TRADABLE FACTOR RISK PREMIA AND ORACLE TESTS OF ASSET PRICING MODELS

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Factor asset pricing models

Goals:

1. Determine which factors price a given cross-section of assets
2. Quantify the compensations for exposures to factor risk $\longleftarrow$ This paper

Relevant facts about this literature

- A zoo of factors and a zoo of estimation methods

- Massive fragility of conventional approaches:

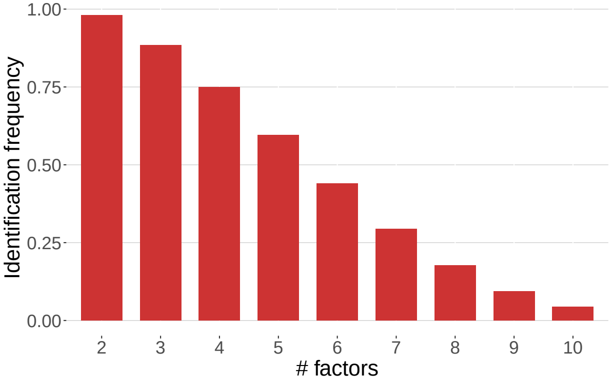
  1. Misspecification
     - Most factor models are still low-dimensional linear $\longrightarrow$ misspecified
  2. Identification
     - Methods rely on a full-rank factor-returns covariance
     - Widely known to be violated in practice: some complicated solutions are available

Identification failures

- Weak or useless factors, i.e., poorly or not correlated with asset returns: (Kan Zhang 99, Kleibergen 09, Gospodinov Kan Robotti 14, 17, Kleibergen Zhan 2023, ...)
  - Undermine identification of risk premia as expected return projections on factor exposures
  - Non-standard inference: conservative on strong and spurious on weak factors
  - Complex interaction with potential asset pricing model misspecification
- General rank-deficiencies in factor exposures, e.g., level factors

Identification frequencies of models from the zoo

β–Rank test (Kleibergen Paap 2006, Chen Feng 2019 test) of models from the zoo



No correction for multiple-testing $\longrightarrow$ conservative figure

This paper: contributions

1. Understanding the economic assumptions leading to identification failure

   - The core of any notion of factor risk premia $\longrightarrow$ $\lambda := -\mathbb{C}ov(\boldsymbol{F}, M)$

   - Identification issues stems from the economic formulation of risk premia

     a) components of the SDF M

     b) SDF estimation strategy (test assets, loss function, etc).

   - Risk premia identification is not a statistical feature!

This paper: contributions

2. Our approach: Rely on a notion of risk premia

    1. Rooted in economic theory with clear interpretation

    2. Not requiring economic assumptions typically leading to identification failure

    3. Point–identified and informative when others are not (not just robust to identification failure)

This paper: contributions

3. Provide a framework for studying the factor zoo with general forms of identification failure:

- Search for a robust subset of economically relevant, full-rank models
  - First tier: MKT, SMB (Fama French 1992), MGMT (Stambaugh Yuan 2017), ICR (He Kelly Manela 2017) and MKT* (Daniel et al. 2020)
  - Second tier: BEH_FIN (Daniel et al. 2020), COMP_ISSUE (Daniel Titman 2006) and CMA (Fama French 2015)
- Identify economic sources of fragility of other notions of risk premia in conditional and unconditional models

FACTOR RISK PREMIA

Factor risk premia

- Risk premia in arbitrage-free frictionless market: $RP(\cdot) := -\mathbb{C}ov[\,\cdot\,, SDF]$
- Conventional approaches for factors (Sharpe 64, Lintner 65, Fama MacBeth 73, ...):
  1. Select a linear SDF projection on some $\boldsymbol{X}$ $\longrightarrow$ $M_X(\gamma) = 1 - \gamma'(\boldsymbol{X} - \mathbb{E}[\boldsymbol{X}])$
  2. Factor $\boldsymbol{F}$ risk premia $\longrightarrow$ $RP(\boldsymbol{F}) := -\mathbb{C}ov[\boldsymbol{F}, M_X]$
- Modelling choices (direct impact on identification):
  1. Which choice of $\boldsymbol{X}$? typically factors $\boldsymbol{F}$
  2. How to project the SDF?

SDF projection on the factor space

- SDF projection on the factors $\longrightarrow$ $M_F(\gamma) = 1 - \gamma'(\boldsymbol{F} - \boldsymbol{\mu}_F)$

- E.g., misspecification-robust approach (Kan Robotti Shanken '13, Fama MacBeth: $\boldsymbol{W} = \boldsymbol{I}$):

$$\gamma_{\boldsymbol{W}} \in \underset{\gamma}{\operatorname{argmin}} \ \mathbb{E}[M(\gamma)\boldsymbol{R}]' \boldsymbol{W} \, \mathbb{E}[M(\gamma)\boldsymbol{R}] \iff \{\gamma : (\boldsymbol{V}_{FR}\boldsymbol{W}\boldsymbol{V}_{RF})\gamma = \boldsymbol{V}_{FR}\boldsymbol{W}\boldsymbol{\mu}_R\}$$

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- Implied misspecification-robust risk premia is identified iff $\boldsymbol{V}_{FR}$ has full-rank

$$\boldsymbol{\lambda}_{\boldsymbol{W}} = -\mathbb{C}ov[\boldsymbol{F}, M_F(\gamma_{\boldsymbol{W}})]$$

  - **Model-dependent**: under misspecification, the population risk premium on Mkt changes if model = (Mkt, SMB) or (Mkt, HML)

SDF projection on the return space

Tradable Factor Risk Premia

Negative factor covariance with the minimum variance SDF projection on returns:

$$M_R = 1 - \boldsymbol{\mu}_R \boldsymbol{V}_R^{-1}(\boldsymbol{R} - \boldsymbol{\mu}_R)$$

$$\boldsymbol{\lambda}^* := -\mathbb{C}ov(\boldsymbol{F}, M_R) = \boldsymbol{V}_{FR}\boldsymbol{V}_R^{-1}\boldsymbol{\mu}_R$$

SDF projection on the return space

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- **Interpretation**: Equals risk premia of factors' orthogonal projection on returns (Balduzzi Kallal 97)

$$\boldsymbol{\lambda}^* = \mathbb{E}[\boldsymbol{V}_{FR}\boldsymbol{V}_R^{-1}\boldsymbol{R}]$$

  - In full-rank models, equals 2-pass risk premium of factor mimicking portfolio (Huberman Kandel Stambaugh 87, Breeden Gibbson Litzenberger 89,...)

Properties: specification

$$\boldsymbol{\lambda}^* := -\mathbb{C}ov(\boldsymbol{F}, M_R) = \boldsymbol{V}_{FR}\boldsymbol{V}_R^{-1}\boldsymbol{\mu}_R$$

- **Model-independence**: No factor–SDF spanning requirement

- **Separability**: Risk premium of factor k is unaffected by the other factors

$$\lambda_k^* = -\mathbb{C}ov[F_k, M_R] = \mathbb{C}ov[F_k, \boldsymbol{R}]\boldsymbol{V}_R^{-1}\boldsymbol{\mu}_R$$

  – E.g., population risk premium on Mkt does not change if model = (Mkt, SMB) or (Mkt, HML)

Properties: identification

$$\boldsymbol{\lambda}^* := -\mathbb{C}ov(\boldsymbol{F}, M_R) = \boldsymbol{V}_{FR}\boldsymbol{V}_R^{-1}\boldsymbol{\mu}_R$$

- **Identification**: Remains point-identified if $\boldsymbol{V}_R$ is invertible, even if $\boldsymbol{V}_{FR}$ is reduced-rank
- Assigns a zero risk premium to useless/weak factors

$$\mathbb{C}ov[F_k, \boldsymbol{R}] = 0 \quad \Longrightarrow \quad \lambda_k^* = 0$$

Other properties

- Equals factor's expectation if factor is in Span($\boldsymbol{R}$)

$$F_k = \gamma' \boldsymbol{R} \qquad \Longrightarrow \qquad \lambda_k^* = \boldsymbol{\gamma}' \boldsymbol{\mu}_R$$

- Invariant to simple repackagings of test assets (Kandel Stambaugh '95)

Generating factor–SDF

- Tradable risk premia as penalized misspecification-robust approach:

$$\gamma_{\boldsymbol{V}_R^{-1}} = \underset{\gamma}{\text{argmin}} \underbrace{\mathbb{E}[M_t(\gamma)\boldsymbol{R}_t]'\boldsymbol{V}_R^{-1}\mathbb{E}[M_t(\gamma)\boldsymbol{R}_t]}_{\text{missp. robust objective}} + \underbrace{\gamma'(\boldsymbol{V}_F - \boldsymbol{V}_{FR}\boldsymbol{V}_R^{-1}\boldsymbol{V}_{RF})\gamma}_{\text{penalty}}$$

  - Unique if $\boldsymbol{V}_F$ and $\boldsymbol{V}_R$ are invertible

- In full-rank models:

  - Non-tradable factors: $\boldsymbol{\lambda}_{\boldsymbol{V}_R^{-1}} = \boldsymbol{V}_F(\boldsymbol{V}_{FR}\boldsymbol{V}_R^{-1}\boldsymbol{V}_{RF})^{-1}\boldsymbol{\lambda}^*$
  - Tradable factors: $\boldsymbol{\lambda}_{\boldsymbol{V}_R^{-1}} = \boldsymbol{\lambda}^*$
  - When factors are spanned by returns, the approaches coincide ($\boldsymbol{V}_F - \boldsymbol{V}_{FR}\boldsymbol{V}_R^{-1}\boldsymbol{V}_{RF} = \boldsymbol{0}$)

ESTIMATION OF TRADABLE FACTOR RISK PREMIA

Estimators of tradable factor risk premia

- Sample estimator:

$$\hat{\boldsymbol{\lambda}}^* := \widehat{\boldsymbol{V}}_{FR}\widehat{\boldsymbol{V}}_R^{-1}\hat{\boldsymbol{\mu}}_R = \frac{1}{T}\sum_{t=1}^{T}\widehat{\boldsymbol{V}}_{FR}\widehat{\boldsymbol{V}}_R^{-1}\boldsymbol{F}_t$$

Estimators of tradable factor risk premia

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$$\hat{\boldsymbol{\lambda}}^* := \widehat{\boldsymbol{V}}_{FR}\widehat{\boldsymbol{V}}_R^{-1}\hat{\boldsymbol{\mu}}_R = \frac{1}{T}\sum_{t=1}^{T}\widehat{\boldsymbol{V}}_{FR}\widehat{\boldsymbol{V}}_R^{-1}\boldsymbol{F}_t$$

- Oracle estimator (robust to weak factors): $\tilde{\boldsymbol{\lambda}}^*$ where

$$\tilde{\lambda}_k^* = \text{hard-threshold}(\hat{\lambda}_k^*; \tau w_k) = \begin{cases} \hat{\lambda}_k^* & |\hat{\lambda}_k^*| > \tau w_k \\ 0 & \text{else} \end{cases}$$

  - penalty parameter $\tau > 0$ and adaptive weights $w_k = 1/\left\|\widehat{\mathbb{C}or}[F_k, \boldsymbol{R}]\right\|_2^2$

Assumptions

- $(\boldsymbol{R}_t, \boldsymbol{F}_t)$ near-epoch dependent with finite fourth moments, $\boldsymbol{V}_R$ is invertible
- Joint covariance of factors and returns:

$$\boldsymbol{V}^{(T)} := \begin{bmatrix} \boldsymbol{V}_R & \boldsymbol{V}_{RF}^{(T)} \\ \boldsymbol{V}_{FR}^{(T)} & \boldsymbol{V}_F \end{bmatrix}, \quad \boldsymbol{V}_{FR}^{(T)} := \boldsymbol{V}_{FR} + \boldsymbol{\Delta}/\sqrt{T}$$

  - $\boldsymbol{V}_{FR}$ can be reduced-rank
  - $\boldsymbol{\Delta}$ has zero rows where $\boldsymbol{V}_{FR}$ has nonzero rows:

$\longrightarrow \lambda_k^{*(T)} = \boldsymbol{V}_{F_k R} \boldsymbol{V}_R^{-1} \boldsymbol{\mu}_R, \qquad F_k = \text{strong}$

$\longrightarrow \lambda_k^{*(T)} = \frac{\boldsymbol{\Delta}}{\sqrt{T}} \boldsymbol{V}_R^{-1} \boldsymbol{\mu}_R, \qquad F_k = \text{weak}$

$\longrightarrow \lambda_k^{*(T)} = 0, \qquad F_k = \text{useless}$

Asymptotic behavior of $\hat{\lambda}^*$ without weak factors

Theorem

If $\boldsymbol{\Delta} = \mathbf{0}$, i.e., there are no weak factors:

$$\sqrt{T}(\hat{\boldsymbol{\lambda}}^* - \boldsymbol{\lambda}^*) \to_d \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}), \quad \boldsymbol{\Sigma} := \lim_{T\to\infty} \frac{1}{T} \sum_{t,s=1}^{T} \mathbb{E}(\boldsymbol{h}_t \boldsymbol{h}_s')$$

$$\boldsymbol{h}_t := \bar{\boldsymbol{F}}_t \bar{\boldsymbol{R}}_t' \boldsymbol{V}_R^{-1} \boldsymbol{\mu}_R - \boldsymbol{V}_{FR} \boldsymbol{V}_R^{-1} \bar{\boldsymbol{R}}_t \bar{\boldsymbol{R}}_t' \boldsymbol{V}_R^{-1} \boldsymbol{\mu}_R + \boldsymbol{V}_{FR} \boldsymbol{V}_R^{-1} \bar{\boldsymbol{R}}_t$$

$$(\bar{\boldsymbol{R}}_t := \boldsymbol{R}_t - \boldsymbol{\mu}_R \, , \; \bar{\boldsymbol{F}}_t := \boldsymbol{F}_t - \boldsymbol{\mu}_F)$$

- **Separability** also in the standard errors of tradable risk premia:

$$h_{kt} = \bar{F}_{kt} \bar{\boldsymbol{R}}_t' \boldsymbol{V}_R^{-1} \boldsymbol{\mu}_R - \boldsymbol{V}_{F_k R} \boldsymbol{V}_R^{-1} \bar{\boldsymbol{R}}_t \bar{\boldsymbol{R}}_t' \boldsymbol{V}_R^{-1} \boldsymbol{\mu}_R + \boldsymbol{V}_{F_k R} \boldsymbol{V}_R^{-1} \bar{\boldsymbol{R}}_t$$

Asymptotic behavior of $\hat{\lambda}^*$ with weak factors

Theorem

$$\sqrt{T}(\hat{\boldsymbol{\lambda}}^* - \boldsymbol{\lambda}^*) \to_d \mathcal{N}(\boldsymbol{\eta}(\boldsymbol{\Delta}), \boldsymbol{\Sigma}(\boldsymbol{\Delta}))$$

$$\boldsymbol{\eta}(\boldsymbol{\Delta}) = \boldsymbol{\Delta} \boldsymbol{V}_R^{-1} \boldsymbol{\mu}_R$$

$$\boldsymbol{\Sigma}(\boldsymbol{\Delta}) = \boldsymbol{\Sigma} - \boldsymbol{\eta}(\boldsymbol{\Delta})\boldsymbol{\eta}'(\boldsymbol{\Delta})$$

- No distortions for strong and useless factors:

$$F_k \text{ not weak} \implies \begin{cases} \boldsymbol{\eta}(\boldsymbol{\Delta})_k = 0 \\ \boldsymbol{\Sigma}(\boldsymbol{\Delta})_{kk} = \boldsymbol{\Sigma}_{kk} \end{cases}$$

- No impact on the estimator's mean squared error

Asymptotic behavior of $\tilde{\lambda}^*$ with weak factors

Theorem

If tuning parameter $\tau\sqrt{T} \to 0$ and $\tau T \to \infty$:

1. **Oracle factor separation**

$$\Pr(\tilde{\mathcal{S}} = \mathcal{S}) \to 1, \quad \text{where} \quad \begin{cases} \mathcal{S} := \{k \; : \; \mathbb{C}or[F_k, \boldsymbol{R}] \neq 0\} \; \to \; \text{not useless/weak factors} \\ \tilde{\mathcal{S}} := \{k \; : \; \tilde{\lambda}^*_k \neq 0\} \; \to \; \text{nonzero estimates} \end{cases}$$

2. **Oracle asymptotic distribution**

$$\sqrt{T}(\tilde{\boldsymbol{\lambda}}^*_{\mathcal{S}} - \boldsymbol{\lambda}^*_{\mathcal{S}}) \to_d \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\mathcal{S}}), \quad \boldsymbol{\Sigma}_{\mathcal{S}} := \lim_{T\to\infty} \frac{1}{T} \sum_{t,m=1}^{T} \mathbb{E}(\boldsymbol{h}_{\mathcal{S}t}\boldsymbol{h}'_{\mathcal{S}m})$$

MONTE CARLO STUDY

Skip simulation

Monte Carlo simulation setting

Simulation of returns and factors with moments calibrated from data:

$$\begin{pmatrix} \boldsymbol{R}_t \\ \boldsymbol{F}_t \end{pmatrix} \sim ii\mathcal{N}\left(\begin{pmatrix} \boldsymbol{\mu}_R \\ \boldsymbol{\mu}_F \end{pmatrix}, \begin{bmatrix} \boldsymbol{V}_R & \boldsymbol{V}_{RF} \\ \boldsymbol{V}_{FR} & \boldsymbol{V}_F \end{bmatrix} \right)$$

42 test assets: 25 ME-BTM + 17 IND

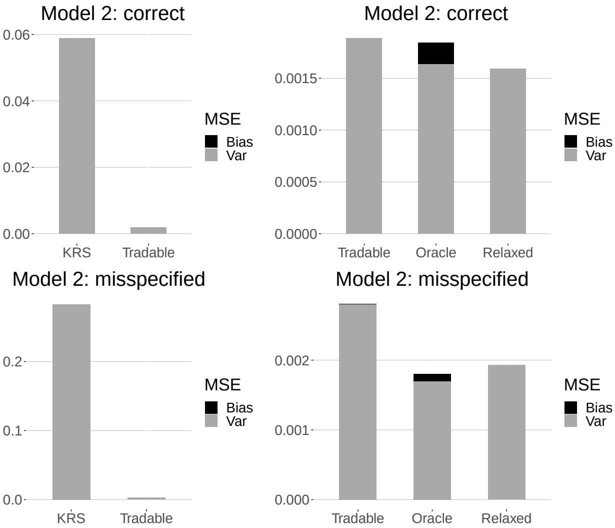
15 factors: 5FF, Mom, q-factor Hou et al. 2015, intermediary factor He et al. (2017), and betting-against beta factor Frazzini and Pedersen (2014)

Correctly specified and misspecified models:

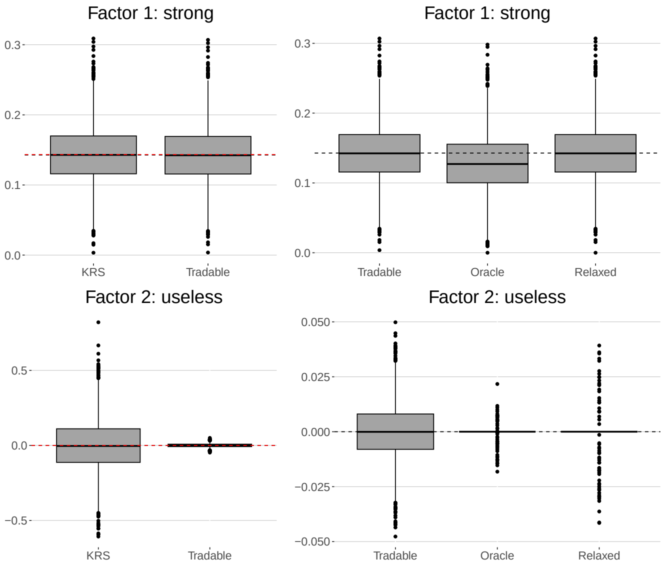
1 S || 1 S, 2 U || 1 S, 2 W || 3 S, 11 U, 1 W

S: strong, U: useless, W: weak

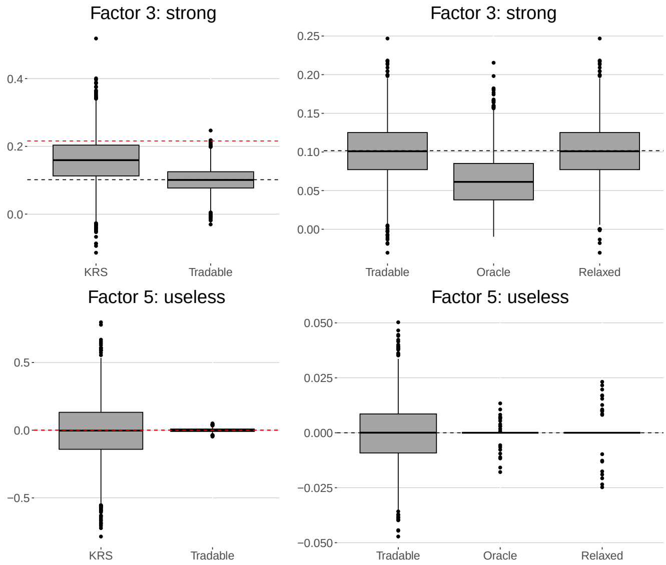
MSE and bias-variance decomposition: Model 2 (1 S, 2 U)



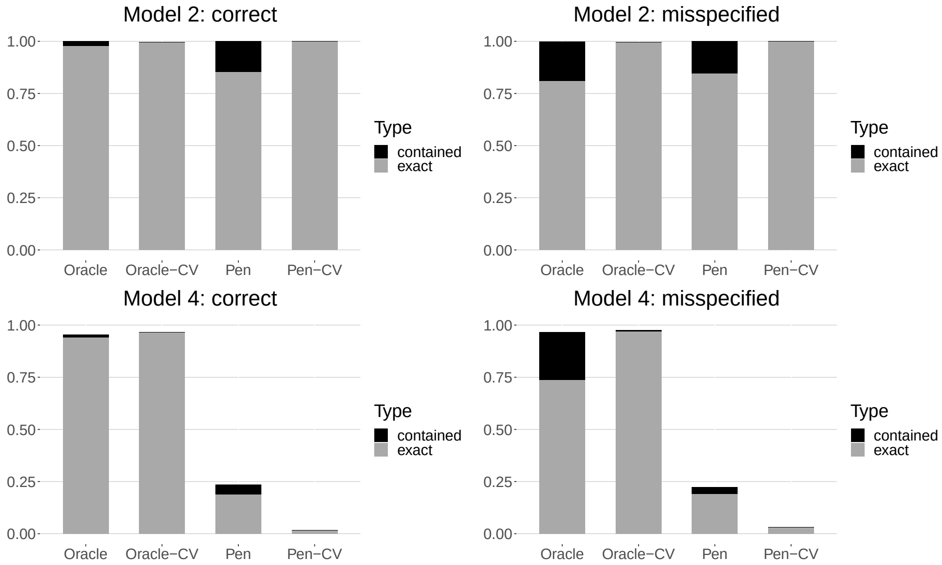
Point estimates distribution: Model 2 (1 S, 2 U) correct



Point estimates distribution: Model 4 (3 S, 11 U, 1 W) correct



Factor selection: contained $\Pr(\mathcal{S} \subset \check{\mathcal{S}})$ and exact $\Pr(\mathcal{S} = \check{\mathcal{S}})$



EMPIRICAL ANALYSIS

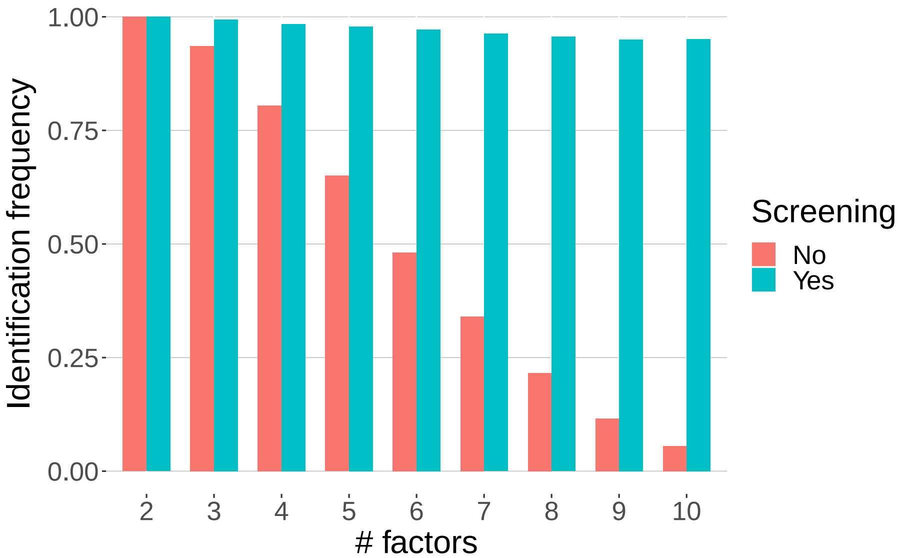
Goal

- Diagnose the factor zoo: detect factors giving rise to full-rank models
- Explore factor space dimension and factor composition of full-rank models
- Quantify the role of weak identification and misspecification for factor risk premia

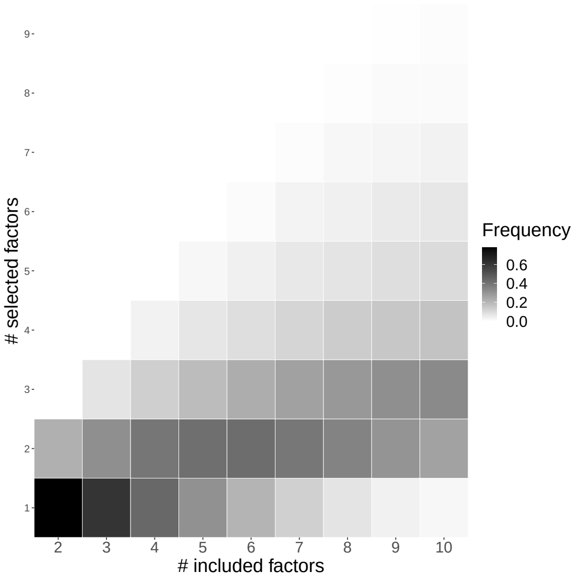
Empirical setting

- 51 factors from Bryzgalova Julliard Huang 2023 and test assets 25 ME/BTM + 17 IND
- Randomized models with 1–10 factors (always including the MKT)
- Study model properties before and after Oracle factor screening
- Test model rank properties via $\beta-$rank test in Kleibergen Paap 2006 and Chen Fang 2019

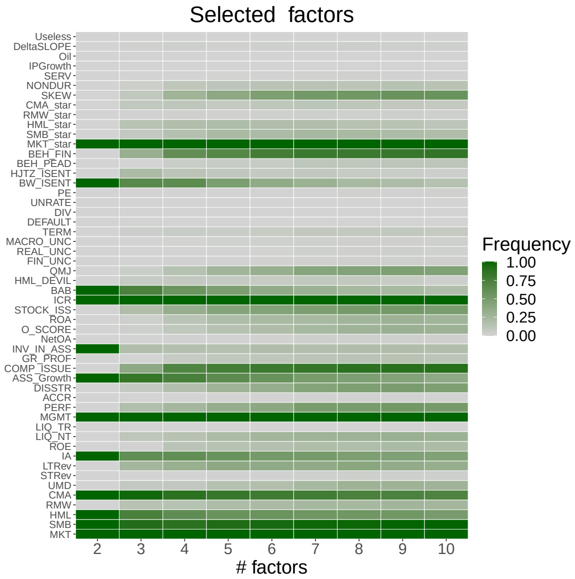
Model identification frequency



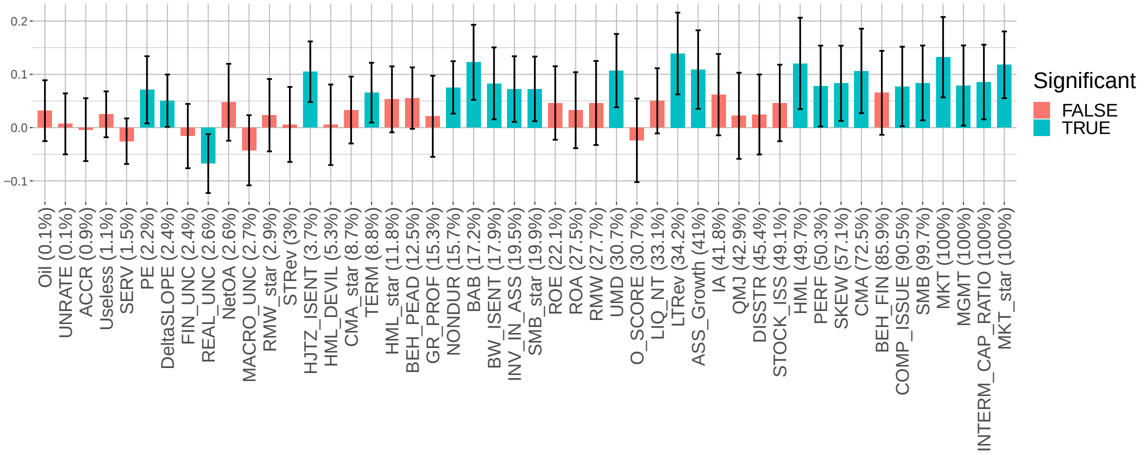
Post-screening factor space dimension



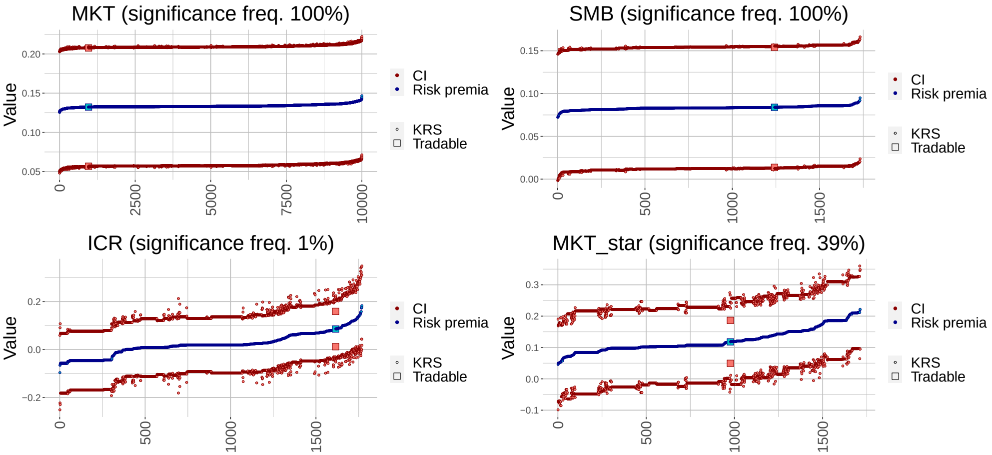
Factor selection frequency



Tradable factor risk premia



Selection of misspecification-robust factor risk premia



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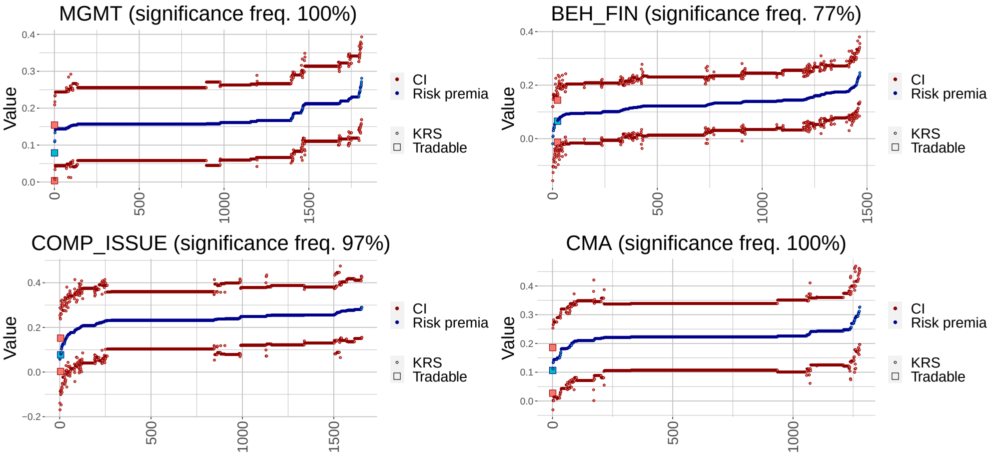
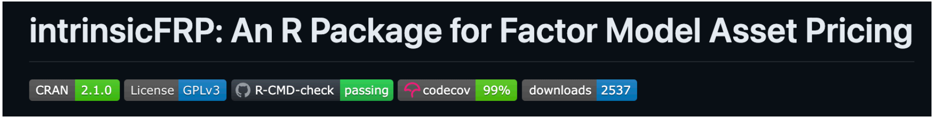


THANK YOU!

Software: R package intrinsicFRP available on CRAN

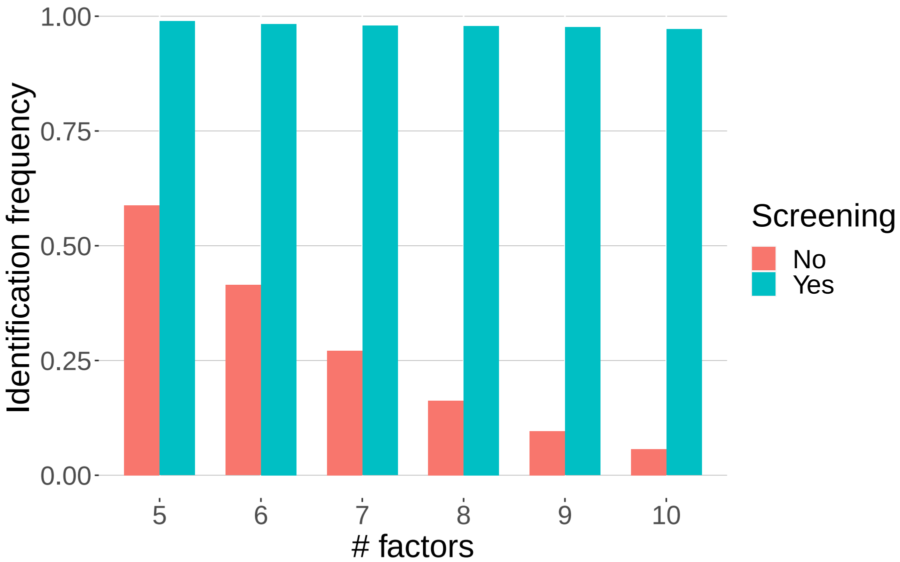


APPENDIX

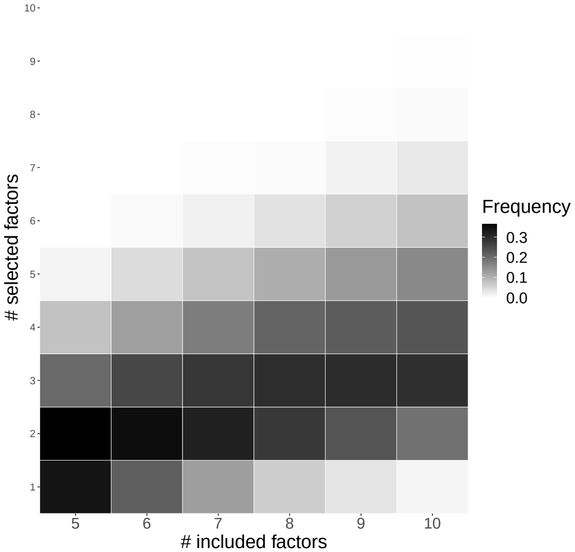
Robustness check

- Test assets: 25 size/book-to-market and 8 PCs
- PCs are extracted from 17 industry and 310 double-sorted portfolios
  - Portfolios sorted on: size, book-to-market, operating profitability, investment, net issuance, beta, variance, accruals, short-term reversal, long-term reversal, and momentum
- Initial model always includes the market

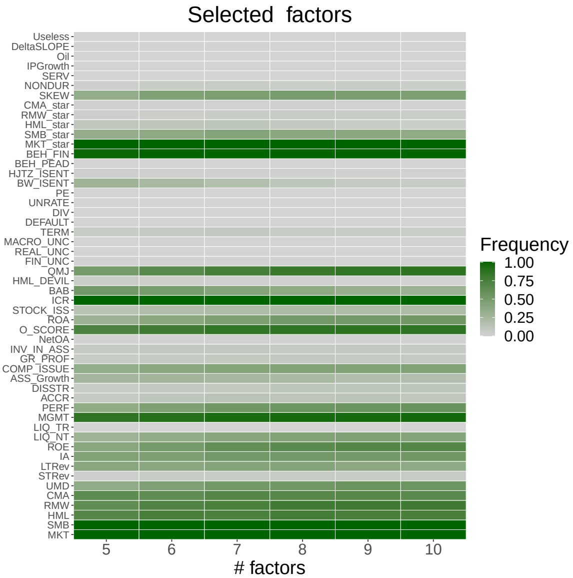
Robustness check: Model identification frequency



Post-screening factor space dimension



Factor selection frequency



Tradable factor risk premia