Estimation and Inference of the Forecast Error Variance Decomposition for Set-Identified SVARs

Francesco Fusari, **Joe Marlow** and Alessio Volpicella
University of Surrey

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Motivation

- For SVARs, there is a classical identification problem: more parameters than equations $\Rightarrow$ researchers impose identifying restrictions.
- Under weak restrictions, the Forecast Error Variance Decomposition (FEVD) is set-identified. It can be only bounded up to an interval $\Rightarrow$ estimation and inference is challenging.

Motivation

- Most of the empirical practice for estimation of set-identified FEVD follows a standard Bayesian approach (Arias et al. 18, *Ecta*).
- Two concerns for this approach under set-identification:
  1. priors are not updated by data, even asymptotically (Baumeister and Hamilton 15, *Ecta*; Poirier 98, *Econom. Theory*).
  2. any prior choice breaks down the asymptotic equivalence between Bayesian and frequentist inference: the Bayesian interval asymptotically lies inside the true identified set (Moon and Schorfheide 12, *Ecta*).

Motivation

- Robust approaches have been proposed accordingly, but they mostly focus on Impulse Response Functions (IRFs) and cannot be easily extended to FEVD (Granziera et al. 18, *QE*; Gafarov et al. 18, *JoE*).
- Significant exception: multiple-prior robust framework in Giacomini and Kitagawa (21, *Ecta*) [GK21], but ... is extremely heavy computationally (its practical feasibility is limited) and frequentist validity is not guaranteed for FEVD.

This Paper

- Estimation and inference toolkit for set-identified $FEVD_{ijh}$ with no need to rely on a prior we cannot revise.
  - We estimate the upper and lower bounds of the set as solution to quadratic programming.
  - We prove differentiability of such bounds.
  - A delta-method confidence interval (adjusted by the length of the set) is proposed.
- Both frequentist and Bayesian interpretation.

Today's Presentation

- Literature.
- Econometric Framework.
- Estimation.
- Differentiability (Very Briefly).
- Inference: frequentist and Bayesian.
- Monte-Carlo Simulation.
- Empirical Application.

Econometric Framework

Econometric Framework

- SVAR(p):

$$\boldsymbol{A}_0 \boldsymbol{y}_t = \boldsymbol{a} + \sum_{j=1}^{p} \boldsymbol{A}_j \boldsymbol{y}_{t-j} + \boldsymbol{\epsilon}_t, \tag{1}$$

$$\boldsymbol{\epsilon}_t | (\boldsymbol{y}_{t-1}, \dots) \sim (\boldsymbol{0}, \boldsymbol{I}_n), \tag{2}$$

  where $\boldsymbol{\theta} = (\boldsymbol{A}_0, \boldsymbol{a}, \boldsymbol{A}_1, \dots, \boldsymbol{A}_p)$ collects the structural parameters and n is the number of variables.

- Reduced-form VAR(p):

$$\boldsymbol{y}_t = \boldsymbol{b} + \sum_{j=1}^{p} \boldsymbol{B}_j \boldsymbol{y}_{t-j} + \boldsymbol{u}_t, \tag{3}$$

$$\boldsymbol{u}_t | (\boldsymbol{y}_{t-1}, \dots) \sim (\boldsymbol{0}, \boldsymbol{\Sigma}), \tag{4}$$

  where $\boldsymbol{\phi} = (\boldsymbol{b}, \boldsymbol{B}_1, \dots, \boldsymbol{B}_p, \boldsymbol{\Sigma}) \in \mathcal{R}^d$ collects the reduced-form parameters.

The Identification Problem

- $dim(\boldsymbol{\theta}) > dim(\boldsymbol{\phi})$: data are not enough to recover structural parameters, we need identifying restrictions.
- $\boldsymbol{\theta} = \boldsymbol{Q} f(\boldsymbol{\phi})$, where $\boldsymbol{Q} \in \boldsymbol{\mathcal{Q}}$ is an orthonormal (rotation) $n \times n$ matrix, with $\boldsymbol{\mathcal{Q}}$ being the set of orthonormal matrices.
- Identification corresponds to pin down $\boldsymbol{Q}$.
- If we are interested in identifying the effect of shock j, this corresponds to pin down the j-th column of $\boldsymbol{Q}$, i.e., $\boldsymbol{q_j}$.

Forecast Error Variance Decomposition

- $FEVD_{ijh} = \boldsymbol{q}_j' \mathbf{Y}_h^i(\boldsymbol{\phi})\boldsymbol{q}_j$ represents the contribution of shock j to explain the fluctuation of variable i at horizon h.
- $0 \leq FEVD_{ijh} \leq 1$, where $\mathbf{Y}_h^i(\boldsymbol{\phi})$ is a $n \times n$ semipositive definite matrix.
- E.g., j is a financial shock, i the GDP growth.

Set-Identification

- It is common to impose sign-restrictions, typically on the Impulse Response Functions (IRFs).
- E.g. negative financial shock decreases GDP growth.
- For a single shock j, the restrictions can collected by $\boldsymbol{S}_j(\boldsymbol{\phi})\boldsymbol{q}_j \geq \mathbf{0}$.
- $\boldsymbol{S}_j(\boldsymbol{\phi})$ is a $s_j \times n$ matrix, with s_j being the number of sign restrictions on shock j.
- This delivers a set for $FEVD_{ijh}$:

$$IS_{FEVD}(\boldsymbol{\phi}) = \{FEVD_{ijh} : \boldsymbol{Q} \in \mathcal{Q}(\boldsymbol{\phi}|\boldsymbol{S}_j(\boldsymbol{\phi})\boldsymbol{q}_j \geq \mathbf{0})\}. \tag{5}$$

We provide a toolkit for estimation and inference of the lower and upper bounds of $IS_{FEVD}(\boldsymbol{\phi})$:

Definition 1

Given a vector of the reduced-form parameters $\boldsymbol{\phi}$, a shock of interest j^*, $l_{ij^*h}(\boldsymbol{\phi})$ and $u_{ij^*h}(\boldsymbol{\phi})$ are the lower- and upper-bound of $IS_{FEVD}(\boldsymbol{\phi})$, respectively:

$$l_{ij^*h}(\boldsymbol{\phi}) \equiv min_{\mathbf{q}_{j^*}} \mathbf{q}'_{j^*} \mathbf{\Upsilon}^i_h(\boldsymbol{\phi}) \mathbf{q}_{j^*} \; s.t. \; \mathbf{S}_{j^*}(\boldsymbol{\phi}) \mathbf{q}_{j^*} \geq \mathbf{0}, \; , \; \mathbf{q}'_{j^*} \mathbf{q}_{j^*} = 1, \quad (6)$$

and

$$u_{ij^*h}(\boldsymbol{\phi}) \equiv max_{\mathbf{q}_{j^*}} \mathbf{q}'_{j^*} \mathbf{\Upsilon}^i_h(\boldsymbol{\phi}) \mathbf{q}_{j^*} \; s.t. \; \mathbf{S}_{j^*}(\boldsymbol{\phi}) \mathbf{q}_{j^*} \geq \mathbf{0} \; , \; \mathbf{q}'_{j^*} \mathbf{q}_{j^*} = 1. \quad (7)$$

Estimation

Some Assumptions

Assumption A1: Non-Emptiness

$IS_{FEVD}(\boldsymbol{\phi})$ *is non-empty at* $\boldsymbol{\phi}$.

Assumption A2: Linear Independence

Given a constrained shock j^*, $\boldsymbol{S}_{j^*}(\boldsymbol{\phi})$ *is linearly independent.*

Estimation

- Let $\boldsymbol{r}(\phi)$ denote the set of active constraints, i.e. $\boldsymbol{r}(\boldsymbol{\phi})\boldsymbol{q}_{j^*} = \boldsymbol{0}$.
- Given $\boldsymbol{S}_{j^*}(\boldsymbol{\phi})$, there are $\sum_{i=0}^{min(n-1,s_{j^*})} \frac{s_{j^*}!}{i!(s_{j^*}-i)!}$ possible combinations of active constraints, i.e. possible ways to construct $\boldsymbol{r}(\boldsymbol{\phi})$.
- For example, if there are 2 inequality constraints, there are 4 ways of constructing $\boldsymbol{r}(\boldsymbol{\phi})$: both constraints are binding, the first one is binding, the second one is binding, neither are binding.

Theorem 1

Suppose that Assumption A1 and A2 hold and a single shock j^* is sign-constrained.

- *Let* $\lambda(\boldsymbol{\phi}, \boldsymbol{r})$ *be the maximum eigenvalue of the matrix* $\boldsymbol{Z}(\boldsymbol{\phi}, \boldsymbol{r}) = [\boldsymbol{I}_n - \boldsymbol{P}(\boldsymbol{\phi}, \boldsymbol{r})]\, \Upsilon_h^i(\boldsymbol{\phi})$, *where* $\boldsymbol{P}(\boldsymbol{\phi}, \boldsymbol{r}) = \boldsymbol{r}(\boldsymbol{\phi})' \left[\boldsymbol{r}(\boldsymbol{\phi})\boldsymbol{r}(\boldsymbol{\phi})'\right]^{-1} \boldsymbol{r}(\boldsymbol{\phi})$.
- $u_{ij^*h}(\boldsymbol{\phi}) = max_{\boldsymbol{r}(\boldsymbol{\phi})} \lambda(\boldsymbol{\phi}, \boldsymbol{r})$ *as long as inactive constraints are satisfied.*

We can obtain the minimum analogously.

Inference

Very Briefly: Differentiability

Assumption A3: Simple Eigenvalues

The algebraic multiplicity of the eigenvalues delivering $u_{ij^*h}(\boldsymbol{\phi})$ *and* $l_{ij^*h}(\boldsymbol{\phi})$ *is equal to* 1.

Assumption A4: Differentiability

$S_{j^*}(\boldsymbol{\phi})$ *is differentiable at* $\boldsymbol{\phi}$.

Under this assumption, we prove the differentiability of $u_{ij^*h}(\boldsymbol{\phi})$ and $l_{ij^*h}(\boldsymbol{\phi})$, i.e. $\frac{\partial u_{ij^*h}(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}}$ and $\frac{\partial l_{ij^*h}(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}}$ exist and can be characterized.

Confidence Interval

We propose a delta-method interval, adjusted by the length of the set:

$$CI_\alpha \equiv \left[l_{ij^*h}(\widehat{\boldsymbol{\phi}}_{OLS}) - c_\alpha \widehat{\sigma}_{l\,ij^*h}/\sqrt{T},\, u_{ij^*h}(\widehat{\boldsymbol{\phi}}_{OLS}) + c_\alpha \widehat{\sigma}_{u\,ij^*h}/\sqrt{T} \right], \quad (8)$$

where

$$\widehat{\sigma}_{lij^*h} = \left[\left(\frac{\partial l_{ij^*h}(\widehat{\boldsymbol{\phi}}_{OLS})}{\partial \widehat{\boldsymbol{\phi}}_{OLS}} \right)' \widehat{\boldsymbol{\Omega}} \left(\frac{\partial l_{ij^*h}(\widehat{\boldsymbol{\phi}})}{\partial \widehat{\boldsymbol{\phi}}_{OLS}} \right) \right]^{\frac{1}{2}}. \quad (9)$$

$\widehat{\boldsymbol{\Omega}}$ is the estimated variance-covariance matrix of $\boldsymbol{\phi}$. $\widehat{\sigma}_{lij^*h}$ is defined analogously.

The critical value c_α is adjusted to take into account the length of the set:

$$\Phi\left(c_\alpha + \frac{\sqrt{T}\widehat{\Delta}_{ij^*h}}{max\{\widehat{\sigma}_{lij^*h}, \widehat{\sigma}_{uij^*h}\}}\right) - \Phi\left(-c_\alpha\right) = 1 - \alpha, \qquad (10)$$

where $\Phi(\bullet)$ is the standard normal cumulative distribution evaluated at $\bullet$ and $\widehat{\Delta}_{ij^*h} = u_{ij^*h}(\widehat{\boldsymbol{\phi}}_{OLS}) - l_{ij^*h}(\widehat{\boldsymbol{\phi}}_{OLS})$ is the estimated length of the identified set.

Frequentist Coverage

Assumption A5: Asymptotic Normality

OLS estimators uniformly satisfy

$$\sqrt{T}(\widehat{\boldsymbol{\phi}}_{OLS} - \boldsymbol{\phi}(P)) \to_d N(\mathbf{0}, \boldsymbol{\Omega}(P)), \tag{11}$$

$$\widehat{\boldsymbol{\Omega}} \to_p \boldsymbol{\Omega}(P), \tag{12}$$

where P is the data-generating process.

We establish the uniform consistency of CI_α:

Theorem 3

In a compact subset for $\boldsymbol{\phi}$, suppose that Assumptions A1-A5 hold and $\frac{\partial u_{ij^*h}(\widehat{\boldsymbol{\phi}}_{OLS})}{\partial \widehat{\boldsymbol{\phi}}_{OLS}}$ and $\frac{\partial l_{ij^*h}(\widehat{\boldsymbol{\phi}}_{OLS})}{\partial \widehat{\boldsymbol{\phi}}_{OLS}}$ are different from zero. We obtain

$$lim_{T\to\infty}\ inf\ \ inf_{FEVD_{ij^*h}(P)\in IS(\boldsymbol{\phi}(P))} Pr\left(FEVD_{ij^*h}(P)\in CI_\alpha\right)=1-\alpha. \quad (13)$$

Sketch of the proof: uniform convergence in distribution of the delta-method under some conditions + length-adjusted confidence interval in Stoye (09, *Ecta*).

Empirical Application

Empirical Application

- Estimation of the effects of credit supply shocks on the US economy.
- $1973Q1 - 2012Q4$.
- Identification strategy from Mumtaz et al. (18, *Int. Econ. Rev.*): sign restrictions from the DSGE model of Gertler and Karadi (11, *JME*).

Restrictions

Variable	Restriction
Lending Rate Spread	≥ 0
Total Lending Growth	≤ 0
Investment Growth	
Consumption Growth	
GDP Growth	≤ 0
CPI Inflation	≤ 0
Three-Month Treasury Bill Rate	≤ 0
Financial Conditions Index	
Economic Uncertainty	

Table: Set-Identification of a Credit Supply Shock

Inference, $1 - \alpha = 0.95$

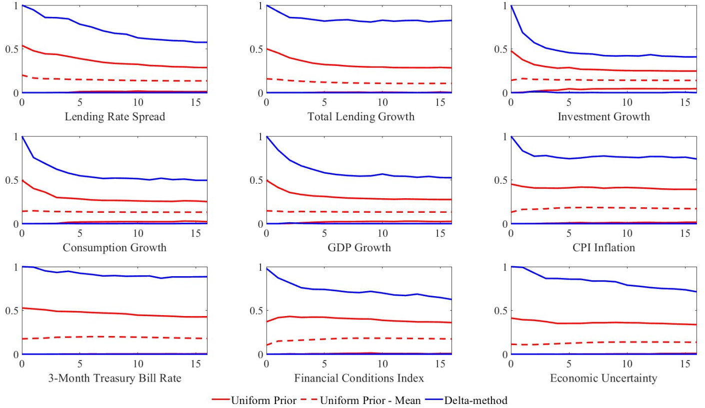

Figure: Our toolkit (blue) vs. Standard Bayesian Credible Region (red)

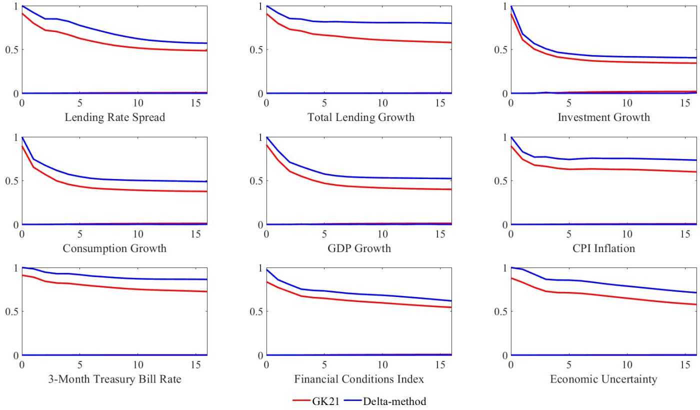

Figure: Confidence Interval (our toolkit, blue) vs. Robust Bayesian Credible Region in GK21 (red)

- Computational gains with respect to the robust Bayesian approach in GK21.
- For this application, our toolkit takes 35s, in comparison to $9,074$s for GK21.

Conclusion

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- We provide a new toolkit for estimation and inference of set-identified FEVD in a SVAR setting.
- This overcomes the well-known problem of non-revising the prior distribution in the standard approach.
- Bounds of the FEVD characterized as solutions to quadratic programming.
- Differentiability of the bounds allows us to propose a length-set-adjusted delta method confidence interval.
- The confidence interval is uniformly consistent in level and has asymptotic robust Bayesian credibility.
- High computational efficiency.

Appendix

Estimation

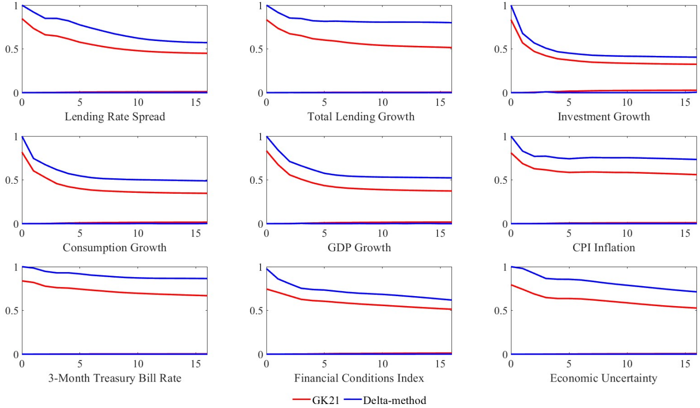
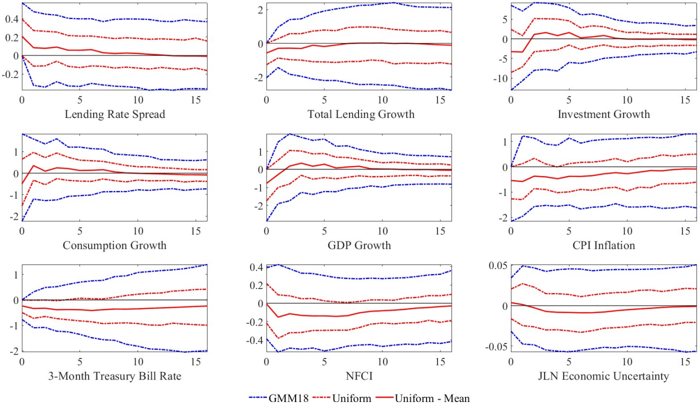


Figure: Estimated Bounds: our toolkit (blue) vs. GK21 (red)

IRFs


Lending Rate Spread
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GMM18
Uniform
Uniform - Mean