de Haan and Ferreira (2006, page 176)). Then, the restriction to $\xi \leq 1/2$ implies that F_V has finite second moments. Examples include Pareto, Student-t, and F distributions. Second, the case with $\xi = 0$ encompasses distributions with unbounded support (i.e., $v_H = \infty$) but with exponential decaying (i.e., "light") right tail and bounded moments of any order. Examples include normal and log-normal distributions. Finally, the case $\xi < 0$ covers distributions with bounded support (i.e., $v_H < \infty$), such as Beta, Uniform, and triangular distributions. In turn, condition (2.1) fails for any distribution that has a probability mass point at the highest value of its support, such as geometric or Poisson distributions; see de Haan and Ferreira (2006, Exercise 1.13).

The significance of (2.1) in our paper is that it allows us to characterize the joint distribution of the ordered valuations for all auctions as the number of bidders diverges. We now introduce the relevant notation to this end. For each auction $j = 1, \ldots, n$, let $\{V_{(i),j} : i = 1, \ldots, K_j\}$ denote the order statistics of $\{V_{i,j} : i = 1, \ldots, K_j\}$ in decreasing order, i.e., $V_{(1),j} \geq V_{(2),j} \geq \cdots \geq V_{(K_j),j}$. Lemma 2.1 provides the joint distribution of the extreme order statistics for all auctions.

Lemma 2.1. *Assume* (2.1) *holds. For any $n \in \mathbb{N}$ and any $d \in \mathbb{N}$, and as $K \to \infty$,*

$$\Big\{ \Big(\frac{V_{(1),j} - b_K}{a_K}, \frac{V_{(2),j} - b_K}{a_K}, \ \ldots \ , \frac{V_{(d),j} - b_K}{a_K} \Big) : j = 1, \ldots, n \Big\} \xrightarrow{d} \\ \Big\{ \Big(H_\xi(E_{1,j}), \ H_\xi(E_{1,j} + E_{2,j}), \ \ldots \ , H_\xi\big(\sum\nolimits_{s=1}^{d} E_{s,j}\big) \Big) : j = 1, \ldots, n \Big\}, \tag{2.3}$$

where $\{(a_K, b_K) \in \mathbb{R}_{++} \times \mathbb{R} : K \in \mathbb{N}\}$ are the normalizing constants in (2.1)*, $\{E_{s,j} : s = 1, \ldots, d, \ j = 1, \ldots, n\}$ are i.i.d. standard exponential random variables, and*

$$H_\xi(x) \equiv \begin{cases} (x^{-\xi} - 1)/\xi & \text{if } \xi \neq 0, \\ -\ln(x) & \text{if } \xi = 0. \end{cases} \tag{2.4}$$

Lemma 2.1 characterizes the asymptotic distribution of the largest order statistics. In this paper, we consider first-price and second-price auction formats, which involve only $V_{(1),j}$ and $V_{(2),j}$, respectively. We assume symmetric equilibrium bidding, enabling us to establish a relationship between private valuations, equilibrium bids, and transaction prices $\mathbf{P} \equiv \{P_j : j = 1, \ldots, n\}$. As a corollary, we can completely describe the asymptotic distribution of transaction prices in terms of the tail index ξ. This, in turn, allows us to perform inference of several objects of economic interest solely based on the transaction prices.

3 Second-price auctions

We begin our analysis with second-price sealed-bid auctions, in which the highest bidder wins the object and pays the second-highest bid. Since we consider a private value framework, second-price auctions are equivalent in a weak sense to open ascending price (or English) auctions (see Krishna (2009, page 4)). By standard arguments (e.g., Krishna (2009, Proposition 2.1)), the weakly dominant strategy for a bidder with valuation v in auction $j = 1, \ldots, n$ is

$$\beta_j(v) \;=\; v. \tag{3.1}$$

Thus, the observed transaction price in auction j equals the second-highest bid, i.e.,

$$P_j \;=\; V_{(2),j}. \tag{3.2}$$

By Lemma 2.1 and (3.2), we conclude that as $K \to \infty$,

$$\left\{\frac{P_j - b_K}{a_K} : j = 1, \ldots, n\right\} \;\xrightarrow{d}\; \{Z_j : j = 1, \ldots, n\}, \tag{3.3}$$

where $\{Z_j : j = 1, \ldots, n\}$ is i.i.d. with $Z_j \equiv H_\xi(E_{1,j} + E_{2,j})$ for each $j = 1, \ldots, n$, and $\{(a_K, b_K) \in \mathbb{R}_{++} \times \mathbb{R} : K \in \mathbb{N}\}$, $\{(E_{1,j}, E_{2,j}) :\ j = 1, \ldots, n\}$, and H_ξ are as in Lemma 2.1.

If the constants $\{(a_K, b_K) \in \mathbb{R}_{++} \times \mathbb{R} : K \in \mathbb{N}\}$ were known, we could use (3.3) to perform inference on functions of the EV index ξ. Unfortunately, these constants are unknown and depend implicitly on the underlying distribution of valuations. To sidestep this issue, we sort the transaction prices across auctions (i.e., $P_{(1)} \leq P_{(2)} \leq \cdots \leq P_{(n)}$), and consider the sorted and self-normalized prices: for $j = 1, \ldots, N \equiv n - 2 \geq 1$,

$$\tilde{P}_j \;\equiv\; \begin{cases} \frac{P_{(j+1)} - P_{(1)}}{P_{(n)} - P_{(1)}} & \text{if } P_{(n)} > P_{(1)} \\ 0 & \text{if } P_{(n)} = P_{(1)}, \end{cases} \tag{3.4}$$

and let $\tilde{\mathbf{P}} = \{\tilde{P}_j : j = 1, \ldots, N\} \in \Sigma \equiv \{h \in [0,1]^N : 0 \leq h_1 \leq \cdots \leq h_N \leq 1\}$. The following result characterizes the asymptotic distribution of $\tilde{\mathbf{P}}$ as $K \to \infty$.

Lemma 3.1. *Assume (2.1) holds. For any $N \in \mathbb{N}$, and as $K \to \infty$,*

$$\tilde{\mathbf{P}} = \{\tilde{P}_j : j = 1, \ldots, N\} \overset{d}{\to} \tilde{\mathbf{Z}} = \{\tilde{Z}_j : j = 1, \ldots, N\}, \tag{3.5}$$

where the joint density of $\tilde{\mathbf{Z}}$ is

$$\begin{aligned} f_{\tilde{\mathbf{Z}}|\xi}(z_1, \ldots, z_N) \;&\equiv\; 1[0 \le z_1 \le \cdots \le z_N \le 1]\ (N+2)!\ \Gamma(2(N+2)) \times \\ &\begin{cases} \int_0^{-1/\xi} s^N \exp\begin{pmatrix} -2(N+2)\ln(\sum_{j=1}^N (1+z_j\xi s)^{-1/\xi} + (1+\xi s)^{-1/\xi}) \\ -(1+2/\xi)(\sum_{j=1}^N \ln(1+z_j\xi s) + \ln(1+\xi s)) \end{pmatrix} ds & \text{if } \xi < 0, \\ \int_0^{\infty} s^N \exp\begin{pmatrix} -2(N+2)\ln(\sum_{j=1}^N \exp(-z_j s) + \exp(-s)) \\ -2s(\sum_{j=1}^N z_j + 1) \end{pmatrix} ds & \text{if } \xi = 0, \\ \int_0^{\infty} s^N \exp\begin{pmatrix} -2(N+2)\ln(\sum_{j=1}^N (1+z_j\xi s)^{-1/\xi} + (1+\xi s)^{-1/\xi}) \\ -(1+2/\xi)(\sum_{j=1}^N \ln(1+z_j\xi s) + \ln(1+\xi s)) \end{pmatrix} ds & \text{if } \xi > 0, \end{cases} \end{aligned} \tag{3.6}$$

and Γ is the standard Gamma function.

Lemma 3.1 reveals that the asymptotic distribution of $\tilde{\mathbf{P}}$ is informative about the tail index ξ. In the next subsections, we show how to use this information to conduct asymptotically valid inference on the tail index ξ and several other important features of these auctions, such as the winner's expected utility and the seller's expected revenue.

3.1 Inference about the winner's expected utility

Our objective is to conduct inference on the average of the winner's expected utility based on the transaction prices. Since each bidder bids their own valuation, the auction is won by the highest bidder, and the transaction price is equal to the second-highest bid, we conclude that the winner's expected utility in auction $j = 1, \ldots, n$ is $E[V_{(1),j} - P_j] = E[V_{(1),j} - V_{(2),j}]$, whose average is

$$\mu_K \;=\; \frac{1}{n}\sum_{j=1}^{n} E[V_{(1),j} - V_{(2),j}]. \tag{3.7}$$

Given transaction prices $\mathbf{P}$, we consider a confidence interval (CI) for μ_K given by

$$U(\mathbf{P}) \;=\; (P_{(n)} - P_{(1)}) \times \tilde{U}(\tilde{\mathbf{P}}), \tag{3.8}$$

where $\tilde{\mathbf{P}} \in \Sigma$ are the sorted and self-normalized transaction prices in (3.4), and $\tilde{U} : \Sigma \to \mathcal{P}(\mathbb{R})$ is a CI defined on $\tilde{\mathbf{P}}$. By (3.8), the CI $U(\mathbf{P})$ is invariant to the sorting and translation of $\mathbf{P}$, and equivariant to their scale.

The remainder of the section denotes $Z_{1,j} = H_\xi(E_{1,j})$, $Z_{2,j} = H_\xi(E_{1,j} + E_{2,j})$, $\{(E_{1,j}, E_{2,j}) : j = 1, \ldots, n\}$ are i.i.d. standard exponential random variables and H_ξ is as in (2.4). We also use

$\mathbf{Z} = \{Z_{2,j} : j = 1, \ldots, n\}$, $Z_{(n)} = \max\{Z_{2,j} : j = 1, \ldots, n\}$, $Z_{(1)} = \min\{Z_{2,j} : j = 1, \ldots, n\}$, for $j = 1, \ldots, N = n-2$,

$$\tilde{Z}_j = \begin{cases} \frac{Z_{(j+1)} - Z_{(1)}}{Z_{(n)} - Z_{(1)}} & \text{if } Z_{(n)} > Z_{(1)}, \\ 0 & \text{if } Z_{(n)} = Z_{(1)}, \end{cases} \tag{3.9}$$

and $\tilde{\mathbf{Z}} = \{\tilde{Z}_j : j = 1, \ldots, N\} \in \Sigma$. Finally, let $Y_\mu \equiv E[Z_{1,1} - Z_{2,1}]/(Z_{(n)} - Z_{(1)})$ and $\kappa_\xi(\tilde{\mathbf{Z}}) = E[Z_{(n)} - Z_{(1)}|\tilde{\mathbf{Z}}]$. The distributions of these random variables are fully characterized by the tail index ξ. In particular, Lemma A.4 shows $E[Z_{1,1} - Z_{2,1}] = \Gamma(1-\xi)$, where Γ is the standard Gamma function. For the remainder of this section, we will use P_ξ and E_ξ to refer to the probability and expectation associated with this distribution. The following result describes the asymptotic properties of the CI in (3.8) as $K \to \infty$.

Theorem 3.1. *Assume (2.1) holds, and that for some $\varepsilon > 0$ with $(1+\varepsilon)\xi < 1$, $E[|V_{i,j}|^{1+\varepsilon}] < \infty$ for all $i = 1 \ldots, K_j$ in auction $j = 1, \ldots, N$. Finally, assume that the CI for μ_K, $U(\mathbf{P})$, is as in (3.8) with $\tilde{U} : \Sigma \to \mathcal{P}(\mathbb{R})$ that satisfies the following conditions:*

(a) $P_\xi(\{Y_\mu, \tilde{\mathbf{Z}}\} \in \partial\{(y,h) \in \mathbb{R} \times \Sigma : y \in \tilde{U}(h)\}) = 0$, where ∂A denotes the boundary of A.

(b) $\lg(\tilde{U}(h)) < \infty$ for any $h \in \Sigma$, where $\lg(A)$ denotes the length of A (i.e., $\lg(A) \equiv \int \mathbf{1}[y \in A]dy$).

(c) For any sequence $\{h_\ell \in \Sigma\}_{\ell \in \mathbb{N}}$ with $h_\ell \to h \in \Sigma$, $\lg(\tilde{U}(h_\ell)) \to \lg(\tilde{U}(h))$.

Then, as $K \to \infty$,

1. $P(\mu_K \in U(\mathbf{P})) \;\to\; P_\xi(Y_\mu \in \tilde{U}(\tilde{\mathbf{Z}}))$,
2. $E[\lg(U(\mathbf{P}))]/a_K \;\to\; E_\xi[\kappa_\xi(\tilde{\mathbf{Z}}) \lg(\tilde{U}(\tilde{\mathbf{Z}}))]$.

Suppose that we consider the class of CIs for μ_K given by (3.8) and conditions (a)-(c) in Theorem 3.1. The finite sample properties of these CIs are unknown. If the number of bidders is large, it is natural to rely on the asymptotic behavior derived in Theorem 3.1 to choose our CI. First, we can guarantee asymptotic validity by imposing

$$P_\xi(Y_\mu \in \tilde{U}(\tilde{\mathbf{Z}})) \;\geq\; 1-\alpha \quad \text{for all } \xi \in \Xi. \tag{3.10}$$

Second, we can seek improvements in statistical power by choosing a CI that has a small asymptotic expected length (scaled by $a_K > 0$). Since the tail index ξ is unknown, we focus on the CI's asymptotic weighted length, given by

$$\int_{\xi \in \Xi} E_\xi[\kappa_\xi(\tilde{\mathbf{Z}}) \lg(\tilde{U}(\tilde{\mathbf{Z}}))]dW(\xi), \tag{3.11}$$

where W is a user-defined weight function. We can combine both objectives by choosing the CI for μ_K that minimizes the asymptotic weighted length in (3.11) subject to asymptotic validity condition in (3.10).

Formally, let $\mathbb{U}$ denote the collection of CIs that satisfy conditions (a)-(c) in Theorem 3.1. Then, we propose choosing $\tilde{U}$ in (3.8) as the solution to the following problem:

$$\arg\min_{\tilde{U}\in\mathbb{U}} \int_{\xi\in\Xi} E_\xi[\kappa_\xi(\tilde{\mathbf{Z}}) \lg(\tilde{U}(\tilde{\mathbf{Z}}))]dW(\xi) \quad \text{s.t.} \;\; P_\xi(Y_\mu \in \tilde{U}(\tilde{\mathbf{Z}})) \;\geq\; 1-\alpha \quad \text{for all } \xi \in \Xi. \qquad (3.12)$$

Following Müller and Wang (2017), we write down (3.12) in its Lagrangian form:

$$\min_{\tilde{U}\in\mathbb{U}} \int_{\xi\in\Xi} E_\xi[\kappa_\xi(\tilde{\mathbf{Z}}) \lg(\tilde{U}(\tilde{\mathbf{Z}}))]dW(\xi) + \int_{\xi\in\Xi} P_\xi(Y_\mu \in \tilde{U}(\tilde{\mathbf{Z}}))d\Lambda(\xi), \qquad (3.13)$$

where the non-negative measure Λ denotes the Lagrangian weights chosen to guarantee the asymptotic size constraint in (3.12). If we ignore the constraints in $\mathbb{U}$, the solution to (3.13) is given by the following set: for every $h \in \Sigma$,

$$\tilde{U}(h) \;=\; \left\{ y \in \mathbb{R} : \int_{\xi\in\Xi} \kappa_\xi(h) f_{\tilde{\mathbf{Z}}|\xi}(h)dW(\xi) \leq \int_{\xi\in\Xi} f_{(Y_\mu,\tilde{\mathbf{Z}})|\xi}(y,h)d\Lambda(\xi) \right\}, \qquad (3.14)$$

where for every $(y,h) \in \mathbb{R}\times\Sigma$, $\kappa_\xi(h)f_{\tilde{\mathbf{Z}}|\xi}(h)$ and $f_{(Y_\mu,\tilde{\mathbf{Z}})|\xi}(y,h)$ are given in Section A.3, and the integrals in (3.14) can be numerically calculated by Gaussian quadrature. We can verify numerically that $\tilde{U}(h)$ in (3.14) takes the form of an interval whose length varies continuously with $h \in \Sigma$.[2] Under these conditions, Lemma A.3 shows that $\tilde{U}(h)$ in (3.14) belongs to $\mathbb{U}$ and, therefore, $\tilde{U}(h)$ in (3.14) solves (3.13).

By combining (3.8), (3.14), and the arguments in the previous paragraph, we propose:

$$U(\mathbf{P}) \;=\; (P_{(n)} - P_{(1)}) \times \left\{ y \in \mathbb{R} : \int_{\xi\in\Xi} \kappa_\xi(\tilde{\mathbf{P}}) f_{\tilde{\mathbf{Z}}|\xi}(\tilde{\mathbf{P}})dW(\xi) \leq \int_{\xi\in\Xi} f_{(Y_\mu,\tilde{\mathbf{Z}})|\xi}(y,\tilde{\mathbf{P}})d\Lambda(\xi) \right\}. \qquad (3.15)$$

Under the conditions in the previous paragraph, Theorem 3.1 implies that (3.15) belongs to the class of asymptotically valid CIs and minimizes the asymptotic expected length (scaled by $a_K > 0$) within this class.

To calculate (3.15), the only remaining challenge is to find appropriate Lagrangian weights Λ that ensure asymptotic validity in the limiting problem, as described in (3.10). We tackle this challenge using a numerical approach developed by Elliott, Müller, and Watson (2015). It is relevant to emphasize that these Lagrangian weights depend solely on the value of n and, as a result, they only need to be computed once. For more information on calculating these weights, please refer to Section A.3.

[2] Given the intricate nature of the densities, we were unable to prove this result analytically.

3.2 Inference about the seller's expected revenue

We now conduct inference on the average of the seller's expected revenue based on transaction prices. Since bidders bid their valuation and the transaction price is the second-highest bid, we get that the seller's expected revenue in auction $j = 1, \ldots, n$ is $E[P_j] = E[V_{(2),j}]$, whose average is

$$\pi_K \; = \; \frac{1}{n}\sum_{j=1}^{n} E[V_{(2),j}]. \tag{3.16}$$

Given transaction prices $\mathbf{P}$, we consider a CI for π_K given by

$$U(\mathbf{P}) \; \equiv \; (P_{(n)} - P_{(1)}) \times \tilde{U}(\tilde{\mathbf{P}}) + P_{(1)}, \tag{3.17}$$

where $\tilde{\mathbf{P}} \in \Sigma$ are the sorted and self-normalized transaction prices in (3.4), and $\tilde{U} : \Sigma \to \mathcal{P}(\mathbb{R})$ is a CI defined on $\tilde{\mathbf{P}}$. By (3.17), the CI $U(\mathbf{P})$ is invariant to the sorting of $\mathbf{P}$, and equivariant to their location and scale.

The remainder of this section denotes $Z_{2,j} = H_\xi(E_{1,j} + E_{2,j})$, $\{(E_{1,j}, E_{2,j}) : j = 1, \ldots, n\}$ are i.i.d. standard exponential random variables and H_ξ is as in (2.4). We also use $\mathbf{Z} = \{Z_{2,j} : j = 1, \ldots, n\}$, $Z_{(n)} = \max\{Z_{2,j} : j = 1, \ldots, n\}$, $Z_{(1)} = \min\{Z_{2,j} : j = 1, \ldots, n\}$, for $j = 1, \ldots, N = n-2$,

$$\tilde{Z}_j = \begin{cases} \frac{Z_{(j+1)} - Z_{(1)}}{Z_{(n)} - Z_{(1)}} & \text{if } Z_{(n)} > Z_{(1)}, \\ 0 & \text{if } Z_{(n)} = Z_{(1)}, \end{cases}$$

and $\tilde{\mathbf{Z}} = \{\tilde{Z}_j : j = 1, \ldots, N\} \in \Sigma$. Finally, let $Y_\pi \equiv E[Z_{2,1}]/(Z_{(n)} - Z_{(1)})$ and $\kappa_\xi(\tilde{\mathbf{Z}}) = E[Z_{(n)} - Z_{(1)} | \tilde{\mathbf{Z}}]$. As in the previous section, the distributions of these random variables are fully characterized by its tail index ξ. In particular, Lemma A.5 shows that $E[Z_{2,1}] = (\Gamma(2-\xi) - 1)/\xi$ if $\xi \neq 0$, and $-1 + \bar{\gamma}$ if $\xi = 0$, where Γ is the standard Gamma function and $\bar{\gamma} \approx 0.577$ is the Euler's constant. For the remainder of this section, we will use P_ξ and E_ξ to refer to the probability and expectation associated with this distribution. The next result provides the asymptotic properties of the CI in (3.17) as $K \to \infty$.

Theorem 3.2. *Assume (2.1) holds, and that for some $\varepsilon > 0$ with $(1+\varepsilon)\xi < 1$, $E[|V_{i,j}|^{1+\varepsilon}] < \infty$ for all $i = 1 \ldots, K_j$ in auction $j = 1, \ldots, N$. Finally, assume that the CI for π_K, $U(\mathbf{P})$, is as in (3.17) with $\tilde{U} : \Sigma \to \mathcal{P}(\mathbb{R})$ that satisfies the following conditions:*

(a) $P_\xi(\{Y_\pi, \tilde{\mathbf{Z}}\} \in \partial\{(y, h) \in \mathbb{R} \times \Sigma : y \in \tilde{U}(h)\}) = 0$, where ∂A denotes the boundary of A.

(b) $\lg(\tilde{U}(h)) < \infty$ for any $h \in \Sigma$, where $\lg(A)$ denotes the length of A.

(c) For any sequence $\{h_\ell \in \Sigma\}_{\ell \in \mathbb{N}}$ with $h_\ell \to h \in \Sigma$, $\lg(\tilde{U}(h_\ell)) \to \lg(\tilde{U}(h))$.

Then, as $K \to \infty$,

1. $P(\pi_K \in U(\mathbf{P})) \;\to\; P_\xi(Y_\pi \in \tilde{U}(\tilde{\mathbf{Z}}))$,

2. $E[\lg(U(\mathbf{P}))]/a_K \;\to\; E_\xi[\kappa_\xi(\tilde{\mathbf{Z}}) \lg(\tilde{U}(\tilde{\mathbf{Z}}))]$.

Following the ideas in Section 3.2, we choose our CI based on the asymptotic behavior in Theorem 3.2. In particular, we propose to choose $\tilde{U}$ in (3.17) to minimize the asymptotic weighted length of the CI subject to asymptotic validity, given by

$$\arg\min_{\tilde{U}\in\mathbb{U}} \int_{\xi\in\Xi} E_\xi[\kappa_\xi(\tilde{\mathbf{Z}}) \lg(\tilde{U}(\tilde{\mathbf{Z}})))]dW(\xi) \quad \text{s.t.} \quad P_\xi(Y_\pi \in \tilde{U}(\tilde{\mathbf{Z}})) \;\geq\; 1-\alpha \quad \text{for all } \xi \in \Xi, \tag{3.18}$$

where $\mathbb{U}$ denotes the collection of CIs that satisfy the conditions in Theorem 3.2 and W is a user-defined weight function. The solution to (3.18) is as follows: for every $h \in \Sigma$,

$$\tilde{U}(h) \;=\; \left\{ y \in \mathbb{R} : \int_{\xi\in\Xi} \kappa_\xi(h) f_{\tilde{\mathbf{Z}}|\xi}(h) dW(\xi) \leq \int_{\xi\in\Xi} f_{(Y_\pi,\tilde{\mathbf{Z}})|\xi}(y,h) d\Lambda(\xi) \right\}, \tag{3.19}$$

where, for every $(y,h) \in \mathbb{R}\times\Sigma$, $\kappa_\xi(h) f_{\tilde{\mathbf{Z}}|\xi}(h)$ and $f_{(Y_\pi,\tilde{\mathbf{Z}})|\xi}(y,h)$ are given in Section A.3.

By combining (3.8), (3.14), and the arguments in the previous paragraph, we propose:

$$\begin{aligned} U(\mathbf{P}) \;=\;& P_{(1)} + (P_{(n)} - P_{(1)}) \\ &\times \left\{ y \in \mathbb{R} : \int_{\xi\in\Xi} \kappa_\xi(\tilde{\mathbf{P}}) f_{\tilde{\mathbf{Z}}|\xi}(\tilde{\mathbf{P}}) dW(\xi) \leq \int_{\xi\in\Xi} f_{(Y_\pi,\tilde{\mathbf{Z}})|\xi}(y,\tilde{\mathbf{P}}) d\Lambda(\xi) \right\}, \end{aligned} \tag{3.20}$$

Our derivations establish that (3.20) belongs to the class of asymptotically valid CIs and minimizes the asymptotic expected length (scaled by $a_K > 0$) within this class.

3.3 Inference about the tail index

A key parameter in our asymptotic framework is the tail index $\xi \in \Xi$. The goal of this section is to conduct inference on this parameter. That is, we are interested in the following hypothesis test:

$$H_0 : \xi \in \xi_0 \quad \text{v.s.} \quad H_1 : \xi \in \Xi_1, \tag{3.21}$$

where ξ_0 is a fixed parameter value in Ξ and $\Xi_1 = \Xi \backslash \{\xi_0\}$.

We divide this section into three subsections. Section 3.3.1 considers the situation where the alternative hypothesis in (3.21) is simple, while Section 3.3.2 explores the case where this alternative hypothesis is composite. Finally, Section 3.3.3 applies the methods in previous sections to test the standard regularity conditions in auction models.

3.3.1 Simple alternative hypothesis

This section considers the following inference problem:

$$H_0 : \xi = \xi_0 \quad \text{v.s.} \quad H_1 : \xi = \xi_1, \tag{3.22}$$

where ξ_0 and ξ_1 are distinct parameter values.

By the Neyman-Pearson Lemma, a natural starting point is the likelihood ratio test for the sorted and self-normalized version of our data. While the likelihood ratio test is unknown in finite samples, its limiting distribution is provided in Lemma 3.1. This idea yields the following testing procedure:

$$\varphi^*(\mathbf{Z}) \equiv 1\big[f_{\tilde{\mathbf{Z}}|\xi_1}(\tilde{\mathbf{Z}})/f_{\tilde{\mathbf{Z}}|\xi_0}(\tilde{\mathbf{Z}}) > q(\xi_0, \xi_1, \alpha)\big], \tag{3.23}$$

where the critical value is given by

$$q(\xi_0, \xi_1, \alpha) \equiv (1 - \alpha)\text{-quantile of } f_{\tilde{\mathbf{Z}}|\xi_1}(\tilde{\mathbf{Z}}_0)/f_{\tilde{\mathbf{Z}}|\xi_0}(\tilde{\mathbf{Z}}_0)$$

and $\tilde{\mathbf{Z}}_0$ is distributed according to $f_{\tilde{\mathbf{Z}}|\xi_0}$. The Neyman-Pearson Lemma implies that (3.23) is the most powerful level-α test in the limiting problem.

Following the guidance of the asymptotic analysis, our candidate for optimal test follows from replacing in (3.23) the limiting random variable $\tilde{\mathbf{Z}}$ with its data analog $\tilde{\mathbf{P}}$, i.e.,

$$\varphi_K^*(\mathbf{P}) \equiv 1\big[f_{\tilde{\mathbf{Z}}|\xi_1}(\tilde{\mathbf{P}})/f_{\tilde{\mathbf{Z}}|\xi_0}(\tilde{\mathbf{P}}) > q(\xi_0, \xi_1, \alpha)\big]. \tag{3.24}$$

By Lemma 3.1 and standard convergence arguments, (3.24) is asymptotically valid, i.e.,

$$\lim_{K\to\infty} E_{\xi_0}[\varphi_K^*(\mathbf{P})] \leq \alpha, \tag{3.25}$$

where E_ξ denotes the expectation with respect to distribution with tail index ξ. In fact, (3.24) is asymptotically level α, i.e., (3.25) holds with equality. More interestingly, we can leverage Müller (2011, Theorem 1) to establish that (3.24) is efficient in the sense of being the asymptotically most powerful test in the class of asymptotically valid and equivariant tests. Formally, for any asymptotically valid test $\varphi_K(\mathbf{P})$ (i.e., (3.25) holds with $\varphi_K^*(\mathbf{P})$ replaced by $\varphi_K(\mathbf{P})$), $\limsup_{K\to\infty} E_{\xi_1}[\varphi_K(\mathbf{P})] \leq \limsup_{K\to\infty} E_{\xi_1}[\varphi_K^*(\mathbf{P})]$. We record these conclusions in the following result.

Theorem 3.3. *Assume (2.1) holds. In the hypothesis testing problem in (3.22), the test defined by (3.24) satisfies the following properties:*

1. *It is asymptotically valid and level α, i.e., (3.25) holds with equality.*

2. *It is asymptotically efficient.*

3.3.2 Composite alternative hypothesis

We now turn our attention to the case in which the alternative hypothesis in (3.21) is composite. Following the ideas used in Section 3.3.1, we consider the feasible version of the efficient test in the limiting problem. Unfortunately, the limiting problem does not lend itself to the usual tools to develop uniformly most powerful tests.[3] For this reason, we consider tests that maximize the weighted average power criterion (WAP), following the approach of Wald (1943); Andrews and Ploberger (1994). To this end, let W denote a user-defined weight function on Ξ_1, which the researcher chooses to reflect the importance attached to the various alternative hypotheses. The weighting function W effectively transforms the composite alternative into a simple one, allowing us to focus on the weighted average power:

$$\text{WAP}_K(\varphi_K) \ \equiv \ \int_{\Xi_1} E_\xi[\varphi_K(\tilde{\mathbf{P}})]dW(\xi).$$

As in the previous section, we use the asymptotic behavior to guide the construction of our hypothesis test. The likelihood ratio test in the limiting problem applied to its data $\tilde{\mathbf{P}}$ is given by

$$\varphi^*_K(\mathbf{P}) \ \equiv \ 1\left[\int_{\Xi_1} f_{\tilde{\mathbf{Z}}|\xi}(\tilde{\mathbf{P}})dW(\xi)/f_{\tilde{\mathbf{Z}}|\xi_0}(\tilde{\mathbf{P}}) > q(\xi_0, W, \alpha)\right], \tag{3.26}$$

where the critical value is

$$q(\xi_0, W, \alpha) \ \equiv \ (1-\alpha)\text{-quantile of} \int_{\Xi_1} f_{\tilde{\mathbf{Z}}|\xi_0}(\tilde{\mathbf{Z}}_\mathbf{0})dW(\xi)/f_{\tilde{\mathbf{Z}}|\xi_0}(\tilde{\mathbf{Z}}_\mathbf{0}),$$

and $\tilde{\mathbf{Z}}_0$ is distributed according to $f_{\tilde{\mathbf{Z}}|\xi_0}$. By standard asymptotic arguments, we can establish that (3.26) is asymptotically valid and level α, i.e.,

$$\lim_{K\to\infty} E_{\xi_0}[\varphi^*_K(\mathbf{P})] \ = \ \alpha. \tag{3.27}$$

Moreover, (3.26) is efficient in the sense of maximizing the asymptotic weighted average power criterion in the class of asymptotically valid and equivariant tests. Formally, for any test $\varphi_K(\mathbf{P})$ that is asymptotically valid (i.e., (3.27) holds with φ^*_K replaced by φ_K), then $\limsup_{K\to\infty} \text{WAP}_K(\varphi_K) \ \leq \ \limsup_{K\to\infty} \text{WAP}_K(\varphi^*_K)$. The next result records these conclusions.

Theorem 3.4. *Assume* (2.1) *holds. In the hypothesis testing problem in* (3.21)*, the test defined by* (3.26) *satisfies the following properties:*

1. *It is asymptotically valid and level* α*, i.e.,* (3.27) *holds.*

2. *It is asymptotically efficient.*

[3] In particular, the likelihood ratio statistic is not monotonic, rendering the results in Lehmann and Romano (2005, Section 3.4) inapplicable.

3.3.3 Testing the regularity conditions in the auction literature

The auctions literature routinely assumes the regularity condition that f_V is continuous, and that there is a finite maximum valuation (i.e., $v_H < \infty$) with $f_V(v_H) > 0$. For examples of this, see Maskin and Riley (1984), Guerre et al. (2000), and Guerre and Luo (2022). Within our asymptotic framework, the next result shows that these regularity conditions imply that (2.1) holds with $\xi = -1$.

Lemma 3.2. *Assume that $f_V(v) \to f_V(v_H) > 0$ as $v \uparrow v_H < \infty$. Then, (2.1) holds with $\xi = -1$.*

In light of Lemma 3.2, we can test the regularity conditions used in the auction literature via the following hypothesis test:

$$H_0 : \xi = -1 \quad \text{v.s.} \quad H_1 : \xi \in \Xi_1, \tag{3.28}$$

where $\Xi_1 = \Xi \backslash \{-1\}$. In the remainder of this section, we will argue that $\Xi = [-1, 0.5]$ is a suitable choice for the parameters space for ξ. Since (3.28) is a special case of (3.21) with $\Xi_1 = (-1, 0.5]$, we can implement this test using the procedure in (3.26). For concreteness, we consider uniform weight $W(\xi) = 1[\xi \in (-1, 0.5]]$. By Theorem 3.4, our proposed test is asymptotically valid, level-α, and efficient.

We now justify choosing $\Xi = [-1, 0.5]$ as the parameter space for the test. We rely on the so-called von Mises' condition to interpret the different values of ξ. de Haan and Ferreira (2006, Theorem 1.1.8) states this condition and shows that it is a sufficient condition for (2.1). Under the von Mises' condition and that f_V' is bounded, we have three possible cases:

1. $f_V(v) \to f_V(v_H) > 0$ as $v \to v_H < \infty$ implies that $\xi = -1$.

2. $f_V(v) \to f_V(v_H) = 0$ as $v \to v_H < \infty$ implies that $\xi \in (-1, 0]$.

3. $f_V(v) \to f_V(v_H) = 0$ as $v \to v_H = \infty$ implies that $\xi > 0$.

A few remarks are in order. First, as expected, Case 1 aligns with the findings of Lemma 3.2. Second, we note that de Haan and Ferreira (2006, Theorem 2.1.2) shows that $v_H < \infty$ if and only if $\xi \leq 0$. Cases 2 and 3 then follow from derivations in de Haan and Ferreira (2006, page 18) and Falk, Hüsler, and Reiss (1994, Theorem 2.1.2). Since these three cases are exhaustive, we deduce that $\xi \geq -1$. Finally, if we also impose that V has second moments, de Haan and Ferreira (2006, page 176) implies that $\xi \leq 1/2$. Combining these restrictions, we conclude that $\Xi = [-1, 0.5]$ is a suitable parameter space for ξ.

Figure 1 presents the asymptotic rejection probabilities of (3.26) for the hypothesis testing problem in (3.28). The proposed test is asymptotically valid under $H_0 : \xi = -1$ and has nontrivial power properties under H_1, with asymptotic rejection rates that increase when either ξ or n increase.

It is worth noting that Figure 1 only shows the asymptotic properties (as $K \to \infty$) of our test. We explore the finite sample properties of our methodology via simulations in Section 6.

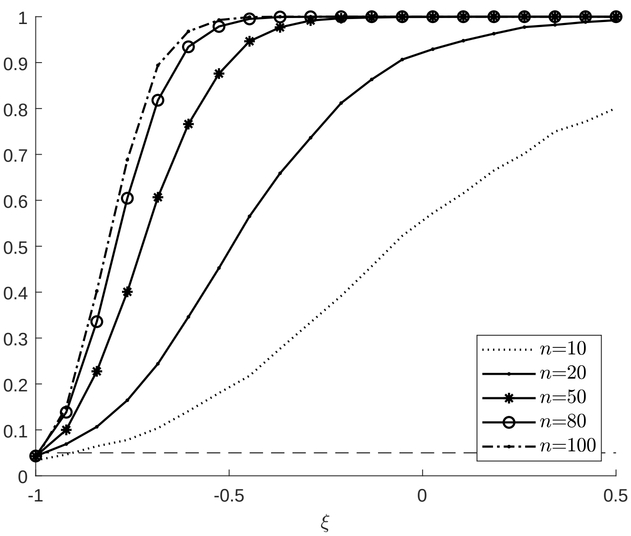

Figure 1: Asymptotic rejection probabilities of the hypothesis testing procedure in (3.26) with $\alpha = 5\%$ for the hypothesis testing problem in (3.28).

4 First-price auctions

We now consider first-price sealed-bid actions, in which the highest bidder gets the object and pays the highest bid. This type of auction is strategically equivalent to an open descending price (or Dutch) auction (see Krishna (2009, page 4)). By standard arguments (e.g., Krishna (2009, Proposition 2.2)), the symmetric equilibrium strategy for a bidder with valuation v in an auction with K_j participants is to bid $\beta_j(v) \equiv E[\check{V}_{(1),j}|\check{V}_{(1),j} < v]$, where $\check{V}_{(1),j}$ denotes the highest bid among the remaining $(K_j - 1)$ participants. Note that

$$\beta_j(v) \overset{(1)}{=} \frac{K_j - 1}{F_V(v)^{K_j-1}} \int_{v_L}^{v} u F_V(u)^{K_j-2} f_V(u) du \overset{(2)}{=} v - \frac{\int_{v_L}^{v} F_V(u)^{K_j-1} du}{F_V(v)^{K_j-1}}, \tag{4.1}$$

where (1) holds by computing $E[\check{V}_{(1),j}|\check{V}_{(1),j} < v]$ using the fact that the remaining $(K_j - 1)$ participants have a common valuation distribution with PDF f_V, and (2) by integration by parts. Since $\beta(v)$ is increasing, the auction is won by the highest-valuation bidder, with a transaction price equal to

$$P_j = V_{(1),j} - \frac{\int_{v_L}^{V_{(1),j}} F_V(u)^{K_j-1} du}{F_V(V_{(1),j})^{K_j-1}}. \tag{4.2}$$

Lemma A.7 uses (4.2) to deduce that

$$\left\{\frac{P_j - b_K}{a_K} : j = 1, \ldots, n\right\} \xrightarrow{d} \{X_j : j = 1, \ldots, n\}, \tag{4.3}$$

where, for each $j = 1, \ldots, n$,

$$X_j \equiv H_\xi(E_{1,j}) - \frac{\int_{-\infty}^{H_\xi(E_{1,j})} G_\xi(h)dh}{G_\xi(H_\xi(E_{1,j}))}, \tag{4.4}$$

with G_ξ is as in (2.2), and $\{(a_K, b_K) \in \mathbb{R}_{++} \times \mathbb{R} : K \in \mathbb{N}\}$, $\{E_{1,j} :\ j = 1, \ldots, n\}$, and H_ξ are as specified in Lemma 2.1.

As in Section 3, the statement in (4.3) cannot be directly used for inference as it requires the unknown normalizing constants $\{(a_K, b_K) \in \mathbb{R}_{++} \times \mathbb{R} : K \in \mathbb{N}\}$. Nevertheless, we can reiterate the idea of considering sorted and self-normalized prices in (3.4); for $j = 1, \ldots, N = n - 2 \geq 1$, let

$$\tilde{P}_j \equiv \begin{cases} \frac{P_{(j+1)} - P_{(1)}}{P_{(n)} - P_{(1)}} & \text{if } P_{(n)} > P_{(1)} \\ 0 & \text{if } P_{(n)} = P_{(1)}, \end{cases} \tag{4.5}$$

and let $\tilde{\mathbf{P}} = \{\tilde{P}_j : j = 1, \ldots, N\} \in \Sigma$. The next result characterizes the asymptotic distribution of $\tilde{\mathbf{P}}$ as $K \to \infty$.

Lemma 4.1. *Assume* (2.1) *holds. For any* $N \in \mathbb{N}$, *and as* $K \to \infty$,

$$\tilde{\mathbf{P}} = \{\tilde{P}_j : j = 1, \ldots, N\} \xrightarrow{d} \tilde{\mathbf{X}} = \{\tilde{X}_j : j = 1, \ldots, N\}, \tag{4.6}$$

where $\tilde{\mathbf{X}} = \{\tilde{X}_j : j = 1, \ldots, N\} \in \Sigma$ *is obtained as follows: for* $j = 1, \ldots, N = n - 2 \geq 1$,

$$\tilde{X}_j \equiv \begin{cases} \frac{X_{(j+1)} - X_{(1)}}{X_{(n)} - X_{(1)}} & \text{if } X_{(n)} > X_{(1)} \\ 0 & \text{if } X_{(n)} = X_{(1)}, \end{cases} \tag{4.7}$$

with $\{X_j : j = 1, \ldots, n\}$ *is i.i.d. according to* (4.4).

Given that the random variable in (4.4) is informative about the tail EV index ξ, Lemma 4.1 implicitly reveals that the asymptotic distribution of $\tilde{\mathbf{P}}$ can be used to conduct inference on functions of ξ. From this point onward, the remainder of this section is analogous to that of Section 3. The main difference is that the explicit PDF of the asymptotic distribution of $\tilde{\mathbf{P}}$ in Lemma 3.1 is replaced by the implicit distribution in Lemma 4.1.

4.1 Inference about the winner's expected utility

Our goal is to conduct inference on the average of the winner's expected utility using the transaction prices. Since bidders act according to (4.1), the auction is won by the bidder with the highest valuation. From here, we conclude that the winner's expected utility in auction $j = 1, \ldots, n$ is $E[V_{(1),j} - P_j]$, whose average is

$$\mu_K = \frac{1}{n}\sum_{j=1}^{n} E\Big[\frac{\int_{-\infty}^{V_{(1),j}} F_V(u)^{K_j-1}du}{F_V(V_{(1),j})^{K_j-1}}\Big]. \tag{4.8}$$

By the Revenue Equivalence Theorem (e.g., Krishna (2009, Section 3)), μ_K coincides with the average of the winner's expected utility in second-price auctions.

The construction of the CI for μ_K closely follows the arguments and derivations presented in Section 3.1. In particular, we propose a CI for μ_K given by

$$U(\mathbf{P}) \;=\; (P_{(n)} - P_{(1)}) \times \Big\{ y \in \mathbb{R} : \int_{\xi\in\Xi} \kappa_\xi(\tilde{\mathbf{P}}) f_{\tilde{\mathbf{X}}|\xi}(\tilde{\mathbf{P}}) dW(\xi) \le \int_{\xi\in\Xi} f_{(Y_\mu,\tilde{\mathbf{X}})|\xi}(y, \tilde{\mathbf{P}}) d\Lambda(\xi) \Big\}, \tag{4.9}$$

where $\tilde{\mathbf{P}} = \{\tilde{P}_j : j = 1, \ldots, N\}$ with $\tilde{P}_j$ as in (4.5), $\tilde{\mathbf{X}} = \{\tilde{X}_j : j = 1, \ldots, N\}$ as in (4.7), $\kappa_\xi(h) = E[X_{(n)} - X_{(1)}|\tilde{\mathbf{X}} = h]$, and $Y_\mu = E[\int_{-\infty}^{X_{1,j}} G_\xi(u)du / G_\xi(X_{1,j})]/(X_{(n)} - X_{(1)})$. By repeating arguments in Section 3.2, (4.9) belongs to the class of asymptotically-valid CIs and minimizes the asymptotic expected length within this class. See Theorem A.1 in the appendix for a statement of this result.

Unfortunately, implementing the CI in (4.9) is considerably more challenging than in the case of second-price auctions. The main reason is that there is no closed-form expression for the PDF of $\tilde{\mathbf{X}}$ available for first-price auctions. Without these, we cannot easily compute $\kappa_\xi(h) f_{\tilde{\mathbf{X}}|\xi}(h)$ and $f_{(Y_\mu,\tilde{\mathbf{X}})|\xi}(y, h)$ for $(y, h) \in \mathbb{R} \times \Sigma$. To sidestep this issue, we use a numerical approximation to the problem based on a Taylor series expansion. See Section A.3 for a detailed explanation.

4.2 Inference about the seller's expected revenue

We now conduct inference on the average of the seller's expected revenue using transaction prices. Since bidders act according to (4.1) and the transaction price equals the highest bid, the average of the seller's expected revenue is given by

$$\pi_K \;=\; \frac{1}{n}\sum_{j=1}^{n} E\Big[V_{(1),j} - \frac{\int_{-\infty}^{V_{(1),j}} F_V(u)^{K_j-1}du}{F_V(V_{(1),j})^{K_j-1}}\Big]. \tag{4.10}$$

Reiterating arguments in Section 3.2, we propose a CI for μ_K given by

$$\begin{aligned} U(\mathbf{P}) \;=\; & P_{(1)} + (P_{(n)} - P_{(1)}) \\ & \times \Big\{ y \in \mathbb{R} : \int_{\xi\in\Xi} \kappa_\xi(\tilde{\mathbf{P}}) f_{\tilde{\mathbf{X}}|\xi}(\tilde{\mathbf{P}}) dW(\xi) \le \int_{\xi\in\Xi} f_{(Y_\pi,\tilde{\mathbf{X}})|\xi}(y, \tilde{\mathbf{P}}) d\Lambda(\xi) \Big\}, \end{aligned} \tag{4.11}$$

where $Y_\pi = E[X_{1,j} - \int_{-\infty}^{X_{1,j}} G_\xi(h)dh/G_\xi(X_{1,j})]/(X_{(n)} - X_{(1)})$ and the rest of the objects are as in Section 4.1. By the same arguments in Section 3.2, (4.9) belongs to the class of asymptotically valid CIs and minimizes the asymptotic expected length within this class. See Theorem A.2 in the appendix for a statement of this result.

Implementing the CI in (4.11) suffers from the computational issues elaborated in Section 4.1, which we avoid via numerical approximations. Once again, see Section A.3 for details.

4.3 Inference about the tail index

Finally, we consider the problem of inference about the tail index using transaction prices from first-price auctions. As in Section 3.3, we are interested in the following hypothesis test:

$$H_0 : \xi \in \xi_0 \quad \text{v.s.} \quad H_1 : \xi \in \Xi_1, \tag{4.12}$$

where ξ_0 is a fixed parameter value in Ξ and $\Xi_1 = \Xi \backslash \{\xi_0\}$. For brevity, we only focus on the case when the alternative hypothesis in (4.12) is composite. By our previous arguments, we propose using the feasible version of the generalized likelihood ratio test in the limiting problem, i.e.,

$$\varphi_K^*(\mathbf{P}) \;\equiv\; 1\left[\int_{\Xi_1} f_{\tilde{\mathbf{X}}|\xi}(\tilde{\mathbf{P}})dW(\xi)/f_{\tilde{\mathbf{X}}|\xi_0}(\tilde{\mathbf{P}}) > q(\xi_0, W, \alpha)\right], \tag{4.13}$$

where $\tilde{\mathbf{P}}$, $\tilde{\mathbf{X}}$, and $\kappa_\xi(h)$ are as in Section 4.1, W is the user-defined weight function on Ξ_1, and $q(\xi_0, W, \alpha)$ is the critical value of the likelihood ratio test in (4.12), i.e.,

$$q(\xi_0, W, \alpha) \;\equiv\; (1-\alpha)\text{-quantile of } \int_{\Xi_1} f_{\tilde{\mathbf{X}}|\xi_0}(\tilde{\mathbf{X}}_0)dW(\xi)/f_{\tilde{\mathbf{X}}|\xi_0}(\tilde{\mathbf{X}}_0).$$

and $\tilde{\mathbf{X}}_0$ is a random vector with PDF $f_{\tilde{\mathbf{X}}|\xi_0}$. By our previous arguments, it follows that (4.13) is asymptotically valid and asymptotically level α, and efficient in the sense of maximizing the asymptotic weighted average power criterion in the class of asymptotically valid and equivariant tests. See Theorem A.3 in the appendix for a statement of this result. The implementation of the hypothesis test in (4.13) suffers from the aforementioned computational issues, and is addressed by the approximation described in Section A.3.

To conclude, we note that the arguments presented in Section 3.3.3 indicate that the hypothesis test in (4.12) can be applied to test the standard regularity conditions in the auctions literature. This can be achieved by conducting the hypothesis test in (4.12) with $\xi_0 = -1$ and $\Xi_1 = (-1, 0.5]$.

5 Extensions

Our analysis thus far considers an IPV setup and does not allow the seller to use a reserve price. This section briefly explains how our analysis can be extended beyond these features.

5.1 Beyond the IPV setup

According to Section 2, the K_j bidders in auction $j = 1, \ldots, n$ have IPV distributed according to a common CDF F_V. The auctions may differ in the number of potential bidders, but these are assumed to coincide asymptotically in the sense that $K_j/K \to 1$ with $K \equiv \min\{K_1, \ldots, K_n\}$.

The IPV assumption may be restrictive in certain empirical settings. For example, consider the case where the K_j bids in auction j depend on an auction-specific feature A_j. In this context, it is plausible that auctions are independent and, conditional on A_j, the K_j bids in auction j are independent and distributed according to a common CDF $F_{V|A_j}$. This is known as the conditional IPV model, and it is an extension of the standard IPV setup that allows for heterogeneity across auctions and (unconditional) dependence among private values within an auction. See Li, Perrigne, and Vuong (2000) for a discussion of the conditional IPV model. As we now explain, it is possible to adapt our methodology to the conditional IPV setup.

First, consider the case in which the auction-specific features $\{A_j : j = 1, \ldots, n\}$ are observed. In this case, one can always apply our analysis to collections of auctions that have the same (or similar) feature value, which we refer to as clusters. One can apply the analysis to each cluster with more than three auctions without any modification. This extension illustrates one of the advantages of our methodology, as it does not require the number of auctions to diverge.

Second, consider the case in which $\{A_j : j = 1, \ldots, n\}$ are unobserved. If so, it is still possible to implement inference based on our asymptotic framework provided that we observe at least three bids from each action. The basic insight is to treat each auction as its own cluster (in the sense of the previous paragraph), which naturally satisfies the IPV model in Section 2. For illustration, we consider the second-price auctions, in which bidders declare their true valuations. Suppose that for auction $j = 1, \ldots, n$ we observe $m \geq 3$ bids $\{P_{(k_1),j}, P_{(k_2),j}, \ldots, P_{(k_m),j}\}$ for known indices $(k_1, k_2, \ldots, k_m) \in \mathbb{N}$, which are in increasing order without loss of generality (i.e., $k_1 < k_2 < \ldots < k_m$). These bids need not be consecutive (i.e., k_{u+1} need not equal $k_u + 1$ for $u = 1, \ldots, m-1$) or include the maximum in the auction (i.e., k_m need not equal m). Given that the bids belong to the same auction, the unobserved auction-specific feature A_j is conditioned upon in these data. By repeating the arguments in Lemma 2.1, it follows that, as $K \to \infty$,

$$\Big(\frac{P_{(k_1),j} - b_{K,j}}{a_{K,j}}, \cdots, \frac{P_{(k_m),j} - b_{K,j}}{a_{K,j}}\Big) \overset{d}{\to} \left(H_{\xi_j}\left(\sum_{s=1}^{k_1} E_{s,j}\right), \cdots, H_{\xi_j}\left(\sum_{s=1}^{k_m} E_{s,j}\right)\right), \tag{5.1}$$

where $\{(a_{K,j}, b_{K,j}) \in \mathbb{R}_{++} \times \mathbb{R} : K \in \mathbb{N}\}$ are auction-specific normalizing constants, $H_{\xi_j}(\cdot)$ is defined in (2.4) with auction-specific EV index $\xi_j \equiv \xi(A_j)$, and $\{E_{s,j} : s = 1, \ldots, k_m\}$ are i.i.d. standard exponential random variables. Given these bids, we can construct the auction-specific self-normalized statistics: for $u = 1, \ldots, M \equiv m - 2 \geq 1$,

$$\tilde{P}_{u,j} \equiv \begin{cases} \frac{P_{(k_{u+1}),j} - P_{(k_1),j}}{P_{(k_m),j} - P_{(k_1),j}} & \text{if } P_{(k_m),j} > P_{(k_1),j} \\ 0 & \text{if } P_{(k_m),j} = P_{(k_1),j}, \end{cases}$$

and let $\tilde{\mathbf{P}}_j = \{\tilde{P}_{u,j} : u = 1, \ldots, M\} \in \Sigma \equiv \{h \in [0,1]^M : 0 \leq h_1 \leq \cdots \leq h_M \leq 1\}$. We can then derive the asymptotic distribution of $\tilde{\mathbf{P}}_j$ as $K \to \infty$ from (5.1). If we observe multiple bids from several independent auctions, we can combine the self-normalized statistics $\tilde{\mathbf{P}}_j$ for $j = 1, \ldots, J$ for further analysis. Since auctions are independent, the limiting distribution of the combined self-normalized statistics is the product of their limiting distributions. We can then test hypotheses about these auctions similar to the one in our paper. In addition, we can test the homogeneity of the J auctions (i.e., $\xi(A_j) = \xi$ for all $j = 1, \ldots, J$) by using a generalized likelihood ratio test.

5.2 Reserve price

Our analysis can be adapted to allow for the presence of a reserve price r set by the seller. For concreteness, we assume that $r \in (v_L, v_H)$. By setting a reserve price, the seller reserves the right not to sell the object if the price determined in the auction is below this price. Krishna (2009, Section 2.5) analyses the effect of the reserve price on the equilibrium bidding behavior. In second-price auctions, bidders still have a weakly dominant strategy to bid their valuation, i.e., (3.1) holds. In first-price auctions, the symmetric equilibrium strategy for a bidder with valuation v in an auction with K_j participants and reserve price r is to bid $\beta_j(v) \equiv E[\max\{\check{V}_{(1),j}, r\} | \check{V}_{(1),j} < v]$, where $\check{V}_{(1),j}$ denotes the highest bid among the remaining $(K_j - 1)$ participants. By repeating arguments used in Section 4, we can show that the resulting formula for $\beta_j(v)$ is as in (4.1) but with v_L replaced by r.

These results imply that reserve prices do not affect the asymptotic distribution of the transaction prices in our asymptotic framework. That is, reserve prices have no effect on Lemmas 3.1 and 4.1, or any of the subsequent asymptotic results. Since v_L does not appear on our asymptotic distributions, these remain unaltered when v_L is replaced by r. Intuitively, the asymptotic behavior of transaction prices with $K \to \infty$ is naturally focused on the upper tail of the valuation distribution (i.e., valuations in the neighborhood of v_H), which remains unaffected by the introduction of a reservation price $r < v_H$.

The fact that reserve prices do not affect the asymptotic analysis is a by-product of our limiting framework in which $K \to \infty$ and $r < v_H$. If one were interested in making reserve prices a salient feature in the asymptotic analysis, one would need to consider an asymptotic framework in which the reserve prices approach v_H as $K \to \infty$. (As with any other asymptotic analysis, this is not meant to be taken literally, but rather as an approximation to a finite-sample situation in which K is large and r is relatively close to v_H). We decided to omit the results under this alternative asymptotic framework for brevity, but these are available upon request.

6 Monte Carlo simulations

This section investigates the finite sample performance of our proposed inference methods using Monte Carlo simulations. We also compare their performance with other inference methods that rely on the standard asymptotic framework with a divergent number of auctions. For brevity, we concentrate on simulations for second-price auctions. The corresponding results for first-price auctions are provided in the supplementary material.

We first consider the problem of inference about the winner's expected utility using only transaction prices. Each simulated dataset consists of n independent second-price auctions where auction $j = 1, \ldots, n$ has K bidders with independent valuations distributed according to a distribution F_V. As explained in Section 3.1, our parameter of interest is the average winner's expected utility across auctions, given by $\mu_K = \frac{1}{n}\sum_{j=1}^{n} E[V_{(1),j} - V_{(2),j}]$. We conduct simulations with four distributions for F_V: (a) the standard uniform distribution over $[0, 3]$, (b) the absolute value of standard Normal, (c) the absolute value of Student-t(20), and (d) the Pareto distribution with exponent 0.25. These distributions satisfy condition (2.1) with $\xi = -1$, 0, 0.05, and 0.25, respectively.

In our simulations, we set the number of auctions to $n = 10$ or $n = 100$. For each one of these n auctions, we set the number of bidders to either $K = 10$, $K = 100$, or to a random number to a uniformly distributed integer in $\{90, 91, \ldots, 110\}$. This generates six possible designs for (n, K), and includes combinations that are better approximated by our asymptotic framework with a growing number of bidders (e.g., $n = 10$ and $K = 100$ or $K \in \{90, 91, \ldots, 110\}$), but also others that are better aligned to the traditional asymptotic framework with an increasing number of auctions (e.g., $n = 100$ and $K = 10$). Our results are based on $S = 500$ independent datasets. Finally, we set the significance level equal to $\alpha = 5\%$ throughout this section.

We now describe the three CIs that we consider for μ_K:

(i) This is our proposed CI in Section 3.1. The method is implemented by constructing $U(\mathbf{P})$ in (3.15) with $\Xi = [-1, 0.5]$ and W equal to the uniform distribution over this interval. As already explained, the validity of this CI is based on asymptotics as $K \to \infty$.

(ii) A CI based on observing the two highest bidders in each auction. We note that this approach is infeasible in our data setting (we only observe transaction prices), but we take it as a benchmark for any method that relies on the traditional asymptotics with $n \to \infty$. These data allow us to compute $D_j \equiv B_{(1),j} - B_{(2),j}$ for each auction $j = 1, \ldots, n$. In turn, this enables us to test $H_0 : \mu_K = b$ using the t-statistic $\sqrt{n}(\bar{D}_n - b)/s_n$ with $\bar{D}_n = \sum_{j=1}^{n} D_j/n$

and $s_n^2 = \sum_{j=1}^n (D_j - \bar{D}_n)^2/(n-1)$. One can then construct a CI by standard asymptotic approximations based on $n \to \infty$.

(iii) A bootstrap-based CI. To implement this idea, we first need to express μ_K as a function of the distribution of transaction prices (further details are provided in Section A.5). By replacing this distribution with a suitable estimator, one can consistently estimate μ_K. In addition, one can repeat this process using bootstrap samples to construct a CI for μ_K. The method is implemented with 500 bootstrap samples. The validity of this method relies on standard asymptotic arguments with $n \to \infty$. For references using this type of CI, see Bajari and Hortaçsu (2003) and Haile and Tamer (2003).

Table 1 presents the average coverage and length of the three CIs. We summarize the findings as follows. First, our proposed CI exhibits excellent coverage properties across all data configurations. As one would expect, in scenarios where K is significantly larger than n, our empirical CI coverage closely approximates the desired level. On the other hand, when n is considerably larger than K, our proposed CI exhibits slight overcoverage, especially when valuations are distributed according to $U(0,3)$. Second, the CI based on the two highest bids and $n \to \infty$ are relatively shorter and tend to suffer from slight undercoverage, especially when n is small or valuations are Pareto distributed. Finally, the bootstrap-based CI tends to suffer significant undercoverage problems. We attribute this to the fact that $n \in \{10, 100\}$ is insufficient to guarantee accuracy based on asymptotic arguments as $n \to \infty$.

Next, we consider the problem of hypothesis testing the value of the tail index based on transaction prices. Motivated by Section 3.3.3, we focus on $H_0 : \xi = -1$ vs. $H_1 : \xi \in (-1, 0.5]$, with W equal to the uniform distribution over this interval. As explained, the validity of this method is based on asymptotics as $K \to \infty$. We consider the same four distributions for F_V: (a) the standard uniform distribution over $[0,3]$, (b) the absolute value of standard Normal, (c) the absolute value of Student-t(20), and (d) the Pareto distribution with exponent 0.25. The first distribution satisfies $H_0 : \xi = -1$, while the other three belong to H_1 with $\xi = 0$, 0.05, and 0.25, respectively.

Table 2 presents the empirical rejection rate of the test proposed in Section 3.3.3 over 500 simulations using a nominal size of 5%. Under H_0 (i.e., when valuations are $U(0,3)$), our proposed test controls size as long as the number of bidders is not too small relative to the sample size. Under H_1 (i.e., when valuations are not $U(0,3)$), our methodology provides reasonable power.

7 Empirical illustration

This section illustrates our new inference method using auction data for vehicle license plates in Hong Kong. Since 1973, the Hong Kong government has employed standard oral ascending auctions to auction license plates. As mentioned in Section 3, this auction format is weakly equivalent to

# Bidders	$K = 10$				$K = 100$				$K \sim U\{90, 91, \ldots, 110\}$			
# Auctions	$n = 10$		$n = 100$		$n = 10$		$n = 100$		$n = 10$		$n = 100$	
Dist.	Cov	Lgth	Cov	Lgth	Cov	Lgth	Cov	Lgth	Cov	Lgth	Cov	Lgth
Method (i): Our proposed CI, asy. with $K \to \infty$												
$U(0, 3)$	0.98	0.72	0.96	0.09	0.98	0.30	0.99	0.13	0.97	0.30	0.98	0.13
$\lvert N(0, 1)\rvert$	0.95	1.54	0.97	0.29	0.94	1.20	0.98	0.23	0.92	1.21	0.97	0.23
$\lvert t(20)\rvert$	0.93	1.74	0.97	0.37	0.94	1.52	0.98	0.35	0.93	1.49	0.99	0.35
$Pa(0.25)$	0.96	1.45	0.99	0.62	0.95	2.62	0.98	1.11	0.95	2.62	0.98	1.10
Method (ii): CI based on two highest bids, asy. with $n \to \infty$												
$U(0, 3)$	0.86	0.28	0.94	0.10	0.83	0.03	0.93	0.01	0.88	0.03	0.88	0.01
$\lvert N(0, 1)\rvert$	0.89	0.46	0.94	0.16	0.88	0.35	0.96	0.12	0.88	0.35	0.95	0.12
$\lvert t(20)\rvert$	0.89	0.55	0.92	0.19	0.86	0.48	0.92	0.17	0.87	0.47	0.93	0.17
$Pa(0.25)$	0.81	0.76	0.90	0.30	0.80	1.34	0.90	0.51	0.78	1.35	0.88	0.51
Method (iii): CI based on bootstrap, asy. with $n \to \infty$												
$U(0, 3)$	0.66	0.22	0.93	0.08	0.67	0.03	0.95	0.01	0.68	0.03	0.94	0.01
$\lvert N(0, 1)\rvert$	0.04	0.22	0.23	0.11	0.05	0.15	0.20	0.08	0.05	0.15	0.18	0.08
$\lvert t(20)\rvert$	0.02	0.24	0.15	0.13	0.04	0.19	0.13	0.11	0.03	0.19	0.14	0.11
$Pa(0.25)$	0.00	0.17	0.06	0.15	0.01	0.31	0.05	0.25	0.00	0.30	0.05	0.25

Table 1: Empirical coverage frequency (Cov) and length (Lgth) of various CIs for the winner's expected utility μ_K in second-price auctions. The results are the average of 500 simulation draws and a nominal coverage level of 95%.

a second-price auction. The dataset was provided by Ng, Chong, and Du (2010), who obtained it from the Hong Kong Transport Department. The data include a detailed description of 292 license plate auctions conducted from 1997 to 2009, resulting in the sale of a total of 40,000 license plates. Given the substantial size of the auction venue and the rapid increase in the number of registered private cars in Hong Kong, it is reasonable to assume that a significant number of bidders participated in these auctions. On average, each one of these auctions sells more than a hundred different plates sequentially. In terms of our asymptotic framework, we treat each one of these license plates as an individual instance of a second-price auction with a large number of bidders.

While the dataset includes numerous individual instances of license plate auctions, there is significant heterogeneity among them. In fact, the main finding in Ng et al. (2010) is that certain plates are more valuable due to superstition. To account for this observed heterogeneity, we focus on ordinary license plates that meet specific criteria: letters are not 'HK', letters are not the same (e.g., not AA, BB, CC, etc.), the numbers on the plate are not in order (e.g., not 1369), or in

# Bidders	$K = 10$		$K = 20$		$K = 100$		$K \sim U\{90, 91, \ldots, 110\}$	
# Auctions	$n = 10$	$n = 100$	$n = 10$	$n = 100$	$n = 10$	$n = 100$	$n = 10$	$n = 100$
$U(0, 3)$	0.06	0.35	0.05	0.14	0.04	0.05	0.03	0.04
$\|N(0, 1)\|$	0.42	1.00	0.46	1.00	0.48	1.00	0.45	1.00
$\|t(20)\|$	0.46	1.00	0.48	1.00	0.54	1.00	0.51	1.00
$Pa(0.25)$	0.73	1.00	0.71	1.00	0.68	1.00	0.66	1.00

Table 2: Empirical rejection rate of the test proposed in Section 3.3.3 for $H_0 : \xi = -1$ in second-price auctions. The results are the average of 500 simulation draws and a nominal size of 5%.

reverse order (e.g., not 9631), the number part of the plate has 4 digits, and none of these digits is an 8 or a 4 (considered fortunate or unfortunate), and the transaction price exceeds the reserve price. After imposing these restrictions, we are left with 318 license plates sold from 1997 until 2008. These data are further divided by year, allowing valuation distributions to vary over time. This results in 12 separate datasets, one for each year from 1997 to 2008, with an average of 26.5 auctions per year. We assume that the selected auctions within each year are homogeneous, with a large and relatively constant number of bidders. Under these conditions, we can implement our methods for each one of these years.

Table 7 displays the confidence intervals for the expected utility of the winner and the p-values of the test (3.26) with a null hypothesis $H_0 : \xi = -1$ using data from each year. Our analysis reveals several interesting empirical observations. First, the expected utility of the winner is economically substantial, with the midpoints of the 95% confidence intervals ranging from approximately 1,500 to 7,000 HKDs (equivalent to 192 to 895 USD at the current exchange rate). Prior to 2006, the average confidence interval had a midpoint of approximately 3,500 HKDs and a width of around 4,500 HKDs. After 2006, these figures were reduced to 1,600 HKDs and 2,300 HKDs, respectively. We speculate that these differences could be attributed to the introduction of special license plates in these auctions in 2006. Third, the hypothesis of $\xi = -1$ is strongly rejected for all years. This suggests that the standard regularity conditions imposed by traditional inference methods in auction literature do not hold in this dataset.

8 Conclusions

This paper considers the problem inference for first-price or second-price sealed-bid auctions with many bidders with symmetric independent private values. In this context, we present a novel asymptotic framework where the number of bidders increases significantly while the number of auctions remains small and fixed. This approach differs from the more conventional approach with a divergent number of auctions, while the number of bidders remains fixed. We argue that our

year	n	95% CI for μ_K	p-value for $H_0 : \xi = -1$	year	n	95% CI for μ_K	p-value for $H_0 : \xi = -1$
1997	26	[2.23 , 9.23]	0.00	2003	46	[0.82 , 2.98]	0.00
1998	27	[1.71 , 7.29]	0.00	2004	12	[0.69 , 5.59]	0.00
1999	32	[1.22 , 5.72]	0.00	2005	22	[0.61 , 3.83]	0.00
2000	29	[1.58 , 6.63]	0.00	2006	17	[0.12 , 1.69]	0.00
2001	34	[1.23 , 4.71]	0.00	2007	31	[0.69 , 3.07]	0.00
2002	18	[1.02 , 6.30]	0.00	2008	24	[0.58 , 3.54]	0.00

Table 3: 95% CIs for the winner's expected utility (in 1,000 HKDs) and p-values for the test in (3.26) for $H_0 : \xi = -1$ in Hong Kong car license plate auctions.

results provide an accurate approximation in auction settings where the number of bidders is large relative to the number of auctions. This framework is especially well-suited for applications with substantial heterogeneity across auctions, where only a few truly homogeneous auctions exist.

Within our novel asymptotic framework, we introduce new inference methods for the expected utility of the auction winner, the expected revenue for the seller, and the tail behavior of the valuation distribution. We show that the latter can serve as a means to test the validity of the regularity conditions typically imposed in auction literature (e.g., bounded valuation support and nonzero density at the upper end of the support). Our data requirements are minimal; our tests only necessitate observing transaction prices from a fixed and finite number of auctions. That is, we do not require observing multiple bids from these auctions or the number of bidders.

Our methodology relies on the fact that, as the number of bidders grows, the transaction prices reveal the tail properties of the valuation distribution. This information is sufficient to provide asymptotically valid inference for the above-mentioned objects of economic interest. Within our asymptotic framework, our inference methods are shown to control size and have desirable power properties. We demonstrate through Monte Carlo simulations that our methodology provides an accurate approximation in finite samples.

A Appendix

The appendix is organized as follows. Section A.1 collects all of the proofs omitted from the main text. Section A.2 gives intermediate results. Section A.3 provides computational details omitted from the main text. Section A.4 presents additional Monte Carlo simulations, focusing on first-price auctions. Finally, Section A.5 provides auxiliary derivations related to the Monte Carlo simulations.

A.1 Proofs of results in the main text

Proof of Lemma 2.1. By independence across auctions, it suffices to prove that for any auction $j = 1, \dots n$ and as $K \to \infty$,

$$\big(\tfrac{V_{(1),j} - b_K}{a_K}, \dots, \ \tfrac{V_{(d),j} - b_K}{a_K}\big) \ \xrightarrow{d} \ \big(H_\xi(E_{1,j}), \dots, \ H_\xi\big(\textstyle\sum_{s=1}^{d} E_{s,j}\big)\big), \tag{A.1}$$

where $\{E_{s,j} : s = 1, \dots, d\}$ are i.i.d. standard exponential random variables.

Let x be any continuity point of G_ξ. Since $K_j \to \infty$ as $K \to \infty$, (2.1) implies that there is a sequence of constants $\{(a_{K_j}, b_{K_j}) \in \mathbb{R}_{++} \times \mathbb{R} : K_j \in \mathbb{N}\}$ such that

$$\lim_{K \to \infty} F(a_{K_j} x + b_{K_j})^{K_j} \ = \ G_\xi(x) \ \text{ as } K \to \infty. \tag{A.2}$$

Under (A.2), de Haan and Ferreira (2006, Theorem 2.1.1), implies that as $K \to \infty$,

$$\Psi_j \equiv \big(\tfrac{V_{(1),j} - b_{K_j}}{a_{K_j}}, \dots, \ \tfrac{V_{(d),j} - b_{K_j}}{a_{K_j}}\big) \ \xrightarrow{d} \ \big(H_\xi(E_{1,j}), \dots, \ H_\xi\big(\textstyle\sum_{s=1}^{d} E_{s,j}\big)\big). \tag{A.3}$$

Note that

$$\big(\tfrac{V_{(1),j} - b_K}{a_K}, \dots, \tfrac{V_{(d),j} - b_K}{a_K}\big) \ = \ \Psi_j \tfrac{a_{K_j}}{a_K} \ + \ \big(\tfrac{b_K - b_{K_j}}{a_K}, \dots, \tfrac{b_K - b_{K_j}}{a_K}\big). \tag{A.4}$$

By (A.3) and (A.4), (A.1) follows from showing that

$$\lim_{K \to \infty} \big(\tfrac{a_{K_j}}{a_K}, \ \tfrac{b_{K_j} - b_K}{a_K}\big) \ = \ (1, \ 0). \tag{A.5}$$

We devote the remainder of this proof to establish (A.5).

Let x be any continuity point of G_ξ. By (A.2) and $K_j/K \to 1$ as $K \to \infty$,

$$\lim_{K \to \infty} F(a_{K_j} x + b_{K_j})^{K} \ = \ G_\xi(x) \ \text{ as } K \to \infty. \tag{A.6}$$

Under (2.1) and (A.6), de Haan (1976, Lemma 1) implies that for all $y \in \mathbb{R}$,

$$G_\xi\big(\lim_{K \to \infty} \big(\tfrac{a_{K_j}}{a_K} y + \tfrac{b_{K_j} - b_K}{a_K}\big)\big) \ = \ G_\xi(y). \tag{A.7}$$

Regardless of ξ, there is a continuum of values for which G_ξ is strictly increasing and, thus, invertible. For any y in this continuum, (A.7) implies that

$$\lim_{K \to \infty} \tfrac{a_{K_j}}{a_K} y + \tfrac{b_{K_j} - b_K}{a_K} \ = \ y. \tag{A.8}$$

From this observation, (A.5) follows. ■

Proof of Lemma 3.1. This result follows from Lemma 2.1, Lemma A.1, and the continuous mapping theorem. ■

Proof of Theorem 3.1. Part 1. By (3.2) and Lemma 2.1, as $K \to \infty$,

$$\left\{\left(\tfrac{V_{(1),j}-P_j}{a_K}, \tfrac{P_j - b_K}{a_K}\right) : j = 1,\ldots,N\right\} \overset{d}{\to} \{(Z_{1,j} - Z_{2,j}, Z_{2,j}) : j = 1,\ldots,N\}. \tag{A.9}$$

Let $\delta > 0$ and $\bar{K} \in \mathbb{N}$ be as in Lemma A.2. Then, for all $j = 1,\ldots,N$ and $K > \bar{K}$,

$$E\left[\left|\tfrac{V_{(1),j}-P_j}{a_K}\right|^{1+\delta}\right] \overset{(1)}{=} E\left[\left|\tfrac{V_{(1),j}-V_{(2),j}}{a_K}\right|^{1+\delta}\right] \leq 2^{\delta}\left(E\left[\left|\tfrac{V_{(1),j}-b_K}{a_K}\right|^{1+\delta} + \left|\tfrac{V_{(2),j}-b_K}{a_K}\right|^{1+\delta}\right]\right) \overset{(2)}{<} \infty, \tag{A.10}$$

where (1) holds by (3.1) and (3.2), and (2) by Lemma A.2. By (3.7), (A.9), and (A.10), as $K \to \infty$,

$$\left\{\tfrac{\mu_K}{a_K}, \left\{\left(\tfrac{P_j - b_K}{a_K}\right) : j = 1,\ldots,N\right\}\right\} \overset{d}{\to} \{E[Z_{1,1} - Z_{2,1}], \{Z_{2,j} : j = 1,\ldots,N\}\}, \tag{A.11}$$

where we have used that $Z_{1,1} - Z_{2,1} \overset{d}{=} Z_{1,j} - Z_{2,j}$ for all $j = 1,\ldots,n$. Define

$$R_K \equiv \begin{cases} \frac{\mu_K}{P_{(n)} - P_{(1)}} & \text{if } P_{(n)} \neq P_{(1)}, \\ 0 & \text{if } P_{(n)} = P_{(1)}. \end{cases} \tag{A.12}$$

By (A.11), (A.12), Lemma A.1, and the continuous mapping theorem, we have that, as $K \to \infty$,

$$\{R_K, \tilde{\mathbf{P}}\} \overset{d}{\to} \{Y_\mu, \tilde{\mathbf{Z}}\}. \tag{A.13}$$

Next, consider the following derivation as $K \to \infty$.

$$\begin{aligned} P(\mu_K \in U(\mathbf{P})) &= P(\{\mu_K \in U(\mathbf{P})\} \cap \{P_{(n)} \neq P_{(1)}\}) + P(\{\mu_K \in U(\mathbf{P})\} \cap \{P_{(n)} = P_{(1)}\}) \\ &\overset{(1)}{=} P(\{\mu_K \in (P_{(n)} - P_{(1)}) \times \tilde{U}(\tilde{\mathbf{P}})\} \cap \{P_{(n)} \neq P_{(1)}\}) + o(1) \\ &\overset{(2)}{=} P(\{R_K \in \tilde{U}(\tilde{\mathbf{P}})\} \cap \{P_{(n)} \neq P_{(1)}\}) + o(1) \\ &\overset{(3)}{=} P(R_K \in \tilde{U}(\tilde{\mathbf{P}})) + o(1) \\ &= P(\{R_K, \tilde{\mathbf{P}}\} \in \{(y,h) \in \mathbb{R} \times \Sigma : y \in \tilde{U}(h)\}) + o(1) \\ &\overset{(4)}{\to} P(\{Y_\mu, \tilde{\mathbf{Z}}\} \in \{(y,h) \in \mathbb{R} \times \Sigma : y \in \tilde{U}(h)\}) \\ &= P(Y_\mu \in \tilde{U}(\tilde{\mathbf{Z}})), \end{aligned}$$

as desired, where (1) holds by (3.8) and Lemma A.1, (2) holds because (A.12) implies that $\{\mu_K \in (P_{(n)} - P_{(1)}) \times \tilde{U}(\tilde{\mathbf{P}})\} = \{R_K \in \tilde{U}(\tilde{\mathbf{P}})\}$ under $P_{(n)} \neq P_{(1)}$, (3) by Lemma A.1, and (4) by (A.13), Assumption (a), and the Portmanteau Theorem.

Part 2. By similar arguments as those used in the proof of Lemma 3.1, we have that, as $K \to \infty$,

$$\{(P_{(n)} - P_{(1)})/a_K, \tilde{\mathbf{P}}\} \overset{d}{\to} \{Z_{(n)} - Z_{(1)}, \tilde{\mathbf{Z}}\}. \tag{A.14}$$

By (A.14), Assumption (c), and the continuous mapping theorem, we have that, as $K \to \infty$,

$$((P_{(n)} - P_{(1)})/a_K) \times \lg(\tilde{U}(\tilde{\mathbf{P}})) \overset{d}{\to} (Z_{(n)} - Z_{(1)}) \times \lg(\tilde{U}(\tilde{\mathbf{Z}})). \tag{A.15}$$

Let $\delta > 0$ and $\bar{K} \in \mathbb{N}$ be as in Lemma A.2. Then,

$$\begin{aligned} |P_{(n)} - P_{(1)}|^{1+\delta} &\leq (\sum_{l,m=1}^{n} |P_l - P_m|)^{1+\delta} \\ &\overset{(1)}{=} (\sum_{l,m=1}^{n} |V_{(2),l} - V_{(2),m}|)^{1+\delta} \\ &\leq (2n)^{1+\delta} \sum_{l=1}^{n} |V_{(2),l} - b_K|^{1+\delta}, \end{aligned} \tag{A.16}$$

where (1) holds by (3.2). Then, for all $K > \bar{K}$,

$$E[|\tfrac{(P_{(n)} - P_{(1)})}{a_K} \lg(\tilde{U}(\tilde{\mathbf{P}}))|^{1+\delta}] \overset{(1)}{\leq} (2n)^{1+\delta} \sum_{l=1}^{n} E[|\tfrac{V_{(2),l} - b_K}{a_K}|^{1+\delta} |\lg(\tilde{U}(\tilde{\mathbf{P}}))|^{1+\delta}] \overset{(2)}{<} \infty, \tag{A.17}$$

where (1) holds by (A.16), and (2) by Assumption (b), $\lg(\tilde{U}(h)) \geq 0$ for all $h \in \Sigma$, and Lemma A.2.

Then, consider the following derivation as $K \to \infty$.

$$\begin{aligned} E[\lg(U(\mathbf{P}))/a_K] &\overset{(1)}{=} E[((P_{(n)} - P_{(1)})/a_K) \times \lg(\tilde{U}(\tilde{\mathbf{P}}))] \\ &\overset{(2)}{\to} E[(Z_{(n)} - Z_{(1)}) \times \lg(\tilde{U}(\tilde{\mathbf{Z}}))] \\ &= E_\xi[\kappa_\xi(\tilde{\mathbf{Z}}) \times \lg(\tilde{U}(\tilde{\mathbf{Z}}))], \end{aligned}$$

as desired, where (1) holds by (3.8), and (2) by (A.15) and (A.17). ■

Proof of Theorem 3.2. This proof follows from the same arguments as in Theorem 3.1. The main difference between the proofs is that the vector $\{(V_{(1),j} - P_j) : j = 1, \ldots, N\}$ is replaced by $\{(P_j - b_K) : j = 1, \ldots, N\}$, which then implies that $(\pi_K - b_K)$ and Y_μ are replaced by μ_K and Y_π, respectively. With these changes in place, the desired result follows from repeating the steps used to prove Theorem 3.1. ■

Proof of Theorem 3.3. This proof follows from applying Müller (2011, Theorem 1) based on the convergence result in Lemma 3.1. To apply these results in that paper, we specify the connection between our object and the relevant objects in that paper: the class of DGPs satisfying (2.1) take the role of $\mathcal{M}$, $K \to \infty$ takes the role of $T \to \infty$, $j = 1, \ldots, n$ the role of $i = 1, \ldots, n$, $\{\xi_0\}$ the role of Θ_0, $\{\xi_1\}$ the role of Θ_1, $\mathbf{P}$ the role of Y_T, $\tilde{\mathbf{P}}$ the role of X_T, Σ the role of S, the self-normalizing transformation $\{(P_j - P_{(1)})/(P_{(n)} - P_{(1)})\}_{j=1}^N : \mathbf{P} \to \Sigma$ the role of h_T, $\tilde{\mathbf{Z}}$ the role of X, $\tilde{\varphi}^*(\tilde{\mathbf{Z}}) \equiv \varphi^*(\mathbf{Z})$ the role of $\varphi_S(Y)$, $\varphi_K^*(\mathbf{P})$ the role of $\hat{\varphi}_T^*(\mathbf{P})$, $\varphi_K(\mathbf{P})$ the role of $\varphi_T(\mathbf{P})$, and $f_{\tilde{\mathbf{Z}}|\xi_0}$ the role of μ_P. Moreover, we set $\mathcal{F}_s$ equal to the set with the zero function and $\pi_0 = 0$, which effectively trivializes Müller (2011, Equations (5)-(6) and (10)-(11)).

By definition, $\tilde{\varphi}^*(\tilde{\mathbf{Z}}) : \Sigma \to [0,1]$ is a level-α test in the limiting problem, i.e., (3.25) holds. According to the Neyman-Pearson Lemma, it maximizes power in the limiting problem. Finally, note that $\tilde{\varphi}^*$ is continuous except for a set of zero $f_{\tilde{\mathbf{Z}}|\xi_0}$-measure. These conditions align with the requirements in Müller (2011, Theorem 1), and so our desired results follow immediately from its application. ■

Proof of Theorem 3.4. This proof follows from the same arguments as in Theorem 3.3. ■

Proof of Lemma 3.2. Fix $x > 0$ arbitrarily. By $v_H < \infty$ and de Haan and Ferreira (2006, Theorem 1.2.1), it suffices show that

$$\lim_{t \downarrow 0} \tfrac{1 - F_V(v_H - tx)}{1 - F_V(v_H - t)} = x.$$

To this end, consider the following derivation.

$$\lim_{t \downarrow 0} \tfrac{1 - F_V(v_H - tx)}{1 - F_V(v_H - t)} \overset{(1)}{=} \lim_{t \downarrow 0} \tfrac{x f_V(v_H - tx)}{f_V(v_H - t)} \overset{(2)}{=} x,$$

as desired, where (1) holds by L'Hôpital rule and (2) by $f_V(v) \to f_V(v_H) > 0$ as $v \uparrow v_H$. ■

Proof of Lemma 4.1. This result follows from Lemma A.1, Lemma A.7, and the continuous mapping theorem. ■

A.2 Auxiliary results

Lemma A.1. *For any $n > 1$ and for first-price or second-price auctions,*

$$P(P_{(n)} = P_{(1)}) \;=\; 0.$$

Proof. Note that

$$\{P_{(n)} = P_{(1)}\} \overset{(1)}{\subseteq} \bigcap_{l,m=1,\ldots,n, l \neq m} \; \bigcup_{i_l = 1,\ldots,K_l, i_m = 1,\ldots,K_m} \{V_{i_l,l} = V_{i_m,m}\}, \qquad \text{(A.18)}$$

where (1) holds by (3.2) and (4.1) for second-price and first-price auctions, respectively. Then,

$$P(P_{(n)} = P_{(1)}) \overset{(1)}{\leq} \min_{l,m=1,\ldots,n,l\neq m} \sum_{i_l=1,\ldots,K_l,i_m=1,\ldots,K_m} P(V_{i_l,l} = V_{i_m,m}) \overset{(2)}{=} 0,$$

as desired, where (1) holds by (A.18) and (2) by the fact that valuations are continuously distributed and l, m refer to different and, thus, independent, auctions. ∎

Lemma A.2. *Assume (2.1) holds. For some $\varepsilon > 0$ with $(1+\varepsilon)\xi < 1$, assume that $E[|V_{i,j}|^{1+\varepsilon}] < \infty$ for all $i = 1\ldots, K_j$ in auction $j = 1, \ldots, N$. Then, $\exists \bar{K} \in \mathbb{N}$ such that for $\delta = \varepsilon/2$ and $K \geq \bar{K}$,*

$$E[|\tfrac{V_{(1),j}-b_K}{a_K}|^{1+\delta}] \ < \ \infty \quad \textit{and} \quad E[|\tfrac{V_{(2),j}-b_K}{a_K}|^{1+\delta}] \ < \ \infty. \tag{A.19}$$

Proof. The first result in (A.19) is a corollary of de Haan and Ferreira (2006, Theorem 5.3.1). Thus, we focus the rest of this proof on the second result in (A.19).

It is convenient to introduce the following notation. For any $\tilde{K} \in \mathbb{N}$, let $U(\tilde{K})$ distributed according to $\max\{V_{i,j} : j = 1, \ldots, \tilde{K}\}$. For any $x \in [v_L, v_H]$, the CDF and PDF of $U(\tilde{K})$ are

$$F_{U(\tilde{K})}(x) \ = \ F_V(x)^{\tilde{K}} \quad \text{and} \quad f_{U(\tilde{K})}(x) \ = \ \tilde{K}(F_V(x))^{\tilde{K}-1} f_V(x). \tag{A.20}$$

Conditional on $V_{(1),j} = x \in [v_L, v_H]$, $V_{(2),j}$ is the highest valuation among the remaining $K_j - 1$ bidders. Thus, for any $t \leq x$ with $x, t \in [v_L, v_H]$, $P(V_{(2),j} \leq t | V_{(1),j} = x) = (F_V(t)/F_V(x))^{K_j-1}$. By setting $\tilde{t} = (t - b_K)/a_K$ and $\tilde{x} = (x - b_K)/a_K$ for any $x, t \in [v_L, v_H]$ with $t \leq x$,

$$P\Big(\tfrac{V_{(2),j}-b_K}{a_K} \leq \tilde{t} \Big| \tfrac{V_{(1),j}-b_K}{a_K} = \tilde{x}\Big) \ = \ \Big(\tfrac{F_V(\tilde{t}a_K+b_K)}{F_V(\tilde{x}a_K+b_K)}\Big)^{K_j-1}$$

and, so, the conditional PDF is

$$f_{\Big(\frac{V_{(2),j}-b_K}{a_K}\Big|\frac{V_{(1),j}-b_K}{a_K}=\tilde{x}\Big)}(\tilde{t}) \ = \ \tfrac{(K_j-1)(F_V(\tilde{t}a_K+b_K))^{K_j-2} f_V(\tilde{t}a_K+b_K)a_K}{F_V(\tilde{x}a_K+b_K)^{K_j-1}} \ = \ \tfrac{f_{U(K_j-1)}(\tilde{t}a_K+b_K)a_K}{F_{U(K_j-1)}(\tilde{x}a_K+b_K)}. \tag{A.21}$$

For any $x \in [v_L, v_H]$, we have $P((V_{(1),j} - b_K)/a_K \leq \tilde{x}) = (F_V(\tilde{x}a_K + b_K))^{K_j}$ and, so,

$$f_{\frac{V_{(1),j}-b_K}{a_K}}(\tilde{x}) \ = \ K_j(F_V(\tilde{x}a_K + b_K))^{K_j-1} f_V(\tilde{x}a_K + b_K)a_K \ = \ f_{U(K_j)}(\tilde{x}a_K + b_K)a_K. \tag{A.22}$$

Consider the following derivation:

$$
\begin{aligned}
&E\big[\big|\tfrac{V_{(2),j}-b_K}{a_K}\big|^{1+\delta}\big] \\
&= E\big[E\big[\big|\tfrac{V_{(2),j}-b_K}{a_K}\big|^{1+\delta}\big|\tfrac{V_{(1),j}-b_K}{a_K}\big]\big] \\
&\overset{(1)}{=} \int_{\tilde{x}=\frac{v_L-b_K}{a_K}}^{\tilde{x}=\frac{v_H-b_K}{a_K}} \Big(\int_{\tilde{t}=\frac{v_L-b_K}{a_K}}^{\tilde{t}=\tilde{x}} |\tilde{t}|^{1+\delta}\tfrac{f_{U(K_j-1)}(\tilde{t}a_K+b_K)a_K}{F_{U(K_j-1)}(\tilde{x}a_K+b_K)}d\tilde{t}\Big) f_{U(K_j)}(\tilde{x}a_K+b_K)a_K d\tilde{x} \\
&\overset{(2)}{=} \int_{\tilde{x}=\frac{v_L-b_K}{a_K}}^{\tilde{x}=\frac{v_H-b_K}{a_K}} \Big(E\big[\big|\tfrac{U(K_j-1)-b_K}{a_K}\big|^{1+\delta}1[U(K_j-1)\leq \tilde{x}a_K+b_K]\big]\Big)\tfrac{f_{U(K_j)}(\tilde{x}a_K+b_K)}{F_{U(K_j-1)}(\tilde{x}a_K+b_K)}a_K d\tilde{x},
\end{aligned}
\tag{A.23}
$$

where (1) holds by (A.21) and (A.22) and (2) by the change of variables $t=\tilde{t}a_K+b_K$.

For any $p,q>1$ with $1/p+1/q=1$, we have

$$
\begin{aligned}
&E\big[\big|\tfrac{U(K_j-1)-b_K}{a_K}\big|^{1+\delta}1[U(K_j-1)\leq \tilde{x}a_K+b_K]\big] \\
&\overset{(1)}{\leq} E\big[\big|\tfrac{U(K_j-1)-b_{K_j-1}}{a_{K_j-1}}\tfrac{a_{K_j-1}}{a_K}+\tfrac{b_{K_j-1}-b_K}{a_K}\big|^{(1+\delta)p}\big]^{1/p}F_{U(K_j-1)}(\tilde{x}a_K+b_K)^{1/q} \\
&\overset{(2)}{\leq} \left\{\begin{array}{c} 2^{\frac{(1+\delta)p-1}{p}}\Big[E\big[\big|\tfrac{U(K_j-1)-b_{K_j-1}}{a_{K_j-1}}\big|^{(1+\delta)p}\big]\big|\tfrac{a_{K_j-1}}{a_K}\big|^{(1+\delta)p}+\big|\tfrac{b_{K_j-1}-b_K}{a_K}\big|^{(1+\delta)p}\Big]^{1/p} \\ \times F_{U(K_j-1)}(\tilde{x}a_K+b_K)^{1/q}\end{array}\right\},
\end{aligned}
\tag{A.24}
$$

where (1) holds by Hölder's inequality and (2) by Minkowski's inequality. Next, let $C > 2^{((1+\delta)p-1)/p}[E(|Z_\xi|^{(1+\delta)p})]^{1/p}$ where Z_ξ has CDF G_ξ, and $p=(1+\varepsilon)/(1+\delta)=(2+2\varepsilon)/(2+\varepsilon)>1$ (since $\delta=\varepsilon/2$). Under these conditions, we have that

$$
\lim_{K\to\infty}\Big[E\big[\big|\tfrac{U(K_j-1)-b_{K_j-1}}{a_{K_j-1}}\big|^{(1+\delta)p}\big]\big|\tfrac{a_{K_j-1}}{a_K}\big|^{(1+\delta)p}+\big|\tfrac{b_{K_j-1}-b_K}{a_K}\big|^{(1+\delta)p}\Big]^{1/p} < C. \tag{A.25}
$$

This result is a corollary of two observations. First, we note that (A.5) implies that $\lim_{K\to\infty}(a_{K_j-1}/a_K,(b_{K_j-1}-b_K)/a_K)=(1,0)$. Second, under $p=(1+\varepsilon)/(1+\delta)$, $(1+\varepsilon)\xi<1$, and $E[|V_{i,j}|^{1+\varepsilon}]<\infty$, de Haan and Ferreira (2006, Theorem 5.3.1) implies that

$$
\lim_{K\to\infty}E(\big|\tfrac{U(K_j-1)-b_{K_j-1}}{a_{K_j-1}}\big|^{(1+\delta)p}) = \lim_{K\to\infty}E(\big|\tfrac{U(K_j-1)-b_{K_j-1}}{a_{K_j-1}}\big|^{1+\varepsilon}) = E[|Z_\xi|^{1+\varepsilon}] < \infty.
$$

Note that (A.24) and (A.25) imply that $\exists\bar{K}\in\mathbb{N}$ such that for all $K>\bar{K}$,

$$
E\big[\big|\tfrac{U(K_j-1)-b_K}{a_K}\big|^{1+\delta}1[U(K_j-1)\leq \tilde{x}a_K+b_K]\big] \leq C\times F_{U(K_j-1)}(\tilde{x}a_K+b_K)^{1/q}. \tag{A.26}
$$

To complete the proof, consider the following argument for all $K > \max\{\bar{K}, 2\}$,

$$
\begin{aligned}
& E[|\tfrac{V_{(2),j}-b_K}{a_K}|^{1+\delta}] \\
& \overset{(1)}{\leq} \ C\int_{\frac{v_L-b_K}{a_K}}^{\frac{v_H-b_K}{a_K}} F_{U(K_j-1)}(\tilde{x}a_K+b_K)^{1/q-1} f_{U(K_j)}(\tilde{x}a_K+b_K)a_K d\tilde{x} \\
& \overset{(2)}{=} \ C\int_{\frac{v_L-b_K}{a_K}}^{\frac{v_H-b_K}{a_K}} F_{U(K_j-1)}(\tilde{x}a_K+b_K)^{1/q-1} K_j(F_V(\tilde{x}a_K+b_K))^{K_j-1} f_V(\tilde{x}a_K+b_K)a_K d\tilde{x} \\
& \overset{(3)}{\leq} 2C\int_{\frac{v_L-b_K}{a_K}}^{\frac{v_H-b_K}{a_K}} F_{U(K_j-1)}(\tilde{x}a_K+b_K)^{1/q-1} (K_j-1)(F_V(\tilde{x}a_K+b_K))^{K_j-2} f_V(\tilde{x}a_K+b_K)a_K d\tilde{x} \\
& \overset{(4)}{=} 2C\int_{\frac{v_L-b_K}{a_K}}^{\frac{v_H-b_K}{a_K}} F_{U(K_j-1)}(\tilde{x}a_K+b_K)^{1/q-1} f_{U(K_j-1)}(\tilde{x}a_K+b_K)a_K d\tilde{x} \\
& \overset{(5)}{=} 2C\int_0^1 y^{1/q-1}dy \ = \ 2C \ < \ \infty,
\end{aligned}
$$

as desired, where (1) holds by (A.23) and (A.26), (2) and (4) by (A.20), (3) by $F_V(\tilde{x}a_K+b_K) \leq 1$ and $K_j \geq K \geq 2$, and (5) by the change of variables $y = F_{U(K_j-1)}(\tilde{x}a_K+b_K)$. ■

Lemma A.3. *For any $h \in \Sigma$, assume that*

$$
\left\{y \in \mathbb{R} : \int_\Xi \kappa_\xi(h) f_{\tilde{\mathbf{Z}}|\xi}(h)dW(\xi) \leq \int_\Xi f_{(Y_\mu,\tilde{\mathbf{Z}})|\xi}(y,h)d\Lambda(\xi)\right\} \ = \ [A(h), B(h)], \tag{A.27}
$$

where $B(h) - A(h) : \Sigma \to \mathbb{R}_+$ is continuous. Then,

$$
\tilde{U}(h) \ = \ \left\{y \in \mathbb{R} : \int_\Xi \kappa_\xi(h) f_{\tilde{\mathbf{Z}}|\xi}(h)dW(\xi) \leq \int_\Xi f_{(Y_\mu,\tilde{\mathbf{Z}})|\xi}(y,h)d\Lambda(\xi)\right\}.
$$

Proof. It suffices to show that $[A(h), B(h)]$ satisfies conditions (a)-(c) of Theorem 3.1.

Part (a). Consider the following derivation.

$$
P(\partial\{(y,h) \in \mathbb{R}\times\Sigma : y \in \tilde{U}(h)\}) \ \overset{(1)}{\leq} \ \int P_\xi(Y_\mu \in \{A(h), B(h)\}|\tilde{\mathbf{Z}} = h) f_{\tilde{\mathbf{Z}}|\xi}(h)dh \ \overset{(2)}{=} \ 0,
$$

where (1) holds by (A.27), which implies that $\partial\{(y,h) \in \mathbb{R}\times\Sigma : y \in \tilde{U}(h)\} \ \subseteq \ \{(A(h), h), (B(h), h) : h \in \Sigma\}$, and (2) because $\{Y_\mu|\tilde{\mathbf{Z}} = h; \xi\}$ is continuously distributed.

Part (b). Fix $h \in \Sigma$ arbitrarily. Since

$$
\lim_{|y|\to\infty} \int_\Xi f_{(Y_\mu,\tilde{\mathbf{Z}})|\xi}(y,h)d\Lambda(\xi) = 0 < \int_\Xi \kappa_\xi(h) f_{\tilde{\mathbf{Z}}|\xi}(h)dW(\xi),
$$

$\exists y(h) \in (0,\infty)$ such that $\max\{|B(h)|, |A(h)|\} \leq y(h)$ for all $|y| > y(h)$. Then, $[A(h), B(h)] \subseteq$

$[-y(h), y(h)]$, and so $\lg([A(h), B(h)]) \leq 2y(h) < \infty$.

Part (c). Under (A.27), $\lg([A(h), B(h)]) = B(h) - A(h)$ which is assumed continuous. ■

Lemma A.4. *Assume that $\xi < 1$. Let $Z_1 = H_\xi(E_1)$ and $Z_2 = H_\xi(E_1 + E_2)$ with E_1, E_2 i.i.d. standard exponential random variables and H_ξ as in (2.4). Then,*

$$E[Z_1 - Z_2] \;=\; \Gamma(1-\xi),$$

where Γ is the standard Gamma function.

Proof. The support of (Z_1, Z_2) is $S_\xi = \{(x_1, x_2) : x_1 \geq x_2,\ \xi x_1 \geq -1,\ \xi x_2 \geq -1\}$. For any $(x_1, x_2) \in S_\xi$, the PDF of (Z_1, Z_2) is

$$f_{Z_1,Z_2|\xi}(x_1, x_2) \;=\; \begin{cases} (1+\xi x_1)^{-1/\xi-1}(1+\xi x_2)^{-1/\xi-1}\exp(-(1+\xi x_2)^{-1/\xi}) & \text{if } \xi \neq 0, \\ \exp(-x_1)\exp(-x_2)\exp(-\exp(-x_2)) & \text{if } \xi = 0, \end{cases}$$

The desired result follows from this formula. We only show the result for $\xi < 0$, as the result for the other two cases is analogous.

$$\begin{aligned} & E[Z_1 - Z_2] \\ = \;& \int_{-\infty}^{-1/\xi}\int_{x_2}^{-1/\xi}(x_1 - x_2)(1+\xi x_1)^{-1/\xi-1}(1+\xi x_2)^{-1/\xi-1}\exp(-(1+\xi x_2)^{-1/\xi})dx_1dx_2 \\ \overset{(1)}{=} \;& \frac{1}{\xi^3}\int_{\infty}^{0}\int_{t_2}^{0}(t_1 - t_2)t_1^{-1/\xi-1}t_2^{-1/\xi-1}\exp(-t_2^{-1/\xi})dt_1dt_2 \\ \overset{(2)}{=} \;& \int_0^\infty(\frac{1}{(1-\xi)})\exp(-v)v^{-\xi+1}dv \\ \overset{(3)}{=} \;& \Gamma(1-\xi), \end{aligned}$$

as desired, where (1) holds by the change of variables $t_1 = 1 + \xi x_1$ and $t_2 = 1 + \xi x_2$, (2) by the change of variables $v = t_2^{-1/\xi}$, and (3) by $(1-\xi)\Gamma(1-\xi) = \Gamma(2-\xi)$. ■

Lemma A.5. *Assume that $\xi < 1$. Let $Z = H_\xi(E_1 + E_2)$ with E_1, E_2 i.i.d. standard exponential random variables and H_ξ as in (2.4). Then,*

$$E[Z] \;=\; \begin{cases} (\Gamma(2-\xi) - 1)/\xi & \textit{if } \xi \neq 0 \\ \bar{\gamma} - 1 & \textit{if } \xi = 0 \end{cases}$$

where Γ is the Gamma function and $\bar{\gamma} \approx 0.577$ is the Euler constant.

Proof. Note that the PDF of Z is

$$f_{Z|\xi}(z) \;=\; \begin{cases} (1+\xi z)^{-(2+\xi)/\xi}\exp(-(1+\xi z)^{-1/\xi})1[1+\xi z \geq 0] & \text{if } \xi \neq 0 \\ \exp(-2z)\exp(-\exp(-z)) & \text{if } \xi = 0. \end{cases}$$

We only show the result for $\xi \neq 0$, as the result for $\xi = 0$ is analogous.

$$\begin{aligned} E[Z] &= \int_{-\infty}^{\infty} z(1+\xi z)^{-\frac{2+\xi}{\xi}}\exp(-(1+\xi z)^{-1/\xi})1[1+\xi z \geq 0]dz \\ &\overset{(1)}{=} \big(\int_{0}^{\infty}(t^{1-\xi}-t)\exp(-t)dt\big)/\xi \\ &\overset{(2)}{=} (\Gamma(2-\xi)-1)/\xi, \end{aligned}$$

where (1) holds by the change of variables $u = (1+\xi z)^{-1/\xi}$ and (2) by $\Gamma(0) = 1$. ■

Lemma A.6. *Assume* (2.1) *holds. Also, assume that* $\xi < 1$ *and* $E[|V|^{1+\varepsilon}] < \infty$ *for some* $\varepsilon > 0$. *For any sequence* $\{x_K : K \in \mathbb{N}\}$ *with* $x_K \to x \in \bar{S}_\xi \equiv \{s : G_\xi(s) > 0\}$, $\lim_{K\to\infty} L_K(x_K) = L(x)$, *where*

$$L_K(x) \;\equiv\; x - \frac{\int_{(v_L-b_K)/a_K}^{x} F_V(ha_K+b_K)^K dh}{F_V(xa_K+b_K)^K} \quad \textit{and} \quad L(x) \;\equiv\; x - \frac{\int_{-\infty}^{x} G_\xi(h)dh}{G_\xi(x)}. \tag{A.28}$$

Proof. Throughout this proof, it is relevant to note that $F_V(ha_K + b_K)^K$ is the CDF of $(V_{(1)} - b_K)/a_K$ and G_ξ is the CDF of $H_\xi(E_1)$, where $V_{(1)}$ denotes the sample maximum of K random draws from F_V, H_ξ and E_1 are as in Lemma 2.1.

As a preliminary step, we show that for any $x \in \mathbb{R}$,

$$\lim_{K\to\infty}\int_{-\infty}^{x}(F_V(ha_K+b_K)^K - G_\xi(h))dh \;=\; 0. \tag{A.29}$$

To this end, consider the following argument for any K.

$$\begin{aligned} &\int_{-\infty}^{x}(F_V(ha_K+b_K)^K - G_\xi(h))dh \\ &\overset{(1)}{=} \left[\begin{array}{c} x(F_V(xa_K+b_K)^K - G_\xi(x)) + \\ E[H_\xi(E_1)1[H_\xi(E_1)\leq x]] - E[\frac{V_{(1)}-b_K}{a_K}1[\frac{V_{(1)}-b_K}{a_K}\leq x]] \end{array}\right]. \end{aligned} \tag{A.30}$$

where (1) holds by $\lim_{h\to-\infty} F_V(ha_K + b_K)^K = \lim_{h\to-\infty} G_\xi(h) = 0$ and integration by parts.

Lemma 2.1 implies that as $K \to \infty$,

$$\frac{V_{(1)}-b_K}{a_K} \;\overset{d}{\to}\; H_\xi(E_1). \tag{A.31}$$

Since $H_\xi(E_1)$ is continuously distributed, (A.31) implies that

$$\lim_{K\to\infty} x(F_V(xa_K+b_K)^K - G_\xi(x)) = 0 \tag{A.32}$$

Also, note that (A.31) and the continuous mapping theorem imply that as $K\to\infty$,

$$\tfrac{V_{(1)}-b_K}{a_K}1[\tfrac{V_{(1)}-b_K}{a_K}\leq x] \ \overset{d}{\to}\ H_\xi(E_1)1[H_\xi(E_1)\leq x]. \tag{A.33}$$

Let $\delta>0$ and $\bar{K}\in\mathbb{N}$ be as in Lemma A.2. Then, for all $K\geq\bar{K}$,

$$E[|\tfrac{V_{(1)}-b_K}{a_K}1[\tfrac{V_{(1)}-b_K}{a_K}\leq x]|^{1+\delta}] \ \leq\ E[|\tfrac{V_{(1)}-b_K}{a_K}|^{1+\delta}] \ <\ \infty. \tag{A.34}$$

Then, (A.33) and (A.34) imply that

$$\lim_{K\to\infty} E[\tfrac{V_{(1)}-b_K}{a_K}1[\tfrac{V_{(1)}-b_K}{a_K}\leq x]] \ =\ E[H_\xi(E_1)1[H_\xi(E_1)\leq x]]. \tag{A.35}$$

Finally, note that (A.29) follows from (A.30), (A.32), and (A.35).

As a second preliminary step, we note that

$$\int_{-\infty}^{x} G_\xi(h)dh \ \overset{(1)}{=}\ xG_\xi(x) - \int_{-\infty}^{x} hg_\xi(h)dh \ \leq\ xG_\xi(x) + E[|H_\xi(E_1)|] \ \overset{(2)}{<}\ \infty, \tag{A.36}$$

where (1) holds by integration by parts and $\lim_{x\to-\infty} xG_\xi(x)=0$, and (2) by $\xi<1$.

We are now ready to show the desired results. For any $x\in\bar{S}_\xi$, consider the following argument:

$$\begin{aligned}|L_K(x_K)-L(x)| &\overset{(1)}{=} \left|\frac{\int_{-\infty}^{x_K} F_V(ha_K+b_K)^K dh}{F_V(x_Ka_K+b_K)^K} - \frac{\int_{-\infty}^{x} G_\xi(h)dh}{G_\xi(x)}\right| \\ &\overset{(2)}{\leq} \frac{\left|\begin{array}{c} |x_K-x| + |\int_{-\infty}^{x}(F_V(ha_K+b_K)^K - G_\xi(h))dh| \\ +(\int_{-\infty}^{x} G_\xi(h)dh)\sup_{y\in\mathbb{R}}|F_V(ya_K+b_K)^K - G_\xi(y)| \\ +(\int_{-\infty}^{x} G_\xi(h)dh)|G_\xi(x_K)-G_\xi(x)| \end{array}\right|}{G_\xi(x)[G_\xi(x)-\sup_{y\in\mathbb{R}}|F_V(ya_K+b_K)^K-G_\xi(y)|-|G_\xi(x_K)-G_\xi(x)|]},\end{aligned} \tag{A.37}$$

where (1) holds because $F_V(v)=0$ for $v<v_L$, and (2) by $\sup_{y\in\mathbb{R}} F_V(ya_K+b_K)^K\leq 1$ and $G_\xi(x)\leq 1$. As $K\to\infty$, we can show that the numerator and the denominator of the right-hand side of (A.37) converge to zero and $G_\xi(x)^2>0$, respectively. This conclusion relies on $x\in\bar{S}_\xi$ (and so $G_\xi(x)^2>0$), $x_K\to x\in\bar{S}_\xi$ as $K\to\infty$, the continuity of G_ξ, (A.29), (A.36), and van der Vaart (1998, Lemma 2.11). From this and (A.37), the desired result follows. ■

Lemma A.7. *Assume (2.1) holds. Assume that $\xi<1$ and that $E[|V_{i,j}|^{1+\varepsilon}]<\infty$ for all $i=1\ldots,K_j$*

in auction $j = 1, \dots, N$ *for some* $\varepsilon > 0$*. For any* $n \in \mathbb{N}$ *and as* $K \to \infty$*,*

$$\{\tfrac{P_j - b_K}{a_K} : j = 1, \dots, n\} \ \stackrel{d}{\to}\ \{H_\xi(E_{1,j}) - \tfrac{\int_{-\infty}^{H_\xi(E_{1,j})} G_\xi(h)dh}{G_\xi(H_\xi(E_{1,j}))} : j = 1, \dots, n\},$$

with $\{(a_K, b_K) \in \mathbb{R}_{++} \times \mathbb{R} : K \in \mathbb{N}\}$*,* $\{E_{1,j} :\ j = 1, \dots, n\}$*, and* H_ξ *as in Lemma 2.1.*

Proof. Since auctions are independent, it suffices to prove that for any auction $j = 1, \dots n$, as $K \to \infty$,

$$\tfrac{P_j - b_K}{a_K} \ \stackrel{d}{\to}\ H_\xi(E_{1,j}) - \tfrac{\int_{-\infty}^{H_\xi(E_{1,j})} G_\xi(h)dh}{G_\xi(H_\xi(E_{1,j}))}, \tag{A.38}$$

where $E_{1,j}$ is a standard exponential random variable. To this end, consider the following argument.

$$\begin{aligned} \tfrac{P_j - b_K}{a_K} &= \big(\tfrac{V_{(1),j} - b_{K_j-1}}{a_{K_j-1}} - \tfrac{(\int_{v_L}^{V_{(1),j}} F_V(u)^{K_j-1}du)/a_{K_j-1}}{F_V(V_{(1),j})^{K_j-1}}\big)\tfrac{a_{K_j-1}}{a_K} + \tfrac{b_{K_j-1} - b_K}{a_K} \\ &\stackrel{(1)}{=} \big(L_{K_j-1}\big(\tfrac{V_{(1),j} - b_{K_j-1}}{a_{K_j-1}}\big)\big)\tfrac{a_{K_j-1}}{a_K} + \tfrac{b_{K_j-1} - b_K}{a_K}, \end{aligned} \tag{A.39}$$

where (1) holds by the change of variables $u = ha_{K_j-1} + b_{K_j-1}$ and L_K as in (A.28).

Since $(K_j - 1)/K \to 1$ as $K \to \infty$, the same argument as Lemma 2.1 implies that

$$\lim_{K\to\infty} \big(\tfrac{a_{K_j-1}}{a_K}, \tfrac{b_{K_j-1} - b_K}{a_K}\big) \ = \ (1, 0). \tag{A.40}$$

Next, notice that Lemma 2.1 and (A.40) imply that as $K \to \infty$,

$$\tfrac{V_{(1),j} - b_{K_j-1}}{a_{K_j-1}} \ = \ \tfrac{V_{(1),j} - b_K}{a_K}\tfrac{a_K}{a_{K_j-1}} + \tfrac{b_K - b_{K_j-1}}{a_K} \ \stackrel{d}{\to}\ H_\xi(E_1). \tag{A.41}$$

In addition, $H_\xi(E_1) \in \bar{S}_\xi \equiv \{x \in \mathbb{R} : G_\xi(x) > 0\}$. By (A.41), Lemma A.6, and the extended continuous mapping theorem (e.g. van der Vaart, 1998, Theorem 1.11.1), as $K \to \infty$,

$$L_{K_j-1}\big(\tfrac{V_{(1),j} - b_{K_j-1}}{a_{K_j-1}}\big) \ \stackrel{d}{\to}\ L(H_\xi(E_1)) \ = \ O_p(1), \tag{A.42}$$

where L is as in (A.28). Then, (A.38) follows from (A.39), (A.40), and (A.42). ■

Theorem A.1. *Assume (2.1) holds, and that for some* $\varepsilon > 0$ *with* $(1 + \varepsilon)\xi < 1$*,* $E[|V_{i,j}|^{1+\varepsilon}] < \infty$ *for all* $i = 1 \dots, K_j$ *in auction* $j = 1, \dots, N$*. Finally, assume that the CI for* μ_K*,* $U(\mathbf{P})$*, is as in (4.9) with* $\tilde{U} : \Sigma \to \mathcal{P}(\mathbb{R})$ *that satisfies the following conditions:*

(a) $P_\xi(\{Y_\mu, \tilde{\mathbf{X}}\} \in \partial\{(y, h) \in \mathbb{R} \times \Sigma : y \in \tilde{U}(h)\}) = 0$*, where* ∂A *denotes the boundary of* A*.*

(b) $\lg(\tilde{U}(h)) < \infty$ *for any* $h \in \Sigma$*, where* $\lg(A)$ *denotes the length of* A *(i.e.,* $\lg(A) \equiv \int \mathbf{1}[y \in A]dy$*).*

(c) For any sequence $\{h_\ell \in \Sigma\}_{\ell\in\mathbb{N}}$ *with* $h_\ell \to h \in \Sigma$*,* $\lg(\tilde{U}(h_\ell)) \to \lg(\tilde{U}(h))$*.*

Then, as $K \to \infty$*,*

1. $P(\mu_K \in U(\mathbf{P})) \;\to\; P_\xi(Y_\mu \in \tilde{U}(\tilde{\mathbf{X}}))$,

2. $E[\lg(U(\mathbf{P}))]/a_K \;\to\; E_\xi[\kappa_\xi(\tilde{\mathbf{X}}) \lg(\tilde{U}(\tilde{\mathbf{X}}))]$.

Proof. This proof follows from the same arguments used to prove Theorem 3.1. The main difference between the proofs is that we replace $\{(P_j - b_K)/a_K : j = 1, \ldots, N\}$ with $\{L_{K_j}((V_{(1),j} - b_K)/a_K) : j = 1, \ldots, N\}$ with L_{K_j} as in (A.28) (instead of $\{(V_{(2),j} - b_K)/a_K : j = 1, \ldots, N\}$ as in second-price auctions). Then, the vectors $\{Z_j : j = 1, \ldots, n\}$ and $\tilde{\mathbf{Z}} = \{\tilde{Z}_j : j = 1, \ldots, N\}$ are replaced by $\{X_j : j = 1, \ldots, N\}$ and $\tilde{\mathbf{X}} = \{\tilde{X}_j : j = 1, \ldots, N\}$, as in (4.4) and (4.7), respectively. With these changes in place, the results follow from repeating the steps used to prove Theorem 3.1. ∎

Theorem A.2. *Assume* (2.1) *holds, and that for some* $\varepsilon > 0$ *with* $(1+\varepsilon)\xi < 1$, $E[|V_{i,j}|^{1+\varepsilon}] < \infty$ *for all* $i = 1 \ldots, K_j$ *in auction* $j = 1, \ldots, N$. *Finally, assume that the CI for* π_K, $U(\mathbf{P})$, *is as in* (4.11) *with* $\tilde{U} : \Sigma \to \mathcal{P}(\mathbb{R})$ *that satisfies the following conditions:*

(a) $P_\xi(\{Y_\pi, \tilde{\mathbf{X}}\} \in \partial\{(y,h) \in \mathbb{R} \times \Sigma : y \in \tilde{U}(h)\}) = 0$, *where* ∂A *denotes the boundary of* A.

(b) $\lg(\tilde{U}(h)) < \infty$ *for any* $h \in \Sigma$, *where* $\lg(A)$ *denotes the length of* A.

(c) For any sequence $\{h_\ell \in \Sigma\}_{\ell \in \mathbb{N}}$ *with* $h_\ell \to h \in \Sigma$, $\lg(\tilde{U}(h_\ell)) \to \lg(\tilde{U}(h))$.

Then, as $K \to \infty$,

1. $P(\pi_K \in U(\mathbf{P})) \;\to\; P_\xi(Y_\pi \in \tilde{U}(\tilde{\mathbf{X}}))$,

2. $E[\lg(U(\mathbf{P}))]/a_K \;\to\; E_\xi[\kappa_\xi(\tilde{\mathbf{X}}) \lg(\tilde{U}(\tilde{\mathbf{X}}))]$.

Proof. This proof follows from the same arguments as in Theorems 3.1 and A.1. ∎

Theorem A.3. *Assume* (2.1) *holds. In the hypothesis testing problem in* (4.12)*, the test defined by* (4.13) *satisfies the following properties:*

1. *It is asymptotically valid and level* α, *i.e.*, $\lim_{K\to\infty} E_{\xi_0}[\varphi_K^*(\mathbf{P})] \;=\; \alpha$.

2. *It is asymptotically efficient.*

Proof. This proof follows from the same arguments as in Theorems 3.3 and 3.4. ∎

A.3 Computational details

A.3.1 Second-price auctions

This section provides computational details for objects introduced in Section 3. Throughout this section, we use $\tilde{\mathbf{z}} = (z_1, \ldots, z_N) \in \Sigma$ and $N = n - 2$.

First, recall that $f_{\tilde{\mathbf{Z}}|\xi}(\tilde{\mathbf{z}})$ is as computed in (3.6). For $\tilde{\mathbf{z}} \in \Sigma$, we can compute:

$$\kappa_\xi(\tilde{\mathbf{z}}) f_{\tilde{\mathbf{Z}}|\xi}(\tilde{\mathbf{z}}) = n!(\Gamma(2n-\xi)) \int_0^{b(\xi)} s^{n-1} \exp\left(\begin{array}{c} (\xi - 2n)\log\left(\sum_{j=1}^n (1+z_j\xi s)^{-1/\xi}\right) \\ -(1+\frac{2}{\xi})\left(\sum_{j=1}^n \log(1+z_j\xi s)\right) \end{array} \right) ds, \quad \text{(A.43)}$$

where $\kappa_\xi(\tilde{\mathbf{z}}) = E[Z_{(n)} - Z_{(1)}|\tilde{\mathbf{Z}} = \tilde{\mathbf{z}}]$ and $b(\xi) = -1/\xi$ for $\xi < 0$, and $b(\xi) = \infty$ otherwise. For $(y, \tilde{\mathbf{z}}) \in \mathbb{R}_+ \times \Sigma$, calculations yield:

$$f_{Y_\mu, \tilde{\mathbf{Z}}|\xi}(y, \tilde{\mathbf{z}}) =$$

$$\frac{n!(\Gamma(1-\xi))^{n-1}}{y^n} \int_{a_\mu(\xi)}^{c_\mu(\xi)} \exp\left(\begin{array}{c} -\sum_{j=1}^n (1+\xi s + (\Gamma(1-\xi))\xi z_j/y)^{-1/\xi} \\ -(1+\frac{2}{\xi})\sum_{j=1}^n \log(1+\xi s+(\Gamma(1-\xi))\xi z_j/y) \end{array} \right) ds, \quad \text{(A.44)}$$

where $a_\mu(\xi)$ and $c_\mu(\xi)$ are defined such that for all $s \in (a_\mu(\xi), c_\mu(\xi))$, we have $1+\xi s+(\Gamma(1-\xi))\xi/y > 0$ and $1+\xi s > 0$. For $(y, \tilde{\mathbf{z}}) \in \mathbb{R} \times \Sigma$, more calculations yield:

$$f_{Y_\pi, \tilde{\mathbf{Z}}|\xi}(y, \tilde{\mathbf{z}}) =$$

$$\frac{n!}{|y|^n} \int_{a_\pi(\xi)}^{c_\pi(\xi)} |\pi - s|^{n-1} \exp\left(\begin{array}{c} -\sum_{j=1}^n (1+\xi(s+z_j(\pi-s)/y)^{-1/\xi} \\ -(1+\frac{2}{\xi})\sum_{j=1}^n \log(1+\xi(s+z_j(\pi-s)/y) \end{array} \right) ds, \quad \text{(A.45)}$$

where $a_\mu(\xi)$ and $c_\mu(\xi)$ are such that for all $s \in (a_\mu(\xi), c_\mu(\xi))$, we have $1+\xi(s+(\pi-s)/y > 0$, $1+\xi s > 0$, and $(\pi - s)/y > 0$. The integrals in (A.43), (A.44), and (A.45) can be approximated by Gaussian quadrature.

The CI's in Section 3 require Lagrange multipliers Λ that we compute using the algorithm developed by Elliott et al. (2015). See Müller and Wang (2017) for another application of this algorithm. We now provide a detailed description of how we implemented this algorithm:

1) Discretize Ξ into a fine grid $\Xi_M \equiv \{\xi_1, \xi_2, \ldots, \xi_M\}$ between $\xi_1 = \inf\{\Xi\}$ and $\xi_{M+1} = \sup\{\Xi\}$ (we use $M = 50$ uniformly located points between ξ_1 and ξ_{M+1}). Set $s = 1$, and define an arbitrary set of initial positive weights $\lambda^{(s)} = \{\lambda_m^{(s)} : m = 1, \ldots, M\}$ over Ξ_M (we use a uniform weights, i.e., $\lambda^{(1)} = \{1/M, \ldots, 1/M\}$).

2) For each $m = 1, \ldots, M$, simulate a large number B of n i.i.d. draws the EV distribution with parameter ξ_m (we use $B = 10{,}000$). For each $m = 1, \ldots, M$ and $b = 1, \ldots, B$, the samples is denoted by $\mathbf{Z}_{\xi_m}(b) = \{Z_{\xi_m,1}(b), \ldots, Z_{\xi_m,n}(b)\}$. By applying (3.9) to each sample, we get $N = n-2$ sorted and normalized draws, denoted by $\tilde{\mathbf{Z}}_{\xi_m}(b) = \{\tilde{Z}_{\xi_m,1}(b), \ldots, \tilde{Z}_{\xi_m,N}(b)\} \in \Sigma$.

3) For each $m = 1, \ldots, M$, use the random draws in step 2) to approximate the limiting coverage probabilities for parameter ξ_m in the following manner:

   (a) For the winner's expected utility, we approximate $P_{\xi_m}(\Gamma(1-\xi_m)/(Z_{(n)} - Z_{(1)}) \in U(\tilde{\mathbf{Z}}))$

with

$$\hat{\mathbb{P}}_m \equiv \frac{1}{B}\sum_{b=1}^{B} 1\big(\tfrac{\Gamma(1-\xi_m)}{Z_{\xi_m,(1)}(b)-Z_{\xi_m,(n)}(b)} \in U(\tilde{\mathbf{Z}}_{\xi_m}(b))\big),$$

where Γ is the standard Gamma function and $U(\tilde{\mathbf{Z}}_{\xi_m}(b))$ is as in (3.14), involving integrals of (A.43) and (A.44).

(b) For the seller's expected revenue, we approximate $P_{\xi_m}((\pi(\xi_m) - Z_{(1)})/(Z_{(n)} - Z_{(1)}) \in U(\tilde{\mathbf{Z}}))$, with

$$\hat{\mathbb{P}}_m \equiv \frac{1}{B}\sum_{b=1}^{B} 1\big(\tfrac{\pi(\xi_m)-Z_{\xi_m,(n)}(b)}{Z_{\xi_m,(1)}(b)-Z_{\xi_m,(n)}(b)} \in U(\tilde{\mathbf{Z}}_{\xi_m}(b))\big),$$

where $\pi(\xi_m) = (\Gamma(2-\xi_m) - 1)/\xi_m$ with Γ is the standard Gamma function if $\xi_m \neq 0$, and $\pi(\xi_m) = -1 + \bar{\gamma}$ if $\xi_m = 0$ with $\bar{\gamma} \approx 0.577$ equal to the Euler's constant, and $U(\tilde{\mathbf{Z}}_{\xi_m}(b))$ is as in (3.19), involving integrals of (A.43) and (A.44).

4) Update the weights by setting $\lambda_m^{(s+1)} = \lambda_m^{(s)} + \varepsilon((1 - \hat{\mathbb{P}}_m) - \alpha)$ for all $m = 1, \ldots, M$, where α is the significance level and $\varepsilon > 0$ is a small step length (we use $\varepsilon = 0.05$). Intuitively, the weight on ξ_m is decreased or increased if the CI has overcoverage and undercoverage, respectively.

5) Repeat steps 3)-4) a large number of times S (we use $S = 2,000$), to get $\lambda^{(S)} = \{\lambda_m^{(S)} : m = 1, \ldots, M\}$. The Lagrange multipliers Λ is obtained by interpolating $\lambda^{(S)}$ from Ξ_M to Ξ.

If the algorithm's tuning parameters are appropriately chosen, the Lagrange multipliers generated by it yield a CI that (a) approximately minimizes asymptotic weighted length (due to the reliance on (3.14) or (3.19)), and (b) achieves approximately size control for all $\xi \in \Xi_M$, which should extend to all $\xi \in \Xi$ by continuity. Finally, it is worth noting that these Lagrange multipliers only need to be determined once for a given value of n and set Ξ. The tables of the Lagrange multipliers and the corresponding MATLAB code are available on our website.

A.3.2 First-price auctions

We now consider analogous expressions for first-price auctions. Let $j = 1, \ldots, N$ denote an arbitrary first-price auction. Relative to the previous section, the main difficulty here is that PDF of X_j in (4.4) does not generally have a closed-form expression and, thus, we do not have an analog of (3.6) for first-price auctions.

To explain this issue, we now provide an implicit formula for the PDF of X_j for any arbitrary auction $j = 1, \ldots, N$. Consider the following argument for $\xi < 0$:

$$
\begin{aligned}
X_j &\overset{(1)}{=} H_1 - \tfrac{1}{G_\xi(H_1)} \int_{-\infty}^{H_1} G_\xi(h) dh \\
&\overset{(2)}{=} H_1 - \exp((1+\xi H_1)^{-1/\xi}) \int_{-\infty}^{H_1} \exp(-(1+\xi h)^{-1/\xi}) dh \\
&\overset{(3)}{=} H_1 - \exp((1+\xi H_1)^{-1/\xi}) \Gamma(-\xi, (1+\xi H_1)^{-1/\xi}) \\
&\overset{(4)}{=} [(E_{1,j})^{-\xi} - 1]/\xi - \exp(E_{1,j}) \Gamma(-\xi, E_{1,j}) \\
&\overset{(5)}{=} [\exp(E_{1,j}) \Gamma(1-\xi, E_{1,j}) - 1]/\xi,
\end{aligned} \tag{A.46}
$$

where (1) holds by (4.4) and denoting $H_1 = H_\xi(E_{1,j})$ with $\{E_{1,j} : j = 1, \ldots, n\}$ and H_ξ as specified in Lemma 2.1, (2) by (2.2), (3) by the change of variables $u = (1+\xi h)^{-1/\xi}$ and using $\Gamma(a, x) = \int_x^\infty u^{a-1} \exp(-u) du$ to denote the upper incomplete Gamma function, (4) by $E_{1,j} = H_\xi^{-1}(H_1)$, and (5) by $\Gamma(1+a, x) = a\Gamma(a, x) + x^a \exp(-x)$ applied to $a = -\xi$ and $x = E_{1,j}$. By a similar derivation for $\xi > 0$ and $\xi = 0$, we get $X_j = e_\xi(E_{1,j})$ with

$$
e_\xi(x) \equiv \begin{cases} [\exp(x)\Gamma(1-\xi, x) - 1]/\xi & \text{if } \xi \neq 0, \\ -\ln(x) - \exp(x)\Gamma(0, x) & \text{if } \xi = 0. \end{cases} \tag{A.47}
$$

We numerically verified that $e_\xi(x)$ is decreasing in x for all ξ, and so the PDF of X_j can be expressed as follows:

$$
f_{X_j|\xi}(x) = -\tfrac{\partial e_\xi^{-1}(x)}{\partial x} \exp(-e_\xi^{-1}(x)). \tag{A.48}
$$

We can use (A.48) to derive an implicit expression for the joint PDF of $\tilde{\mathbf{X}} = \{\tilde{X}_j : j = 1, \ldots, N\} \in \Sigma$ with $\tilde{X}_j$ as in (4.7). The main difficulty relative to the second-price auction is that neither e_ξ in (A.47) nor its inverse have closed-form expressions, and so evaluating them repeatedly is computationally challenging. For this reason, we do not have a closed-form analog of (3.6) for first-price auctions.

To deal with the aforementioned computational issues, we propose a numerical approximation based on a series expansion of the incomplete Gamma function in Abramowitz and Stegun (1964, Page 263): for x sufficiently large,

$$
\Gamma(1-\xi, x) \approx x^{-\xi} \exp(-x)[1 + \tfrac{(-\xi)}{x} + \tfrac{(-\xi)(-\xi-1)}{x^2} + \tfrac{(-\xi)(-\xi-1)(-\xi-2)}{x^3} + \ldots]. \tag{A.49}
$$

A first-order approximation based on (A.49) gives $\ln(\exp(x)\Gamma(1-\xi, x)) \approx -\xi \ln x$. To allow for approximation errors, we propose the following equation:

$$
\ln(\exp(x)\Gamma(1-\xi, x)) \approx r_2(\xi) - r_1(\xi) \ln x, \tag{A.50}
$$

where $r_1(\xi)$ and $r_2(\xi)$ are functions of ξ.[4] To find suitable values for these functions we fit an OLS regression of $\ln(\Gamma(1-\xi,x)\exp(x))$ on $\ln x$ and a constant for a grid of 50,000 equally-spaced values of x between 10^{-6} and $1-10^{-6}$ quantiles of the exponential distribution. We use the OLS coefficients as our values for $(-r_1(\xi), r_2(\xi))$. By combining (A.47) and (A.50), we get

$$e_\xi(x) \approx \begin{cases} [x^{-r_1(\xi)}\exp(r_2(\xi)) - 1]/\xi & \text{if } \xi \neq 0, \\ (r_1(0)-1)\ln x - r_2(0) & \text{if } \xi = 0. \end{cases} \tag{A.51}$$

By (A.51) and using $r_3(\xi) = \exp(r_2(\xi)/r_1(\xi))$, we get

$$e_\xi^{-1}(x) \approx \begin{cases} r_3(\xi)(1+\xi x)^{-1/r_1(\xi)} & \text{if } \xi \neq 0, \\ \exp[(x+r_2(0))/(r_1(0)-1)] & \text{if } \xi = 0, \end{cases}$$
$$\frac{\partial e_\xi^{-1}(x)}{\partial x} \approx \begin{cases} -(1+\xi x)^{-(1/r_1(\xi)+1)}\xi r_3(\xi)/r_1(\xi) & \text{if } \xi \neq 0, \\ \exp[(x+r_2(0))/(r_1(0)-1)]/(r_1(0)-1) & \text{if } \xi = 0. \end{cases} \tag{A.52}$$

By combining (A.48) and (A.52), we obtain an approximation of the joint distribution of $\tilde{\mathbf{X}}$ and related functions. Ignoring the case $\xi = 0$, then for $\tilde{\mathbf{x}} \in \Sigma$ and $N = n-2$ we get

$$f_{\tilde{\mathbf{X}}|\xi}(\tilde{\mathbf{x}}) = \frac{n!\Gamma(n)|r_1(\xi)|}{|\xi|} \int_0^{b(\xi)} s^{n-2} \exp\begin{pmatrix} -n\ln(r_3(\xi)\sum_{j=1}^n (1+\xi\tilde{x}_j s)^{-1/r_1(\xi)}) \\ -(1/r_1(\xi)+1)\sum_{j=1}^n \ln(1+\xi\tilde{x}_j s) \\ +n\ln(|\xi r_3(\xi)/r_1(\xi)|) \end{pmatrix} ds, \tag{A.53}$$

where $b(\xi) = -1/\xi$ for $\xi < 0$, and $b(\xi) = \infty$ otherwise. This expression is the analog of (3.6) for first-price auctions. Analogously, we have

$$\kappa_\xi(\tilde{\mathbf{x}}) f_{\tilde{\mathbf{X}}|\xi}(\tilde{\mathbf{x}}) =$$
$$\frac{n!|r_1(\xi)|\Gamma(n-r_1(\xi))}{|\xi|} \int_0^{b(\xi)} s^{n-1} \exp\begin{pmatrix} (r_1(\xi)-n)\ln(r_3(\xi)\sum_{j=1}^n (1+\xi\tilde{x}_j s)^{-1/r_1(\xi)}) \\ -(1/r_1(\xi)+1)\sum_{j=1}^n \ln(1+\xi\tilde{x}_j s) \\ +n\ln(|\xi r_3(\xi)/r_1(\xi)|) \end{pmatrix} ds, \tag{A.54}$$

where $\kappa_\xi(\mathbf{x}) = E[X_{(n)} - X_{(1)}|\tilde{\mathbf{X}} = \mathbf{x}]$. For $(y, \tilde{\mathbf{x}}) \in \mathbb{R}_+ \times \Sigma$, we also have

$$f_{Y_\pi,\tilde{\mathbf{X}}|\xi}(y,\tilde{\mathbf{x}}) =$$
$$\frac{1}{|y|^n} \int_{a_\pi(\xi)}^{c_\pi(\xi)} |\pi - s|^{n-1} \exp\begin{pmatrix} -(1/r_1(\xi)+1)\sum_{j=1}^n \ln(1+\xi(s+\tilde{x}_j(\pi-s)/y)) \\ -r_3(\xi)\sum_{j=1}^n (1+\xi(s+\tilde{x}_j(\pi-s)/y))^{-1/r_1(\xi)} \\ +n\ln(|\xi r_3(\xi)/r_1(\xi)|) \end{pmatrix} ds, \tag{A.55}$$

where $a_\pi(\xi)$ and $c_\pi(\xi)$ are such that for all $s \in (a_\pi(\xi), c_\pi(\xi))$, we have $1 + \xi(s + (\pi - s)/y) > 0$,

[4] If the first-order approximation was exact, we would have $r_1(\xi) = \xi$ and $r_2(\xi) = 0$.

$1 + \xi s > 0$, and $(\pi - s)/y > 0$. The integrals in (A.53), (A.54), and (A.55) can be approximated by Gaussian quadrature. Given these expressions, we can use the algorithm in Section A.3.1 to compute the CI.

A.4 Additional Monte Carlo simulations

This section provides Monte Carlo simulations for first-price auctions. We first consider the problem of inference on the winner's expected utility using only transaction prices. The parameters of the simulations are as in Section 6.

We consider three CIs for μ_K that are analogous to those used in Section 6:

(i) This is our proposed CI in Section 4.1. As in Section 6, this method is implemented by constructing $U(\mathbf{P})$ in (4.9) with $\Xi = [-1, 0.5]$ and W equal to the uniform distribution over this interval. As before, the validity of this CI is based on asymptotics as $K \to \infty$.

(ii) A CI based on observing the transaction price and the highest valuation for each auction. These data are infeasible in empirical applications of first-price auctions (valuations are unobserved), but we take it as a benchmark for any method that relies on the traditional asymptotics with $n \to \infty$. These data allows us to compute $D_j \equiv V_{(1),j} - P_j$ for each auction $j = 1, \ldots, n$. In turn, this information enables us to test $H_0 : \mu_K = b$ using the t-statistic $\sqrt{n}(\bar{D}_n - b)/s$, where $\bar{D}_n = \sum_{j=1}^{n} D_j/n$ and $s^2 = \sum_{j=1}^{n}(D_j - \bar{D}_n)^2/(n-1)$. Under H_0, standard asymptotic arguments imply that the t-statistic converges to a standard normal distribution as $n \to \infty$. One can then construct a CI by inverting the aforementioned hypothesis tests. In contrast to the first CI, the validity of this method relies on standard asymptotic arguments with $n \to \infty$.

(iii) A bootstrap-based CI based on observing the highest valuations for each auction. This approach is obviously infeasible in using data from first-price auctions, but it could be feasible with bid data from second-price auctions; cf. Menzel and Morganti (2013). Observing the highest valuations across auctions allows us to identify the distribution of valuations which, in turn, allows us to identify μ_K (further details are provided in Section A.5). By replacing identified objects with suitable estimators, one can consistently estimate μ_K. In addition, one can repeat this process using bootstrap samples to construct a CI for μ_K. The method is implemented with 500 bootstrap samples. The validity of this method relies on standard asymptotic arguments with $n \to \infty$.

Table 4 presents the coverage and length of the three CIs. Our proposed CI suffers from a slight undercoverage when valuations are $U(0, 3)$ or when n is significantly larger than K. As our theory predicts, this undercoverage should diminish as K increases. On the other hand, our proposed CI demonstrates excellent size control for the non-$U(0, 3)$ valuations and K is significantly larger than

n. The other two methods perform analogously to Table 1. The CI based on observing the highest valuation and the transaction price and the asymptotics $n \to \infty$ are relatively shorter and tend to suffer from undercoverage, especially when n is small or valuations are Pareto distributed. In turn, the bootstrap-based CI tends to suffer significant undercoverage problems.

# Bidders	$K = 10$				$K = 100$				$K \sim U\{90, 91, \ldots, 110\}$			
# Auctions	$n = 10$		$n = 100$		$n = 10$		$n = 100$		$n = 10$		$n = 100$	
Dist.	Cov	Lgth	Cov	Lgth	Cov	Lgth	Cov	Lgth	Cov	Lgth	Cov	Lgth
Method (i): Our proposed CI, asy. w. $K \to \infty$												
$U(0,3)$	0.89	1.20	0.89	0.29	0.91	0.22	0.98	0.12	0.91	0.22	0.98	0.13
$\|N(0,1)\|$	0.94	1.28	0.84	0.40	0.97	1.46	0.98	0.33	0.98	1.47	0.98	0.33
$\|t(20)\|$	0.95	1.29	0.86	0.46	0.98	1.30	0.97	0.42	0.98	1.34	0.98	0.43
$Pa(0.25)$	0.97	1.47	0.87	0.54	0.97	2.12	0.95	0.99	0.97	2.12	0.96	1.00
Method (ii): CI based on highest valuations, asy. w. $n \to \infty$												
$U(0,3)$	0.89	0.03	0.95	0.01	0.85	0.00	0.95	0.00	0.90	0.00	0.96	0.00
$\|N(0,1)\|$	0.86	0.31	0.94	0.11	0.86	0.24	0.93	0.09	0.86	0.25	0.93	0.09
$\|t(20)\|$	0.88	0.40	0.93	0.14	0.83	0.37	0.92	0.13	0.85	0.36	0.94	0.13
$Pa(0.25)$	0.81	0.69	0.88	0.28	0.76	1.22	0.91	0.47	0.75	1.21	0.90	0.47
Method (iii): CI based on bootstrap & highest valuations, asy. w. $n \to \infty$												
$U(0,3)$	0.35	0.19	0.89	0.11	0.36	0.02	0.89	0.01	0.36	0.02	0.88	0.01
$\|N(0,1)\|$	0.28	0.29	0.84	0.12	0.28	0.21	0.85	0.09	0.27	0.21	0.86	0.09
$\|t(20)\|$	0.29	0.34	0.85	0.14	0.27	0.28	0.82	0.12	0.26	0.28	0.86	0.12
$Pa(0.25)$	0.20	0.34	0.69	0.20	0.23	0.61	0.69	0.35	0.23	0.61	0.69	0.35

Table 4: Empirical coverage frequency (Cov) and length (Lgth) of various CIs for the winner's expected utility μ_K in first-price auctions. The results are the average of 500 simulation draws and a nominal coverage level of 95%.

Finally, we consider the problem of hypothesis testing about the tail index using only transaction prices. We focus on testing $H_0 : \xi = -1$ vs. $H_1 : \xi \in \Xi/\{-1\}$, with $\Xi = [-1, 0.5]$ and W equal to the uniform distribution over this interval. Table 5 shows the average rejection rate of the test proposed in Section 4.3 over 500 simulations using a desired nominal size of 5%. Under H_0 (i.e., when valuations are $U(0,3)$), our proposed test controls size as long as the number of bidders is not too small relative to the sample size. Under H_1 (i.e., when valuations are $|N(0,1)|$, $|t(20)|$, and $Pa(0.25)$), our proposed test exhibits adequate power.

# Bidders	$K=10$		$K=20$		$K=100$		$K \sim U\{90,91,\ldots,110\}$	
# Auctions	$n=10$	$n=100$	$n=10$	$n=100$	$n=10$	$n=100$	$n=10$	$n=100$
$U(0,3)$	0.06	0.09	0.04	0.06	0.04	0.05	0.03	0.09
$\|N(0,1)\|$	0.25	1.00	0.22	1.00	0.28	1.00	0.29	1.00
$\|t(20)\|$	0.27	1.00	0.29	1.00	0.34	1.00	0.36	1.00
$Pa(0.25)$	0.54	1.00	0.55	1.00	0.55	1.00	0.62	1.00

Table 5: Empirical rejection rate of the test proposed in Section 4.3 for $H_0 : \xi = -1$ in first-price auctions. The results are the average of 500 simulation draws and a nominal size of 5%.

A.5 Auxiliary derivations related to the Monte Carlo simulations

We first provide auxiliary derivations related to the third CI considered in Section 6. In a second-price auction $j = 1, \ldots, n$ with K bidders, the CDF of the transaction price satisfies:

$$F_{P_j}(x) \overset{(1)}{=} F_{V_{(2),j}}(x) \overset{(2)}{=} F_V(x)^K + KF_V(x)^{K-1}(1 - F_V(x)), \tag{A.56}$$

where (1) holds by (3.2) and (2) by the fact that bidders' valuations are i.i.d. The previous equation expresses the CDF of the transaction price as a function of the CDF of valuations. By inverting this mapping, we can derive a relationship between the distribution of transaction prices and the distribution of valuations. Furthermore, we have the following connection between the distribution of valuations and the winner's expected utility:

$$\begin{aligned}
\mu_K &\overset{(1)}{=} E[V_{(1),j} - P_j] \\
&\overset{(2)}{=} E[V_{(1),j} - V_{(2),j}] \\
&\overset{(3)}{=} K \int_0^\infty x F_V(x)^{K-1} f_V(x) dx - (K-1)K \int_0^\infty x (f_V(x)(1 - F_V(x)) F_V(x)^{K-2}) dx \\
&\overset{(4)}{=} K \left(\int_0^\infty (1 - F_V^K(x)) dx - \int_0^\infty (1 - F_V^{K-1}(x)) dx \right),
\end{aligned}$$

where (1) holds because the auctions are i.i.d. (which holds in all of our Monte Carlo designs), (2) holds by (3.2), (3) by the fact that bidders are i.i.d and valuations are nonnegative (which holds in all of our Monte Carlo designs), and (4) by integration by parts.

To conclude, we provide auxiliary derivations related to the third CI considered in Section A.4. Recall that this method presumes that we observe the highest valuation $V_{(1),j}$ for each auction $j = 1, \ldots, n$. Given that there are K bidders, the CDF of the highest valuation satisfies:

$$F_{V_{(1)}}(x) \overset{(1)}{=} F_V(x)^K, \tag{A.57}$$

where (1) holds by the fact that bidders' valuations are i.i.d. and $V_{(1)} = \max\{V_1, V_2, \ldots, V_K\}$. Furthermore, we can connect the distribution of valuations and the winner's expected utility in the following fashion:

$$\begin{aligned}
\mu_K &\overset{(1)}{=} E\left[\frac{\int_0^{V_{(1)}} F_V(u)^{K-1}du}{F_V(V_{(1)})^{K-1}}\right] \\
&\overset{(2)}{=} \int_0^{\infty}\left(\int_0^{v} KF_V^{K-1}(x)dx\right)f_V(v)dv \\
&\overset{(3)}{=} K\int_0^{\infty}(1-F_V(v))F_V(v)^{K-1}dv,
\end{aligned}$$

where (1) holds by (4.8) and by the fact that valuations are nonnegative and auctions are i.i.d. (which holds in all of our Monte Carlo designs), (2) by (A.57), and (3) by integration by parts.

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