Intergenerational Consequences of Rare Disasters

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Motivation

- In the last twenty years we saw large economic downturns, such as the Great Recession and the Covid-19 pandemic, leading to
    - declines in labor income
    - declines in asset prices
- Exposure to shocks is heterogeneous across households
- We focus on age heterogeneity
    - portfolio composition varies over the lifecycle
    - exposure of labor income to business cycle fluctuations varies with age

This paper

- Question: what are the intergenerational consequences of rare but large disasters?
  - how do the economic mechanisms contribute?
  - what is the impact of change in social security?
- Approach: we build and calibrate a life-cycle model with rare disasters, three assets: houses, equity, bonds, and equilibrium prices
- Equilibrium models with many assets, borrowing, and large shocks are computationally challenging
  - we introduce two complementary innovations to deep learning based solution methods
- Finding:
  - young and very old households suffer the most
    - young: largest decline in labor income and borrowing constraint
    - very old: decline in asset prices
  - old households, shortly before retirement, suffer the least
    - more stable income
    - long in risk free assets
    - still live long

Literature

- Intergenerational consequences of the great recession:
  Glover et al. (2020), Hur (2018)
  - → study a general equilibrium model (relative to Hur (2018))
  - → distinguish bonds, housing and equity (relative to Glover et al. (2020), Hur (2018))
  - → have assets of different liquidity (relative to Glover et al. (2020))
  - → model borrowing constraints (relative to Glover et al. (2020))
- Rare disasters:
  Rietz (1988), Barro (2006), Barro and Ursúa (2008), Gourio (2012), Nakamura et al. (2013)
  - → we study the distributional consequences of rare disasters (relative to representative agent studies)
- Deep learning based solution methods:
  Azinovic et al. (2022); Maliar et al. (2021); Kase et al. (2023); Gu et al. (2023); Kahou et al. (2021); Han et al. (2022); Valaitis and Villa (2024); Fernández-Villaverde et al. (2023); Barnett et al. (2023); Gopalakrishna et al. (2024)
  - → market clearing neural network architectures
  - → step-wise solution procedure for model with multiple assets

Model

Model: Technology

Following Gourio (2012)

- Representative firm

$$Y_t = K_t^{\alpha}(z_t L)^{1-\alpha}$$

- Two regimes: normal and disaster.
- Productivity

$$\log(z_t) = \underbrace{\log(z_t^p)}_{\text{permanent}} + \underbrace{\log(z_t^r)}_{\text{transitory}}$$

$$\log(z_t^p) = \log(z_{t-1}^p) + \mu + \epsilon_t + \underbrace{\theta_t}_{\text{permanent}}$$

$$\log(z_t^r) = \rho^r \log(z_{t-1}^r) + \underbrace{\phi_t}_{\text{transitory}} - \theta_t$$

- Shocks to capital depreciation: $\xi_t := \mu + \epsilon_t + \theta_t$
- Amount of capital after production: $K_t(1-\delta^k)e^{\xi_t}$
- Probability of disaster in the next period:
  - during disaster: $1 - p^{\text{exit}}$
  - during normal times: $\log(p_t) = \rho_p \log(p_{t-1}) + (1-\rho_p)\log(\bar{p}) + \epsilon_t^p$

Model: Demographics and labor income

- Life-cycle model with $H = 18$ cohort, one period corresponds to 4 years
- One representative agent per cohort, age-group indexed by $h \in \{1, \dots, H\}$
- Age-dependent household size e^h, survival probability Γ^h, mass distribution μ^h
- Efficient labor units l_t^h with age-dependent exposure to aggregate fluctuations ζ^h

$$\log(l_t^h) = \log(l^h) + \underbrace{\zeta_h \log\left(\frac{Y_t}{Y_{t-1}}\right)}_{\text{heterogeneous exposure}} + \underbrace{\text{normalization}_t}_{\text{such that } L_t = 1}$$

- Retirees receive defined benefit pay-as-you-go social security s_t^h ▸ normalization
- Time separable preferences over consumption and housing following Huo and Ríos-Rull (2016) ▸ preferences, warm-glow bequest utility

Model: Asset markets

Households can invest in three assets

- Bonds:
  - price: p_t^b
  - risk free payout of one
  - liquid
  - can be sold short, subject to posting housing as collateral ▸ more details
- Equity:
  - modeled as leveraged capital
  - price: $p_t^e = q_t - \lambda p_t^b$
  - risky payout: $(1 - \delta^K)e^{\xi_{t+1}} q_{t+1} + r_{t+1}^K + \pi_{t+1}^K - \lambda$
  - illiquid: subject to quadratic adjustment costs
  - no short-selling
- Housing:
  - price: p_t^H
  - risky payout: $(1 - \delta^H)p_{t+1}^H + \pi_{t+1}^H$
  - illiquid: subject to quadratic adjustment costs
  - no short-selling

Capital and housing are produced by intermediary firms following Bayer et al. (2019) ▸ more details

Model: Household problem

- t: time, a: age, Γ^a: survival probability
- $k_t^a, k_t^{\text{end},a}$ / $b_t^a, b_t^{\text{end},a}$ / $h_t^a, h_t^{\text{end},a}$: beginning and end of period capital / bond / housing

$$V_t^a = \max_{\left\{k_t^{\text{end},a}, h_t^{\text{end},a}, b_t^{\text{end},a}\right\}} \underbrace{u(c_t^{\text{eff},a}) + \psi^{\text{housing}} v(h_t^{\text{eff},a})}_{\text{util. from cons. and housing}} + \beta \mathrm{E} \left[(1-\Gamma^a) \underbrace{\psi^{\text{bequest motive}} u(w_t^{\text{eff},a})}_{\text{bequest}} + \Gamma^a \underbrace{V_{t+1}^{a+1}}_{\text{cont. val.}} \right]$$

Model: Household problem

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subject to:

$$c_t^a = \underbrace{l_t^a(1-\tau_t)w_t + s_t^a}_{\text{labor and ret. inc.}} + \underbrace{b_t^a + k_t^a((1-\delta)e^{\xi_t}q_t + \pi_t^{K,\text{inter}} + r_t - \lambda) + h_t^a((1-\delta^H)p_t^H + \pi_t^{H,\text{inter}})}_{\text{payout from assets}}$$

$$\underbrace{-p_t b_t^{\text{end},a} - (q_t - \lambda p_t^b)k_t^{\text{end},h} - p_t^H h_t^{\text{end},a}}_{\text{expenses on new assets}} \underbrace{- \frac{\psi_h}{z_{p,t}}(h_t^{\text{end},a} - h_t^a)^2 - \frac{\psi_k}{z_{p,t}}(k_t^{\text{end},a} - k_t^a)^2}_{\text{expenses on adjustment costs}}$$

$$0 \leq b_t^{\text{end},a} + \tilde{X}_t^{\kappa,a} \tilde{X}_t^{P^H,a} h_t^{\text{end},a}, \forall a \in \{1, \ldots, h^{\text{retirement}} - 1\}$$

$$0 \leq b_t^{\text{end},a}, \forall a \in \{h^{\text{retirement}}, \ldots, H\}$$

$$0 \leq k_t^{\text{end},a}$$

Equilibrium

- State of the economy

$$\mathbf{x}_t := [x_t, z_t^p, z_t^r, p_t, Y_{t-1}, X_{t-1}^C, X_{t-1}^\kappa, X_{t-1}^{p^H}, \mathbf{h}_t, \mathbf{k}_t, \mathbf{b}_t] \in \{0,1\} \times \mathbb{R}^{8+3\times H}. \tag{1}$$

- Functional rational expectations equilibrium
  - bond policies $\mathbf{b}^{\text{end}}(\mathbf{x}_t) \in \mathbb{R}^H$
  - capital policies $\mathbf{k}^{\text{end}}(\mathbf{x}_t) \in \mathbb{R}^H$
  - housing policies $\mathbf{h}^{\text{end}}(\mathbf{x}_t) \in \mathbb{R}^H$
  - bond price $p^b(\mathbf{x}_t)$

  such that
  - households optimize ($3 \times H$ Karush Kuhn Tucker conditions)
  - markets clear (for the household bond market)

Equilibrium

- State of the economy

$$\mathbf{x}_t := [x_t, z_t^p, z_t^r, p_t, Y_{t-1}, X_{t-1}^C, X_{t-1}^\kappa, X_{t-1}^{p^H}, \mathbf{h}_t, \mathbf{k}_t, \mathbf{b}_t] \in \{0,1\} \times \mathbb{R}^{8+3\times H}. \quad (1)$$

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  such that
  - households optimize ($3 \times H$ Karush Kuhn Tucker conditions)
  - markets clear (for the household bond market)
- Challenges
  - high-dimensional state space ⇒ challenging for grid based methods
  - non-linear policy functions ⇒ need a flexible function approximator
  - large shocks ⇒ global method
  - many continuous shocks, many endogenous aggregate state variables and rich asset distribution ⇒ Krusell and Smith (1998) very challenging

Numerical method

Starting point for the method

- Deep learning based solution methods: Azinovic et al. (2022) (DEQN), Kahou et al. (2021); Maliar et al. (2021); Kase et al. (2023); Gu et al. (2023); Han et al. (2022); Valaitis and Villa (2024); Fernández-Villaverde et al. (2023); Barnett et al. (2023).
  - deep neural networks as an approximator for equilibrium functions of the economy
  - trained to minimize equilibrium conditions error on a simulated ergodic set
- DEQN can handle stochastic models with many state variables, however, two pain points remain:
  - portfolio choice
  - market clearing

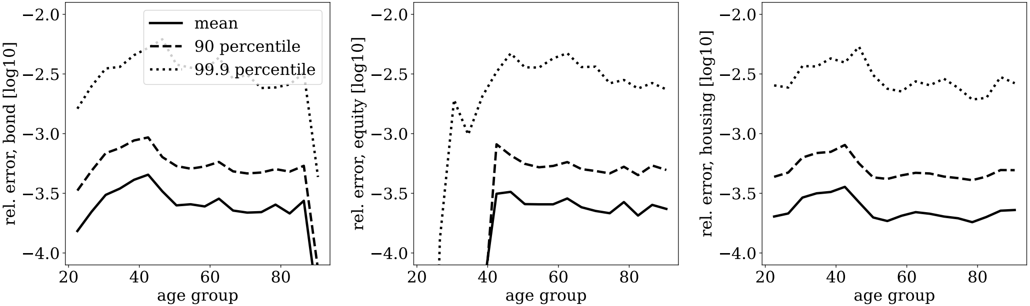
▸ more details on DEQN

Methodological contribution

- Two complementary innovations
    1. market clearing layers
    2. step-wise model transformations for portfolio choice and asset prices
- Market clearing layers: neural network predictions are consistent with market clearing by design
- Step-wise model tranformations: robustly solve models with multiple assets

▸ more details on MCL and step-wise algorithm

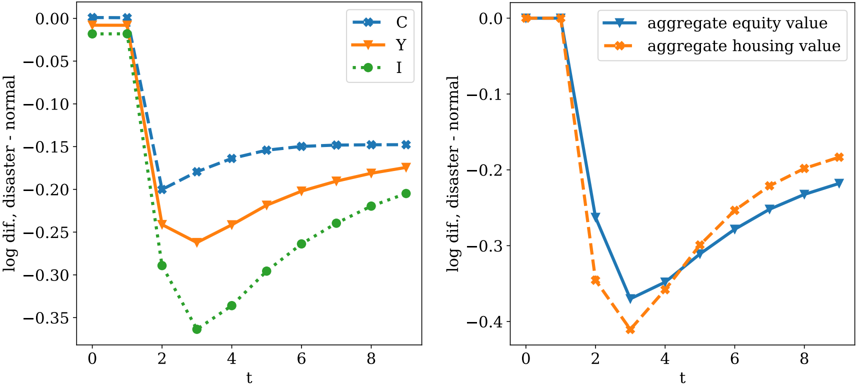
Accuracy of the solution to the benchmark model



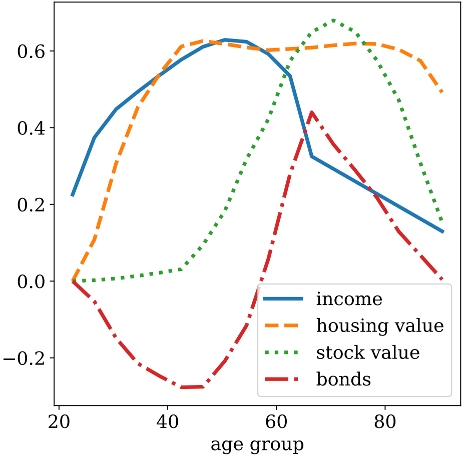
Results

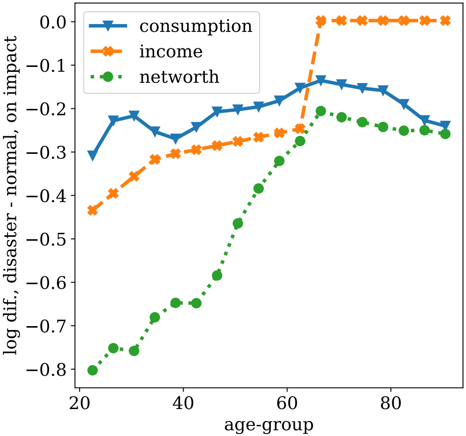
▸ Calibration

Aggregate consequences of an average rare disaster



Intergenerational consequences of a rare disaster

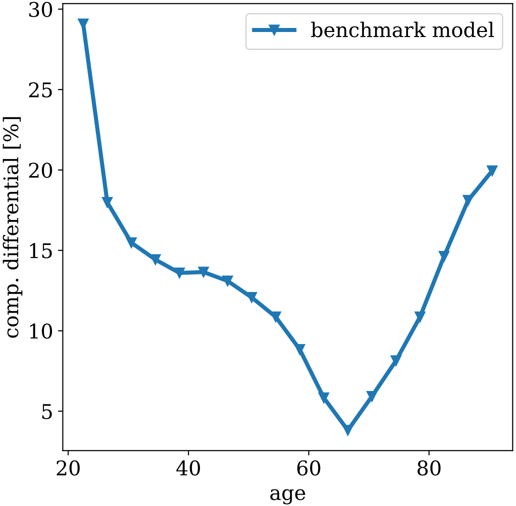




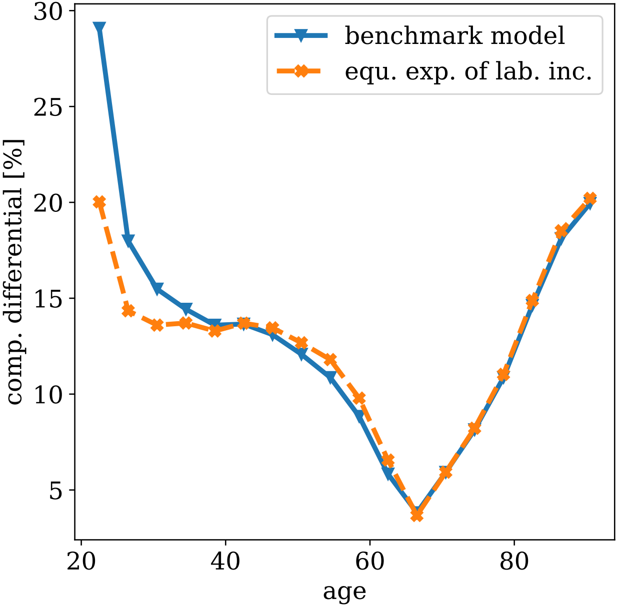
▸ more details

Intergenerational consequences of a rare disaster

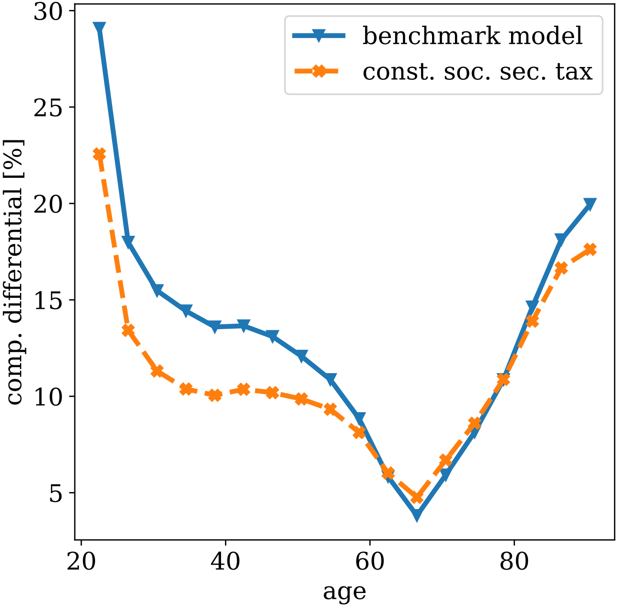
We compute consumption equivalent compensating differentials.



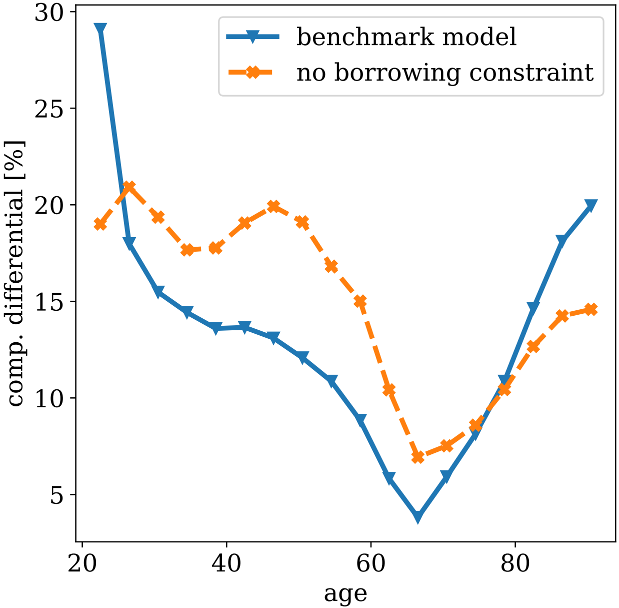
Experiment: equal exposure of labor income



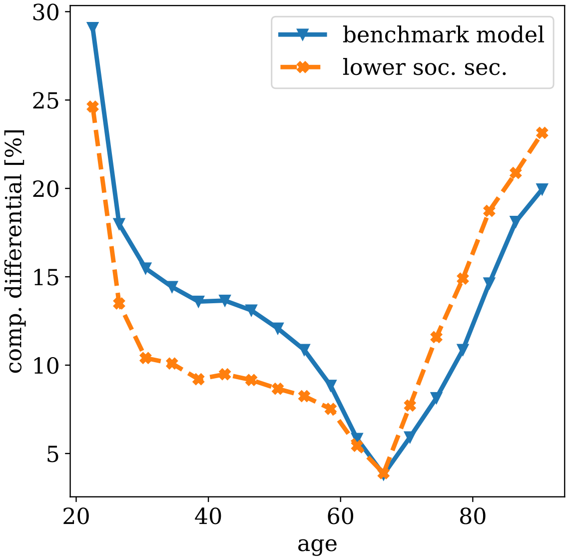
Experiment: constant social security tax



Experiment: no borrowing constraints



Experiment: 25% lower social security payments



▸ Depreciation disaster ▸ What are Neural Nets? ▸ Why use Neural Nets?

Conclusion

Conclusion

- We analyze a quantitative life-cycle model with disaster risk, housing, equity and bonds in general equilibrium
- Our results show that disasters hit young and very old households hardest. Relative winners are households shortly before retirement.
- To solve our model, we develop a deep learning solution method tailored for solving large stochastic models with portfolio choice
- Two key innovations
    - market clearing layers, an economics-inspired neutral network architecture
    - step-wise model transformation procedure to guide network training with many assets

Thank you!

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Appendices

Details on production of housing and capital

- Households accumulate capital and housing
- Following Bayer et al. (2019), we model two intermediaries, one for housing and one for capital
- Transform the consumption good into capital or housing subject to quadratic adjustment costs
- Equilibrium price for capital q_t, equilibrium supply K_{t+1}, supply elasticity parameter $\xi^{K,\text{adj}}$
- Equilibrium price for housing p_t^H, equilibrium supply H_{t+1}, supply elasticity parameter $\xi^{H,\text{adj}}$

▸ more details ▸ back

Details on the intermediaries

- Intermediaries can transform investment I_t^K into capital $\Delta K_t = K_{t+1} - K_t$ and investment I_t^H housing $\Delta H_t = H_{t+1} - H_t$ subject to adjustment costs

$$\frac{\Delta K_t}{K_t} = \frac{I_t^K}{K_t} - \frac{\xi^{K,\text{adj}}}{2}\left(\frac{\Delta K_t}{K_t}\right)^2, \quad \frac{\Delta H_t}{H_t} = \frac{I_t^H}{H_t} - \frac{\xi^{H,\text{adj}}}{2}\left(\frac{\Delta H_t}{H_t}\right)^2$$

- maximize profits

$$\Pi_t^{K,\text{inter}} = q_t \Delta K_t - I_t^K, \quad \Pi_t^{H,\text{ inter}} = p_t^H \Delta H_t - I_t^H$$

- in equilibrium prices are given by

$$q_t = 1 + \xi^{K,\text{adj}} \frac{\Delta K_t}{K_t}, \quad p_t^H = 1 + \xi^{H,\text{adj}} \frac{\Delta H_t}{H_t}$$

▸ back

Details on preferences I

- Time separable expected utility preferences. Instantaneous utility

$$u(c_t^{\text{eff},a}) + \psi^{\text{housing}} v(h_t^{\text{eff},a}) + \beta \underbrace{(1-\Gamma^a)}_{\text{prob. to die}} \psi^{\text{bequest}} u(w_t^{\text{eff},a})$$

  over effective consumption $c_t^{\text{eff},a}$, housing $h_t^{\text{eff},a}$, and bequeathing wealth $w_t^{\text{eff},a}$
- Effective values are normalized by household size and an exponential moving average of aggregate consumption
- CRRA utility from consumption and bequests
- Utility from housing $v(\cdot)$ follows Huo and Ríos-Rull (2016) and is a combination of two CRRA utility functions, such that the marginal utility from housing decreases faster for $h_t^{\text{eff},a} > h^{cut}$
- Inheritance: bequests are inherited by the age-group 30 years younger

▸ more details ▸ back

Details on preferences II

- preferences over effective consumption, housing, and bequests

$$c_t^{\text{eff},a} := \frac{c_t^a}{e^a X_{t-1}^C}, h_t^{\text{eff},a} := \frac{h_t^a}{e^a X_{t-1}^C}.$$

- the economy is growing, X_t^C is an aggregate consumption habit: $X_t^C = \rho^{X_C} X_{t-1}^C + (1 - \rho^{X_C}) C_t$.
- time separable expected utility, utility from consumption $u(c_t^{\text{eff},h})$, CRRA
- time separable expected utility, utility from housing following Huo and Ríos-Rull (2016)

$$v(h_t^{\text{eff},a}) := w_1(h_t^{\text{eff},a}) v^1(h_t^{\text{eff},a}) + \left(1 - w_1(h_t^{\text{eff},a})\right) v^2(h_t^{\text{eff},a})$$

$$w_1(h_t^{\text{eff},a}) := \text{smooth step function from 0 to 1,} = 0.5 \text{ at } h_t^{\text{eff},a} = h^{cut}$$

marginal utility from housing decreases faster for v^2 than for v^1
$\Rightarrow$ marginal utility from housing decreases faster for $h_t^{\text{eff},a} > h^{cut}$

back

Model: Collateral requirement

- Housing can serve as collateral to short-sell the bond
- Simplest form with LTV requirement $\kappa_t \in \{\kappa^{\text{normal}}, \kappa^{\text{disaster}}\}$:

$$b_t^{\text{end},j} + \kappa_t P_t^H h_t^{\text{end},j} \geq 0,$$

- But most mortgages have long duration, hence roll-over risk is limited
  - → apply collateral requirement $b_t^{\text{end},j} + \kappa_t P_t^H h_t^{\text{end},j} \geq 0$ to the share of newly purchased houses
  - → apply the collateral constraint with an exponential moving average the house price $X_t^{P^H}$ and the LTV ratio X_t^{κ}: $b_t^{\text{end},j} + X_t^{\kappa} X_t^{P^H} h_t^{\text{end},j} \geq 0$ on previously purchased houses

▸ more details ▸ back

Details on the collateral requirement

- housing can serve as collateral to borrow in the bond
- simplest form with LTV requirement $\kappa_t \in \{\kappa^{\text{normal}}, \kappa^{\text{disaster}}\}$:

$$b_t^{\text{end},j} + \kappa_t P_t^H h_t^{\text{end},j} \geq 0,$$

- but most mortgages have long duration, hence roll-over risk is limited
- we want to apply this constraint only on new housing and new debt, hence

$$b_t^{\text{end},j} + \tilde{X}_t^{\kappa,j} \tilde{X}_t^{P^H,j} h_t^{\text{end},j} \geq 0,$$

where

$$\begin{aligned}
w_t^{\text{new, j}} &:= \max\left\{0, \frac{h_t^{\text{end},j} - h_t^j}{h_t^{\text{end},j}}\right\} \\
\tilde{X}_t^{\kappa,j} &:= w_t^{\text{new, j}} \kappa_t + (1 - w_t^{\text{new, j}}) X_t^{\kappa} \\
\tilde{X}_t^{p^H,j} &:= w_t^{\text{new, j}} p_t^H + (1 - w_t^{\text{new, j}}) X_t^{p^H} \\
X_t^{\kappa} &= \rho^{X^{\kappa}} X_{t-1}^{\kappa} + (1 - \rho^{X^{\kappa}}) \kappa_t \\
X_t^{p^H} &= \rho^{X^{p^H}} X_{t-1}^{p^H} + (1 - \rho^{X^{p^H}}) p_t^H
\end{aligned}$$

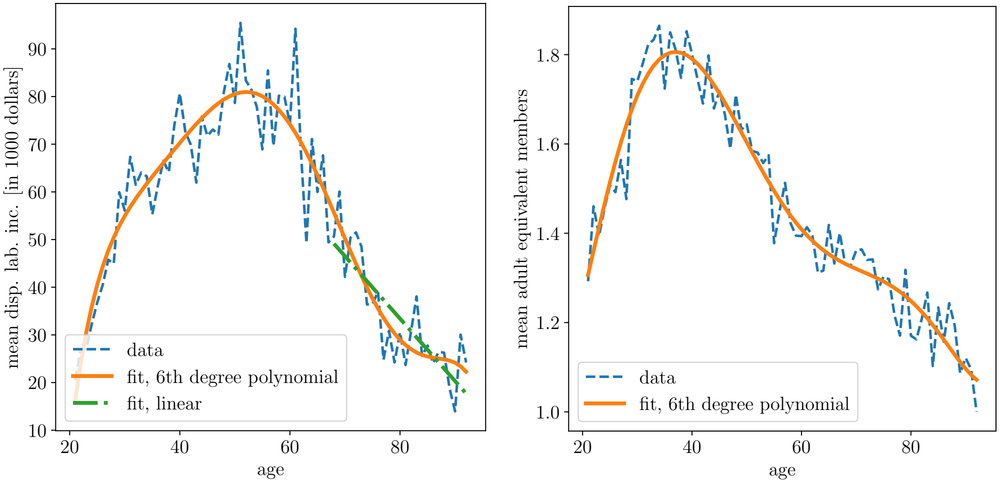
▸ back

Calibration

▸ Results

Lifecycle

based on the SCF (2007)

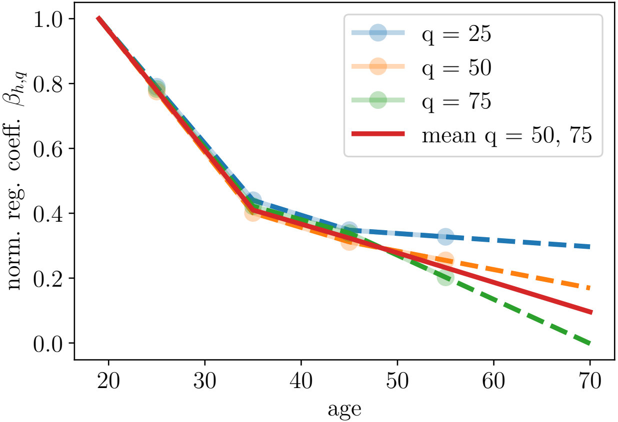


Mortality: life-tables (US 2007)

Exposure to aggregate fluctuations

Based on real earnings data by age and income percentile, provided by Guvenen et al. (2014). Specify

$$\Delta l_{h,q,t} = \alpha_{h,q} + \beta_{h,q}\Delta Y_t + \epsilon_{h,q,t}.$$



▸ not normalized

Exogenously chosen

Parameter	Value	Meaning
		Preferences
σ_C	6	risk aversion consumption
σ_H	6	risk aversion housing
$\rho^{X^{P_H}}$	0.8	pers. of the exp. moving average for the house price relevant in the LTV constraint (half life time of roughly 12 years)
		Technology and policy
α	0.3	capital share in production
δ_K	0.344	depreciation of capital (10% yearly)
δ_H	0.252	maintenance costs for housing (7% yearly)
κ_{normal}	0.5	LTV ratio in normal times
κ_{disaster}	0.4	LTV ratio in disasters
λ^{firm}	0.5	leverage of capital
ρ^{X_C}	0.95	pers. of the aggregate consumption habit (half life time of roughly 50 years)
$\rho^{X^{\kappa}}$	0.8	pers. of the exp. moving average for the LTV requirement (half life time of roughly 12 years)
		Shocks
μ	0.08	trend growth (as in Gourio (2012))
σ_ϵ	0.04	std. dev. of growth shocks during all times (as in Gourio (2012))
p_{exit}	$\frac{2}{3}$	prob. to remain in the disaster state (estimated by Nakamura et al. (2013))
ρ_p	0.185	persistence of the disaster probability during normal times (match average prob. estimated by Nakamura et al. (2013))
σ_p	2.75	std. dev. of shocks to the disaster probability during normal times (match average prob. estimated by Nakamura et al. (2013))
$\bar{p}$	0.025	probability of disaster in the absence of disaster probability shocks (match average prob. estimated by Nakamura et al. (2013))
μ_θ	-0.10	mean permanent shock during dis. (estimated by Nakamura et al. (2013))
σ_θ	0.06	std. dev. of disaster-specific permanent shocks
σ_ϕ	0.08	std dev. of disaster-specific transitory shocks

Calibrated parameters and moments

Parameter	Value	Meaning	Associated model moments
		Preferences	
β	1.07	patience	wealth to income ratio
ψ^{housing}	0.35	preference for housing	housing share in networth
ψ^{bequest}	10	bequest motive	share of net-worth held by old households
$h^{\text{eff, cut}}$	1	start of quicker utility decrease	life-cycle profile of home ownership
		Shocks	
μ_ϕ	-0.40	mean of transitory shock during disasters	impact response of agg. cons. to an average disaster shock
ρ_z	0.35	persistence of the transitory shock	response of agg. cons. in the second subsequent disaster period
		Technology and policy	
R^{ss}	2.6	level of social security	income of old relative to middle aged households
ψ_k	0.10	hh. level adjustment costs on equity	none, chosen as $\frac{2}{3} \times \psi_h$
ψ_h	0.15	hh. level adjustment costs on housing	0.5% of adjusted value on average
$\xi^{K,\text{adj}}$	8	agg. adjustment costs on capital	rel. volatility of aggregate consumption growth
$\xi^{H,\text{adj}}$	12	agg. adjustment costs on housing	none, chosen as $\frac{3}{2} \times \xi^{K,\text{adj}}$

Calibration: Aggregate Consumption IRF Disaster

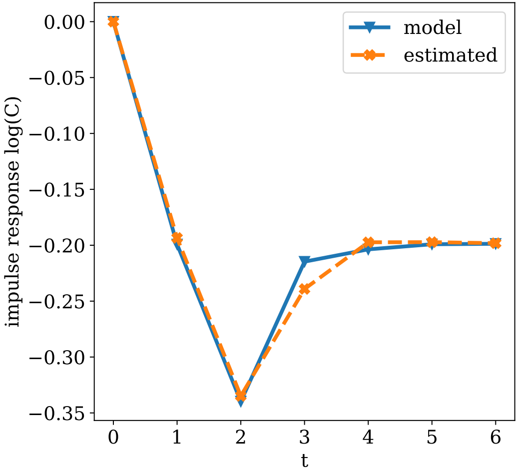
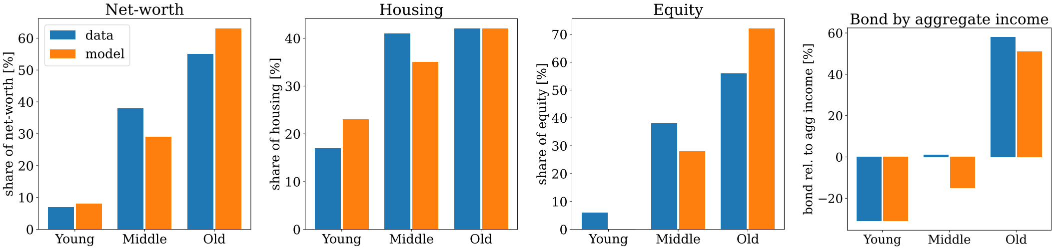


Figure: Model implied impulse response of aggregate consumption (solid blue line) and impulse response estimated by Nakamura et al. (2013). The impulse response corresponds to a disaster realizing at $t = 1$ and lasting for two periods, corresponding to 8 calendar years.

Wealth Distribution

	net-worth to inc. ratio	housing share	equity share	bond share
Model (normal times)	1.78	67%	33%	0%
Data (2007)	1.77	66%	31%	3%



▸ Results

Illustrative Model

▸ Back

Illustrative OLG model with capital and bond

- Representative firm produces with

$$\begin{aligned} F(z_t, K_t, L) &= z_t K_t^{\alpha} L^{1-\alpha} \\ w_t &= \alpha z_t K_t^{\alpha-1} L^{1-\alpha} \\ r_t &= z_t(1-\alpha) K_t^{\alpha} L^{\alpha} \end{aligned}$$

- Uncertainty in TFP z_t, and depreciation of capital δ_t

$$\begin{aligned} \log(z_{t+1}) &= \rho_z \log(z_t) + \sigma_z \epsilon_t \\ \epsilon_t &\sim N(0,1) \\ \delta_t &= \delta \frac{2}{1+z} \end{aligned}$$

- Assets
  - one period bond with price p_t in aggregate supply B
  - risky capital K_t
  - borrowing constraints on both assets

$$\begin{aligned} b_t^h &\geq 0 \\ k_t^h &\geq 0 \end{aligned}$$

- Households
  - $H = 32$ age-groups, indexed with $h \in \mathcal{H} := \{1, \ldots, 32\}$
  - supply labor units l_t^h inelastically
  - adjustment costs on capital

$$\begin{aligned} \Delta_{k,t}^h &:= k_{t+1}^{h+1} - k_t^h \\ \text{adj. costs} &= \psi \left(\Delta_{k,t}^h \right)^2 \end{aligned}$$

  - budget constraint

$$\begin{aligned} c_t^h = l^h w_t + b_{t-1}^{h-1} + k_{t-1}^{h-1}(1 - \delta_t + r_t) \\ - p_t^b b_t^h + k_t^h - \psi \left(\Delta_{k,t}^h \right)^2 \end{aligned}$$

  - maximize

$$\mathrm{E}\left[\sum_{i=h}^{H} \beta^{i-h} u(c_{t+i}^{h+i}) \right]$$

$$u(c) := \frac{c^{1-\gamma} - 1}{1-\gamma}$$

Equilibrium conditions

- **Market clearing**:

$$K_t := \sum_{h \in \mathcal{H}} k_t^h$$

$$B = \sum_{h \in \mathcal{H}} b_t^h \Leftrightarrow \epsilon_t^B := B - \sum_{h \in \mathcal{H}} b_t^h = 0$$

- **Firms optimize**:

$$w_t := \alpha z_t K_t^{\alpha-1} L^{1-\alpha}$$

$$r_t := z_t(1-\alpha) K_t^{\alpha} L^{\alpha}$$

- **Households optimize**:
  - H sets of Karush Kuhn Tucker conditions for bond
    $\Rightarrow$ single equation using the Fisher-Burmeister equation
    $\Rightarrow$ H errors $\epsilon_t^{k,i}$
  - H sets of Karush Kuhn Tucker conditions for capital
    $\Rightarrow$ single equation using the Fisher-Burmeister equation
    $\Rightarrow$ H errors $\epsilon_t^{h,i}$

▸ details

Approximation with standard DEQN

- State of the economy

$$\mathbf{x}_t = [\underbrace{z_t}_{\text{ex. shock}}, \underbrace{k_t^1, \ldots, k_t^{32}}_{\text{dist. of cap.}}, \underbrace{b_t^1, \ldots, b_t^{32}}_{\text{dist. of bonds}}]$$

- Equilibrium policies

$$\mathbf{f}(\mathbf{x}_t) = [\underbrace{k_{t+1}^1, \ldots, k_{t+1}^{32}}_{\text{capital policy}}, \underbrace{b_{t+1}^1, \ldots, b_{t+1}^{32}}_{\text{bond policy}}, \underbrace{p_t^b}_{\text{bond price}}]$$

- Neural network approximates

$$\mathcal{N}_\rho(\mathbf{x}_t) = [\underbrace{\hat{k}_{t+1}^1, \ldots, \hat{k}_{t+1}^{32}}_{\text{capital policy}}, \underbrace{\hat{b}_{t+1}^1, \ldots, \hat{b}_{t+1}^{32}}_{\text{bond policy}}, \underbrace{\hat{p}_t^b}_{\text{bond price}}] \approx \mathbf{f}(\mathbf{x}_t)$$

- Loss function

$$\ell_\rho(\mathbf{x}_t) := \underbrace{w_{hh,k}}_{\text{weight}} \underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon_t^{k,h}\right)^2\right)}_{\text{opt. cond. cap.}} + \underbrace{w_{hh,b}}_{\text{weight}} \underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon_t^{b,h}\right)^2\right)}_{\text{opt. cond. bond}} + \underbrace{w_{mc,B}}_{\text{weight}} \underbrace{\left(\epsilon_t^B\right)^2}_{\text{market clearing}}$$

Innovation 1: Market clearing layers

- Neural network first predicts

$$\mathcal{N}^{\text{pre}}_{\boldsymbol{\rho}}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, \tilde{b}^1_{t+1}, \ldots, \tilde{b}^{32}_{t+1}, \hat{p}^b_t]$$

- Apply transformation $m(\ldots, \cdot)$

$$[\hat{b}^1_{t+1}, \ldots, \hat{b}^{32}_{t+1}] = m\left(\mathcal{N}^{\text{pre}}_{\boldsymbol{\rho}}(\mathbf{x}_t), B\right) \text{ such that } B = \sum_{h=1}^{32} \hat{b}^h_{t+1}$$

- Put together

$$\mathcal{N}_{\boldsymbol{\rho}}(\mathbf{x}_t) := [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, \hat{b}^1_{t+1}, \ldots, \hat{b}^{32}_{t+1}, \hat{p}^b_t]$$

- Loss function now

$$\ell_{\boldsymbol{\rho}}(\mathbf{x}_t) := \underbrace{w_{hh,k}}_{\text{weight}} \underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon^{k,h}_t\right)^2\right)}_{\text{opt. cond. cap.}} + \underbrace{w_{hh,b}}_{\text{weight}} \underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon^{b,h}_t\right)^2\right)}_{\text{opt. cond. bond}} + \cancelto{= 0}{\underbrace{w_{mc,B}}_{\text{weight}} \underbrace{\left(\epsilon^B_t\right)^2}_{\text{market clearing}}}$$

Innovation 1: Market clearing layers

- Neural network first predicts

$$\mathcal{N}_{\boldsymbol{\rho}}^{\text{pre}}(\mathbf{x}_t) = [\hat{k}_{t+1}^{1}, \ldots, \hat{k}_{t+1}^{32}, \tilde{b}_{t+1}^{1}, \ldots, \tilde{b}_{t+1}^{32}, \hat{p}_t^{b}]$$

- Apply transformation $m(\ldots, \cdot)$

$$[\hat{b}_{t+1}^{1}, \ldots, \hat{b}_{t+1}^{32}] = m\left(\mathcal{N}_{\boldsymbol{\rho}}^{\text{pre}}(\mathbf{x}_t), B\right) \text{ such that } B = \sum_{h=1}^{32} \hat{b}_{t+1}^{h}$$

- Put together

$$\mathcal{N}_{\boldsymbol{\rho}}(\mathbf{x}_t) := [\hat{k}_{t+1}^{1}, \ldots, \hat{k}_{t+1}^{32}, \hat{b}_{t+1}^{1}, \ldots, \hat{b}_{t+1}^{32}, \hat{p}_t^{b}]$$

- Loss function now

$$\ell_{\boldsymbol{\rho}}(\mathbf{x}_t) := \underbrace{w_{hh,k}}_{\text{weight}} \underbrace{\left(\sum_{h=1}^{H-1}\left(\epsilon_t^{k,h}\right)^2\right)}_{\text{opt. cond. cap.}} + \underbrace{w_{hh,b}}_{\text{weight}} \underbrace{\left(\sum_{h=1}^{H-1}\left(\epsilon_t^{b,h}\right)^2\right)}_{\text{opt. cond. bond}} + \cancelto{=0}{\underbrace{w_{mc,B}}_{\text{weight}} \underbrace{\left(\epsilon_t^{B}\right)^2}_{\text{market clearing}}}$$

1. no need to learn economics we already know ex-ante
2. remaining loss easier to interpret
3. states simulated from the policy are always consistent with market clearing ▸ details
4. see Gopalakrishna et al. (2024) on how to ensure market clearing in continuous time models

Innovation 2: Stabilizing step-wise model transformations

- Single asset models are easy

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    - portfolio choice only pinned down at low errors in equilibrium conditions
    - but how do we get there?

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  2. start with single asset model

$$\mathcal{N}^1_{\boldsymbol{\rho}}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \dots, \hat{k}^{32}_{t+1}, 0 \times \hat{b}^1_{t+1}, \dots, 0 \times \hat{b}^{32}_{t+1}, \hat{p}^b_t], B^1 = 0$$

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  3. solve the model

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  4. train the neural network to predict the bond price (supervised, from zero liquidity limit)

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  3. solve the model
  4. train the neural network to predict the bond price (supervised, from zero liquidity limit)
  5. slowly introduce the second asset (such that the error remains low)

$$\mathcal{N}^2_{\boldsymbol{\rho}}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 1\times\hat{b}^1_{t+1}, \ldots, 1\times\hat{b}^{32}_{t+1}, \hat{p}^b_t], B^2 = 0.1$$

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  3. solve the model
  4. train the neural network to predict the bond price (supervised, from zero liquidity limit)
  5. slowly introduce the second asset (such that the error remains low)

$$\mathcal{N}^4_{\boldsymbol{\rho}}(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 1 \times \hat{b}^1_{t+1}, \ldots, 1 \times \hat{b}^{32}_{t+1}, \hat{p}^b_t], B^4 = 0.3$$

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  3. solve the model
  4. train the neural network to predict the bond price (supervised, from zero liquidity limit)
  5. slowly introduce the second asset (such that the error remains low)

$$\mathcal{N}^{\cdots}_{\boldsymbol{\rho}}(\mathbf{x}_t) = [\hat{k}^{1}_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 1 \times \hat{b}^{1}_{t+1}, \ldots, 1 \times \hat{b}^{32}_{t+1}, \hat{p}^{b}_{t}], B^{\cdots} = \ldots$$

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$$\mathcal{N}_{\boldsymbol{\rho}}^{1}(\mathbf{x}_t) = [\hat{k}_{t+1}^{1}, \ldots, \hat{k}_{t+1}^{32}, 0\times\hat{b}_{t+1}^{1}, \ldots, 0\times\hat{b}_{t+1}^{32}, \hat{p}_t^b], B^1 = 0$$

  3. solve the model
  4. train the neural network to predict the bond price (supervised, from zero liquidity limit)
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$$\mathcal{N}_{\boldsymbol{\rho}}^{100}(\mathbf{x}_t) = [\hat{k}_{t+1}^{1}, \ldots, \hat{k}_{t+1}^{32}, 1\times\hat{b}_{t+1}^{1}, \ldots, 1\times\hat{b}_{t+1}^{32}, \hat{p}_t^b], B^{100} = 10$$

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$$\mathcal{N}_{\boldsymbol{\rho}}^{100}(\mathbf{x}_t) = [\hat{k}_{t+1}^{1}, \ldots, \hat{k}_{t+1}^{32}, 1\times\hat{b}_{t+1}^{1}, \ldots, 1\times\hat{b}_{t+1}^{32}, \hat{p}_t^{b}], B^{100} = 10$$

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$$\mathcal{N}^{100}_{\boldsymbol{\rho}}(\mathbf{x}_t) = [\hat{k}^{1}_{t+1}, \dots, \hat{k}^{32}_{t+1}, 1\times\hat{b}^{1}_{t+1}, \dots, 1\times\hat{b}^{32}_{t+1}, \hat{p}^{b}_{t}], B^{100} = 10$$

  6. equilibrium errors always remain low

Application to our illustrative model

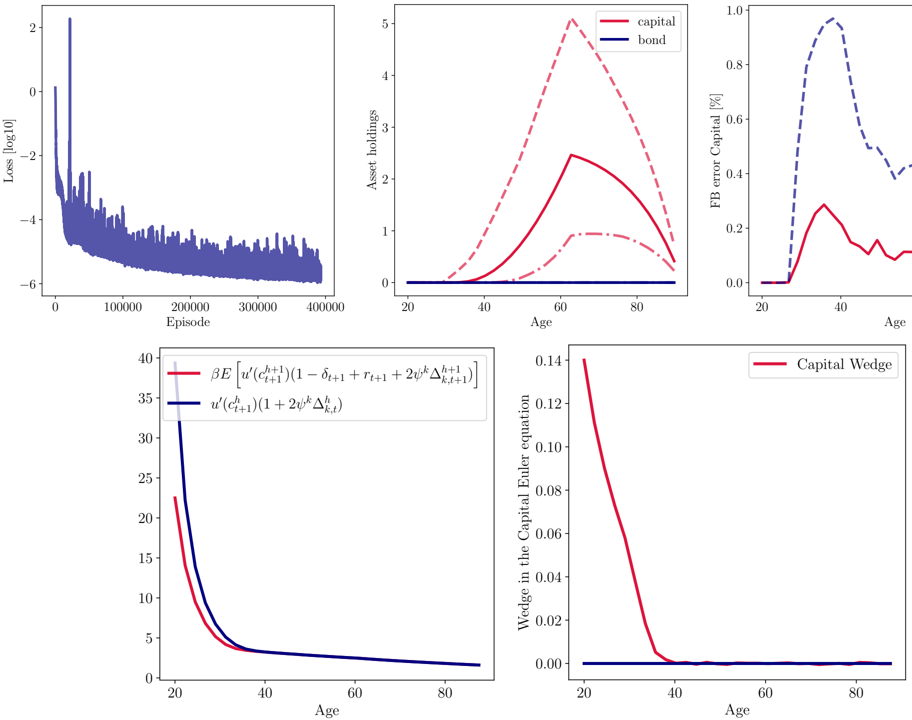
Step 1: Solve single asset model

- Borrowing constraint $\underline{b} = 0$, net-supply $B = 0$
- Neural network predicts

$$\mathcal{N}_{\boldsymbol{\rho}}^{\text{pre}}(\mathbf{x}_t) = [\hat{k}_{t+1}^{1}, \ldots, \hat{k}_{t+1}^{32}, 0 \times \tilde{b}_{t+1}^{1}, \ldots, 0 \times \tilde{b}_{t+1}^{32}, \hat{p}_t^b]$$
$$\Rightarrow \mathcal{N}_{\boldsymbol{\rho}}(\mathbf{x}_t) = [\hat{k}_{t+1}^{1}, \ldots, \hat{k}_{t+1}^{32}, 0, \ldots, 0, \hat{p}_t^b]$$

- Loss function

$$\ell_{\boldsymbol{\rho}}(\mathbf{x}_t) := 1 \times \underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon_t^{k,h} \right)^2 \right)}_{\text{opt. cond. cap.}} + \underbrace{0 \times \underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon_t^{b,h} \right)^2 \right)}_{\text{opt. cond. bond}}}_{=0}$$


Loss [log10]
2
0
−2
−4
−6
0
100000
200000
300000
400000
Episode
Asset holdings
capital
bond
0
1
2
3
4
5
20
40
60
80
Age
FB error Capital [%]
mean
99 percentile
0.0
0.2
0.4
0.6
0.8
1.0
20
40
60
80
Age
$\beta E\left[u'(c_{t+1}^{h+1})(1-\delta_{t+1}+r_{t+1}+2\psi^k\Delta_{k,t+1}^{h+1})\right]$
$u'(c_{t+1}^{h})(1+2\psi^k\Delta_{k,t}^{h})$
0
5
10
15
20
25
30
35
40
20
40
60
80
Age
Wedge in the Capital Euler equation
Capital Wedge
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
20
40
60
80
Age

Step 2: Pre-train bond price in the capital only model

- Keep borrowing constraint $\underline{b} = 0$, net-supply $B = 0$, and neural network masks

$$\mathcal{N}_\rho(\mathbf{x}_t) = [\hat{k}^1_{t+1}, \ldots, \hat{k}^{32}_{t+1}, 0, \ldots, 0, \hat{p}^b_t]$$

- In equilibrium we know that

$$p^b_t \geq \frac{\beta \mathrm{E}\left[u'(c^{h+1}_{t+1})\right]}{u'(c^h_t)}$$

  with equality for unconstrained agents.

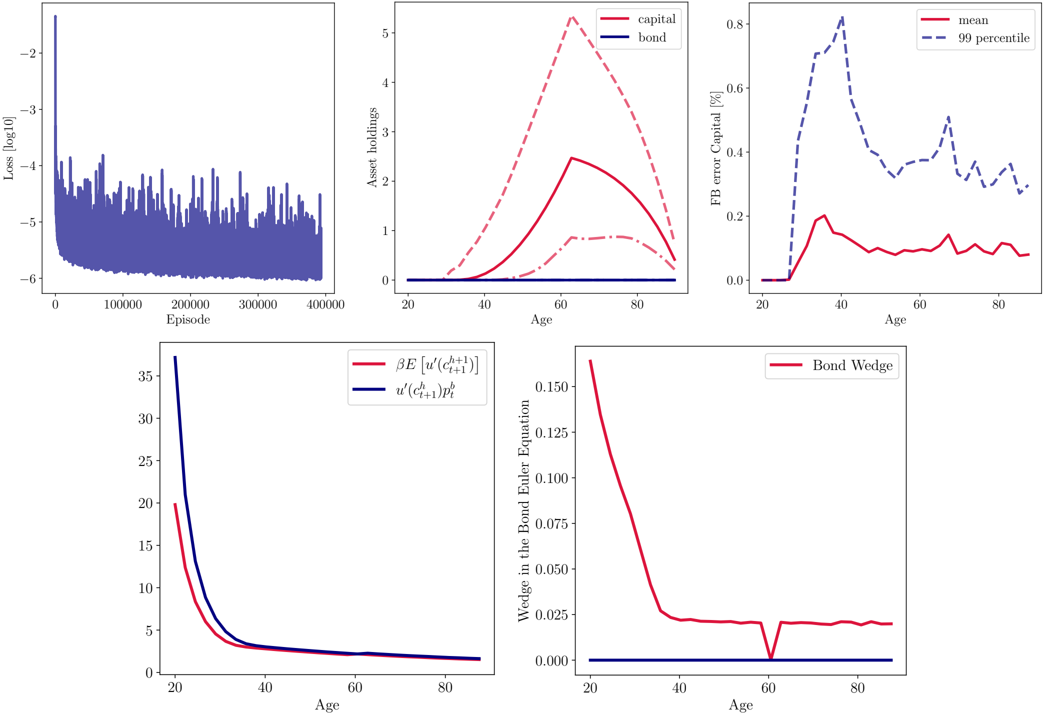
- With market clearing policies, we have a closed form expression for the bond price and can define pre-train price and error

$$p^{b,\text{pre-train}}_t := \max_{h \in \mathcal{H}} \left\{ \frac{\beta \mathrm{E}\left[u'(c^{h+1}_{t+1})\right]}{u'(c^h_t)} \right\}$$

$$\epsilon^{\text{pre-train}}_t := p^{b,\text{pre-train}}_t - \hat{p}^b_t$$

- Loss function

$$\ell_\rho(\mathbf{x}_t) := 1 \times \underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon^{k,h}_t\right)^2 \right)}_{\text{opt. cond. cap.}} + \underbrace{0 \times \underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon^{b,h}_t\right)^2 \right)}_{\text{opt. cond. bond}}}_{=0} + 1 \times \underbrace{\left(\epsilon^{\text{pre-train}}_t\right)^2}_{\substack{\text{price pre-train error} \\ \text{train supervised}}}$$


Loss [log10]
−2
−3
−4
−5
−6
0
100000
200000
300000
400000
Episode
Asset holdings
capital
bond
0
1
2
3
4
5
20
40
60
80
Age
FB error Capital [%]
mean
99 percentile
0.0
0.2
0.4
0.6
0.8
20
40
60
80
Age
$\beta E\left[u'(c_{t+1}^{h+1})\right]$
$u'(c_{t+1}^{h})p_t^b$
0
5
10
15
20
25
30
35
20
40
60
80
Age
Wedge in the Bond Euler Equation
Bond Wedge
0.000
0.025
0.050
0.075
0.100
0.125
0.150
20
40
60
80
Age

Step 3: Slowly increase bond supply

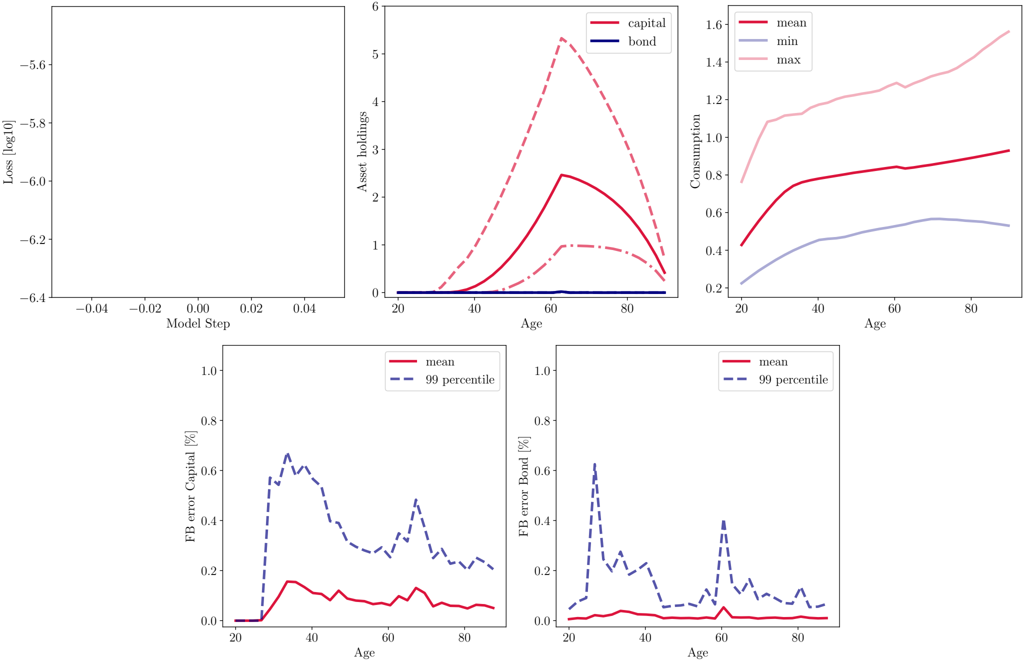
- Borrowing constraint $\underline{b} = 0$, increase net-supply from $B = 0.1$ to $B = 10$
- Neural network predicts

$$\mathcal{N}_{\boldsymbol{\rho}}^{\text{pre}}(\mathbf{x}_t) = [\hat{k}_{t+1}^1, \ldots, \hat{k}_{t+1}^{32}, \underbrace{0.01 \times \tilde{b}_{t+1}^1, \ldots, 0.01 \times \tilde{b}_{t+1}^{32}}_{\text{bond policies active}}, \hat{p}_t^b]$$

$$\Rightarrow \mathcal{N}_{\boldsymbol{\rho}}(\mathbf{x}_t) = [\hat{k}_{t+1}^1, \ldots, \hat{k}_{t+1}^{32}, \underbrace{\hat{b}_{t+1}^1, \ldots, \hat{b}_{t+1}^{32}}_{\text{always add up the B}}, \hat{p}_t^b]$$

- Loss function

$$\ell_{\boldsymbol{\rho}}(\mathbf{x}_t) := 1 \times \underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon_t^{k,h} \right)^2 \right)}_{\text{opt. cond. cap.}} + \underbrace{1 \times}_{\text{bond equ. cond. active}} \underbrace{\left(\sum_{h=1}^{H-1} \left(\epsilon_t^{b,h} \right)^2 \right)}_{\text{opt. cond. bond}}$$


Loss [log10]
−5.6
−5.8
−6.0
−6.2
−6.4
−0.04
−0.02
0.00
0.02
0.04
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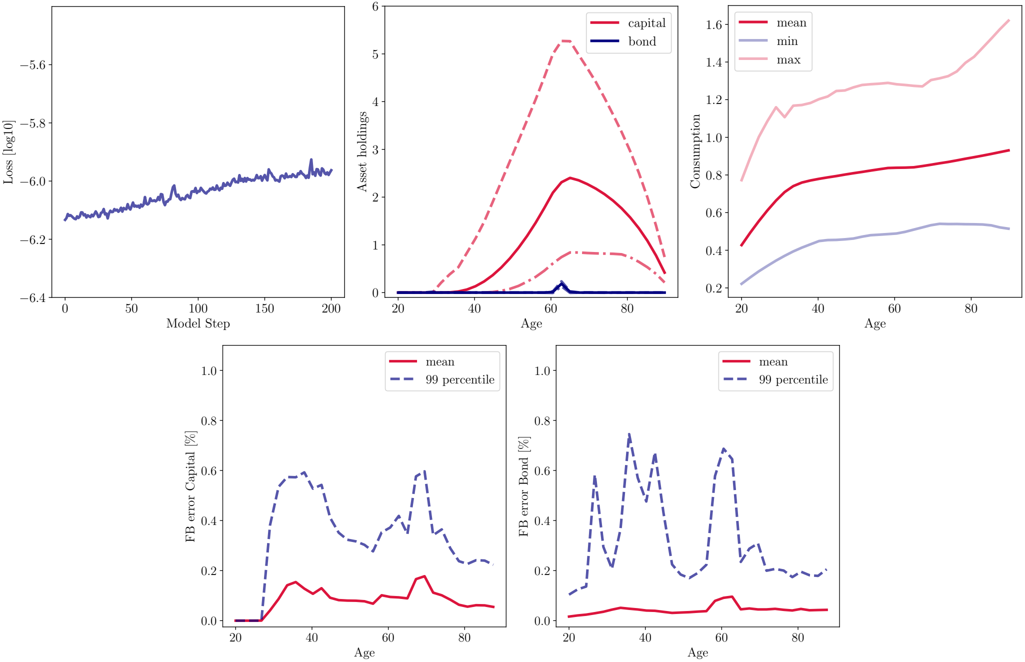
▸ Back


Loss [log10]
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FB error Bond [%]
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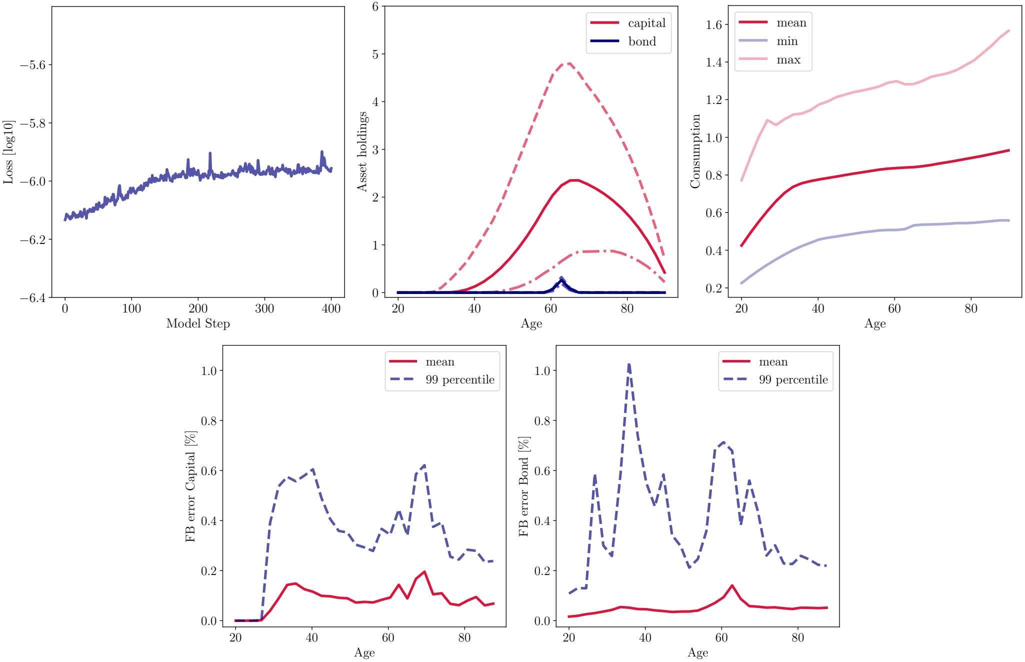
▸ Back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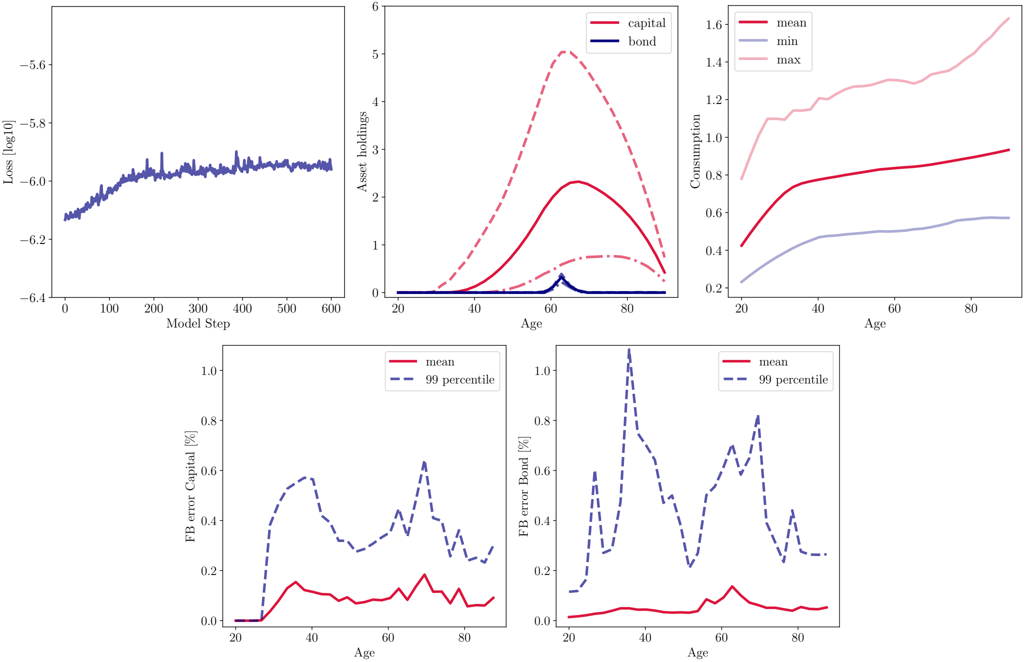
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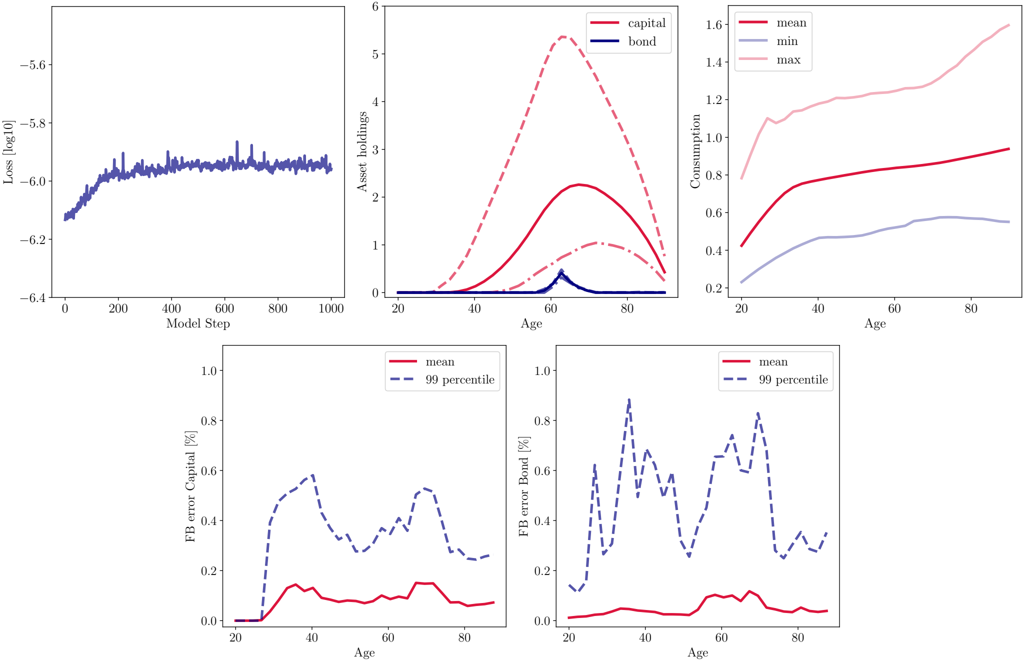
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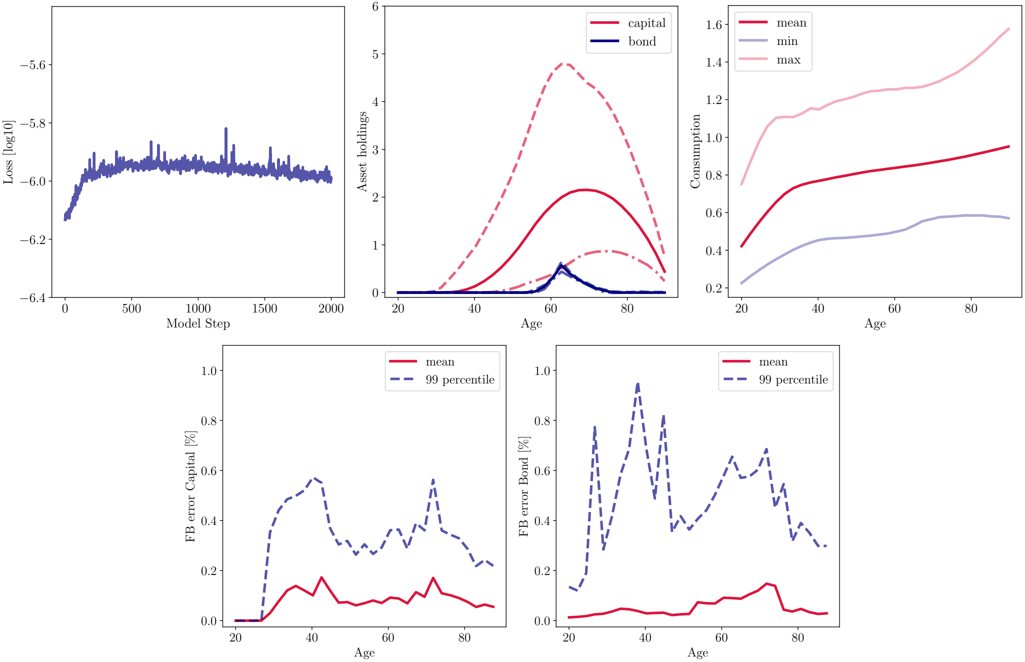
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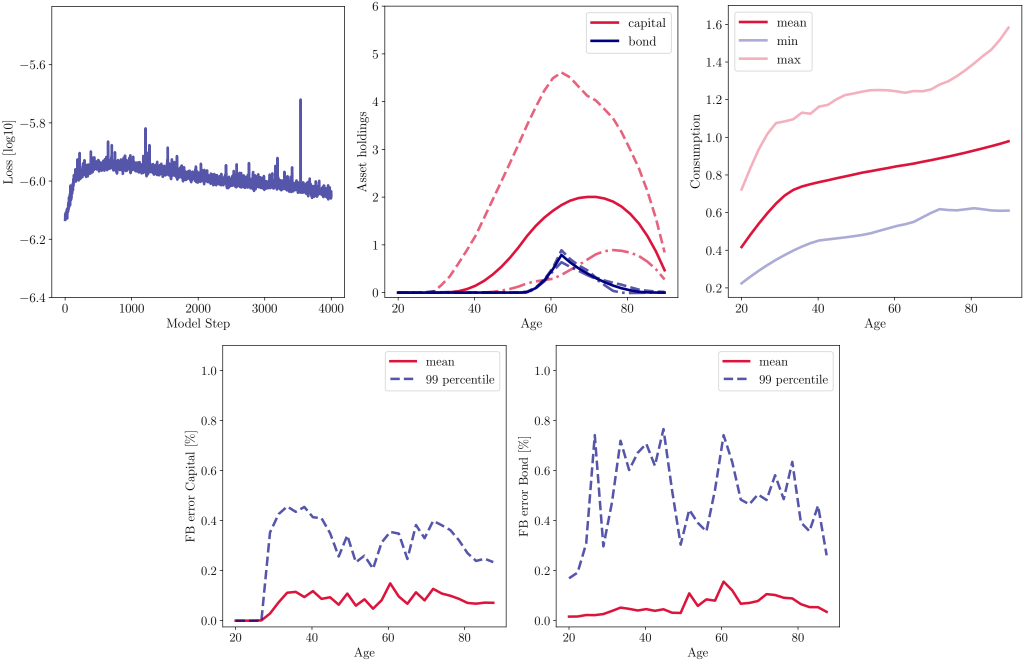
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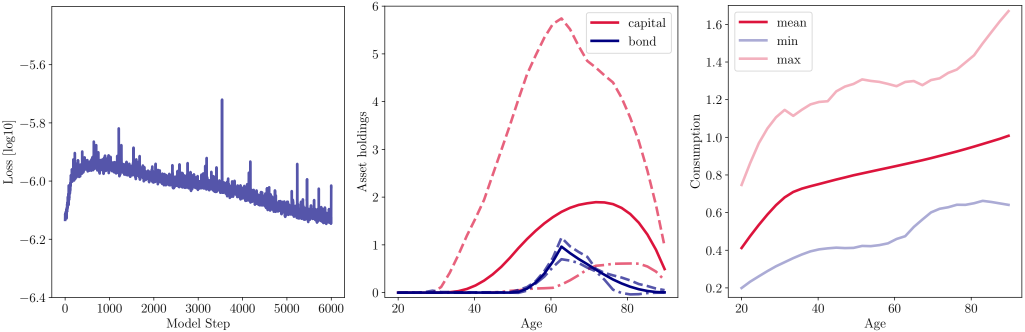
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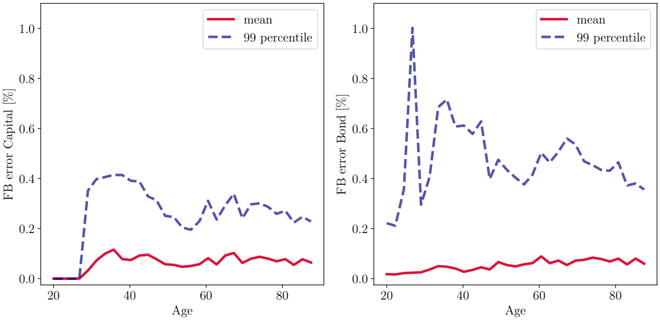
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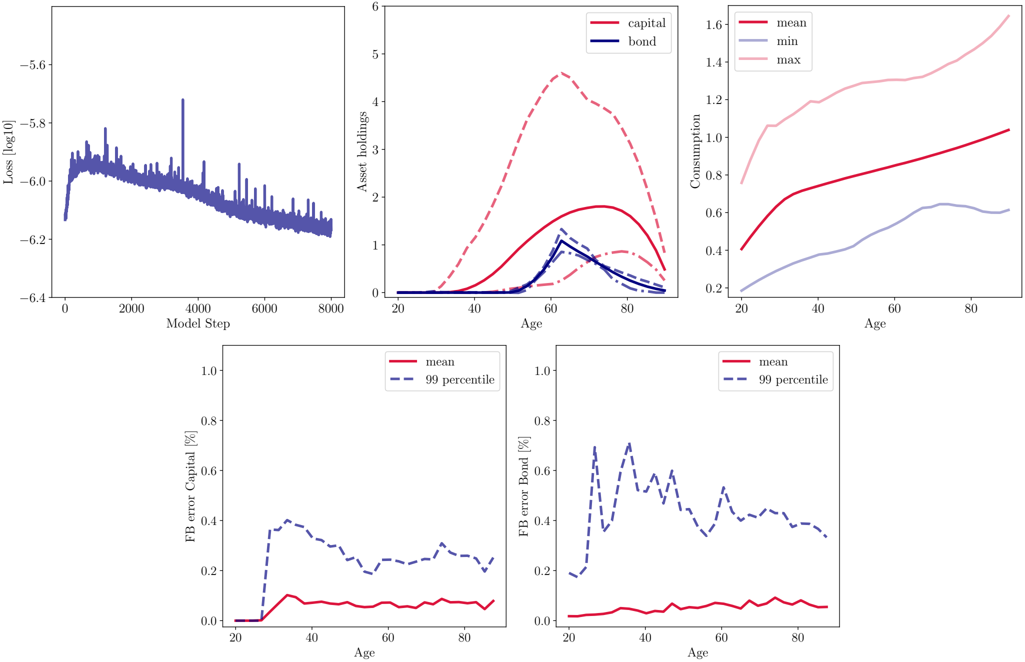

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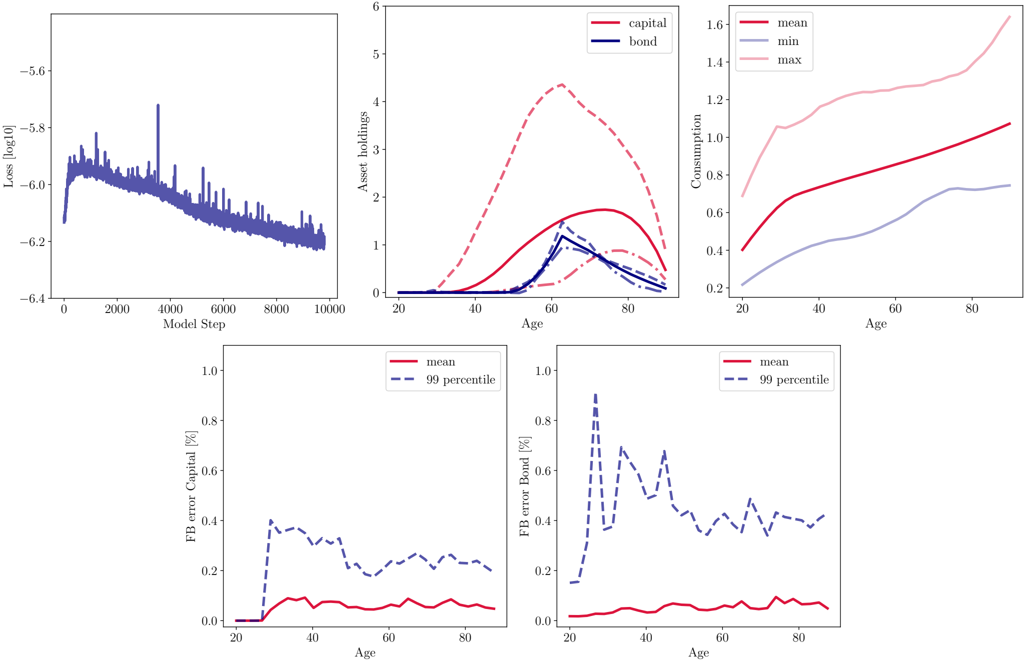
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▸ Back

Deep Equilibrium Nets

Violations of equilibrium conditions as loss function

Basic idea in Azinovic et al. (2022): write equilibrium conditions as

$$\mathbf{G}(\mathbf{x}, \mathbf{f}) = 0 \ \forall \mathbf{x}$$

$$\begin{aligned} \mathbf{G} &: \text{ equilibrium conditions: FOC's, market clearing, Bellman equations, } \ldots \\ \mathbf{x} &: \text{ state of the economy} \\ \mathbf{f} &: \text{ equilibrium functions.} \end{aligned}$$

Approximate **f** by neural network $\mathcal{N}_{\boldsymbol{\rho}}$

$$\mathcal{N}_{\boldsymbol{\rho}}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x})$$

Given network parameters $\boldsymbol{\rho}$, we define a loss function

$$\ell_{\boldsymbol{\rho}} := \frac{1}{N_{\text{path length}}} \sum_{\mathbf{x}_i \text{ on sim. path}} \left(\mathbf{G}(\mathbf{x}_i, \mathcal{N}_{\boldsymbol{\rho}})\right)^2$$

If $\ell_{\boldsymbol{\rho}} \approx 0$, then $\mathcal{N}_{\boldsymbol{\rho}}(\mathbf{x})$ gives us a good approximation of $\mathbf{f}(\mathbf{x})$.

▸ What are Neural Nets? ▸ Why use Neural Nets?

Training DEQNs

1. Simulate a sequence of states $\mathcal{D}^i_{\text{train}} \leftarrow \{\mathbf{x}^i_1, \mathbf{x}^i_2, \ldots, \mathbf{x}^i_T\}$ from the policy encoded by the network parameters $\boldsymbol{\rho}^i$.
2. Evaluate the errors of the equilibrium conditions on the newly generated set $\mathcal{D}_{\text{train}}$.
3. If the error statistics are not low enough:
   - 3.1 update the parameters of the neural network with a gradient descent step (or a variant):
   $$\rho^{i+1}_k = \rho^i_k - \alpha_{\text{learn}} \frac{\partial \ell_{\mathcal{D}^i_{\text{train}}}(\boldsymbol{\rho}^i)}{\partial \rho^i_k}.$$
   - 3.2 set new starting states for simulation: $\mathbf{x}^{i+1}_0 = \mathbf{x}^i_T$.
   - 3.3 increase i by one and go back to step 1.

▸ back

Deep Neural Networks

What is a deep neural net?

Consider:

$$\textbf{input} := \mathbf{x} \to W_{\boldsymbol{\rho}}^{1}\mathbf{x} + \mathbf{b}_{\boldsymbol{\rho}}^{1} =: \textbf{hidden 1}$$

What is a deep neural net?

Consider:

$$\begin{aligned} &\textbf{input} := \mathbf{x} \to W_{\boldsymbol{\rho}}^{1}\mathbf{x} + \mathbf{b}_{\boldsymbol{\rho}}^{1} =: \textbf{hidden 1} \\ \to\ &\textbf{hidden 1} \to W_{\boldsymbol{\rho}}^{2}(\textbf{hidden 1}) + \mathbf{b}_{\boldsymbol{\rho}}^{2} =: \textbf{hidden 2} \end{aligned}$$

What is a deep neural net?

Consider:

$$\begin{aligned}
&\textbf{input} := \mathbf{x} \to W_{\boldsymbol{\rho}}^{1}\mathbf{x} + \mathbf{b}_{\boldsymbol{\rho}}^{1} =: \textbf{hidden 1} \\
&\to \textbf{hidden 1} \to W_{\boldsymbol{\rho}}^{2}(\textbf{hidden 1}) + \mathbf{b}_{\boldsymbol{\rho}}^{2} =: \textbf{hidden 2} \\
&\to \textbf{hidden 2} \to W_{\boldsymbol{\rho}}^{3}(\textbf{hidden 2}) + \mathbf{b}_{\boldsymbol{\rho}}^{3} =: \textbf{output}
\end{aligned}$$

What is a deep neural net?

Consider:

$$\begin{aligned} \textbf{input} := \mathbf{x} &\to W^1_{\boldsymbol{\rho}}\mathbf{x} + \mathbf{b}^1_{\boldsymbol{\rho}} =: \textbf{hidden 1} \\ \to \textbf{hidden 1} &\to W^2_{\boldsymbol{\rho}}(\textbf{hidden 1}) + \mathbf{b}^2_{\boldsymbol{\rho}} =: \textbf{hidden 2} \\ \to \textbf{hidden 2} &\to W^3_{\boldsymbol{\rho}}(\textbf{hidden 2}) + \mathbf{b}^3_{\boldsymbol{\rho}} =: \textbf{output} \end{aligned}$$

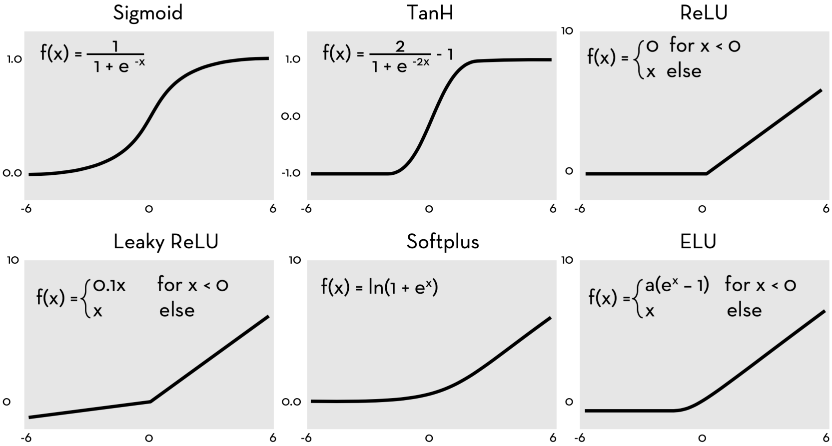
The parameters $\boldsymbol{\rho}$ of this procedure are the entries of the matrices $(W^1_{\boldsymbol{\rho}},\ W^2_{\boldsymbol{\rho}},\ W^3_{\boldsymbol{\rho}})$ and vectors $(\mathbf{b}^1_{\boldsymbol{\rho}},\ \mathbf{b}^2_{\boldsymbol{\rho}},\ \mathbf{b}^3_{\boldsymbol{\rho}})$.

What is a deep neural net? (cont.)

So far we have a concatenation of affine maps and therefore an afffine map.

What is a deep neural net? (cont.)

So far we have a concatenation of affine maps and therefore an afffine map.
Next ingredient: activation functions ϕ^1, ϕ^2, ϕ^3. Activation functions could be any function, but popular are:



What is a deep neural net? (cont.)

Now we get:

$$\begin{aligned}\textbf{input} := \mathbf{x} &\to \phi^1(W^1_{\boldsymbol{\rho}}\mathbf{x} + \mathbf{b}^1_{\boldsymbol{\rho}}) =: \textbf{hidden 1}\\ \to \textbf{hidden 1} &\to \phi^2(W^2_{\boldsymbol{\rho}}(\textbf{hidden 1}) + \mathbf{b}^2_{\boldsymbol{\rho}}) =: \textbf{hidden 2}\\ \to \textbf{hidden 2} &\to \phi^3(W^3_{\boldsymbol{\rho}}(\textbf{hidden 2}) + \mathbf{b}^3_{\boldsymbol{\rho}}) =: \textbf{output}\end{aligned}$$

The neural net is then given by the choice of activation functions and the parameters $\boldsymbol{\rho}$.

▸ back

Why neural networks?

Approximation method	High-dimensional input	Can resolve local features accurately	Irregularly shaped domain	Large amount of data
Polynomials	✓	✗	✓	✓
Splines	✗	✓	✗	✓
Adaptive (sparse) grids	✓	✓	✗	✓
Gaussian processes	✓	✓	✓	✗
Deep neural networks	✓	✓	✓	✓

Table: Taken from Azinovic et al. (2022).

▸ back

Innovation 1: Details on the market clearing transformation function

- Simple market clearing layer: subtract excess demand ED_t from initial predictions

$$ED_t := \sum_{h \in \mathcal{H}} \tilde{b}^h_{t+1} - B$$

$$\hat{b}^h_{t+1} := \tilde{b}^h_{t+1} - \frac{1}{H} ED_t$$

- Why this adjustment?

→ we try to minimize the modification to the initial predictions $\{\tilde{b}^h_{t+1}\}_{h \in \mathcal{H}}$.

- Final predictions $\{\hat{b}^h_{t+1}\}_{h \in \mathcal{H}}$ solve

$$\underset{\{x^h_{t+1}\}_{h \in \mathcal{H}}}{\arg\min} \sum_{h \in \mathcal{H}} \left(x^h_{t+1} - \tilde{b}^h_{t+1} \right)^2$$

subject to

$$\sum_{h \in \mathcal{H}} x^h_{t+1} = B$$

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subject to

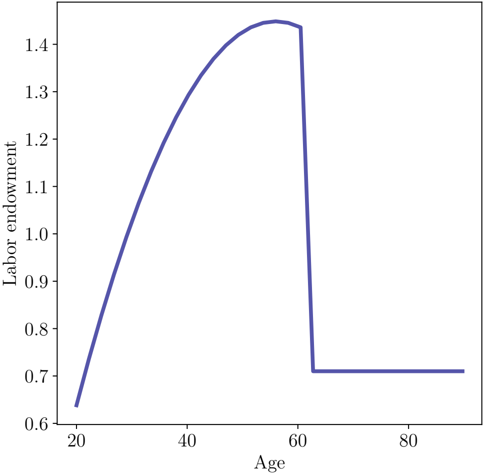
$$\sum_{h \in \mathcal{H}} x^h_{t+1} = B$$

- In the paper: enforcing market clearing & borrowing constraints using implicit layer

▸ back

Parameters

Parameters	H	β	γ	ψ	ρ	σ	α
Values	32	0.912	4	0.1	0.693	0.052	0.333
Meaning	num. age groups	patience	RRA	adj. costs	pers. tfp	std. innov. tfp	cap. share



Households' optimality conditions

$$\left.\begin{aligned} 1 &= \frac{\beta \mathrm{E}\left[u'(c_{t+1}^{h+1})(1-\delta_{t+1}+r_{t+1}+2\psi^k\Delta_{k,t+1}^{h+1}\right]+\mu_t^h}{(1+2\psi^k\Delta_{k,t}^h)u'(c_t^h)} \\ k_t^h &\geq 0 \\ \mu_t^h &\geq 0 \\ k_t^h\mu_t^h &= 0 \end{aligned}\right\} \Leftrightarrow \epsilon_t^{k,h} := \psi^{FB}\left(\frac{u'^{-1}\left(\beta \mathrm{E}\left[u'(c_{t+1}^{h+1})\frac{(1-\delta_{t+1}+r_{t+1}+2\psi^k\Delta_{k,t+1}^{h+1}}{(1+2\psi^k\Delta_{k,t}^h)}\right]\right)}{c_t^h}-1, \frac{k_t^h}{c_t^h}\right)$$

$$\left.\begin{aligned} 1 &= \frac{\beta \mathrm{E}\left[u'(c_{t+1}^{h+1})\right]+\lambda_t^h}{p_t^b u'(c_t^h)} \\ b_t^h - \underline{b} &\geq 0 \\ \lambda_t^h &\geq 0 \\ (b_t^h - \underline{b}^h)\lambda_t^h &= 0 \end{aligned}\right\} \Leftrightarrow \epsilon_t^{b,h} := \psi^{FB}\left(\frac{u'^{-1}\left(\beta \mathrm{E}\left[\frac{1}{p_t^b}u'(c_{t+1}^{h+1})\right]\right)}{c_t^h}-1, \frac{b_t^h-\underline{b}}{c_t^h}\right)$$

where

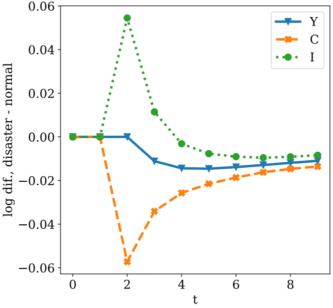
$$\psi^{FB}(a,b) := a+b-\sqrt{a^2+b^2}$$

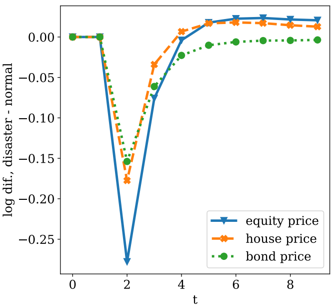
▸ back

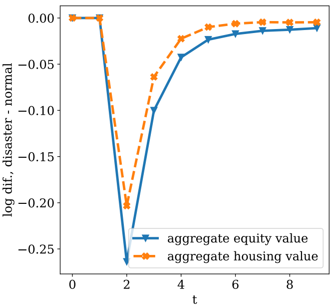
Pure depreciation disasters

Depreciation disaster: aggregate response

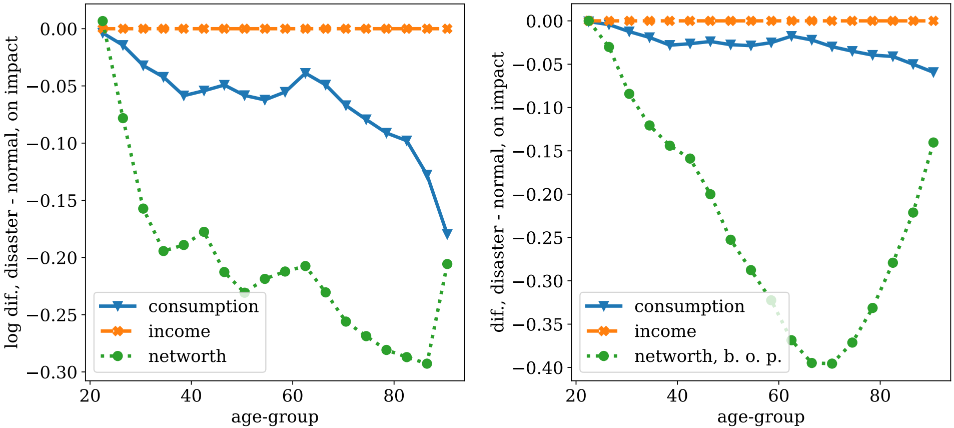
- So far we considered TFP disaster in the spirit of Barro (2006)
- Now we consider a disaster, in which the depreciation of capital increases by 50%



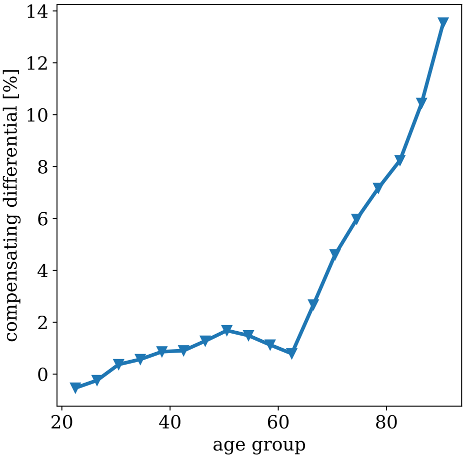




Depreciation disaster: intergenerational impact



Depreciation disaster: welfare



▸ back

Details on the normalization

We assume

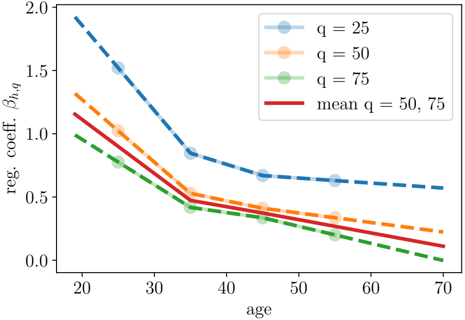
$$l_t^h = \begin{cases} l_{ss}^h \frac{\left(\frac{Y_{t+1}}{Y_t}\right)^{\zeta_h}}{\sum_h \mu_h l_{ss}^h \left(\frac{Y_{t+1}}{Y_t}\right)^{\zeta_h}} & \text{for } h < h^{\text{retirement}} \\ 0 & \text{for } h \geq h^{\text{retirement}}, \end{cases}$$

where ζ_h captures the age-dependent exposure of labor income to aggregate fluctuations.

▸ back

Details on the data by Guvenen et al. (2014)

- Guvenen et al. (2014) provide publicly available data on real earnings by age and income percentile constructed from the U.S. Social Security Administration's Master Earnings file
- We look at annual real earnings growth between 1979 and 2010
- We look at age-groups 25, 35, 45, and 55
- We look at income percentiles 25, 50, and 75
- Data for real output per capita is obtained from the FRED database



back

Inspecting the mechanism

▸ back