Debt Sustainability in a Stochastic $r - g$ Economy*

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Preliminary and Incomplete:

The Latest Version Is Available Here.

Abstract

This study presents a novel approach for assessing debt sustainability by analyzing the asymptotic behavior of the debt-GDP ratio in discrete-time frameworks that account for uncertainties in the interest rate-growth differential $(r-g)$ and primary deficit. Low government bond yields typically lead to a right-skewed power-law distribution of the debt-GDP ratio, with the upper tail index determined by the stochastic properties of $r-g$. Our general equilibrium model shows that the intertemporal government budget constraint is sufficiently met when the asymptotic distribution of the debt-GDP ratio has a finite mean. We propose a particle filter-based technique to estimate tail risk magnitude, finding that persistent fluctuations in $r-g$ undermine debt sustainability, while fiscal reaction rules enhance it.

Key Words: Debt sustainability, Debt-to-GDP ratio, Interest rate-growth differential, Tail risk, Estimation, Particle filter

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1 Introduction

Yields on government bonds (r_G) have predominantly been lower than nominal GDP growth (g), suggesting that fiscal deficits do not necessarily cause an explosive rise in the debt-to-GDP ratio (Blanchard, 2019). However, the $r_G - g$ differential fluctuates significantly. During the 1980s and 1990s in the United States, r_G exceeded g, leading to a sharp increase in the debt-to-GDP ratio (See Figure 1). Indeed, Hall and Sargent (2011) show that of the 28.3 percentage point rise in the U.S. debt-to-GDP ratio from 1981 to 1993, 17.8 points were due to a higher budget deficit, with the rest from fluctuations in $r_G - g$ caused by changes in nominal returns, inflation, and GDP growth. These observations highlight the need for a stochastic framework to assess debt sustainability, accounting for uncertainties in $r_G - g$ and the possibility of $r_G - g > 0$.

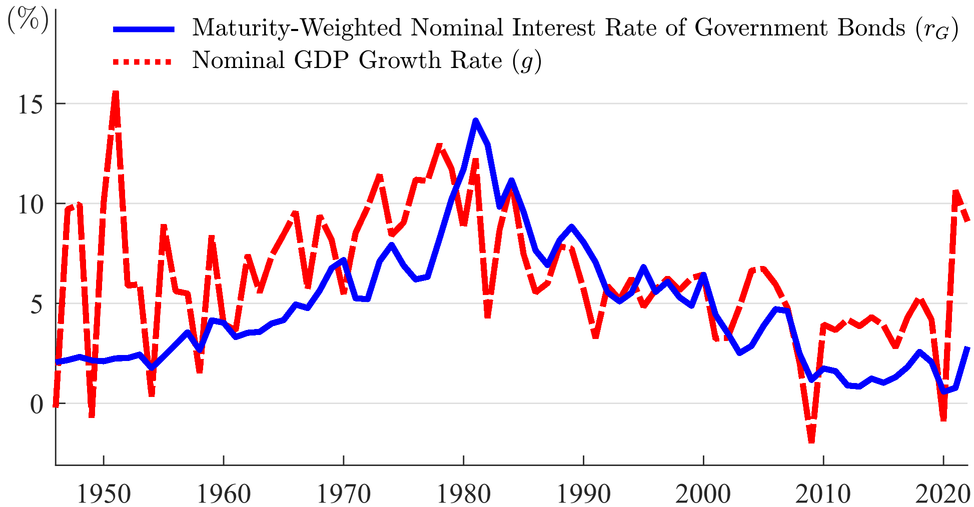
This study investigates the dynamics and sustainability of debt under uncertainty through both theoretical and empirical lenses. We employ a discrete-time stochastic model centered on the fundamental equation for the evolution of the debt-to-GDP ratio (b_t):

$$b_{t+1} = \left(\frac{1 + r_{G,t+1}}{1 + g_{t+1}} \right) b_t - s_{t+1}, \tag{1}$$

where s_t denotes the ratio of primary surplus to GDP. We begin with a theoretical analysis of the asymptotic distribution of the debt-to-GDP ratio, focusing on its upper tail behavior. We then develop a general equilibrium model linking the asymptotic behavior of the debt-to-GDP ratio to debt sustainability. Finally, leveraging our theoretical findings, we introduce an innovative methodology for empirically assessing debt sustainability using historical debt-to-GDP ratio data.

Our theoretical analysis begins by examining the fundamental equation (1), which displays the characteristics of a Kesten multiplicative process when the primary deficit remains positive—though this condition is relaxed in later analyses for broader applicability. Drawing on the foundational work of Kesten (1973) and Goldie (1991), we find that if $r_G - g$ is negative on average but occasionally turns positive, the debt-to-GDP ratio converges to a stationary power-law distribution. This distribution is right-skewed with a Pareto upper tail, indicating that the upper tail risks of the debt-to-GDP ratio increase disproportionately over time. Moreover, the stochastic nature of $r_G - g$ significantly affects the tail index,

Figure 1: r_G versus g in the United States: 1946-2022



Note: All series are annual. The solid line represents the maturity-weighted nominal interest rate of U.S. government bonds, while the dashed line depicts the U.S. nominal GDP growth rate. The maturity-weighted nominal interest rate data is sourced from the database of Blanchard (2019) covering the period from 1946 to 2017 and has been extended by the authors to include data until 2022.

emphasizing the need to consider the tail-risk magnitude arising from fluctuations in $r_G - g$ when assessing debt sustainability.

We refine this analysis by exploring a generalized version of the Kesten process to account for the observed persistence in fluctuations of $r_G - g$, as extensively documented in empirical research (e.g., Ball, Elmendorf, and Mankiw, 1998; Barrett, 2018).[1] Drawing on insights from de Saporta (2005) and Benhabib, Bisin, and Zhu (2011), we demonstrate that increased persistence in $r_G - g$ results in a distribution with an even heavier tail—sometimes so heavy that the mean becomes infinite.

We also explore the implications of fiscal policies that follow reaction rules as proposed by Bohn (1998), leading to the generalized equation:

$$b_{t+1} = a_{t+1} b_t + \eta_{t+1}, \tag{2}$$

where fiscal policies modulate the stochastic processes of $a_t \approx 1 + r_{G,t} - g_t - \phi$ with ϕ reflecting how the primary surplus responds to an increase in debt and η_t that represents the ratio of the discretionary primary deficit to GDP. In the generalized fundamental equation (2), the

[1] In their work, Ball, Elmendorf, and Mankiw (1998) describe the statistical properties of $r_G - g$ in the United States, estimating autoregressive coefficients between 0.21 and 0.59.

dynamics of a_t are crucial in defining the tail index, enabling an analysis of how shifts in fiscal policies affect the tail-risk magnitude the debt-to-GDP ratio.

To examine how the magnitude of tail risk is related to debt sustainability, we present a general equilibrium nominal exchange economy that incorporates the liquidity and convenience benefits of government bonds (Woodford, 1990; Krishnamurthy and Vissing-Jorgensen, 2012; Jiang, Lustig, Van Nieuwerburgh, and Xiaolan, 2024). This model provides a microfoundation for the stochastic processes governing a_t, resulting in a $\log(a_t)$ process that follows a stationary autoregressive-moving average (ARMA) process. Within this infinite-horizon general equilibrium model, we define debt sustainability as the fulfillment of the intertemporal government budget constraint (IGBC)—namely, the alignment of the value of outstanding debt with the expected present value of future primary surpluses and the convenience premia of government bonds.

Given this definition, we first demonstrate that the existence of a stationary distribution of the debt-to-GDP ratio does *not* ensure debt sustainability.[2] This implies that a negative average value of $r_G - g$ does not necessarily indicate debt sustainability in a stochastic environment. Then, by considering a power-law asymptotic distribution of the debt-to-GDP ratio, we show that the fulfillment of the IGBC is sufficiently ensured if the asymptotic distribution has a finite mean. This means that debt is considered sustainable if the distribution's Pareto tail index exceeds one.[3] In summary, when taking tail risk into account, whether the asymptotic distribution has a tail index exceeding one could be a reasonable condition for debt sustainability.

We propose a novel method to estimate the tail index using debt-to-GDP ratio data for empirically evaluating debt sustainability. Our approach involves developing a nonlinear, non-Gaussian state-space model based on the fundamental equation (2) and the stochastic processes for the latent variables a_t and η_t, with the debt-to-GDP ratio serving as the observable variable. By applying historical debt-to-GDP ratio data and utilizing a particle filter technique, we estimate the stochastic processes for a_t and η_t and determine the tail index of

[2]The existence of a stationary distribution of the debt-to-GDP ratio has been used as a practical definition of debt sustainability in empirical studies (e.g., Hamilton and Flavin, 1986; Trehan and Walsh, 1988, 1991; Quintos, 1995).

[3]When the tail index is below one, resulting in a divergent asymptotic mean of the debt-to-GDP ratio, debt sustainability depends on the relationship between the divergence speed of the debt-to-GDP ratio and debt holders' discount rates. We analytically derive the divergence speed of the debt-to-GDP ratio when the ratio follows (2) and a_t follows an ARMA process.

the asymptotic distribution of the debt-to-GDP ratio.[4]

We apply this estimation method to data from the United States and the United Kingdom. The results reveal that the unconditional mean value of a_t is below 1 for both countries, indicating the presence of a Pareto stationary distribution for the debt-to-GDP ratio. Despite significant variance and high autocorrelation in a_t, both countries exhibit stationary distributions with a Pareto tail index greater than one, suggesting a finite mean and, thus, sustainable debt. However, in the United States, the estimated tail index is close to one, indicating that even minor relaxations in fiscal stabilization policies could quickly lead to unsustainable debt trajectories. Our estimates of latent variable trajectories suggest that since the 1990s in the United Kingdom and the 2000s in the United States, the value of a_t has risen to levels comparable to those observed during the World Wars. Recent trends show multiple instances where a_t exceeded 1, implying that the recent increase in the debt-to-GDP ratio is driven not only by temporary fiscal deficits from the Great Recession and the COVID-19 pandemic but also by a sustained increase in the interest rate-growth differential.

Our research adds to the extensive body of theoretical and empirical work on public debt sustainability. Numerous empirical studies have investigated debt sustainability by testing specific stationarity and co-integration conditions on fiscal variables such as debt and primary deficits (e.g., Hamilton and Flavin, 1986; Trehan and Walsh, 1988, 1991; Quintos, 1995; Bohn, 2007).[5] Building on the work of Bohn (1998), who highlighted the importance of the fiscal rule in stabilizing the debt-to-GDP ratio, several empirical studies have estimated the form of the fiscal reaction function (e.g., Mendoza and Ostry, 2008; Ghosh, Kim, Mendoza, Ostry, and Qureshi, 2013). In our theoretical analysis, we revisit the implications of the fiscal stabilization rule for debt sustainability in more complex settings.

Our study is also related to the evolving theoretical literature on the implications of $r_G < g$ for public debt and fiscal policies. Various modeling approaches have been employed to explore scenarios where $r_G < g$, including models with overlapping generations in a production economy (e.g., Bullard and Russell, 1999; Chalk, 2000; Blanchard, 2019), infinitely-lived agents subject to uninsurable idiosyncratic risks (e.g., Kaas, 2016; Kocherlakota, 2023;

[4]Given that historical data indicate substantial primary deficits during periods of war or severe recessions, our estimation model incorporates disaster shocks to account for significant stochastic fluctuations in η_t.

[5]Bohn (2007) demonstrate that assuming the government bond rate and the discount rate are identical and constant, the IGBC is satisfied as long as the debt series is integrated at any finite order. For comprehensive reviews, see D'Erasmo, Mendoza, and Zhang (2016) and Reis (2022).

Miao and Su, forthcoming), and infinitely-lived agents with a preference for liquid and safe assets (Mehrotra and Sergeyev, 2021; Mian, Straub, and Sufi, 2022). Among them, the study most closely aligned with ours is that of Mehrotra and Sergeyev (2021), who explore debt dynamics using a continuous-time stochastic differential equation approach. They demonstrate that under specific conditions, such as the presence of a lower reflection barrier in the debt level, the debt-to-GDP ratio can exhibit a power-law distribution in the long run. While our analysis primarily focuses on discrete-time stochastic processes, the continuous-time counterparts of our model nest those examined by Mehrotra and Sergeyev (2021) as special cases.[6] Additionally, our findings indicate that the asymptotic distribution of the debt-to-GDP ratio follows a power-law distribution in more general scenarios, even without reflection barriers, especially when a permanent fiscal primary deficit is present. Furthermore, the use of discrete-time stochastic models enhances empirical tractability, enabling the estimation of tail risk magnitude from historical debt-to-GDP data.[7]

The power-law distribution is a well-recognized characteristic in the distribution of income, wealth, and firm sizes. Recently, several studies have developed equilibrium models that lead to power-law distributions asymptotically (e.g., Gabaix, 1999; Nirei and Souma, 2007; Luttmer, 2007, 2011; Benhabib, Bisin, and Zhu, 2011, 2015; Toda, 2014; Gabaix, Lasry, Lions, Moll, and Qu, 2016; Stachurski and Toda, 2019). We establish that in equilibrium models incorporating commonly-used fiscal structures, the debt-to-GDP ratio can follow the (generalized) Kesten process, resulting in a fat-tailed long-run distribution, similar to those observed in distributions of income, wealth, or firm sizes.

The structure of this study is as follows: Section 2 introduces the stochastic dynamic equation for the debt-to-GDP ratio and provides a definition of debt sustainability. Section 3 examines debt sustainability in a deterministic context. Section 4 explores the asymptotic behavior of the debt-to-GDP ratio under uncertainty, with a focus on upper-tail risk. Section 5 presents the equilibrium model underlying the stochastic dynamic equation and derives the conditions for debt sustainability. Section 6 applies the model to empirical data. Section 7 addresses relevant discussions. Finally, Section 8 provides concluding remarks.

[6] See Supplementary Appendix D for an extensive discussion.

[7] Our theoretical framework closely aligns with those employed in empirical research on public finance (e.g., Bohn, 1998; Ball, Elmendorf, and Mankiw, 1998; Mendoza and Ostry, 2008; Hall and Sargent, 2011) and stochastic debt sustainability analysis (SDSA), a key instrument for policymakers in assessing debt sustainability under uncertain conditions (see e.g., Celasun, Ostry, and Debrun, 2006; Barrett, 2018; Blanchard, Leandro, and Zettelmeyer, 2021).

2 Debt Dynamics and Definition of Debt Sustainability

This section presents the stochastic dynamic equation for the debt-to-GDP ratio and defines the concept of debt sustainability as used in this paper.

2.1 Debt Dynamics

In this study, we employ a discrete-time setting, with periods labeled $t = 0, 1, 2 \ldots .$ Throughout this paper, let Y_t denote the real GDP in period t, B_t denote the real value of government *debt* at the end of period t, and S_t denote the real primary fiscal surplus during period t. The evolution of B_t is governed by

$$B_{t+1} = (1 + r_{G,t+1})B_t - S_{t+1}, \tag{3}$$

where $r_{G,t+1}$ is the real interest rate on government bonds due in period $t+1$ and the initial real stock of debt is assumed to be $B_0 > 0$.

Let $r_{M,t+j}$ represent the random variable for the debt holder's discount rate on payoffs j periods ahead in period t. Note that our study does *not* assume the discount rate equals the government bond rate. By iterating (3) forward and taking expectations, we obtain[8]

$$B_t = \lim_{k\to\infty} \begin{bmatrix} \mathbb{E}_t \sum_{j=1}^{k} \left(\prod_{i=1}^{j} \frac{1}{1 + r_{M,t+i}} \right) \left[S_{t+j} + \left(r_{M,t+j} - r_{G,t+j} \right) B_{t+j-1} \right] \\ + \mathbb{E}_t \left(\prod_{j=1}^{k} \frac{1}{1 + r_{M,t+j}} \right) B_{t+k} \end{bmatrix}. \tag{4}$$

Debt-to-GDP Ratio By dividing (3) by real GDP, Y_t, we derive the law of motion for the debt-to-GDP ratio, $b_t = B_t/Y_t$:

$$b_{t+1} = \left(\frac{1 + r_{G,t+1}}{1 + g_{t+1}} \right) b_t - s_{t+1},$$

[8] See Supplementary Appendix C.1 for the derivation. In deriving (4), we have assumed that limits and expectations are interchangeable.

where g_{t+1} is the net GDP growth rate between t and $t+1$ and $s_t = S_t/Y_t$ is the primary surplus-to-GDP ratio. This equation, referred to as the fundamental equation for the debt-to-GDP ratio, forms the basis of our analysis.

Fiscal Policy We consider a fiscal policy where the primary surplus-to-GDP ratio follows:

$$s_t = \phi b_{t-1} - \eta_t, \tag{5}$$

where the first component represents a rule-based fiscal policy with ϕ indicating how the primary surplus-to-GDP ratio adjusts in response to fluctuations in the debt-to-GDP ratio, and the latter accounts for the ratio of the discretionary fiscal deficit to GDP. We note that ϕ may assume a negative value; in such a case, the government opts to elevate the primary *deficit*-to-GDP ratio in response to an increase in the debt-to-GDP ratio.

Given the fiscal rule in (5), the law of motion for the debt-to-GDP ratio is given by

$$b_{t+1} = a_{t+1} b_t + \eta_{t+1}, \tag{6}$$

where

$$a_{t+1} = \frac{1 + r_{G,t+1}}{1 + g_{t+1}} - \phi \approx r_{G,t+1} - g_{t+1} - \phi. \tag{7}$$

Intertemporal Government Budget Constraint The intertemporal government budget constraint (IGBC) stipulates that the value of debt must equal the expected present value of future government net revenues. Under the conventional assumption that the investors' discount rate equals the government bond rate, the IGBC is satisfied if the value of debt matches the expected present value of the primary fiscal surplus. However, our analysis incorporates the convenience yield of government bonds, arising from liquidity and safety premia, resulting in the government bond rate r_G being lower than the investors' discount rate r_M. Below, we define the IGBC considering the convenience yield of government bonds.

Definition 1 (Intertemporal Government Budget Constraint)**.** The IGBC is satisfied if

$$B_t = \mathbb{E}_t \sum_{j=1}^{k} \left(\prod_{i=1}^{j} \frac{1}{1 + r_{M,t+i}} \right) \left[S_{t+j} + (r_{M,t+j} - r_{G,t+j}) B_{t+j-1} \right] \in [0, \infty). \tag{8}$$

According to definition 1, the IGBC stipulates that the sum of the discounted present value of the stream of primary surpluses and the convenience yields of government bonds must equal the value of the outstanding government bonds. Notably, given (4), Definition 1 implies that the IGBC is equivalent to

$$\lim_{k\to\infty} \mathbb{E}_t \left(\prod_{j=1}^{k} \frac{1}{1 + r_{M,t+j}} \right) B_{t+k} = 0, \tag{9}$$

which means that the discounted present value of future government debt converges to zero.

2.2 Debt Sustainability: Definition

In this study, we use the conventional definition of debt sustainability as employed in a large body of work on debt sustainability and empirical tests of fiscal solvency:

Definition 2 (Debt Sustainability)**.** Debt is considered sustainable if the IGBC is satisfied.

Definition 2 implies that debt sustainability is satisfied is condition (9) holds. In later section, we show that condition (9) corresponds to the transversality condition (TVC) for the infinitely-lived investors in a complete market model with the stochastic discount factor $M_{t,t+1} = 1/(1 + r_{M,t+1})$. Therefore, unless (9) holds, the government cannot find buyers for its debt, and the debt will be unsustainable.

3 Debt Sustainability under Certainty

The central aim of this paper is to investigate debt sustainability in uncertain environments. Before addressing the primary topic, we first examine debt dynamics and sustainability under *certainty*, where the government bond rate, the discount rate, and the GDP growth rate are all constant: $r_{G,t} = r_G$, $r_{M,t} = r_M$, and $g_t = g$, and the ratio of the discretionary fiscal deficit to GDP η_t is constant: $\eta_t = \eta > 0$. For empirical relevance, we restrict our attention to the following case:

$$r_M > g > r_G, \qquad \text{and} \qquad \eta > 0.$$

Specifically, (i) the debt holders' discount rate is higher than the growth rate, (ii) the government bond rate is lower than the growth rate, and (iii) the discretionary fiscal deficit is positive.

In this deterministic environment, the IGBC simplifies to:

$$B_t = \sum_{j=1}^{\infty} \left(\frac{1}{1+r_M} \right)^j \left[S_{t+j} + (r_M - r_G) B_{t+j-1} \right],$$

accompanied by the terminal condition:

$$\lim_{k \to \infty} \left(\frac{1}{1+r_M} \right)^k B_{t+k} = \lim_{k \to \infty} \left(\frac{1+g}{1+r_M} \right)^k b_{t+k} = 0. \tag{10}$$

This illustrates that the terminal condition is met if the debt-to-GDP ratio grows at a rate below $(1+r_M)/(1+g) > 1$, indicating that stabilizing the debt-to-GDP ratio at a finite level is sufficient for meeting the IGBC. The law of motion for the debt-to-GDP ratio (6) is given by

$$b_{t+1} = ab_t + \eta; \qquad a = \frac{1+r_G}{1+g} - \phi > 0$$

This confirms that (10) is satisfied if and only if

$$a < \frac{1+r_M}{1+g}, \quad \text{or} \quad \frac{1+r_G}{1+g} - \phi < \frac{1+r_M}{1+g}.$$

Note that in a deterministic context, under $r_M > g > r_G$, debt is sustainable even without fiscal stabilization policies ($\phi = 0$).[9] This implies that the government can sustain unbacked deficit finance as long as $r_M > g > r_G$ (Blanchard, 2019; Blanchard, Leandro, and Zettelmeyer, 2021; Blanchard, 2023). The following sections explore these dynamics in a stochastic context where $r_M > g > r_G$ holds on average.

4 Debt Dynamics under Uncertainty

In this section, we explore the dynamics of debt under uncertainty. As outlined in Section 2.1, the evolution of the debt-to-GDP ratio is given by

$$b_{t+1} = a_{t+1} b_t + \eta_{t+1},$$

[9] In this case, the debt-to-GDP ratio converges to $-\eta(1+g)/(r_G - g) > 0$.

where a_t and η_t are random variables.

4.1 Tail Thickness

To examine the tail dynamics of the debt-to-GDP ratio, we begin by rigorously defining the thickness of the tails in probability distributions. Consistent with its established definition, we characterize a heavy-tailed distribution as follows:

Definition 3 (Heavy tail)**.** A random variable X has a *heavy* upper tail if the moment generating function of X, $M_X(\vartheta)$, is infinite for all $\vartheta > 0$, i.e., $\mathbb{E}\left[\mathrm{e}^{\vartheta X}\right] = \infty$ for all $\vartheta > 0$. Otherwise, X has a light upper tail.

Definition 3 indicates that heavily-tailed distributions feature tail probabilities $\mathrm{Prob}(X > x)$ that decay more slowly than any exponential function, i.e., $\lim_{x\to\infty} \mathrm{e}^{\vartheta x}\mathrm{Prob}(X > x) = \infty$ for all $\vartheta > 0$. Conversely, light-tailed distributions exhibit tail probabilities that decay at a rate faster than or equal to that of an exponential function. Examples of light-tailed distributions include bounded distributions, Gaussian distributions, and exponential distributions.

We further provide a formal definition of a fat-tailed distribution:

Definition 4 (Fat tail)**.** We say that the distribution of X has a *fat* upper tail if there is a positive exponent $\alpha > 0$ such that $\mathrm{Prob}(X > x) \sim x^{-\alpha}$ as $x \to \infty$.

Since the decay rate of a fat-tailed distribution is slower than that of any exponential function, every fat-tailed distribution is heavy-tailed, but not vice versa.[10]

Remark 1. A log-normal distribution is a heavy-tailed but not a fat-tailed distribution.

4.2 Debt-to-GDP Ratio in the Long Run

This section presents a detailed examination of the existence, uniqueness, and characteristics of the asymptotic distribution of b_t. In our study, we establish the following assumptions for the random variables a_t and η_t.

Assumption 1. a_t is positive for all t, i.e., $a_t > 0$ for all t.

Assumption 2. η_t is positive and exhibits a light upper tail.

[10] See, for example, Stachurski and Toda (2019) for the formal proof.

Let $\mathbb{E}_t$ denote the conditional expectation based on information available up to t. We posit a mild constraint on the upper bound of a_t:

Assumption 3. $\mathbb{E}_t[a_{t+1} \mid g_{t+1}] < \infty$.

For the analysis below, we mainly focus on the case where the logarithm of a_t follows a stationary first-order autoregressive and first-order moving-average (ARMA(1,1)) process

$$\log(a_t) = (1-\rho_a)\mu_a + \rho_a \log(a_{t-1}) + \varepsilon_{a,t} + \theta_a \varepsilon_{a,t-1}, \tag{11}$$

where $|\rho_a| < 1$ and $\varepsilon_{a,t}$ is an independently and identically distributed (IID) random variable. Parameter μ_a presents the unconditional mean of $\log(a_t)$: $\mathbb{E}[\log(a_t)] = \mu_a$.

4.2.1 a_t Is IID: Kesten Multiplicative Process

We begin our analysis with a specific case where a_t is an IID random variable, aligning with the case of $\rho_a = \theta_a = 0$ in (11). In this situation, the stochastic process (6) is recognized as the Kesten multiplicative process, whose asymptotic behavior is well-documented by the Kesten–Goldie theorem (Kesten, 1973; Goldie, 1991). Specifically, under Assumptions 1 and 2, the asymptotic distribution of b_t is characterized as follows:

(i) If $\mathbb{E}[\log(a_t)] \geq 0$, the asymptotic distribution is nonstationary and $\mathbb{E}[b_t]$ increases explosively.[11]

(ii) If $\mathbb{E}[\log(a_t)] < 0$, a unique stationary distribution of b_t exists, which:

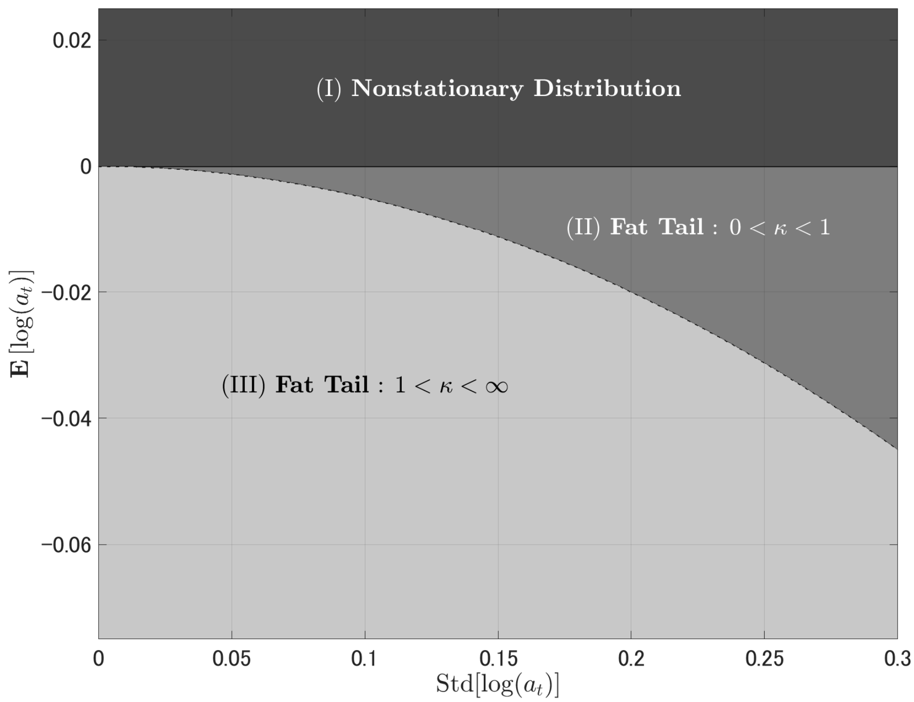
  (ii-a) Is fat-tailed with a Pareto tail index κ, satisfying $\mathbb{E}[a_t^\kappa] = 1$ if $\sup(a_t) > 1$.

  (ii-b) Is light-tailed if $\sup(a_t) \leq 1$.

Consequently, the Kesten–Goldie theorem indicates that under the conditions: (i) a negative average net growth rate of b (i.e., $\mathbb{E}[\log(a_t)] < 0$), (ii) the existence of a positive probability of $a_t > 1$ ($\exists \kappa > 0$ such that $\mathbb{E}[a_t^\kappa] = 1$), and (iii) a positive primary deficit-to-GDP ratio for

[11] In this case, the asymptotic distribution is explicitly written as $b_t = b_0 \exp\left(\mu_a t + \sqrt{t\nu} Z + o(\sqrt{t})\right)$, where $\mu_a = \mathbb{E}[\log(a_t)]$, $\nu = \mathrm{Var}(\log(a_t))$, $Z \sim N(0,1)$, and $o(\sqrt{t})/\sqrt{t} \to 0$ as $t \to \infty$. See Hitczenko and Wesołowski (2011) and Forbes and Grosskinsky (2021).

Figure 2: Characteristics of the asymptotic distribution of b_t: IID



all t (i.e., $\eta_t > 0$), the debt-to-GDP ratio dynamics converges to a stationary distribution with a Pareto law:

$$\mathrm{Prob}\left(b_t > \bar{b}\right) \sim k\bar{b}^{-\kappa}, \qquad k > 0.$$

Here, the tail index κ depends exclusively on the a_t process, independent of the η_t process.[12]

For instance, assume that $\log(a_t)$ is independently and normally distributed with mean μ_{iid} and standard deviation $\sigma_{iid} > 0$, i.e., $\log(a_t) \overset{iid}{\sim} N(\mu_{iid}, \sigma_{iid}^2)$. In this case, $\log(a_t)$ is unbounded above (i.e. it has no upper bound), $\sup(a_t) > 1$ clearly (see Section 4.2.3 for the case with bounded processes). Therefore, according to the Kesten–Goldie theorem, if $\mu_{iid} < 0$, b_t asymptotically converges to the Pareto distribution with the tail index condition satisfying

$$1 = \mathbb{E}\left[a_t^{\kappa}\right] = \mathbb{E}\left[\exp(\kappa \log(a_t))\right] = \exp\left(\kappa\mu_{iid} + \frac{\kappa^2\sigma_{iid}^2}{2}\right).$$

[12]More precisely, the Kesten–Goldie theorem indicates that the stationary distribution has fat tails characterized by a Pareto tail index κ, satisfying $\mathbb{E}\left[a_t^{\kappa}\right] = 1$ if $\mathbb{E}\left[\eta_t^{\kappa}\right] < \infty$. In our study, we assume that η_t has a light upper tail (Assumption 2), ensuring that $\mathbb{E}\left[\eta_t^{\kappa}\right] < \infty$ for all $\kappa > 0$.

Namely, the tail index $\kappa > 0$ is given by

$$\kappa = -\frac{\mu_{iid}}{\sigma_{iid}^2/2}.$$

Figure 2 illustrates the nature of the asymptotic distribution. We highlight the asymptotic distribution can be categorized into three distinct types:

(I) Nonstationary asymptotic distribution if $\mu_{iid} \geq 0$ (Region (I) in Figure 2): The debt-to-GDP ratio grows explosively.

(II) Fat-tailed stationary distribution with an *infinite* mean if $-\sigma_{iid}^2/2 < \mu_{iid} < 0$ (Region (II) in Figure 2): A unique stationary distribution exists with a power-law upper tail index $0 < \kappa < 1$

(III) Fat-tailed stationary distribution with a *finite* mean if $\mu_{iid} \leq -\sigma_{iid}^2/2$ (Region (III) in Figure 2): A unique stationary distribution exists with a power-law upper tail index $1 \leq \kappa$.

Consequently, we note that although the condition for the existence of a stationary distribution is $\mu_{iid} < 0$, the requirement for this distribution to have a finite mean (i.e., $1 \leq \kappa$) is more stringent, specified as

$$\mu_{iid} + \frac{\sigma_{iid}^2}{2} \leq 0.$$

Furthermore, the following proposition describes the growth rate of mean of the asymptotic debt-to-GDP ratio:

Theorem 1. *As $t \to \infty$, the mean of the debt-to-GDP ratio $\mathbb{E}[b_t]$ is infinite if $\mu_{iid}+\sigma_{iid}^2/2 > 0$. In this case, $\mathbb{E}[b_t]$ grows at the rate of $\mu_{iid} + \sigma_{iid}^2/2 > 0$.*

Proof of Theorem 1. See Supplementary Appendix B.1. □

This theorem indicates that the mean of the debt-to-GDP ratio converges to a certain value if $\mu_{iid} + \sigma_{iid}^2/2 \leq 0$, while it grows at the rate of $\mu_{iid} + \sigma_{iid}^2/2$ in long run otherwise. In conclusion, depending of μ_{iid} and σ_{iid}, the debt-to-GDP ratio follows three distinct paths, as summarized in Table 1.

Table 1: The asymptotic distribution of b_t: $\log(a_t) \overset{iid}{\sim} N(\mu_{iid}, \sigma^2_{iid})$

Condition	Stationarity	Tail index	mean of b_t	Growth rate of mean of b_t	Figure 2
$\mu_{iid} \geq 0$	No	—	∞	$\mu_{iid} + \frac{\sigma^2_{iid}}{2}$	Region (I)
$-\frac{\sigma^2_{iid}}{2} < \mu_{iid} < 0$	Yes	$0 < \kappa < 1$	∞	$\mu_{iid} + \frac{\sigma^2_{iid}}{2}$	Region (II)
$\mu_{iid} \leq -\frac{\sigma^2_{iid}}{2}$	Yes	$\kappa \geq 1$	Finite	0	Region (III)

4.2.2 Generalized Processes for a_t

Expanding our scope, we move beyond the assumption of an IID process for a_t, adopting a more empirically relevant stochastic process where $\log(a_t)$ follows the stationary ARMA(1,1) process (11). Intriguingly, even when $\log(a_t)$ deviates from an IID random variable, the criteria for the existence and uniqueness of its stationary distribution still align with the Kesten-Goldie theorem (Brandt, 1986). As evidenced by de Saporta (2005) and Benhabib, Bisin, and Zhu (2011), when $\log(a_t)$ is a non-IID random variable, the prerequisite for the Pareto tail index is modified as:

$$\lim_{n \to +\infty} \frac{1}{n} \log \left(\mathbb{E} \left[\left(\prod_{t=0}^{n} a_t \right)^{\kappa} \right] \right) = 0. \tag{12}$$

The lemma below specifies the Pareto tail index condition when $\log(a_t)$ follows the stationary ARMA(1,1) process.

Lemma 1. *Under the stationary ARMA(1,1) process (11) for $\log(a_t)$, the tail condition (12) is formulated as:*

$$\mathbb{E} \left[\exp \left(\kappa \left(\frac{1+\theta_a}{1-\rho_a} \varepsilon_{a,t} + \mu_a \right) \right) \right] = 1. \tag{13}$$

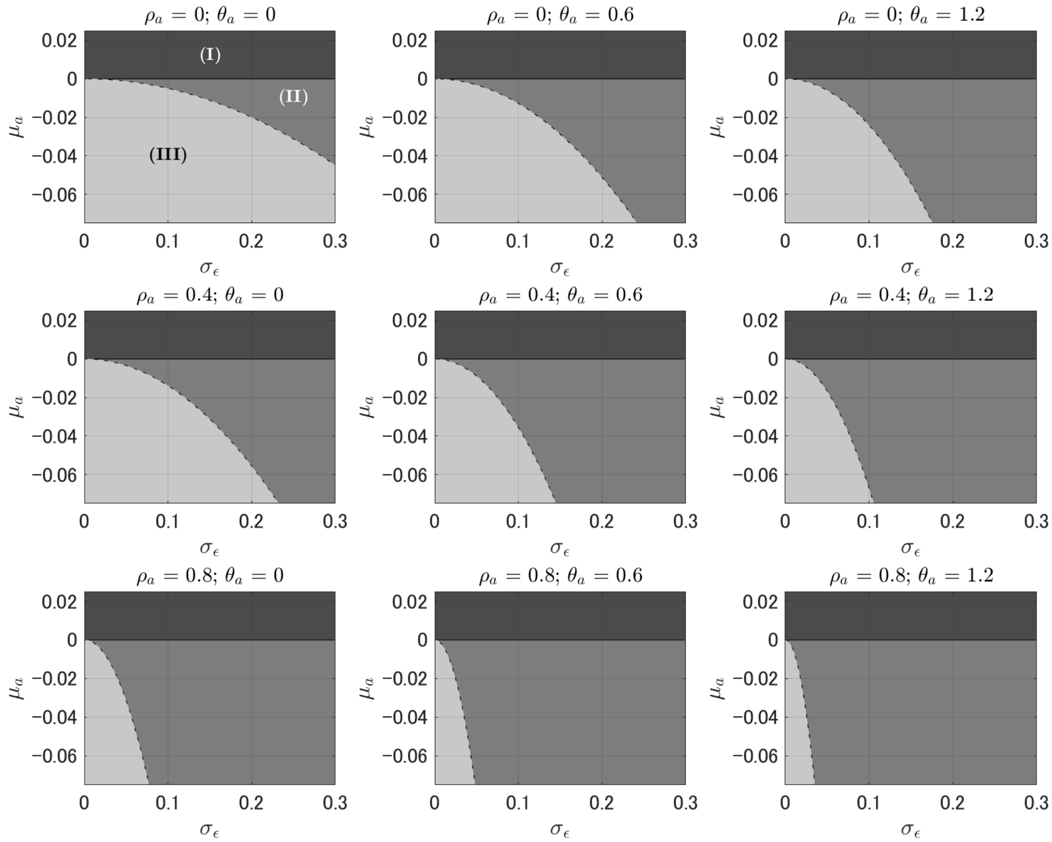
Proof of Lemma 1. See Supplementary Appendix C.2.[13] □

Notably, when $\rho_a = \theta_a = 0$ (implying that $\log(a_t)$ is IID), the tail condition (13) reduces to $\mathbb{E}\left[\exp(\kappa \log(a_t))\right] = \mathbb{E}\left[a_t^{\kappa}\right] = 1$, aligning with the tail condition in the Kesten-Goldie theorem. Furthermore, Lemma 1 implies that when $\log(a_t)$ follows an ARMA(1,1) process with Gaussian innovations $\varepsilon_{a,t} \overset{iid}{\sim} N(0, \sigma^2_{\epsilon})$, the Pareto tail index is given by

$$\kappa = -\left(\frac{1-\rho_a}{1+\theta_a} \right)^2 \left(\frac{\mu_a}{\sigma^2_{\epsilon}/2} \right) > 0. \tag{14}$$

[13]See also Benhabib, Bisin, and Zhu (2011, Proposition 4).

Figure 3: Characteristics of the asymptotic distribution of b_t: ARMA(1,1) process



Note: Each panel illustrates the characteristics of the asymptotic distribution of b_t when $\log(a_t)$ follows an ARMA(1,1) process (11).

This formulation indicates that the asymptotic behavior of the debt-to-GDP ratio depends on parameters μ_a, σ_ϵ, ρ_a, and θ_a. In particular, (14) implies that, given the distributional assumptions for $\varepsilon_{a,t}$, the Pareto tail index κ diminishes (indicating augmented tail risk) as ρ_a and θ_a increase. Consequently, increased persistence in the $\log(a_t)$ process leads to a heavier tail in the asymptotic debt-to-GDP ratio, which in turn suggests that more persistent fluctuations in the interest rate-growth differential can significantly elevate the risk of extreme debt levels.

Figure 3 depicts the nature of the asymptotic distribution. The condition for achieving

Table 2: The asymptotic distribution of b_t: $\log(a_t)$ follows ARMA(1,1)

Condition	Stationarity	Tail index	mean of b_t	Growth rate of mean of b_t	Figure 3
$\mu_a \geq 0$	No	—	∞	$\mu_a + \left(\frac{1+\theta_a}{1-\rho_a}\right)^2 \frac{\sigma_\epsilon^2}{2}$	Region (I)
$-\left(\frac{1+\theta_a}{1-\rho_a}\right)^2 \frac{\sigma_\epsilon^2}{2} < \mu_a < 0$	Yes	$0 < \kappa < 1$	∞	$\mu_a + \left(\frac{1+\theta_a}{1-\rho_a}\right)^2 \frac{\sigma_\epsilon^2}{2}$	Region (II)
$\mu_a \leq -\left(\frac{1+\theta_a}{1-\rho_a}\right)^2 \frac{\sigma_\epsilon^2}{2}$	Yes	$\kappa \geq 1$	Finite	0	Region (III)

Note: $\log(a_t)$ follows an ARMA(1,1) process: $\log(a_t) = (1-\rho_a)\mu_a + \rho_a \log(a_{t-1}) + \varepsilon_{a,t} + \theta_a \varepsilon_{a,t-1}$, where $\varepsilon_{a,t} \overset{iid}{\sim} N(0, \sigma_\epsilon^2)$.

an asymptotic finite mean ($\kappa \geq 1$) is:

$$\mu_a + \left(\frac{1+\theta_a}{1-\rho_a}\right)^2 \frac{\sigma_\epsilon^2}{2} \leq 0,$$

showing that higher values of ρ_a or θ_a expand Region (II), characterized by an infinite mean, and compress Region (III), where the mean is finite. Similar to Theorem 1, the long-run growth rate of the mean of the debt-to-GDP ratio is given by

$$\Delta \log(\mathbb{E}[b_t]) = \begin{cases} \mu_a + \left(\frac{1+\theta_a}{1-\rho_a}\right)^2 \frac{\sigma_\epsilon^2}{2} & \text{if } \ \mu_a + \left(\frac{1+\theta_a}{1-\rho_a}\right)^2 \frac{\sigma_\epsilon^2}{2} > 0 \\ 0 & \text{otherwise} \end{cases},$$

as $t \to \infty$. Table 2 summarizes the asymptotic distribution of b_t.

4.2.3 Further Discussions

In Sections 4.2.1 and 4.2.2, we discussed the asymptotic behavior of the debt-to-GDP ratio using examples with Gaussian innovations. Here, we briefly describe the generality of these arguments.

Firstly, we note that the debt-to-GDP ratio exhibits a power-law stationary distribution even if $\log(a_t)$ is bounded, as long as $\sup(a_t) > 1$ and $\mathbb{E}[\log(a_t)] < 0$. This imply that the tail risk exists even with a discrete-space Markov chain for a_t. In Supplementary Appendix E, we provide a detailed discussion for the case of bounded $\log(a_t)$, considering a uniform random variable.

Secondly, we have assumed that a_t and η_t are independent random variables. Recent studies regarding Kesten process have shown that a power-law stationary distribution exists even when a_t and η_t are correlated under some regularity conditions (de Saporta, 2005). We will detail the impact of the correlation on the tail risk magnitude in Section 7.

5 Debt Sustainability under Uncertainty

This section analyzes sustainability of public debt in the presence of aggregate shocks. To define sustainability of public debt and study the linkage between the tail risk magnitude of debt-to-GDP ratio and sustainability of public debt, we develop an equilibrium model that provide a microfoundation of the fundamental equation (6) and the stochastic process (11).

5.1 Model

We introduce a general equilibrium model of an infinite-horizon, complete-market exchange economy with government sector. A distinctive feature of this model is its incorporation of safety and liquidity benefits associated with holding government bonds (Sidrauski, 1967; Krishnamurthy and Vissing-Jorgensen, 2012; Fisher, 2015). This results in a convenience yield for government bonds.

Let P_t denote the nominal price of final goods. The growth rate of price level, i.e., inflation rate, is assumed to follow an exogenous process:

$$\frac{P_{t+1}}{P_t} = \exp\left(\pi - \frac{\sigma_\pi^2}{2} + \sigma_\pi \varepsilon_{\pi,t+1}\right), \tag{15}$$

where $\pi = \log(\mathbb{E}[P_{t+1}/P_t])$, $\sigma_\pi > 0$ and $\varepsilon_{\pi,t}$ is an IID standard Gaussian random variable.

Household There exists a unit mass of identical households, each earning an income of Y_t in each period t. The growth rate of income is assumed to follow:

$$1 + g_{t+1} \equiv \frac{Y_{t+1}}{Y_t} = \exp\left(\hat{g} - \frac{\sigma_g^2}{2} + \sigma_g \varepsilon_{g,t+1}\right),$$

where $\hat{g} = \log(\mathbb{E}[Y_{t+1}/Y_t])$, $\sigma_g > 0$ is the standard deviation of the earnings growth rate, and $\varepsilon_{g,t}$ is an IID standard Gaussian random variable. We assume $\hat{g} > \sigma_g^2$, which ensures that the average growth rate $1+g \equiv \mathbb{E}[(1+g_t)^{-1}]^{-1} = \exp\left(\hat{g}-\sigma_g^2\right) > 1$.

In every period, each household determines its consumption C_t, nominal government bond holdings B_t^n, and safe bond holdings A_t to maximize the expected utility function:

$$\mathbb{E}_t\left[\sum_{k=0}^{\infty}\exp(-\rho)^k\left[u(C_{t+k})+\nu\left(\frac{B_{t+k}^n}{P_{t+k}}\right)\right]\right],$$

where $\rho > 0$ is the subject discount rate. The utility from consumption $u(C_t)$ is given by

$$u(C_t) = \frac{C_t^{1-\gamma}-1}{1-\gamma}; \qquad \gamma > 1,$$

and $\nu(B_t^n/P_t)$ denotes the liquidity benefits derived from holding government bonds. We posit a linear utility from holding bonds with exogenous shifts in the marginal utility:[14]

$$\nu\left(\frac{B_t^n}{P_t}\right) = \varrho_t\frac{B_t^n}{P_t}; \qquad \varrho_t = u'(\bar{C}_t)\left(1-\exp(z_t)^{-1}\right),$$

where $\bar{C}_t$ represents the mean consumption among households at time t and z_t represents an exogenous preference shifter. We assume that z_t follows

$$z_t = (1-\rho_z)\mu_z + \rho_z z_{t-1} + \sigma_z\varepsilon_{z,t},$$

where $\mu_z > 0$, $|\rho_z| < 1$, $\sigma_z > 0$, and $\varepsilon_{z,t}$ is an IID standard Gaussian random variable. We impose the following parametric assumption:

Assumption 4 (Average convenience benefits)**.** The value for μ_z is sufficiently high so that

$$\mu_z > \rho + (\gamma-1)\hat{g} + \left[1-\frac{\gamma(\gamma+1)}{2}\right]\sigma_g^2.$$

The household's budget constraint is given by

$$C_t + \frac{B_t^n}{P_t} + A_t \leq Y_t - T_t + \left(1+r_{G,t-1}^n\right)\frac{B_{t-1}^n}{P_t} + \left(1+r_{F,t-1}\right)A_{t-1},$$

where T_t is the lump-sum tax, and $r_{G,t-1}^n$ is the nominal interest rate on government bond,

[14] See Kocherlakota (2023) for a mircofoundation of a linear utility from holding bonds.

and $r_{F,t-1}$ is the real interest rates on safe assets.

In this study, to ensure a positive equilibrium risk-free rate, we impose

Assumption 5 (Small income volatility)**.** The variance of the earnings growth rate σ_g^2 is small so that

$$(\gamma+1)\frac{\sigma_g^2}{2} < \hat{g}.$$

Government In each period t, the government finances its spending G_t by lump-sum tax T_t and issuing short-term government bonds, B_t. The government debt's evolution is described by

$$\frac{B_t^n}{P_t} = (1+r_{G,t-1}^n)\frac{B_{t-1}^n}{P_t} - S_t, \tag{16}$$

where $S_t = T_t - G_t$ is the real primary surplus.

The government keeps government spending as a share of GDP constant $G_t/Y_t = \gamma_g$ and adjusts the primary surplus-to-GDP ratio $s_t = S_t/Y_t$ according to the fiscal rule presented in Bohn (1998): $s_t = \phi b_{t-1} - \eta_t$ where $\eta_t = \eta + \varepsilon_{\eta,t}$ represents the *discretionary* primary deficit-to-GDP ratio and $\varepsilon_{\eta,t}$ is an IID shock. We make the assumption:

Assumption 6 (Discretionary Primary Deficit)**.** $\eta > 0$ and $\inf \varepsilon_{\eta,t} \geq -\eta$, indicating the discretionary primary deficit is positive $\eta_t = \eta + \varepsilon_{\eta,t} > 0$ for all t.

Equilibrium conditions The net supply of safe bonds is zero: $A_t = 0$. In equilibrium, all final goods are used for private consumption C_t and for government spending G_t, subject to the resource constraint $C_t + G_t = Y_t$. The growth rate of consumption equals the growth rate of GDP, i.e., $C_{t+1}/C_t = Y_{t+1}/Y_t$ since G_t/Y_t is constant.

5.2 Equilibrium Analysis

The household's optimal conditions, along with the principle of aggregation ($C_t = \bar{C}_t$) and the equilibrium conditions, determine the equilibrium interest rates as follows:

$$r_{F,t} \approx \log(1+r_{F,t}) = \rho + \gamma\hat{g} - \frac{\gamma(\gamma+1)}{2}\sigma_g^2 \equiv r_F > 0, \tag{17}$$

and

$$r_{G,t}^n - \pi \approx \log\big(1+r_{G,t}^n\big) - \pi = r_F - \sigma_\pi^2 - z_t. \tag{18}$$

From (17), it follows that the equilibrium real risk-free rate is constant. Conversely, (18) indicates that the equilibrium expected real interest rate of government bonds, $r^n_{G,t} - \pi$, fluctuates over time and averages lower than the risk-free rate r_F, given that $\mathbb{E}[z_t] = \mu_z > 0$.

The transversality condition is given by

$$\lim_{k\to\infty} \mathbb{E}_t \left[M_{t,t+k} B_{t+k}\right] = 0, \tag{19}$$

where $B_t = B^n_t/P_t$ represents the real stock of government bonds and $M_{t,t+k}$ is the stochastic discount factor between period t and $t+k$. For convenience, we define the stochastic discount rate $r_{M,t+1}$ as $M_{t,t+1} = 1/(1 + r_{M,t+1})$. Then, we can obtain the following:

Lemma 2. *In equilibrium, it holds that*

$$M_{t,t+k} = \prod_{j=t}^{k} \frac{1}{1 + r_{M,t+j}}; \quad \textit{where} \quad 1 + r_{M,t+1} = \exp\left(\rho + \gamma\left(\hat{g} - \frac{\sigma_g^2}{2} + \sigma_g \varepsilon_{g,t+1}\right)\right).$$

The average discount rate r_M, defined as the harmonic mean of $1 + r_{M,t}$, is given by

$$1 + r_M \equiv (\mathbb{E}[1 + r_{M,t}]^{-1})^{-1} = 1 + r_F.$$

Proof of Lemma 2. See Supplementary Appendix C.3. □

The government budget constraint (16) induces

$$B_{t+1} = (1 + r_{G,t+1})B_t - S_{t+1},$$

where $1 + r_{G,t+1} \equiv (1 + r^n_{G,t})P_t/P_{t+1}$ represents the (realized) gross real interest rate of government bond. From (15) and (18), $1 + r_{G,t+1} = \exp(r_F - z_t - \sigma_\pi^2/2 - \sigma_\pi \varepsilon_{\pi,t+1})$. Thus, the average real interest rate of government bond, defined as $1 + r_G \equiv \mathbb{E}[(1 + r_{G,t})^{-1}]^{-1}$, is given by $1 + r_G = \exp(r_F - \mu_z - \sigma_\pi^2 - (\sigma_z^2/2)/(1 - \rho_z))$.

Proposition 1. *The debt-to-GDP ratio $b_t = B_t/Y_t$ follows*

$$b_{t+1} = a_{t+1} b_t + \eta_{t+1},$$

where

$$a_t = \frac{1 + r_{G,t}}{1 + g_t} - \phi \approx r_{G,t} - g_t - \phi. \tag{20}$$

Furthermore, $\log(a_t)$ *also follows an ARMA(1,1) process:*

$$\log(a_{t+1}) = (1 - \rho_a)\mu_a + \rho_a \log(a_t) + \varepsilon_{a,t+1} + \theta_a \varepsilon_{a,t}.$$

where $\mu_a = \rho + (\gamma - 1)\hat{g} + [1 - \gamma(\gamma + 1)]\,\sigma_g^2/2 - \sigma_\pi^2/2 - \mu_z - \phi < 0$, $\rho_a = \rho_z$, $\theta_a = \sqrt{\sigma_z^2 + \rho_z^2(\sigma_g^2 + \sigma_\pi^2)}/\sqrt{\sigma_g^2 + \sigma_\pi^2}$ *and* $\varepsilon_{a,t} \sim N(0, \sigma_\epsilon^2)$ *with* $\sigma_\epsilon = \sqrt{\sigma_g^2 + \sigma_\pi^2}$.

Proof of Proposition 1. See Supplementary Appendix C.3. □

The following proposition summarizes the relationship between the average discount rate, growth rate, interest rate on government bond.

Proposition 2. *Under* $\gamma > 1$ *and Assumptions 4 and 5, (i) the average discount rate is equal to the average interest rate on private bonds, (ii) the average discount rate is strictly greater than the average growth rate, and (iii) the average growth rate is strictly greater than the average interest rate on government bond. Namely,* $r_M = r_F > g > r_G$.

Proof of Proposition 2. See Supplementary Appendix C.4. □

5.3 Sustainability of Debt

According to Definition 2, debt sustainability is achieved when (8) is met. As discussed in Section 2, satisfying (8) is equivalent to satisfying (19).

The following theorem provides the sufficient condition for debt sustainability

Theorem 2. *When* a_t *is a random variable following an ARMA(1,1) process (20), debt is sustainable according to Definition 2 if*

$$\mu_a + \left(\frac{1 + \theta_a}{1 - \rho_a}\right)^2 \frac{\sigma_\epsilon^2}{2} < \rho + (\gamma - 1)\left(\hat{g} - \gamma\frac{\sigma_g^2}{2}\right).$$

Proof of Theorem 2. See Appendix B.2. □

Given that $\rho + (\gamma - 1)\left(\hat{g} - \gamma\sigma_g^2/2\right) > 0$ (Assumption 5), Theorem 2 implies that, debt sustainability is definitely satisfied if

$$\mu_a + \left(\frac{1+\theta_a}{1-\rho_a}\right)^2 \frac{\sigma_\epsilon^2}{2} \leq 0. \tag{21}$$

Recalling the arguments in Section 4.2.2, this condition corresponds to the condition of achieving an asymptotic finite mean ($\kappa \geq 1$). Hence, satisfying (21), which implies Pareto tail index $\kappa \geq 1$, can be considered a sufficient condition for debt sustainability.[15]

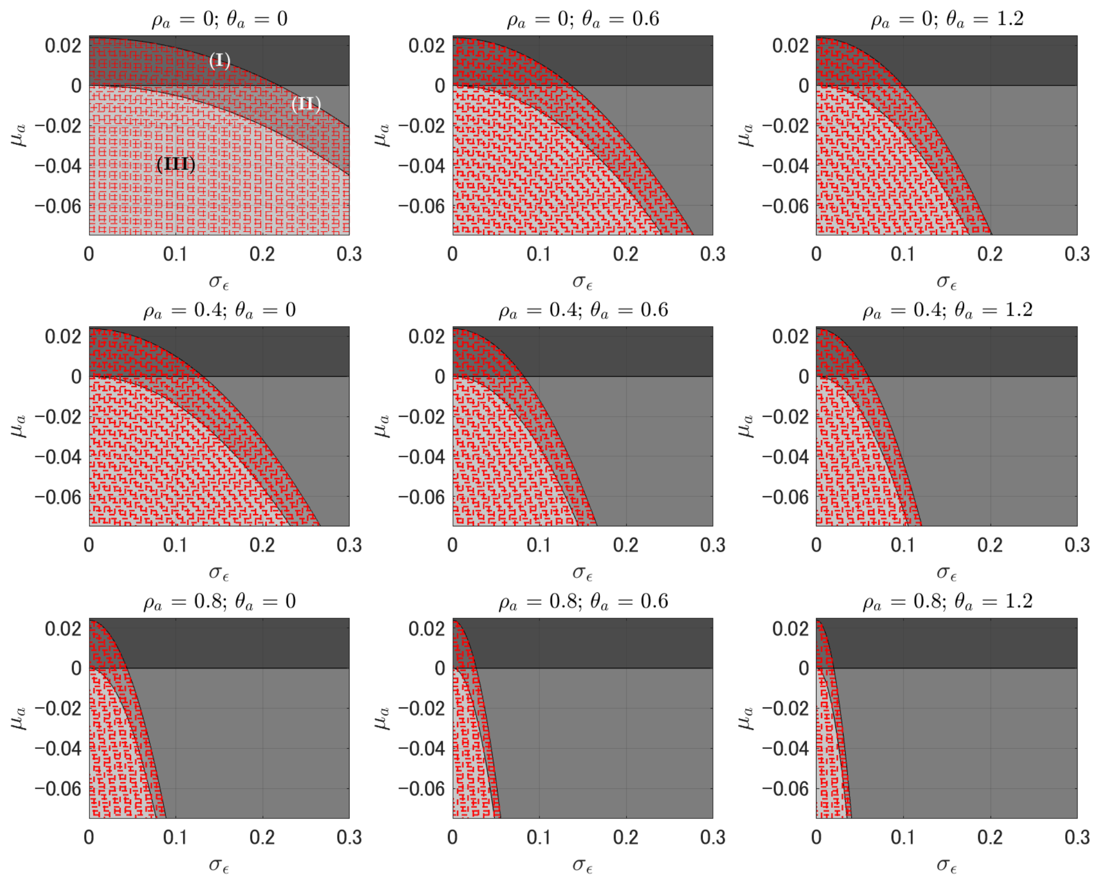
Figure 4 illustrates the condition for debt sustainability, along with the characteristics of the asymptotic distribution of b_t. It shows that region (III), where the asymptotic distribution is stationary with a finite mean, satisfies the conditions for debt sustainability. Additionally, debt sustainability can be satisfied in parts of region (I), where the asymptotic distribution is non-stationary, and region (II), where the asymptotic distribution is stationary with an infinite mean. This is because, even if the mean of b_t diverges, the IGBC can be met as long as the asymptotic growth rate of the mean debt-to-GDP ratio is less than the threshold value (see Theorem 2). Notably, when $\log(a_t)$ is more persistent (i.e., ρ_a is larger), debt sustainability is less likely to be achieved in regions (I) and (II). Therefore, achieving an asymptotic finite mean ($\kappa \geq 1$) is a reasonable condition for debt sustainability when $\log(a_t)$ is highly persistent.

6 Estimation and Empirical Analyses

In the previous sections, we have investigated the asymptotic distribution of the debt-to-GDP ratio that is governed by the stochastic process described by (6) and (11). This section employs historical data on the debt-to-GDP ratio from both the United States and the United Kingdom to estimate the statistical model and evaluate the sustainability of public debt of these countries.

[15] More precisely, Theorem 2 states that debt sustainability is satisfied if the asymptotic growth rate of the mean debt-to-GDP ratio is less than $\rho + (\gamma - 1)\left(\hat{g} - \gamma\sigma_g^2/2\right) > 0$. However, the threshold depends on the time preference rate and the elasticity of intertemporal substitution (EIS), in addition to the parameters underlying the GDP growth rate process. Notably, the range of estimates for the EIS is quite broad. Specifically, when EIS equals 1, the threshold is ρ, which is very small. Therefore, a finite average (i.e., zero asymptotic growth rate) is considered a reasonable condition for debt sustainability.

Figure 4: The Asymptotic Distribution of b_t and Debt Sustainability



Note: The IGBC is satisfied in the red dotted region. We set $\rho = 0.01$, $\hat{g} = 0.0284$, and $\gamma = 1.5$, and $\sigma_g = 0.0290$.

6.1 Estimation Methodology

We first describe the estimation strategy. The model we estimate is as follows:

$$\begin{cases} \log(a_t) = (1-\rho_a)\,\mu_a + \rho_a \log(a_{t-1}) + \varepsilon_{a,t} + \theta_a \varepsilon_{a,t-1}; \qquad |\rho_a| < 1, \mu_a < 0 \\ b_t = a_t b_{t-1} + \eta_t, \end{cases}$$

where $\varepsilon_{a,t}$ represents an IID Gaussian random variable with a mean of zero and a standard deviation of σ_ϵ and η_t an error term embodying discretionary fiscal deficits. We assert:

$$\eta_t = \bar{\eta}_t + \xi_t,$$

where $\bar{\eta}_t$ is an IID exponential random variable with an mean of $\bar{\eta} > 0$, indicating the normal-times ratio of discretionary fiscal deficits to GDP as $\eta_t \sim \exp(1/\bar{\eta})$. Meanwhile, ξ_t, termed as a disaster shock, is a discrete IID random variable, representing infrequent but extraordinary fiscal expansions, such as wars, financial crises, and disasters. Specifically, we posit the following probability mass function for ξ_t:

$$p(\xi) = \begin{cases} 1-(p_1+p_2) & \xi = 0 \\ p_1 & \xi = \xi_d > 0 \\ p_2 & \xi = \hat{\xi}_d > \xi_d \end{cases}.$$

We note that the debt-to-GDP ratio b_t is observable, while a_t, $\bar{\eta}_t$, and ξ_t are latent variables. This specification allows us to derive the following nonlinear and non-Gaussian state-space representation:

$$\begin{bmatrix} \log(a_t) \\ \varepsilon_{a,t} \\ 1 \end{bmatrix} = \begin{bmatrix} \rho_a & \theta_a & (1-\rho_a)\,\mu_a \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \log(a_{t-1}) \\ \varepsilon_{a,t-1} \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \varepsilon_{a,t},$$

and

$$b_t = \begin{bmatrix} b_{t-1} & 0 & 0 \end{bmatrix} \begin{bmatrix} \exp(\log(a_t)) \\ \varepsilon_{a,t} \\ 1 \end{bmatrix} + \bar{\eta}_t + \xi_t.$$

Using the state-space model described above, we obtain the maximum likelihood estimator of $\Theta = (\mu_a, \rho_a, \theta_a, \sigma_\epsilon, \bar{\eta})'$. Specifically, in the estimation, we externally calibrate the values for the disaster shock parameters $(p_1, p_2, \xi_d, \hat{\xi}_d)$ to address the identification issue.

Given these disaster shock parameters, we apply particle filter methods with $25,000$ particles to the state-space model to approximate the conditional likelihood $L(b_k \mid b_{1:k-1}; \Theta)$.

6.2 Data

For our estimation, we use historical debt-to-GDP ratio data from the United States and the United Kingdom. Figure 5 depicts the debt held by the public as a percentage of GDP. Detailed historical overviews and datasets for the debt-to-GDP ratios in these countries are provided in Supplementary Appendix F. The U.S. sample covers the period from 1856 to 2022, as our estimation method is not applicable for the years before 1856 when there was no outstanding debt. For the United Kingdom, the analysis spans the entire dataset from 1700 to 2022.

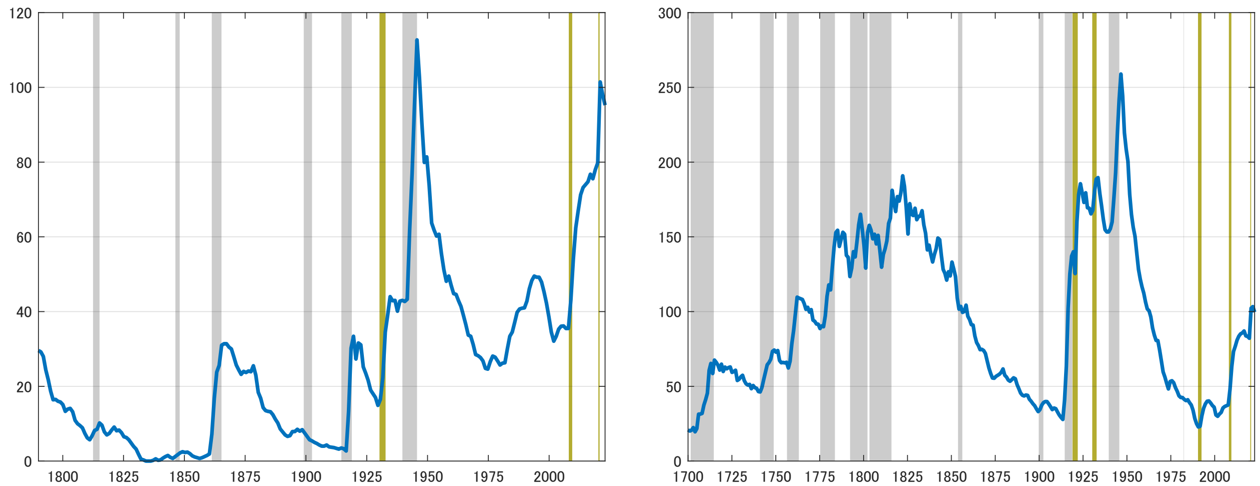
6.3 Estimate Results

Table 3 presents the estimated values alongside the externally assigned disaster shock parameters. The parameters μ_a, ρ_a, θ_a, and σ_ϵ describe the statistical properties of $\log(a_t)$. In both the United States and the United Kingdom, the mean of $\log(a_t)$ is negative ($\mu_a < 0$), indicating that the asymptotic distribution of the debt-to-GDP ratio is stationary and follows a power-law distribution. The United States shows a significantly lower mean of $\log(a_t)$ compared to the United Kingdom, suggesting a more favorable long-term interest rate-growth differential. The autoregressive coefficient ρ_a and the moving-average coefficient θ_a vary between the two countries, reflecting differences in persistence and moving-average effects. The estimates suggest that interest rate-growth differentials in the United States are more persistent than in the United Kingdom. Additionally, the standard deviation of innovations to $\log(a_t)$ (σ_ϵ) is similar across both countries, indicating comparable volatility in $\varepsilon_{a,t}$.

The average ratio of discretionary fiscal deficits to GDP during normal periods ($\bar{\eta}$) is estimated to be slightly higher for the United States compared to the United Kingdom. The externally assigned parameters, including the probabilities of a disaster (p_1) and a large disaster (p_2), as well as the additional fiscal deficits during these events (ξ_d and $\hat{\xi}_d$), are assumed to be identical for both countries. In our benchmark scenario, we assume that a disaster shock leading to a fiscal deficit-to-GDP ratio of 10% occurs once every 20 years on

Figure 5: Debt Held by the Public as Percentage of GDP (%)

(a) United States: 1790-2022 **(b)** United Kingdom: 1700-2022



Source: Federal Debt Held by the Public, 1790 to 2000 (Congressional Budget Office, 2010); the Debt to the Penny dataset (the U.S. Department of the Treasury).

Source: Bank of England's 'A Millennium of Macroeconomic Data' (Thomas and Dimsdale, 2017); Public sector finances data (the Office for National Statistics).

Note: The debt level is assessed based on the face value of the total federal debt held by the public at the end of each calendar year. Shaded areas indicate the duration of major wars and deep recessions: In Panel (a), these are the War of 1812 (1812–15), Mexican-American War (1846–48), American Civil War (1861-65), Philippine–American War (1899–1902), World War I (1914–18), Great Depression (1930-32), World War II (1939–45), Great Recession (2008-09), and COVID-19 recession (2020). In Panel (b), these periods include the War of the Spanish Succession (1701-14), War of the Austrian Succession (1740-48), Seven Years' War (1756-63), American War of Independence (1775-83), French Revolutionary Wars (1792-1802), Napoleonic Wars (1803–15), Crimean War (1853–56), Second Boer War (1899–1902), World War I (1914–18), post-WWI recession (1919-21), Great Depression (1930-32), World War II (1939–45), Falklands War (1982), early 1990s recession (1990-92), Great Recession (2008-09), and COVID-19 recession (2020).

average, and a large disaster shock causing a 20% ratio occurs about once every 60 years. In Section 6.6, we assess the robustness of estimation results, particularly examining the impact of the selected disaster shock parameters.

The bottom of Table 3 displays the implied Pareto tail index (κ) for the asymptotic distribution of the debt-to-GDP ratio. As previously discussed, the asymptotic distribution has a finite mean, and debt sustainability is assured if $\kappa \geq 1$. Our estimates reveal that while the tail index exceeds 1 in both the United States and the United Kingdom, it is closer to 1 in the United States ($\kappa = 1.59$). This finding indicates a higher risk of extreme values in the debt-to-GDP ratio distribution for the United States compared to the United Kingdom. In summary, our results show that although the United States has benefited from a low long-term interest rate-growth differential, the highly persistent fluctuations in this

Table 3: Parameter Value and Implied Pareto Index

Parameter	**Description**	**Values** US	UK
Estimated Parameters			
μ_a	mean of $\log(a_t)$	-0.0703	-0.0168
ρ_a	first-order autoregressive coefficient of $\log(a_t)$	0.8333	0.5556
θ_a	moving-average coefficient of $\log(a_t)$	-0.0278	-0.0167
σ_ϵ	standard deviation of innovations to $\log(a_t)$	0.0468	0.0489
$\bar{\eta}$	mean of the ratio of discretionary fiscal deficits to GDP in normal times	0.0100	0.0073
Externally Assigned Parameters			
p_1	probability of a disaster	0.050	0.050
p_2	probability of a large disaster	0.017	0.017
ξ_d	additional fiscal deficit during a disaster	0.10	0.10
$\hat{\xi}_d$	additional fiscal deficit during a large disaster	0.20	0.20
Tail Index			
κ	Pareto tail index	1.887	2.870

Note: The Pareto tail index is given by $\kappa = -\left(\frac{1-\rho_a}{1+\theta_a}\right)^2 \left(\frac{\mu_a}{\sigma_\epsilon^2/2}\right)$.

differential increase the tail risk in debt accumulation.

6.4 Historical Path of $\log(a_t)$ and η_t

We estimate the historical trajectories of the latent variables $\log(a_t)$ and η_t using the particle smoothing method.[16] Recall that fluctuations in $\log(a_t)$ can arise due to changes in the interest rate-growth differential or fiscal stabilization rules (see (7)).

The United States Figure 6 displays the estimated historical paths of $\log(a_t)$ (Panel (a)) and η_t (Panel (b)) for the United States. It can be seen that from the Civil War until World War I (WWI), the United States did not experience significant discretionary budget deficits (Panel (b)), and the value of a_t consistently remained below 1 (Panel (a)), indicating low interest rates relative to the growth rate and effective fiscal stabilization policies. During WWI, the debt-to-GDP ratio surged with high discretionary budget deficits (Panel (b)), but a_t remained low (Panel (a)). This pattern indicates that, in the WWI period, the United States benefited from high GDP growth rates, enabling it to finance substantial deficits at relatively low costs. This advantageous debt-financing environment helped reduce the debt-

[16] In implementing particle smoothing, we use algorithms proposed by Klaas, Briers, De Freitas, Doucet, Maskell, and Lang (2006).

to-GDP ratio post-WWI. From the Great Depression through WWII, fiscal deficits escalated, and a_t increased significantly, suggesting periods where $a_t > 1$ (Panel (a)). Post-WWII to the early 1970s, the growth rate mostly exceeded interest rates, reducing the debt-to-GDP ratio. In the late 1970s, interest rates began to exceed growth rates, raising a_t around or above 1 until the early 1990s (Panel (a)). After temporarily falling below 1, the Great Recession pushed a_t to unprecedented levels. Since then, a_t has remained around 1. During the COVID-19 recession, the United States incurred discretionary fiscal deficits similar to those in WWI and WWII.

The United Kingdom Figure 7 displays the estimated historical paths of $\log(a_t)$ (Panel (a)) and η_t (Panel (b)) for the United Kingdom. Panel (a) shows that while $\log(a_t)$ was predominantly negative (i.e., $a_t < 1$), it experienced significant fluctuations, notably surging during wartime and major recessions, resulting in substantial periods of positive $\log(a_t)$. Remarkably, a_t reached exceptionally high levels during WWI and the Great Recession. Our estimates suggest that a_t remained chronically high during three distinct periods: the 18th century, the World Wars era (1914-45), and from the 1990s onwards, coinciding with uptrends in the debt-to-GDP ratio. In contrast, during the 1950s-70s, when the ratio was significantly reduced, a_t remained well below 1. Panel (b) shows that while the discretionary fiscal deficit typically remained under 1 percent of GDP during normal periods, disaster shocks occurred during significant historical events such as the War of the Spanish Succession, the Napoleonic Wars and their aftermath, WWI and the ensuing recession, the Great Recession, and the COVID-19 recession. In summary, the United Kingdom has repeatedly experienced explosive debt-to-GDP dynamics ($a_t > 1$) and is thus at tail risk given the historical fiscal environment. Particularly, the last 30 years have increased the likelihood of this risk materializing.

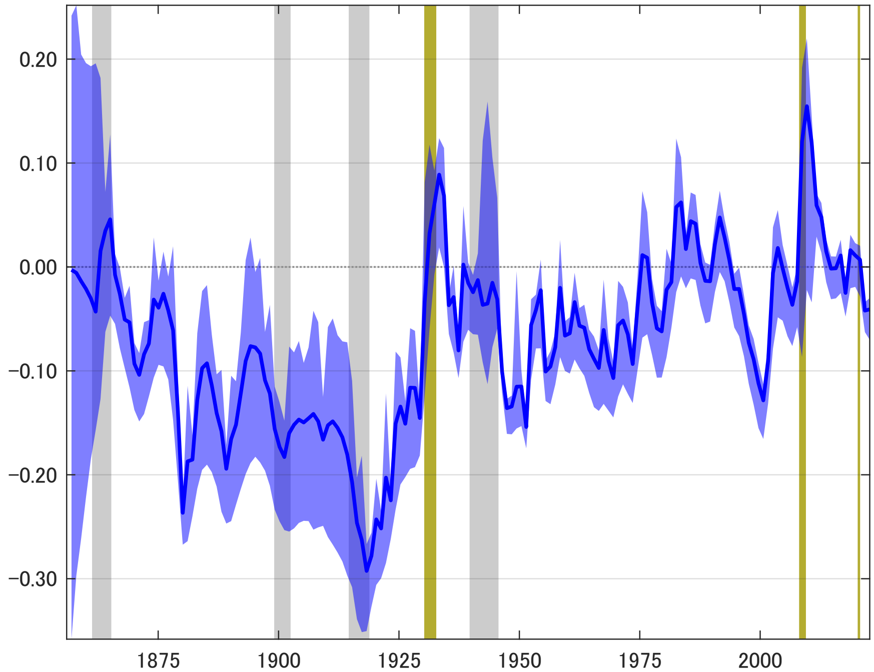
6.5 Future Projection of the Debt-to-GDP Ratio

Figure 8 presents the future projection of the debt-to-GDP ratio for the United States (Panel (a)) and the United Kingdom (Panel (b)) over the period from 2023 to 2053. The projections are illustrated using a fan chart analysis.

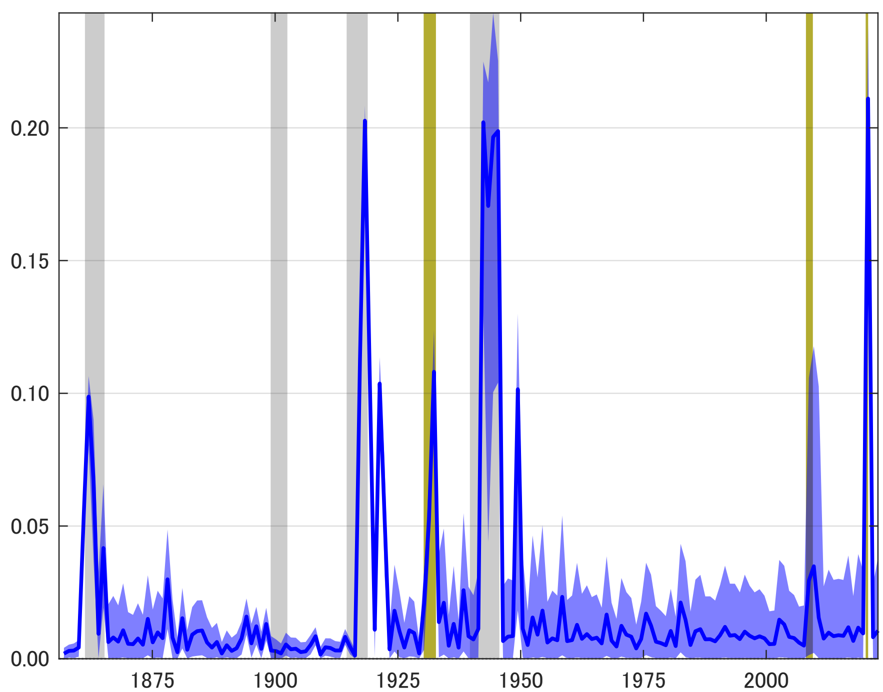
For the United States, the median forecast indicates a steady decline in the debt-to-GDP ratio, while the 95% coverage range reveals a potential risk of the ratio rising to 250%

Figure 6: Estimates of $\log(a_t)$ and η_t in the United States

(a) Estimates of $\log(a_t)$



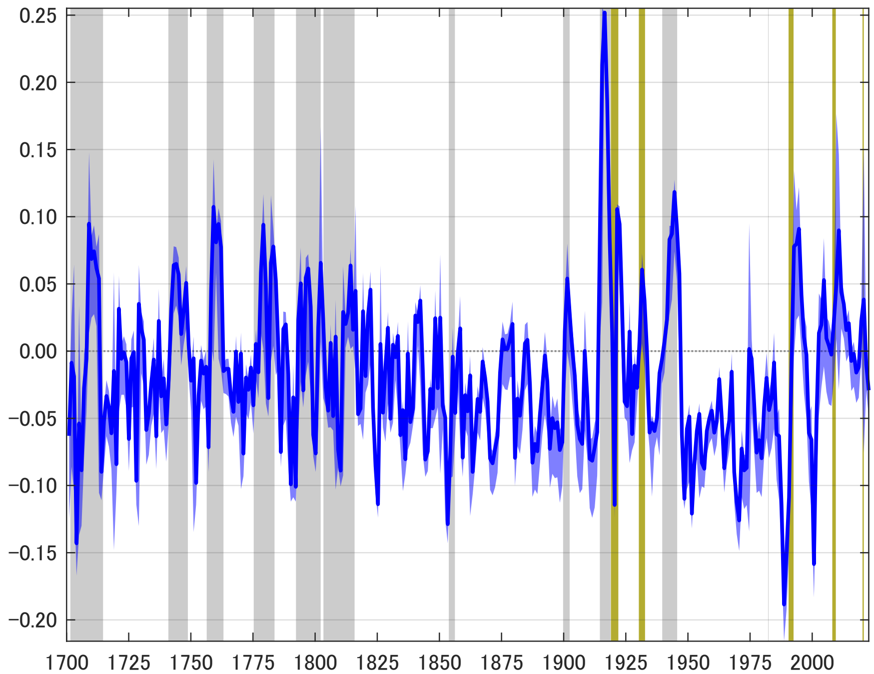
(b) Estimates of η_t (the Ratio of Discretionary Fiscal Deficits to GDP)



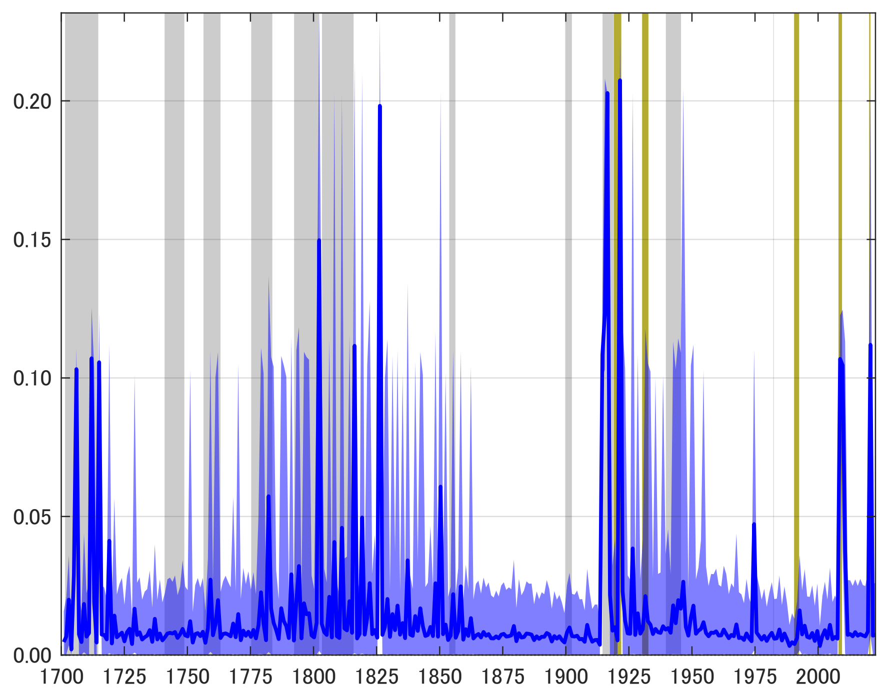
Note: Each panel presents estimates obtained through particle smoothing using 25,000 particles. The solid line represents the mean value, while the accompanying band illustrates the 95% interval, spanning from the 2.5th to the 97.5th percentiles. Shaded areas indicate the duration of major wars and deep recessions: War of 1812 (1812–15); Mexican-American War (1846–48); American Civil War (1861–65); Philippine–American War (1899–1902); WWI (1914–18); Great Depression (1930–32); WWII (1939–45); Great Recession (2008–09); and COVID-19 recession (2020).

Figure 7: Estimates of $\log(a_t)$ and η_t in the United Kingdom

(a) Estimates of $\log(a_t)$

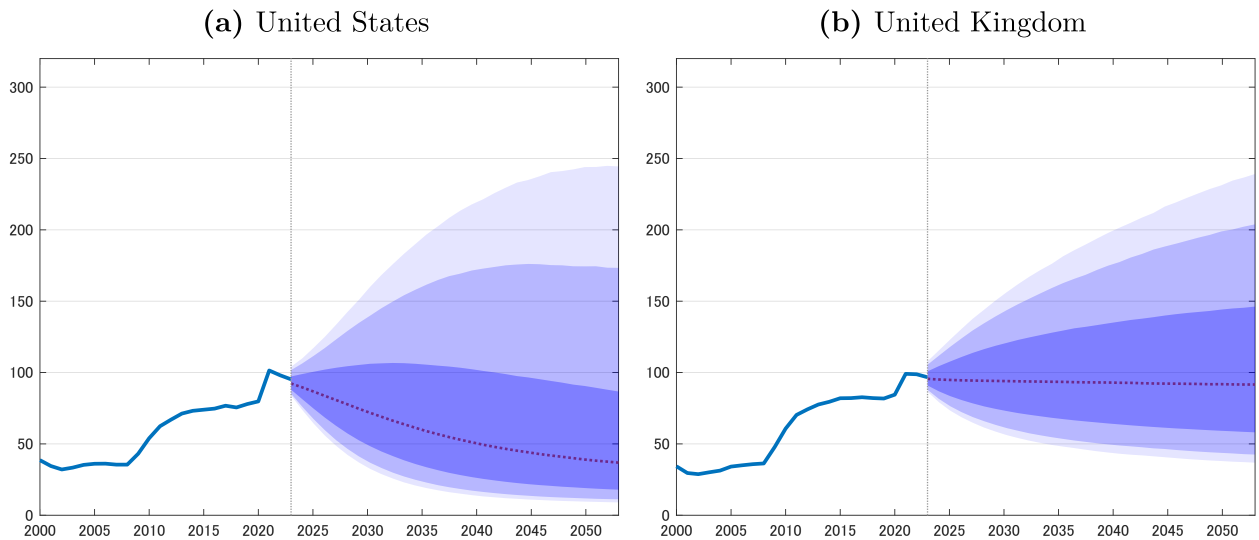


(b) Estimates of η_t (the Ratio of Discretionary Fiscal Deficits to GDP)



Note: Each panel presents estimates obtained through particle smoothing using 25,000 particles. The solid line represents the mean value, while the accompanying band illustrates the 95% interval, spanning from the 2.5th to the 97.5th percentiles. Shaded regions mark the periods of major wars and deep recessions: the War of the Spanish Succession (1701-14); the War of the Austrian Succession (1740-48); Seven Years' War (1756-63); American War of Independence (1775-83); the French Revolutionary Wars (1792-1802); the Napoleonic Wars (1803–15); Crimean War (1853–56); Second Boer War (1899–1902); World War I (1914–18); the post–World War I recession (1919-21); Great Depression (1930-32); World War II (1939–45); Falklands War (1982); Early 1990s recession (1990-92); Great Recession (2008-09); COVID-19 recession (2020).

Figure 8: Debt-to-GDP Ratio (%): Future Projection

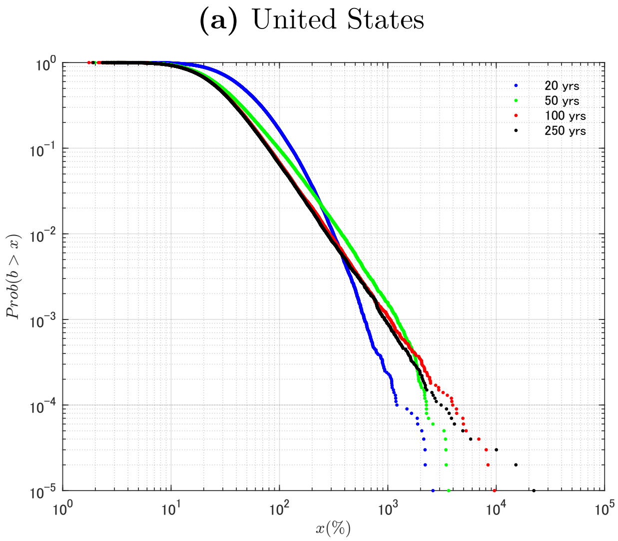


Note: The dotted line indicates the median forecast, the dark band illustrates the forecast within the 68% coverage range, the middle band represents the forecast within the 90% coverage range, and the light band denotes the forecast within the 95% coverage range.

by 2050. For the United Kingdom, the median forecast suggests a stable debt-to-GDP ratio, with the 95% coverage range indicating a potential risk of the ratio rising to 250% by 2050. Notably, the distributional forecasts exhibit an asymmetry between upside and downside risks, which becomes more pronounced as the coverage range increases, resulting in disproportionately escalating upward risks over time. This asymmetric risk reflects the distribution of the debt-to-GDP ratio approaching a right-skewed, fat-tailed distribution.

Figure 9 presents the complementary cumulative distribution function (CDF) of the simulated debt-to-GDP ratio on a log-log scale. The complementary CDF exhibits a linear log-log relationship characteristic of a Pareto distribution in the long run, with the slope flattening over time for both countries. This indicates that while the simulated debt-to-GDP ratio initially has thin tails, its upper tail gradually thickens. Notably, the convergence rate to the Pareto stationary distribution is relatively swift, particularly for the United States, suggesting that a tail event could materialize within this century.

Figure 9: Complementary CDF of the Simulated Debt-to-GDP Ratio



Note: The horizontal axis represents the debt-to-GDP ratio, while the vertical axis shows the corresponding complementary cumulative distribution function.

6.6 Robustness of Estimation Results

This section evaluates the robustness of the estimation results. Table A.1 in Appendix A presents the parameter and tail index estimates using alternative disaster shock parameters from those in Table 3. The results demonstrate that the estimates remain robust across a range of plausible disaster shock scenarios.

7 Discussions

This section addresses additional relevant topics.

7.1 Continuous-Time Framework

In this study, we developed a discrete-time model for debt dynamics. In Supplementary Appendix D, we introduce a continuous-time counterpart, building on the work of Mehrotra and Sergeyev (2021). We demonstrate the broad applicability of our model and, using Luttmer (2016)'s theorem, show that the debt-to-GDP ratio follows a power-law asymptotic distribution under specific assumptions, similar to Assumptions 2 and 4.

7.2 Correlation of $r_G - g$ and Fiscal Deficit

While various stochastic processes for a_t have been examined, we have assumed that they are independent of η_t. This section relaxes this assumption. To apply the mathematical theorem de Saporta (2005), we follow Benhabib, Bisin, and Zhu (2011) and consider a Markov modulated chain, defined as

Definition 5 (Markov Modulated Chain)**.** The stochastic process $(a_t, \eta_t)_t$ is a real, aperiodic, irreducible, stationary Markov chain with finite state space $\bar{\mathbf{a}} \times \bar{\boldsymbol{\eta}} := \{\bar{a}_1, \ldots, \bar{a}_m\} \times \{\bar{\eta}_1, \ldots, \bar{\eta}_l\}$. It is a Markov modulated chain if $\text{Prob}\,(a_t, \eta_t \mid a_{t-1}, \eta_{t-1}) = \text{Prob}\,(a_t, \eta_t \mid a_{t-1})$ where $\text{Prob}\,(a_t, \eta_t \mid a_{t-1}, \eta_{t-1})$ denotes the conditional probability of (a_t, η_t) given (a_{t-1}, η_{t-1}).

Markov modulated chains allow for autocorrelation of η_t and correlation between a_t and η_t. Benhabib, Bisin, and Zhu (2011) demonstrate that a modulated chain $(a_t, \eta_t)_t > 0$ satisfying the following conditions results in an asymptotic Pareto distribution with an upper tail index determined by (12): (i) $\mathbb{E}\,[a_t \mid a_{t-1}] < 1$ for any a_{t-1}, (ii) $\bar{a}_i > 1$ for some $i = 1, \ldots, m$, and (iii) the diagonal elements of the transition matrix of a_t are positive.[17] This implies that the autocorrelation in η_t and the correlation between a_t and η_t do not influence the tail index of b_t. Table 4 confirms this through stochastic simulations of Markov modulated chains with varying autocorrelation in η_t and correlation between a_t and η_t.[18]

7.3 Fiscal Stabilization Rule

This section examines the roles of the activeness of the fiscal stabilization rule ϕ in debt dynamics and sustainability. We first explore the deterministic environment and then proceed to the stochastic environment. Recall that, in the absence of uncertainty, (6) is given by $b_{t+1} = ab_t + \eta$ with $a = (1 + r_G)/(1 + g) - \phi$. The following proposition summarizes the interconnection between the fiscal reaction function, the stabilization of the debt-to-GDP ratio, and the IGBC.

Proposition 3 (Bohn's Condition under Certainty)**.** *Under $r_M > g > r_G$ and $\eta > 0$, the IGBC is satisfied if the growth rate of the debt-to-GDP ratio $(1 + r_G)/(1 + g) - \phi$ does not*

[17] See Appendix B in Benhabib, Bisin, and Zhu (2011).

[18] Table 4 reports the ratio of the top 0.01 percentile to the top 0.1 percentile of the 250,000 simulated debt-to-GDP ratios over 1,000 years. For a Pareto distribution with an upper tail index κ, this ratio is $10^{1/\kappa}$.

Table 4: Stochastic Simulation Results: Markov Modulated Chain

The United States							
$\mathbb{E}[a_t]$	$\sigma(a_t)$	corr(a_t, a_{t-1})	$\mathbb{E}[\eta_t]$	$\sigma(\eta_t)$	corr(η_t, η_{t-1})	corr(a_t, η_t)	0.01%/0.1%
0.94	0.08	0.82	0.02	0.04	0.00	0.00	2.84
0.94	0.08	0.82	0.02	0.04	0.05	0.15	2.92
0.94	0.08	0.82	0.02	0.04	0.18	0.30	2.94
0.94	0.08	0.82	0.02	0.04	0.39	0.44	2.97
0.94	0.08	0.82	0.02	0.04	0.68	0.58	3.00
The United Kingdom							
$\mathbb{E}[a_t]$	$\sigma(a_t)$	corr(a_t, a_{t-1})	$\mathbb{E}[\eta_t]$	$\sigma(\eta_t)$	corr(η_t, η_{t-1})	corr(a_t, η_t)	0.01%/0.1%
0.99	0.06	0.54	0.01	0.03	0.00	-0.00	2.35
0.99	0.06	0.54	0.01	0.03	0.02	0.09	2.34
0.99	0.06	0.54	0.02	0.03	0.09	0.19	2.34
0.99	0.06	0.54	0.02	0.03	0.21	0.28	2.32
0.99	0.06	0.54	0.02	0.03	0.37	0.37	2.33

Note: 0.01%/0.1% represents the ratio of the top 0.01 percentile to the top 0.1 percentile value. For a Pareto distribution $F(\alpha, \kappa)$, where $\alpha > 0$ is the scale parameter and $\kappa > 0$ is the shape parameter, this ratio is calculated as $10^{1/\kappa}$.

exceed $(1 + r_S)/(1 + g)$, *or*

$$\phi > -\frac{r_M - r_G}{1 + g}. \tag{22}$$

In addition, the debt-to-GDP ratio converges to a finite value if

$$\phi > -\frac{g - r_G}{1 + g}. \tag{23}$$

Proof of Proposition 3. See Section 3. □

We emphasize that condition (22) indicates that when the discount rate equals the government bond rate, $r_M = r_G$ (a conventional assumption in existing literature), a positive ϕ is necessary to satisfy the IGBC (Bohn, 1998).[19] Conversely, if the government bond rate is lower than the discount rate, it is possible to meet the IGBC even with a negative ϕ. Similarly, (23) implies that when $g > r_G$, the stabilization of the debt-to-GDP ratio can achieved even with a negative ϕ. In sum, under $r_M > g > r_G$, even seemingly irresponsible

[19]This condition is often imposed to obtain a stable or unique determinate solution of locally approximated dynamic stochastic general equilibrium models.

policies to increase the deficit in response to rising debt can guarantee the IGBC.

In a stochastic environment, we have demonstrated that fluctuations in $\log(a_t) \approx r_{G,t} - g_t - \phi$ determine the magnitude of tail risk in debt accumulation, impacting debt sustainability. Specifically, when $\log(a_t)$ follows an ARMA(1,1) process, a higher fiscal policy parameter ϕ decreases the mean of $\log(a_t)$, $\mu_a = \rho + (\gamma-1)\hat{g} + [1-\gamma(\gamma+1)]\,\sigma_g^2/2 + \sigma_\pi^2/2 - \mu_z - \phi$ (see Proposition 1). The IGBC is satisfied when $\mu_a + \left(\frac{1+\theta_a}{1-\rho_a}\right)^2 \frac{\sigma_\epsilon^2}{2} < \rho + (\gamma-1)\left(\hat{g} - \gamma\frac{\sigma_g^2}{2}\right)$ (see Theorem 2). Therefore, the condition for ϕ to satisfy the IGBC is:

Proposition 4 (Bohn's Condition under Uncertainty)**.**

$$\phi > -(r_M - r_G) + \frac{1}{1-\rho_z}\frac{\sigma_z^2}{2} + \left[1 + 2\gamma + \left(\frac{1+\theta_a}{1-\rho_a}\right)^2\right]\frac{\sigma_g^2}{2} - \left[1 - \left(\frac{1+\theta_a}{1-\rho_a}\right)^2\right]\frac{\sigma_\pi^2}{2}.$$

Proof of Proposition 4. See Supplementary Appendix C.5. □

7.4 Fiscal Limit

This paper has examined debt sustainability based on the fulfillment of the IGBC. As Reis (2022) described, however, recent studies such as Mendoza, Tesar, and Zhang (2014), D'Erasmo, Mendoza, and Zhang (2016), and Mian, Straub, and Sufi (2022) focus on the feasible maximum value of the right-hand side of the IGBC (8), leading to threshold levels of the debt-to-GDP, called "fiscal limit". Here, we briefly discuss how the fiscal limit relates to our analysis.

Let $\bar{s}$ denote the maximum value of the primary surplus-to-GDP ratio, and assume $\bar{s} = 0$. Since the discretionary primary deficit-to-GDP ratio η is positive from Assumption 2, $\bar{s} = 0$ if the coefficient of the fiscal reaction function $\phi = 0$. In this case, $a_t = \frac{1+r_{G,t}}{1+g_t}$. As long as a_t satisfies Assumptions 1, 3, the results of our study for the asymptotic properties of the debt-to-GDP ratio do not change.[20] As shown in Bohn (1991) and Mehrotra and Sergeyev (2021), however, if the primary surplus-to-GDP ratio is bounded above, threshold levels of the debt-to-GDP ratio exist. In this case, having a finite mean of debt-to-GDP ratio would not be sufficient to ensure debt sustainability. The unconditional mean of the debt-to-GDP ratio is no longer an issue, and the probability of reaching a threshold debt level is calculated

[20] It is possible to consider cases where η is negative or where the fiscal reaction function is nonlinear, but it will not be possible to show the properties of the asymptotic distribution analytically.

for assessing debt sustainability in the presence of fiscal limits.[21] Even in such an analysis, our finding of asymmetric expansion of upside risk to the debt-to-GDP ratio over time would be useful.

8 Concluding Remarks

This study explores debt sustainability under uncertainty arising from the interest rate-growth differential and fiscal primary surplus within discrete-time stochastic frameworks. Our theoretical analysis shows that the debt-to-GDP ratio can follow significantly different paths depending on the stochastic nature of the interest rate-growth differential. Specifically, we demonstrate that while a stationary asymptotic distribution is present when the growth rate generally exceeds the government bond rate, the risk of a temporary reversal—such as that experienced in the United States during the 1980s and 1990s—can lead to a right-skewed Pareto distribution. This outcome can result in an upper tail so heavy that the mean debt-to-GDP ratio diverges.

To assess how tail risk influences debt sustainability, we develop a general equilibrium model that incorporates the liquidity and convenience benefits of holding government debt. This model provides a microfoundation for the stochastic process of the interest rate-growth differential discussed earlier, enabling us to evaluate debt sustainability based on the intertemporal government budget constraint (IGBC). We derive a sufficient condition, demonstrating that the magnitude of tail risk, as measured by the Pareto index, is crucial for meeting these conditions. Notably, the IGBC may not be satisfied even if the debt-to-GDP ratio has a stationary asymptotic distribution; however, it is certainly satisfied when the mean of the ratio converges. Based on this analysis, we propose that a finite mean of the long-run debt-to-GDP ratio should be the criterion for debt sustainability.

On the empirical side, we introduce a novel method to estimate tail risk and assess debt sustainability using historical debt-to-GDP data. By transforming the dynamic equation of the debt-to-GDP ratio with stochastic shocks into a non-linear, non-Gaussian state-space model, we apply particle filter techniques for maximum likelihood estimation. Our analysis of 200-300 years of U.S. and U.K. data indicates that both countries' debts generally meet

[21] See, Davig, Leeper, and Walker (2010), Bi (2012), Bi, Leeper, and Leith (2013), Matsuoka (2015)

the finite mean condition for sustainability. However, in the United States, this condition is only marginally satisfied, with significant tail risk. Moreover, since the Great Recession, refinancing costs have escalated to levels comparable to those during the World Wars, exacerbating recent debt challenges in both countries.

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Appendices

APPENDIX A Robustness of Estimation Results

Table A.1: Parameter Estimates: Robustness

	The United States						The United Kingdom					
μ_a	-0.059	-0.078	-0.067	-0.078	-0.052	-0.081	-0.014	-0.017	-0.009	-0.017	-0.007	-0.024
ρ_a	0.780	0.833	0.780	0.807	0.780	0.833	0.473	0.556	0.556	0.556	0.514	0.473
θ_a	0.156	0.094	0.217	0.217	0.094	-0.028	-0.017	-0.017	-0.017	-0.078	0.044	-0.017
σ_ϵ	0.047	0.047	0.044	0.050	0.050	0.047	0.053	0.049	0.049	0.049	0.053	0.049
$\bar{\eta}$	0.010	0.010	0.010	0.010	0.007	0.015	0.007	0.008	0.006	0.007	0.007	0.012
p_1	0.025	0.075	0.050	0.050	0.050	0.050	0.025	0.075	0.050	0.050	0.050	0.050
p_2	0.008	0.025	0.025	0.013	0.017	0.017	0.008	0.025	0.025	0.013	0.017	0.017
ξ_d	0.100	0.100	0.100	0.100	0.050	0.150	0.100	0.100	0.100	0.100	0.050	0.150
$\hat{\xi}_d$	0.200	0.200	0.200	0.200	0.100	0.300	0.200	0.200	0.200	0.200	0.100	0.300
κ	1.966	1.646	2.298	1.569	1.681	2.185	2.891	2.868	1.614	3.261	1.064	5.788
$\log(L(\Theta))$	385.471	386.483	387.833	386.741	377.870	371.508	508.633	508.885	509.523	510.060	496.226	503.793

Note: $\log(L(\Theta))$ represents the log likelihood.

Supplementary Appendix (Not for Publication)

APPENDIX B Proof of Theorems

B.1 Proof of Theorem 1

As discussed in Section 4.2.1, the distribution of b_t converges to a Pareto distribution with a finite mean if $\mu_{iid} + \sigma_{iid}^2/2 \leq 0$. This clearly indicates that $\Delta \log\left(\mathbb{E}\left[b_t\right]\right) \to 0$ as $t \to \infty$ under the condition $\mu_{iid} + \sigma_{iid}^2/2 \leq 0$. Otherwise, $\mathbb{E}[b_t]$ diverges to infinity as $t \to \infty$. We examine the long-run growth rate of $\mathbb{E}[b_t]$ for the case $\mu_{iid} + \sigma_{iid}^2/2 > 0$ by dividing it into two cases.

First, we consider the case where $\mu_{iid} \geq 0$. In this scenario, as demonstrated by Hitczenko and Wesołowski (2011) and Forbes and Grosskinsky (2021), the asymptotic distribution of b_t is nonstationary and can be explicitly expressed as:

$$b_t = b_0 \exp\left(\mu_a t + \sqrt{t\nu} Z + o(\sqrt{t})\right),$$

where $\mu_a = \mathbb{E}\left[\log(a_t)\right]$, $\nu = \mathrm{Var}(\log(a_t))$, $Z \sim N(0,1)$, and $o(\sqrt{t})/\sqrt{t} \to 0$ as $t \to \infty$. Thus, as $t \to \infty$, $\mathbb{E}[b_t] = b_0 \mathbb{E}\left[\exp\left(\mu_{iid} t + \sqrt{t\sigma_{iid}^2} Z\right)\right] = b_0 \exp\left((\mu_{iid} + \sigma_{iid}^2/2)t\right)$. This implies $\Delta \log(\mathbb{E}[b_t]) = \mu_{iid} + \sigma_{iid}^2/2$.

Second, we examine the case where $-\sigma_{iid}^2/2 < \mu_{iid} < 0$ (i.e., $\mu_{iid} < 0$ and $\mu_{iid} + \sigma_{iid}^2/2 > 0$), in which the asymptotic distribution of b_t is stationary and Pareto-tailed with an infinite mean. To compute the long-run growth rate of $\mathbb{E}[b_t]$ for this case, following the approach proposed by Sornette and Cont (1997), we introduce a new variable $\tilde{b}_t$ defined as

$$\tilde{b}_t = \exp(-\zeta t) b_t; \qquad \zeta > 0. \tag{B.1}$$

Using $\tilde{b}_t$, (6) can be written by

$$\tilde{b}_{t+1} = \tilde{a}_{t+1} \tilde{b}_t + \tilde{\eta}_{t+1}; \qquad \tilde{a}_{t+1} = \exp(-\zeta) a_{t+1}, \quad \tilde{\eta}_{t+1} = \exp(-\zeta(t+1)) \eta_{t+1}. \tag{B.2}$$

We note that (B.2) is also a Kesten process with $0 < \mathbb{E}[\tilde{a}_t] < \mathbb{E}[a_t]$ as $\exp(-\zeta) \in (0,1)$.

As $\mathbb{E}[\tilde{a}_t^{\tilde{\kappa}}] = \mathbb{E}[\exp(\tilde{\kappa} \log(\tilde{a}_t))] = \mathbb{E}[\exp(\tilde{\kappa}(\log(a_t) - \zeta))] = \exp(\tilde{\kappa}(\mu_{iid} + \tilde{\kappa}\sigma_{iid}^2/2 - \zeta))$, the tail index for the asymptotic distribution of $\tilde{b}_t$ is given by

$$\tilde{\kappa} = -\frac{\mu_{iid} - \zeta}{\sigma_{iid}^2/2}.$$

This implies $\lim_{t\to\infty} \mathbb{E}[\tilde{b}_t] < \infty$ if $\zeta \geq \mu_{iid} + \sigma_{iid}^2/2 > 0$. In particular, setting $\zeta = \mu_{iid} + \sigma_{iid}^2/2$

implies $\lim_{t\to\infty}\mathbb{E}[\tilde{b}_t] < \infty$ and $\mathbb{E}[\tilde{a}_t] = 1$. Since $\mathbb{E}[\tilde{\eta}_t] \to 0$ as $t \to \infty$, $\Delta \log\Big(\mathbb{E}[\tilde{b}_t]\Big) \to 0$. Hence, using (B.1) that implies that $b_t = \exp(\zeta t)\tilde{b}_t$, the mean of b_t grows exponentially at the rate of $\zeta = \mu_{iid} + \sigma^2_{iid}/2$.

B.2 Proof of Theorem 2

Let L be defined as $L = \lim_{t\to\infty}\exp(-\rho t)\mathbb{E}\left[(Y_t/Y_0)^{-\gamma} B_{t+1}\right]$. Assuming $Y_0 = 1$ without loss of generality, it follows that:

$$L = \lim_{t\to\infty}\exp(-\rho t)\mathbb{E}\left[\prod_{n=0}^{t-1}\left(\frac{Y_{n+1}}{Y_n}\right)^{1-\gamma}\frac{Y_{t+1}}{Y_t}b_{t+1}\right],$$

where the debt-to-GDP ratio is given by:

$$b_{t+1} = a_{t+1}b_t + \eta_{t+1} = \prod_{n=0}^{t} a_{n+1}b_0 + \sum_{n=1}^{t}\left(\prod_{m=n}^{t} a_{m+1}\right)\eta_n + \eta_{t+1}.$$

Following the proof of Theorem 1, we introduce a new variable $\tilde{b}_t$ defined as

$$\tilde{b}_t = \exp(-\zeta t)b_t; \qquad \zeta \geq 0.$$

Using $\tilde{b}_t$, (6) can be rewritten as:

$$\tilde{b}_{t+1} = \tilde{a}_{t+1}\tilde{b}_t + \tilde{\eta}_{t+1},$$

where $\tilde{a}_t = \exp(-\zeta)a_t$ and $\tilde{\eta}_t = \exp(-\zeta t)\eta_t$. Note that $\tilde{a}_t \leq a_t$ as $\exp(-\zeta) \in (0, 1]$. With these variables, L can be rewritten as:

$$L = \lim_{t\to\infty}\exp(-(\rho-\zeta)t)\exp(\zeta)\mathbb{E}\left[\prod_{n=0}^{t-1}\left(\frac{Y_{n+1}}{Y_n}\right)^{1-\gamma}\frac{Y_{t+1}}{Y_t}\tilde{b}_{t+1}\right], \tag{B.3}$$

where

$$\tilde{b}_{t+1} = \prod_{n=0}^{t}\tilde{a}_{n+1}b_0 + \sum_{n=1}^{t}\left(\prod_{m=n}^{t}\tilde{a}_{m+1}\right)\tilde{\eta}_n + \tilde{\eta}_{t+1}. \tag{B.4}$$

As shown above, by setting ζ such that

$$\zeta \geq \max\left\{0, \mu_a + \left(\frac{1+\theta_a}{1-\rho_a}\right)^2\frac{\sigma_\epsilon^2}{2}\right\},$$

there exists $\kappa > 1$ for which

$$\lim_{n\to+\infty} \frac{1}{n} \log \left(\mathbb{E}\left[\left(\prod_{n=0}^{t-1} \tilde{a}_{n+1} \right)^{\kappa} \right] \right) = 0. \tag{B.5}$$

This implies that under the condition

$$\mu_a + \left(\frac{1+\theta_a}{1-\rho_a} \right)^2 \frac{\sigma_\epsilon^2}{2} < \rho + (\gamma - 1) \left(\hat{g} - \gamma \frac{\sigma_g^2}{2} \right),$$

there exists a $\zeta < \rho + (\gamma-1)\left(\hat{g} - \gamma\sigma_g^2/2\right)$ such that $\tilde{a}_t = \exp(-\zeta)a_t$ satisfies (B.5). Below, we consider such ζ.

By substituting (B.4) into (B.3), decompose L into $L = L_1 + L_2 + L_3$, where

$$L_1 = \lim_{t\to\infty} \exp(-(\rho-\zeta)t)\mathbb{E}\left[\prod_{n=0}^{t-1} \left(\frac{Y_{n+1}}{Y_n} \right)^{1-\gamma} \frac{Y_{t+1}}{Y_t} \prod_{n=0}^{t} \tilde{a}_{n+1} \right] b_0 \geq 0,$$

$$L_2 = \lim_{t\to\infty} \exp(-(\rho-\zeta)t)\mathbb{E}\left[\prod_{n=0}^{t-1} \left(\frac{Y_{n+1}}{Y_n} \right)^{1-\gamma} \frac{Y_{t+1}}{Y_t} \sum_{k=1}^{t} \left(\prod_{m=k}^{t} \tilde{a}_{m+1} \right) \tilde{\eta}_k \right] \geq 0,$$

$$L_3 = \lim_{t\to\infty} \exp(-(\rho-\zeta)t)\mathbb{E}\left[\prod_{n=0}^{t-1} \left(\frac{Y_{n+1}}{Y_n} \right)^{1-\gamma} \frac{Y_{t+1}}{Y_t} \tilde{\eta}_{t+1} \right] \geq 0.$$

We proceed to show that $L_1 = L_2 = L_3 = 0$ if

$$\mu_a + \left(\frac{1+\theta_a}{1-\rho_a} \right)^2 \frac{\sigma_\epsilon^2}{2} < \rho + (\gamma - 1) \left(\hat{g} - \gamma \frac{\sigma_g^2}{2} \right).$$

For later use, we present two lemmas:

Lemma B.1. *For any $q > 0$,*

$$\exp(-\rho t) \left(\mathbb{E}\left[\left(\prod_{n=0}^{t-1} \left(\frac{Y_{n+1}}{Y_n} \right)^{1-\gamma} \right)^{q} \right] \right)^{\frac{1}{q}} = \exp(-vt)$$

where

$$v = \rho + (\gamma-1)\hat{g} - \frac{\gamma(\gamma-1)}{2}\sigma_g^2 > 0.$$

Proof.

$$\begin{aligned}\mathbb{E}_t\left[\left(\frac{Y_{t+1}}{Y_t}\right)^{1-\gamma}\right] &= \exp\left((1-\gamma)\left(\hat{g}-\frac{\sigma_g^2}{2}\right)\right)\mathbb{E}\left[\exp((1-\gamma)\sigma_g\varepsilon_{g,t+1})\right] \\ &= \exp\left((1-\gamma)\left(\hat{g}-\frac{\sigma_g^2}{2}\right)\right)\exp\left(\frac{(1-\gamma)^2\sigma_g^2}{2}\right) \\ &= \exp\left(-(\gamma-1)\left(\hat{g}-\gamma\frac{\sigma_g^2}{2}\right)\right).\end{aligned}$$

□

Lemma B.2. *If there exists $\kappa > 1$ for which*

$$\lim_{n\to+\infty}\frac{1}{n}\log\left(\mathbb{E}\left[\left(\prod_{n=0}^{t-1}a_{n+1}\right)^{\kappa}\right]\right) = 0, \tag{B.6}$$

there exist a $p \geq 1$ such that

$$\left(\mathbb{E}\left[\left(\prod_{n=0}^{t-1}a_{n+1}\right)^{p}\right]\right)^{\frac{1}{p}} < t.$$

Proof. (B.6) implies that for any $\tilde{\kappa} \in (0,\kappa)$,

$$\lim_{n\to+\infty}\frac{1}{n}\log\left(\mathbb{E}\left[\left(\prod_{n=0}^{t-1}a_{n+1}\right)^{\tilde{\kappa}}\right]\right) < 0 \quad\Longleftrightarrow\quad \lim_{n\to+\infty}\frac{1}{n}\mathbb{E}\left[\left(\prod_{n=0}^{t-1}a_{n+1}\right)^{\tilde{\kappa}}\right] < 1.$$

Therefore, for large t, a $p \in [1,\kappa)$ exists such that $\mathbb{E}\left[\left(\prod_{n=0}^{t-1}a_{n+1}\right)^{p}\right] < t$, leading to $\left(\mathbb{E}\left[\left(\prod_{n=0}^{t-1}a_{n+1}\right)^{p}\right]\right)^{\frac{1}{p}} < t^{\frac{1}{p}} < t$. □

We start with showing $L_3 = 0$. Given that $\eta_t > 0$ is an IID random variable with $\mathbb{E}[\eta_t] < \infty$, and Lemma B.1, we have

$$\begin{aligned}L_3 &= \lim_{t\to\infty}\exp(-(\rho-\zeta)t)\mathbb{E}\left[\prod_{n=0}^{t-1}\left(\frac{Y_{n+1}}{Y_n}\right)^{1-\gamma}\frac{Y_{t+1}}{Y_t}\tilde{\eta}_{t+1}\right] \\ &= \lim_{t\to\infty}\exp(-\rho t)\prod_{n=0}^{t-1}\mathbb{E}\left[\left(\frac{Y_{n+1}}{Y_n}\right)^{1-\gamma}\right]\mathbb{E}\left[\frac{Y_{t+1}}{Y_t}\right]\mathbb{E}\left[\eta_{t+1}\right] \\ &= \lim_{t\to\infty}\exp(-vt)\exp(g)\mathbb{E}\left[\eta_{t+1}\right] = 0.\end{aligned}$$

We proceed to show that $L_1 = 0$. Consider the expression

$$\begin{aligned}
L_1 &= \lim_{t\to\infty} \exp(-(\rho-\zeta)t)\mathbb{E}\left[\prod_{n=0}^{t-1}\left(\frac{Y_{n+1}}{Y_n}\right)^{1-\gamma}\frac{Y_{t+1}}{Y_t}\prod_{n=0}^{t}\tilde{a}_{n+1}\right]b_0 \\
&= \lim_{t\to\infty} \exp(-(\rho-\zeta)t)\mathbb{E}\left[\prod_{n=0}^{t-1}\left(\tilde{a}_{n+1}\left(\frac{Y_{n+1}}{Y_n}\right)^{1-\gamma}\right)\mathbb{E}_t\left[\left(\frac{Y_{t+1}}{Y_t}\right)\tilde{a}_{t+1}\right]\right]b_0 \\
&\leq \lim_{t\to\infty} \exp(-(\rho-\zeta)t)\mathbb{E}\left[\prod_{n=0}^{t-1}\left(\tilde{a}_{n+1}\left(\frac{Y_{n+1}}{Y_n}\right)^{1-\gamma}\right)\right]Sb_0,
\end{aligned}$$

where $S \equiv \sup \mathbb{E}_t\left[(Y_{t+1}/Y_t)\,\tilde{a}_{t+1}\right] < \infty$ by Assumption 3. Applying Hölder's inequality, we obtain

$$\mathbb{E}\left[\prod_{n=0}^{t-1}\left(a_{n+1}\left(\frac{Y_{n+1}}{Y_n}\right)^{1-\gamma}\right)\right] \leq \left(\mathbb{E}\left[\left(\prod_{n=0}^{t-1}\tilde{a}_{n+1}\right)^p\right]\right)^{\frac{1}{p}}\left(\mathbb{E}\left[\left(\prod_{n=0}^{t-1}\left(\frac{Y_{n+1}}{Y_n}\right)^{1-\gamma}\right)^q\right]\right)^{\frac{1}{q}}$$

for $1 \leq p, q \leq \infty$ with $\frac{1}{p}+\frac{1}{q}=1$. Hence, using Lemma B.1, we obtain

$$L_1 \leq \lim_{t\to\infty}\exp(-(v-\zeta)t)\left(\mathbb{E}\left[\left(\prod_{n=0}^{t-1}\tilde{a}_{n+1}\right)^p\right]\right)^{\frac{1}{p}}$$

for any $1 \leq p$. Using Lemma B.2, for large t, there exists a $p \in (1,\kappa)$ such that

$$\left(\mathbb{E}\left[\left(\prod_{n=0}^{t-1}\tilde{a}_{n+1}\right)^p\right]\right)^{\frac{1}{p}} < t.$$

Therefore, we have:

$$L_1 \leq \lim_{t\to\infty}\exp(-(v-\zeta)t)tSb_0.$$

Recall that we have assumed $\tilde{a}_t = \exp(-\zeta)a_t$ such that $\zeta < \rho + (\gamma-1)\left(\hat{g}-\gamma\sigma_g^2/2\right) = v$. Hence, $\exp(-(v-\zeta)t)t = 0$ as $t\to\infty$, and therefore $L_1 \leq 0$.

We finally show that $L_2 = 0$. Observe that

$$\begin{aligned}
L_2 &= \lim_{t\to\infty} \exp(-(\rho-\zeta)t)\mathbb{E}\left[\prod_{n=0}^{t-1}\left(\frac{Y_{n+1}}{Y_n}\right)^{1-\gamma}\frac{Y_{t+1}}{Y_t}\sum_{k=1}^{t}\left(\prod_{m=k}^{t}\tilde{a}_{m+1}\right)\tilde{\eta}_k\right] \\
&= \lim_{t\to\infty} \exp(-(\rho-\zeta)t)\mathbb{E}\left[\prod_{n=0}^{t-1}\left(\frac{Y_{n+1}}{Y_n}\right)^{1-\gamma}\sum_{k=1}^{t-1}\left(\prod_{m=k}^{t-1}\tilde{a}_{m+1}\right)\tilde{\eta}_k\mathbb{E}_t\left[\left(\frac{Y_{t+1}}{Y_t}\right)\tilde{a}_{t+1}\right]\right] \\
&\leq \lim_{t\to\infty} \exp(-(\rho-\zeta)t)\sum_{k=1}^{t-1}\mathbb{E}\left[\prod_{n=0}^{t-1}\left(\frac{Y_{n+1}}{Y_n}\right)^{1-\gamma}\left(\prod_{m=k}^{t-1}\tilde{a}_{m+1}\right)\right]\sum_{k=1}^{t-1}\mathbb{E}[\tilde{\eta}_k]S \\
&\leq \lim_{t\to\infty} \exp(-(\rho-\zeta)t)\sum_{k=1}^{t-1}\mathbb{E}\left[\prod_{n=0}^{t-1}\left(\frac{Y_{n+1}}{Y_n}\right)^{1-\gamma}\left(\prod_{m=k}^{t-1}\tilde{a}_{m+1}\right)\right]\frac{\mathbb{E}[\eta]}{1-\exp(-\zeta)}S \\
&\leq \lim_{t\to\infty} \exp(-(v-\zeta)t)\sum_{k=1}^{t-1}\left(\mathbb{E}\left[\left(\prod_{m=k}^{t-1}\tilde{a}_{m+1}\right)^{p}\right]\right)^{\frac{1}{p}}\frac{\mathbb{E}[\eta]}{1-\exp(-\zeta)}S
\end{aligned}$$

for any $1 \leq p$. Using Lemma B.2, for large t, there exists a $p \in (1,\kappa)$ such that $\mathbb{E}\left[\left(\prod_{n=0}^{t-1}\tilde{a}_{n+1}\right)^{p}\right] < t$, resulting in $\left(\mathbb{E}\left[\left(\prod_{n=0}^{t-1}\tilde{a}_{n+1}\right)^{p}\right]\right)^{\frac{1}{p}} < t^{\frac{1}{p}}$ and

$$\sum_{k=1}^{t-1}\left(\mathbb{E}\left[\left(\prod_{m=k}^{t-1}\tilde{a}_{m+1}\right)^{p}\right]\right)^{\frac{1}{p}} < \sum_{k=1}^{t-1}k^{\frac{1}{p}} < \sum_{k=1}^{t-1}k = \frac{t(t-1)}{2}.$$

As a result, we have

$$L_2 \leq \lim_{t\to\infty}\exp(-(v-\zeta)t)\frac{t(t-1)}{2}S\frac{\mathbb{E}[\eta]}{1-\exp(-\zeta)} = 0.$$

Recall that we have assumed $\tilde{a}_t = \exp(-\zeta)a_t$ such that $\zeta < \rho + (\gamma-1)\left(\hat{g} - \gamma\sigma_g^2/2\right) = v$. Hence, $\exp(-(v-\zeta)t)t(t-1) \to 0$ as $t \to \infty$, and therefore $L_2 \leq 0$.

APPENDIX C Proofs and Derivations

C.1 Derivation of Equation (4)

We begin with the government budget constraint:

$$B_{t+1} = \left(1 + r_{G,t+1}\right)B_t - S_{t+1}.$$

By adding and subtracting $r_{M,t+1}B_t$ from the right-hand side, we obtain

$$B_{t+1} = (1 + r_{M,t+1})B_t + (r_{G,t+1} - r_{M,t+1})B_t - S_{t+1}$$

Hence, we have

$$\begin{aligned}
B_t =& \frac{1}{1 + r_{M,t+1}} \left[S_{t+1} + (r_{M,t+1} - r_{G,t+1})B_t\right] + \frac{1}{1 + r_{M,t+1}} B_{t+1} \\
=& \frac{1}{1 + r_{M,t+1}} \left[S_{t+1} + (r_{M,t+1} - r_{G,t+1})B_t\right] \\
&+ \frac{1}{1 + r_{M,t+1}} \frac{1}{1 + r_{M,t+2}} \left[S_{t+2} + (r_{M,t+2} - r_{G,t+2})B_{t+1}\right] \\
&+ \frac{1}{1 + r_{M,t+1}} \frac{1}{1 + r_{M,t+2}} B_{t+2}.
\end{aligned}$$

Iterating forward, we obtain

$$B_t = \lim_{k\to\infty} \left[\begin{array}{l} \sum_{j=1}^{k} \left(\prod_{i=1}^{j} \frac{1}{1 + r_{M,t+i}} \right) \left[S_{t+j} + \left(r_{M,t+j} - r_{G,t+j}\right) B_{t+j-1}\right] \\ + \left(\prod_{j=1}^{k} \frac{1}{1 + r_{M,t+j}} \right) B_{t+k} \end{array} \right].$$

Taking conditional expectations and assuming that limits and expectations are interchangable, we have

$$B_t = \lim_{k\to\infty} \left[\begin{array}{l} \mathbb{E}_t \sum_{j=1}^{k} \left(\prod_{i=1}^{j} \frac{1}{1 + r_{M,t+i}} \right) \left[S_{t+j} + \left(r_{M,t+j} - r_{G,t+j}\right) B_{t+j-1}\right] \\ + \mathbb{E}_t \left(\prod_{j=1}^{k} \frac{1}{1 + r_{M,t+j}} \right) B_{t+k} \end{array} \right].$$

C.2 Proof of Lemma 1

We have

$$\lim_{n\to+\infty} \frac{1}{n} \log \left(\mathbb{E} \left(\prod_{t=0}^{n} a_t \right)^{\kappa} \right) = \lim_{n\to+\infty} \frac{1}{n} \log \left(\mathbb{E} \exp \left(\kappa \sum_{t=0}^{n} \log(a_t) \right) \right). \tag{C.1}$$

We obtain the expression for $\sum_{t=0}^{n}\log(a_t)$. Using

$$\begin{aligned}\log(a_t) &= (1-\rho_a)\,\mu_a\sum_{s=0}^{t}\rho_a^s + \varepsilon_{a,t} + \sum_{s=1}^{t}\rho_a^{s-1}\left(\rho_a+\theta_a\right)\varepsilon_{a,t-s} + \rho_a^{t+1}\log\left(a_{-1}\right)\\ &=\mu_a(1-\rho_a^t) + \rho_a^{t+1}\log\left(a_{-1}\right) + \varepsilon_{a,t} + \left(\rho_a+\theta_a\right)\sum_{k=0}^{t-1}\rho_a^{t-k-1}\varepsilon_{a,k},\end{aligned}$$

we have

$$\begin{aligned}\sum_{t=0}^{n}\log(a_t) =& \mu_a\left(n+\frac{1-\rho_a^n}{1-\rho_a}\right) + \frac{\rho_a(1-\rho_a^n)}{1-\rho_a}\log(a_{-1}) + \sum_{t=0}^{n}\varepsilon_{a,n} + \sum_{t=0}^{n-1}\left(\rho_a+\theta_a\right)\sum_{k=0}^{n-1-t}\rho_a^k\varepsilon_{a,t}\\ =& \mu_a\left(n+\frac{1-\rho_a^n}{1-\rho_a}\right) + \frac{\rho_a(1-\rho_a^n)}{1-\rho_a}\log(a_{-1}) + \varepsilon_{a,n} + \sum_{t=0}^{n-1}\left[1+\left(\rho_a+\theta_a\right)\sum_{k=0}^{n-1-t}\rho_a^k\right]\varepsilon_{a,t}\\ =& \mu_a\left(n+\frac{1-\rho_a^n}{1-\rho_a}\right) + \frac{\rho_a(1-\rho_a^n)}{1-\rho_a}\log(a_{-1}) + \varepsilon_{a,n} + \sum_{t=0}^{n-1}\left[\frac{1+\theta_a-(\rho_a+\theta_a)\rho_a^{n-t}}{1-\rho_a}\right]\varepsilon_{a,t}.\end{aligned}$$

Substituting the above expression into (C.1) yields

$$\begin{aligned}&\lim_{n\to+\infty}\frac{1}{n}\log\left(\mathbb{E}\left(\prod_{t=0}^{n}a_t\right)^{\kappa}\right)\\ &\quad=\lim_{n\to+\infty}\frac{1}{n}\log\left(\mathbb{E}\exp\left(\kappa\sum_{t=0}^{n}\log(a_t)\right)\right)\\ &\quad=\lim_{n\to+\infty}\frac{1}{n}\log\mathbb{E}\exp\left(\kappa\mu_a\left(n+\frac{1-\rho_a^n}{1-\rho_a}\right)+\kappa\sum_{t=0}^{n-1}\frac{(1+\theta_a-(\theta_a+\rho_a)\rho_a^{n-t})}{1-\rho_a}\varepsilon_{a,t}\right)\\ &\quad=\log(\exp(\kappa\mu_a))+\lim_{n\to+\infty}\frac{1}{n}\sum_{t=0}^{n-1}\log\mathbb{E}\exp\left(\frac{(1+\theta_a-(\theta_a+\rho_a)\rho_a^{n-t})}{1-\rho_a}\kappa\varepsilon_{a,t}\right)\\ &\quad=\log\left(\mathbb{E}\left[\exp(\kappa\mu_a)\exp\left(\frac{1+\theta_a}{1-\rho_a}\kappa\varepsilon_{a,t}\right)\right]\right)\\ &\quad=\log\left(\mathbb{E}\left[\exp\left(\kappa\left(\frac{1+\theta_a}{1-\rho_a}\varepsilon_{a,t}+\mu_a\right)\right)\right]\right),\end{aligned}$$

where the fourth line is obtained since $\varepsilon_{a,t}$ is independent.

C.3 Proofs of Lemma 2 and Proposition 1

Recall that P_t and Y_t follow

$$\log(1+\pi_{t+1}) \equiv \log\left(\frac{P_{t+1}}{P_t}\right) = \pi - \frac{\sigma_\pi^2}{2} + \sigma_\pi \varepsilon_{\pi,t+1}.$$

$$\log(1+g_{t+1}) \equiv \log\left(\frac{Y_{t+1}}{Y_t}\right) = \hat{g} - \frac{\sigma_g^2}{2} + \sigma_g \varepsilon_{g,t+1}.$$

The optimization problem of the representative household is given by

$$\max_{\{C_{t+k}, B^n_{t+k}, A_{t+k}\}} \quad \mathbb{E}_t \left[\sum_{k=0}^{\infty} \exp(-\rho)^k \left(\frac{C_{t+k}^{1-\gamma} - 1}{1-\gamma} + \nu \left(\frac{B^n_{t+k}}{P_{t+k}} \right) \right) \right]; \quad \text{with} \;\; \nu\left(B^n_t / P_t\right) = \varrho_t B_t / P_t,$$

subject to

$$C_t + \frac{B^n_t}{P_t} + A_t \leq Y_t - T_t + \left(1 + r^n_{G,t-1}\right) \frac{B^n_{t-1}}{P_t} + \left(1 + r_{F,t-1}\right) A_{t-1}.$$

The first order necessary conditions are given by

$$\begin{aligned}
&C_t : \exp(-\rho t) C_t^{-\gamma} - \lambda_t = 0, \\
&A_t : -\lambda_t + \left(1 + r_{F,t}\right) \mathbb{E}_t \lambda_{t+1} = 0, \\
&B_t : \exp(-\rho t) \varrho_t - \lambda_t + \left(1 + r^n_{G,t}\right) \mathbb{E}_t \left[\lambda_{t+1} \frac{P_t}{P_{t+1}} \right] = 0.
\end{aligned}$$

Euler equation for risk-free asset A_t is given by

$$\exp(-\rho)\left(1 + r_{F,t}\right) = \left[\mathbb{E}_t \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right]^{-1}.$$

In equilibrium, the growth rate of consumption is

$$\frac{C_{t+1}}{C_t} = \frac{Y_{t+1}}{Y_t} = \exp\left(\hat{g} - \frac{\sigma_g^2}{2} + \sigma_g \varepsilon_{g,t+1} \right) = \exp\left(\hat{g} - \frac{\sigma_g^2}{2} \right) \exp(\sigma_g \varepsilon_{g,t+1}).$$

Thus, we obtain

$$\mathbb{E}_t \left[\left(\frac{Y_{t+1}}{Y_t} \right)^{-\gamma} \right] = \exp\left(-\gamma \left(\hat{g} - \frac{\sigma_g^2}{2} \right) \right) \exp\left(\frac{\gamma^2 \sigma_g^2}{2} \right) = \exp\left(-\gamma \left(\hat{g} - (1+\gamma) \frac{\sigma_g^2}{2} \right) \right).$$

Hence, for all t, we have $r_{F,t} = r_F$ such that

$$1 + r_F = \exp\left(\rho + \gamma\left(\hat{g} - (\gamma+1)\frac{\sigma_g^2}{2}\right)\right).$$

Taking log

$$r_F \approx \log(1 + r_F) = \rho + \gamma\left(\hat{g} - (\gamma+1)\frac{\sigma_g^2}{2}\right).$$

Euler equation for government bond B_t^n is given by

$$\exp(-\rho)\left(1 + r_{G,t}^n\right)\mathbb{E}_t\left[C_{t+1}^{-\gamma}\frac{P_t}{P_{t+1}}\right] = C_t^{-\gamma} - \bar{C}_t^{-\gamma}(1 - \exp(-z_t))$$

In equilibrium, $C_t = \bar{C}_t$ and $C_{t+1}/C_t = Y_{t+1}/Y_t$,

$$\begin{aligned}
1 + r_{G,t}^n =& \exp(-z_t)\exp(\rho)\left(\mathbb{E}_t\left[\left(\frac{Y_{t+1}}{Y_t}\right)^{-\gamma}\frac{P_t}{P_{t+1}}\right]\right)^{-1} \\
=& \exp(-z_t)\exp(\rho)\left[\mathbb{E}_t\left(\frac{Y_{t+1}}{Y_t}\right)^{-\gamma}\right]^{-1}\left[\mathbb{E}_t\left(\frac{P_{t+1}}{P_t}\right)^{-1}\right]^{-1} \\
=& \exp(-z_t)\left(1 + r_F\right)\left[\mathbb{E}_t\exp\left(-\pi + \frac{\sigma_\pi^2}{2} - \sigma_\pi\varepsilon_{\pi,t+1}\right)\right]^{-1} \\
=& (1 + r_F)\exp\left(\pi - \sigma_\pi^2 - z_t\right).
\end{aligned}$$

Thus, the real interest rate of government bond is given by

$$\begin{aligned}
1 + r_{G,t+1} =& (1 + r_{G,t}^n)\left(\frac{P_t}{P_{t+1}}\right) \\
=& (1 + r_{G,t}^n)\exp\left(-\left(\pi - \frac{\sigma_\pi^2}{2} + \sigma_\pi\varepsilon_{\pi,t+1}\right)\right) \\
=& (1 + r_F)\exp\left(-z_t - \frac{\sigma_\pi^2}{2} - \sigma_\pi\varepsilon_{\pi,t+1}\right).
\end{aligned}$$

- The derivation of a_{t+1}. Recall that

  $$a_{t+1} = \frac{1 + r_{G,t+1}}{1 + g_{t+1}} \implies \log(a_{t+1}) = \log(1 + r_{G,t+1}) - \log(1 + g_{t+1}).$$

  Hence,

  $$\log(a_{t+1}) = m_0 - z_t - \sigma_\pi\varepsilon_{\pi,t+1} - \sigma_g\varepsilon_{g,t+1},$$

  where $m_0 \equiv \log(1 + r_F) - \hat{g} + (\sigma_g^2 - \sigma_\pi^2)/2$. Let us define $\varepsilon_{g\pi,t} \equiv \sigma_\pi\varepsilon_{\pi,t} + \sigma_g\varepsilon_{g,t}$. As

$\varepsilon_{g\pi,t} \sim N(0, \sigma_g^2 + \sigma_\pi^2)$, we can write $\varepsilon_{g\pi,t} = \sqrt{\sigma_g^2 + \sigma_\pi^2}\varepsilon_t$ where $\varepsilon_t \sim N(0,1)$. Using this, we obtain

$$\log(a_{t+1}) = m_0 - z_t - \sqrt{\sigma_g^2 + \sigma_\pi^2}\varepsilon_{t+1},$$

and

$$\rho_z \log(a_t) = \rho_z m_0 - \rho_z z_{t-1} - \rho_z\sqrt{\sigma_g^2 + \sigma_\pi^2}\varepsilon_t.$$

Taking difference, we obtain

$$\begin{aligned}\log(a_{t+1}) - &\rho_z \log(a_t) \\ &= (1-\rho_z)\mu_a - \sqrt{\sigma_g^2 + \sigma_\pi^2}\varepsilon_{t+1} - \sigma_z \varepsilon_{z,t} + \rho_z\sqrt{\sigma_g^2 + \sigma_\pi^2}\varepsilon_t.\end{aligned}$$

where $\mu_a = m_0 - \mu_z$. As $\varepsilon_{z,t} \sim N(0,1)$ and $\varepsilon_t \sim N(0,1)$, $-\sigma_z\varepsilon_{z,t} + \rho_z\sqrt{\sigma_g^2 + \sigma_\pi^2}\varepsilon_t \sim N(0, \sigma_z^2 + \rho_z^2(\sigma_g^2 + \sigma_\pi^2))$. As a result, $\log(a_t)$ follows an ARMA(1,1) process:

$$\log(a_{t+1}) = (1-\rho_z)\mu_a + \rho_z \log(a_t) + \varepsilon_{a,t+1} + \theta_a \varepsilon_{a,t},$$

with $\mu_a = \rho + (\gamma - 1)\hat{g} + [1 - \gamma(\gamma+1)]\,\sigma_g^2/2 - \sigma_\pi^2/2 - \mu_z$, $\theta_a = \sqrt{\sigma_z^2 + \rho_z^2(\sigma_g^2 + \sigma_\pi^2)}/\sqrt{\sigma_g^2 + \sigma_\pi^2}$ and $\varepsilon_{a,t} \sim N(0, \sigma_g^2 + \sigma_\pi^2)$. Assumption 4 ensures that $\mu_a < 0$.

- The stochastic discount factor is given by

$$\begin{aligned} M_{t,t+1} &= \exp(-\rho)\left(\frac{Y_{t+1}}{Y_t}\right)^{-\gamma} \\ &= \exp\left(-\rho - \gamma\left(\hat{g} - \frac{\sigma_g^2}{2} + \sigma_g \varepsilon_{g,t+1}\right)\right) \\ &= \frac{1}{\exp\left(\rho + \gamma\left(\hat{g} - \frac{\sigma_g^2}{2} + \sigma_g \varepsilon_{g,t+1}\right)\right)}. \end{aligned}$$

Hence,

$$\mathbb{E}\left[M_{t,t+1}\right]^{-1} = 1 + r_F.$$

C.4 Proof of Proposition 2

Using Lemma 2 and (17),

$$\log(1 + r_M) = \log(1 + r_F) = \rho + \gamma\hat{g} - \frac{\gamma(\gamma+1)}{2}\sigma_g^2.$$

As $\log(1+g) = \hat{g} - \sigma_g^2$,

$$\log(1+r_F) - \log(1+g) = \rho + (\gamma-1)\hat{g} + \left(1 - \frac{\gamma(\gamma+1)}{2}\right)\sigma_g^2 > 0$$

by Assumption 5. Finally, $\log(1+r_G)$ is given by

$$\log(1+r_G) = \rho + \gamma\hat{g} - \frac{\gamma(\gamma+1)}{2}\sigma_g^2 - \sigma_\pi^2 - \mu_z - \frac{\sigma_z^2/2}{1-\rho_z}.$$

So,

$$\begin{aligned}\log(1+g) - \log(1+r_G) =& -\left[\rho + (\gamma-1)\hat{g} + \left(1 - \frac{\gamma(\gamma+1)}{2}\right)\sigma_g^2\right] + \mu_z + \sigma_\pi^2 + \frac{\sigma_z^2/2}{1-\rho_z} \\ >& \mu_z - \left[\rho + (\gamma-1)\hat{g} + \left(1 - \frac{\gamma(\gamma+1)}{2}\right)\sigma_g^2\right] > 0\end{aligned}$$

by Assumption 4.

C.5 Proof of Proposition 4

The condition for debt sustainability is given by

$$\mu_a + \left(\frac{1+\theta_a}{1-\rho_a}\right)^2 \frac{\sigma_\epsilon^2}{2} < \rho + (\gamma-1)\left(\hat{g} - \gamma\frac{\sigma_g^2}{2}\right), \tag{C.2}$$

where

$$\begin{aligned}\mu_a &= \rho + (\gamma-1)\hat{g} + [1 - \gamma(\gamma+1)]\frac{\sigma_g^2}{2} - \frac{\sigma_\pi^2}{2} - \mu_z - \phi, \\ \sigma_\epsilon^2 &= \sigma_g^2 + \sigma_\pi^2.\end{aligned}$$

Thus, (C.2) is given by

$$\left(\frac{1+\theta_a}{1-\rho_a}\right)^2 \frac{\sigma_\epsilon^2}{2} + \frac{\sigma_g^2}{2} - \frac{\sigma_\pi^2}{2} - \mu_z - \phi < [-(\gamma-1)\gamma + \gamma(\gamma+1)]\frac{\sigma_g^2}{2},$$

and hence

$$\phi > -\mu_z + \left[1 + 2\gamma + \left(\frac{1+\theta_a}{1-\rho_a}\right)^2\right]\frac{\sigma_g^2}{2} - \left[1 - \left(\frac{1+\theta_a}{1-\rho_a}\right)^2\right]\frac{\sigma_\pi^2}{2}.$$

As

$$\mu_z = \log(1+r_M) - \log(1+r_G) - \frac{1}{1-\rho_z}\frac{\sigma_z^2}{2} \approx (r_M - r_g) - \frac{1}{1-\rho_z}\frac{\sigma_z^2}{2},$$

we obtain

$$\phi > -(r_M - r_G) + \frac{1}{1-\rho_z}\frac{\sigma_z^2}{2} + \left[1 + 2\gamma + \left(\frac{1+\theta_a}{1-\rho_a}\right)^2\right]\frac{\sigma_g^2}{2} - \left[1 - \left(\frac{1+\theta_a}{1-\rho_a}\right)^2\right]\frac{\sigma_\pi^2}{2}.$$

APPENDIX D Continuous-Time Framework

In Section 5, we established that discrete-time stochastic debt models with fiscal deficits lead to an asymptotically power-law-distributed debt-to-GDP ratio. Extending our analysis, this section generalizes these stochastic debt models, following the approach in Mehrotra and Sergeyev (2021), to demonstrate that this phenomenon also occurs in continuous-time stochastic debt models.

We consider a representative household with preferences given by:

$$\mathbb{E}\int_0^\infty \mathrm{e}^{-\rho t}\left[\frac{C(t)^{1-\gamma}-1}{1-\gamma}+\bar{C}(t)^{-\gamma}\alpha_\nu B(t)\right]dt,$$

where $C(t)$ denotes the household's consumption, $\bar{C}(t)$ the average consumption, and $B(t)$ the household's bond holdings. The budget constraint is:

$$dH(t)=[r_F(t)A(t)+r_G(t)B(t)-C(t)-T(t)]\,dt+dy(t),$$

with $H(t)$ representing household wealth, $A(t)$ the holdings of risk-free assets, $T(t)$ lump-sum taxes, and $y(t)$ an endowment following a stochastic process:

$$dy(t)=\left(g-\frac{\sigma_g^2}{2}\right)y(t)dt+\sigma_y y(t)dW(t);\qquad g\geq 0,\ \sigma_g>0. \tag{D.1}$$

Here, $W(t)$ is a Wiener process with $\mathbb{E}[dW(t)]=0$ and $\mathbb{E}[dW(t)^2]=dt$. Denoting aggregate income or GDP by $Y(t)$, the endowment process (D.1) suggests that GDP experiences random growth with drift $g-\sigma_g^2/2$:

$$\frac{dY(t)}{Y(t)}=\left(\hat{g}-\frac{\sigma_g^2}{2}\right)dt+\sigma_y dW(t).$$

The Euler equations are:

$$r_F(t)-\rho=\gamma g,$$

and

$$r_G(t)-\rho+\alpha_\nu=\gamma g.$$

The government determines the net lump-sum transfer following the fiscal rule:

$$S(t)\equiv T(t)-G(t)$$
$$G(t)=\gamma_g Y(t)$$

Thus, the fiscal primary surplus-to-GDP ratio $s(t) = S(t)/Y(t)$ is given by[22]

$$s(t) = \phi b(t) - \gamma_g$$

The time evolution equation for the real outstanding government debt is given by

$$\frac{dB(t)}{dt} = r_G(t)B(t) - S(t).$$

Using Itô's lemma, the dynamic equation for the debt-to-GDP ratio ($b(t) = B(t)/Y(t)$) is given by

$$db(t) = \left[(r_G(t) - g - \phi)\, b(t) + \left(\gamma_g + \frac{\sigma_y^2}{2} \right) \right] dt - \sigma_y b(t) dW(t)$$

The equilibrium conditions $C(t) = Y(t)$ and $A(t) = 0$ imply the equilibrium interest rate for government bonds

$$r_G(t) = \rho + \gamma g - \alpha_\nu.$$

Hence, the dynamic equation for the equilibrium debt-to-GDP ratio is given by

$$db(t) = \left[\{\rho - \alpha_\nu + (\gamma - 1)\, g - \phi\}\, b(t) + \left(\gamma_g + \frac{\sigma_y^2}{2} \right) \right] dt - \sigma_y b(t) dW(t). \qquad \text{(D.2)}$$

Note that under $\phi \geq 0$, Assumption 4 implies $\rho - \alpha_\nu + (\gamma - 1)\, g \leq 0$. Applying Itô's lemma, we have

$$d\log(b(t)) = \left[\rho - \alpha_\nu + (\gamma - 1)\, g - \phi + \left(\gamma_g + \frac{\sigma_y^2}{2} \right) \frac{1}{b(t)} \right] dt - \sigma_y dW(t).$$

Lemma D.1 (Characteristics of the asymptotic debt-to-GDP ratio in continuous-time model)**.** *The asymptotic distribution of the debt-to-GDP ratio induced by* (D.2) *depends on* (γ_g, τ_y)*:*

- *If* $2\gamma_g + \sigma_y^2 = 0$*,* $b(t)$ *follows the geometric Brownian motion:*

$$\begin{aligned} db(t) &= \mu_b b(t) dt + \sigma_b b(t) dW(t); \\ \mu_b &= \rho - \alpha_\nu + (\gamma - 1)\, g - \phi \\ \sigma_b &= -\sigma_y \end{aligned}$$

  In this case, without some "frictions" (e.g., reflecting walls), there is no stationary distribution for the debt-to-GDP ratio.

[22] Mehrotra and Sergeyev (2021) assume the deficit-to-GDP-ratio drift $\gamma_g = 0$.

- *If $2\gamma_g + \sigma_y^2 \neq 0$, $b(t)$ follows the (continuous-time version of) Kesten process:*

$$\begin{aligned} db(t) &= \theta_b(\mu_b - b(t))dt + \sigma_b b(t) dW(t) \\ \theta_b &= -\rho + \alpha_\nu - (\gamma - 1)\, g + \phi \\ \mu_b &= \frac{-2\gamma_g - \sigma_y^2}{2\left(\rho - \alpha_\nu + (\gamma - 1)\, g - \phi\right)} \\ \sigma_b &= -\sigma_y \end{aligned}$$

  Following Luttmer (2016), the stationary distribution is

$$f(b) \propto \frac{1}{b^{2+\alpha} e^{\beta/b}}, \;\; \alpha = \frac{\theta_b}{\sigma_b^2/2}, \;\; \beta = \alpha \mu_b$$

  In this case, the right tail of the stationary distribution is

$$\kappa = 1 + \frac{\theta_b}{\sigma_b^2/2} > 0$$

APPENDIX E Additional Results in Section 4

In this appendix section, we consider a bounded random variable for a_t to demonstrate that even if $\log(a_t)$ is bounded, the debt-to-GDP ratio exhibits a power-law stationary distribution as long as $\sup(a_t) > 1$ and $\mathbb{E}\left[\log(a_t)\right] < 0$. For instance, consider $\log(a_t)$ independently drawn from a uniform distribution with mean μ_{iid} and support $(\mu_{iid} - \sigma_{iid}, \mu_{iid} + \sigma_{iid})$, i.e., $\log(a_t) \sim U(\mu_{iid} - \sigma_{iid}, \mu_{iid} + \sigma_{iid})$. This is a specific case of (11) with $\rho_a = 0$, $\theta_a = 0$, and $\varepsilon_{a,t} \sim U(m, M)$ where $m \equiv \mu_{iid} - \sigma_{iid}$ and $M \equiv \mu_{iid} + \sigma_{iid}$. In this scenario, the tail index condition, $\mathbb{E}\left[a_t^\kappa\right] = 1$, is defined as:

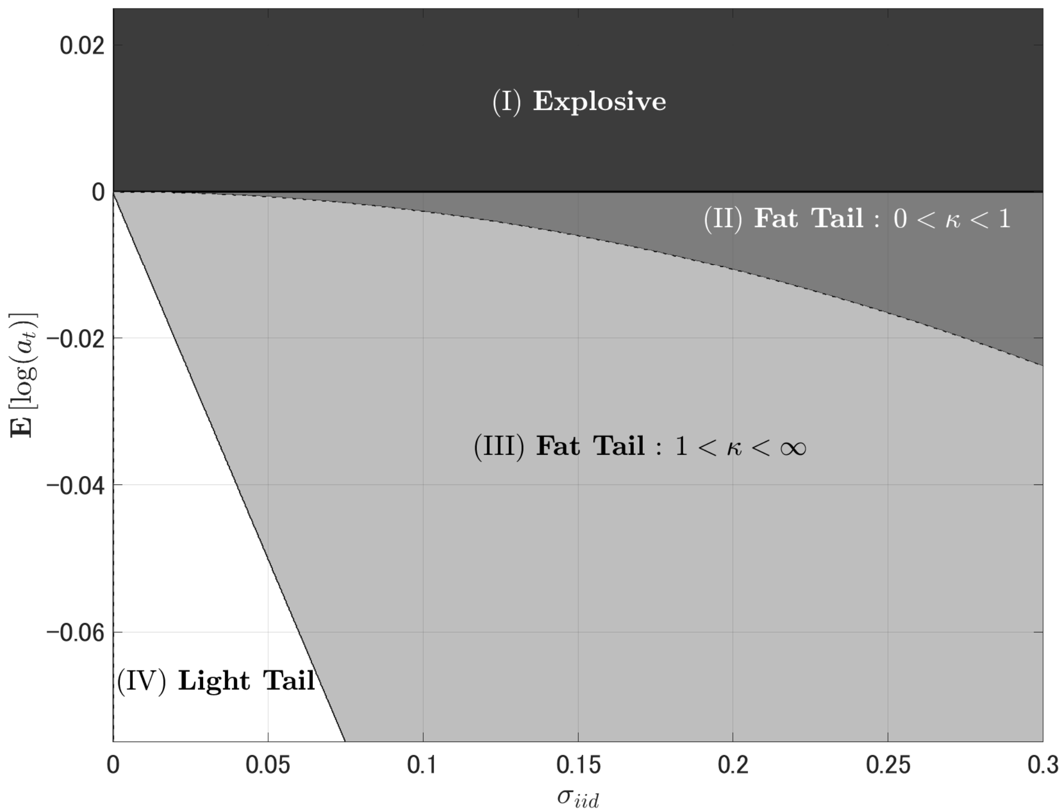
$$\kappa = \frac{1}{2\sigma_{iid}} \left[\exp(\kappa M) - \exp(\kappa m)\right].$$

Therefore, the asymptotic debt-to-GDP ratio is determined by parameters μ_{iid} and σ_{iid}. As illustrated in Figure E.1, the asymptotic distribution can be categorized into four distinct types:

(I) Explosive debt-to-GDP ratio (Region (I) in Figure E.1): The ratio grows indefinitely without a stationary distribution

(II) Fat-tailed stationary distribution with an *infinite* mean (Region (II) in Figure 2): A unique stationary distribution exists with a power-law upper tail and an infinite mean.

Figure E.1: Characteristics of the asymptotic distribution of b_t: IID



(III) Fat-tailed stationary distribution with a *finite* mean (Region (III) in Figure 2): A unique stationary distribution exists with a power-law upper tail and a finite mean.

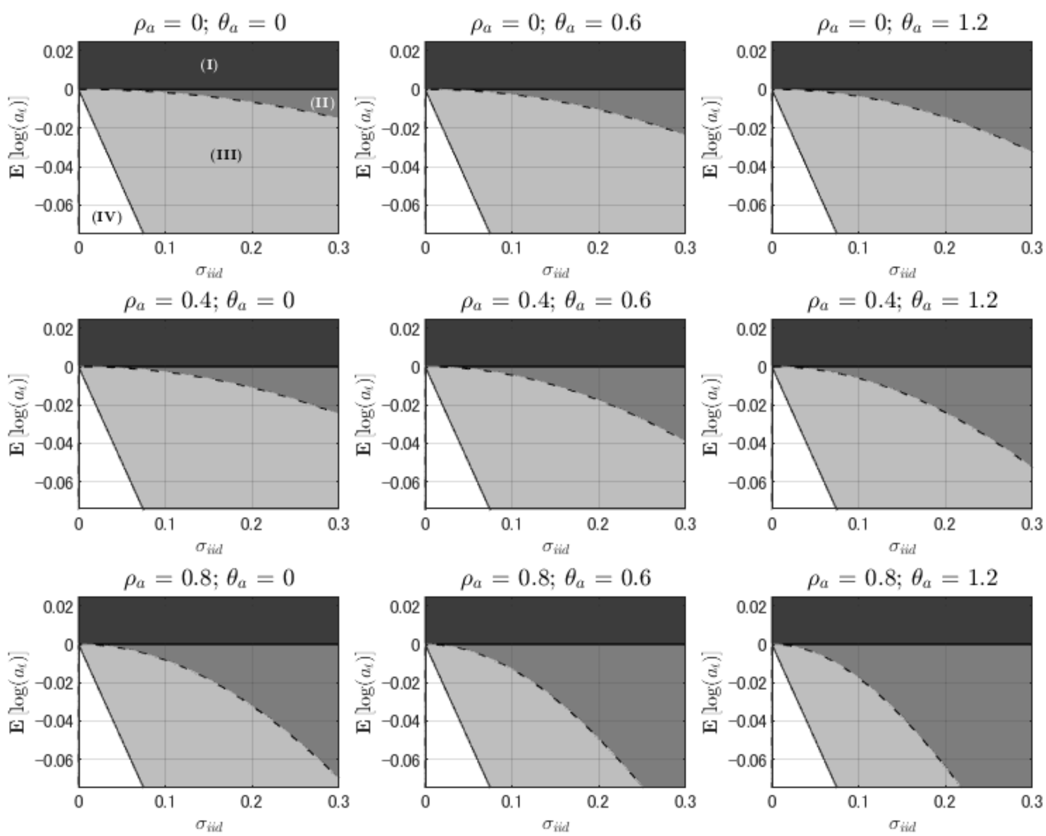
(IV) Light-tailed stationary distribution (Region (IV) in Figure 2): A unique stationary distribution exists with light tails.

The theorem concludes that if $\mu_{iid} = \mathbb{E}[\log(a_t)] \geq 0$, the ratio falls into Region (I); if $\mu_{iid} < 0$ and $\mu_{iid} + \sigma_{iid} = \sup(\log(a_t)) < 0$, it falls into Region (IV); Otherwise, it falls into Region (II) when $0 < \kappa < 1$ and (III) when $\kappa > 1$.

Expanding our scope, we consider $\log(a_t)$ following the stationary ARMA(1,1) process (11). Specifically, assume $\varepsilon_{a,t} \sim U(m, M)$ with $m \equiv \mu_{iid} - \sigma_{iid}$ and $M \equiv \mu_{iid} + \sigma_{iid}$, ensuring that $\log(a_t)$ remains bounded (see Lemma E.1). In this scenario, applying the proposition by Benhabib, Bisin, and Zhu (2011, Proposition 4) to our equation yields the Pareto tail condition:

$$\kappa \frac{1+\theta_a}{1-\rho_a} = \frac{1}{2\sigma_{iid}} \left[\exp\left(\kappa \frac{1+\theta_a}{1-\rho_a} M \right) - \exp\left(\kappa \frac{1+\theta_a}{1-\rho_a} m \right) \right]$$

Figure E.2: Characteristics of the asymptotic distribution of b_t: ARMA(1,1) process



Note: Each panel illustrates the characteristics of the asymptotic distribution of b_t when $\log(a_t)$ follows an ARMA(1,1) process (11).

The asymptotic behavior of the debt-to-GDP ratio is influenced not only by parameters μ_{iid} and σ_{iid} but also by ρ_a and θ_a. In Figure E.2, we depict the characteristics of the asymptotic distribution of the debt-to-GDP ratio when $\log(a_t)$ follows a ARMA(1,1) process. It is observed that as ρ_a or θ_a increases, the area of Region (II) widens while that of Region (III) narrows. In essence, greater persistence in the $\log(a_t)$ process leads to a thicker tail in the asymptotic debt-to-GDP ratio, implying that more persistent fluctuations in the interest rate-growth differential heighten the risk of extreme debt accumulation.

Lemma E.1. *a_t is bounded if $\varepsilon_{a,t}$ is bounded.*

Proof of Lemma E.1. a_t is clearly bounded below since $\log(a_t)$ and thus $a_t > 0$. Hence, it is sufficient to show there exists an upper bound $A > 0$ such that $a_t < \bar{A}$, i.e., $\log(a_t) < \log(\bar{A})$

for all t. From (11), we obtain that

$$\begin{aligned}\log(a_t) =&(1-\rho_\alpha)\mu_a \sum_{s=0}^{t} \rho_a^s + \sum_{s=0}^{t} \rho_a^s(\varepsilon_{a,t-s} + \theta_a \varepsilon_{a,t-s-1}) + \rho_a^{t+1}\log(a_{-1}) \\ =&(1-\rho_\alpha)\mu_a \sum_{s=0}^{t} \rho_a^s + \varepsilon_{a,t} + \sum_{s=1}^{t} \rho_a^{s-1}(\rho_a + \theta_a)\varepsilon_{a,t-s} + \rho_a^{t+1}\log(a_{-1})\end{aligned}$$

Since $\mu_a > 0$ and $\rho_a \in (0,1)$, we have

$$(1-\rho_\alpha)\mu_a \sum_{s=0}^{t} \rho_a^s \leq (1-\rho_\alpha)\mu_a \sum_{s=0}^{\infty} \rho_a^s = \mu_a,$$

and

$$\rho_a^t \log(a_{-1}) \leq \max\{0, \log(a_{-1})\},$$

for all $t \geq 0$. Furthermore, since $\varepsilon_{a,t}$ is bounded, let $\bar{\varepsilon} = \sup(\varepsilon_{a,t}) < \infty$ and $\underline{\varepsilon} = \inf(\varepsilon_{a,t}) > -\infty$. This clearly implies that

$$\bar{\varrho} \equiv \sup\left\{(\theta_a + \rho_a)\bar{\varepsilon}, (\theta_a + \rho_a)\underline{\varepsilon}\right\} < \infty.$$

Hence,

$$\begin{aligned}\log(a_t) \leq&\mu_a + \bar{\varepsilon} + \sum_{s=0}^{t} \rho_a^s \bar{\varrho} + \max\{0, \log(a_{-1})\} \\ &= \mu_a + \bar{\varepsilon} + \frac{\bar{\varrho}}{1-\rho_a} + \max\{0, \log(a_{-1})\} < \infty.\end{aligned}$$

Therefore, let $\log\left(\bar{A}\right) = \mu_a + \bar{\varepsilon} + \frac{\bar{\varrho}}{1-\rho_a} + \max\{0, \log(a_{-1})\}$, $a_t < \bar{A}$ for all t.

Since $\log a_t$ is bounded, therefore a_t is bounded. $\square$

APPENDIX F Detailed Descriptions of Historical Data

In this section, employing historical data from the United States and the United Kingdom, we overview the historical debt-to-GDP ratio in both countries.

The United States For the United States, we employ a time series that traces the ratio of federal debt held by the public to GDP from 1790 to 2022. To achieve this, we reference the dataset 'Federal Debt Held by the Public, 1790 to 2000 (Percentage of Gross Domestic Product)' as documented in Congressional Budget Office (2010b) (refer also to Congressional Budget Office, 2010a, Figure 1), and have extended the dataset to include data through 2022.[23] Specifically, for the period post-2000, we calculate the ratio by dividing the federal debt held by the public, sourced from the Debt to the Penny dataset provided by the U.S. Department of the Treasury, by nominal GDP.[24]

Figure 5a shows the public debt-to-GDP ratio from 1790 to 2022. After the Treasury Department was established in 1789, state debts from the American Revolution were federalized in 1790, resulting in a debt-to-GDP ratio of about 50 percent. Despite a spike due to the War of 1812, fiscal management eliminated the national debt by 1835. The ratio remained low until the Civil War in 1861. It rose significantly during the Civil War, World War I (WWI), the Great Depression, and World War II (WWII), with reductions in other periods. Post-WWII, the ratio declined until the 1970s, then increased sharply in the 1980s and early 1990s. It decreased in the late 1990s and 2000s but surged with the 2008 financial crisis and Great Recession, continuing to rise in the 2010s, reaching post-WWII levels during the COVID-19 pandemic in 2020.

The United Kingdom For the U.K.'s historical data, we rely on the 'A Millennium of Macroeconomic Data' dataset provided by the Bank of England.[25] This comprehensive dataset includes consolidated records of the U.K.'s national debt since 1690 and nominal GDP estimates for U.K. territories, dating back to 1700. Although the historical dataset is

[23] Congressional Budget Office (2010b) compiles these historical statistics by using federal debt data from the Department of the Treasury and the Board of Governors of the Federal Reserve System, in conjunction with GDP estimates from the Bureau of the Census, Berry (1978), Gallman (2000) and Balke and Gordon (1989).

[24] The Debt to the Penny dataset can be accessed at `https://fiscaldata.treasury.gov/datasets/debt-to-the-penny/debt-to-the-penny`. The nominal GDP data is sourced from National Income and Product Accounts (NIPAs), produced by the Bureau of Economic Analysis (BEA): `https://apps.bea.gov/histdata/fileStructDisplay.cfm?HMI=7&DY=2023&DQ=Q3&DV=Third&dNRD=December-22-2023`.

[25] The dataset is available at `https://www.bankofengland.co.uk/statistics/research-datasets`. This dataset, compiled by the Bank of England, contains various macroeconomic and financial data dating back to 1086.

updated through 2016 as per Thomas and Dimsdale (2017), we have extended the debt and GDP data up to 2022, following their approach.

Specifically, from the Bank of England's historical dataset, we use 'A23-Bank of England Balance Sheet' and 'A29-The National Debt' to compile a series representing the value of debt held by the public. This value is computed by subtracting the total debt held by public sector banks from the overall gross debt value for 1700-1974, as detailed below. For 1975-2022, we use Public Sector Net Debt (PSND) from the Office for National Statistics (ONS).[26]

The "National Debt" dataset comprises multiple measures of the total par value of the U.K. national debt dating back to 1700. From 1700 to 1835, the debt value includes the United Kingdom's total funded and unfunded debt, along with the estimated capital value of terminable annuities. From 1836 to 1899, the debt value comprises the aggregate of the U.K.'s total funded and unfunded debt and the capital value of terminable annuities.[27] For the period from 1900 to 1949, the Gross National Liabilities value as reported in Pember and Boyle (1950) was used to represent the debt amount. For the years 1950 to 1974, the national debt figure documented in Pember and Boyle (1976) was employed. Additionally, this database provides the estimated value of the outstanding debt at the end of each calendar year.[28]

Regarding the "Bank of England Balance Sheet," this dataset chronicles the institution's extensive balance sheet history, including the government debt within the assets section. For the period 1700-1974 covered here, the balance sheet dates annually as follows: 1700-1764 - balance sheet at the end of August; 1766-1844 - balance sheet at the end of February; 1845-1857 - bank report published on the final Saturday in February; 1858-1966 - bank report

[26]The dataset is available at https://www.ons.gov.uk/economy/governmentpublicsectorandtaxes/publicsectorfinance/articles/widermeasuresofpublicsectornetdebt/december2018. It contains time series data on public sector finances and key fiscal aggregates. In this analysis, we use Public sector net debt excluding public sector banks (PSND ex, ONS code: HF6W). PSND ex includes bonds (debt securities), loans, deposits, and currency, and liquid financial assets. The dataset records first quarter debt from 1975 to 1992, and from 1993 onwards, debt is recorded on a monthly basis. Therefore, from 1993 onwards, the debt at the end of December is used to correspond to the calendar year.

[27]Details about funded debt values from 1700 to 1899 can be found in Pember and Boyle (1950), while those on unfunded debt are provided in Mitchell (1988). Additionally, data regarding payment amounts and discount rates were employed to estimate the capital value of terminable annuities.

[28]For example, the outstanding debt at the end of the calendar year 1964, when the financial year ends on March 31, is calculated as follows:

$$\hat{B}_{1964} = \frac{1}{4} \times B_{1963/64} + \frac{3}{4} \times B_{1964/65},$$

where $\hat{B}_{1964}$ denotes the outstanding debt at the end of the calendar year 1964, $B_{1963/64}$ denotes the outstanding debt at the end of the financial year 1963/64, and $B_{1964/65}$ denotes the outstanding debt at the end of the financial year 1964/65.

released on the last Wednesday in February; 1966-1974 - bank reports issued on the third Wednesday in February. Due to the unavailability of balance sheets for 1765 and 1774, the data for these years represent the average of the figures from adjacent years.[29]

For GDP data, we integrate 'A9-Nominal GDP (A)' from the Bank of England's historical dataset with nominal GDP data sourced from the ONS. The U.K. nominal GDP represents the nominal gross output of Great Britain (England, Scotland, and Wales) from 1700-1800, expands to include Ireland from 1801-1920, and encompasses Great Britain plus Northern Ireland post-1920. From 1700 to 1800, the nominal GDP is calculated the sum of the GDP factor cost, documented by Broadberry, Campbell, Klein, Overton, and Van Leeuwen (2015), and the value of indirect taxes, based on tax revenue data recorded in fiscal sheet. The market value of nominal GDP from 1801 to 1920 was calculated by calculating the value of the factor cost of GDP and indirect taxes as measured by the Compromise/Balanced measure and recording the sum of the two as the market value. The market value of nominal GDP from 1921 to 1947 was calculated using data from Sefton and Weale (1995). After 1948, we use Gross Domestic Product at market prices sourced from the ONS.[30]

Figure 5b illustrates the debt-to-GDP ratio of the United Kingdom from 1700 to 2022. Over the last 320 years, the UK experienced three significant periods of increase in this ratio. The first was during the 18th century, marked by numerous large-scale European wars, culminating in a debt-to-GDP ratio of approximately 200 percent following the Napoleonic Wars. The subsequent period, spanning the 19th and early 20th centuries, saw efforts to reduce the debt. However, the two World Wars and the severe recessions of the interwar years drove the ratio above 250 percent immediately after WWII. Thereafter, the ratio consistently decreased until around 1990, dropping well below 50 percent. The trend reversed with the

[29] We use the same methodology to calculate the amount of government debt held by the Bank of England at the end of the calendar year. For example, the amount of government debt held by the Bank of England at the end of calendar year 1964 (the balance sheet date is the third Wednesday in February) is calculated as follows:

$$g_{1964} = \frac{1}{6} \times G_{1964} + \frac{5}{6} \times G_{1965}$$

where g_{1964} denotes the amount of government debt held by the Bank of England at the end of calendar year 1964, G_{1964} denotes the amount of government debt held by the Bank of England on the balance sheet date in 1964, and G_{1965} denotes the amount of government debt held by the Bank of England on the balance sheet date in 1965. The value of the public net debt excluding public sector banks for 1964 can be calculated as follows:

$$PB_{1964} = \hat{B}_{1964} - g_{1964}$$

where $\hat{B}_{1964}$ denotes the outstanding debt at the end of the calendar year 1964 and g_{1964} denotes the amount of government debt held by the Bank of England.

[30] The dataset is available at https://www.ons.gov.uk/economy/grossdomesticproductgdp/timeseries/ybha/pn2.

Early 1990s recession, and further escalated due to the Great Recession and the COVID-19 Recession, bringing the current ratio to approximately 100 percent.

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