Debt Sustainability in a Stochastic $\mathbf{r - g}$ Economy

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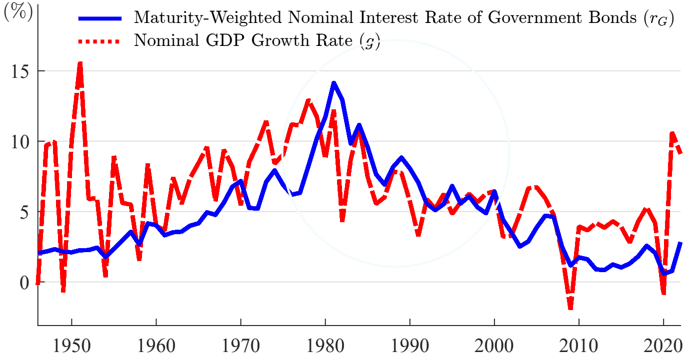
Motivations

- Dynamics of $r - g$ is important for debt-GDP ratio
- In his influential book, Olivier Blanchard emphasizes
  - $r - g < 0$ ($r < g$) on average in advanced countries
  - $r - g$ fluctuates a lot, and $r > g$ is possible



Need for the debt sustainability analysis under stochastic $r - g$

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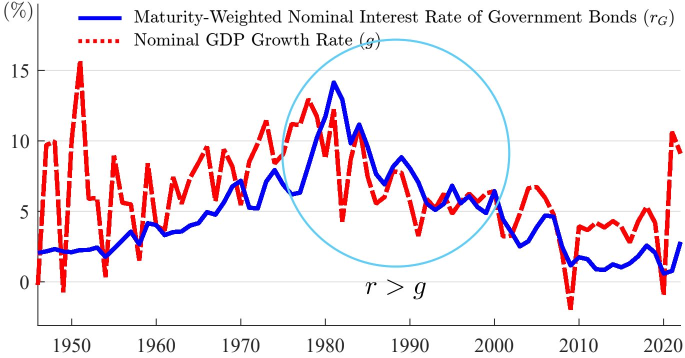

- Need for the debt sustainability analysis under **stochastic** $r - g$

What We Did

- Explore the asymptotic distribution of debt-GDP ratio ($b_t = B_t/Y_t$)

(1) $$b_t = (1 + (r_t - g_t))b_{t-1} + \eta_t \quad (\eta_t = \text{primary deficit-GDP ratio})$$

  - $r_t - g_t$ and η_t are random variables: (1) follows "**Kesten process**"
  - Distribution for b_t converges to a fat-tail Pareto dist. $\Rightarrow$ Tail risk in b

- Microfounded model to study how the tail risk affects sustainability
  - The Pareto tail index of b is related to IGBC.

- A novel empirical method to estimate the tail risk magnitude
  - We can assess debt sustainability from historical debt-GDP data.

Literature

- Debt sustainability under low interest rates
  - Blanchard (2019, 2023). Mian, Straub, & Sufi (2024); Miao & Su (2024); Kocherlakota (2022); Reis (2022); Mehrotra & Sergeyev (2021)
- Tail risk in debt accumulation
  - Mehrotra & Sergeyev (2021)
- Wealth inequality
  - Kesten process has been used to explain the observed Pareto tailed wealth distribution (Benhabib & Bisin).
- Policy Analysis
  - SDSA (fan chart analysis based on future projection)

Asymptotic distribution of b_t

- Asymptotic behavior of b_t that follows so-called "Kesten process"

$$b_t = a_t b_{t-1} + \eta_t,$$

where $a_t \approx 1 + (r_t - g_t)$ and $\eta_t > 0$ are random variables.

- Kesten-Goldie theorem (Kesten, 1973; Goldie, 1991): If
  - $\mathbb{E}\left[\log(a_t)\right] < 0$
  - $\sup\left(\log(a_t)\right) > 0$

The asymptotic dist of b_t is stationary and has a Pareto upper-tail:

$$\text{Prob}\left(b_t > x\right) = cx^{-\kappa}, \qquad c > 0 \text{ as } t \to \infty$$

  - Tail index $\kappa > 0$ is determined by process of a_t, i.e., $r_t - g_t$

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- Kesten-Goldie theorem (Kesten, 1973; Goldie, 1991): If
  - $\mathbb{E}[\log(a_t)] < 0 \iff \mathbb{E}[r_t - g_t] < 0$
  - $\sup(\log(a_t)) > 0 \iff$ Prob $(r_t - g_t > 0) > 0$

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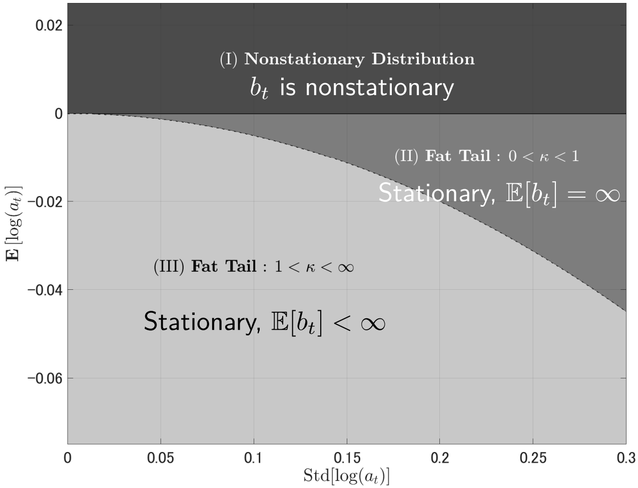
  - **Tail index $\kappa > 0$ is determined by process of a_t, i.e., $r_t - g_t$**

Asymptotic distribution of b_t: IID ($\mathbb{E}[a_t^\kappa] = 1$)

$$b_t = a_t b_{t-1} + \eta_t, \qquad \log(a_t) \overset{\text{iid}}{\sim} N(\mu_a, \sigma^2_{iid}), \quad a_t \perp\!\!\!\perp \eta_t$$

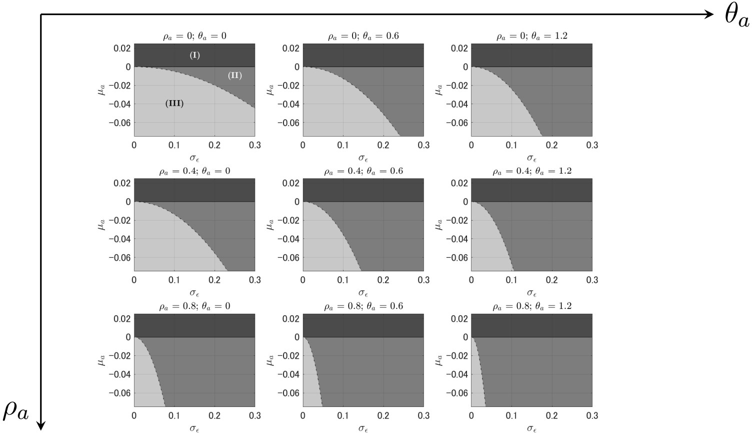


Asymptotic distribution of b_t: ARMA(1,1)

- Recent math allows for *Persistent fluctuations in* $\log(a_t) \approx r_t - g_t$

$$b_t = a_t b_{t-1} + \eta_t$$

$$\log(a_t) = (1 - \rho_a)\mu_a + \rho_a \log(a_{t-1}) + \varepsilon_{a,t} + \theta_a \varepsilon_{a,t-1}, \quad \varepsilon_{a,t} \overset{\text{iid}}{\sim} N(0, \sigma_\epsilon^2)$$



As $\log(a_t)$ (i.e., $r_r - g_t$) becomes more persistent, the tail of b_t is fatter.

Model

- General equilibrium model that yields

$$b_t = a_t b_{t-1} + \eta_t, \qquad \log(a_t) \text{ follows ARMA(1,1)}$$

  - Microfoundation for (generalized) Kesten process for debt-GDP ratio

- Intertemporal government budget constraint (IGBC) to evaluate sustainability of debt

Model

- Exchange economy with the representative household
  - Random growth of household endowment

$$\log\left(\frac{Y_{t+1}}{Y_t}\right) = g - \frac{\sigma_g^2}{2} + \sigma_g \varepsilon_{g,t+1}, \qquad \varepsilon_{g,t} \overset{\text{iid}}{\sim} N(0,1)$$

  - Household preference and budget

$$\mathbb{E}_t\left[\sum_{k=0}^{\infty} \exp(-\rho)^k \left(\frac{C_{t+k}^{1-\gamma}-1}{1-\gamma} + \underbrace{\nu\left(B_{t+k}\right)}_{\text{convenience benefits}}\right)\right],$$

$$C_t + B_t + A_t \leq Y_t - T_t + \left(1 + r_{G,t-1}\right) B_{t-1} + \left(1 + r_{F,t-1}\right) A_t,$$

- Two risk-free assets
  - A_t private bonds (net supply =0): r_F is discount rate for asset pricing
  - B_t government bonds: $r_G < r_F$ due to convenience benefits

Model: Equilibrium Condition

- Convenience benefits: linear random coefficient $\nu(B_t) = \varrho_t B_t$ with $\varrho_t = u'(\bar{C}_t)\left(1 - \exp(z_t)^{-1}\right)$; $z_t = (1-\rho_z)\mu_z + \rho_z z_{t-1} + \sigma_z \varepsilon_{z,t}$,

- Equilibrium interest rates

$$r_{F,t} = \rho + \gamma g - \frac{\gamma(\gamma+1)}{2}\sigma_g^2,$$

$$r_{G,t} = r_{F,t} - z_t.$$

- If $\gamma > 1$ and μ_z is sufficiently large, $r_G < g < r_F$ in steady state.

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- If $\gamma > 1$ and μ_z is sufficiently large, $r_G < g < r_F$ in steady state.

- Debt-GDP ratio: $b_t = a_t b_{t-1} + \eta_t$ where

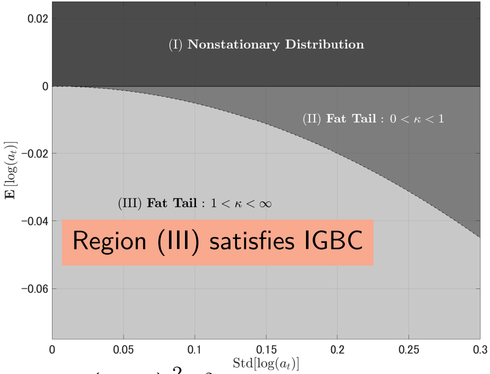
$$\log(a_{t+1}) = (1-\rho_z)\bar{m} + \rho_z \log(a_t) + \tilde{\varepsilon}_{t+1} + \theta\tilde{\varepsilon}_t; \quad \text{(See paper)}$$

- IGBC is satisfied: $\lim_{k\to\infty} \mathbb{E}_t\left[e^{-\rho t}\left(\frac{Y_{n+k}}{Y_t}\right)^{1-\gamma} B_{t+k}\right] = 0$ (TVC)

Model: Debt Sustainability

Theorem

IGBC is satisfied if the Pareto index κ of asymptotic distribution of debt-GDP ratio is $\kappa \geq 1$. (i.e,. $\lim_{t\to\infty} \mathbb{E}[b_t] < \infty \Rightarrow IGBC$)



Region (III) iff $\mu_a + \left(\frac{1+\theta_a}{1-\rho_a}\right)^2 \frac{\sigma_\epsilon^2}{2} \leq 0$. Estimate $(\mu_a, \rho_a, \theta_a, \sigma_\epsilon)$ & check it.

Model: Debt Sustainability

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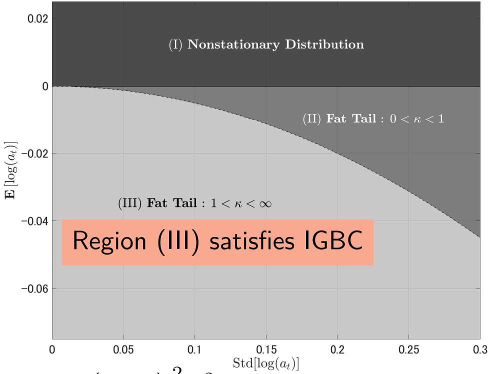
IGBC is satisfied if the Pareto index κ of asymptotic distribution of debt-GDP ratio is $\kappa \geq 1$. (i.e,. $\lim_{t\to\infty} \mathbb{E}[b_t] < \infty \Rightarrow IGBC$)



The IGBC can still be satisfied even if $\lim_{t\to\infty} \mathbb{E}[b_t] = \infty$, provided that *the growth rate of $\mathbb{E}[b_t]$ remains small*.

Region (III) iff $\mu_a + \left(\frac{1+\theta_a}{1-\rho_a}\right)^2 \frac{\sigma_\epsilon^2}{2} \leq 0$. Estimate $(\mu_a, \rho_a, \theta_a, \sigma_\epsilon)$ & check it.

Estimation Method

- Estimate stochastic process for $r_G - g$ **without** $r_G - g$ **data**
  - r_G is difficult to measure due to a various forms of debt
- Nonlinear Non-Gaussian State-Space Model

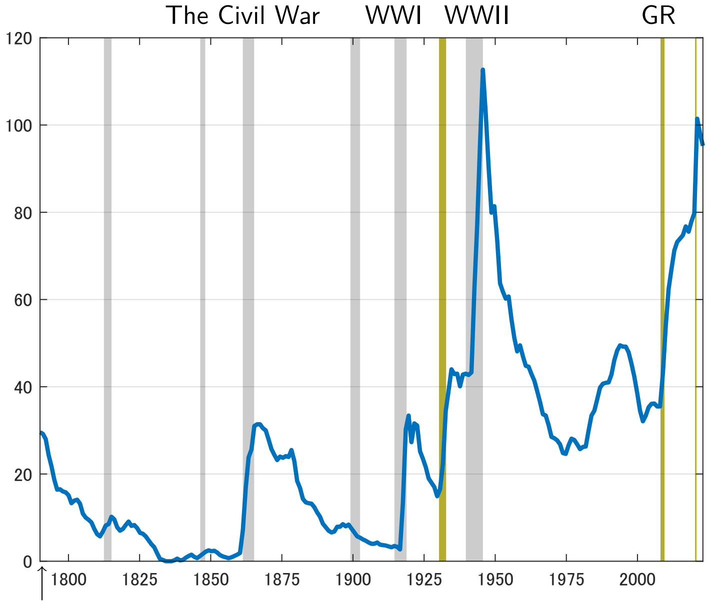
$$\begin{cases} \log(a_t) = (1-\rho_a)\mu_a + \rho_a \log(a_{t-1}) + \varepsilon_{a,t} + \theta_a \varepsilon_{a,t-1} \\ b_t = a_t b_{t-1} + \eta_t, \end{cases}$$

  - Observable variable: b_t (Debt-GDP ratio)
  - Latent state variable: $a_t \approx 1 + (r_{G,t} - g_t)$
  - Shock: $\varepsilon_{a,t} \sim N(0, \sigma_\epsilon)$
  - Measurement error: $\eta_t > 0$ (primary deficit GDP ratio)

$$\eta_t = \bar{\eta}_t + \xi_t > 0$$

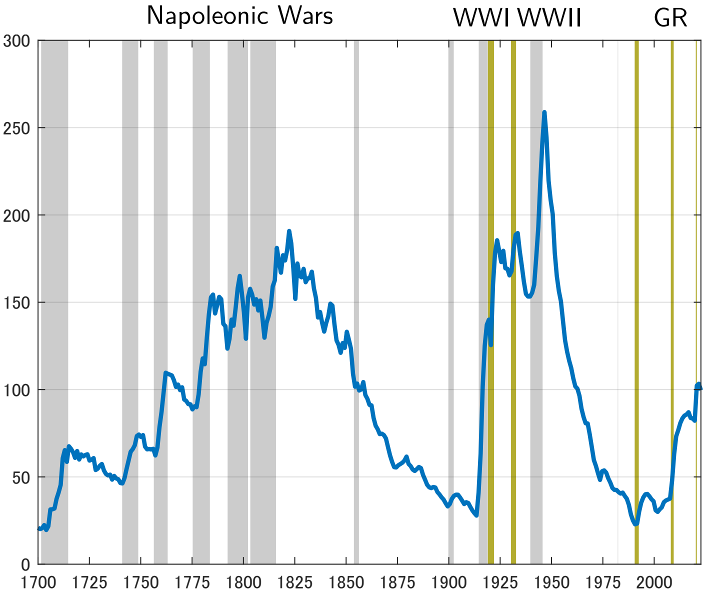
    - $\bar{\eta}_t \sim exp(1/\bar{\eta})$ (normal times); ξ_t disaster shocks
- Particle filter for maximum likelihood estimation: $(\bar{\eta}, \mu_a, \rho_a, \theta_a, \sigma_\epsilon)$

Data: Debt-GDP ratio in US: 1790-2023



US Treasury dept. Created in 1790

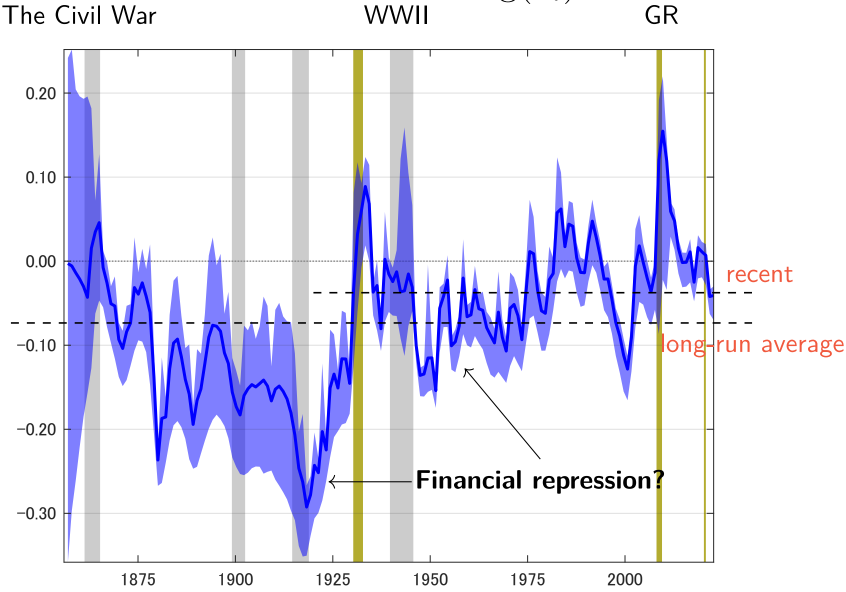
Data: Debt-GDP ratio in UK: 1700-2023



Results: Parameter Estimate

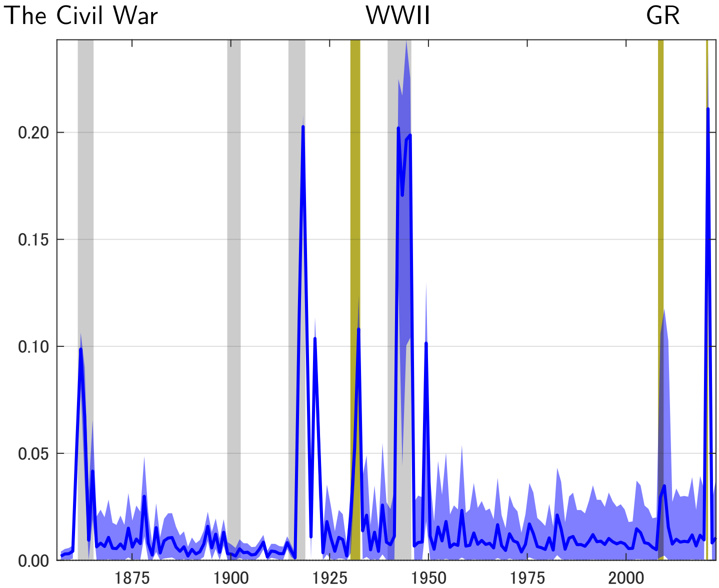
Parameter	Description	Values US	Values UK
Estimated Parameters			
μ_a	mean of $\log(a_t)$	-0.0703	-0.0168
ρ_a	first-order autoregressive coefficient of $\log(a_t)$	0.8333	0.5556
θ_a	moving-average coefficient of $\log(a_t)$	-0.0278	-0.0167
σ_ϵ	standard deviation of innovations to $\log(a_t)$	0.0468	0.0489
$\bar{\eta}$	mean of the ratio of discretionary fiscal deficits to GDP	0.0100	0.0073
Externally Assigned Parameters			
p_1	probability of a disaster	0.050	0.050
p_2	probability of a large disaster	0.017	0.017
ξ_d	additional fiscal deficit during a disaster	0.10	0.10
$\hat{\xi}_d$	additional fiscal deficit during a large disaster	0.20	0.20
Tail Index			
κ	Pareto tail index	**1.887**	**2.870**

Results: Smoother Estimate of $\log(a_t)$ in US

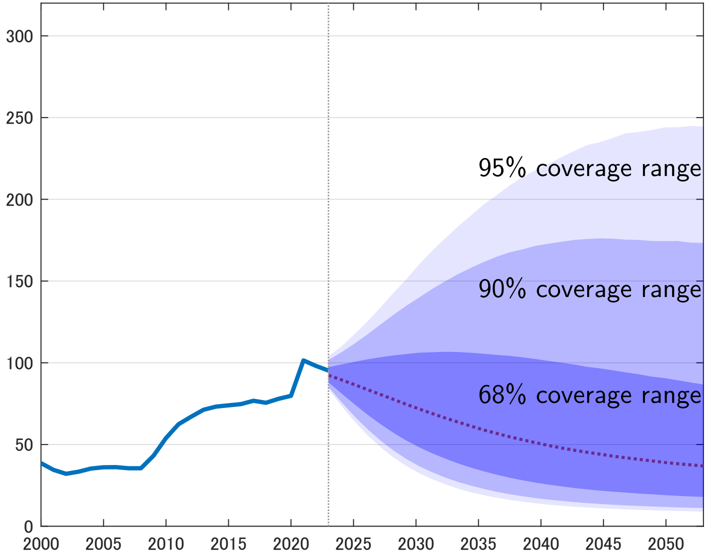


The tail-risk magnitude has increased in recent years!

Results: Smoother Estimate of η_t in US

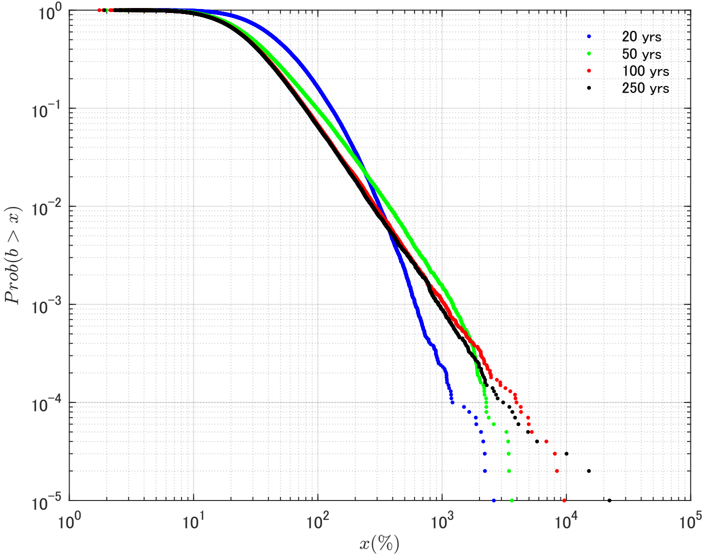


Future Projections for Debt-GDP Ratio (%)

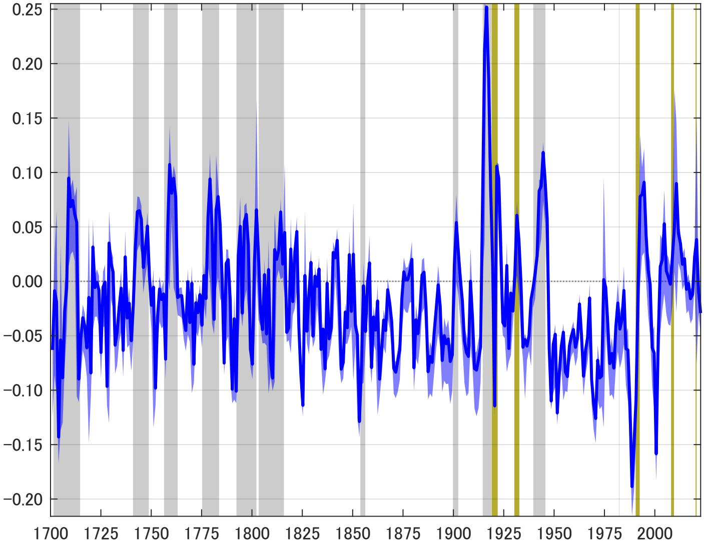


The upper risk of the debt-to-GDP ratio disproportionately escalates!

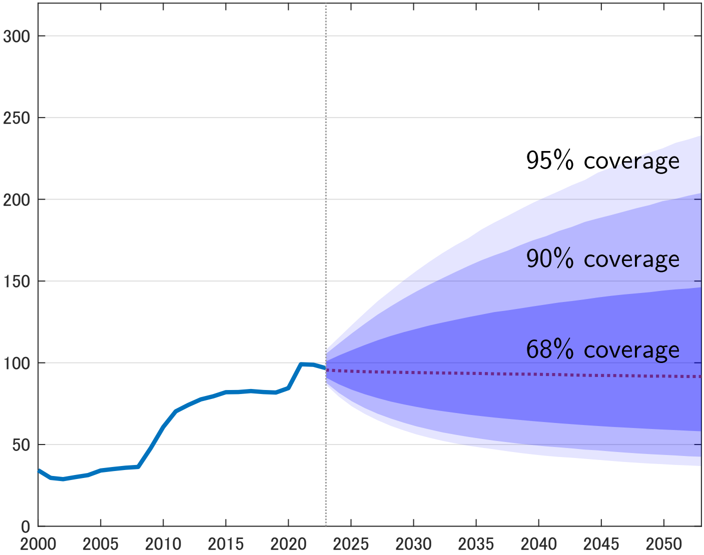
Future Debt-GDP Ratio: Upper-Tail Probability



Results: Smoother Estimate of $\log(a_t)$ in UK



Future Projections for Debt-GDP Ratio (UK)



Summary

- A novel empirical method to evaluate public debt sustainability based on Kesten process:

$$b_t = a_t b_{t-1} + \eta_t \qquad (a_t = 1 + r_t - g_t)$$

- It considers tail risk in debt accumulation due to risks of $r > g$.
- Estimate the Pareto tail index of the debt-to-GDP ratio using the historical data of the ratio.