On the Pettis Integral Approach to Large Population Games

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Abstract

Large population games with incomplete information often entail integration of a continuum of random variables. We showcase the usefulness of the integral notion à la Pettis (1938) to study such models. We present several results on Pettis integral relevant to game-theoretic applications, including convenient sufficient conditions for Pettis integrability and Fubini-like exchangeability formulae, illustrated through a running example. We further investigate an equation involving Pettis integral, which has a stochastic process as the unknown, motivated by its use in equilibrium characterization. The solvability of the equation is linked to its spectral properties.

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1 Introduction

When analyzing economic models involving uncertainties associated with a large number of economic entities, we encounter the demand for aggregating a continuum of random variables. For instance, in the beauty contest model introduced by Morris and Shin (2002), each player best responds to an aggregated economic variable—an "integral" of actions taken by a continuum of opponents—by choosing an action equal to a convex combination of her best estimates regarding the aggregated action and an exogenous payoff-relevant state. As opponents' actions are stochastic under incomplete information, the aggregated action takes the form of the integral of a "stochastic process," which are indexed by the players' identities.

Integrating stochastic processes comes with mathematical complications when the process contains purely idiosyncratic components: As pointed out by Judd (1985), a typical sample path of an i.i.d. process is terribly discontinuous, hampering the modeler to define the integral in a realized path-wise manner, which is one of the most natural approaches. Moreover, while one may expect a certain law of large numbers (LLN) to apply when aggregating a continuum of i.i.d. random variables, path-wise integration lacks this property: if we assume the idiosyncratic components disappear in the aggregate for all subintervals of a population, the stochastic process must essentially be constant, limiting the validity of the LLN to only trivial cases.[1]

The source of these problems can be attributed to a way we interpret the integral of a stochastic process. In this paper, instead, we advocate interpreting aggregation differently by appealing to the integral notion à la Pettis (1938) and demonstrates the usefulness of "Pettis integral" in the analysis of large population games.[2] The Pettis integral is defined for an abstract process that assigns to each input in a measurable space an output that admits a value in a normed space. The definition leverages the duality of the outcome space. Especially, when the process is valued in a Hilbert space, the core idea of Pettis can be roughly described as "the inner product of the integral coincides with the integral of inner products." In a probabilistic context, this implies that the "the expectation of the integral coincides with the integral of expectations," which bears similarity to Fubini's exchangeability. In this regard, we can think of Pettis integral as an integral notion that is designed to retain the desirable property of usual integral notions, holding for a continuum of random variables.

To clarify the relevance to game-theoretic analysis, let us consider the following sym-

[1] See Theorem 1 in Uhlig (1996), as well as Proposition 2.1 in Sun (2006).

[2] Some authors including Al-Najjar (1995) and Uhlig (1996) also consider Pettis integral as a remedy for the measurability problem of Judd (1985).

metric linear-quadratic-Gaussian (LQG) game of Bergemann and Morris (2013).

Example 1. Each player i in a closed interval $[0,1]$ chooses an action $a_i \in \mathbb{R}$ as a function of a private signal x_i and a public signal y, which are correlated with a payoff-relevant random variable θ, referred to as a *state*. We assume that these random variables are normally distributed as follows:

$$\theta \sim \mathcal{N}\Big(\mu_\theta,\, \sigma_\theta^2\Big), \quad \epsilon_i \sim \mathcal{N}\Big(0,\, \sigma_x^2\Big), \quad \epsilon \sim \mathcal{N}\Big(0,\, \sigma_y^2\Big), \quad x_i = \theta + \epsilon_i, \quad y = \theta + \epsilon,$$

where ϵ_i and ϵ are independently distributed with respect to each other and to the state θ. A player's payoff is determined by her own action a_i, the aggregated action $A \coloneqq \int_0^1 a_j \mathrm{d}j$, and the state θ. Specifically, we posit that in an equilibrium, player i sets her action a_i equal to a linear combination of her best estimates regarding A and θ,

$$a_i = r\,\mathbb{E}\left[A \mid x_i,\, y\right] + s\,\mathbb{E}\left[\theta \mid x_i,\, y\right] + k, \tag{1}$$

where $r, s, k \in \mathbb{R}$ are the parameters of the best response function. The beauty contest is a special case that sets $r \in (0,1)$, $s = 1 - r$, and $k = 0$.

In this example, defining the aggregated action A entails the measurability issue: Each individual action a_j contains idiosyncratic randomness stemming from independent noise in private signals. Our starting point is interpreting A as the Pettis integral of a strategy profile. Unlike other integral notions, Pettis integrability requires only a notably weak measurability condition, which can be met even when the presence contains i.i.d. components. In Proposition 1, we outline easy-to-verify sufficient conditions for a process to be Pettis integrable. In probabilistic contexts, these conditions are expressed solely in terms of the first and second moments of a stochastic process, enabling us to readily verify the well-definedness of A.

As argued in Angeletos and Pavan (2007), two pivotal variables in the welfare analysis of LQG games are *volatility* and *dispersion*, defined respectively as the variance of aggregated action $V \coloneqq \mathbb{V}\mathrm{ar}\,[A]$ and that of the idiosyncratic difference $D \coloneqq \mathbb{V}\mathrm{ar}\,[A - a_i]$. Ui and Yoshizawa (2015) show that the expected welfare in any symmetric LQG game can be expressed as a linear combination of V and D, thus, calculating these variables holds particular importance for welfare evaluations. Earlier analyses perform this task by relying on the intuition that idiosyncratic parts of individual actions cancel out through aggregation. However, in Proposition 2, we provide a formal demonstration that this kind of LLN can be derived directly from the definition of Pettis integral.

We also establish results on Pettis integral that are instrumental in characterizing equilibrium in large population games. In Proposition 3, we present a "conditional" version of Fubini-like formula, justifying the interchange of conditional expectation and integration. This result facilitates the computation of an agent's conditional expectation concerning the aggregated action, $\mathbb{E}[A \mid x_i, y]$, a crucial step in deriving a closed form of an equilibrium strategy using the "matching coefficient" method.

When players have different payoff functions or signal distributions, the matching coefficient method is not practical to derive equilibrium strategies since these ought to be asymmetric across players. Nonetheless, even in this case, an equilibrium can be characterized as a solution to an equation akin to (1) that involves the (weighted) Pettis integral of others players' strategies and the state. Motivated by this, we investigate the nature of such a *Pettis integral equation*, which has a stochastic process as the unknown, from a general mathematical standpoint. In Proposition 4, we offer regularity conditions on primitives, under which the equation enjoys a desirable property of *compactness*. Leveraging the Riesz–Fredholm theory of functional analysis, this result enables us to establish connections between the solvability of the equation and the spectral properties of the corresponding integral operator. In our related work Miyashita and Ui (2024), we employ this finding to characterize the existence, uniqueness, and stability of equilibrium in large population games, accommodating general payoff and information structures, beyond the symmetric case illustrated in Example 1.

The rest of this paper is organized as follows. In Section 2, we offer the definition of Pettis integral and establish some basic results on it. In Section 3, we delve into the analysis of Pettis integral equations. In Section 4, we conclude the paper by discussing some potential benefits and cautions of the Pettis-integral approach in analyzing large population games. Throughout the paper, Example 1 serves as a running example to illustrate the relevance of our results to economic contexts. All proofs omitted from the main body are relegated to Appendix.

2 Pettis Integral

Notations. For a (real) normed space X, denote its norm by $\|\cdot\|_X$ and the dual space of bounded linear functionals by X^*. We write as $\langle\cdot,\cdot\rangle_X$ when X is equipped with an inner product. The subscript "$_X$" may be omitted for simplicity. Denote by $x_n \xrightarrow{\|\cdot\|} x$ when a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converges to $x \in X$ in norm. Also, denote by $x \xrightarrow{\mathrm{w}} x$ when $\{x_n\}_{n\in\mathbb{N}}$ weakly converges to x. For a subset $E \subseteq X$, denote its closure by $\overline{E}$, the set of finite linear combinations of its elements by $\operatorname{span}(E)$, and the orthogonal complement by $E^\perp$. Let $\mathcal{B}_\mathbb{R}$ be the Borel σ-algebra on $\mathbb{R}$. For a semiring $\mathcal{S}$ on a set T, $\sigma(\mathcal{S})$ denote the smallest σ-algebra containing $\mathcal{S}$. When $\mathcal{T}$ is a σ-algebra on T, $\mathcal{T}\otimes\mathcal{T} = \sigma(\mathcal{T}\times\mathcal{T})$ denotes the product σ-algebra.

2.1 Pettis Integral of General Hilbert-valued Processes

An *input space* is given as a finite measure space $(T, \mathcal{T}, \nu)$, which is normalized as $\nu(T) = 1$. For example, T can be the unit interval $[0, 1]$ with the Lebesgue measure. An *output space* is given as a Hilbert space X, equipped with the inner product $\langle \cdot, \cdot \rangle_X$. That X is Hilbertian is needed to prove Proposition 1, while the notion of Pettis integral can be defined in a more general setting.[3]

A *process* is meant by any function $f : T \to X$ that assigns an output $f(t) \in X$ to each input $t \in T$. Our primary interest lies in the integration of a process f with respect to ν. In stating the next definition, and throughout the paper, the integral of any real-valued function shall be understood in the sense of Lebesgue. The integral range will be suppressed when performed over the entire space T.

Definition 1. A process $f : T \to X$ is *weakly measurable* if the mapping $t \mapsto \langle y, f(t) \rangle$ is $(\mathcal{T}, \mathcal{B}_{\mathbb{R}})$-measurable for every $y \in X$. Moreover, f is *Pettis integrable* if it is weakly measurable, and if there exists $x \in X$ such that

$$\langle y, x \rangle_X = \int \langle y, f(t) \rangle_X \, \mathrm{d}\nu(t), \quad \forall y \in X. \tag{2}$$

In this case, x is called the *Pettis integral* of f, which is written as w-$\int f(t)\mathrm{d}\nu(t)$, or simply, $\int f(t)\mathrm{d}\nu(t)$ when there is no risk of confusion.

Two standard references on weak measurability and Pettis integral are the books Diestel and Uhl (1977) and Talagrand (1984). Here, we mention a few properties that are immediate from the definition. First, the Pettis integral is linear: For two Pettis integrable processes f and g, their point-wise linear combination $\alpha f + \beta g$ is Pettis integrable for any $\alpha, \beta \in \mathbb{R}$, and we have

$$\int (\alpha f + \beta g)(t)\mathrm{d}\nu(t) = \alpha \int f(t)\mathrm{d}\nu(t) + \beta \int g(t)\mathrm{d}\nu(t).$$

Second, the Pettis integral is consistent with convergence in norm: For any sequence $\{f_n\}_{n\in\mathbb{N}}$ of Pettis integrable processes, if $f_n(t)$ converges to $f(t)$ in norm ν-almost everywhere (ν-a.e.), then f is Pettis integrable. Moreover, the sequence of the Pettis integrals $\int f_n(t)\mathrm{d}\nu(t)$ converges to $\int f(t)\mathrm{d}\nu(t)$ in norm.

There are several known sufficient conditions for Pettis integrability. For example, Huff (1986) reports that a process f is Pettis integrable if it is weakly measurable, and if the operator $y \mapsto \langle y, f(\cdot) \rangle_X$ acts weak-to-weak continuously from X to the Banach space of ν-integrable real-valued functions. However, these conditions may not be very tractable

[3] See Chapter 11.10 of Aliprantis and Border (2006).

from applied standpoints since they involve arbitrary elements of the dual space, most of which are orthogonal to the process f itself. In light of this, we provide simpler sufficient conditions, which can be stated solely in terms of the moments of the process f, without invoking duality.

Proposition 1. *Consider the following conditions on a process $f : T \to X$.*

(P1). *The mapping $t \mapsto \langle f(s), f(t) \rangle$ is $(\mathcal{T}, \mathcal{B}_{\mathbb{R}})$-measurable for every $s \in T$.*

(P2). *The mapping $t \mapsto \|f(t)\|$ is $(\mathcal{T}, \mathcal{B}_{\mathbb{R}})$-measurable and $\int \|f(t)\| \mathrm{d}\nu(t) < \infty$.*

If (and only if) f satisfies (P1)*, it is weakly measurable. If, in addition, f satisfies* (P2)*, then it is Pettis integrable.*

Remark 1. By the symmetry of inner products, f satisfies (P1) if and only if $s \mapsto \langle f(s), f(t) \rangle$ is $(\mathcal{T}, \mathcal{B}_{\mathbb{R}})$-measurable for every $t \in T$. Thus, (P1) is equivalent to that $(s, t) \mapsto \langle f(s), f(t) \rangle$ is *separately measurable*, according to Definition 4.47 of Aliprantis and Border (2006).

Since we can take any $f(s)$ for y in Definition 2, weak measurability implies (P1). A crucial part of Proposition 1 is the converse implication. To establish this, we slightly generalize the arguments presented in Al-Najjar (1995), relying on Hilbert space geometry.[4] In this regard, having values in a Hilbert space is a necessary assumption for the proposition, although this restriction is fairly innocuous as long as we consider a space of square-integrable random variables.

A process is said to be *strongly measurable* when it is the norm-limit of a sequence of "simple" processes that admit at most finitely many values. For any strongly measurable process, (P2) is necessary and sufficient for the process to be "Bochner integrable," an integral notion stronger than Pettis.[5] On the other hand, (P2) is sufficient but not necessary for a weakly measurable process to be Pettis integrable. This is illustrated in Example 2 in Appendix, which is a variant of Birkhoff's example, showing that a continuum of mutually uncorrelated random variables are Pettis-integrated to zero no matter what values are taken by each individual variance.[6]

The same example illustrates that the mapping $(s, t) \mapsto \langle f(s), f(t) \rangle$ may fail jointly measurable for Pettis integrable processes f. While one can interpret this as an indication that the notion of Pettis integral is generous and widely applicable, a stronger measurability condition turns out useful in applications. Specifically, strengthening (P1) to joint measurability will be instrumental when we define a linear space of Pettis integrable processes in

[4]See Footnote 7 and Section A.1 of Al-Najjar (1995).

[5]See Theorem 2 in p. 45 of Diestel and Uhl (1977).

[6]For the standard Birkhoff example, see Example 5 in p. 43 of Diestel and Uhl (1977).

Section 3. Additionally, we consider a slightly stronger version of (P2), requiring a process to have square-integrable norms.

(Q1). The mapping $(s,t) \mapsto \langle f(s), f(t) \rangle$ is $(\mathcal{T} \otimes \mathcal{T}, \mathcal{B}_{\mathbb{R}})$-measurable.

(Q2). The mapping $t \mapsto \|f(t)\|^2$ is $(\mathcal{T}, \mathcal{B}_{\mathbb{R}})$-measurable and $\int \|f(t)\|^2 \mathrm{d}\nu(t) < \infty$.

The above (Q1) simply requires that $(s,t) \mapsto \langle f(s), f(t) \rangle$ is a *jointly measurable* mapping. Thus, by Theorem 4.48 of Aliprantis and Border (2006), we confirm that (Q1) is stronger than (P1) and weak measurability. Additionally, we can prove the next lemma, which will be useful when we introduce an inner product space of Pettis integrable processes and an integral equation defined over it.

Lemma 1. *If processes f and g satisfy* (Q1)*, then $t \mapsto \langle f(t), g(t) \rangle$ is $(\mathcal{T}, \mathcal{B}_{\mathbb{R}})$-measurable.*

Remark 2. Since this lemma implies that $t \mapsto \|f(t)\|^2$ is measurable whenever f satisfies (Q1), an essential content of (Q2) is the square-integrability $\int \|f(t)\|^2 \mathrm{d}\nu(t) < \infty$ in the presence of (Q1).

2.2 Pettis Integral of Stochastic Processes

In the remainder of Section 2, we specify each output $x \in X$ as a random variable with a finite second moment. Specifically, let $(\Omega, \Sigma, \mathbb{P})$ be an arbitrary probability space, and let X be a collection of random variables $x : \Omega \to \mathbb{R}$ such that $\mathbb{E}\,|x|^2 = \int |x(\omega)|^2 \mathrm{d}\mathbb{P}(\omega) < \infty$. The inner product between random variables $x, y \in X$ is defined as

$$\langle x, y \rangle_X \coloneqq \mathbb{E}\,[xy] = \int_{\Omega} x(\omega) y(\omega) \mathrm{d}\mathbb{P}(\omega).$$

For any $x, y \in X$, we denote the covariance by $\mathbb{C}\mathrm{ov}\,[x,y] = \mathbb{E}\,[xy] - \mathbb{E}\,[x]\,\mathbb{E}\,[y]$, the variance by $\mathbb{V}\mathrm{ar}\,[x] = \mathbb{C}\mathrm{ov}\,[x,x]$, and the standard deviation by $\mathbb{S}\mathrm{d}\,[x] = \mathbb{V}\mathrm{ar}\,[x]^{1/2}$.

In this probabilistic context, a *(stochastic) process* is meant by a function $f : T \to X$ that assigns a random variable $f(t) \in X$ to each input $t \in T$. As an immediate implication of Proposition 1, a stochastic process is Pettis integrable if it has measurable covariance and integrable mean and standard deviation.

Corollary 1. *Let f be a process such that $\mathbb{C}\mathrm{ov}\,[f(s), f(\cdot)]$ is measurable for every $s \in T$, and that $\mathbb{E}[f(\cdot)]$ and $\mathbb{S}\mathrm{d}\,[f(\cdot)]$ are integrable. Then, f is Pettis integrable.*

Example 1 (Continued)**.** By leveraging the result of Ui and Yoshizawa (2013), Proposition 4 of Bergemann and Morris (2013) states the unique existence of an equilibrium under the condition that $r < 1$. This equilibrium, symmetric across all players, is characterized by a linear function of signals, given by $a_i = \alpha_0^* + \alpha_x^* x_i + \alpha_y^* y$, where the coefficients $(\alpha_0^*, \alpha_x^*, \alpha_y^*)$ are determined by $(r, s, k, \mu_\theta, \sigma_\theta^2, \sigma_x^2, \sigma_y^2)$.

The Pettis integrability of this equilibrium strategy can be verified as follows: Firstly, the mean is computed as $\mathbb{E}[a_i] = \alpha_0^* + (\alpha_x^* + \alpha_y^*)\mu_\theta$, which is constant across agents, thus, integrable. Moreover, the covariance is computed as

$$\mathbb{C}\mathrm{ov}[a_i, a_j] = \left(\alpha_x^* + \alpha_y^*\right)^2 \sigma_\theta^2 + \alpha_y^{*2}\sigma_y^2 + \alpha_x^{*2}\sigma_x^2 \cdot \mathbf{1}\{i = j\},$$

which is a step function, taking constant values on the on-diagonal and off-diagonal subsets of $[0,1]^2$, respectively. Hence, $\mathbb{C}\mathrm{ov}[a_i, a_j]$ is measurable. Consequently, by Corollary 1, it follows that $\{a_i\}_{i\in[0,1]}$ is Pettis integrable.

The definition of Pettis integral reminds us of a Fubini-like property that justifies exchanging the order of expectation over $\omega \in \Omega$ and integration over $t \in T$. Indeed, by taking a non-zero constant random variable in the position of y in (2), we confirm the following holds true:

$$\mathbb{E}\left[\int f(t)\mathrm{d}\nu(t)\right] = \int \mathbb{E}[f(t)]\mathrm{d}\nu(t). \tag{3}$$

In relation to this formula, we provide two results pertaining to the exchangeability of probabilistic operations and integration on the input space. The next result justifies the interchange of covariance and integration.

Proposition 2. *For any Pettis integrable stochastic process f, it holds that*

$$\mathbb{C}\mathrm{ov}\left[x, \int f(t)\mathrm{d}\nu(t)\right] = \int \mathbb{C}\mathrm{ov}[x, f(t)]\mathrm{d}\nu(t), \quad \forall x \in X. \tag{4}$$

In particular, we have

$$\mathbb{V}\mathrm{ar}\left[\int f(t)\mathrm{d}\nu(t)\right] = \int\int \mathbb{C}\mathrm{ov}[f(s), f(t)]\mathrm{d}\nu(s)\mathrm{d}\nu(t). \tag{5}$$

This proposition yields an important implication when $f(t)$ are pairwise uncorrelated and have a common mean across t. In this case, the right-hand side of (5) becomes 0, suggesting that the Pettis integral coincides with the common mean in mean square, which is exactly the Pettis integral version of LLN, known as Theorem 3 of Uhlig (1996). However, as pointed out in pp. 551–552 of Khan and Sun (1999), we have to be careful to interpret this result because its probabilistic content is quite different from the classical LLN, stating that a sample average of countably many i.i.d. random variables converges to the mean for almost every sample path. This issue is elaborated in Section 4 by referring to their discussion more in details.

Example 1 (Continued)**.** For a symmetric strategy profile, Bergemann and Morris (2013) argue that the volatility and dispersion are computed as follows:

$$V = \mathbb{C}\text{ov}\,[a_i, a_j] \quad \text{and} \quad D = \mathbb{V}\text{ar}\,[a_i] - \mathbb{C}\text{ov}\,[a_i, a_j],$$

where $i, j \in [0, 1]$ are any distinct pair of representative agents. These formulae can be readily derived as a consequence of Proposition 2.

Next, we develop a "conditional" version of the Fubini formula (3) that arises when we replace the unconditional expectation by the conditional expectation with respect to a sub-σ-algebra of Σ.

Proposition 3. *Let $\hat{\Sigma} \subseteq \Sigma$ be a sub-σ-algebra, f be a Pettis integrable stochastic process, and $\hat{f} \equiv \mathbb{E}[f \mid \hat{\Sigma}]$. If $\hat{f}$ is Pettis integrable, then we have*

$$\mathbb{P}\Big(\mathbb{E}\left[\int f(t)\mathrm{d}\nu(t) \mid \hat{\Sigma}\right] = \int \mathbb{E}\left[f(t) \mid \hat{\Sigma}\right]\mathrm{d}\nu(t)\Big) = 1. \tag{6}$$

In particular, if f satisfies (P2) *and $\hat{f}$ satisfies* (P1)*, then $\hat{f}$ is Pettis integrable.*

According to Proposition 3, the interchange of conditional expectation and integration over T is justified if the "conditional" stochastic process $\hat{f}$ is Pettis integrable. In evaluating the Pettis integrability of $\hat{f}$, we observe that $\mathbb{E}[\hat{f}] = \mathbb{E}[f]$ and $\mathbb{S}\text{d}[\hat{f}] \leq \mathbb{S}\text{d}[f]$, which are confirmed by the law of iterated expectations and Jensen's inequality, respectively. Hence, given that the initial process f meets all assumptions of Corollary 1, our task simplifies to verifying the measurability of the covariance of $\hat{f}$. We are going to illustrate this by focusing on a class of Gaussian processes.

2.3 Pettis Integral of Gaussian Processes

Many economic applications, including Example 1, focus on normally distributed random variables for tractability. To model a continuum of such random variables, the concept of Gaussian processes is useful.

We say that a stochastic process $f : T \to X$ is a *Gaussian process* if any finite selection from the collection $\{f(t)\}_{t \in T}$ is jointly normally distributed. By normality, the joint distribution of f can be summarized by the mean and covariance functions,

$$\mu(t) \coloneqq \mathbb{E}\,[f(t)] \quad \text{and} \quad \sigma(s, t) \coloneqq \mathbb{C}\text{ov}(f(s), f(t)), \quad \forall s, t \in T.$$

Notice that by definition, σ necessarily satisfies the statistical property that for any $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in \mathbb{R}$, the matrix $[\sigma(t_i, t_j)]_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}$ is symmetric and positive semidefinite. In fact, this property of σ is sufficient for the existence of a Gaussian process, which has σ exactly as its covariance function (see, e.g., Chapter 12 of Dudley, 1989). No restriction on μ is needed.

Corollary 1 readily provides sufficient conditions on μ and σ for the Pettis integrability of Gaussian processes. Moreover, when Proposition 3 is narrowed down to Gaussian processes, we see that no further assumption, other than the joint measurability of σ, is needed to obtain the conditional Fubini formula.

Corollary 2. *Let $f : T \to X$ be a Gaussian process that satisfies all assumptions of Corollary 1. If σ is jointly measurable, then for any $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in T$, the process $\hat{f} \equiv \mathbb{E}[f \mid f(t_1, \ldots, t_n)]$ is Pettis integrable, and it holds that*

$$\mathbb{P}\left(\mathbb{E}\left[\int f(s)\mathrm{d}\nu(s) \mid f(t_1, \ldots, t_n)\right] = \int \mathbb{E}[f(s) \mid f(t_1, \ldots, t_n)]\mathrm{d}\nu(s)\right) = 1. \tag{7}$$

To verify this result, we can simply use the conditional Gaussian formula to obtain

$$\mathbb{C}\mathrm{ov}\left[\hat{f}(s), \hat{f}(t)\right] = \begin{bmatrix} \sigma(s,t_1) \\ \vdots \\ \sigma(s,t_n) \end{bmatrix}^\top \begin{bmatrix} \sigma(t_1,t_1) & \cdots & \sigma(t_1,t_n) \\ \vdots & \ddots & \vdots \\ \sigma(t_n,t_1) & \cdots & \sigma(t_n,t_n) \end{bmatrix}^{-1} \begin{bmatrix} \sigma(t,t_1) \\ \vdots \\ \sigma(t,t_n) \end{bmatrix}.$$

Given that $\sigma : T^2 \to \mathbb{R}$ is jointly measurable, it should be evident that the conditional process $\hat{f}$ has a jointly measurable covariance function. Therefore, by the discussion after Proposition 3, we confirm that (7) holds true.

Example 1 (Continued)**.** A common way to get a closed form of an equilibrium is based on matching coefficients. Namely, we firstly guess a (symmetric) equilibrium strategy $a_i = \alpha_0 + \alpha_x x_i + \alpha_y y$, which is linear in signals, and then substitute it into the best-response formula (1) to express $(\alpha_0, \alpha_x, \alpha_y)$ in terms of parameters. In doing so, we need to calculate each agent's conditional expectation of the aggregated action A. This step is facilitated by postulating the exchangeability of conditional expectation and integration,

$$\mathbb{E}[A \mid x_i,\, y] = \int_0^1 \mathbb{E}[a_j \mid x_i,\, y]\mathrm{d}j.$$

By means of Proposition 3, the interchange is justified if the process $j \mapsto \mathbb{E}[a_j \mid x_i,\, y]$ is Pettis integrable. Indeed, by the linearity of a_j in (x_j, y) and the normality of information, one can see that $\mathbb{E}[a_j \mid x_i,\, y]$ is expressed as a linear combination of x_i and y, with weights identical across agents. Consequently, the conditional process maintains the needed conditions for Pettis integrability.

3 Pettis Integral Equation

In this section, we investigate a linear equation that involves a Pettis integral of an unknown process. Specifically, we are interested in an equation of the form

$$f(t) = \text{w-}\int K(t,t')f(t')\mathrm{d}\nu(t') + g(t), \quad \nu\text{-a.e.} \quad t \in T. \tag{8}$$

Here, f is an unknown process that we want to derive, g is a known process, and $K : T^2 \to \mathbb{R}$ is a real-valued function, termed an (integral) kernel, that specifies wights put on $f(t')$ when computing the Pettis integral in the right-hand side. Our main interests lie in the existence and uniqueness issues of the above Pettis integral equation.

This type of equation naturally arises when characterizing an equilibrium in a large population game with linear best responses. In this context, each input $t \in T$ is interpreted as a player. An outcome $f(t)$ is viewed as an equilibrium strategy of player t, which is a random variable measurable with respect to t's private information. The equation (8) then requires that for (almost) every player t, the best response strategy $f(t)$ sets equal to the weighted aggregate $\int K(t,t')f(t')\mathrm{d}\nu(t')$ of others' equilibrium strategies, defined as the Pettis integral, plus some exogenous component g. Here, an integral kernel K captures payoff interactions among agents by quantitatively evaluating "whose behavior affect whom and how much."

Example 1 (Continued)**.** Let us generalize the running example to accommodate payoff asymmetry across players.[7] Specifically, we postulate that player i's best-response strategy is now given as

$$a_i = \mathbb{E}\left[\int_0^1 r_{ij}a_j\mathrm{d}j \mid x_i,\, y\right] + s_i\,\mathbb{E}\left[\theta \mid x_i,\, y\right] + k_i, \tag{9}$$

where $r_{ij}, s_i, k_i \in \mathbb{R}$ are the parameters that can vary across agents. In this context, an integral kernel K corresponds to $\{r_{ij}\}_{i,j\in[0,1]}$, indicating how player i adjusts her strategy in response to changes in j's strategy. For example, in network games, we often consider the specification such as $r_{ij} = r \cdot \mathbf{1}\{(i,j) \in G\}$, where $r > 0$ is interpreted as the degree of peer effects, and G is a subset of T^2, called a *graph* or *network structure*, that represents the connections among players. An exogenous process g corresponds to $\{s_i\theta + k_i\}_{i\in[0,1]}$.

[7]This generalized case falls in the model of Miyashita (2022).

Having heterogeneous g enables us to model player's base action levels and responses to the state that differ across players.

Throughout, we assume that K satisfies jointly measurability and bounded. These assumptions are satisfied in most economic applications.

Definition 2. A *kernel* is given as a jointly measurable function $K : T^2 \to \mathbb{R}$. In particular, we say that K is *bounded* if

$$\|K\|_\infty \coloneqq \sup_{s,t \in T} |K(s,t)| < \infty$$

To formally discuss the solvability of the equation (8), we need to introduce a space of processes, serving as a set of candidate solutions. To this end, we recall the strengthening of weak measurability that makes an inner product between processes be definable. Specifically, denote by $\mathcal{F}$ the linear space of processes $f : T \to X$ that satisfy (Q1) and (Q2), equipped with point-wise additions and multiplications. Then, by means of Lemma 1, the inner product between two processes $f, g \in \mathcal{F}$ can be defined by

$$\langle f, g \rangle_{\mathcal{L}_2} \coloneqq \int \langle f(t), g(t) \rangle_X \mathrm{d}\nu(t). \tag{10}$$

Additionally, we define the induced norm as $\|f\|_{\mathcal{L}_2} \coloneqq \langle f, f \rangle_{\mathcal{L}_2}^{1/2}$. Notice that (Q2) ensures that $\|f\|_{\mathcal{L}_2} < \infty$ for all $f \in \mathcal{F}$.

We say that $f, f' \in \mathcal{F}$ are *equivalent* if $\|f - f'\|_{\mathcal{L}_2} = 0$. Denote by $[f]_\sim$ the induced equivalent class of f, while we often abuse the notation to write $f = [f]_\sim$ as usual. Then, let $\mathcal{L}_2$ be the space of (equivalent classes of) processes $f \in \mathcal{F}$. By routine arguments, we can show that $\mathcal{L}_2$ is Hilbertian, which will serve as the space in which all processes appearing in the equation (8) reside.

Lemma 2. *$\mathcal{L}_2$ equipped with the inner product (10) constitutes a Hilbert space space.*

Now, given a bounded kernel K, we define an integral operator $\mathbf{K}$ by

$$\mathbf{K}f \coloneqq \text{w-}\int K(\cdot, t) f(t) \mathrm{d}\nu(t)$$

for each $f \in \mathcal{L}_2$. It should be checked that $\mathbf{K}$ is a well-defined linear operator acting from $\mathcal{L}_2$ to itself, which is accomplished in the next proposition. Moreover, the proposition shows that $\mathbf{K}$ enjoys a nice property of compactness, enabling us to deal with $\mathbf{K}$ in a way similar to a finite-dimensional matrix.

Proposition 4. *If K is a bounded kernel, then $\mathbf{K}$ is a bounded linear operator acting from $\mathcal{L}_2$ to itself. Moreover, $\mathbf{K}$ is compact.*

Now, by using the current notation, we can rewrite (8) as

$$(\mathbf{I} - \mathbf{K})(f) = g,$$

where $\mathbf{I}$ stands for the identity operator on $\mathcal{L}_2$. As mentioned earlier, the compactness of $\mathbf{K}$ allows us to treat the equation akin to a finite-dimensional matrix equation. Specifically, the equation admits a unique solution in $\mathcal{L}_2$ if and only if the operator $(\mathbf{I} - \mathbf{K})$ has a bounded inverse. Moreover, by virtue of the Riesz–Fredholm theory, this is the case if and only if $(\mathbf{I} - \mathbf{K})$ is *injective*, which is further equivalent to saying that 1 is *not* an eigenvalue of $\mathbf{K}$. Formally, by applying Theorem 3.4 of Kress (2014), we establish the following result as a corollary to Proposition 4.

Corollary 3. *Let $g \in \mathcal{L}_2$ and K be a bounded kernel. Then, (8) has a unique solution in $\mathcal{L}_2$ if and only if 1 is not an eigenvalue of $\mathbf{K}$.*

We remark that there is a missing component in the equation (9) of our example that does not appear in (8): That is, the conditional expectation operator $\mathbb{E}[\cdot \mid x_i, , y]$, which depends on each player's private information, and may influence equilibrium properties. This aspect is addressed in our complementary work Miyashita and Ui (2024), filling the mentioned gap by uncovering the mathematical nature of the role that information structures play in network games with incomplete information. In this work, we delve into the relationship between equilibrium properties and the spectral properties of $\mathbf{K}$ by providing sufficient conditions that guarantee the existence, uniqueness, and stability of an equilibrium under an *arbitrary* information structure, and further show that these conditions are, in some sense, necessary as well.

4 Discussion

Several papers on large LQG games, such as Bergemann and Morris (2013), can be read without prior knowledge of Pettis integral, and their economic implications remain valid even when integration is interpreted naively. Therefore, the contribution of this paper lies in providing a solid mathematical foundation for the economic insights derived from earlier works, rather than in identifying errors or incorrect conclusions, by employing the Pettis-integral approach. We believe that this approach is among the simplest and least technically demanding ones, as most results in this paper are proven using standard technical machinery in economic theory. The implications of our findings, demonstrated through the running example, may align with economists' intuitions. In light of this, we propose that this approach can serve as a useful instrument for future studies of large population games, offering a means to avoid tricky issues related to the aggregation of a continuum of random variables.

As cautioned in Khan and Sun (1999), the Pettis-integral approach can "avoid" the measurability issue, but it does not "resolve" the issue and leaves some interpretation challenges.[8] To illustrate their discussion, consider a collection of random variables $\{f(t)\}_{t\in T}$ Pettis-integrated to the aggregated random variable $\bar{f} \equiv \int f(t)\mathrm{d}\nu(t)$. While $\bar{f}$ has an ex-post realization $\bar{f}(\omega)$ for each ω in the underlying probability space, it may lack meaningful relations to the realized sample path $\{f(t,\omega)\}_{t\in T}$, which may not be integrable across t. In the context of Example 1, the Pettis integral approach allows us to define the aggregate A from an ex-ante standpoint, but it does not reveal how the realization of A is related to specific realizations of the state, signals, and individual actions. Relatedly, Proposition 2 cannot be given probabilistic interpretations in the same way as the traditional LLN, but rather, it should be understood as calculating the moment of the aggregate from an ex-ante standpoint before the resolution of uncertainty about ω.[9]

Under what circumstances would the Pettis integral approach be appropriate for modeling large strategic situations with incomplete information? In our perspective, the combination of two ingredients—aggregative and Bayesian elements—is crucial. In the running example, the aggregated action is the sole strategic argument in players' payoff functions that matter their incentives. The notion of Pettis integral relies on the interchange of integration and probabilistic operations, taken unconditionally, which implies its conditional counterpart, Proposition 3. Applying this result to the best-response formula (1), we observe a specific way of how Bayesian players evaluate strategic uncertainty: The aggregated action is best estimated as the total sum of the individual uncertainties associated with each opponent's strategic decision. If one agrees that viewing aggregated uncertainty as the sum of individual uncertainties guides ex-ante payoff maximization, then the Pettis integral approach becomes appealing, as the notion is essentially defined by embracing this idea as an axiom. On the other hand, the Pettis integral approach may lack clear economic interpretations when the modeler's interest lies in ex-post realizations of the aggregate or sample path.[10]

Finally, applying the Pettis-integral approach to games requires some regularity con-

[8] The lack of ex-post interpretations of Pettis integral is well recognized and discussed in Al-Najjar (1995) and Uhlig (1996), who firstly advocate for this approach as modeling tools of large economies.

[9] Sun (2006) proposes a different approach to integrate a sample path by considering an extension of the measurable space $(T \times \Omega, \mathcal{T} \otimes \Sigma, \nu \otimes \mathbb{P})$, on which the real-valued function $f : T \times \Omega \to \mathbb{R}$ acts measurably. On this extended measurable space, he establishes the "exact" LLN, stating that the sample mean $\int_T f(t,\omega)\mathrm{d}\nu(t)$ is equal to the ex-ante mean of the process $\mathbb{P}$-almost surely. He defines the extension by appealing to the Fubini property, the interchange of the integrals over T and Ω, which has a similar spirit to Pettis integral.

[10] Khan and Sun (1999) reach a similar conclusion to us; see the last paragraph of their Section 7.

ditions on primitives, such as the measurability of network structures. More crucially, we need to restrict attention to a well-posed class of strategy profiles, satisfying (Q1) and (Q2), to properly define the aggregate. As pointed out by Al-Najjar (2008), since any measurable function is approximately continuous, measurability implicitly assume some degree of similarity among players. Thus, imposing similarity of strategies could be an undesirable restriction, as an equilibrium is typically perceived as a consequence of decentralized strategic decisions. On the other hand, it would be natural to anticipate that rational choices of players tend to be similar if they share similar primitives, such as payoff functions, private information, or sets of neighbours. In Miyashita and Ui (2024), we show that by imposing certain measurability conditions on primitives, there exists an equilibrium wherein players' strategies are measurable in the same degree. Our existence result can be viewed as a positive result, affirming such predictions.

A Appendix

Proof of Proposition 1

Clearly, f satisfies (P1) if it is weakly measurable. Conversely, assume that f satisfies (P1). Let $F \coloneqq \overline{\operatorname{span}}\{f(t) : t \in T\}$. Then, given any $y \in X$, there exists a unique orthogonal decomposition $(z, y^{\perp}) \in F \times F^{\perp}$ such that $y = z + y^{\perp}$ by Theorem 3 in p. 55 of Lax (2002). Since $\langle y, f(t)\rangle = \langle z, f(t)\rangle$ holds for every $t \in T$, it is without loss of generality to assume that $y \in F$ to show that f is weakly measurable.

By the construction of F, we can find a sequence $\{y_n\}_{n\in\mathbb{N}}$ in $\operatorname{span}\{f(t) : t \in T\}$, converging to y in norm, such that each y_n is expressed as a finite linear combination

$$y_n = \sum_{k=1}^{\bar{k}_n} \beta_k^n f(t_k^n) \tag{11}$$

with some $\bar{k}_n \in \mathbb{N}$, $\beta_1^n, \dots, \beta_{\bar{k}_n}^n \in \mathbb{R}$, and $t_1^n, \dots, t_{\bar{k}_n}^n \in T$. Then, by setting $\bar{y}_n = \frac{1}{n}\sum_{m=1}^{n} y_m$, we observe that

$$\|y - \bar{y}_n\| \leq \frac{1}{n}\sum_{m=1}^{n} \|y - y_m\|.$$

Since the right-hand side is the Cesàro mean of $\|y - y_m\|$ through $m = 1, \dots, n$, and since $y_m \xrightarrow{\|\cdot\|} y$, it follows that $\bar{y}_n \xrightarrow{\|\cdot\|} y$.

For each $n \in \mathbb{N}$, we define a function $\bar{\phi}_n : T \to \mathbb{R}$ by

$$\bar{\phi}_n(t) = \frac{1}{n} \sum_{m=1}^{n} \phi_m(t) \quad \text{where} \quad \phi_m(t) = \sum_{k=1}^{\bar{k}_m} \beta_k^m \left\langle f(t_k^m), f(t) \right\rangle, \quad \forall t \in T.$$

Note that $\phi_m(t) = \langle y_m, f(t) \rangle$ holds by (11), whereas we have $\lim_{m\to\infty} \langle y_m, f(t) \rangle = \langle y, f(t) \rangle$ since norm convergence implies weak convergence. Thus, for every $t \in T$, the real sequence $\{\phi_m(t)\}_{m\in\mathbb{N}}$ is convergent, and its limit equal to $\langle y, f(t) \rangle$. Moreover, by the fact that $\bar{\phi}_n(t)$ is the Cesàro mean of $\phi_m(t)$, it follows that

$$\lim_{n\to\infty} \bar{\phi}_n(t) = \langle y, f(t) \rangle \quad \forall t \in T.$$

Notice that $\bar{\phi}_n$ is written as a finite linear combination of functions $t \mapsto \langle f(t_k^m), f(t) \rangle$, each of which is measurable by (P1). Thus $\bar{\phi}_n$ is measurable. Moreover, since the mapping $t \mapsto \langle y, f(t) \rangle$ is given as the pointwise limit of $\bar{\phi}_n$, it is measurable by Lemma 4.29 of Aliprantis and Border (2006). Since y is arbitrary, we have established the weak measurability of f.

Next, we show that f is Pettis integrable when it satisfies both (P1) and (P2). To this end, by the Cauchy–Schwartz inequality, observe that

$$\int |\langle y, f(t) \rangle| \mathrm{d}\nu(t) \leq \|y\| \int \|f(t)\| \mathrm{d}\nu(t),$$

where the right-hand side is finite by (P2). Now, take any sequence $\{y_n\}_{n\in\mathbb{N}} \subseteq X$ such that $y_n \xrightarrow{\mathrm{w}} y$. Let us show that $\langle y_n, f(\cdot) \rangle$ converges to $\langle y, f(\cdot) \rangle$ in L_1, i.e.,

$$\lim_{n\to\infty} \int |\langle y_n - y, f(t) \rangle| \mathrm{d}\nu(t) = 0. \tag{12}$$

Since any weakly convergent sequence is norm-bounded, we can find some $c \in \mathbb{R}_{++}$ such that $c > \|y\| \vee \sup_{n\in\mathbb{N}} \|y_n\|$. For each $t \in T$, by the Cauchy–Schwartz inequality, we have

$$|\langle y_n - y, f(t) \rangle| \leq \|y_n - y\| \cdot \|f(t)\| < 2c\|f(t)\|,$$

whereas $t \mapsto 2c\|f(t)\|$ is integrable by (P2). Hence, by the dominated convergence theorem, it follows that

$$\lim_{n\to\infty} \int |\langle y_n - y, f(t) \rangle| \mathrm{d}\nu(t) = \int \lim_{n\to\infty} |\langle y_n - y, f(t) \rangle| \mathrm{d}\nu(t) = 0,$$

where the second equality holds by $y_n \xrightarrow{\mathrm{w}} y$. Thus, (12) is verified. Therefore, Proposition 1 of Huff (1986) implies that f is Pettis integrable. *Q.E.D.*

While weak measurability is equivalent to (P1), the following example shows that a Pettis integrable process can violate (P2).

Example 2. Let ν be the Lebesgue measure on $T = [0, 1]$, and let X be a non-separable Hilbert space that has an uncountable orthonormal system $\{e(t)\}_{t \in T}$. Then, let $f(t) = \gamma(t)e(t)$, where $\gamma : T \to \mathbb{R}$ is any function. In particular, the mapping $t \mapsto |\gamma(t)|$ need be neither measurable nor integrable. Regardless of γ, however, f is weakly measurable since $\langle f(s), f(t)\rangle = 0$ whenever $s \neq t$. Let us show that f is Pettis integrated to zero. Given any element $y \in X$, consider its orthogonal decomposition $y = z + y^{\perp}$ with $z \in E := \overline{\text{span}}\{e(t)\}_{t \in T}$ and $y^{\perp} \in E^{\perp}$. Notice that $\langle y, f(t)\rangle = \langle z, f(t)\rangle$ holds for every $t \in T$. Moreover, by the construction of E, there exists a sequence $\{z_n\}_{n \in \mathbb{N}}$, converging to z in norm, such that each z_n is a finite linear combination of elements of $\{e(t)\}_{t \in T}$. Now, for each n, we have $\langle z_n, f(t)\rangle = 0$ except for at most finitely many t, and thus, $\int \langle z_n, f(t)\rangle \mathrm{d}\nu(t) = 0$ holds. In particular, this implies that the mapping $t \mapsto |\langle z_n, f(t)\rangle|$ is bounded ν-a.e. uniformly across all n, so the dominated convergence theorem implies

$$0 = \lim_{n \to \infty} \int \langle z_n, f(t)\rangle \mathrm{d}\nu(t) = \int \langle z, f(t)\rangle \mathrm{d}\nu(t) = \int \langle y, f(t)\rangle \mathrm{d}\nu(t).$$

Since y is arbitrary, we have shown that w-$\int f(t)\mathrm{d}\nu(t) = 0$. △

Proof of Lemma 1

For any process f, we denote $c_f : (s, t) \mapsto \langle f(s), f(t)\rangle$ and $d_f : t \mapsto \|f(t)\|^2$. Also, let $\delta : T \to T^2$ be defined by $\delta(t) = (t, t)$. It is clear that δ is $(\mathcal{T}, \mathcal{T} \otimes \mathcal{T})$-measurable. Now, assume that f and g satisfy (Q1) so that c_f and c_g are jointly measurable. Since $d_f = c_f \circ \delta$, by Lemma 4.22 of Aliprantis and Border (2006), we see that d_f is $(\mathcal{T}, \mathcal{B}_R)$-measurable, and so is true for d_g. Moreover, since $f + g$ satisfies (Q1), notice that $d_{f+g} : t \mapsto \|f(t) + g(t)\|^2$ is $(\mathcal{T}, \mathcal{B}_R)$-measurable. We then observe that

$$\langle f(t), g(t)\rangle = \frac{\|f(t) + g(t)\|^2 - \|f(t)\|^2 - \|g(t)\|^2}{2},$$

indicating that the mapping $t \mapsto \langle f(t), g(t)\rangle$ is a linear combination of d_f, d_g, and d_{f+g}, and hence, it is $(\mathcal{T}, \mathcal{B}_R)$-measurable. *Q.E.D.*

Proof of Proposition 2

By using the definition of the Pettis integral, we observe that

$$\begin{aligned}\mathbb{C}\mathrm{ov}\left[x, \int f(t)\mathrm{d}\nu(t)\right] &= \mathbb{E}\left[x\int f(t)\mathrm{d}\nu(t)\right] - \mathbb{E}\left[x\right]\mathbb{E}\left[\int f(t)\mathrm{d}\nu(t)\right] \\ &= \int \mathbb{E}\left[xf(t)\right]\mathrm{d}\nu(t) - \mathbb{E}\left[x\right]\int \mathbb{E}\left[f(t)\right]\mathrm{d}\nu(t) \\ &= \int \left(\mathbb{E}\left[xf(t)\right] - \mathbb{E}\left[x\right]\mathbb{E}\left[f(t)\right]\right)\mathrm{d}\nu(t) \\ &= \int \mathbb{C}\mathrm{ov}\left[x, f(t)\right]\mathrm{d}\nu(t),\end{aligned}$$

where the first and last equalities are nothing more than the definition of the covariance of random variables. Thus, we have established (4). Moreover, (5) is obtained by taking $\int f(s)\mathrm{d}\nu(s)$ in the position of x above and by applying (4) twice. *Q.E.D.*

Proof of Proposition 3

Suppose that $\hat{f}(\cdot) = \mathbb{E}[f(\cdot) \mid \hat{\Sigma}]$ is Pettis integrable. To verify (6), fix any event $E \in \hat{\Sigma}$. By the definition of conditional expectation (see, e.g., Chapter 10 of Dudley, 1989), it suffices to show that

$$\mathbb{E}\left[\chi_E \int f(t)\mathrm{d}\nu(t)\right] = \mathbb{E}\left[\chi_E \int \mathbb{E}\left[f(t) \mid \hat{\Sigma}\right]\mathrm{d}\nu(t)\right], \tag{13}$$

where χ_E is a binary random variable such that $\chi(\omega) = 1$ if $\omega \in E$ and $\chi(\omega) = 0$ otherwise. Since $\mathbb{E}\left|\chi_E\right|^2 = \mathbb{P}(E) \leq 1$ is finite, we have

$$\mathbb{E}\left[\chi_E \int f(t)\mathrm{d}\nu(t)\right] = \int \mathbb{E}\left[\chi_E f(t)\right]\mathrm{d}\nu(t). \tag{14}$$

On the other hand, by the definition of Pettis integral, we have

$$\begin{aligned}\mathbb{E}\left[\chi_E \int \mathbb{E}\left[f(t) \mid \hat{\Sigma}\right]\mathrm{d}\nu(t)\right] &= \int \mathbb{E}\left[\chi_E \mathbb{E}\left[f(t) \mid \hat{\Sigma}\right]\right]\mathrm{d}\nu(t) \\ &= \int \mathbb{E}\left[\mathbb{E}\left[\chi_E f(t) \mid \hat{\Sigma}\right]\right]\mathrm{d}\nu(t) = \int \mathbb{E}\left[\chi_E f(t)\right]\mathrm{d}\nu(t),\end{aligned} \tag{15}$$

where the second equality holds true since χ_E is $f(t)$-measurable, and the third equality comes from the law of iterated expectation. Combining (14) and (15), we confirm that (13) holds true. *Q.E.D.*

Proof of Proposition 4

Let $K : T^2 \to \mathbb{R}$ be a bounded kernel. Given any $f \in \mathcal{L}_2$ and $s \in T$, let us show that the process $t \mapsto K(s,t)f(t)$ satisfies (Q1) and (Q2) so that it is Pettis integrable. By the homogeneity of inner products, we have

$$\langle K(s,t)f(t), K(s,t')f(t')\rangle = K(s,t)K(s,t')\langle f(t), f(t')\rangle.$$

Since K is jointly measurable, and since f satisfies (Q1), the above expression is measurable in (t,t') as being the product of three measurable functions. Thus $t \mapsto K(s,t)f(t)$ satisfies (Q1). Moreover, by the boundedness of K, it holds that

$$\int \|K(s,t)f(t)\|^2 \mathrm{d}\nu(t) = \int |K(s,t)|^2 \cdot \|f(t)\|^2 \mathrm{d}\nu(t) \leq \|K\|_\infty^2 \cdot \|f\|_{\mathcal{L}_2}^2 < \infty,$$

which confirms (Q2). Thus, by Proposition 1, the Pettis integral $\mathbf{K}f(s) = \int K(s,t)f(t)\mathrm{d}\nu(t)$ exists for all $s \in T$.

Next, we show that $s \mapsto \mathbf{K}f(s)$ satisfies (Q1) and (Q2) whenever $f \in \mathcal{L}_2$. Given any $s \in T$, by applying the definition of Pettis integral twice, we have

$$\begin{aligned}
\|\mathbf{K}f(s)\| &= \left\langle \int K(s,t)f(t)\mathrm{d}\nu(t), \int K(s,t')f(t')\mathrm{d}\nu(t') \right\rangle \\
&= \int\int \langle K(s,t)f(t), K(t,t')f(t')\rangle \,\mathrm{d}\nu(t)\mathrm{d}\nu(t') \\
&\leq \|K\|_\infty^2 \cdot \int\int \langle f(t), f(t')\rangle \,\mathrm{d}\nu(t)\mathrm{d}\nu(t')
\end{aligned}$$

Thus, given that $s \mapsto \|\mathbf{K}f(s)\|$ is measurable, by using the Cauchy–Schwartz and Jensen's inequalities, we can bound the norm of $\mathbf{K}f$ as

$$\begin{aligned}
\|\mathbf{K}f\|_{\mathcal{L}_2}^2 &= \int \|\mathbf{K}f(s)\|^2 \mathrm{d}\nu(s) \\
&\leq \|K\|_\infty^2 \cdot \left(\int\int \langle f(t), f(t')\rangle \,\mathrm{d}\nu(t)\mathrm{d}\nu(t') \right)^2 \\
&\leq \|K\|_\infty^2 \cdot \int\int |\langle f(t), f(t')\rangle|^2 \mathrm{d}\nu(t)\mathrm{d}\nu(t') \\
&\leq \|K\|_\infty^2 \cdot \left(\int \|f(t)\|^2 \mathrm{d}\nu(t) \right)^2,
\end{aligned}$$

from which

$$\|\mathbf{K}f\|_{\mathcal{L}_2} \leq \|K\|_\infty \cdot \|f\|_{\mathcal{L}_2}^2 < \infty. \tag{16}$$

This confirms $\mathbf{K}f$ satisfies (Q2), and particularly, we have $\mathbf{K}f' \in [\mathbf{K}f]_\sim$ whenever $f' \in [f]_\sim$. We need to show that $s \mapsto \|\mathbf{K}f(s)\|$ is measurable, and that $\mathbf{K}f$ satisfies (Q1). The next lemma deals with these measurability issues.[11]

Lemma 3. *For any $f \in \mathcal{L}_2$, $\mathbf{K}f$ satisfies* (Q1)*, i.e., the mapping $(s,t) \mapsto \langle \mathbf{K}f(s), \mathbf{K}f(t)\rangle$ is $(\mathcal{T}\otimes\mathcal{T}, \mathcal{B}_\mathbb{R})$-measurable. In particular, $s \mapsto \|\mathbf{K}f(s)\|$ is $(\mathcal{T}, \mathcal{B}_\mathbb{R})$-measurable.*

Proof. Consider the semiring $\mathcal{S} \coloneqq (\mathcal{T}\otimes\mathcal{T}) \times (\mathcal{T}\otimes\mathcal{T})$ on T^4, and let $\tau : \mathcal{S} \to [0,1]$ be a set function given by

$$\tau(A \times B) = (\nu\otimes\nu(A))(\nu\otimes\nu(B)), \quad \forall A, B \in \mathcal{T}\otimes\mathcal{T}.$$

By Theorem 10.45 of Aliprantis and Border (2006), we know that τ is a measure on $\mathcal{S}$. Let $\mathcal{S}_\tau$ be the collection of τ-measurable subsets of T^4. By Theorem 10.23 of Aliprantis and Border (2006), $\mathcal{S}_\tau$ is a σ-algebra on T^4, satisfying $\sigma(\mathcal{S}) \subseteq \mathcal{S}_\tau$. In particular, since $T^4 \subseteq \mathcal{S}$, we have $\sigma(T^4) \subseteq \mathcal{S}_\tau$. This implies that any jointly measurable function $\phi : T^4 \to \mathbb{R}$ is τ-measurable.

Fix any $f \in \mathcal{L}_2$. Since

$$\gamma(s,t,s',t') \coloneqq \langle K(s,s')f(s'), K(t,t')f(t')\rangle = K(s,s')K(t,t')\langle f(s'), f(t')\rangle,$$

is jointly measurable in (s,t,s',t'), γ above defines a τ-measurable function by $\sigma(T^4) \subseteq \mathcal{S}_\tau$. Moreover, by the boundedness of K and the Cauchy–Schwartz inequality, we have

$$|\gamma(s,t,s',t')| \leq \|K\|_\infty^2 \cdot \|f(s')\| \cdot \|f(t')\|.$$

Thus, it follows that

$$\int_{T^4} |\gamma(s,t,s',t')| \mathrm{d}\tau(s,t,s',t') \leq \|K\|_\infty^2 \cdot \left(\int \|f(t')\| \mathrm{d}\nu(t')\right)^2 \leq \|K\|_\infty^2 \cdot \|f\|_{\mathcal{L}_2}^2 < \infty,$$

from which γ is τ-integrable. Therefore, by the standard Fubini theorem (Theorem 11.27 of Aliprantis and Border, 2006), the iterated integral $\int_{T^2}\int_{T^2} \gamma(s,t,s',t')\mathrm{d}\nu\otimes\nu(s',t')\mathrm{d}\nu\otimes\nu(s,t)$ exists, and particularly, this implies that the function

$$(s,t) \mapsto \int_{T^2} \gamma(s,t,s',t')\mathrm{d}\nu\otimes\nu(s',t')$$

is $(\nu\otimes\nu)$-measurable. Moreover, since every set in $\mathcal{T}\otimes\mathcal{T}$ is $(\nu\otimes\nu)$-measurable by

[11] The proof of the lemma invokes some concepts pertaining to Carathéodory extension, such as measurability of sets or functions with respect to a measure defined on a semiring. All our definitions are consistent with those of Aliprantis and Border (2006).

Corollary 10.28 of Aliprantis and Border (2006), by the construction of γ, it follows that $(s,t) \mapsto \langle \mathbf{K}f(s), \mathbf{K}f(t) \rangle$ is $(\mathcal{T} \otimes \mathcal{T}, \mathcal{B}_{\mathbb{R}})$-measurable.

Having established the measurability of $(s,t) \mapsto \langle \mathbf{K}f(s), \mathbf{K}f(t) \rangle$, that of $s \mapsto \|\mathbf{K}f(s)\|$ follows from Lemma 1. □

By means of this lemma, we have shown that $\mathbf{K}$ is an operator acting from $\mathcal{L}_2$ to itself. Clearly, $\mathbf{K}$ is linear by the linearity of Pettis integral. So, our remaining task is establishing the compactness of $\mathbf{K}$. To this end, take any bounded sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}_2$. Since $\mathcal{L}_2$ is Hilbertian, the Banach–Alaoglu theorem implies that $\{f_n\}_{n\in\mathbb{N}}$ has a weakly convergent subsequence $f_{n(i)} \xrightarrow{\text{w}} f \in \mathcal{L}_2$. Abusing the notation, we write as $\{f_n\}_{n\in\mathbb{N}}$ to represent the subsequence. It is without loss of generality to let $f_n \xrightarrow{\text{w}} 0$, since otherwise, we can consider an alternative sequence $f_n - f$ without changing subsequent arguments.

We want to show that the sequence $\{\mathbf{K}f_n\}_{n\in\mathbb{N}}$ converges to 0 in norm. First, notice that the sequence is uniformly bounded, as (16) implies

$$\sup_{n\in\mathbb{N}} \|\mathbf{K}f_n\|_{\mathcal{L}_2} \leq \|K\|_\infty \cdot \left(\sup_{n\in\mathbb{N}} \|f_n\|^2_{\mathcal{L}_2} \right) < \infty. \tag{17}$$

By Theorem 9 and Theorem 10 in pp. 60–61 of Lax (2002), we can take a (possibly, uncountable) orthonormal basis $\{e_i\}_{i\in I}$ of $\mathcal{L}_2$ so that each $\mathbf{K}f_n$ is written as

$$\mathbf{K}f_n = \sum_{i\in I} \langle \mathbf{K}f_n, e_i \rangle_{\mathcal{L}_2} \cdot e_i. \tag{18}$$

Since $\|\mathbf{K}f_n\|_{\mathcal{L}_2} < \infty$, there are at most countably many indexes $j \in I$ for which $\langle \mathbf{K}f_n, e_i \rangle_{\mathcal{L}_2} \neq 0$. For each n, we denote the set of such indexes by J_n, and let $J \coloneqq \bigcup_{n\in\mathbb{N}} J_n \subseteq I$. Note that J is countable, and the expression (18) now implies the following Parseval's equality:

$$\|\mathbf{K}f_n\|^2_{\mathcal{L}_2} = \sum_{j\in J} \left| \langle \mathbf{K}f_n, e_j \rangle_{\mathcal{L}_2} \right|^2. \tag{19}$$

By the Cauchy–Schwartz inequality and (17), we have

$$\sup_{n\in\mathbb{N},\, j\in J} \left| \langle \mathbf{K}f_n, e_i \rangle_{\mathcal{L}_2} \right|^2 \leq \left(\sup_{n\in\mathbb{N}} \|\mathbf{K}f_n\|^2_{\mathcal{L}_2} \right) \cdot \underbrace{\left(\sup_{j\in J} \|e_j\|^2_{\mathcal{L}_2} \right)}_{=1} < \infty.$$

Hence, by letting $n \to \infty$ in (19), the dominated convergence theorem yields

$$\lim_{n\to\infty} \|\mathbf{K}f_n\|_{\mathcal{L}_2}^2 = \sum_{j\in J} \Big(\lim_{n\to\infty} |\langle \mathbf{K}f_n, e_j\rangle_{\mathcal{L}_2}|^2 \Big). \tag{20}$$

Now, consider the adjoint operator $\mathbf{K}^*$ of $\mathbf{K}$. Since $\mathcal{L}_2$ is Hilbertian, $\mathbf{K}^*$ acts from $\mathcal{L}_2$ to itself and satisfies

$$\langle \mathbf{K}g, h\rangle_{\mathcal{L}_2} = \langle g, \mathbf{K}^* h\rangle_{\mathcal{L}_2}, \quad \forall g, h \in \mathcal{L}_2.$$

Since $f_n \xrightarrow{\text{w}} 0$, we then observe that

$$\lim_{n\to\infty} |\langle \mathbf{K}f_n, e_j\rangle_{\mathcal{L}_2}|^2 = \lim_{n\to\infty} |\langle f_n, \mathbf{K}^* e_j\rangle_{\mathcal{L}_2}|^2 = 0,$$

By this and (20), we conclude that $\mathbf{K}f_n \xrightarrow{\|\cdot\|} 0$, as desired. *Q.E.D.*

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