How Likely is it that Omitted Variable Bias will Overturn your Results?

Deepankar Basu

Department of Economics
University of Massachusetts Amherst

August 26–30, 2024
ESEM 2024

Question and Outline

Main question

- Question: Can we quantify the possibility of omitted variable bias overturning reported results in a linear model?
  - Result is understood to be overturned if zero is contained in the confidence interval for the true parameter
- I provide an affirmative answer by building on Cinelli and Hazlett (2020)
- This paper connects to the literature on sensitivity analysis: Frank, 2000; Krauth, 2016; Ding and VanderWeele, 2016; VanderWeele and Ding, 2017; Oster, 2019; Cinelli and Hazlett (2020); Diegert et al., 2023,

Question and Outline

Outline of talk

- Omitted variable bias using partial R^2
- Benchmark covariate (or group of covariates) + 2 sensitivity parameters, k_D, k_Y
- Cinelli and Hazlett (2020) show how to compute bias-adjusted estimate and confidence interval for specific values of k_D, k_Y
- Easier to justify a range, rather than specific values of k_D, k_Y
- Once range of k_D, k_Y is chosen, we can compute a probability of OVB overturning result
- Discuss an example

Set Up

- Full model:

$$Y = \hat{\tau} D + X\hat{\beta} + \hat{\gamma} Z + \hat{\varepsilon}_{\text{full}}$$

- Restricted model:

$$Y = \hat{\tau}_{\text{res}} D + X\hat{\beta}_{\text{res}} + \hat{\varepsilon}_{\text{res}}$$

- Omitted variable bias:

$$\widehat{\text{bias}} = \hat{\tau}_{\text{res}} - \hat{\tau}$$

  - Y: outcome variable
  - D: treatment variable (scalar)
  - X: vector of control variables
  - Z: unobserved confounder (omitted variable)

Set Up

Using the FWL theorem and definitions of partial R^2, Cinelli and Hazlett (2020) show that

- Expression for the absolute value of the omitted variable bias:

$$\left|\widehat{\text{bias}}\right| = \widehat{\text{se}}\left(\hat{\tau}_{\text{res}}\right)\sqrt{\frac{\text{df} \times R^2_{Y\sim Z|D,X} \times R^2_{D\sim Z|X}}{1 - R^2_{D\sim Z|X}}}$$

- Expression for standard error of the true estimate, $\hat{\tau}$:

$$\widehat{\text{se}}\left(\hat{\tau}\right) = \widehat{\text{se}}\left(\hat{\tau}_{\text{res}}\right)\sqrt{\frac{1 - R^2_{Y\sim Z|D,X}}{1 - R^2_{D\sim Z|X}} \times \frac{\text{df}}{\text{df} - 1}}$$

  - $R^2_{Y\sim Z|D,X}$: partial R^2 of Y and Z, conditional on D, X
  - $R^2_{D\sim Z|X}$: partial R^2 of D and Z, conditional on X
  - *se*: standard error
  - *df*: degrees of freedom of restricted regression

What is partial R^2?

The definition of the partial R^2 is

$$R^2_{Y\sim Z|X} = \frac{R^2_{Y\sim Z+X} - R^2_{Y\sim X}}{1 - R^2_{Y\sim X}}$$

- Numerator: increment in total R^2 when Z is added to a regression of Y on X
- Denominator: 1 minus total R^2 in a regression of Y on X

Main Proposal

- If we know $R^2_{Y\sim Z|D,X}$ and $R^2_{D\sim Z|X}$, we could
  - compute $\left|\widehat{\text{bias}}\right|$ and $\widehat{\text{se}}\left(\hat{\tau}\right)$
  - and use it to construct bias-adjusted treatment effect and bias-adjusted confidence intervals
- Case 1: When estimated treatment effect, $\hat{\tau}_{\text{res}}$, is positive
  - bias-adjusted estimate: $\hat{\tau} = \hat{\tau}_{\text{res}} - |\widehat{\text{bias}}|$
  - bias-adjusted confidence interval (significance level α) for τ (true value):

$$\left(\hat{\tau}_{\text{res}} - |\widehat{\text{bias}}| - |t^*_{\alpha/2}|\widehat{\text{se}}\left(\hat{\tau}\right), \hat{\tau}_{\text{res}} - |\widehat{\text{bias}}| + |t^*_{\alpha/2}|\widehat{\text{se}}\left(\hat{\tau}\right)\right)$$

    where $|t^*_{\alpha/2}|$ is the absolute magnitude of the critical value.
- Case 2: When estimated treatment effect, $\hat{\tau}_{\text{res}}$, is negative
  - bias-adjusted estimate: $\hat{\tau} = \hat{\tau}_{\text{res}} + |\widehat{\text{bias}}|$
  - bias-adjusted confidence interval (significance level α) for τ:

$$\left(\hat{\tau}_{\text{res}} + |\widehat{\text{bias}}| - |t^*_{\alpha/2}|\widehat{\text{se}}\left(\hat{\tau}\right), \hat{\tau}_{\text{res}} + |\widehat{\text{bias}}| + |t^*_{\alpha/2}|\widehat{\text{se}}\left(\hat{\tau}\right)\right)$$

Main Proposal

- We do not know $R^2_{Y\sim Z|D,X}$ and $R^2_{D\sim Z|X}$ because Z is unobserved
- What do we do?
  - We choose a (group of) *benchmark* covariate(s) from X (set of included regressors): X_j
  - We define two sensitivity parameters, k_D, k_Y
    - k_D: captures relative strength of association of $Z^{\perp X}$ (part of omitted variable orthogonal to covariates) with D (treatment variable) compared to benchmark covariate(s) with D
    - k_Y: captures relative strength of association of $Z^{\perp X}$ (part of omitted variable orthogonal to covariates) with Y (outcome variable) compared to benchmark covariate(s) with Y
  - Strength of association can be measured in three ways
    - Total R^2
    - Partial R^2 without conditioning on D
    - Partial R^2 with conditioning on D

Main Proposal

- Total R^2-based benchmarking:

$$k_D := \frac{R^2_{D \sim Z^{\perp X}}}{R^2_{D \sim X_j}}, \quad k_Y := \frac{R^2_{Y \sim Z^{\perp X}}}{R^2_{Y \sim X_j}}$$

- Partial R^2-based benchmarking without conditioning on treatment variable:

$$k_D := \frac{R^2_{D \sim Z^{\perp X}|X_{-j}}}{R^2_{D \sim X_j|X_{-j}}}, \quad k_Y := \frac{R^2_{Y \sim Z^{\perp X}|X_{-j}}}{R^2_{Y \sim X_j|X_{-j}}}$$

- Partial R^2-based benchmarking with conditioning on treatment variable (for k_Y):

$$k_D := \frac{R^2_{D \sim Z^{\perp X}|X_{-j}}}{R^2_{D \sim X_j|X_{-j}}}, \quad k_Y := \frac{R^2_{Y \sim Z^{\perp X}|X_{-j},D}}{R^2_{Y \sim X_j|X_{-j},D}}$$

Main Proposal

For any value of k_D, k_Y, we can compute what we need

- For total R^2-based benchmarking, we have

$$R^2_{D\sim Z|X} = \frac{k_D R^2_{D\sim X_j}}{1 - R^2_{D\sim X}}, \quad R^2_{Y\sim Z|X} = \frac{k_Y R^2_{Y\sim X_j}}{1 - R^2_{Y\sim X}}.$$

- For partial R^2-based benchmarking without conditioning on treatment variable, we have

$$R^2_{D\sim Z|X} = \frac{k_D R^2_{D\sim X_j|X_{-j}}}{1 - R^2_{D\sim X_j|X_{-j}}}, \quad R^2_{Y\sim Z|X} = \frac{k_Y R^2_{Y\sim X_j|X_{-j}}}{1 - R^2_{Y\sim X_j|X_{-j}}}$$

- In both cases, we then compute

$$R^2_{Y\sim Z|D,X} = \frac{\left(\left|R_{Y\sim Z|X}\right| - \left|R_{Y\sim D|X}\right|\left|R_{D\sim Z|X}\right|\right)^2}{\left(1 - R^2_{Y\sim D|X}\right)\left(1 - R^2_{D\sim Z|X}\right)}.$$

Main Proposal

- For partial R^2-based benchmarking with conditioning on treatment variable:

$$R^2_{D\sim Z|X} = \frac{k_D R^2_{D\sim X_j|X_{-j}}}{1 - R^2_{D\sim X_j|X_{-j}}}, \quad R^2_{Y\sim Z|D,X} = \eta^2 f^2_{Y\sim X_j|X_{-j},D}$$

where

$$\eta = \frac{\sqrt{k_Y} + \left| f_{k_D} \times f_{D\sim X_j|X_{-j}} \right|}{\sqrt{1 - f^2_{k_D} \times f^2_{D\sim X_j|X_{-j}}}}$$

and

$$f^2_{D\sim X_j|X_{-j}} = \frac{R^2_{D\sim X_j|X_{-j}}}{1 - R^2_{D\sim X_j|X_{-j}}} \qquad f_{k_D} = \frac{\sqrt{k_D R^2_{D\sim X_j|X_{-j}}}}{\sqrt{1 - k_D R^2_{D\sim X_j|X_{-j}}}}$$

Main Proposal

- Since $0 \leq R^2_{D\sim Z|X} \leq 1$ and $0 \leq R^2_{Y\sim Z|D,X} \leq 1$, this gives us permissible values of k_D and k_Y.
- Example: For total R^2-based benchmarking, we have

$$R^2_{D\sim Z|X} = \frac{k_D R^2_{D\sim X_j}}{1 - R^2_{D\sim X}}, \quad R^2_{Y\sim Z|X} = \frac{k_Y R^2_{Y\sim X_j}}{1 - R^2_{Y\sim X}}$$

and so, we have

$$0 \leq k_D \leq \frac{1 - R^2_{D\sim X}}{R^2_{D\sim X_j}} = k_D^{\max}$$

and

$$0 \leq k_Y \leq \frac{1 - R^2_{Y\sim X}}{R^2_{Y\sim X_j}} = k_Y^{\max}$$

Main Proposal: Summary

- For any value of k_D, k_Y, we compute $R^2_{Y\sim Z|D,X}$ and $R^2_{D\sim Z|X}$
- And we get bias-adjusted estimate and confidence intervals for our choice of k_D, k_Y
- Then we compute
  - contour plots of lower (or upper) boundary of confidence interval over *all permissible* values of $k_D, k_Y \Rightarrow$ look at level 0
    - if estimated effect is positive, look at lower boundary
    - if estimated effect is negative, look at upper boundary
  - probability that OVB can overturn reported result
    - When estimated treatment effect is positive, this is

$$1 - \left(\frac{\text{area where lower boundary of conf int is} > 0}{\text{valid area of contour area}} \right)$$

    - When estimated treatment effect is negative, this is

$$1 - \left(\frac{\text{area where lower boundary of conf int is} < 0}{\text{valid area of contour area}} \right)$$

Example

$$\text{PeaceIndex}_i = \beta_0 + \beta_1 \text{DirectHarm}_i + \text{Controls}_i + \varepsilon_i,$$

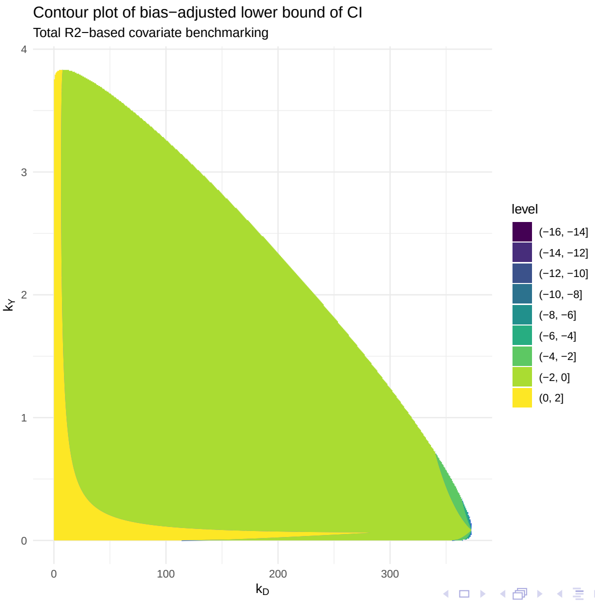
- i index for individual
- PeaceIndex (outcome, Y): index of attitude towards peace efforts
- DirectHarm (treatment, D): measures the exposure to violence
- Control variables, X: gender of the individual, age, whether the individual was a farmer, herder, merchant or trader, household size, whether or not the individual voted in the past, and village-level fixed effects.
- Omitted variable, Z: wealth
- Benchmark, X_j: gender of the individual

Bias-adjusted estimate and confidence interval

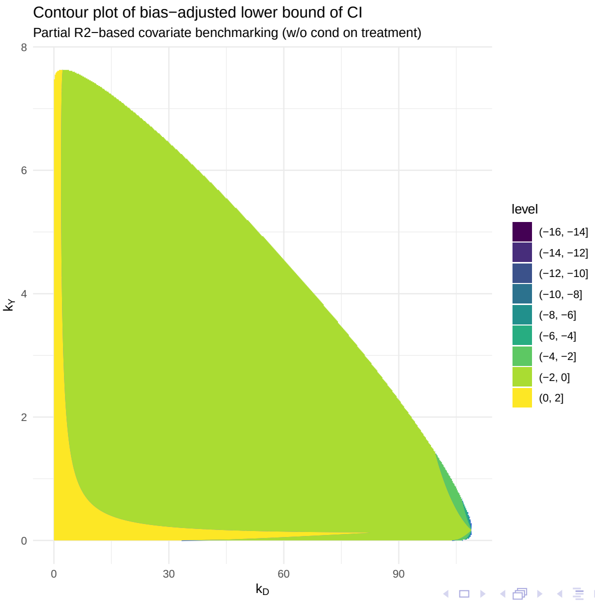
	Panel A: $k_D = 1, k_Y = 1$			**Panel B:** $k_D = 3, k_Y = 3$		
	Total	Partial 1	Partial 2	Total	Partial 1	Partial 2
$R^2_{Y\sim D\|X}$	0.022	0.022	0.022	0.022	0.022	0.022
$R^2_{D\sim Z\|X}$	0.003	0.009	0.009	0.008	0.027	0.027
$R^2_{Y\sim Z\|D,X}$	0.259	0.125	0.125	0.781	0.381	0.374
Estimate	0.097	0.097	0.097	0.097	0.097	0.097
Bias-Adj Estimate	0.080	0.075	0.075	0.046	0.030	0.030
Bias-Adj Standard Error	0.020	0.022	0.022	0.011	0.019	0.019
Lwr Bdary of Bias-Adj Conf. Int.	0.041	0.032	0.032	0.024	-0.007	-0.006
Upr Bdary of Bias-Adj Conf. Int.	0.120	0.118	0.118	0.067	0.066	0.067

- Total: total R^2-based covariate benchmarking
- Partial 1: partial R^2-based covariate benchmarking without conditioning on treatment
- Partial 2: partial R^2-based covariate benchmarking with conditioning on treatment

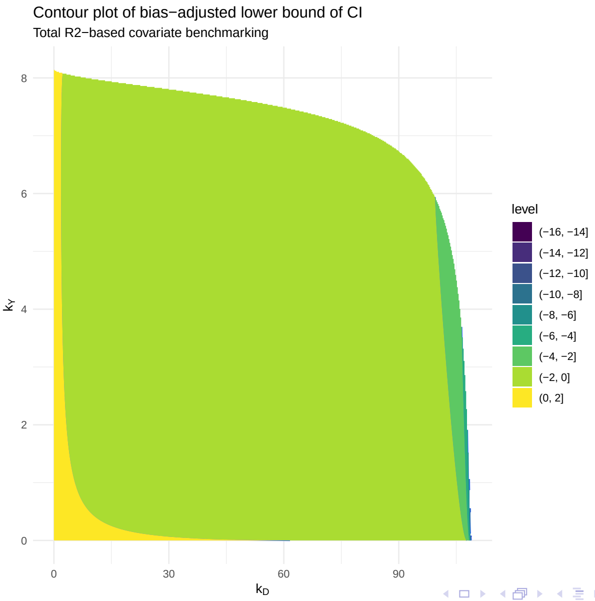
Contour plot (Total R^2)



Contour plot (Partial R^2 without conditioning)



Contour plot (Partial R^2 with conditioning)



Probability of reported result being overturned

	Total	Partial 1	Partial 2
Panel A: Full range:			
$0 \leq k_D \leq k_D^{\max}; 0 \leq k_Y \leq k_Y^{\max}$	0.925	0.925	0.960
Panel B: Absolute upper bounds:			
$0 \leq k_D \leq 1; 0 \leq k_Y \leq 1$	0.000	0.000	0.000
$0 \leq k_D \leq 3; 0 \leq k_Y \leq 3$	0.000	0.021	0.019
$0 \leq k_D \leq 5; 0 \leq k_Y \leq 5$	0.000	0.336	0.335
Panel C: Relative upper bounds:			
$0 \leq k_D \leq 0.1k_D^{\max}; 0 \leq k_Y \leq 0.1k_Y^{\max}$	0.046	0.046	0.116
$0 \leq k_D \leq 0.25k_D^{\max}; 0 \leq k_Y \leq 0.25k_Y^{\max}$	0.628	0.628	0.701
Memo:			
$k_D^{\max}$	373.134	109.119	109.119
$k_Y^{\max}$	3.839	7.643	8.155

Probability of reported result being overturned

- Important message: result depends on choice of $k_D^{\max}$ and $k_Y^{\max}$
- Full permissible range of k_D, k_Y gives very conservative answer
- Can we do better?
- Can we choose upper bounds for k_D, k_Y that are lower than $k_D^{\max}$ and $k_Y^{\max}$?
- Two possibilities
  - Absolute bounds
  - Relative bounds: better because it is based on the sample (which determines $k_D^{\max}$ and $k_Y^{\max}$)
- Some simulation evidence for a possible answer using relative bounds

Simulation Set-up

- X^s: standardized k-dimensional multivariate Gaussian of size N with mean $0_{k\times 1}$ and covariance matrix $A'A$ (with A coming from k^2 draws from uniform(0,1))
- I generate the scalar treatment variable, D, as

$$D = a_0 + X_1^s a_1 + X_2^s a_2 + \cdots + X_{k-1}^s a_{k-1} + X_k^s a_k + u_D,$$

where $a = (a_0, a_1, \ldots, a_k)$ is a $(k+1)$-vector formed by drawing $k+1$ random numbers from a uniform distribution over $(-1, 1)$ and $u_D \sim \text{i.i.d.} N\left(0, \sigma_{u_D}^2\right)$.

- I generate the scalar outcome variable, Y, as

$$Y = b_0 + X_1^s b_1 + X_2^s b_2 + \cdots + X_{k-1}^s b_{k-1} + X_k^s b_k + u_Y,$$

where $b = (b_0, b_1, \ldots, b_k)$ is a $(k+1)$-vector formed by drawing $k+1$ random numbers from a uniform distribution over $(-1, 1)$ and $u_Y \sim \text{i.i.d.} N\left(0, \sigma_{u_Y}^2\right)$.

Simulation Set-up

- Y, D, X^s comprise the simulated data set.
- Using this data set, I estimate the following model using OLS,

$$Y = \beta_0 + D\tau + X_1^s\beta_1 + X_2^s\beta_2 + \cdots + X_{k-1}^s\beta_{k-1} + \varepsilon,$$

- I treat the k-th column of X^s as the unobserved confounder (the omitted variable) and *leave it out of the estimated model*, i.e. $Z = X_k^s$.
- I compute k_D and k_Y and $k_D^{\max}$ and $k_Y^{\max}$
- I repeat this 1000 times
- Look at the empirical distribution of $k_D/k_D^{\max}$ and $k_Y/k_Y^{\max}$

Simulation results

Table: 90-th percentile of the empirical distribution of k_d/k_d^{max} and k_y/k_y^{max}

N	$k=10$		$k=25$		$k=50$		$k=100$	
	$\frac{k_d}{k_d^{max}}$	$\frac{k_y}{k_y^{max}}$	$\frac{k_d}{k_d^{max}}$	$\frac{k_y}{k_y^{max}}$	$\frac{k_d}{k_d^{max}}$	$\frac{k_y}{k_y^{max}}$	$\frac{k_d}{k_d^{max}}$	$\frac{k_y}{k_y^{max}}$
250	0.408	0.398	0.206	0.213	0.126	0.106	0.074	0.067
500	0.433	0.435	0.212	0.210	0.115	0.109	0.061	0.057
1000	0.424	0.413	0.222	0.214	0.125	0.119	0.062	0.059
2500	0.385	0.413	0.190	0.179	0.107	0.113	0.057	0.053
5000	0.421	0.419	0.204	0.200	0.094	0.094	0.064	0.056
10000	0.422	0.417	0.202	0.190	0.119	0.115	0.056	0.064

- I have used $\sigma_{u_Y} = \sigma_{u_D} = 1$
- Main takeaways:
  - result depends on k but not on N
  - k_d/k_d^{max} and k_y/k_y^{max} fall as k increases

Simulation results

Table: 90-th percentile of the empirical distribution of k_d/k_d^{max} and k_y/k_y^{max}

	$k = 10$		$k = 25$		$k = 50$		$k = 100$	
	$\frac{k_d}{k_d^{max}}$	$\frac{k_y}{k_y^{max}}$	$\frac{k_d}{k_d^{max}}$	$\frac{k_y}{k_y^{max}}$	$\frac{k_d}{k_d^{max}}$	$\frac{k_y}{k_y^{max}}$	$\frac{k_d}{k_d^{max}}$	$\frac{k_y}{k_y^{max}}$
$\sigma_{u_Y} = \sigma_{u_D}$	0.42	0.44	0.20	0.21	0.11	0.11	0.06	0.06
$\sigma_{u_Y} = 3\sigma_{u_D}$	0.43	0.08	0.21	0.03	0.10	0.02	0.06	0.01
$3\sigma_{u_Y} = \sigma_{u_D}$	0.07	0.42	0.03	0.19	0.02	0.12	0.01	0.06
$\sigma_{u_Y} = 5\sigma_{u_D}$	0.42	0.03	0.20	0.01	0.11	0.01	0.06	0.01
$5\sigma_{u_Y} = \sigma_{u_D}$	0.03	0.41	0.01	0.20	0.01	0.11	0.01	0.05

- Benchmark case with $\sigma_{u_Y} = \sigma_{u_D}$ is most conservative
- When $\sigma_{u_Y} > \sigma_{u_D}$: same k_D/k_D^{max} but lower k_Y/k_Y^{max} than in benchmark case
- When $\sigma_{u_Y} < \sigma_{u_D}$, same k_Y/k_Y^{max} but lower k_D/k_D^{max} than in benchmark case

Probability of reported result being overturned

	Total	Partial 1	Partial 2
Panel A: Full range:			
$0 \leq k_D \leq k_D^{\max}; 0 \leq k_Y \leq k_Y^{\max}$	0.925	0.925	0.960
Panel B: Absolute upper bounds:			
$0 \leq k_D \leq 1; 0 \leq k_Y \leq 1$	0.000	0.000	0.000
$0 \leq k_D \leq 3; 0 \leq k_Y \leq 3$	0.000	0.021	0.019
$0 \leq k_D \leq 5; 0 \leq k_Y \leq 5$	0.000	0.336	0.335
Panel C: Relative upper bounds:			
$0 \leq k_D \leq 0.1k_D^{\max}; 0 \leq k_Y \leq 0.1k_Y^{\max}$	0.046	0.046	0.116
$0 \leq k_D \leq 0.25k_D^{\max}; 0 \leq k_Y \leq 0.25k_Y^{\max}$	0.628	0.628	0.701
Memo:			
$k_D^{\max}$	373.134	109.119	109.119
$k_Y^{\max}$	3.839	7.643	8.155