Multi-unit auctions with uncertain supply and single-unit demand*

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Abstract

We study multi-unit auctions where bidders have single-unit demand and asymmetric information. For symmetric equilibria, we identify circumstances where uniform-pricing is better for the auctioneer than pay-as-bid pricing. In this case, the auctioneer also benefits from disclosing information before the auction, including its traded volume. But an issue with the uniform-price auction is that seemingly collusive equilibria can also exist. We show that such outcomes are less likely if the traded volume is uncertain. If bidders are asymmetric ex-ante, then both a price floor and a price cap are normally needed to get a unique equilibrium, which is well behaved.

Key words: Uniform price auction, asymmetric information, publicity effect

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1 INTRODUCTION

Commodities and financial instruments are often traded in multi-unit auctions. For example, treasury auctions in the U.S. and wholesale electricity markets around the world are cleared by a uniform-price auction. Most treasury auctions in Europe, on the other hand, use pay-as-bid auctions. In electric-power systems, pay-as-bid auctions are often used when procuring ancillary services.[1] The traded volume is uncertain in wholesale electricity markets and in some treasury auctions.[2] In this paper, we compare the two auction formats, taking into account that information can be asymmetric and that the traded volume can be uncertain. Under our assumptions, uniform-pricing is better for the auctioneer and the auctioneer benefits from disclosing its information,[3] including the volume that it will trade, as long as bidders select well-behaved equilibria.

A concern with uniform-price auctions is that they can result in outcomes with prices at the collusive level. This is known from theoretical studies of divisible-good auctions, such as Wilson (1979), Klemperer & Meyer (1989), von der Fehr & Harbord (1993), and Vives (2011). In practice, seemingly collusive equilibria in uniform-price auctions have for example been observed in the procurement of electric-generation capacity (Schwenen, 2015).[4] Another of our contributions is that we identify circumstances for which well-behaved equilibria can be ensured in the uniform-price auction.

[1] Ancillary services are for example used to control the frequency and voltage in the power system.

[2] In electricity markets, producers submit offers before the level of demand and amount of available production capacity are fully known. In this case, the traded volume of strategic producers is uncertain due to demand shocks and intermittent output from non-strategic, renewable energy sources. In Mexico, Finland and Italy, the treasury sometimes reduces the quantity of issued bonds after the bids have been received (McAdams, 2007). Some treasury auctions in U.S. have an uncertain amount of non-competitive bids from many small non-strategic investors (Wang and Zender, 2002).

[3] Disclosure of information is for example a topical issue for European electricity markets. EU has improved transparency of these markets during the last years. In March 2023, EU proposed that transparency of electricity markets should increase even further.

[4] The purpose of the capacity payment is to increase the production capacity in the market, which lowers the risk of having blackouts. Several electricity markets in the U.S. have had problems with seemingly collusive prices when procuring capacity (Aagaard & Kleit, 2022).

We prove our results by extending the single-object settings studied by Milgrom & Weber (1982) and Blume & Heidhues (2004), respectively, to a multi-unit setting, where the traded volume of the auctioneer can be uncertain. Similar to them, we consider a sales auction, but results are analogous for procurement auctions. We make the simplification that bidders have single-unit demand, i.e. each bidder buys at most one good.[5] Hence, the market is competitive in the sense that strategic demand reduction is not an issue. Similar simplifications of multi-unit auctions have been made by Vickrey (1961), Milgrom (1981), Weber (1983) and Krishna (2010).

Each bidder receives private information in the form of a signal and it is used to estimate the value of the traded good. In our setting a signal could be correlated with the auctioneer's supply. We allow the auctioneer to also receive a private signal, which it can choose to disclose to all bidders. The auctioneer could also disclose the number of units that it will sell. We study a single-shot game, but in practice the auctioneer could disclose aggregated bid data and traded volumes for previous auctions.

We start by studying symmetric equilibria for bidders that are symmetric ex ante, before private information has been received. Values of bidders are interdependent. This means that a bidder has an estimate of the good's value, the bidder's signal, and that it would get a (weakly) better estimate if it could also observe the signals of the competitors and the auctioneer. All of these signals are assumed to be affiliated[6]. The information structure is similar to Milgrom & Weber (1982), but we allow the number of sold units, Z, to be larger than one and uncertain, and make the additional assumption that $-Z$ is affiliated with the signals. This means that Z and signals are required to be negatively correlated (or Z may be independent of the signals). This would for example be the case if the auctioneer and other sellers (outside the model) tend to have a shortage of the good at the same time, so that prices increase in alternative markets, which increases the value of the good. We also make the

[5] Each bidder could have a limited storage capacity, be liquidity constrained or have preferences, such that at most one unit would be bought. In an analogous procurement auction each bidder has capacity to produce at most one unit of the good.

[6] Affiliation is a strong version of positive correlation. One implication of the affiliation property is that if the observed signal of a bidder increases, then, conditional on this increase, the expected value of the other signals will also increase.

technical assumption that signals remain affiliated if we condition on the value of Z.[7] Many of the results in Milgrom & Weber (1982) can be generalized to multi-unit auctions with single-unit demand and uncertain demand as long as our assumptions are satisfied.

For the symmetric model, we show that there is a well-behaved, symmetric equilibrium in the uniform-price auction. This equilibrium is efficient and each of the bids is continuous and monotonically increasing with respect to the signal. If Z is fixed, we can show that it is the only symmetric equilibrium. But we also present an example showing that if demand uncertainty does not satisfy the technical assumption that signals remain affiliated if we condition on the value of Z, then it is possible that symmetric, monotonic equilibria do not exist and that a non-monotonic and discontinuous symmetric equilibrium exists instead.

For well-behaved symmetric equilibria, we show that an auctioneer will find it beneficial (its expected revenue weakly increases) to ex ante disclose its own signal. This is sometimes referred to as a publicity effect. Moreover, we find that uniform pricing gives a (weakly) higher revenue for the auctioneer in comparison to pay-as-bid pricing. The ranking result is proved by means of a linkage-principle argument that we have generalized to a multi-unit auction with single-unit demand and uncertain supply, which could be correlated with the signals. Related results have been proved when the auctioneer's volume is preannounced (Weber, 1983)[8] or independent of signals (Holmberg & Wolak, 2018).[9] Perry & Reny (1999) show by an example that the publicity effect does not always hold in a Vickrey-Clarke-Groves (VCG) auction with multi-unit demand, even if signals are affiliated.

A new type of result in our study is that the auctioneer also benefits from disclosing its traded volume, or other information that help bidders to predict sales of the auctioneer. Pycia & Woodward (2021) find a somewhat similar result, but it is driven

[7] If Z were a continuous random variable this property would follow automatically when $-Z$ (or Z) is affiliated with the other signals. But this is not necessarily the case in our model where Z is a discrete random variable.

[8] Weber (1983) ranks uniform and pay-as-bid auctions, but does not study the publicity effect.

[9] Holmberg & Wolak (2018) consider a divisible-good market with flat bids. This setting is more complicated as a bid can be partly accepted. Hence, their study is restricted to duopoly markets and a restrictive information structure.

by effects related to strategic demand reduction and/or equilibrium selection. They consider a uniform-price auction where values are common knowledge among bidders. In our setting, disclosure of information, including the traded volume, would not have any effect if each bidder knows its valuation of the good, as long as bids are in accordance with a well-behaved symmetric equilibrium.

A problem with uniform-price auctions is that ill-behaved and inefficient equilibria can exist. We study this issue in detail, and how such equilibria can be prevented. To make progress, we simplify the information structure by assuming that each bidder has full information of its valuation of the good, but is not informed of competitors' valuations. Bidders can be asymmetric ex ante. We solve for all pure-strategy Bayesian Nash equilibria, including asymmetric equilibria.[10]

For the asymmetric model, we show that inefficient equilibria with prices at the collusive level can exist in uniform-price auctions, even if we assume single-unit demand so that bidders have limited market power. We refer to them as high-low equilibria, because some bidders always bid high and others always bid low.[11] The high bids are above the maximum valuation of the low bidders, and the low bids are below the minimum valuation of the high bidders. The high bids are always accepted and the low bids are always rejected. The highest bid among the low bids sets the price. For the case with at least two objects and at least two more bidders than objects, we show that any uncertainty in the auctioneer's supply will knock out high-low equilibria in uniform-price auctions.

If all bidders have the same value range, then there is a unique equilibrium, which is well-behaved, if there is any doubt what the auctioneer's supply will be. If supply is certain, then an effective price floor or price cap, which is binding with a positive probability, will give a unique equilibrium. Hence, there are circumstances where introducing a maximum price can actually increase the revenue of an auctioneer that is selling goods, by eliminating the high-low equilibrium. Single-object, second-price

[10] Our methodology would allow us to also characterize all mixed-strategy equilibria. But this would make the presentation more complicated and technical. Also, allowing for such equilibria would not change our conclusions for the private-value model in any substantial way.

[11] Blume et al. (2009) refer to the high-low equilibrium as the second class of equilibria in Vickrey auctions.

auctions are special.[12] In this case, there is a range of partial high-low equilibria, where bidders play the high-low strategy when observing signals below some threshold (Blume & Heidhues, 2004).[13] In this case, an effective price floor gives uniqueness (Blume & Heidhues, 2004), but we find that a price cap does not ensure uniqueness. Another special case is when there is exactly one more bidder than the number of objects. In this case, it is the other way around. There exists a range of partial high-low equilibria where bidders play the high-low strategy when observing signals above some threshold. In this case, an effective price cap gives a unique equilibrium, but not a price floor.

We also study a setting where bidders have different value ranges. In this case, a range of partial high-low equilibria exist, even if there are at least two objects and at least two more bidders than objects. For sufficiently low bids, below the lower value bound of $\overline{Z}-1$ bidders, where $\overline{Z}$ is the maximum supply, there is at most one bid in the range that can be accepted. There are inefficient equilibria for which the high-low strategy is played in that price range, which is somewhat similar to the partial high-low equilibria in the single-object auction. Analogously, there is a range of partial high-low equilibria where the high-low strategy is played at sufficiently high prices, where at most one bid in that range is rejected. This corresponds to the inefficient equilibria that exist in the reflected version of the single-object auction. If bidders have different bounds both at the top and bottom, then both an effective price floor and an effective price cap are needed to get a unique equilibrium. When unique equilibria occur they are well-behaved.

Existence of high-low equilibria in a one-shot game implies that a collusive agreement is self enforcing once it has been established, so that a bidding ring would be stable. Hence, our results imply that uncertainty in the auctioneer's supply, an effective price cap, an effective price floor and less heterogenous bidders reduce the risk of getting inefficient outcomes in a one-shot game and also reduce the risk of tacit collusion in repeated uniform-price auctions.

Our analysis of uniform-price auctions shows that this auction format has an

[12] We consider a uniform-price auction with single-unit demand, where the price is set by the highest rejected bid. In the single-object case this corresponds to a second-price auction.

[13] Blume et al. (2009) refer to this as the first-class of equilibria in Vickrey auctions.

invariance/equivalence property. If there is an equilibrium in an auction with n bidders and Z goods, then there is a corresponding equilibrium in a transformed auction with n bidders and $n-Z$ goods, if the signs of all values and bids are reversed. We refer to the transformed auction as the reflected auction. As an example, the reflected version of a single-object, sales auction with n bidders is a sales auction with n bidders and $n-1$ goods. This explains why a price floor is needed to get uniqueness in the single-object auction and a price cap in the reflected auction.

The equivalence property of the reflected auction significantly simplified some of our proofs, and improves the intuitive understanding of our asymmetric equilibrium results. We have not seen similar results in the previous auction literature, but in spirit this result is somewhat related to game transformations and strategic equivalence in games studied by Thompson (1952) and Morris & Ui (2004).

The remainder of the article is organized as follows. Our symmetric, uniform-price model with affiliated values is introduced in Section 2. In Section 3, we show that this model has a well-behaved symmetric equilibrium, which we characterize. In Section 4, we extend the linkage principle to a multi-unit auction with single-unit demand and uncertain supply. In Section 5 we study an auction with pay-as-bid pricing, and how it compares with the uniform-price auction. Section 6 proves the equivalence property of the reflected auction. In Section 7, we analyse the private-value model, which allows for asymmetric bidders. The proofs of Propositions and Theorems are in the Appendix. Proofs of some technical lemmas are in the on-line Appendix.

2 AFFILIATED-SIGNALS MODEL

Our assumptions for the symmetric model are similar to Milgrom & Weber (1982) and Weber (1983). We assume that signals are affiliated, that bidders are symmetric ex ante (before signals are received) and that they have single-unit demand and are risk-neutral. Another similarity with Milgrom & Weber (1982) is that, in Sections 3-5, we focus on pure-strategy equilibria that are symmetric ex ante. Asymmetric equilibria will be considered in Sections 6-7. Our analysis extends Milgrom & Weber (1982) in that we allow bid functions to be non-monotonic with respect to signals. Also, we allow the auctioneer to sell more than one unit and we allow the auctioneer's supply to be uncertain. We let Z be the number of items that are auctioned, with

$\underline{Z} \leq Z \leq \overline{Z}$. In the case that $\underline{Z} < \overline{Z}$, Z may take more than one value, and Z is unknown to the bidders. Moreover we assume that each value Z with $\underline{Z} \leq Z \leq \overline{Z}$ has a positive probability, irrespective of signals observed by bidders.

Each bidder $i \in \{1, \ldots, n\}$ receives a private signal X_i which has information about the value of an object. Similar to Milgrom & Weber (1982) one could think of the signal as a value estimate. We allow the signal to be correlated with Z. Hence, it could potentially be used to predict Z. Let $\boldsymbol{X}$ be a vector with all private signals and let $\boldsymbol{X}_{-i}$ be a vector with all private signals except for X_i. In addition, we let $\boldsymbol{S} = [S_1, \ldots, S_m]$ be a vector with m signals, which are informative of the value of the good. Some of these additional signals might be observed by the auctioneer, who might consider disclosing this information to all bidders.[14] Sometimes we find it convenient to write $\widetilde{\boldsymbol{X}}$ for the set of affiliated signals, (excluding X_1), $\widetilde{\boldsymbol{X}} = \{S_1, ..., S_m, X_2, ...X_n\}$. The value $V_i = u_i(\boldsymbol{S}, \boldsymbol{X})$ of the object to bidder i will depend on all signals, including signals that are not observed by the bidder. Note that the value does not directly depend on Z. We make the following standing assumptions for the affiliated-values model:

Assumption 1: There is a function u on $\mathbb{R}^{m+n}$ such that for all i, $u_i(\boldsymbol{S}, \boldsymbol{X}) = u(\boldsymbol{S}, X_i, \boldsymbol{X}_{-i})$, where u is symmetric in its last $n-1$ arguments. Hence, all of the bidders' valuations depend on $\boldsymbol{S}$ in the same manner, and bidders valuations are symmetric with respect to the private signals of competitors.

Assumption 2: The function u is non-negative, is continuous and non-decreasing in its variables, and is strictly increasing in X_i.[15]

Assumption 3: For each bidder i, the expected value $\mathbb{E}[V_i]$ conditioned on a subset of signals S and X is defined (i.e. bounded) no matter what subset is chosen.[16]

In our model $f(\boldsymbol{S}, \boldsymbol{X}, Z)$ denotes the joint probability density of the signals and

[14] Another purpose with these additional signals is to allow for shocks that neither bidders nor the auctioneer has any prior knowledge of.

[15] Note that our Assumption 2 is slightly more restrictive compared to Milgrom and Weber (1982). We assume that the function u is strictly increasing in X_i, whereas they assume that it is weakly increasing. Our stricter assumption is useful when proving Lemma 1 and its implications.

[16] Milgrom & Weber (1982) assume that $\mathbb{E}[V_i]$ is bounded for each i. We need a stronger assumption as we comnsider non-monotonic symmetric equilibria.

Z. This function is common knowledge among bidders. We can condition on the possible values of $Z \in \{1, 2, ..., n-1\}$ and we write $f_k(\boldsymbol{S}, \boldsymbol{X})$, $k = 1, 2, ..., n-1$, for the joint probability density of the signals conditional on $Z = k$. We assume that $f_k(\boldsymbol{S}, \boldsymbol{X})$ is continuous with respect to $\boldsymbol{X}$. We define the set $K_Z = \{k | \underline{Z} \leq k \leq \overline{Z}\}$ as the set of Z values that occur with positive probability.

We further assume that each signal X_i has possible values given by the interval $Q_X = (a_L, a_U)$ where $a_L \in \mathbb{R} \cup \{-\infty\}$ and $a_U \in \mathbb{R} \cup \{+\infty\}$. Similarly the signals $\mathbf{S}$ have possible values Q_S, which may be unbounded. We will want to consider only signals that occur with a positive density, so we assume that for every $\boldsymbol{S} \in Q_S$, and for each $x \in Q_X$, the density $f_k(\boldsymbol{S}, x, \boldsymbol{X}_{-1}) > 0$ for some $\boldsymbol{X}_{-1}$ and k.

Consider bidder 1 and let Y_k denote the kth highest signal of its competitors, and $\boldsymbol{Y}$ be a vector of those signals. Due to symmetry properties and Assumption 1, we can write the value of bidder 1 as follows:

$$V_1 = u(\boldsymbol{S}, X_1, \boldsymbol{Y}).$$

Moreover, we make the following standing assumptions for the affiliated-signals model.

Assumption 4: f_k is symmetric in its last n arguments.

Assumption 5: The variables $S_1, \ldots, S_m, X_1, \ldots, X_n, -Z$ are affiliated and $S_1, \ldots, S_m, X_1, \ldots, X_n$ remain affiliated if we condition on the value of Z.[17]

Assumptions 1 and 4 are similar to Milgrom & Weber (1982) and ensure that the game is symmetric ex ante, before bidders observe any signals. Assumption 5 is consistent with Milgrom & Weber (1982) for the special case when $Z = 1$ with certainty.

A strategy for bidder i is a function mapping its value estimate X_i into a bid $b = b_i(X_i) \geq 0$.[18] The auctioneer accepts the Z highest bids. In case of ties, acceptance is determined randomly such that each bid at the clearing price has the same chance of being accepted. We will solve for a pure-strategy Bayesian Nash Equilibrium (BNE), where each bidder chooses an optimal bid conditional on the bidding strategies of

[17] In particular Assumption 5 will hold when the signals $S_1, \ldots, S_m, X_1, \ldots, X_n$ are affiliated and Z is certain or independent of these signals.

[18] Recall that values are assumed to be non-negative (Assumption 2).

the competitors and conditional on the information that it observes. Without loss of generality, we will focus on the bidding decision by bidder 1. We solve for Bayesian Nash Equilibrium (BNE), where each bidder maximizes its expected profit for every observed private signal.[19]

Affiliation and related concepts are formally defined in Milgrom & Weber (1982). We repeat some of those definitions here. A subset L of $\mathbb{R}^k$ is a *sublattice* if u and v in L imply that the vectors $u \vee v$ and $u \wedge v$ are also in L (where $(u \vee v)_i = \max(u_i, v_i)$, $(u \wedge v)_i = \min(u_i, v_i)$, $i = 1, ..., k$). A subset A of $\mathbb{R}^k$ is *increasing* if its indicator function I_A is non-decreasing. In other words if $x \in A$ and $y_i \geq x_i$, $i = 1, ..., k$ then $y \in A$. Let $X = (X_1, ..., X_k)$ be a random vector. $X_1, ..., X_k$ are *associated* if for all increasing sets A and B, $\Pr(A \cap B) \geq \Pr(A) \Pr(B)$, and $X_1, ..., X_k$ are *affiliated* if for all increasing sets A and B, and every sublattice L, $\Pr(A \cap B \mid L) \geq \Pr(A \mid L) P(B \mid L)$, i.e., if the variables are associated conditional on any sublattice. In the case where there are densities, Milgrom & Weber (1982) show that if variables $X_1, ..., X_k$ are affiliated for any vectors x and x' that are possible realisations of the X_i, then

$$f(x \wedge x') f(x \vee x') \geq f(x) f(x') \tag{1}$$

where f is the joint density of the variables $X_1, ..., X_k$.

We define Y_Z to be the Zth highest signal of the competitors, a random variable that is determined from the values taken by the random variables $\mathbf{X}_{-i}$ and Z. Y_Z plays a central role in our analysis, similar to Y_1 in Milgrom & Weber (1982), who assume that $Z = 1$ with certainty. In Proposition 1 below we show that our assumptions are sufficient to ensure that Y_Z is affiliated with the other signals, which is important for our analysis.

Proposition 1 *Under our assumptions $S_1, ..., S_m$,X_1, $Y_1, ...Y_{n-1}$, Y_Z are affiliated.*

[19] The standard definition of the BNE would allow players to act irrationally for a finite number of events that occur with measure zero. Such events have no influence on payoffs or the best response of competitors. Our simplification makes the analysis less technical and only changes equilibrium outcomes for events that occur with measure zero.

The highest rejected bid sets the clearing price, so it corresponds to a Vickrey auction. We will establish existence of an equilibrium that is symmetric, continuous and monotonically increasing. This corresponds to the equilibria that Milgrom & Weber (1982) solve for in a second-price auction. But we also consider non-monotonic, piece-wise continuous bid functions. We say that a bid function $b(x)$ defined on Q_X is *regular* if there are a finite number of break points $a^{(1)} < a^{(2)} < ... < a^{(M+1)}$ where we take $a^{(1)} = a_L \in \mathbb{R} \cup \{-\infty\}$ and $a^{(M+1)} = a_U \in \mathbb{R} \cup \{+\infty\}$, and $b(x)$ is either (i) continuous and strictly increasing; or (ii) continuous and strictly decreasing; or (iii) constant; in each of the intervals $(a^{(\ell)}, a^{(\ell+1)})$ for $\ell = 1, 2, ..., M$. We assume that the value of the bid functions at the points $a^{(\ell)}$, $\ell = 1, 2, ..., M$, are defined by continuity either on the right or left. We say that an equilibrium is regular if the bid functions used are regular.

Lemma 1 below is useful when proving that bid functions are monotonic. It is a slightly stronger version of Theorem 5 in Milgrom & Weber (1982) who show that a bidder's expected value is non-decreasing with respect to its own signal. We are able to prove this result as our Assumption 2 is slightly stronger than in Milgrom & Weber (1982).

Lemma 1 *For any sublattice L, the functions $E[V_1 | X_1 = x, \widetilde{\mathbf{X}} \in L]$ and $E[V_1 | X_1 = x, Y_Z = y, \widetilde{\mathbf{X}} \in L]$ are strictly increasing in x for $x \in Q_X$, and non-decreasing in Y_Z.*

Without loss of generality, we focus on bidder 1 and write W_Z for the Zth highest bid amongst the other bidders (whereas Y_Z is the Zth highest signal). A marginal change in the bid only matters when the bid is on the margin of being accepted, i.e. when the bid b equals W_Z. Hence, this is the event that bidder 1 will condition on when it optimizes its bid. In the proof of Theorem 1, we show that a necessary condition for a symmetric equilibrium is that $b(x) = v_W(x, b(x))$, where

$$v_W(x, b) = \mathbb{E}[V_1 | X_1 = x, W_Z = b].$$

For strictly increasing, continuous bid functions it can be shown that

$$\begin{aligned} v_W(x, b(y)) &= \mathbb{E}[V_1 | X_1 = x, b(y) = W_Z] = \mathbb{E}[V_1 | X_1 = x, Y_Z = y] \\ &= v(x, y). \end{aligned} \quad (2)$$

Theorem 1 *There is a symmetric equilibrium with a strictly increasing bid function $b^*(x) = v(x,x)$ and, except possibly at isolated points, this is the only symmetric equilibrium with regular, strictly increasing bid functions. If Z is fixed at k, then bid functions are continuous and $b^*(x) = v(x,x)$ is the only symmetric equilibrium with regular bid functions.*

3.1 *Example 1*

In this example we demonstrate that when Y_Z is not affiliated then we may have symmetric non-monotonic equilibria (and no symmetric monotonic equilibria). Moreover we show that having $-Z$ affiliated with the signals is not enough to ensure that the signals are affiliated if we condition on Z, [20] which is why Assumption 5 fails here and we cannot rely on Theorem 1.

There are three bidders, each receives a private signal $X_i \in (0,2)$. The common value of the object is determined from the signals and is given by $V = \sum X_i$. Signals are uniformly distributed on $(0,2)$ and are independent. The number of items auctioned, Z, is also determined by the signals. If two or three of the signals are in the range $(0,1)$ then $Z = 2$, and otherwise $Z = 1$. It can be shown that $-Z$ is affiliated with the signals.[21] However the signals are not affiliated if we condition on $Z = 2$, since taking $x = (0.5, 1.5, 0.5)$ and $x' = (1.5, 0.5, 0.5)$ and writing f_2 for the density conditional on $Z = 2$ shows that $f_2(x)$ and $f_2(x')$ are positive but $f_2(x \vee x') = f(1.5, 1.5, 0.5) = 0$ thus contradicting the affiliation condition (1).

If there were a symmetric monotonic equilibrium we get a contradiction by showing that the value of $\mathbb{E}[V \mid X_1 = x, Y_Z = x]$ jumps down at $x = 1$. Details are in the on-line appendix. But there is a symmetric non-monotonic equilibrium given by $b^*(x) = 1 + \frac{5x}{2}$, for $x \in (0, \frac{1}{2})$, $b^*(x) = h_1^{-1}(x)$ for $x \in (\frac{1}{2}, 1)$, $b^*(x) = h_2^{-1}(x)$ for $x \in (1, \frac{3}{2})$ and $b^*(x) = \frac{5x}{2}$ for $x \in (\frac{3}{2}, 2)$. Here the functions $h_1(b)$ and $h_2(b)$ are

[20] If Z were a continuous variable this would be true automatically as is shown by an argument in Milgrom & Weber(1982).

[21] This is proved in Anderson & Holmberg (2023).

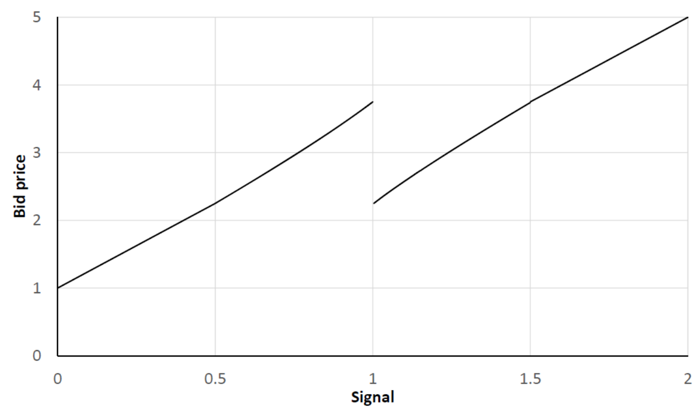

Figure 1: A symmetric non-monotonic equilibrium for Example 1.

determined from the two simultaneous equations:

$$b = \frac{1}{2\left(2-h_1(b)\right)}\left(2h_2(b)^2-2h_1(b)h_2(b)-2h_2(b)-5h_1(b)^2+10h_1(b)+4\right), \quad (3)$$

$$b = h_1(b)+\frac{5}{2}h_2(b)+\frac{h_1(b)}{h_2(b)}-\frac{h_1(b)^2}{h_2(b)}-1, \quad (4)$$

which are easy to solve for any given b. If h_1 and h_2 satisfy (3) and (4) then we show in the online Appendix that the corresponding bid function b^* (shown in Figure 1) is an equilibrium. For bids that are in the range $(2.25, 3.75)$ there is an overlap with two different signals giving the same bid (one signal less than 1, the other more than 1).

The non-monotonic equilibrium can be explained as follows. For a symmetric equilibrium it is optimal for a bidder to bid $\mathbb{E}\left[V_1 | X_1 = x, W_Z = b\right]$. If a bidder observes a signal below 1, then there is a high probability that $Z = 2$. In this case, conditioning on $W_Z = b$ means that one competitor bids at b and the other competitor bids above b, which is positive for the expected common value. But this changes when a bidder observes a signal above 1, so that $Z = 1$ occurs with a high probability. In this case conditioning on $W_Z = b$ means that the other competitor bids below b, which is bad for the expected common value. Hence, $\mathbb{E}\left[V_1 | X_1 = x, W_Z = b\right]$ and the bid drops at $x = 1$.

3.2 *Publicity effect*

Now, assume that the seller, the auctioneer, has a signal X_0 that it might want to disclose to all bidders. We assume that it is one of the S signals, and make the additional assumption that it is a continuous random variable. Similar to Milgrom & Weber (1982), we use the superscript $\mathbb{N}$ for markets where X_0 is not disclosed and the superscript $\mathbb{I}$ when the auctioneer's information is made public.

Proposition 2 *For strictly increasing, symmetric equilibria in the uniform-price auction, the expected price and the expected revenue of the auctioneer weakly increases when it discloses its signal X_0, and do not change in the case when X_0 is independent of $X_1, \ldots, X_n$ and Z. Similarly, the auctioneer benefits from disclosing Z.*

It can also be shown (see Anderson & Holmberg, 2023) that it is optimal for an auctioneer to always and fully disclose its information before the auction starts, instead of disclosing it partly or sometimes.

4 THE LINKAGE PRINCIPLE

The linkage principle essentially says the following: The more closely the winning bidder's payment is linked to its signal, the greater the expected revenue will be for the auctioneer. The linkage principle was first introduced by Milgrom & Weber (1982) and was further developed by Holmberg & Wolak (2018) for divisible-good markets with flat bids when the traded volume is independent of signals. Here we make an additional extension, so that the linkage principle can be used for multiple bidders and when the number of traded units is correlated with the signals $\boldsymbol{X}$.

Consider a symmetric equilibrium, where competitors submit strictly increasing bids $b(x)$. We allow the considered bidder (bidder 1) to deviate and act as if observing a signal $\widetilde{x}$, i.e. it can make an offer $b(\widetilde{x})$, although it actually observes the signal x. Conditional on having the bid accepted, let $J(\widetilde{x}, x) \geq 0$ be the expected payment when bidder 1 observes $x \in Q_X = (a_L, a_U)$ and bids as if observing $\widetilde{x} \in Q_X$. Bidder 1 needs to act as if having a higher signal than Y_Z to have its bid accepted. Thus the expected payment to the auctioneer from bidder 1 and bidder 1's expected utility of the good are given by :

$$K(\widetilde{x},x) = J(\widetilde{x},x)\Pr(Y_Z \le \widetilde{x}|\, x) \quad (5)$$
$$U(\widetilde{x},x) = \mathbb{E}[V_1|\, Y_Z \le \widetilde{x};x]\Pr(Y_Z \le \widetilde{x}|\, x). \quad (6)$$

From (5) we have that

$$\lim_{\widetilde{x}\searrow a_L} K(\widetilde{x},x) = 0, \text{ for } x \in Q_X. \quad (7)$$

For the moment, we assume that $J(\widetilde{x},x)$, $U(\widetilde{x},x)$ and $K(\widetilde{x},x)$ are bounded and differentiable functions with respect to both arguments, and we use subscripts $\widetilde{x}$ and x for these partial derivatives.[22]

Lemma 2 *If for two auction designs* $\mathbb{A}$ *and* $\mathbb{B}$ $\lim_{x\searrow a_L} J^{\mathbb{A}}(x,x) \ge \lim_{x\searrow a_L} J^{\mathbb{B}}(x,x)$ *and* $J_x^{\mathbb{A}}(x,x) \ge J_x^{\mathbb{B}}(x,x)$ *for* $x \in Q_X$*, then* $J^{\mathbb{A}}(x,x) \ge J^{\mathbb{B}}(x,x)$ *and* $K^{\mathbb{A}}(x,x) \ge K^{\mathbb{B}}(x,x)$ *for* $x \in Q_X$*.*

Hence, an auction design that increases the linkage between a bidder's private signal and its expected payment to the auctioneer, conditional on acceptance, increases the expected revenue of the auctioneer.

5 PAY AS BID AUCTION

In this section, we will consider a well-behaved symmetric equilibrium in a pay-as-bid auction and use the linkage principle to make comparisons with the uniform-price auction. In addition to Assumption 1-5, the following simplifying assumption is used in this section.

Assumption 6: $f_k(\boldsymbol{S},\boldsymbol{X})$ and $V_1 = u(\boldsymbol{S},X_1,\boldsymbol{X}_{-1}) = u(\boldsymbol{S},X_1,\boldsymbol{Y})$ are both differentiable with respect to $\boldsymbol{X}$ and $f_k(\boldsymbol{S},\boldsymbol{X}) > 0$ if $X_i \in Q_X$ for $i = 1,\ldots,n$ and $k \in K_Z$.

Assume that each competitor $i \ne 1$ observes a signal x_i and bids in accordance with the strategy $b^*(x_i)$, which is a strictly increasing (and invertible) function. The

[22] Unlike Milgrom & Weber (1982), we assume that signals are in open sets. This simplifies the equilibrium analysis, but we need to use limits when stating the linkage principle result.

expected payoff for bidder 1 observing signal x and bidding b in a pay-as-bid auction is then:

$$\Pi(b,x) = \mathbb{E}\left[(V_1 - b)\, 1_{W_Z<b} \middle| X_1 = x\right] = \int_{a_L}^{b^{*-1}(b)} (v(x,\alpha) - b)\, f_{Y_Z}(\alpha | x)\, d\alpha, \qquad (8)$$

where $f_{Y_Z}(\cdot| x)$ is the probability density of Y_Z (and $F_{Y_Z}(\cdot| x)$ is the corresponding cdf) conditional on bidder 1 observing x.

Proposition 3 *In a pay-as-bid auction, there is a symmetric equilibrium with the bid function*

$$\begin{aligned} b^*(x) &= v(x,x) - \int_{a_L}^{x} \exp\left(-\int_{\alpha}^{x} \frac{f_{Y_Z}(s| s)}{F_{Y_Z}(s| s)} ds\right) dt(\alpha) \\ t(\alpha) &= v(\alpha,\alpha), \end{aligned} \qquad (9)$$

which is continuous and strictly increasing for $x \in Q_X$.

A working paper version of this paper (Anderson & Holmberg, 2023) shows that under Assumption 6 and under the assumption that bidders play the equilibrium in Proposition 3 (or a symmetric, well-behaved equilibrium in the uniform-price auction), then $J(\widetilde{x}, x)$, $U(\widetilde{x}, x)$ and $K(\widetilde{x}, x)$ are differentiable functions with respect to both arguments. Hence, the linkage principle can be applied to rank auctions. The result below extends Milgrom & Weber's (1982) ranking of first- and second-price auctions, and generalizes the ranking result in Weber (1983) for an auctioneer with a fixed supply.

Proposition 4 *The expected revenue of the auctioneer in a uniform-price auction is at least as large as for the pay-as-bid auction. The expected revenue is the same in the two auctions if $X_1, \ldots, X_n$ and Z are all independent.*

Holmberg & Wolak (2018) prove a similar result for a divisible-good auction with two bidders, a traded volume that is independent of signals and a restrictive information structure. Previous work (Ausubel et al., 2014; Baisa & Burkett, 2018; Fabra et al., 2006; Holmberg, 2009; Pycia & Woodward, 2021) show that ranking of uniform-price and pay-as bid auctions can be different if there is strategic demand reduction or if bidders select less well-behaved equilibria.

In Anderson & Holmberg (2023), we use the linkage principle to verify that the publicity effect also holds for the pay-as-bid auction.

6 THE REFLECTED AUCTION

In this section, we prove the equivalence property of a reflected uniform-price auction. This property will simplify the analysis in the next section, where we study asymmetric equilibria in the uniform-price auction. The equivalence property is general. It does not rely on Assumptions 1-6, and in particular, bidders can be asymmetric ex-ante, and have different value ranges. Values could be interdependent. Signals could be correlated, and are not necessarily affiliated.

Definition 1 *Given an auction $\mathbb{A}$ with realisation of signals $\boldsymbol{X}$ and $\boldsymbol{S}$ and quantity Z, and with values $V_i = u_i(\boldsymbol{S}, \boldsymbol{X})$, realisations in the reflected auction $\mathbb{B}$ are obtained using the same realisation of signals $\boldsymbol{X}$ and $\boldsymbol{S}$, with the quantity $n - Z$, and with values $V_i = -u_i(\boldsymbol{S}, \boldsymbol{X})$.*

Proposition 5 *Suppose that bidder i, $i = 1, 2, ...n$, makes a bid given by the function $b_i(x)$ for signal $X_i = x$ in a uniform-price auction. This is an equilibrium in auction $\mathbb{A}$ if and only if there is an equilibrium in the reflected auction $\mathbb{B}$ in which bidder i with with signal $X_i = x$, makes the bid $-b_i(x)$, $i = 1, 2, ...n$.*

In essence this result shows that uniform-price auctions in which all values are negated, and where the supply Z is replaced by $n - Z$ have the same equilibrium as before, we simply negate all the previous bids. If a bid was accepted in the first auction it will be rejected in the reflected auction for the corresponding realisation, and vice versa. The intuition is as follows. In auction $\mathbb{A}$, the competitor with the Zth highest bid will be the marginal competitor of bidder 1. If the bid by bidder 1 is accepted it will pay the bid of the marginal competitor, which we can call W_A. Transforming the number of goods to $n - Z$ makes sure that the marginal competitor in auction $\mathbb{A}$ will also be the marginal competitor in auction $\mathbb{B}$, where the sign of all bids have been reversed. If the bid of bidder 1 is above W_A in auction $\mathbb{A}$, bidder 1 will bid below $-W_A$ in auction $\mathbb{B}$. In this case, the payoff of bidder 1 is $V_1 - W_A$ in auction $\mathbb{A}$ and 0 in auction $\mathbb{B}$. If the bid of bidder 1 is below W_A in auction $\mathbb{A}$, it will be above the bid of the marginal competitor in auction $\mathbb{B}$. In this case, the payoff of bidder 1 is 0 in auction $\mathbb{A}$ and $-V_1 + W_A$ in auction $\mathbb{B}$, where also the sign of values have been reversed. Hence, the difference in the payoff between auction $\mathbb{A}$ and $\mathbb{B}$ is always $V_1 - W_A$ for bidder 1, irrespective of which bid bidder 1 made in auction $\mathbb{A}$. This implies that if a bid of bidder 1 is optimal in auction $\mathbb{A}$, then the negative of that bid will be optimal in auction $\mathbb{B}$. The argument can be repeated for any bidder. In case the original auction has a price floor then it will be transformed into a price cap in the reflected auction, and vice versa.

It is well-known from the previous literature that equilibria in an auction can be transformed to equilibria in a new auction by a positive affine map. So if a bidder had a previous value of v it has a new value of $\alpha v + \beta$, for some constants $\alpha > 0$ and β (with the same constants for each bidder), then an equilibrium amongst bids in the old auction, is transformed into an equilibrium in the new auction when an old bid of y in auction is translated into a new bid of $\alpha y + \beta$. Proposition 5 makes it possible to transform equilibria in uniform-price auctions also for negative affine transformations, where $\alpha < 0$.

7 PRIVATE VALUE MODEL

In Sections 3-5, we studied symmetric BNE, and we showed that such equilibria are well behaved. In particular we showed that all symmetric equilibria are well behaved if Z is preannounced. But from practice we know that uniform-price auctions can have prices at the collusive level. In this section, we will solve for all pure-strategy equilibria in the uniform-price auction with private values, including asymmetric equilibria. We will show that inefficient asymmetric equilibria with prices at the collusive level may exist, but we also show how they can be avoided.

7.1 *Partial high-low equilibria in the single-object auction and its reflection*

Before studying asymmetric equilibria of multi-unit auctions in detail, we will summarize results that Blume & Heidhues (2004) have found for single-object auctions and, by means of Proposition 5, show what the equilibria looks like in the reflected version of a single-object auction.

Blume & Heidhues (2004) solve for all equilibria in a second-price auction, which corresponds to a uniform-price auction with $Z = 1$, under the following assumption.

Assumption B&H: $N \geq 3$ bidders have single-unit demand and independent private values. Distributions F_i, $i = 1, \ldots, N$, of valuations have positive densities f_i

on the common support $[0, v_h]$.

The result in Blume & Heidhues (2004) can be summarized as follows: there is an equilibrium where for low values $[0, \hat{b}) \subset [0, v_h]$, a single bidder bids high (at the threshold $\hat{b}$) and the rest low (at 0). The high bid is accepted and the low bids set the price, if all bidders have values below the threshold $\hat{b}$. All bid their value above $\hat{b}$. We refer to this as a high-low equilibrium at the bottom. We refer to the degenerate case where $\hat{b} = v_h$ as a high-low equilibrium. In this case, one bid is high and the others low for the whole range of values, so that the price is always zero (at the collusive level). The other degenerate case where $\hat{b} = 0$ implies that all bid their value for the whole range of values. This corresponds to the well-behaved equilibrium. Blume & Heidhues (2004) show that other equilibria can be ruled out for their setting.

Applying Proposition 5 to their result (and adding v_h to all bids and values) immediately gives us a reflected version of the single-object auction, which, as far as we know, has not been studied in the previous literature.

Corollary 1 *Under Assumption B&H, in a uniform-price auction with $N-1$ objects a strategy profile is a Nash equilibrium if there is a $\hat{b}$ such that:*

1. *any bidder with valuation $v < \hat{b}$ bids his valuation,*
2. *if $\hat{b} > 0$, then there is one bidder who bids at $\hat{b}$ whenever his valuation v satisfies $v > \hat{b}$, and if $\hat{b} \leq 0$ then there is one bidder who bids at or below 0 for any valuation v,*
3. *all other bidders bid v_h whenever their valuation v is in $(\hat{b}, v_h]$.*

Again all other equilibria can be ruled out for the setting in Corollary 1. The high-low equilibrium at the bottom in the single-object auction becomes a high-low equilibrium at the top in the reflected auction. The degenerate cases, the well-behaved equilibrium and the high-low equilibrium, remain in the reflected auction. We refer to high-low equilibria at the top and bottom as partial high-low equilibria.

One might wonder why a high-low equilibrium at the bottom only occurs in the single-object auction and a high-low equilibrium at the top only occurs in the reflected version of the single-object auction.[23] The reason is that two or more bidders can only bid at the same price with a positive probability if those bids are either accepted with certainty or rejected with certainty. Otherwise one of those bidders would find it profitable to deviate and bid slightly higher. Hence, if, with a positive probability, there are bids at the threshold $\hat{b}$ in the middle of the range of equilibrium bids, then those bids must come from the same bidder. This is the case in a single-object auction, where there is a single high bidder at the threshold, and in the reflected version, where there is a single low bidder at the threshold. Hence, these two auctions are special cases. Our analysis below will focus on multi-unit auctions where there might be high-low equilibrium, but not partial high-low equilibria, at least not if bidders have the same value ranges. We will also show that partial high-low equilibria can exist in such auctions if bidders have different value ranges.

7.2 *Assumptions for the private-value model*

Now we will drop Assumption B&H. The signal of a bidder is allowed to be correlated with signals of competitors, but we will assume that the value of a bidder, conditional on its signal, is independent of competitor's signals. This corresponds to the private-value case. It does not matter whether a bidder directly observes its valuation of the object or works with an expected valuation $u_i(\boldsymbol{S}, X_i)$, as long as the latter is independent of competitors' signals. This also means that we can drop the vector of unobserved signals $\boldsymbol{S}$ from the analysis. As before we assume that the value function is increasing and continuous. Hence, signals and values are equivalent and we assume that each bidder observes a signal equal to its private value.

We suppose that the values for bidder i are in the open interval $(\underline{V}_i, \overline{V}_i)$ with a positive probability density throughout this range, without any single points where there is a positive probability mass. These open intervals can vary between bidders. We allow Z and signals to be drawn from a general probability distribution. Signals could for example be positively correlated or negatively correlated, so there is no assumption of affiliated signals or ex-ante symmetry in this section. Hence,

[23] In case of a single-object auction with two bidders, the reflected auction would also be a single-object auction with two bidders. In this special case, high-low equilibria at the top and bottom can occur in the same auction.

Assumptions 1-6 are also dropped.

Moreover, we assume that the range of possible values and positivity of densities remain unchanged no matter the value of Z, or what signals are observed by competitors, and that these ranges are common knowledge, even though the joint distribution of signals may not be common knowledge. The bidders might have different beliefs about the joint distribution. We restrict bids to regular bids, as for the affiliated case.

We will introduce some new notation and some additional restrictions on the number of bidders and the value ranges. The following notation will be useful in our analysis. Choose a set Ω_U having size $\underline{Z}$ and containing the bidders with the highest values of $\overline{V}_i$. Note that the set Ω_U may not be uniquely defined when there are common values of $\overline{V}_i$. We identify V_U as $\max\{\overline{V}_i : i \notin \Omega_U\}$ so that $\overline{V}_i \geq V_U$ for $i \in \Omega_U$ and $\overline{V}_i \leq V_U$ for $i \notin \Omega_U$. We define $\Omega_U' = \Omega_U \cup \{j_U\}$ where $j_U \notin \Omega_U$ is a bidder with $\overline{V}_{j_U} = V_U$. We also define $V_U' = \max\{\overline{V}_i : i \notin \Omega_U'\}$, so that $V_U' \leq V_U$.

Similarly we choose a subset Ω_L having size $\overline{Z}$ and containing the bidders with the highest values of $\underline{V}_i$. Then we identify V_L as $\min\{\underline{V}_i : i \in \Omega_L\}$ so that $\underline{V}_i \geq V_L$ for $i \in \Omega_L$ and $\underline{V}_i \leq V_L$ for $i \notin \Omega_L$. We also define $\Omega_L' = \Omega_L \backslash \{j_L\}$, where $j_L \in \Omega_L$ is a bidder with $\underline{V}_{j_L} = V_L$, and $V_L' = \min\{\underline{V}_i : i \in \Omega_L'\}$. Thus $V_L' \geq V_L$ and they may be the same.

Note that even if the sets Ω_U etc. are not uniquely defined, due to common upper or lower bounds of value ranges, the values of V_U, V_U', V_L, V_L' are still fixed. These values are illustrated in Figure 2. This shows a set of bids (on the vertical axis) for four bidders with different ranges of values (on the horizontal axis). We will show later that this is an equilibrium set of bids if $v_0 \in [V_L, V_L']$, $v_1 \in [V_U', V_U]$, and any price cap or price floor is non-restrictive.

Blume & Heidhues (2004) consider the case with at least three bidders in a single-object auction. In our setting this corresponds to the case where at least two bids are rejected in every auction. We also make the reflected version of this assumption.

Assumption A: $n \geq \overline{Z} + 2$ and $\underline{Z} \geq 2$.

We allow bidders to have different value ranges, and therefore need to make sure that all of these n bidders are relevant, i.e. that they have the possibility to compete at relevant prices. Hence, we also make assumptions on the value ranges of the n

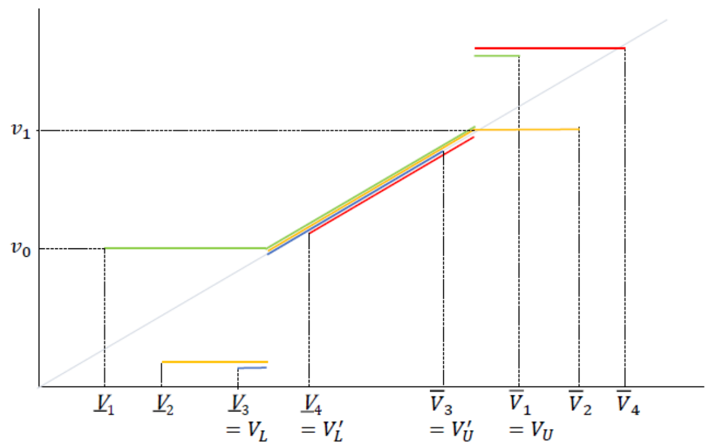

Figure 2: Example where bidders have asymmetric value ranges. A possible equilibrium is shown for the case $\underline{Z} = \overline{Z} = 2$.

bidders. We let $V''_L = \min\{\overline{V}_i\}$ and $V''_U = \max\{\underline{V}_i\}$, and we assume:

Assumption B: $V''_L > V'_L$ and $V''_U < V'_U$.

By definition $V''_L \leq V'_U$ and $V''_U \geq V'_L$ so Assumption B establishes the ordering $V_L \leq V'_L < V'_U \leq V_U$. Note that a bidder with $\overline{V}_i \leq V_L$ would not take part in the auction. Also a bidder with $\underline{V}_i \geq V_U$ will always have a value high enough to be accepted, and could be allocated a unit before the auction starts. Assumption B rules out such outcomes.

We write X_x for the set of bidders which can have a signal x, i.e. $X_x = \{i : x \in (\underline{V}_i, \overline{V}_i)\}$, and we will assume

Assumption C: The set X_x contains at least three bidders for any value $x \in (V_L, V_U)$, with potentially different bidders for different x values.

Ex-post optimality

In the private value model, optimality of bids would not be changed even if other player's bids are known (i.e. ex-post optimality). The only outcomes that would make a bid not ex-post optimal involve ties that occur with measure zero.[24]

[24] The result is true also for single-object auctions and for the reflected version of single-object auctions.

Lemma 3 *With private values, each bid is almost surely ex-post optimal in an equilibrium.*

Ex-post optimal equilibria are robust. They do not depend on the probability distribution of signals nor on risk aversion of the bidders.

High-low equilibrium

If the auctioneer's supply is certain, then there is a high-low equilibrium.

Proposition 6 *If the auctioneer's supply is certain so that $Z = \underline{Z} = \overline{Z}$, and a set Ω of exactly Z high bidding bidders have bids at least as high as $\max\{\overline{V}_i : i \notin \Omega\}$ and the remaining low bidding bidders all have bids that are no larger than $\min\{\underline{V}_i : i \in \Omega\}$, then this is a BNE (the high-low equilibrium).*

Under our assumptions, it can be shown that there is no other equilibrium where the bid of a bidder is rejected with probability 1 for its highest signal or where the bid of a bidder is accepted with probability 1 for its lowest signal. Moreover, high-low equilibria only occur when the auctioneer's supply is certain.

Lemma 4 *If the equilibrium bid of a bidder is rejected with probability* 1 *for the highest signals, then supply must be certain and the equilibrium must be of the high-low type. If the equilibrium bid of a bidder is accepted with probability* 1 *for the lowest signals, then supply must be certain and the equilibrium must be of the high-low type.*

Partial high-low equilibria

We will show that equilibria that are different from a high-low equilibrium are well-behaved or partly well-behaved. For these equilibria, all bidders will bid their value for some mid-range of signals (v_0, v_1). But bidding might be ill behaved near the edges, in the intervals $[V_L, V_L']$ and $[V_U', V_U]$, where partial high-low bidding can occur.

Theorem 2 *If an equilibrium is not of high-low type, then it must have the following properties: (a) There are points $v_0 \in [V_L, V_L']$ and $v_1 \in [V_U', V_U]$ such that for all signals in (v_0, v_1) all the bidders bid at their values. (b) When $v_0 = V_L$ then all bidders bid at v_0 or lower for signals strictly less than v_0. (c) When $v_0 > V_L$ there is* *a single bidder, i_X, which bids at v_0 for signals less than v_0, while the other bidders bid at a value $\underline{V}_{i_X}$ or lower for signals strictly less than v_0. (d) When $v_1 = V_U$ then all bidders bid at any value at v_1 or higher for signals greater than v_1. (e) When $v_1 < V_U$, then one bidder i_Y with $\overline{V}_{i_Y} > v_1$ bids at v_1 for almost all signals in $(v_1, \overline{V}_{i_Y})$ while the other bidders bid at any value at $\overline{V}_{i_Y}$ or higher for signals strictly greater than v_1. Moreover, a set of bid functions that satisfies properties (a) - (e) constitutes an equilibrium.*

For equilibria with $v_0 > V_L$, a single bidder i_X bids high, at v_0, for signals less than v_0. Other bidders bid low, at or below $\underline{V}_{i_X}$, for signals less than v_0. Hence, for signals less than v_0, bidding is similar to the high-low equilibrium at the bottom. If $v_1 < V_U$ then there is a high-low equilibrium at the top. In this case, a single bidder i_y bids low, at v_1, for signals higher than v_1. Other bidders bid high, at or above $\overline{V}_{i_Y}$, for signals higher than v_1. Figure 2 illustrates the partial high-low equilibria at the top and bottom. Partial high-low equilibria have outcomes where a bid is accepted and another bid is rejected even if the latter bidder has a higher valuation, which is inefficient. But inefficiencies only occur for values in the ranges $[V_L,\ v_0]$ and $[v_1, V_U]$, where bidders do not bid their value.

In case bidders have the same value range, as assumed in Blume & Heidhues (2004), we get $v_0 = V_L = V_L' = \underline{V}$ and $v_1 = V_U' = V_U = \overline{V}$, so that we can conclude the following from Theorem 2 and Proposition 6:

Corollary 2 *If all bidders have the same range of private values $\left(\underline{V}, \overline{V}\right)$, then there is an efficient equilibrium where every bidder bids at its value for each signal. This is the unique equilibrium if Z can take more than one value. But in the case when Z is fixed, then high-low equilibria also exist.*

Extension: Price floor and price cap

There is no price cap and no price floor (reservation price) in the model analysed above. In this extension we will argue that they can be used to give a unique equilibrium, when supply is certain or if bidders have heterogeneous value ranges. We omit formal proofs, but arguments below could be formalized using minor variations of the proofs of results in Section 7.2.

We will first consider the case where all bidders have the same range of private values $\left(\underline{V}, \overline{V}\right)$. According to Corollary 2 the equilibrium is unique, unless supply is certain, in which case high-low equilibria exist. It can be shown that any price floor $\underline{p}$ in the range $\left(\underline{V}, \overline{V}\right)$ would give uniqueness. The reason is that the market price is at least $\underline{p}$, so it is no longer profitable for high bidders to bid high, and get accepted with certainty, for signals in the range $\left(\underline{V}, \underline{p}\right)$. This also means that low bidders, which have zero profit in a high-low equilibrium, will be accepted with a positive probability and make a positive profit if they deviate and bid their value for signals in the range $\left(\underline{p}, \overline{V}\right)$.

An effective price cap would also give uniqueness if all bidders have the same range of private values $\left(\underline{V}, \overline{V}\right)$. Any price cap $\overline{p}$ in the range $\left(\underline{V}, \overline{V}\right)$ would knock out high-low equilibria, also when the volume Z is certain. The reason is that if high bidders bid at $\overline{p}$ (or lower), then it would be profitable for low bidders to deviate, and bid at $\overline{p}$, when observing signals in the range $(\overline{p}, \overline{V})$. Such a bid would be accepted with a positive probability and give a positive payoff in case of acceptance. Knocking out the high-low equilibrium implies that there are circumstances where introducing a maximum price would actually increase the revenue of the auctioneer.

The discussion above and the analysis in Section 7.2 is based on Assumption A, which rules out single-object auctions and its reflected version. In a single-object auction a price cap is not sufficient to get uniqueness. The problem is that there exists a range of partial high-low equilibria where all bidders bid their value above the threshold $\widehat{b}$, and such equilibria will not be knocked out by a price cap just below $\overline{V}$. But an effective price floor gives uniqueness, as proved by Blume & Heidhues (2004). It is the other way around for the reflected version of the single-object auction. It has a range of partial high-low equilibria at the top that will not be knocked out by a price floor just above $\underline{V}$. In this case an effective price cap is needed to get uniqueness.

It gets more complicated when bidders have different value ranges. As shown in Theorem 2, high-low equilibria can occur at the bottom if bidders have different lower bounds on their values. One needs a price floor to knock out this partial high-low equilibrium. Any price floor $\underline{p}$ in the range (V_L, V_L') would prevent i_X from bidding high, at v_0, for signals less than $\underline{p}$, and the partial high-low equilibrium falls apart.

Such a price floor would also knock out any high-low equilibrium.

If bidders have different upper bounds on their values, then it follows from Theorem 2 that partial high-low equilibria can occur at high prices. Such an equilibrium can be knocked out by a price cap. Any price cap in the range (V'_U, V_U) would prevent bidders from bidding high, at or above $\overline{V}_{i_Y}$, for signals above $\overline{p}$. Such a price cap would also knock out any high-low equilibrium. If bidders have both different upper and lower bounds on their values, then both a price floor and a price cap are needed to get uniqueness.

In Anderson & Holmberg (2023), we make a similar argument showing that reducing the supply by one unit at high and low prices has an effect that is similar to a price cap and price floor, respectively.

APPENDIX

Proof. (Proposition 1) We write T for the $m+n$-tuple $S_1, ..., S_m, X_1, Y_1, ...Y_{n-1}$, then from Milgrom & Weber Theorem 2 (and Assumption 5) the variables in T are affiliated. We want to show that T, Y_Z is affiliated. To do this we will consider arbitrary increasing sets A and B in R^{m+n+1}, as well as an arbitrary sublattice L in R^{m+n+1}.

We define the maps $\eta_i : R^{m+n} \to R^{m+n+1}$ for $i = 1, ..., n-1$ by $\eta_i(u) = (u, u_{m+i+1})$, so that when $u = (S_1, ..., S_m, X_1, Y_1, ...Y_{n-1})$ then

$$\eta_i(u) = (S_1, ..., S_m, X_1, Y_1, ...Y_{n-1}, Y_i).$$

We define for any set U in R^{m+n+1}

$$U'_i = \{u \in \mathbb{R}^{m+n} \ : \ \eta_i(u) \in U\}.$$

for $i = 1, ..., n-1$. Note that with this definition we have that A'_i and B'_i are increasing sets in R^{m+n} and also $L'_i = \{u \in R^{m+n} \ : \ \eta_i(u) \in L\}$ is a sublattice in R^{m+n}.

Now $(T, Y_i) \in A$ if and only if $T \in A'_i$. Thus we can use the affiliation property for T conditional on Z to show that

$$\begin{aligned} \Pr((T, Y_i) \in A \cap B \mid L, Z = i) &= \Pr(T \in A'_i \cap B'_i \mid L'_i, Z = i) \\ &\geq \Pr(T \in A'_i \mid L'_i, Z = i) \Pr(T \in B'_i \mid L'_i, Z = i) \qquad (10) \\ &= \Pr((T, Y_i) \in A \mid L, Z = i) \Pr((T, Y_i) \in B \mid L, Z = i). \end{aligned}$$

In this inequality we have both left and right hand sides equal to zero in the case that $L_i' = \emptyset$. Now

$$\Pr((T, Y_Z) \in A \cap B \mid L) = \sum_{i=1}^{n-1} \Pr(Z = i | L) \Pr((T, Y_i) \in A \cap B \mid L, Z = i).$$

Thus using (10), we deduce

$$\Pr((T, Y_Z) \in A \cap B \mid L) \geq \sum_{i=1}^{n-1} \Pr(Z = i | L) \Pr((T, Y_i) \in A \mid L, Z = i) \Pr((T, Y_i) \in B \mid L, Z = i). \tag{11}$$

In order to use Lemma 8 on this sum, we require $\Pr((T, Y_i) \in A \mid L, Z = i)$ decreasing in i. To show this note that if $(T, Y_{i+1}) \in A$ then $(T, Y_i) \in A$, because $Y_i \geq Y_{i+1}$, and A is increasing. Similarly, the indicator function I_A is increasing on T, Y_{i+1}. Hence, it follows from M&W Theorem 5 that

$$\Pr((T, Y_{i+1}) \in A \mid L, Z = i) = \mathbb{E}[I_A \mid L, Z = i] \geq \mathbb{E}[I_A \mid L, Z = i + 1],$$

since $T, Y_{i+1}, -Z$ are affiliated (M&W Theorem 2 and Assumption 5). Hence

$$\Pr((T, Y_i) \in A \mid L, Z = i) \geq \Pr((T, Y_{i+1}) \in A \mid L, Z = i) \geq \Pr((T, Y_{i+1}) \in A \mid L, Z = i+1).$$

Similarly $\Pr((T, Y_i) \in B \mid L, Z = i) \geq \Pr((T, Y_{i+1}) \in B \mid L, Z = i + 1)$. Then we apply Lemma 8 and obtain from (11):

$$\begin{aligned}
\Pr((T, Y_Z) \in A \cap B \mid L) &\geq \sum_{i=1}^{n-1} \underbrace{\Pr(Z = i | L)}_{q_i} \underbrace{\Pr((T, Y_i) \in A \mid L, Z = i)}_{a_i} \underbrace{\Pr((T, Y_i) \in B \mid L, Z = i)}_{b_i} \\
&\geq \left(\sum_{i=1}^{n-1} \Pr(Z = i | L) \Pr((T, Y_i) \in A \mid L, Z = i) \right) \left(\sum_{i=1}^{n-1} \Pr(Z = i | L) \Pr((T, Y_i) \in B \mid L, Z = i) \right) \\
&= \Pr((T, Y_Z) \in A \mid L) \Pr((T, Y_Z) \in B \mid L).
\end{aligned}$$

Since A and B are arbitrary increasing sets in $\mathbb{R}^{m+n+2}$, this demonstrates the inequality we need to show that the variables $S_1, ..., S_m, X_1, Y_1, ...Y_{n-1}, Y_Z$ are affiliated.

■

Proof. (Theorem 1) For the first part we begin by noting that Lemma 1 implies that $v(x, y) = \mathbb{E}[V_1 | X_1 = x, Y_Z = y]$ is strictly increasing in x and non-decreasing in y. Hence if we define the bid function $b^*(x) = v(x, x)$ it will be strictly increasing.

Next we show that $b^*(x) = v(x,x)$ is an equilibrium. The proof corresponds to the proof of Theorem 6 in Milgrom & Weber (1982). We will show that $b^*(x)$ is an optimal response when all other players use the strategy $b^*(x)$. Using that the bid function is strictly monotonic, it follows that bidder 1's conditional expected profit when it bids b is:

$$\begin{aligned} & \mathbb{E}\left[\left(V_1 - b^*(Y_Z)\right) 1_{b^*(Y_Z)<b} \right| X_1 = x \Big] \\ = \; & \mathbb{E}\left[\left(v(X_1, Y_Z) - v(Y_Z, Y_Z)\right) 1_{b^*(Y_Z)<b} \right| X_1 = x \Big] . \end{aligned}$$

Due to the monotonicity properties of v, the difference $v(X_1, Y_Z) - v(Y_Z, Y_Z)$ is non-negative for $X_1 \geq Y_Z$ and non-positive for $X_1 \leq Y_Z$. Hence, it follows that the expression above is maximized when the indicator function is one for $X_1 \geq Y_Z$, and zero for $X_1 < Y_Z$. This is the case when bidder 1 bids $b^*(X_1)$.

For the remainder of the proof we consider an arbitrary symmetric regular equilibrium and make deductions about its form for the special case where Z is fixed at k. Suppose that there is a symmetric equilibrium with bid functions $\widetilde{b}$, where $\widetilde{b}$ is a regular function having break points $a^{(1)}, a^{(2)}, ...a^{(M+1)}$. We will need to use the fact that for every $x \in Q_X$ and $\delta > 0$ there is a positive probability of every bidder receiving a signal in $(x - \delta, x + \delta)$. This is established in Lemma 7 in the online appendix. The proof proceeds in several stages, where each stage proves a property of the equilibrium when Z is fixed at k, (though this assumption is not needed for step 4).

Step 1. $v_W(x,b) = \mathbb{E}[V_1 | X_1 = x, W_Z = b]$ *is strictly increasing in* x. When $W_Z = b$, then we may, by symmetry with respect to competitors, suppose that bidder 2 has the kth highest bid amongst the bidders, and hence is the bidder with bid b. Thus

$$\mathbb{E}[V_1 | X_1 = x, W_Z = b] = \mathbb{E}[V_1 | X_1 = x, \mathbf{X}_{-1} \in H_k(b)], \tag{12}$$

where

$$H_k(b) = \{(x_2, ..., x_n) \; : \; \widetilde{b}(x_2) = b, \; \widetilde{b}(x_i) \geq b, i = 3, ..., k+1, \; \widetilde{b}(x_j) \leq b, j = k+2, ..., n\},$$

and we also use symmetry in assuming that bidders $3, .., k+1$ have bids greater than b. Since $H_k(b)$ is a sublattice we can use Lemma 1, to show $\mathbb{E}[V_1 | X_1 = x, W_Z = b]$ is strictly increasing in x.

Step 2. *There cannot be an interval in which $\widetilde{b}$ is constant.* Suppose otherwise and $\widetilde{b} = b_0$ on the interval $(q_0 - \delta, q_0 + \delta)$. From Step 1 we know that $\mathbb{E}[V_1 | \, q_0 - \delta \leq X_1 < q_0, W_Z = b_0] < \mathbb{E}[V_1 | \, q_0 < X_1 \leq q_0 + \delta, W_Z = b_0]$. Since bids are at b_0 throughout this region, we must either have (A) a positive expected payoff on the right hand part of the range, i.e. $\mathbb{E}[V_1 | \, q_0 < X_1 \leq q_0 + \delta, W_Z = b_0] > b_0$ or (B) a negative expected payoff in the left hand part of the range i.e. $\mathbb{E}[V_1 | \, q_0 - \delta \leq X_1 < q_0, W_Z = b_0] < b_0$. In case (A) we get an improvement by increasing the bid from b_0 to $b_0 + \varepsilon$ for signals in the range $(q_0, q_0 + \delta)$. First note from our earlier observation there is a non-zero probability of all the bidders having signals in $(q_0 - \delta, q_0 + \delta)$ and making the same bid b_0. Thus if $X_1 \in (q_0, q_0 + \delta)$, there is a positive probability that enough players bid at b_0 for W_Z to remain at b_0 after the marginal increase. Since the old bid is accepted only some of the time while the new bid is always accepted, there will be an improvement in payoff. For small enough ε this will guarantee an improvement in expected payoff, contradicting the fact that $\widetilde{b}$ is an optimal bid for bidder 1. The same type of argument also applies in case (B) where player 1 receives negative expected payoff for signals in the range $[q_0 - \delta, q_0)$ when $W_Z = b_0$. In this case we can show an improvement by changing the bid function to $b_0 - \varepsilon$ for signals in the range $[q_0 - \delta, q_0)$.

Step 3. *$v_W(x, b) = \mathbb{E}[V_1 | \, X_1 = x, W_Z = b]$ is continuous in b, except for prices in the set G, which is the set of bid values at possible break points.* Thus $G = \{y : \lim_{\delta \to 0} \widetilde{b}(a^{(k)} + \delta) = y$ or $\lim_{\delta \to 0} \widetilde{b}(a^{(k)} - \delta) = y$, for some $k = 1, 2, ..., M + 1\}$. We define $A(b) = \{x : \widetilde{b}(x) \geq b\}$, $B(b) = \{x : \widetilde{b}(x) \leq b\}$ so that these are collections of intervals defined by the bid function $\widetilde{b}$. Then we see that we can write

$$H_k(b) = \{(x_2, x_3, ...x_n) \ : \ \widetilde{b}(x_2) = b, \ x_3, ..., x_{k+1} \in A(b), \ x_{k+2}, ...x_n \in B(b)\}.$$

Moreover because b is not in G, and because the segments of $\widetilde{b}$ are either continuous increasing or continuous decreasing (segments where $\widetilde{b}$ is constant were ruled out in step 2), a small change in b implies a small change in the end points of the intervals that make up $A(b)$ and $B(b)$. From this it follows that $\mathbb{E}[V_1 | \, X_1 = x, W_Z = b]$, defined from (12), is continuous in b.

Step 4. *In the interior of an interval where $\widetilde{b}$ is continuous, we have $\widetilde{b}(x) = v_W(x, \widetilde{b}(x))$.* Since from Step 2 there are no segments of constant value in $\widetilde{b}$, the probability that $W_Z = b$ is zero. Thus the expected profit to bidder 1 from a bid b given a signal x is $\Pi_1(b,x) = \mathbb{E}\left[(V_1 - W_Z) 1_{W_Z<b} \middle| X_1 = x\right]$. Write $\widetilde{b}_x = \widetilde{b}(x)$. Since $\widetilde{b}(x)$ is optimal, $\Pi_1(\widetilde{b}_x, x) \geq \Pi_1(\widetilde{b}_x - \delta, x)$ so

$$\begin{aligned} & \mathbb{E}\left[(V_1 - W_Z) 1_{W_Z<\widetilde{b}_x} - (V_1 - W_Z) 1_{W_Z<\widetilde{b}_x-\delta} \middle| X_1 = x\right] \\ = \; & \mathbb{E}\left[(V_1 - W_Z) 1_{\widetilde{b}_x-\delta\leq W_Z<\widetilde{b}_x} \middle| X_1 = x\right] \geq 0. \end{aligned}$$

From our observations above there is a non-zero probability of all signals being in a range where bids are in the interval $(\widetilde{b}_x - \delta, \widetilde{b}_x)$. Hence $\Pr(\widetilde{b}_x - \delta < W_Z < \widetilde{b}_x) > 0$. Since

$$\begin{aligned} & \mathbb{E}\left[\left(V_1 - \widetilde{b}_x + \delta\right) 1_{\widetilde{b}_x-\delta\leq W_Z<\widetilde{b}_x} \middle| X_1 = x\right] \\ = \; & \Pr(\widetilde{b}_x - \delta < W_Z < \widetilde{b}_x)\mathbb{E}\left[\left(V_1 - \widetilde{b}_x + \delta\right) \middle| X_1 = x, \widetilde{b}_x - \delta \leq W_Z < \widetilde{b}_x\right], \end{aligned}$$

we can deduce that

$$\mathbb{E}\left[\left(V_1 - \widetilde{b}_x + \delta\right) \middle| X_1 = x, \widetilde{b}_x - \delta \leq W_Z < \widetilde{b}_x\right] \geq 0. \tag{13}$$

Similarly we can deduce that

$$\mathbb{E}\left[\left(V_1 - \widetilde{b}_x - \delta\right) \middle| X_1 = x, \widetilde{b}_x \leq W_Z < \widetilde{b}_x + \delta\right] \leq 0. \tag{14}$$

Now $\mathbb{E}\left[V_1 \middle| X_1 = x, \widetilde{b}_x - \delta \leq W_Z < \widetilde{b}_x\right]$ approaches $v_W\left(x, \widetilde{b}_x\right)$ as $\delta \to 0$, since v_W is continuous with respect to b (Step 3). Thus the left hand side of (13) approaches $v_W\left(x, \widetilde{b}_x\right) - \widetilde{b}_x$ as $\delta \to 0$. But the left hand side of (14) approaches the same limit as $\delta \to 0$, and this implies that $v_W(x, \widetilde{b}_x) - \widetilde{b}_x = 0$, as required.

Step 5. *The bid function $\widetilde{b}$ is strictly increasing for $x \in Q_X$.* Consider a potential bid b_0 made when the signal is x_0 and compare the profit made if the bid is increased by δ. Then

$$\begin{aligned} & \Pi_1(b_0 + \delta, x_0) - \Pi_1(b_0, x_0) = \mathbb{E}\left[(V_1 - W_Z) 1_{W_Z<b_0+\delta} - (V_1 - W_Z) 1_{W_Z<b_0} \middle| X_1 = x_0\right] \\ = \; & \mathbb{E}\left[(V_1 - W_Z) 1_{b_0<W_Z<b_0+\delta} \middle| X_1 = x_0\right] \\ \geq \; & \mathbb{E}\left[(V_1 - b_0 - \delta) 1_{b_0<W_Z<b_0+\delta} \middle| X_1 = x_0\right] \\ = \; & \Pr(b_0 < W_Z < b_0 + \delta) | X_1 = x_0))\mathbb{E}\left[V_1 - b_0 - \delta \middle| X_1 = x_0, b_0 < W_Z < b_0 + \delta\right]. \end{aligned}$$

Now, provided $b_0 \notin G$, then $v_W(x_0, b_0)$ is continuous in b_0 from Step 3, and hence as δ approaches zero $\mathbb{E}[\, V_1 - b_0 - \delta |\, X_1 = x_0, b_0 < W_Z < b_0 + \delta]$ approaches $v_W(x_0, b_0) - b_0$. Thus the change in profit for a small increase in b_0 has the same sign as $v_W(x_0, b_0) - b_0$. Thus the optimal choice $\widetilde{b}(x_0)$ is greater than b_0 if $v_W(x_0, b_0) > b_0$.

Consider two signals x_B and x_A with $x_B > x_A$. If x_A is in the interior of an interval where $\widetilde{b}$ is continuous, and with $x_B > x_A$ then from Step 1 and Step 4 $v_W(x_B, \widetilde{b}(x_A)) > v_W(x_A, \widetilde{b}(x_A)) = \widetilde{b}(x_A)$. Applying the observation above shows that $\widetilde{b}(x_B) > \widetilde{b}(x_A)$. The argument can be easily extended to show that $\widetilde{b}$ cannot jump down at the end of a segment, using continuity of $\widetilde{b}$ within segments.

Step 6. *The bid function $\widetilde{b}$ is continuous for $x \in Q_X$.* Define the set

$$J_k(x) = \{(x_2, x_3, ...x_n) \; : \; x_2 = x, \; x_3, x_4, ..., x_{k+1} > x, \; x_{k+2}, x_{k+3}, ...x_n < x\}. \quad (15)$$

From Step 5 we know that $\widetilde{b}$ is strictly increasing. This (and symmetry of the equilibrium) means that $W_Z = \widetilde{b}(x)$ is equivalent to having k signals greater than x and one signal exactly equal to x. Thus

$$v_W(x, \widetilde{b}(x)) = \mathbb{E}\left[V_1 |\, X_1 = x, W_Z = \widetilde{b}(x) \right] = \mathbb{E}\left[V_1 |\, X_1 = x, \mathbf{X}_{-1} \in J_k(x) \right]$$

where symmetry allows us to specify exactly which signals are above or below x. But $\mathbb{E}[V_1 |\, X_1 = x, \mathbf{X}_{-1} \in J_k(x)]$ is a continuous function of x (see Lemma 9 in the online Appendix), and hence $v_W(x, \widetilde{b}(x))$ is continuous and thus, from Step 4, we have continuity of $\widetilde{b}$.

Thus we have established, in the case that Z is fixed at k, that a regular symmetric equilibria $\widetilde{b}$ must be continuous and strictly increasing and have $\widetilde{b}(x) = v_W(x, \widetilde{b}(x))$. Hence from (2) $b^*(x) = v(x, x)$ is the only such equilibria.

For the case with uncertain Z and strictly increasing symmetric regular bid functions, it follows from step 4 and (2) that $\widetilde{b}(x) = v_W(x, \widetilde{b}(x)) = v(x, x)$ wherever the bid function is continuous. ■

Proof. (**Proposition** 2) If X_0 is disclosed to all bidders, then private signals will be drawn from a new probability density that is conditional on X_0. It can be shown that the remaining signals (i.e. signals except for X_0) are still affiliated after conditioning on X_0 (Milgrom & Weber, 1982). Thus results in Section 3 will also hold for the new conditional distribution. For example, there will be a symmetric BNE where each bidder $i \in \{1, \ldots, n\}$ has a strictly increasing bid function $\hat{b}^*(x; x_0) = \hat{v}(x, x; x_0)$, where

$$\hat{v}(x, y; x_0) = \mathbb{E}[V_1 | X_1 = x, Y_Z = y, X_0 = x_0]. \tag{16}$$

It follows from Theorem 5 in Milgrom & Weber (1982) that $\hat{v}$ is non-decreasing in its arguments. Let

$$\begin{aligned} P^{\mathbb{I}}(X_1) &= \mathbb{E}[\hat{v}(Y_Z, Y_Z; X_0) | X_1 > Y_Z] \\ P^{\mathbb{N}}(X_1) &= \mathbb{E}[v(Y_Z, Y_Z) | X_1 > Y_Z] \end{aligned}$$

be the expected payment from bidder 1 when it observes X_1 and gets the offer accepted. The ranking of revenues is more relevant for an auctioneer selling multiple items. Hence, we also define the following expected payment to the auctioneer from bidder 1, or any bidder, observing the signal x:

$$R(x) = \mathbb{E}[P(x) 1_{Y_Z < x} | X_1 = x]. \tag{17}$$

We have[25]

$$R(x) = \mathbb{E}[P(x) 1_{Y_Z < x} | X_1 = x] = P(x) \Pr(Y_Z < x | X_1 = x) \tag{18}$$

for both $\mathbb{I}$ and $\mathbb{N}$ cases and we have dropped the superscript. The probability $\Pr(Y_Z < x | x)$ is the same irrespective of whether X_0 is disclosed or not. Hence, it is sufficient to rank the expected payment $P(x)$ from bidder 1 when it observes X_1 and gets the bid accepted. It follows from Lemma 10 in the on-line Appendix that $P^{\mathbb{I}}(x) \geq P^{\mathbb{N}}(x)$. The proof of the lemma is inspired by the proof of Theorem 8 in Milgrom & Weber (1982). One difference is that we replace Y_1 by Y_Z. Another difference is that we prove the inequality for each realization of X_1, as we need this

[25] Note that this expression is also valid when X_0 is disclosed.

$$\begin{aligned} R^{\mathbb{I}}(x) &= \mathbb{E}[\hat{v}(Y_z, Y_z, X_0) 1_{Y_z < x} | X_1 = x] = \mathbb{E}[\hat{v}(Y_z, Y_z, X_0) | X_1 = x, Y_z < x] \Pr(Y_z < x | X_1 = x) \\ &= P^{\mathbb{I}}(x) \Pr(Y_z < x | X_1 = x) \end{aligned}$$

property to rank revenues of an auctioneer selling multiple units.
To prove the result for Z, which is a discrete random variable, we need Assumption 5, so that signals are affiliated also when conditioning on Z. Apart from this, the argument proving the publicity effect for Z is the same as the argument made for X_0.
■

Proof. (Proposition 3) Let $\tau(b) = b^{*-1}(b)$. Bidder 1 will choose its bid optimally. Thus we differentiate $\Pi(b,x)$ in (8) with respect to b. Leibniz' rule gives us:

$$\begin{aligned}\frac{\partial \Pi(b,x)}{\partial b} &= \tau'(b)\left(v(x,\tau(b)) - b\right) f_{Y_Z}\left(\tau(b)|\, x\right) \\ &\quad - F_{Y_Z}\left(\tau(b)|\, x\right).\end{aligned} \tag{19}$$

Hence, the symmetric equilibrium candidate $b^*(x)$, where $\tau(b) = x$ and $b = b^*(x)$, can be found from the following differential equation

$$0 = \frac{1}{b^{*\prime}(x)}\left(v(x,x) - b^*(x)\right) f_{Y_Z}(x|\, x) - F_{Y_Z}(x|\, x), \tag{20}$$

which can be simplified to the differential equation:

$$b^{*\prime}(x) = \left(v(x,x) - b^*(x)\right)\frac{f_{Y_Z}(x|\, x)}{F_{Y_Z}(x|\, x)}. \tag{21}$$

The solution to this type of Ordinary Differential Equation (ODE) can be found in Milgrom & Weber (1982), and is given in (9). The solution implies that $v(x,x) > b^*(x)$. We have that $F_{Y_Z}(x|\, x) > 0$ for $x \in Q_X$. Hence, it follows from (21) that $b^{*\prime}(x) > 0$ and finite. Moreover, it follows that $b^*(x)$ is continuous for a given $x \in Q_X$. Next, we want to verify that $b^*(x)$ is the best response of bidder 1. It follows from (19) that:

$$\frac{\partial \Pi(b,x)}{\partial b} = f_{Y_Z}\left(\tau(b)|\, x\right)\tau'(b)\left(v(x,\tau(b)) - b - \frac{1}{\tau'(b)}\frac{F_{Y_Z}\left(\tau(b)|\, x\right)}{f_{Y_Z}\left(\tau(b)|\, x\right)}\right). \tag{22}$$

We have from Proposition 1 that Y_Z is affiliated. It follows (see Lemma 1 in Milgrom & Weber (1982)) that $\frac{F_{Y_Z}(\tau(b)|x)}{f_{Y_Z}(\tau(b)|x)}$ is non-increasing with respect to x. Moreover, $v(x,\tau(b))$ is non-decreasing with respect to x. We have $\frac{\partial \Pi(b,x)}{\partial b} = 0$ for $x = \tau(b)$. Thus, it follows from (22) that $\frac{\partial \Pi(b,x)}{\partial b} \geq 0$ for $x \geq \tau(b)$ and $\frac{\partial \Pi(b,x)}{\partial b} \leq 0$ for $x \leq \tau(b)$. Equivalently, $\frac{\partial \Pi(b,x)}{\partial b} \geq 0$ for $b^*(x) \geq b$ and $\frac{\partial \Pi(b,x)}{\partial b} \leq 0$ for $b^*(x) \leq b$. Thus we can conclude that $b^*(x)$ is the best response of bidder 1. ■

Proof. (Proposition 4) We use the superscripts $\mathbb{U}$ and $\mathbb{P}$ to denote a uniform-price and pay-as-bid auction, respectively. Conditional on acceptance, $J(\widetilde{x}, x)$ is the expected payment when the bidder observes $x \in Q_X$ and bids as if observing $\widetilde{x} \in Q_X$. In the uniform-price auction the price is set by the bid $b^{\mathbb{U}}(Y_Z)$ if bidder 1's bid is accepted. Hence, it follows from Theorem 1 that

$$J^{\mathbb{U}}(\widetilde{x}, x) = \mathbb{E}\left[b(Y_Z) \mid X_1 = x, Y_Z \leq \widetilde{x}\right] = \mathbb{E}\left[v(Y_Z, Y_Z) \mid X_1 = x, Y_Z \leq \widetilde{x}\right]. \qquad (23)$$

We know that v is non-decreasing in its arguments. Hence, it follows from Theorem 5 in Milgrom & Weber (1982) that $J_x^{\mathbb{U}} \geq 0$. In the special case where $X_1, \ldots, X_n, Z$ are all independent it follows that $J_x^{\mathbb{U}} = 0$. Moreover, we have

$$\lim_{x \searrow a_L} J^{\mathbb{U}}(x, x) = \lim_{x \searrow a_L} \mathbb{E}\left[v(Y_Z, Y_Z) \mid X_1 = x, Y_Z = x\right] = \lim_{x \searrow a_L} v(x, x).$$

For the pay-as-bid auction, we have $J^{\mathbb{P}}(\widetilde{x}, x) = b^{\mathbb{P}}(\widetilde{x})$, so $J_x^{\mathbb{P}} = 0$. Moreover, we have from Proposition 3 that $\lim_{x \searrow a_L} J^{\mathbb{P}}(x, x) = \lim_{x \searrow a_L} v(x, x)$. The statement now follows from the above and Lemma 2. ■

Proof. (**Proposition 5**) Because applying reflection twice brings us back to the original auction we only need to prove this implication in one direction. Suppose that the bids $b_i(x)$ are an equilibrium in auction $\mathbb{A}$. Consider a realisation in auction $\mathbb{A}$ with signals $\{S_1, ..., S_m, X_1, ...X_n\}$ and with quantity Z_A. Consider bidder 1 with value V_1. Let W_A be the Z_Ath highest bid amongst the other bids $b_i(X_i)$, $i = 2, 3, ..., n$. Then the profit for bidder 1 in this realisation is $V_1 - W_A$ when $W_A < b_1(X_1)$ and $p_A(V_1 - W_A)$ when $b_1(X_1) = W_A$ (where $p_A < 1$ is the probability of being accepted in case of multiple bids at the clearing price) and zero otherwise. Recall that in case of ties, we have assumed that acceptance is determined randomly, such that each bid at the clearing price has the same chance of being accepted.

Now we consider the profit made by bidder 1 in the reflected auction at this realisation. Firm 1 has value $-V_1$ in the reflected auction and bids $-b_1(X_1)$. Let W_B be the Z_Bth highest bid amongst bidders $j \neq 1$ in the reflected auction, where $Z_B = n - Z_A$. Because the order of bids is reversed, if bidder j bids at W_A (and has the Z_Ath highest bid) in auction $\mathbb{A}$, this bidder will bid at W_B and have the Z_Bth highest bid in the reflected auction. Hence $W_B = -W_A$. A bid strictly above W_A

in auction $\mathbb{A}$ is accepted, but after reversing the order, the bid is strictly below W_B (and rejected) in auction $\mathbb{B}$, and vice versa. If bidder 1 bids at W_A it will also bid at W_B in the reflected auction. We will show that the acceptance probability for a rationed bid at W_A in auction $\mathbb{A}$ becomes the rejection probability for a rationed bid at W_B in auction $\mathbb{B}$. Suppose $b_1(X_1) = W_A$ in auction $\mathbb{A}$ and that there are k bids at W_A of which ℓ are accepted, so that $p_A = \ell/k$. Then in the reflected auction there will be $k - \ell$ accepted from this set, meaning that there will be a probability of acceptance for bidder 1 of $(k - \ell)/k = 1 - p_A$. Hence in the reflected auction the profit for bidder 1 with signal X_1 is: $-V_1 - W_B = -V_1 + W_A$ when $W_B < -b_1(X_1)$ and $(1 - p_A)(-V_1 - W_B)$ when $-b_1(X_1) = W_B$ and zero otherwise.

The expected profit for bidder 1 in auction $\mathbb{A}$ conditional on the signal $X_1 = x$ is

$$\mathbb{E}[\Pi_1 \mid X_1 = x] = \mathbb{E}[(u_1(\boldsymbol{S}, \boldsymbol{X}) - W_A) I_{W_A < b_1(x)} + p_A (u_1(\boldsymbol{S}, \boldsymbol{X}) - W_A) I_{W_A = b_1(x)} \mid X_1 = x].$$

At an equilibrium this is maximized by the bid $b_1(x)$.

The expected profit for bidder 1 in the reflected auction observing the same signal is

$$\begin{aligned}
&= \mathbb{E}[(-u_1(\boldsymbol{S}, \boldsymbol{X}) - W_B) I_{W_B < -b_1(x)} + (1 - p_A)(-u_1(\boldsymbol{S}, \boldsymbol{X}) - W_B) I_{W_B = -b_1(x)} \mid X_1 = x] \\
&= \mathbb{E}[(-u_1(\boldsymbol{S}, \boldsymbol{X}) + W_A) I_{W_A > b_1(x)} + (1 - p_A)(-u_1(\boldsymbol{S}, \boldsymbol{X}) + W_A) I_{W_A = b_1(x)} \mid X_1 = x] \\
&= \mathbb{E}[\Pi_1 \mid X_1 = x] - \mathbb{E}[u_1(\boldsymbol{S}, \boldsymbol{X}) - W_A \mid X_1 = x].
\end{aligned}$$

The term $\mathbb{E}[u_1(\boldsymbol{S}, \boldsymbol{X}) - W_A \mid X_1 = x]$ is conditional on the signal X_1 but is independent of bidder 1's bid. Hence, the payoff of bidder 1 is maximized in the reflected auction if (and only if) its profit is maximized in the original auction. Thus the reflected bids in the reflected auction must be an equilibrium. ■

Proof. (Proposition 6) First we note that $\max\{\overline{V}_i : i \notin \Omega\} > \min\{\underline{V}_i : i \in \Omega\}$, because Assumption B and our definitions imply that $\min\{\overline{V}_i\} > V'_L \geq V_L \geq \min\{\underline{V}_i : i \in \Omega\}$. Hence, bids from high-bidding bidders are always accepted and bids from the low-bidding bidders are always rejected. Low bidding bidders not in Ω have zero profit, but cannot improve unless they are accepted with a positive probability. This would require them to bid at a higher value than $\max\{\overline{V}_i : i \notin \Omega\}$. Doing so would imply $Z + 1$ bidders with bids above $\max\{\overline{V}_i : i \notin \Omega\}$ so the price paid would be at least this value leading to a loss. On the other hand a high bidding bidder is surely accepted at a price that is lower than $\min\{\underline{V}_i : i \in \Omega\}$ and hence always lower than its value. This price is unaltered by its actions (unless its bid is rejected), and thus the bidder cannot improve. So we have established that high-low bidding will be a BNE. ■

Before proving Theorem 2, we need some additional notation. We write B_i for the set of bids for bidder i. There is a distribution of bid values on B_i and because of our assumptions on the structure of bids this will consist of some intervals with a positive density and some points where there is a positive probability mass (occurring where the bid function is constant on an interval). From our assumption on the characteristics of the distribution of signals, the intervals with positive density and the points of positive probability mass on B_i remain the same no matter what signals are received by other bidders, or the value of Z.

In an equilibrium, a bid y made by bidder i is called active if it is accepted and rejected with positive probabilities. This property is determined by the characteristics of the probability distribution of the other bidders' bids. Hence from our assumptions the activity status of a bid at some price y is independent of the bidder's signal x. If there are more than one signal leading to a bid at y, then all of those bids will have the same activity status.

A bid at y by bidder i is not active if there are guaranteed to be at least $\overline{Z}$ bidders with bids strictly higher than y, since then the bid y is always rejected. We say bidder j strictly dominates y if there is a probability of 1 of bids by bidder j above y. So the condition for almost surely rejection of the bid y is that there are $\overline{Z}$ or more bidders other than i, which strictly dominate y. Similarly a bid at y by bidder i is not active if there are guaranteed to be at least $n - \underline{Z}$ bidders with bids strictly lower than y, since then the bid y is always accepted. We say that y strictly dominates bidder j if there is a probability of 1 of bids below y by bidder j. So the condition for almost surely acceptance of the bid y is that y strictly dominates $n - \underline{Z}$ or more bidders other than i. Thus a bid of y by bidder i is active if there are strictly less than $\overline{Z}$ bidders j (with $j \neq i$) which strictly dominate y, and strictly less than $n - \underline{Z}$ bidders j (with $j \neq i$) that y strictly dominates. In the on-line Appendix, we prove the following result that is related to ex-post optimality.

Lemma 5 *If a bid differs from the bidder's valuation, there cannot be a positive probability of a bid from a competitor that is either equal to this bid or strictly between this bid and the bidder's valuation if either the bidder's bid or the competitor's bid is active.*

It is helpful to define critical values a^- and a^+ which determine which bids are active in an equilibrium. We let $p_{[j]} = \inf\{B_j\}$ and let a^- be the $\overline{Z}$th element $p_{[j]}$ if these are put in non-increasing order. Similarly we let $p^{[j]} = \sup\{B_j\}$ and let a^+ be the $n - \underline{Z}$ element $p^{[j]}$ if these are put in non-decreasing order. The following properties of active bids are proved in the on-line Appendix.

Lemma 6 *If the inequality $a^- \leq a^+$ holds, then any bid y with $a^- < y < a^+$ is active. Moreover a bid at $a^- < a^+$ by bidder i is active if there are less than $\overline{Z}$ bidders j (with $j \neq i$), either with $p_{[j]} > a^-$, or with $p_{[j]} = a^-$ and not having an accumulation of bids at a^-. Also a bid at $a^+ > a^-$by bidder i is active if there are less than $n - \underline{Z}$ bidders j (with $j \neq i$), either with $p^{[j]} < a^+$, or with $p^{[j]} = a^+$ and not having an accumulation of bids at a^+. If $a^- = a^+$ then a bid by bidder i at this common price is active if the conditions for both a^- and a^+ hold. A bid of y with $y < a^-$ is rejected with certainty, and is not active. A bid of y with $y > a^+$ is accepted with certainty, and is not active.*

Proof. (**Theorem 2**) We start by establishing some inequalities: (i) $a^- < a^+$; (ii) $V_L < a^+$; and (iii) $V_U > a^-$.

(i) $a^- < a^+$. Since the equilibrium is not of high-low type, then from Lemma 4 for any bidder the bid is accepted with positive probability for the highest signals, and rejected with positive probability for the lowest signals. If $a^- > a^+$ then since there are always $\overline{Z}$ bids above or at a^- the $n - \underline{Z}$ bidders bidding at or below a^+ have bids rejected with certainty even for the highest signal, and we get a contradiction. Hence $a^- \leq a^+$.

Suppose that $a^- = a^+$, and write a^0 for this common price. Then since all bidders have bids rejected with positive probability for the lowest signal, the $\overline{Z}$ bidders bidding above or at a^0 must all have bids at a^0 with positive probability. Similarly, since the $n - \underline{Z}$ bidders bidding below or at a^0 have bids accepted with positive probability for the highest signal, these bidders must all bid at a^0 with positive probability. Thus every bidder has positive probability of bidding at a^0 and all these bids are active. To avoid a contradiction from Lemma 5 bidders with values strictly above a^0 must bid strictly above a^0 and bidders with values strictly below a^0 must bid strictly below a^0. Thus the $\overline{Z}$ bidders bidding at a^0 or above have values at a^0 or above, and the $n-\underline{Z}$ bidders bidding at or below a^0 have values at a^0 or below. But this contradicts $V_L < V_U$ which is an implication of Assumption B. Thus we have $a^- < a^+$.

(ii) $V_L < a^+$. Suppose otherwise and $V_L \geq a^+$. Then there are $\overline{Z}$ bidders, for which all values are at a^+ or above. Since $\overline{Z} \geq \underline{Z} \geq 2$ (Assumption A), there are at least two such bidders. Since each of these two have bids rejected with positive probability, then the two bidders have ranges of values, (say $[v_x, v_y]$ for the first with $a^+ < v_x < v_y$, and $[u_x, u_y]$ for the second with $a^+ < u_x < u_y$) for which these bidders bid at a^+ or below (according to Lemma 6) and are rejected with a positive probability. But if any bid of the first bidder has a positive probability of being accepted for signals in $[v_x, v_y]$ then this bid is active and we obtain a contradiction from Lemma 5 which can be applied both when the second bidder has bids from $[u_x, u_y]$ at or below the active bid of the first bidder and also when these bids are in between the active bid of the first bidder and a^+. Thus for signals in $[v_x, v_y]$ the first bidder bids low enough (at or below a^-) to have zero probability of acceptance. It follows from the definition of a^- that there are always $\overline{Z}$ bids at or above a^-, and that at most $\overline{Z}-1$ bidders always bid above $a^-+\varepsilon$ for any $\varepsilon > 0$. This implies that, with a positive probability, the clearing price is at a^- or below. We have $a^- < a^+ < v_x < v_y$, so this gives a contradiction to Lemma 3 and ex-post optimality for the first bidder for values in the range $[v_x, v_y]$.

(iii) $V_U > a^-$. This is established using a similar argument. If this does not hold and $V_U \leq a^-$ then there are at least $n - \underline{Z}$ bidders with all values less than or equal to a^-. We have $n - \underline{Z} \geq n - \overline{Z}$, so Assumption A implies that there are at least two such bidders. Since each of these two have bids accepted with a positive probability for their high bids, then the two bidders have ranges of values (say $[v_x, v_y]$ for the first with $v_x < v_y < a^-$, and $[u_x, u_y]$ for the second with $u_x < u_y < a^-$) for which these bidders bid at a^- or above (according to Lemma 6) and are accepted with positive probability. But if any bid of the first bidder has a positive probability of being rejected for signals in $[v_x, v_y]$ then this bid is active and we have a contradiction from Lemma 5 which can be applied both when the second bidder has bids from $[u_x, u_y]$ at or below the active bid of the first bidder and also when these bids are above the active bid of the first bidder. Thus for signals $[v_x, v_y]$ the first bidder bids high enough (at or above a^+) to have zero probability of being rejected. It follows from the definition of a^+ that there are $n - \underline{Z}$ bidders that always bid at or below a^+, and that at most $n - \underline{Z} - 1$ bidders always bid below $a^+ - \varepsilon$ for any $\varepsilon > 0$. This implies that, with a positive probability, the clearing price is at a^+ or above. Since $v_x < v_y < a^- < a^+$ this gives a contradiction to Lemma 3 and ex-post optimality for the first bidder for values in the range $[v_x, v_y]$.

We begin by proving part (a) which we accomplish in a number of steps. Define $v_0 = \max(a^-, V_L)$ and $v_1 = \min(a^+, V_U)$. We have established that $a^- < V_U$ and it follows from Assumption B that $V_L < V_U$, so we have $v_0 < V_U$. But we also have $V_L < a^+$ and $a^- < a^+$, so $v_0 < a^+$ and we deduce $v_0 < v_1$.

Step 1. We show there are bids in the intervals just above a^- and just below a^+.

We will prove this for a^- (the proof is analogous for a^+). Recall that by definition, a^- is the infimum of bids made by bidder $[\overline{Z}]$. If this bidder has no accumulation of bids at a^-, then it follows from the construction of regular bids that bidder $[\overline{Z}]$ must have bids in the intervals just above a^-, and we are done. The same argument can be made for any other bidder that has a^- as the lower bound on its bids. Hence, if there are no bids in the interval just above a^-, then bidder $[\overline{Z}]$, and any other bidder that has a^- as its lower bound on bids, must have an accumulation of bids at a^-. From the ordering of the bidders, it follows that at most $\overline{Z} - 1$ bidders bid strictly above a^- with certainty, so the accumulated bids at a^- are accepted with a positive probability. Moreover, we have assumed that all bidders have bids accepted with positive probability for the highest signals. Hence, all bidders bid at or above a^- with a positive probability. Thus bids at a^- are active since they are rejected and accepted with positive probabilities. A similar argument can be made for a^+. If there is an accumulation of bids at a^+, then those bids must be active.

Let g_1 be the infimum of the bids in (a^-, a^+) and take $g_1 = a^+$ if there are no such bids. We want to show that $g_1 = a^-$. Suppose this fails and $g_1 > a^-$, so there is an interval (a^-, g_1) without bids. If $g_1 < a^+$ then it follows from Lemma 6 that there are active bids at g_1 or just above. The definition of a^+ implies that there is an accumulation of bids at a^+ or bids just below a^+. Hence, if $g_1 = a^+$ then there must be an accumulation of bids at a^+, which (according to our result above) are active. From assumption A and the definition of a^- there are (at least) two bidders j_1 and j_2 (indexed by $\overline{Z}+1$, and $\overline{Z}+2$ when $p_{[j]}$ are put in non-increasing order) with $\inf\{B_{j_1}\} \leq a^-$ and $\inf\{B_{j_2}\} \leq a^-$. Because their high bids are accepted with positive probability, these two bidders both have bids greater than or equal to a^- with positive probability. Thus they each have a range of values for which they bid at a^- or above. If for either bidder this range of values includes values strictly less than a^- then since there are active bids at a^- from bidder $[\overline{Z}]$, we get a contradiction from Lemma 5. Thus the range of values for each bidder j_1 and j_2 has a supremum strictly greater than a^-.Consider the bids for these two bidders for values in the range (a^-, g_1). Since there are active bids at a^- from bidder $[\overline{Z}]$, we get a contradiction from Lemma 5 if either of these bidders bid at a^- or below for some subinterval of values in this range. We know that there is a positive probability of active bids in the range $[g_1, g_1 + \delta)$. Hence, there would be a contradiction from Lemma 5 if both j_1 and j_2 bid at or above g_1 for subintervals of values (which could differ between the two bidders) in the range (a^-, g_1). Hence we have established that $g_1 = a^-$.

Step 2. We show that there are bids throughout (v_0, v_1).

We suppose there is a gap (g_3, g_4) with no bid in this interval, with $a^- \leq g_3 < g_4 \leq a^+$. We choose g_3 as small as possible and g_4 as large as possible subject to there being no bid in the interval (g_3, g_4) (so there are bids at g_3 or just below, and at g_4 or just above). The result of step 1 implies that this definition has $a^- < g_3$ and $g_4 < a^+$. Suppose that there is an intersection between (g_3, g_4) and the range $(v_0, v_1) \subseteq (V_L, V_U)$. Then from Assumption C there are at least three bidders with values in a subinterval X_A of (g_3, g_4). These bidders cannot bid in (g_3, g_4) for values in X_A, so either at least two of them bid at g_3 or below, or at least two of them bid at g_4 or above for values in X_A. Suppose two bid at g_3 or below. We know that there is a bid at g_3 or just below, which may be a bid from one of these bidders. Moreover this bid is active since $a^- < g_3 < a^+$. This gives a contradiction from Lemma 5 both in the case of bids at g_3 and also when the bids are strictly less than g_3. In the same way if two bidders bid at g_4 or above for some values in X_A, we get a contradiction since there is an active bid at g_4 or just above (possibly from one of these two bidders). Thus we have shown that any gap (g_3, g_4) is outside the interval (v_0, v_1).

Step 3. We show that each bidder bids at value for values in the range (v_0, v_1) and bids at or below v_0 for values below v_0 and at or above v_1 for values above v_1.

The bids throughout (v_0, v_1) are all active. If a bidder with value at say $y \in (v_0, v_1)$ makes a bid that is not equal to y there will be active bids either just below y or just above y between the bid and the bidder's value. This gives a contradiction to Lemma 5. For similar reasons, a bidder cannot bid above v_0 for values below v_0 and a bidder cannot bid below v_1 for values above v_1.

Step 4. We show that $V_L \leq a^-$ so that $v_0 = a^-$

Suppose that $V_L > a^-$ and we will obtain a contradiction. Let p_L be the infimum of bids made by bidders in the set Ω_L, which has size $\overline{Z}$, and let $i_L \in \Omega_L$ have $\inf\{B_{i_L}\} = p_L$. We have $\overline{Z} - 1$ other bidders j in Ω_L with $\inf\{B_j\} \geq p_L$ and so from the definition of a^- we can deduce $p_L \leq a^-$. It follows from the definition of V_L that the bidder i_L has values greater than or equal to V_L and so from our assumption on the regularity of bids, there is a range of values (a_L, b_L), where $a_L > V_L$, for which bids are below V_L. But $V_L > a^-$ implies that $v_0 = V_L$, so we get a contradiction from the result in step 3 (and we are done).

Step 5. We show that $a^- \leq V_L'$

We suppose that $a^- > V_L'$ and obtain a contradiction. The definition of V_L' means that any set of $\overline{Z}$ bidders contains at least two with $\underline{V}_j \leq V_L'$. We consider the $\overline{Z}$ bidders with the highest values of $p_{[j]}$. Amongst this set of bidders, which by definition have $p_{[j]} \geq a^-$, there are at least two with $\underline{V}_j \leq V_L' < a^-$. Both these bidders bid above their values for a range of signals below a^-. If either of them bid strictly above a^- for signals in this range then there is a contradiction from Lemma 5, since we know (Step 1) that there are active bids just above a^-. Thus both have an accumulation of bids at a^- which are active from our observation earlier. But this also gives a contradiction from Lemma 5. Note that this upper bound on a^- is stronger than the bound we proved above in (iii) that $a^- < V_U$ since, from Assumption B, $V_L' < V_U$.

Step 6 We complete the proof of part (a)

At this point we have shown that $V_L \leq a^- \leq V_L'$. We can use this to establish that $V_U' \leq a^+ \leq V_U$. We consider the reflected equilibrium. The value of a^- in the reflected equilibrium is $-a^+$, and the values of V_L and V_L' in the reflected equilibrium are $-V_U$ and $-V_U'$ respectively. Hence we can apply what we have shown to the reflected equilibrium to establish that $-V_U \leq -a^+ \leq -V_U'$ and $v_1 = a^+$ Thus we have established that it is necessary for an equilibrium that is not of the high-low type to satisfy the properties described in part (a).

Now we turn to part (b). When $v_0 = V_L$ then $a^- = V_L$. It follows from Step 3 that all bidders bid at v_0 or lower for signals strictly less than v_0.

For part (c) We suppose $v_0 > V_L$ and hence $v_0 = a^- > V_L$. We consider the $\overline{Z}$ bidders with the highest values of $p_{[j]}$, and hence with all bids at or above a^-. From the definition of V_L we see that there is at least one bidder i_X in this set with $\underline{V}_{i_X} \leq V_L$, which bids above value for values in the range $(\underline{V}_{i_X}, a^-)$. It follows from Step 3 that bidder i_X must bid at a^- for values in the range $(\underline{V}_{i_X}, a^-)$. The condition $a^- > V_L$ implies that at most $\overline{Z} - 1$ bidders have all values above a^-. It follows from Step 3 that other bidders bid at a^- or lower with a positive probability. This means that bids at a^- are active. But to avoid a contradiction from Lemma 5 no competitor of bidder i_X can bid in the range $\left(\underline{V}_{i_X}, a^-\right]$ with a positive probability. Hence, except for bidder i_X, all bidders must bid at $\underline{V}_{i_X}$ or lower for values below a^-.

It is straightforward to show that parts (d) and (e) follow from parts (b) and (c) applied to the reflected auction (in the same way that we did for Step 6 of part (a)).

Finally we show that a set of bids satisfying these conditions must be an equilibrium, since bids are all ex-post optimal. Suppose that a bidder has a signal between v_0 and v_1 and bids its value, then these bids are always ex-post optimal, and thus we only need to consider bids not at value outside this range. Consider the case that $v_0 = V_L$ and suppose that a bidder observes a signal less than v_0 then, since all bids between v_0 and v_1 are made at value, with probability 1 there are $\overline{Z}$ bids strictly above v_0. Thus every bid at or below v_0 is rejected with certainty, and so bidding at $y \leq v_0$ for a signal less than v_0 is ex-post optimal. Suppose that $V_L < v_0 \leq V_L'$ and we consider bidder i_X bidding at v_0 for values below v_0. This is the only bidder with an accumulation of bids at this price. If the bid is rejected, the price will be at v_0 or above. If the bid is accepted then the clearing price is with probability 1 set by one of the bidders with values below v_0 all of whom bid below $\underline{V}_{i_X}$, so that the clearing price cannot be higher than the value for i_X and we have ex-post optimality for bidder i_X. Now consider bidders bidding below $\underline{V}_{i_X}$ for the case $V_L < v_0 \leq V_L'$. Since there are $\overline{Z}-1$ bidders who bid above V_L' with probability 1 and bidder i_x bids at v_0 or above, bidders bidding below $\underline{V}_{i_X}$ have their bids rejected. Since the last accepted bid is at v_0 or above, which is higher than their value, then these bids are also ex-post optimal. We can use the reflected equilibrium to show that there is also no improvement possible at the top of the bidding range, where v_1 takes the place of v_0. ■

REFERENCES

Aagard, T.S. and A.N. Kleit (2022). *Electricity Capacity Markets.* Cambridge (UK): Cambridge University Press.

Anderson, E., and P. Holmberg (2023). "Multi-unit auctions with uncertain supply and single-unit demand.", Working paper CWPE2339, Department of Economics, Cambridge University.

Ausubel, L.M., P. Cramton, M. Pycia, M. Rostek, and M. Weretka (2014). "Demand reduction and inefficiency in multi-unit auctions", *Review of Economic Studies* 81: 1366-1400.

Baisa, B. and J. Burkett (2018). "Large multi-unit auctions with a large bidder." *Journal of Economic Theory* 174: 1-15.

Blume, A. and P. Heidhues (2004). "All equilibria of the Vickrey auction.", *Journal of Economic Theory* 114: 170–177.

Blume, A., P. Heidhues, J. Lafky, J. Münster and M. Zhang (2009). "All equilibria of the multi-unit Vickrey auction.", *Games and Economic Behavior* 66(2): 729-741.

Fabra, N., N-H. M. von der Fehr and D. Harbord (2006). "Designing electricity auctions.", *RAND Journal of Economics* 37: 23-46.

von der Fehr, N-H. M. and D. Harbord (1993). "Spot market competition in the UK electricity industry.", *Economic Journal* 103: 531-46.

Holmberg, P. (2009). "Supply function equilibria of pay-as-bid auctions.", *Journal of Regulatory Economics* 36: 154-177.

Holmberg, P. and F.A. Wolak (2018). "Comparing auction designs where suppliers have uncertain costs and uncertain pivotal status.", *RAND Journal of Economics* 49(4): 995-1027.

Klemperer, P.D. and M.A. Meyer (1989). "Supply function equilibria in oligopoly under uncertainty.", *Econometrica* 57(6): 1243-1277.

Krishna, V. (2010). *Auction Theory*, London: Academic Press.

McAdams, D. (2007). "Adjustable supply in uniform price auctions: Non-commitment as a strategic tool." *Economics Letters* 95: 48-53.

Milgrom, P.R. (1981). "Rational expectations, information acquisition, and competitive bidding." *Econometrica* 49(4): 921–943.

Milgrom, P.R. and R.J. Weber (1982). "A theory of auctions and competitive bidding.", *Econometrica* 50: 1089-1122.

Morris, S., and T. Ui (2004). "Best response equivalence." *Games and Economic Behavior* 49(2): 260-287.

Perry, M., and P.J. Reny (1999). "On the failure of the linkage principle in multi-unit auctions.", *Econometrica* 67(4): 895-900.

Pycia, M., and K. Woodward. (2021). "Auctions of homogeneous goods: A case for pay-as-bid.", Centre for Economic Policy Research(CEPR), Discussion Paper 15656.

Schwenen, S. (2015). "Strategic bidding in multi-unit auctions with capacity constrained bidders: the New York capacity market.", *RAND Journal of Economics*, 46(4): 730-750.

Thompson, F. B. (1952). "Equivalence of games in extensive form." RM 759, The Rand Corporation.

Vickrey, W. (1961). "Counterspeculation, auctions, and competitive sealed tenders.", *The Journal of Finance* 16(1): 8-37.

Vives, X. (2011). "Strategic supply function competition with private information.", *Econometrica* 79(6): 1919–1966.

Wang, J. J., and J.F. Zender (2002). "Auctioning divisible goods.", *Economic theory* 19: 673–705.

Weber, R. (1983). Multiple-object auctions. In *Auctions, Bidding, and Contracting: Uses and Theory*, Engelbrecht-Wiggans R, Shubik M, Stark R (eds). New York University Press: New York.

Wilson, R. (1979). Auctions of shares. *The Quarterly Journal of Economics* 93(4): 675-689.

ON-LINE APPENDIX

Analysis of Example 1

There are three bidders, each receives a private signal $X_i \in (0,2)$. Each bidder has a value for the object determined from the signals and given by $V = \sum X_i$. Signals are uniformly distributed on $(0,2)$ and are independent. The number of items auctioned, Z, is also determined by the signals. If two or three of the signals are in the range $(0,1)$ then $Z=2$, and otherwise $Z=1$. It can be shown that $-Z$ is affiliated with the signals (Anderson & Holmberg, 2023).

First we show that there cannot be an increasing equilibrium. Assume that bids are increasing. We evaluate $\mathbb{E}[V \mid X_1 = x, Y_Z = x]$ at $x = 1+\delta, \delta > 0$ and small. Assume without loss of generality that $X_2 < X_3$. If both $X_2, X_3 \in (0,1)$ then $Z = 2$, and $Y_Z < X_1$. Otherwise $Z = 1$ and $Y_Z = x$ implies $X_3 = x$ and $X_2 < x$. This gives an expected value where X_2 is uniform on $(0, 1+\delta)$, so

$$\mathbb{E}[V \mid X_1 = x, Y_Z = x] = 2(1+\delta) + (1+\delta)/2 = \frac{5}{2} + \frac{5}{2}\delta.$$

Next we evaluate $\mathbb{E}[V \mid X_1 = x, Y_Z = x]$ at $x = 1-\delta, \delta > 0$ and small. In the cases where $Z = 1$, we have X_2 and $X_3 \in (1,2)$ so we do not have $Y_Z = X_3 = X_1$. So consider cases where $X_2 \in (0,1)$ and $X_3 \in (X_2, 2)$ and $Z = 2$, and $Y_Z = X_2$. Thus $Y_Z = X_1 = 1-\delta$ implies $X_3 \in (1-\delta, 2)$ and is equally likely to take any value in this range. So

$$\mathbb{E}[V \mid X_1 = x, Y_Z = x] = 2(1-\delta) + \frac{3-\delta}{2} = \frac{7}{2} - \frac{5}{2}\delta.$$

Hence the value of $\mathbb{E}[V \mid X_1 = x, Y_Z = x]$ jumps down at $x = 1$. This gives rise to two different signals that can have the same bid value.

The equilibrium bids are given by $b^*(x) = 1 + \frac{5x}{2}$, for $x \in (0, \frac{1}{2})$, $b^*(x) = h_1^{-1}(x)$ for $x \in (\frac{1}{2}, 1)$, $b^*(x) = h_2^{-1}(x)$ for $x \in (1, \frac{3}{2})$ and $b^*(x) = \frac{5x}{2}$ for $x \in (\frac{3}{2}, 2)$. Here the functions $h_1(b)$ and $h_2(b)$ are determined from the two simultaneous equations:

$$b = \frac{1}{2\left(2 - h_1(b)\right)} \left(2h_2(b)^2 - 2h_1(b)h_2(b) - 2h_2(b) - 5h_1(b)^2 + 10h_1(b) + 4\right), \tag{24}$$

$$b = h_1(b) + \frac{5}{2}h_2(b) + \frac{h_1(b)}{h_2(b)} - \frac{h_1(b)^2}{h_2(b)} - 1. \tag{25}$$

Our aim is to show that the first order conditions hold for this solution i.e. that $b^*(x) = \mathbb{E}[V \mid X_1 = x, W_Z = b^*(x)]$. We will continue to assume that $X_2 < X_3$.

First consider $X_1 = x \leq 0.5$. Then if $W_Z = b^*(x)$ at least two signals are less than 1 and we have $Z = 2$. Hence we have $X_2 = x$ and X_3 is equally likely to have any value in $(x, 2)$. Thus

$$\mathbb{E}[V \mid X_1 = x, W_Z = b^*(x)] = 2x + \frac{2+x}{2} = 1 + \frac{5}{2}x$$

as required.

Second consider $X_1 = x \geq 1.5$. Then if $W_Z = b^*(x)$ there are at least two signals greater than 1 and $Z = 1$. Thus $X_3 = x$ and X_2 is equally likely to have any value in $(0, x)$. Thus

$$\mathbb{E}[V \mid X_1 = x, W_Z = b^*(x)] = 2x + \frac{x}{2} = \frac{5}{2}x$$

as required.

For the overlapping solution we define the function $g(x) = h_2(h_1^{-1}(x))$ for $x \in (0.5, 1)$. Thus $b^*(g(x)) = b^*(x)$, and $g(x) > 1$. Now consider $X_1 = x \in (0.5, 1)$. Then $W_Z = b^*(x)$ if either $X_2 = x$ and $X_3 \in (x, 1) \cup (g(x), 2)$ (implying $Z = 2$) or $X_3 = g(x)$ and $X_2 \in (1, g(x))$ (implying $Z = 1$). The probabilities of these events are proportional to the lengths of the sets involved: $3 - x - g(x)$ and $g(x) - 1$. Also the average signal value over $(x, 1) \cup (g(x), 2)$ is

$$\frac{(1-x)(1+x) + (2-g(x))(2+g(x))}{2(3-x-g(x))} = \frac{5 - x^2 - g(x)^2}{2(3-x-g(x))},$$

so

$$\begin{aligned}
\mathbb{E}[V \mid X_1 &= x, W_Z = b^*(x)] \\
&= \frac{3-x-g(x)}{2-x}\left(2x + \frac{5-x^2-g(x)^2}{2(3-x-g(x))}\right) + \frac{g(x)-1}{2-x}\left(x + g(x) + \frac{1+g(x)}{2}\right) \\
&= \frac{1}{2\,(2-x)}\left(2g(x)^2 - 2xg(x) - 2g(x) - 5x^2 + 10x + 4\right).
\end{aligned}$$

For a given value of b we have $x = h_1(b)$ and $g(x) = h_2(b)$ so we deduce from (24) that $\mathbb{E}[V \mid X_1 = x, W_Z = b^*(x)] = b^*(x)$ as required.

For $X_1 = x \in (1, 1.5)$, we have $W_Z = b^*(x)$ if either $X_2 = g^{-1}(x)$ and $X_3 \in (g^{-1}(x), 1)$ (implying $Z = 2$) or $X_3 = x$ and $X_2 \in (0, g^{-1}(x)) \cup (1, x)$ (implying $Z = 1$). The probabilities of these events are proportional to the lengths of the sets involved: $1 - g^{-1}(x)$ and $g^{-1}(x) + x - 1$. So

$$\begin{aligned}
\mathbb{E}[V \mid X_1 &= x, W_Z = b^*(x)] \\
&= \frac{1 - g^{-1}(x)}{x}(x + g^{-1}(x) + \frac{1 + g^{-1}(x)}{2}) \\
&\quad + \frac{(x - 1 + g^{-1}(x))}{x}(2x + \frac{(x+1)(x-1) + g^{-1}(x)^2}{2(x - 1 + g^{-1}(x))}) \\
&= g^{-1}(x) + \frac{5}{2}x + \frac{g^{-1}(x)}{x} - \frac{g^{-1}(x)^2}{x} - 1.
\end{aligned}$$

For a given value of $W_Z = b$ we have $x = h_2(b)$ and $g^{-1}(x) = h_1(b)$ so we deduce from (25) that $\mathbb{E}[V \mid X_1 = x, W_Z = b^*(x)] = b^*(x)$ as required. Finally we can show that the choice $b^*(x)$ is optimal given that the other players use this bidding strategy. When making this argument, we allow bidder 1 to choose a bid $b \neq b^*(x)$ when observing signal x. For $b < 3.75$ we define $\widetilde{h}_1(b) = b^{*-1}(b)$ restricted to $x \in (0, 1)$, so that $\widetilde{h}_1(b) = h_1(b)$ for $b \geq 2.25$. Then for $x \in (0, 1)$,

$$\begin{aligned}
\mathbb{E}[V - W_Z \mid X_1 &= x, W_Z = b] \\
&= x - b + \mathbb{E}[X_2 + X_3 \mid X_1 = \widetilde{h}_1(b), W_Z = b] \\
&= x - b - \widetilde{h}_1(b) + \mathbb{E}[V \mid X_1 = \widetilde{h}_1(b), W_Z = b] = x - \widetilde{h}_1(b).
\end{aligned}$$

Here the first equality follows because conditioning on different values of X_1 that are both less than 1, makes no difference to the values of Z, X_2 and X_3. The third equality follows from the fact that the first-order condition implies that $\mathbb{E}[V \mid X_1 = \widetilde{h}_1(b), W_Z = b] = b$. The optimal choice of b arises from maximising the profit:

$$\begin{aligned}
\Pi_1(b, x) &= \mathbb{E}\left[(V - W_Z)\, 1_{W_Z < b} \middle| X_1 = x\right] = \int_{-\infty}^{b} \mathbb{E}[V - W_Z \mid X_1 = x, W_Z = s] f_W(s) ds \\
&= \int_{-\infty}^{b} (x - \widetilde{h}_1(s)) f_W(s) ds
\end{aligned}$$

where f_W is the density for the distribution of W_Z given that other bidders are using b^* bids. Now since $x - h_1(s)$ changes from positive to negative at $s = b^*(x)$ this is maximised at $b = b^*(x)$. We can use a similar argument for $x \in (1, 2)$ to establish the optimality of bidding at $b^*(x)$ throughout the range $(0, 2)$.

Technical proofs of Section 2

The following lemma shows a diagonal property of the density function which will be useful in some of our proofs.

Lemma 7 *If $f_k(\boldsymbol{S}, q_0, x_2, x_3, ..., x_n) > 0$ for some values $x_2, x_3, ..., x_n$ and $\boldsymbol{S}$, then $f_k(\boldsymbol{S}, q_0, q_0, ..., q_0) > 0$. Moreover for every $\mathbf{S} \in Q_S$ and $x \in Q_X$ there is a k such that $f_k(\mathbf{S}, x, x, ..., x) > 0$.*

Proof. Given $f_k(\boldsymbol{S}, q_0, x_2, x_3, ..., x_n) > 0$ then by symmetry of f_k in Assumption 4, we deduce that $f_k(\boldsymbol{S}, x_2, q_0, x_3, ..., x_n) > 0$ and hence from the affiliation property (1), $f_k(\boldsymbol{S}, q_0, q_0, x_3, ..., x_n) > 0$. Then by symmetry $f_k(\boldsymbol{S}, q_0, x_3, q_0, x_4, ..., x_n) > 0$ and again the affiliation property shows that $f_k(\boldsymbol{S}, q_0, q_0, q_0, x_4, ..., x_n) > 0$. Continuing in this way we can establish that $f_k(\boldsymbol{S}, q_0, q_0, ..., q_0) > 0$. The final statement follows from this and our assumption that for every $\mathbf{S} \in Q_S$ and $x \in Q_X$, the density $f_k(\mathbf{S}, x, \mathbf{X}_{-1}) > 0$ for some $\mathbf{X}_{-1}$ and k. ■

Lemma 8 is a technical result, which will be useful when proving Proposition 1.

Lemma 8 *If $a_1 \geq a_2 \geq ...a_k$ and $b_1 \geq b_2 \geq ...b_k$ then for any set of probabilities q_j, $j = 1, ..., k$ with $q_j \geq 0$ and $\sum_{j=1}^{k} q_j = 1$,*

$$\sum_{j=1}^{k} q_j a_j b_j \geq \left(\sum_{j=1}^{k} q_j a_j \right) \left(\sum_{j=1}^{k} q_j b_j \right).$$

Proof. Let $\overline{a} = \sum_{j=1}^{k} q_j a_j$ and choose h such that $a_h \geq \overline{a} \geq a_{h+1}$. Then for $j = h$, $(a_j - \overline{a})(b_j - b_h) = 0$, while for $j < h$, both $a_j - \overline{a}$ and $b_j - b_h$ are positive, and for $j > h$ both $a_j - \overline{a}$ and $b_j - b$ are negative. Thus we deduce that

$$\sum_{j=1}^{k} q_j (a_j - \overline{a})(b_j - b_h) \geq 0$$

Hence

$$\sum_{j=1}^{k} q_j a_j b_j - b_h \sum_{j=1}^{k} q_j a_j - \overline{a} \left(\sum_{j=1}^{k} q_j b_j \right) + b_h \overline{a} \left(\sum_{j=1}^{k} q_j \right) \geq 0.$$

Two of the terms are equal to $b_h \overline{a}$ and $-b_h \overline{a}$, respectively. They cancel out, which gives the inequality we require. ■

Technical proofs of Section 3

Lemma 9 *Suppose we have sets C_i, $i = 1, 2, ..., n$ where C_i is a collection of h_i disjoint intervals in Q_X each of which is either $(a_j^{(i)}, b_j^{(i)})$, $[a_j^{(i)}, b_j^{(i)})$, $(a_j^{(i)}, b_j^{(i)}]$ or*

$[a_j^{(i)}, b_j^{(i)}]$ *with* $a_j^{(i)} \leq b_j^{(i)} < a_{j+1}^{(i)}$ *for* $j = 1, 2, ..., h_i$ *and where* $a_1^{(i)} \in \mathbb{R} \cup \{-\infty\}$ *and* $b_{h_i}^{(i)} \in \mathbb{R} \cup \{+\infty\}$*, and we let* $\boldsymbol{C} = C_1 \times C_2 ... \times C_n$*. Suppose that for signal* $\boldsymbol{S}$ *there is* $\mathbf{x}^0 \in \boldsymbol{C}$*, with* $f_k(\boldsymbol{S}, \mathbf{x}^0) > 0$ *for some* k*. Then,* $\mathbb{E}[\, V_1 | \, \boldsymbol{S}, \boldsymbol{X} \in \boldsymbol{C}]$ *is continuous in each of the arguments* $a_j^{(i)}, b_j^{(i)}$*,* $j = 1, 2, ..., h_i$*,* $i = 1, 2, ..., n$*, if* $a_j^{(i)} < b_j^{(i)}$*, and continuous in* $\overline{a}_j^{(i)}$ *if* $a_j^{(i)} = b_j^{(i)} = \overline{a}_j^{(i)}$*. Moreover the limit of* $\mathbb{E}[V_1 | \, \boldsymbol{S}, \boldsymbol{X} \in \boldsymbol{C}]$ *as* $a_j^{(i)} \to b_j^{(i)}$ *is equal to its value when* $a_j^{(i)} = b_j^{(i)}$*. These results also hold when conditioning on* $Z = k_0$ *provided that there is* $\boldsymbol{x}^0 \in C$*, with* $f_{k_0}(\mathbf{S}, \boldsymbol{x}^0) > 0$*.*

Proof. First consider the case where $\Pr(\boldsymbol{S}, \boldsymbol{X} \in \boldsymbol{C}) > 0$. Then

$$\mathbb{E}[\, V_1 | \, \boldsymbol{S}, \boldsymbol{X} \in \boldsymbol{C}] = \frac{\sum_{k \in K_Z} \Pr(Z = k) \int_{\boldsymbol{X} \in \boldsymbol{C}} u\left(\boldsymbol{S}, X_1, \boldsymbol{X}_{-1}\right) df_k\left(\boldsymbol{S}, \boldsymbol{X}\right)}{\sum_{k \in K_Z} \Pr(Z = k) \int_{\boldsymbol{X} \in \boldsymbol{C}} df_k\left(\boldsymbol{S}, \boldsymbol{X}\right)}.$$

We have continuity of $\mathbb{E}[\, V_1 | \, \boldsymbol{S}, \boldsymbol{X} \in \boldsymbol{C}]$ when considered as a function of $a_j^{(i)}$ and $b_j^{(i)}$ since both $\int_{\boldsymbol{X} \in \boldsymbol{C}} u\left(\boldsymbol{S}, X_1, \boldsymbol{X}_{-1}\right) df_k\left(\boldsymbol{S}, \boldsymbol{X}\right)$ and $\int_{\boldsymbol{X} \in \boldsymbol{C}} df_k\left(\boldsymbol{S}, \boldsymbol{X}\right)$ are continuous functions of these parameters. In the case that $\Pr(\boldsymbol{S}, \boldsymbol{X} \in \boldsymbol{C}) = 0$, and so $\int_{\boldsymbol{X} \in \boldsymbol{C}} df_k\left(\boldsymbol{S}, \boldsymbol{X}\right) = 0$ for each k, then we must have either zero probability of a signal with $\mathbf{x}_i^0 \in [a_j^{(i)}, b_j^{(i)}]$ or $a_j^{(i)} = b_j^{(i)} = \overline{a}_j^{(i)}$ for all $j = 1, 2, ..., h_i$ for i in some set I_0. In this case we replace integrals with sums for X_i when $i \in I_0$ in the definition of $\mathbb{E}[\, V_1 | \, \boldsymbol{S}, \boldsymbol{X} \in \boldsymbol{C}]$. For example when $n = 2$ and $a_j^{(1)} = b_j^{(1)} = \overline{a}_j^{(1)}$ for all $j = 1, 2, ..., h_1$ we have

$$\mathbb{E}[\, V_1 | \, \boldsymbol{S}, \boldsymbol{X} \in \boldsymbol{C}] = \frac{\sum_{k \in K_Z} \Pr(Z = k) \sum_{j=1}^{h_1} \int_{(\overline{a}_j^{(1)}, X_2) \in \boldsymbol{C}} u\left(\boldsymbol{S}, a_j^{(1)}, X_2\right) df_k\left(\boldsymbol{S}, a_j^{(1)}, X_2\right)}{\sum_{k \in K_Z} \Pr(Z = k) \sum_{j=1}^{h_1} \int_{(\overline{a}_j^{(1)}, X_2) \in \boldsymbol{C}} df_k\left(\boldsymbol{S}, a_j^{(1)}, X_2\right)}.$$

Note that the integrals here are with respect to X_2 with $\overline{a}_j^{(1)}$ fixed. Thus from our assumption that there is $\mathbf{x}^0 \in \mathbf{C}$, with $f_k(\mathbf{S}, \mathbf{x}^0) > 0$ for some k we know that at least one of these integrals has non-zero value. When $I_0 = \{1, 2, \ldots, N\}$ then there are no integrals in the expression and the denominator becomes the sum over the non-zero $f_k(\boldsymbol{S}, \boldsymbol{x})$ values for $\boldsymbol{x} \in \boldsymbol{C}$. The conclusion on continuity still holds in this case. Also the result on the limit of $\mathbb{E}[V_1 | \, S, X \in \mathbf{C}]$ as $a_j^{(i)} \to b_j^{(i)}$ follows directly from the continuity of f_k and u.

The argument is exactly the same in the case that we consider $\mathbb{E}[V_1 | \, \mathbf{S}, \mathbf{X} \in \mathbf{C}, Z = k_0]$. ■

Proof. (Lemma 1) Theorem 5 in Milgrom and Weber (1982) establishes that for V_1 a non-decreasing function of a set of affiliated variables, its expected value conditioning on any subset of the affiliated variables is nondecreasing in the conditioning variables. This result can be straightforwardly extended to include conditioning on a sublattice (see Anderson & Holmberg, 2023). Thus, for any sublattice L, $\mathbb{E}[V_1| X_1 = x, \widetilde{\mathbf{X}} \in L]$ is non-decreasing in x.

To establish strictly increasing in x, take any two possible signals $x_L < x_U$ and both in Q_X. Define V_1^L from V_1 by setting $V_1^L(\mathbf{S}, \mathbf{X}) = V_1(\boldsymbol{S}, x_L, X_2, ..., X_n)$ for $X_1 \in [x_L, x_U]$ and $V_1^L = V_1$ otherwise. Thus using the fact that u is strictly increasing, we have $V_1^L < V_1$ for $X_1 = x_U$. Then

$$\begin{aligned}
\mathbb{E}\left[V_1 | X_1 = x_L, \widetilde{\mathbf{X}} \in L\right] &= \mathbb{E}\left[V_1^L \middle| X_1 = x_L, \widetilde{\mathbf{X}} \in L\right] \\
&\leq \mathbb{E}\left[V_1^L \middle| X_1 = x_U, \widetilde{\mathbf{X}} \in L\right] \\
&< \mathbb{E}\left[V_1 | X_1 = x_U, \widetilde{\mathbf{X}} \in L\right]
\end{aligned}$$

which proves the result for $\mathbb{E}[V_1| X_1 = x, \widetilde{\mathbf{X}} \in L]$. Since Y_Z is also an affiliated variable, the argument for $\mathbb{E}[V_1| X_1 = x, Y_Z = y, \widetilde{\mathbf{X}} \in L]$ is similar. ■

Lemma 10 *For the uniform-price auction we have $P^{\mathbb{I}}(x) \geq P^{\mathbb{N}}(x)$, for strictly increasing symmetric equilibria. We have $P^{\mathbb{I}}(x) = P^{\mathbb{N}}(x)$ if X_0 is independent of $X_1, \ldots, X_n$ and Z.*

Proof. First, note that

$$v(x, y) = \mathbb{E}[\hat{v}(x, y; X_0)].$$

Moreover, it follows from Theorem 5 in Milgrom & Weber (1982) that:

$$v(u, u) = \mathbb{E}[\hat{v}(u, u; X_0)| X_1 = u, Y_Z = u] \leq \mathbb{E}[\hat{v}(u, u; X_0)| X_1 = x, Y_Z = u], \quad (26)$$

if $x \geq u$. The inequality arises because if the expected value is conditioned on a higher X_1 signal, then this tends to give a higher X_0 signal, which in its turn increases the value of $\hat{v}$. This effect disappears if X_0 is independent of $X_1, \ldots, X_n$ and Z. In this special case, the inequality above becomes an equality. This is also true for the inequalities in the rest of the proof.

If we let u be a random variable, which we set equal to Y_Z, and if we condition both sides of (26) on $X_1 = x$, and $u < x$ then we get (in expectation):

$$\mathbb{E}\left[v\left(Y_Z, Y_Z\right)\middle| X_1 = x, Y_Z < x\right] \leq \mathbb{E}\left[\hat{v}\left(Y_Z, Y_Z; X_0\right)\middle| X_1 = x, Y_Z < x\right]. \qquad (27)$$

Now, using Theorem 1 and the inequality above, we get:

$$\begin{aligned} P^{\mathbb{N}}(x) &= \mathbb{E}\left[b\left(Y_Z\right)\middle| X_1 = x, Y_Z < x\right] \\ &= \mathbb{E}\left[v\left(Y_Z, Y_Z\right)\middle| X_1 = x, Y_Z < x\right] \\ &\leq \mathbb{E}\left[\hat{v}\left(Y_Z, Y_Z; X_0\right)\middle| X_1 = x, Y_Z < x\right] \\ &= \mathbb{E}\left[\hat{b}^*\left(Y_Z; X_0\right)\middle| X_1 = x, Y_Z < x\right] \\ &= P^{\mathbb{I}}(x). \end{aligned}$$

■

Technical proofs of Section 4

Our next lemma is a technical result which shows what the ranking of J_x implies for the ranking of J. Related results have been proved by Milgrom & Weber (1982) for single-object auctions. The argument is slightly more involved for multi-unit auctions. We compare two auction designs indicated by superscripts $\mathbb{A}$ and $\mathbb{B}$.

Lemma 11 *For two auction designs we have* $J^{\mathbb{A}}(x,x) \geq J^{\mathbb{B}}(x,x)$ *for* $x \in Q_X$ *if* $J_x^{\mathbb{A}}(x,x) \geq J_x^{\mathbb{B}}(x,x)$ *for* $x \in Q_X$ *and* $\lim_{x \searrow a_L} J^{\mathbb{A}}(x,x) \geq \lim_{x \searrow a_L} J^{\mathbb{B}}(x,x)$.

Proof. (Lemma 11) If the considered bidder observes the private signal x and acts as if observing $\widetilde{x}$, then its expected payoff is given by:

$$U(\widetilde{x}, x) - K(\widetilde{x}, x).$$

In equilibrium, we have that it is optimal for the bidder to choose $\widetilde{x} = x$, so

$$K_{\widetilde{x}}(x,x) = U_{\widetilde{x}}(x,x), \qquad (28)$$

where $U_{\widetilde{x}}(x,x)$, and consequently also $K_{\widetilde{x}}(x,x)$, is independent of the auction design. We have from (5) and (28) that:

$$U_{\widetilde{x}}(x,x) = K_{\widetilde{x}}(x,x) = J_{\widetilde{x}}(x,x)\Pr(Y_Z \le x|\, x) + J(x,x)\left.\frac{d\Pr(Y_Z \le \widetilde{x}|\, x)}{d\widetilde{x}}\right|_{\widetilde{x}=x} \quad (29)$$

$$J_{\widetilde{x}}(x,x) = \frac{U_{\widetilde{x}}(x,x) - J(x,x)\left.\frac{d\Pr(Y_Z\le\widetilde{x}|x)}{d\widetilde{x}}\right|_{\widetilde{x}=x}}{\Pr(Y_Z \le x|\, x)}, \quad (30)$$

where $\Pr(Y_Z \le x|\, x) > 0$ for all $x \in Q_X$. Moreover, we have

$$\frac{dJ(x,x)}{dx} = J_{\widetilde{x}}(x,x) + J_x(x,x). \quad (31)$$

From the inequality $J_x^{\mathbb{A}}(x,x) \ge J_x^{\mathbb{B}}(x,x)$ we now get

$$\begin{aligned} \frac{d\left(J^{\mathbb{A}}(x,x) - J^{\mathbb{B}}(x,x)\right)}{dx} &= J_{\widetilde{x}}^{\mathbb{A}}(x,x) + J_x^{\mathbb{A}}(x,x) - J_{\widetilde{x}}^{\mathbb{B}}(x,x) - J_x^{\mathbb{B}}(x,x). \\ &\ge \frac{\left(J^{\mathbb{B}}(x,x) - J^{\mathbb{A}}(x,x)\right)\left.\frac{d\Pr(Y_Z\le\widetilde{x}|x)}{d\widetilde{x}}\right|_{\widetilde{x}=x}}{\Pr(Y_Z \le x|\, x)}. \end{aligned}$$

Hence,

$$\frac{d\left(J^{\mathbb{B}}(x,x) - J^{\mathbb{A}}(x,x)\right)}{dx} \le -\frac{\left(J^{\mathbb{B}}(x,x) - J^{\mathbb{A}}(x,x)\right)\left.\frac{d\Pr(Y_Z\le\widetilde{x}|x)}{d\widetilde{x}}\right|_{\widetilde{x}=x}}{\Pr(Y_Z \le x|\, x)}.$$

Thus, it follows from Grönwall's lemma that:

$$J^{\mathbb{B}}(x,x) - J^{\mathbb{A}}(x,x) \le \left(J^{\mathbb{B}}(a,a) - J^{\mathbb{A}}(a,a)\right)\exp\left(-\int_a^x \frac{\left.\frac{d\Pr(Y_Z\le\widetilde{x}|v)}{d\widetilde{x}}\right|_{\widetilde{x}=v}}{\Pr(Y_Z \le v|\, v)}dv\right), \quad (32)$$

for $a, x \in Q_X$. Note that

$$0 \le \exp\left(-\int_a^x \frac{\left.\frac{d\Pr(Y_Z\le\widetilde{x}|v)}{d\widetilde{x}}\right|_{\widetilde{x}=v}}{\Pr(Y_Z \le v|\, v)}dv\right) \le 1 \quad (33)$$

for $a, x \in Q_X$. This follows from observing that $\left.\frac{d\Pr(Y_Z\le\widetilde{x}|v)}{d\widetilde{x}}\right|_{\widetilde{x}=v} \ge 0$, and hence the integrand, is non-negative. Moreover, we have by assumption that $\lim_{x\searrow a_L} J^{\mathbb{B}}(x,x) \le \lim_{x\searrow a_L} J^{\mathbb{A}}(x,x)$. Hence, by choosing an a value sufficiently close to a_L, we can use (32) and (33) to find a contradiction for any $x \in Q_X$ such that $J^{\mathbb{B}}(x,x) > J^{\mathbb{A}}(x,x)$. Thus we can conclude that $J^{\mathbb{A}}(x,x) \ge J^{\mathbb{B}}(x,x)$ for $x \in Q_X$. ■

Proof. (Lemma 2) It follows directly from Lemma 11 that $J^{\mathbb{A}}(x,x) \geq J^{\mathbb{B}}(x,x)$ for $x \in Q_X$. Next, we want to show that this implies that $K^{\mathbb{A}}(x,x) \geq K^{\mathbb{B}}(x,x)$. From (5) we can deduce that

$$\begin{aligned} K_x(x,x) &= J_x(x,x)\Pr(Y_Z \leq x|\, x) + J(x,x) \left.\frac{d\Pr(Y_Z \leq \widetilde{x}|\, x)}{dx}\right|_{\widetilde{x}=x} \quad (34) \\ &= J_x(x,x)\Pr(Y_Z \leq x|\, x) + \frac{K(x,x)}{\Pr(Y_Z \leq x|\, x)} \left.\frac{d\Pr(Y_Z \leq \widetilde{x}|\, x)}{dx}\right|_{\widetilde{x}=x}. \end{aligned}$$

Moreover, we have from (28)

$$\frac{dK(x,x)}{dx} = K_{\widetilde{x}}(x,x) + K_x(x,x) = U_{\widetilde{x}}(x,x) + K_x(x,x). \quad (35)$$

Hence we have from (34) and the assumption $J_x^{\mathbb{A}}(x,x) - J_x^{\mathbb{B}}(x,x) \geq 0$ for $x \in Q_X$ that,

$$\begin{aligned} &\frac{d\left(K^{\mathbb{A}}(x,x) - K^{\mathbb{B}}(x,x)\right)}{dx} \\ = &\left(J_x^{\mathbb{A}}(x,x) - J_x^{\mathbb{B}}(x,x)\right)\Pr(Y_Z \leq x|\, x) + \left(K^{\mathbb{A}}(x,x) - K^{\mathbb{B}}(x,x)\right) \frac{\left.\frac{d\Pr(Y_Z \leq \widetilde{x}|x)}{dx}\right|_{\widetilde{x}=x}}{\Pr(Y_Z \leq x|\, x)} \\ \geq &\left(K^{\mathbb{A}}(x,x) - K^{\mathbb{B}}(x,x)\right) \frac{\left.\frac{d\Pr(Y_Z \leq \widetilde{x}|x)}{dx}\right|_{\widetilde{x}=x}}{\Pr(Y_Z \leq x|\, x)}. \end{aligned}$$

The inequality can be written as follows:

$$\frac{d\left(K^{\mathbb{B}}(x,x) - K^{\mathbb{A}}(x,x)\right)}{dx} \leq \left(K^{\mathbb{B}}(x,x) - K^{\mathbb{A}}(x,x)\right) \frac{\left.\frac{d\Pr(Y_Z \leq \widetilde{x}|x)}{dx}\right|_{\widetilde{x}=x}}{\Pr(Y_Z \leq x|\, x)}.$$

It follows from Proposition 1 that Y_Z is affiliated with the signals, so $\left.\frac{d\Pr(Y_Z \leq \widetilde{x}|v)}{dv}\right|$ is non-positive. Moreover, we have from (7) that $\lim_{x \searrow a_L} K^{\mathbb{A}}(x,x) = \lim_{x \searrow a_L} K^{\mathbb{B}}(x,x) = 0$. Using a similar argument as in the proof of Lemma 11, it can be shown that $K^{\mathbb{A}}(x,x) \geq K^{\mathbb{B}}(x,x)$ for $x \in Q_X$. ■

Technical proofs of Section 7

Proof. (Lemma 3) Ex-post optimality implies that a bid is not accepted at a price above the bidder's value or rejected when the lowest accepted bid is below its value. Hence, a bid at value is always ex-post optimal. First consider the case where a bidder's equilibrium bid is accepted at a price above its value. Hence, the bid is above the bidder's private value and one or more competitor's bids are price setting and strictly between these values or equal to this bid. It follows from our assumption on regular bids that this occurs with a positive probability, with a possible exception for ties.[26] The bidder can avoid unprofitable outcomes by reducing its bid to its valuation. The bid is never price setting when it is accepted in a uniform-price auction. Hence, such a change will not influence the payoff from profitable outcomes where the original bid was accepted at a price below the valuation. The expected profit would be improved by such a change, contradicting the assumption that this is an equilibrium.

Next consider the case where the bidder's equilibrium bid is rejected when the lowest accepted bid is below its value with a positive probability. Hence, the bid is below the value of the bidder and a set of competitor's bids are accepted either strictly between this bid and the bidder's value or at this bid. Hence, the bidder can increase its payoff by increasing its bid to its valuation. ■

Proof. (Lemma 4) If the bid of bidder i is rejected with probability 1 for a range of signals $(x_i, \overline{V}_i)$ then there must be a set Ω of at least $\overline{Z}$ competitors that always bid $\overline{V}_i$ or higher. Otherwise there is some bid value $\overline{V}_i - \varepsilon$ that is accepted with positive probability and a range of signals close enough to $\overline{V}_i$ where the bidder's value is above $\overline{V}_i - \varepsilon$ and so the bidder would find it profitable to bid at $\overline{V}_i - \varepsilon$, as these bids would be accepted with a positive probability and generate a positive expected payoff. A competitor bidder $j \in \Omega$ bidding at or above $\overline{V}_i$ for all signals is only possible if the clearing price is $\underline{V}_j$, or lower, with probability 1, also for the lowest realization of Z, $\underline{Z}$. Otherwise there is a positive probability of the clearing price being at level $\underline{V}_j + \varepsilon$ or higher and the competitor would find it profitable to deviate and lower the bid for the signals in some range $(\underline{V}_j, \underline{V}_j + \varepsilon)$ to avoid that the bid is accepted at a clearing price above the value. Hence, at least $n - \underline{Z}$ bidders in a set Ω_0 must, with probability 1, bid at $\min\{\underline{V}_j : j \in \Omega\}$, or lower. Moreover since Assumption B and our definitions imply that $\overline{V}_i > V_L$ and $V_L = \min\{\underline{V}_j : j \in \Omega_L\} \geq \min\{\underline{V}_j : j \in \Omega\}$, we deduce that the bids in Ω are strictly higher than any bids in Ω_0. Thus the two sets do not overlap

[26] It cannot be ruled out that the firm's bid is accepted, that the clearing price is set by a competitor's bid at the same price, and that such an event occurs with measure zero.

giving a minimum of $n-\underline{Z}+\overline{Z}$ bidders and hence $\overline{Z}=\underline{Z}$ and Ω_0 is the set complement of Ω. Since all the bidders in Ω_0 have bids that are rejected with probability 1, we deduce that the bids for bidders in set Ω must bid at $\max\{\overline{V}_i : i \in \Omega_0\}$ or higher. Thus we have established all the features of the high-low equilibrium.

We can use a reflected-auction argument to show that supply must be certain and that the equilibrium must be a high-low equilibrium if the bid of bidder i is accepted with probability 1 for a range of signals $(\underline{V}_i, x_i)$. ■

Lemma 12 *An active bid is price setting (the highest rejected bid) with a positive probability and the lowest bid that is accepted with a positive probability.*

Proof. (Lemma 12) Consider an active bid by bidder i at a price y. Thus there are k_1 bidders j (with $j \neq i$) which strictly dominate y, and k_2 bidders j (with $j \neq i$) that y strictly dominates, and $k_1 < \overline{Z}$, $k_2 < n - \underline{Z}$. To show this bid is price setting we want a Z so that with a positive probability there are exactly Z other bidders bidding above or at y. There are $n-1-k_2$ other bidders that may bid at or above y, and k_1 that bid above y with probability 1. Thus we may choose any Z value between k_1 and $n-1-k_2$, and find a positive probability of exactly that number of bids at or above y. Note that $n-1-k_2$ is at least as large as k_1 since $n-1 \geq k_1+k_2$ as bidders cannot be both strictly dominated by y and strictly dominate y. But $k_1 < \overline{Z}$, and $n-1-k_2 \geq \underline{Z}$ and so this range includes values in the range $\underline{Z}$ to $\overline{Z}$, and hence a value of Z that occurs with positive probability. It follows from our assumptions that this probability is positive independent of the signals that the bidders observe. Also we can show that the bid at y is the lowest bid that is accepted with a positive probability. This is the case if there are exactly $Z-1$ other bidders bidding above or at y with a positive probability. For any Z value between k_1+1 and $n-k_2$ we will have exactly that number of bids at or above y with a positive probability. We have $k_1+1 \leq \overline{Z}$ and $n-k_2 > \underline{Z}$, so as before this range of values must include a value of Z that occurs with positive probability. ■

The result in Lemma 5, below, may perhaps seem obvious, given our ex-post optimality result in Lemma 3. But there is a subtlety that complicates the argument. Properties of a competitor's active bid could change once we condition on a bidder making a bid at a specific price, such as y. In this case, a competitor's active bid below y may not be accepted with a positive probability and a competitor's active bid above y may not be rejected with a positive probability.

Proof. (Lemma 5) Below we prove two properties of active bids. If we condition on a bidder making a bid at y, then 1) a competitor's active bid below y is price setting with a positive probability and 2) a competitor's active bid above y is the lowest bid that is accepted with a positive probability. These two properties are sufficient for the case when the bid from the competitor is active. As explained below, the result for the case when the bid of the bidder is active follows straightforwardly from Lemma 12.

We start by considering the case where bidder i observes the value x and makes an equilibrium bid at $y > x$. Suppose that bidder j has, with positive probability, an active bid between x and y, or at y. Thus we consider the event that some bidder $j \neq i$ bids at a with $x < a \leq y$ and there are k_1 bidders k (with $k \neq j$) which strictly dominate a, and k_2 bidders k (with $k \neq j$) that a strictly dominates, where $k_1 < \overline{Z}$ and $k_2 < n - \underline{Z}$ (because the bid is active).

When $y > a$ and there are exactly Z bidders bidding above a (including the bid by bidder i), a will be the clearing price, bidder i's bid is accepted and we get a contradiction from ex-post optimality. If bidder i strictly dominates a then such a contradictory outcome is possible if $n - 1 - k_2 \geq Z \geq k_1$. Since $n - 1 - k_2 \geq \underline{Z}$ and $k_1 < \overline{Z}$, this range must intersect with a possible value of Z occurring with a positive probability. In the case that bidder i does not strictly dominate a then we get a contradiction if there are exactly $Z - 1$ other bidders, in addition to bidder i (we condition on x), bidding above a. Excluding bidder i there is a positive probability of a bid greater than a by exactly ℓ bidders for any $\ell \geq k_1$ and $\ell \leq n - 2 - k_2$. We get a contradiction with a positive probability for $n - 2 - k_2 \geq Z - 1 \geq k_1$. Again these inequalities must be satisfied for a possible value of Z between $\underline{Z}$ and $\overline{Z}$. Thus in either case we have a contradiction.

When $y = a$, if we cannot use the argument above for $y > a$ then there is a positive probability of a competitor's bid being at $a = y$. We want to show that, conditioning on bidder i's bid at y, there is a positive probability of bidder i's bid being accepted with a clearing price at a, which would not be ex-post optimal. This will happen if amongst bidders not i or j there are at least $Z-1$ bidders which bid at a or above (i.e. no more than $n-Z-1$ bidding strictly below a) and no more than $Z-1$ bidders bidding strictly above a. Notice that bidders i and j do not strictly dominate a and are not strictly dominated by a, so that we know $k_1+k_2 \leq n-2$. We require $k_2 \leq n-Z-1$ and $k_1 \leq Z-1$, i.e. $n-1-k_2 \geq Z \geq k_1+1$. Since $n-1-k_2 \geq \underline{Z}$ and $k_1+1 \leq \overline{Z}$, this range must intersect with a possible value of Z occurring with a positive probability, which gives the contradiction we require.

A similar argument applies when the bidder bids below its value. Ex-post optimality shows that there cannot be a positive probability of the lowest accepted bid being between the bid and its value, and the argument proceeds similarly.

Now consider the case where the bid itself is active. We start with the case where the bid at y is above the bidder's value and the competitor's bid is between this bid and the bidder's value. From Lemma 12, there is a positive probability that the bidder's bid is the lowest accepted bid, then (with a positive probability) the price is at the competitor's bid or above and is above the bidder's value. Thus an improvement is possible. Similarly if the bid is below the bidder's value and is active then there is a positive probability that this bid is the highest rejected bid. This will lead to an improvement from raising this bid to the bidder's value. ■

Proof. (Lemma 6) Consider a bid at y by bidder i. If $y < a^-$ then the bids from all the bidders with $p_{[j]} \geq a^-$ are always higher than y. There are at least $\overline{Z}$ such bidders and hence the bid y is certain to be rejected and is therefore not active. The situation with $y = a^-$ is more complicated since we may have more than one bidder with $p_{[j]} = a^-$ (including bidder i itself). A bid at a^- is rejected with probability 1 if there are at least $\overline{Z}$ bidders other than i, either with $p_{[j]} > a^-$ or with $p_{[j]} = a^-$ and not having an accumulation of bids at a^-. When $y = a^-$ the bid is accepted with positive probability if there are less than $\overline{Z}$ bidders other than i, either with $p_{[j]} > a^-$ or with $p_{[j]} = a^-$ and not having an accumulation of bids at a^-. If $y > a^-$ then the bid is accepted with positive probability.

If $y > a^+$ then the bids from all the bidders with $p^{[j]} \leq a^+$ are always lower than y. There are at least $n - \underline{Z}$ such bidders and hence the bid y is certain to be accepted, and is not active. A bid at a^+ is accepted with probability 1 if there are at least $n - \underline{Z}$ bidders other than i, either with $p^{[j]} < a^+$ or with $p^{[j]} = a^+$ and not having an accumulation of bids at a^+. When $y = a^+$ the bid is rejected with positive probability if there are less than $n - \underline{Z}$ bidders other than i, either with $p^{[j]} < a^+$ or with $p^{[j]} = a^+$ and not having an accumulation of bids at a^+. If $y < a^+$ then there is a positive probability that the bid is rejected. Together with the earlier observation on $y > a^-$ implying a positive probability of acceptance, we have shown a bid y with $a^- < y < a^+$ must be active. Combining all these implications gives the statement of the Lemma. ■