Solving Open Economy Models with Incomplete Markets by Perturbation

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Abstract

This paper proposes to solve open economy models with incomplete markets by first approximating the local dynamics of a tractable auxiliary model, and then applying regular perturbation to some of the parameters to reach the model of interest. The method is easy to implement with available solution packages, and can approximate models around a large subset of the state-space, including the stochastic steady-state. The lead application extends the two-period, multi-asset model of Coeurdacier and Gourinchas (2016, JME) to an infinite horizon setup. The calibrated model with bonds and equities delivers a large level of equity home bias, and a natural link between trade and financial openness with external asset positions comparable to the data. On the downside, the model generates excessive risk-sharing, and counterfactual co-movements of gross capital flows.

Keywords: Equity home bias, incomplete markets, international capital flows, open economy DSGE models, perturbation, portfolio allocations, solution methods.

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1 Introduction

Perturbation methods are a widely popular tool for solving Dynamic Stochastic General Equilibrium (DSGE) models. They tend to be fast, locally accurate as one increases the order of approximation, and they can handle many state variables without suffering from the curse of dimensionality. By far, their most common application consists of building Taylor series approximations to the policy rules of the model around its deterministic steady-state using implicit-function theorems (Fernández-Villaverde, Rubio-Ramírez, and Schorfheide, 2016).

Nevertheless, standard regular perturbation falls short in dealing with many open economy models of interest. This follows because, under incomplete markets, the properties of these models in a stochastic setting can differ radically from their deterministic counterparts. A classic example is the small open economy with a stochastic endowment. While this model is dependent on initial conditions and displays non-stationary dynamics in a deterministic setup (Schmitt-Grohé and Uribe, 2003), a precautionary-savings motive induces local stability and stationarity in a stochastic context (Chamberlain and Wilson, 2000).[1] A related problem arises with DSGE portfolio-choice models: in a deterministic arbitrage-free equilibrium all asset returns must be identical, leading to arbitrary portfolio allocations that makes the analysis of these models intractable with the conventional perturbation approach (Devereux and Sutherland, 2011; Tille and van Wincoop, 2010).

The first contribution of this paper is to propose a small generalization of the standard perturbation procedure to bypass these technical difficulties. The method involves using two perturbation parameters, ε and σ, to link a family of models. While σ is the conventional parameter scaling random innovations, the new parameter ε interacts with a subset of the model parameters. Consider a setup where the model of interest $(\varepsilon, \sigma) = (1, 1)$ is well-behaved (e.g. a small open economy model with a stochastic endowment), but intractable with $(\varepsilon, \sigma) = (1, 0)$ (the deterministic version of this model). However, the auxiliary model with portfolio adjustment costs $(\varepsilon, \sigma) = (0, 0)$ has good local properties. Then, we can obtain approximations to the policy rules of the model of interest around the deterministic steady-state of the auxiliary model by applying the same algorithm used in standard perturbation. As such, simultaneous perturbation of the pair (ε, σ) allows us to effectively reach the model of interest starting from the nearby auxiliary one.

In principle, one could obtain the same result with a single-parameter perturbation model where σ interacts with both the random innovations and the subset of model parameters.[2] The main advantage of a two-parameter approach is eminently practical: popular solution toolboxes (e.g. Dynare) only allow for the classical perturbation scheme. But thanks to the variable-parameter duality of perturbation objects, we can treat the new parameter ε as an

[1]This issue is pervasive in open-economy macroeconomics. For example, Corsetti, Dedola and Leduc (2023) analyze a DSGE one-bond economy that delivers non-stationary wealth dynamics because, for simplicity, they only consider the perfect-foresight case.

[2]For example, Mertens and Judd (2018) solve an Heterogeneous-agent model a là Krusell-Smith with perturbation methods where a factor $(1 - \sigma)$ scales a penalty function ensuring that all households hold the same amount of capital in the auxiliary deterministic model.

exogenous state that is constant over time.[3] Thus, at the very cheap cost of augmenting the auxiliary model with an additional exogenous state, we can compute approximations to the model of interest using available toolboxes.

Furthermore, the two-parameter perturbation model allows to obtain local solutions around a large subset of points of the state-space, while still relying on the implicit function theorems ensuring convergence of the Taylor series. This result follows because, by properly interacting ε with structural parameters of the auxiliary model, we can impose different parameter values to those intended for the model of interest, leading to a different deterministic steady-state than the one implied by the model of interest (which might even fail to exist). Perturbation of ε then corrects for any discrepancy of parameter values between models.

Exploiting this property, the paper develops a simple fixed-point algorithm, akin to standard calibration routines of DSGE models, that approximates the stochastic steady-state of the model, and builds the perturbation solution around it.[4] A proposition ensures that, under standard regularity conditions, the output of the algorithm converges to the true stochastic steady-state of the model as the order of the Taylor series goes to infinity. In the applications, the algorithm delivers the same formulas for zero-order portfolios obtained with bifurcation methods (Judd, 1998), and a stochastic steady-state of the small open economy identical to the one obtained with global methods. The accuracy of the local approximations follows the same pattern emphasized by Devereux and Sutherland (2010) and Tille and van Wincoop (2010). Specifically, a second-order solution might suffice to approximate the stochastic steady-state (provided that innovations are symmetric), but we require at least a third-order solution to capture well portfolio dynamics.

The second contribution of the paper consists of applying two-parameter perturbation to analyze hedging motives of equity home bias in a two-country DSGE model with bonds and equities. To do so, I extend the two-period model by Coeurdacier and Gourinchas (2016, from hereon CG) to an infinite-horizon setting, thus allowing rigorous quantitative work. The baseline model consists of an endowment economy with two symmetric countries (Home and Foreign), two goods, up to four assets, and different sources of risk (income, redistributive, and preference shocks) that makes financial markets incomplete. With two-parameter perturbation, what the literature considered a very challenging task now becomes standard DSGE work. In particular, introducing portfolio adjustment costs in the auxiliary model suffices to obtain a third-order solution to the model of interest around the stochastic steady-state that is highly accurate (average Euler errors below -6), and very fast to compute.[5]

A first set of quantitative results confirms the original message from CG: bonds still matter. Specifically, a model with only trade in equities fails to reproduce the large level of equity home bias found in the data. This was an open question because, as CG showed in their Appendix, sufficiently volatile redistributive risk can generate a sizable home bias. While this theoretical

[3] Levintal (2017) exploits this duality to build an efficient solution algorithm for the standard perturbation model.

[4] Coeurdacier, Rey and Winant (2011) call this point the risky steady-state.

[5] See https://github.com/ghausmann/two_parameter_perturbation.git for a repository with all the replication files to solve the main application and examples of this paper, using both Dynare and the solution package by Levintal (2017).

result holds in the present framework, a calibration set to match key moments of advanced economies rules out this possibility. At best, the equities-only model delivers a near perfectly diversified portfolio; at worst, it reproduces a worse-than-you-think-puzzle scenario (Baxter and Jermann, 1997). In contrast, the same model incorporating both bonds and equities has the potential to deliver a large home bias. But to generate external equity positions consistent with the data, an incomplete markets framework with all sources of risk active is essential. Otherwise, domestic investors find it optimal to hold all domestic equity to eliminate redistributive risk, just as CG found in their two-period model.

A second set of results accounts for the observed patterns of trade and financial globalization in the last decades, both across countries and over time. Firstly, the calibrated model predicts large, leveraged gross debt positions comparable to the data that are long in the home currency, and short in the foreign one (Lane and Shambaugh, 2010; Maggiori, Neiman and Schreger, 2020). Secondly, the model predicts a strong positive relationship between trade and financial openness (Collard et al. 2007; Heathcote and Perry, 2013) that is mostly driven by a large increase in the external debt position of countries (Khalil, 2019). The mechanism that generates this positive link is also novel. In the model, holding leveraged debt positions delivers risk-sharing transfers because the returns of Home and Foreign bonds change differently in response to the same income shocks. But as deeper trade integration synchronizes the returns, countries find it optimal to leverage up their debt positions just to maintain the same degree of risk-sharing.

A strength of two-parameter perturbation is that the quantitative analysis naturally yields implications for the dynamics of equilibrium portfolios and international capital flows. As in Sauzet (2022a, 2022b), the allocation of time-varying wealth across countries emerges as a key driver of portfolio reallocation, an aspect largely ignored by previous literature. Since relative wealth is a slow moving object, this dependence generates a strong portfolio inertia despite the absence of financial frictions in the model of interest.[6] However, the model produces counterfactual co-movements of gross capital flows. Stochastic simulations and impulse responses reveal an almost perfect negative correlation between capital inflows and outflows, implying that gross capital flows are much less volatile than net capital flows. As documented by Broner et al. (2013) and Davis and van Wincoop (2018), both predictions are at odds with international time-series data.

The present model also encounters challenges (common in many previous studies) when attempting to replicate key features of time-series data. Among other issues, the simulations struggle to capture the large volatility and persistence of the real exchange rate, and cannot fully explain its co-movement with relative consumption, despite a considerable quantitative improvement compared to complete-markets models strongly rejected by the data (Backus and Smith, 1993). Another symptom of excessive risk-sharing is the model's inability to account for the consumption correlation puzzle (Backus, Kehoe, and Kydland, 1992), since the cross-correlation of Home and Foreign consumption is too strong, both in comparison to the data

[6]See Bacchetta, Davenport and van Wincoop (2022), and Bacchetta, van Wincoop and Young (2023) for recent contributions introducing portfolio frictions in DSGE models to generate this inertia.

and relative to the cross-correlation of GDP.[7]

Taking further advantage of the capabilities of two-parameter perturbation, a final round of experiments investigate the behavior of international portfolios under long-run global imbalances. Specifically, I follow Gourinchas, Rey, and Govillot (2017), Sauzet (2022a) and Stepanchuk and Tsyrennikov (2015) in introducing a structural asymmetry by making the Home country less risk-averse than Foreign, as a simple way of rationalizing the international role of the United States as a global banker. Similar to these papers, this new source of heterogeneity induces the Home country to adopt a long-run riskier portfolio with less home bias, and to enjoy, on average, a larger consumption stream. In return, Home's consumption becomes significantly more volatile than Foreign's. Nevertheless, in this model, the less risk-averse country is also a net creditor running a long-run positive net foreign asset position, a feature at odds with U.S. data, but consistent with recent empirical findings by Zhang (2023), who shows that creditor countries tend to exhibit more diversified international portfolios.

Related literature. Firstly, the paper contributes to the existing body of literature on solution methods for portfolio-choice models that relies on local approximations.[8] Within this literature, it aligns closely with the methodological breakthroughs of Devereux and Sutherland (2010, 2011) and Tille and van Wincoop (2010), who build approximations based on the so-called zero-order portfolios (those held by investors in a symmetric deterministic steady-state as risk goes to zero). Alternatives to this approach include contributions based on bifurcation theorems (Judd, 1998; Judd and Guu, 2001; Winant, 2014) or continuous-time approximations (Campbell and Viceira, 1999; Evans and Hnatkovska, 2012; Bacchetta, Davenport and van Wincoop, 2022). To this literature, I bring a solution method fully based on regular perturbation, which benefits from all the theoretical results and efficient, user-friendly algorithms available for this mathematical tool.[9] In addition, approximations based on zero-order portfolios suffer from two main limitations. First, they can become highly inacurate for models with asymmetric countries (Rabitsch, Stepanchuk and Tsyrennikov, 2015). Second, being based on a second-order approximation to the portfolio Euler equations, they cannot account for common features in DSGE macro-finance models involving higher-order moments of the shocks, such as non-symmetric distributions or stochastic volatility. By building approximations centered at the stochastic steady-state, two-parameter perturbation provides a satisfactory solution to these shortcomings.

Secondly, the paper contributes to a methodological literature proposing approximations of DSGE models around the stochastic steady-state. Examples include Coeurdacier, Rey and Winant (2011), de Groot (2013), Juillard (2011), Hausmann-Guil (2022), and Lopez, Lopez-

[7]Since the model does not include features such as recursive preferences, stochastic volatility, or disaster risk, it also fails to reproduce many stylized asset-pricing facts (see e.g. Andreasen (2012)). On a positive note, two-parameter perturbation can easily handle all these extensions.

[8]See Bacchetta, van Wincoop and Young (2023), Cao, Luo and Nie (2023), Sauzet (2022b), and Stepanchuk and Tsyrennikov (2015) for examples of global solution methods. Compared to these algorithms, the usual trade-offs between global and local methods apply (higher global accuracy and ability to handle strong non-linearities, versus acute curse of dimensionality and sizable entry costs).

[9]See Jin and Judd (2002), Schmitt-Grohé and Uribe (2004), Andreasen (2012), and Levintal (2017), among others.

Salido and Vazquez-Grande (2022). Relative to these studies, the algorithm presented here is more general in scope, easier to implement, and backed by theoretical results. Indeed, the ability of two-parameter perturbation to solve models around a point different than the deterministic steady-state relates the paper to the Taylor-projection method by Levintal (2018), but without having to rely on non-linear solvers to compute the coefficients of the polynomials approximating decision rules.

Thirdly, the paper contributes to the vast literature that, following Schmitt-Grohé and Uribe (2003), closes small open-economy models by introducing technical modifications with the only purpose of generating stable dynamics around a well-defined steady-state. The problem with this approach is twofold. First, the *ad hoc* modifications prevent researchers from studying the determinants of long-run imbalances in small open economies, since the steady-state of net foreign assets is a direct function of the modification involved. Second, these extra assumptions are not innocuous for the numerical results (Seoane, 2015; de Groot, Durdu and Mendoza, 2023). With two-parameter perturbation, both problems are gone: now we can think of these modifications as excellent auxiliary devices to be shut down at the model of interest.[10]

Finally, the main application relates the paper to the international finance literature that builds general equilibrium models to account for the international diversification puzzle, and more generally, for the determinants of gross external positions. Seminal contributions include Baxter and Jermann (1997), Obstfeld and Rogoff (2000), Engel and Matsumoto (2009), Heathcote and Perry (2013), and Coeurdacier and Gourinchas (2016).[11] Apart from the obvious connection to the later, the present work strongly relates to Heathcote and Perry (2013) and Khalil (2019) in generating a strong link between trade and financial integration, and to Coeurdacier, Kollmann and Martin (2010) and Sauzet (2022a) in delivering quantitative predictions for the dynamics of capital flows. To my knowledge, this is the first study delivering a solution for a model with two (potentially asymmetric) countries, two goods, four assets and incomplete markets with implications for gross capital flows.

2 A simple example: portfolio choices for small risks

To help build the intuition of why two-parameter perturbation is a useful tool to deal with incomplete market models, I start by solving the same example used by Judd (1998, Chapter 15) to introduce bifurcation. It consists of a one-period model whose approximation can be worked out using pen and paper, and it examines the portfolio choice problem of an individual investor with access to a safe and a risky asset. The auxiliary model introduces portfolio adjustment costs (PAC) to pin down the proportion of wealth allocated in the risky asset. Remarkably, two-parameter perturbation yields the same approximation formulas obtained by Judd.

[10]The small open-economy example of this paper shows that, regardless of the auxiliary modification involved, two-parameter perturbation delivers highly accurate approximations (Average Euler errors below -5 with a third-order solution, and well below -6 with a fifth-order one) with a calibration matching key moments of Mexican data.

[11]See Coeurdacier and Rey (2013) for a thorough review of the literature on the equity home bias.

An investor has one unit of wealth to invest in two assets. The safe asset yields 1 dollar per dollar invested, and the risky asset yields Z dollars per dollar invested, where Z is stochastic and given by

$$Z = 1 + \pi + z \tag{2.1}$$

Here $\pi \geq 0$ is the premium paid by the risky asset, and z a random variable with moments $\mathbb{E}\{z\} = 0$, $\mathbb{E}\{z^2\} = \mu_2 > 0$, and $\mathbb{E}\{z^3\} = \mu_3$. Let ω be the proportion of wealth invested in the risky asset, and $Y = (1-\omega) + \omega Z$ the investor's final wealth. The investor has CRRA preferences, and her optimization problem is

$$\max_{\omega} U = \mathbb{E}\left\{\frac{Y^{1-\gamma}-1}{1-\gamma}\right\},$$

with first-order condition

$$0 = \mathbb{E}\left\{\frac{\pi+z}{Y^{\gamma}}\right\}. \tag{2.2}$$

The goal is to approximate the proportion ω^* that solves (2.2). Regular perturbation around the deterministic case $z = 0$ does not work because a unique finite solution for ω^* does not exist: if $\pi > 0$, ω^* should be infinity, and if $\pi = 0$ we find that (2.2) is satisfied for all ω. To deal with this issue, Judd (1998) considers an auxiliary model with $Z(\sigma) = 1 + \sigma^2\pi + \sigma z$ depending on the perturbation parameter σ, and then applies a bifurcation theorem to approximate ω^*.

Instead, consider the following auxiliary model in terms of perturbation parameters ε and σ. We have

$$Z(\varepsilon,\sigma) = 1 + \varepsilon^2\pi + \sigma z.$$

The investor faces quadratic costs of holding a proportion ω different from some given value $\overline{\omega}$, as follows:

$$Y(\omega,\varepsilon,\sigma) = (1-\omega) + \omega Z(\varepsilon,\sigma) - \frac{\psi}{2}\left(1-\varepsilon^2\right)(\omega-\overline{\omega})^2,$$

where $\psi > 0$ is an auxiliary parameter. Note that if $(\varepsilon,\sigma) = (1,1)$ we recover the model of interest.[12] The first-order condition of the auxiliary model is

$$0 = \mathbb{E}\left\{\frac{\varepsilon^2\pi + \sigma z - \psi\left(1-\varepsilon^2\right)(\omega-\overline{\omega})}{\left[Y(\omega,\varepsilon,\sigma)\right]^{\gamma}}\right\} = G(\omega,\varepsilon,\sigma). \tag{2.3}$$

Now we can use regular perturbation. First, at the deterministic case $(\varepsilon,\sigma) = (0,0)$ we find a unique and finite solution $\omega^* = \overline{\omega}$. Second, the Implicit function theorem applies and there exists a well-defined function $\omega^* = \omega(\varepsilon,\sigma)$ around $(\varepsilon,\sigma) = (0,0)$ that satisfies (2.3). Thus, we can write:

$$G(\omega(\varepsilon,\sigma),\varepsilon,\sigma) = 0. \tag{2.4}$$

[12] Following Judd (1998), I use second powers of ε to ensure that, whenever $\sigma = \varepsilon$, risk premia is proportional to variance.

The second-order approximation to ω^* is:

$$\omega^* = \overline{\omega} + \omega_\varepsilon \varepsilon + \omega_\sigma \sigma + \frac{1}{2}\left(\omega_{\varepsilon\varepsilon}\varepsilon^2 + \omega_{\sigma\sigma}\sigma^2 + 2\omega_{\varepsilon\sigma}\varepsilon\sigma\right). \tag{2.5}$$

We can obtain the derivatives of $\omega(\varepsilon, \sigma)$ by taking partial derivatives of G with respect to σ and ε and equating them to zero. The procedure yields $\omega_\varepsilon = \omega_\sigma = \omega_{\varepsilon\sigma} = 0$, and

$$\begin{aligned} \omega_{\varepsilon\varepsilon} &= 2\frac{\pi}{\psi}, \\ \omega_{\sigma\sigma} &= -2\left(\frac{\gamma\mu_2}{\psi}\right)\overline{\omega}. \end{aligned}$$

Plugging these results into (2.5) and evaluating at the model of interest $(\varepsilon, \sigma) = (1, 1)$ gives

$$\psi\left(\omega^* - \overline{\omega}\right) = \pi - \gamma\mu_2\overline{\omega}.$$

Then, imposing $\omega^* = \overline{\omega}$ (a condition analogous to the stochastic steady-state of infinite-horizon models) delivers a solution that depends on second moments:

$$\omega^* = \frac{\pi}{\gamma\mu_2},$$

which is the same formula obtained by Judd (1998). Intuitively, if the approximation point $\overline{\omega}$ is already the optimal proportion of wealth, the sum of the perturbation components $\omega_{\varepsilon\varepsilon}$ and $\omega_{\sigma\sigma}$ must equal zero, which pins down ω^*.

We can proceed further and use a third-order approximation to obtain a solution that depends on both μ_2 and μ_3:

$$\omega^* = \overline{\omega} + \frac{1}{2}\left(\omega_{\varepsilon\varepsilon}\varepsilon^2 + \omega_{\sigma\sigma}\sigma^2\right) + \frac{1}{6}\left(\omega_{\varepsilon\varepsilon\varepsilon}\varepsilon^3 + 3\omega_{\varepsilon\varepsilon\sigma}\varepsilon^2\sigma + 3\omega_{\varepsilon\sigma\sigma}\varepsilon\sigma^2 + \omega_{\sigma\sigma\sigma}\sigma^3\right). \tag{2.6}$$

The new derivatives are $\omega_{\varepsilon\varepsilon\varepsilon} = \omega_{\varepsilon\varepsilon\sigma} = \omega_{\sigma\sigma\varepsilon} = 0$, and

$$\omega_{\sigma\sigma\sigma} = \left(\frac{\gamma(\gamma+1)\mu_3}{\psi}\right)3\left(\overline{\omega}\right)^2.$$

Plugging the results for $\omega_{\sigma\sigma}$, $\omega_{\varepsilon\varepsilon}$ and $\omega_{\sigma\sigma\sigma}$ into (2.6), evaluating at $(\varepsilon, \sigma) = (1, 1)$, and imposing $\omega^* = \overline{\omega}$ gives

$$0 = \pi - \gamma\mu_2\omega^* + \left(\frac{\gamma(\gamma+1)}{2}\right)\mu_3\left(\omega^*\right)^2.$$

While formally a quadratic equation in ω^* , we can use implicit differentiation to obtain the following expression for ω^* around $\mu_3 = 0$:

$$\omega^* = \frac{\pi}{\gamma\mu_2} + \frac{(\gamma+1)}{2}\left(\frac{\mu_3}{\mu_2}\right)\left(\frac{\pi}{\gamma\mu_2}\right)^2.$$

Again, the second term proportional to μ_3 is the same one obtained by Judd (1998).

3 The two-parameter perturbation model

This section presents an extension to the canonical perturbation model where the new perturbation parameter ε interacts with a subset of the model parameters. The structure and notation of the model follows closely the expositions by Schmitt-Grohé and Uribe (2004), and Fernández-Villaverde, Rubio-Ramírez, and Schorfheide (2016).

3.1 The framework

Consider an auxiliary perturbation model with a set of equilibrium conditions of the form

$$\mathbb{E}_t \mathbf{f}\left(\mathbf{y_{t+1}}, \mathbf{y_t}, \mathbf{x_{t+1}}, \mathbf{x_t}, \varepsilon\right) = 0. \tag{3.1}$$

Here, $\mathbb{E}_t$ denotes conditional expectations at time $t = 0, 1, 2, ...,$ $\mathbf{y_t}$ is a vector of n_y control variables, $\mathbf{x_t}$ is a vector of n_x state variables, and $\varepsilon \geq 0$ is a perturbation parameter that interacts with a subset of the model parameters. The function $\mathbf{f}$ maps $\mathbb{R}^{2(n_y+n_x)+1}$ into $\mathbb{R}^{(n_y+n_x)}$. In turn, the state vector can be partitioned as $\mathbf{x_t} = [\mathbf{x_{1,t}}; \mathbf{x_{2,t}}]$, where $\mathbf{x_{1,t}}$ consists of the n_{x1} endogenous states, and $\mathbf{x_{2,t}}$ of the n_{x2} exogenous states.[13] The latter follows a stochastic process of the form

$$\mathbf{x_{2,t+1}} = \mathbf{C}(\mathbf{x_{2,t}}) + \sigma \eta_u \mathbf{u}_{t+1},$$

where $\mathbf{C} : \mathbb{R}^{n_{x2}} \rightarrow \mathbb{R}^{n_{x2}}$ is a differentiable function that generates stationary dynamics, $\mathbf{u}_t$ is a vector of n_u zero-mean innovations, and η_u is a known matrix with dimensions $n_{x2} \times n_u$ scaled by the standard perturbation parameter $\sigma \geq 0$.

The solution to the model consists of a set of policy functions $\mathbf{g}$ and $\mathbf{h}$ for the control and state variables:

$$\begin{aligned} \mathbf{y_t} &= \mathbf{g}(\mathbf{x_t}, \varepsilon, \sigma), \\ \mathbf{x_{t+1}} &= \mathbf{h}(\mathbf{x_t}, \varepsilon, \sigma) + \sigma \eta \mathbf{u}_{t+1}, \end{aligned} \tag{3.2}$$

where $\mathbf{g}$ maps $\mathbb{R}^{(n_x+2)}$ into $\mathbb{R}^{n_y}$, $\mathbf{h}$ maps $\mathbb{R}^{(n_x+2)}$ into $\mathbb{R}^{n_x}$, and $\eta = [0_{n_{x1} \times n_u}; \eta_u]$ is a $n_x \times n_u$ matrix. Note that, under this formulation, the function $\mathbf{C}$ is the lower block of $\mathbf{h}$, and the only one updated by innovations.

The deterministic steady-state (DSS) of the model consists of vectors $(\mathbf{x_d}, \mathbf{y_d})$ that satisfy

$$\mathbf{f}\left(\mathbf{y_d}, \mathbf{y_d}, \mathbf{x_d}, \mathbf{x_d}, 0\right) = 0. \tag{3.3}$$

Assume that the DSS exists and that it is unique. Note that, by construction, we have that

$$\begin{aligned} \mathbf{x_d} &= \mathbf{h}(\mathbf{x_d}, 0, 0), \\ \mathbf{y_d} &= \mathbf{g}(\mathbf{x_d}, 0, 0), \end{aligned} \tag{3.4}$$

[13]Throughout the text I will also refer to the exogenous states as shocks.

that is, the vector $\mathbf{x_d}$ (which can be partitioned as $\mathbf{x_d} = [\mathbf{x_{1,d}}; \mathbf{x_{2,d}}]$) is the fixed-point of $\mathbf{h}$ conditional on $(\varepsilon, \sigma) = (0, 0)$.

The goal of perturbation is to approximate the functions $\mathbf{g}$ and $\mathbf{h}$ with Taylor series around $\mathbf{x_t} = \mathbf{x_d}$. To find their partial derivatives, plug-in the unknown policy functions on $\mathbf{f}$, and define the new operator $\mathbf{F} : \mathbb{R}^{n_x+2} \to \mathbb{R}^{n_x+n_y}$:

$$\mathbf{F}\left(\mathbf{x_t}, \varepsilon, \sigma\right) \equiv \mathbb{E}_t \mathbf{f}\left(\mathbf{g}\left(\mathbf{h} + \sigma\eta\mathbf{u}_{t+1}\right), \mathbf{g}, \mathbf{h} + \sigma\eta\mathbf{u}_{t+1}, \mathbf{x_t}, \varepsilon\right) = 0.$$

Since $\mathbf{F}\left(\mathbf{x_t}, \varepsilon, \sigma\right) = 0$ must hold for all values of $(\mathbf{x_t}, \varepsilon, \sigma)$, any of its partial derivatives evaluated at $(\mathbf{x_d}, 0, 0)$ must also equal zero:

$$\mathbf{F}_{x_i^j \varepsilon^l \sigma^m}\left(\mathbf{x_d}, 0, 0\right) = 0,$$

where $\mathbf{F}_{x_i^j \varepsilon^l \sigma^m}$ is the partial derivative of $\mathbf{F}$ with respect to the i-th component of $\mathbf{x_t}$ taken j times, with respect to ε taken l times, and with respect to σ taken m times. It follows that, by taking the proper number of derivatives, we end up with a system of equations to solve for the coefficients of the Taylor series of $\mathbf{g}$ and $\mathbf{h}$ up to a given order k.

3.2 Implementation

The two-parameter perturbation model can be implemented using any algorithm designed to solve the standard perturbation model. To show this, I follow Levintal (2017) and exploit the variable-parameter duality of perturbation objects to rewrite the model with a new notation that treats the perturbation parameter ε as a state variable.

To start, let ε_t denote an exogenous state variable that is constant over time: $\varepsilon_{t+1} = \varepsilon_t$. From the theoretical side, having a unit root is valid because ε_t is a purely exogenous state that only depends on itself. From the practical side, popular solution packages such as Dynare and the MATLAB function `gx_hx.m` by Schmitt-Grohé and Uribe (2004) accept the equation $\varepsilon_{t+1} = \rho_\varepsilon \varepsilon_t$ with $\rho = 1$ without problems (the latter once we change the default threshold of unit eigenvalues).[14] Alternatively, we can set a value $\rho_\varepsilon < 1$ before executing the solution algorithm, and impose $\rho_\varepsilon = 1$ in the solution matrices right after executing the solver. In practice, this procedure delivers numerical results identical to the unit root case if ρ_ε is sufficiently close to one.

Next, define the new vectors:

$$\widehat{\mathbf{f}} = \begin{pmatrix} \mathbf{f} \\ -\varepsilon_{t+1} + \varepsilon_t \end{pmatrix}, \ \widehat{\mathbf{x}}_\mathbf{t} = \begin{pmatrix} \mathbf{x_t} \\ \varepsilon_t \end{pmatrix}, \ \widehat{\eta} = \begin{pmatrix} \eta \\ 0_{1\times n_u} \end{pmatrix}.$$

Using this notation, the new set of equilibrium conditions is:

$$\mathbb{E}_t \widehat{\mathbf{f}}\left(\mathbf{y_{t+1}}, \mathbf{y_t}, \widehat{\mathbf{x}}_\mathbf{t+1}, \widehat{\mathbf{x}}_\mathbf{t}\right) = 0, \tag{3.5}$$

[14] In the online repository, the function `gx_hx.m` already has a modified default threshold for eigenvalues that is larger than (but extremely close to) one. I thank Oren Levintal for this suggestion.

and we can restate the solution to the model as:

$$\begin{aligned} \mathbf{y_t} &= \widehat{\mathbf{g}}(\mathbf{x_t}, \sigma), \\ \widehat{\mathbf{x}}_{\mathbf{t+1}} &= \widehat{\mathbf{h}}(\widehat{\mathbf{x}}_{\mathbf{t}}, \sigma) + \sigma \widehat{\eta} \mathbf{u}_{t+1}, \end{aligned} \tag{3.6}$$

where $\widehat{\mathbf{g}}$ and $\widehat{\mathbf{h}}$ are the policy functions in the new notation. Finally, let $\widehat{\mathbf{x}}_{\mathbf{d}} = [\mathbf{x_d}; 0]$. The new DSS consists of vectors $(\widehat{\mathbf{x}}_{\mathbf{d}}, \mathbf{y_d})$ that satisfy

$$\widehat{\mathbf{f}}\left(\mathbf{y_d}, \mathbf{y_d}, \widehat{\mathbf{x}}_{\mathbf{d}}, \widehat{\mathbf{x}}_{\mathbf{d}}\right) = 0. \tag{3.7}$$

It follows that any algorithm capable of solving a model with the form of (3.5)-(3.7) will also solve the two-parameter perturbation model.

3.3 Applications

Without loss of generality, we can normalize the parameters of the model such that the relevant cases are when σ and ε take values of either zero or one. Thus, there are two cases with equally valid interpretations of what constitutes the model of interest. The first is to interpret $(\varepsilon, \sigma) = (0, 1)$ as the model of interest. In such scenario, we are dealing with a stochastic model (since $\sigma = 1$ activates the random innovations) whose approximation is identical to the standard perturbation model (since $\varepsilon = 0$ shuts down any term that interacts with this parameter in the Taylor series). A potential application of the two-parameter model is to scale a model parameter (say the discount factor β of the growth model) by $(1 + \varepsilon)$, and think of small departures from $\varepsilon = 0$ as the solution to the model of interest for different values of β, without having to recalculate the DSS and the derivatives of $\mathbf{g}$ and $\mathbf{h}$.

The alternative is to interpret $(\varepsilon, \sigma) = (1, 1)$ as the model of interest. In this case, we are still dealing with a stochastic model, but in general the approximation will be different than the one obtained with standard perturbation, because $\varepsilon = 1$ activates all terms interacting with this parameter in the Taylor series. This new feature allows for two interesting (and complementary) applications:

(i) We can approximate the policy functions $\mathbf{g}$ and $\mathbf{h}$ around a point $\mathbf{x_t} = [\overline{\mathbf{x}}; \mathbf{x_{2,d}}]$ such that $\overline{\mathbf{x}}$ is different than the DSS of the endogenous state variables implied by the model of interest (the $(\varepsilon, \sigma) = (1, 0)$ case), *and still rely on the theorems ensuring convergence of the Taylor series*.[15] This can be achieved by properly interacting ε with some parameters of the auxiliary model. Specifically, we can introduce up to $n_{x1} \times 1$ auxiliary parameters $\overline{\boldsymbol{\psi}}$ that affect the DSS, and scale them by factors such as $(1 - \varepsilon)$ or $\left(1 - \varepsilon^2\right)$. Since at the DSS $\varepsilon = 0$, we can solve for values of $\overline{\boldsymbol{\psi}}$ such that $\mathbf{x_{1,d}} = \overline{\mathbf{x}}$ satisfies the DSS conditions (3.3). Then, evaluating the Taylor series of $\mathbf{g}$ and $\mathbf{h}$ at $(\varepsilon, \sigma) = (1, 1)$ removes any effect of $\overline{\boldsymbol{\psi}}$ on the equilibrium dynamics, and all that is left are policy functions centered around the desired point $\overline{\mathbf{x}}$. The stochastic growth model of

[15] See in particular Theorem 6 in Jin and Judd (2002).

Appendix B.1 provides a clear-cut example.[16]

(ii) We can introduce modifications in the auxiliary model that get shut down at the model of interest. This way, we can build an approximate solution to a DSGE economy with incomplete markets by starting from the solution of a nearby model whose solution we have access to, *even if the deterministic version of the original economy is intractable with standard perturbation*. Again, a judicious interaction of ε with the parameters of the auxiliary model will do the job. For example, one can introduce Portfolio adjustment costs (PAC) in an incomplete-markets model with the only purpose of generating stationary dynamics around a well-defined DSS in the auxiliary model, and then get rid of them by evaluating $\mathbf{g}$ and $\mathbf{h}$ at $(\varepsilon, \sigma) = (1, 1)$. This is the technique applied in the two-country DSGE model of Section 5, and in the small open economy of Appendix B.2.

3.4 Approximating the stochastic steady-state

This subsection describes a simple procedure to approximate the stochastic steady-state (SSS) of the model of interest such that the perturbation is centered at this point. The procedure is akin to standard calibration routines, and it builds on imposing enough conditions such that the DSS of the endogenous state variables in the auxiliary model coincides with their SSS in the model of interest. If the auxiliary model incorporates the proper modifications, the algorithm will approximate the true SSS even if the DSS implied by the model of interest (the $(\varepsilon, \sigma) = (1, 0)$ case) is ill-defined.

To start, define the SSS of the model of interest as the vectors $\mathbf{x_t} = \mathbf{x_s}$ and $\mathbf{y_t} = \mathbf{y_s}$ that satisfy

$$\mathbf{x_s} = \mathbf{h}(\mathbf{x_s}, 1, 1), \tag{3.8}$$

$$\mathbf{y_s} = \mathbf{g}(\mathbf{x_s}, 1, 1) \tag{3.9}$$

In a way analogous to Eq. (3.4), $\mathbf{x_s}$ is the fixed point of $\mathbf{h}$ conditional to $(\varepsilon, \sigma) = (1, 1)$, assumed to be unique. As with the DSS, $\mathbf{x_s}$ can be partitioned as $\mathbf{x_s} = [\mathbf{x_{1,s}}; \mathbf{x_{2,s}}]$, where $\mathbf{x_{1,s}}$ is the SSS of the endogenous states, and $\mathbf{x_{2,s}}$ the SSS of the exogenous states. Note that, while $\mathbf{x_{2,s}} = \mathbf{x_{2,d}}$ by construction, $\mathbf{x_{1,s}}$ does not need to coincide with $\mathbf{x_{1,d}}$ (due to, for example, a precautionary savings effect). In general, finding the true $\mathbf{x_{1,s}}$ requires previous knowledge of $\mathbf{h}$, a function we do not have access to.

[16] A limitation to this technique is that the magnitude of $\overline{\boldsymbol{\psi}}$ must be small enough to ensure that $(\varepsilon, \sigma) = (1, 1)$ falls within the radius of convergence. As in standard DSGE applications, solving for values of $\overline{\boldsymbol{\psi}}$ that satisfy the DSS conditions can be often done analytically. If not possible, one can use a standard nonlinear solver.

3.4.1 The stochastic steady-state conditions

Let $\mathbf{h}_1^k : \mathbb{R}^{n_x+2} \to \mathbb{R}^{n_{x1}}$ be the k-order Taylor series around $(\mathbf{x_d}, 0, 0)$ for the upper block of $\mathbf{h}$. Formally:

$$\mathbf{h}_1^k(\mathbf{x_t}, \varepsilon, \sigma) = \mathbf{x_{1,d}} + \sum_{i=1}^{k} \frac{1}{i!} H_i \begin{pmatrix} \mathbf{x_t} - \mathbf{x_d} \\ \varepsilon \\ \sigma \end{pmatrix}^{\otimes i}, \tag{3.10}$$

where each H_i is a $n_{x1} \times (n_x + 2)^i$ matrix of order i derivatives, and $^{\otimes i}$ denotes a "Kronecker power", that is $z^{\otimes i} = z \otimes ... \otimes z$ i times. Similarly, the k-order Taylor series of $\mathbf{g}$ is:

$$\mathbf{g}^k(\mathbf{x_t}, \varepsilon, \sigma) = \mathbf{y_d} + \sum_{i=1}^{k} \frac{1}{i!} G_i \begin{pmatrix} \mathbf{x_t} - \mathbf{x_d} \\ \varepsilon \\ \sigma \end{pmatrix}^{\otimes i}. \tag{3.11}$$

On what follows, think of $\mathbf{x_s}$ as the fixed point of $\mathbf{h}^k(\mathbf{x_t}, 1, 1)$. Assume that a $n_{x1} \times 1$ vector $\overline{\mathbf{x}}$ completely pins down the DSS of the endogenous states: $\mathbf{x_{1,d}} = \overline{\mathbf{x}}$. As explained in Section 3.3, this can be accomplished by introducing up to $n_{x1} \times 1$ auxiliary parameters $\overline{\boldsymbol{\psi}}$ that affect the DSS scaled by factors such as $(1-\varepsilon)$ or $(1-\varepsilon^2)$.[17] Since we are looking for an approximation centered at the SSS, it must be the case that $\mathbf{x_{1,s}} = \mathbf{x_{1,d}} = \overline{\mathbf{x}}$. Imposing this condition in (3.10) and evaluating at $(\mathbf{x_t}, \varepsilon, \sigma) = (\mathbf{x_d}, 1, 1)$ delivers:

$$\sum_{i=1}^{k} \frac{1}{i!} H_i \begin{pmatrix} 0_{n_x \times 1} \\ 1 \\ 1 \end{pmatrix}^{\otimes i} = 0. \tag{3.12}$$

Since each H_i is itself a function of $\overline{\mathbf{x}}$, the last expression constitutes a system of n_{x1} equations to solve for each of the components of $\overline{\mathbf{x}}$. Intuitively, if $\overline{\mathbf{x}}$ is a fixed-point of $\mathbf{h}^k(\mathbf{x_t}, 1, 1)$, the sum of all coefficients proportional to powers of ε and/or σ must equal zero for each of its rows. Appendix A shows that, under standard regularity assumptions, the solution to (3.12) converges to the true $\mathbf{x_{1,s}}$ as k goes to infinity.

Once a candidate solution to (3.12) has been found, we can combine Eqs. (3.9) and (3.11) to retrieve a candidate solution for $\mathbf{y_s}$:

$$\mathbf{y_s} = \mathbf{y_d} + \sum_{i=1}^{k} \frac{1}{i!} G_i \begin{pmatrix} 0_{n_x \times 1} \\ 1 \\ 1 \end{pmatrix}^{\otimes i}.$$

Example: Consider the SOE model of Appendix B.2. The endogenous state is foreign bonds $\mathbf{x_{1,t}} = b_t$, and the exogenous states are (the logs of) income and returns $\mathbf{x_{2,t}} = [y_t; z_t]$. Using a

[17] Sometimes we will require less than n_{x1} auxiliary parameters if the SSS of some endogenous states is a function of the others. The baseline symmetrical scenario of the two-country model provides an example.

second-order approximation to h, the law of motion for bonds is:

$$\begin{aligned} b_{t+1} &= \bar{b} + h_b\left(b_t - \bar{b}\right) + h_y y_t + h_z z_t \\ &+ \frac{1}{2}\left(h_{bb}\left(b_t - \bar{b}\right)^2 + h_{yy} y_t^2 + h_{zz} z_t^2\right) \\ &+ h_{by}\left(b_t - \bar{b}\right) y_t + h_{bz}\left(b_t - \bar{b}\right) z_t + h_{yz} y_t z_t \\ &+ \frac{1}{2}\left(h_{\varepsilon\varepsilon}\varepsilon^2 + h_{\sigma\sigma}\sigma^2\right), \end{aligned}$$

where $b_d = \bar{b}$ is the DSS of bonds.[18] Then, evaluating at $(b_t, y_t, z_t, \varepsilon, \sigma) = \left(\bar{b}, 0, 0, 1, 1\right)$ and imposing $b_s = b_d = \bar{b}$ delivers the equation that pins down $\bar{b}$:

$$h_{\varepsilon\varepsilon} + h_{\sigma\sigma} = 0.$$

3.4.2 Numerical implementation

The system of equations (3.12) can be solved numerically with a standard nonlinear solver. All that is needed is to code a residual function that takes the vector $\bar{\mathbf{x}}$ as an argument and returns the left hand-side of (3.12). The nonlinear solver will call this function and search for a candidate $\bar{\mathbf{x}}$ that makes it zero. Here I describe in detail the pseudo-code for this residual function.

I assume that the user will implement the algorithm with an available DSGE solution toolbox. For this reason, it is convenient to treat ε_t as an exogenous state and follow the notation of Section 3.2. Therefore, the k-order Taylor series for the upper block of $\widehat{\mathbf{h}}$ is:

$$\widehat{\mathbf{h}}_1^k(\widehat{\mathbf{x}}_\mathbf{t} - \widehat{\mathbf{x}}_\mathbf{d}, \sigma) = \mathbf{x}_{\mathbf{1},\mathbf{d}} + \sum_{i=1}^{k} \frac{1}{i!} H_i \begin{pmatrix} \widehat{\mathbf{x}}_\mathbf{t} - \widehat{\mathbf{x}}_\mathbf{d} \\ \sigma \end{pmatrix}^{\otimes i}$$

where the first argument $(\widehat{\mathbf{x}}_\mathbf{t} - \widehat{\mathbf{x}}_\mathbf{d})$ are deviations of $\widehat{\mathbf{x}}_\mathbf{t} = [\mathbf{x}_\mathbf{t}; \varepsilon_t]$ from the extended DSS $\widehat{\mathbf{x}}_\mathbf{d} = [\mathbf{x}_\mathbf{d}; 0]$.

For any given $\bar{\mathbf{x}}$, the residual function will recalculate the new DSS together with the implied auxiliary parameter values, and call an external algorithm (the DSGE toolbox) to calculate the matrices of derivatives $H_1, ..., H_k$. Using these, the function will construct $\widehat{\mathbf{h}}_1^k$, evaluate it at the model of interest (that is, $\widehat{\mathbf{x}}_\mathbf{t} - \widehat{\mathbf{x}}_\mathbf{d} = [0_{n_x \times 1}; 1]$ and $\sigma = 1$), and return the output $y = \widehat{\mathbf{h}}_1^k - \bar{\mathbf{x}}$. Let $\mathbf{S}$ be all the structure arrays required by the external algorithm.[19] The pseudocode summarizing these steps is:

[18] As with σ, the only non-zero derivative of ε is $h_{\varepsilon\varepsilon}$, because this perturbation parameter enters as ε^2 in the equilibrium conditions.

[19] If the DSGE toolbox is Dynare, $\mathbf{S}$ includes the arrays M_, options_, and oo_. The external Dynare function resol.m takes these as inputs to compute the matrices $H_1, ..., H_k$.

Algorithm 1 Residual SSS function

```
function y = eval_sss(x̄;S,k)

STEP 1: Given x̄, recalculate the new DSS and implied auxiliary
parameter values, modifying S when necessary.
STEP 2: Using the new DSS and S, call an external algorithm
 to compute the matrices H_1,...,H_k.
STEP 3: Using the matrices H_1,...,H_k, evaluate ĥ_1^k at x̂_t − x̂_d =
[0_{n_x×1};1] and σ = 1.

y = h_1^k([0_{n_x×1};1],1) − x̄.
```

Steps 2 and 3 can be automated conditional on the external algorithm. To help with this task, the Github repository includes MATLAB functions that construct and evaluate $\widehat{\mathbf{h}}^k$ and $\widehat{\mathbf{g}}^k$ for Dynare and the perturbation algorithm by Levintal (2017).

Calibration mode. We can easily adapt the previous residual function to solve for values of a $n_{x1} \times 1$ vector $\bar{\theta}$ of parameters of the model of interest that target the desired $\mathbf{x_{1,s}} = \overline{\mathbf{x}}$. In this case, the main argument of the residual function is $\bar{\theta}$, the vector $\overline{\mathbf{x}}$ is kept fixed, and the nonlinear solver will search for a candidate $\bar{\theta}$ that zeroes the left hand-side of (3.12). Appendix B.2.3 uses this algorithm to calibrate the exogenous discount factor that targets the observed long-run level of net foreign assets of the SOE. We can also implement a hybrid approach to solve for a subset of $\bar{\theta}$ and $\overline{\mathbf{x}}$ while keeping the other subset fixed. For example, the baseline application with bonds and equities of Section 5 calibrates the standard deviation of preference shocks consistent with the observed level of Equity home bias, and solves for the long-run level of bond holdings. Finally, we can integrate the SSS algorithm into a more complex one targeting a variety of observables. It all comes down to ensuring that $\overline{\mathbf{x}}$ zeroes the left hand-side of (3.12).

3.5 Pruning

We need at least third-order approximations of $\mathbf{h}$ and $\mathbf{g}$ to ensure that their derivatives with respect to $\mathbf{x_t}$ incorporate the effects of second moments of the innovations, via corrections scaled by σ (Andreasen, 2012; Levintal, 2017). The same result applies to corrections of the auxiliary model driven by ε if this parameter enters as ε^2 in the equilibrium conditions. Thus, only Taylor series of order $k \geq 3$ are valid approximations to study the dynamics of incomplete-markets models.

Working with high-order perturbation solutions comes with the technical difficulty of dealing with the spurious explosive paths they sometimes generate. To resolve this issue for orders $k \geq 3$, we can implement the pruning method by Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2018). However, their pruning method treats σ as a state variable, and therefore it only affects second and higher-order effects. This treatment can be problematic for the two-perturbation model, because it imposes that first-order effects —the main driver of dynamics in a pruned state-space system— are the ones from the auxiliary model. [20]

The solution here is to modify the pruning method by treating ε and σ (or at the very least just σ) as parameters. To do so, impose $\varepsilon = \sigma = 1$, and rewrite the Taylor series (3.10) and (3.11) as follows:

$$\begin{aligned}
\mathbf{h}_1^k(\mathbf{x_t}) &= \widehat{H}_0 + \sum_{i=1}^{k} \frac{1}{i!} \widehat{H}_i \left(\mathbf{x_t} - \mathbf{x_d} \right)^{\otimes i} \\
\mathbf{g}^k(\mathbf{x_t}) &= \widehat{G}_0 + \sum_{i=1}^{k} \frac{1}{i!} \widehat{G}_i \left(\mathbf{x_t} - \mathbf{x_d} \right)^{\otimes i}
\end{aligned}$$

where $\widehat{H}_0$ and $\widehat{G}_0$ are vectors of dimension $n_{x1} \times 1$ and $n_y \times 1$, respectively.[21] From here, one can follow Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2018) and construct the laws of motions for each n-order effect using the rewritten Taylor series. The Github repository includes MATLAB functions that implement the modified pruning method (third-order only) for users of Dynare, and the algorithm by Levintal (2017).[22]

3.6 Practical advice

This last subsection provides practical advice on various topics based on my experience using two-parameter perturbation to solve DSGE models.

Order of approximations. I have already stressed the need of computing third or higher-order solutions to approximate the dynamics of the model of interest. However, a second-order solution is often accurate enough to approximate the SSS if the innovations are symmetrical (see the SOE model of Appendix B.2). To exploit this insight, one can first approximate the SSS with second-order approximations, and then build a third-order solution centered around it, as I do in the two-country DSGE model of Section 5. Dynare's second-order solver is fast enough to make it an efficient option for the algorithm of Section 3.4. For higher orders, I strongly recommend the algorithm by Levintal (2017).

Auxiliary modifications. Very often, more than one modification to the model of interest will be available as an auxiliary device. Which one to choose, then? There are several criteria to consider here. First, if the modifications are doing their job, their approximations evaluated at $(\varepsilon, \sigma) = (1, 1)$ should converge, as shown by the proposition in Appendix A, and illustrated in the SOE example. It is therefore good practice to compare solutions obtained with different

[20]Incidentally, this problem also arises in small open economy models with modifications controlled by parameters set to very small values (e.g. Fernández-Villaverde et al., 2011), as these lead to nearly unit-root first-order dynamics that only get corrected by third-order terms.

[21]Note that, if the approximation is done around the SSS, $\widehat{H}_0 = \mathbf{x_{1,d}}$.

[22]For Dynare, I implement a very mild modification to the original Dynare function simul_.m that only treats σ as a parameter.

modifications and approximation orders, and take the absence of convergence as a warning sign. Second, one should prioritize modifications that are consistent with the model of interest.[23] Third, auxiliary modifications are only valuable as a mathematical tool, and thus their microfoundations (if any) are of no real concern. In that sense, one is allowed to prioritize the simplest modification available. Fourth, as a rule of thumb any model parameter interacting with ε should be relatively small, but depending on the application, smaller is not always necessarily better. Thus, I recommend to always test the accuracy of the approximation (e.g. by computing average Euler equation errors) and, when it comes to auxiliary parameters controlling modifications to be shut down at the model of interest, check if different values lead to sizable gains in accuracy.

4 Practical examples

Appendix B provides two practical examples of DSGE models that fit in the general form of Section 2. The Github repository includes the codes that solve these two examples, using both Dynare and the algorithm by Levintal (2017) to calculate derivatives of the policy rules.

The first example (Appendix B.1) implements two-parameter perturbation to approximate the policy rules of the neoclassical growth model around the SSS, and explores what gains in accuracy can be achieved relative to standard perturbation around the deterministic steady-state. The results indicate that two-parameter perturbation is about four times more accurate in scenarios combining high levels of volatility and risk-aversion, thanks to approximations built around a point much closer to the dynamic fluctuations of the model.

The second example (Appendix B.2) considers the open economy version of the classical income fluctuation problem as in Coeurdacier, Rey, and Winant (2011). Using two different auxiliary models (Uzawa preferences and PAC), I approximate policy rules centered around the SSS. Regardless of the auxiliary model involved, two-parameter perturbation does an excellent job in approximating the true solution of the model. In a simple environment with only income shocks, the method matches the same SSS obtained with the endogenous grid method. In a more complex setting that includes interest rate shocks and is calibrated with aggregate mexican data, the method delivers a well-defined ergodic distribution of net foreign assets with Euler errors below -5 using a third-order approximation, and well below -6 using a fifth-order one.

5 Equity home bias in a two-country DSGE model

The model is the infinite-period version of Coeurdacier and Gourinchas (2016) (from hereon CG). Specifically, I consider a version of the model with endowment, redistributive and preference shocks (Appendix A.4 of their paper) that makes financial markets incomplete. As I

[23]For example, if the model of interest is a SOE with a constant interest rate, it makes little sense to use a Debt elastic interest rate (DEIR) as the auxiliary modification, as perturbation may struggle to eliminate spurious dynamics of the interest rate entirely.

will show in the results, incomplete markets are essential to deliver external asset positions consistent with the data.

5.1 Model description

Consider a World with two symmetric countries, Home (H) and Foreign (F). Below I describe the model from Home's perspective, but keep in mind that Foreign will be its mirror image. When necessary, I will distinguish Foreign's variables with an asterisk.

Preferences. A representative household at Home maximizes the following utility function:

$$\mathbb{E}_t\left[\sum_{t=0}^{\infty}\theta_t\frac{C_t^{1-\gamma}-1}{1-\gamma}\right].$$

The risk-aversion parameter γ is the same for both countries in the baseline scenario, but I will also explore the consequences of relaxing this assumption (which breaks perfect symmetry across countries). The consumption index C_t is a constant-elasticity aggregator over consumption of Home and Foreign goods $c_{H,t}$ and $c_{F,t}$:

$$C_t = \left[\alpha^{\frac{1}{\phi}}\left(q_t c_{H,t}\right)^{(\phi-1)/\phi} + (1-\alpha)^{\frac{1}{\phi}}\left(q_t^* c_{F,t}\right)^{(\phi-1)/\phi}\right]^{\phi/(\phi-1)},$$

where $\phi > 0$ is the elasticity of substitution between the two goods, and the weight $\alpha \in (\frac{1}{2}, 1)$ captures Home bias towards $c_{H,t}$. In turn, q_t and q_t^* are worldwide shocks to the preference (or quality) for Home and Foreign goods.[24]

The Uzawa discount factor θ_t evolves over time according to

$$\begin{aligned}\theta_0 &= 1,\\ \theta_{t+1} &= \theta_t\beta\left(\frac{\widetilde{C}_t}{\overline{C}}\right)^{-\kappa}.\end{aligned}$$

where $\widetilde{C}_t$ denotes Home average consumption, and $\overline{C} > 0$, $\beta \in (0,1)$ and $\kappa \geq 0$ are parameters. In contrast to the SOE model of Appendix B.2, the endogenous discount factor is not a purely auxiliary modification to be shut down at the model of interest, but rather a device that helps to match the persistence of the trade balance and the current account found in the data.[25] Since $\overline{C}$ will be equal to the DSS of C_t in the auxiliary model, the endogenous discount factor does not play a role in pinning down the DSS.

Let the Home good be the numeraire, and denote the relative price of Foreign by $p_{F,t}$. Then,

[24] See Pavlova and Rigobon (2010) and Khalil (2019) for examples of DSGE portfolio-choice models with a similar type of shock.

[25] If $\kappa = 0$, the discount factor becomes fully exogenous and equal to β. Under the baseline calibration of Section 5.3, this leads to a model of interest with asset dynamics extremely close to a random walk.

the implied consumer price indices for Home and Foreign are:

$$P_t = \left[\alpha\left(\frac{1}{q_t}\right)^{1-\phi} + (1-\alpha)\left(\frac{p_{F,t}}{q_t^*}\right)^{1-\phi}\right]^{1/(1-\phi)}, \tag{5.1}$$

$$P_t^* = \left[(1-\alpha)\left(\frac{1}{q_t}\right)^{1-\phi} + \alpha\left(\frac{p_{F,t}}{q_t^*}\right)^{1-\phi}\right]^{1/(1-\phi)}. \tag{5.2}$$

In turn, optimal intra-temporal allocation yields the following relative demands for the two goods:

$$\begin{aligned} c_{H,t} &= \alpha P_t^{\phi} (q_t)^{\phi-1} C_t, \\ c_{F,t} &= (1-\alpha)\left(\frac{P_t}{p_{F,t}}\right)^{\phi} (q_t^*)^{\phi-1} C_t, \end{aligned} \tag{5.3}$$

for Home, and

$$\begin{aligned} c_{H,t}^* &= (1-\alpha)(P_t^*)^{\phi} (q_t)^{\phi-1} C_t^*, \\ c_{F,t}^* &= \alpha\left(\frac{P_t^*}{p_{F,t}}\right)^{\phi} (q_t^*)^{\phi-1} C_t^*, \end{aligned} \tag{5.4}$$

for Foreign.

Asset trading and budget constraint. In each country there is a Lucas tree whose supply is normalized to unity. At each period, the Home tree delivers a endowment Y_t of the Home good. Out of this endowment, a Home share $\delta_t \in (0,1)$ is distributed to shareholders as dividends, while the remaining fraction $(1-\delta_t)$ is distributed to Home households as non-financial income. Similarly, the Foreign tree delivers an endowment of Foreign goods Y_t^*, so that shareholders receive a financial income of $\delta_t^* p_{F,t} Y_t^*$ as dividends (with $\delta_t^* \in (0,1)$), while Foreign households retaining a non-financial income of $(1-\delta_t^*)\, p_{F,t} Y_t^*$. Let $z_{H,t}^S$ and $z_{F,t}^S$ denote the prices of Home and Foreign equity, and $S_{H,t}$ and $S_{F,t}$ the start-of-period portfolio of Home and Foreign stocks.

Households can also trade Home and Foreign one-period bonds in zero net supply. Let $z_{H,t}^B$ and $z_{F,t}^B$ denote the prices of Home and Foreign bonds, and $B_{H,t}$ and $B_{F,t}$ the start-of-period portfolio of Home and Foreign bonds. Following CG, bonds of each country pays one unit of their respective consumption index *not* adjusted for preference shocks.[26] That is, the cash-flows of one unit of the Home and Foreign bond purchased at $t-1$ are:

$$\widetilde{P}_t = \left[\alpha + (1-\alpha)(p_{F,t})^{1-\phi}\right]^{1/(1-\phi)}, \tag{5.5}$$

$$\widetilde{P}_t^* = \left[(1-\alpha) + \alpha(p_{F,t})^{1-\phi}\right]^{1/(1-\phi)}. \tag{5.6}$$

[26] CG introduce this assumption in their two-period model to generate a wedge between relative bond returns and changes in the real exchange rate that makes the inclusion of preference shocks meaningful. Since I have verified that the result holds in this infinite-horizon setup, I keep it as they have it.

Putting all together, Home's representative household faces a budget constraint given by

$$\begin{aligned} & P_t C_t + z^S_{H,t} S_{H,t+1} + z^S_{F,t} S_{F,t+1} + z^B_{H,t} B_{H,t+1} + z^B_{F,t} B_{F,t+1} \\ &= (1-\delta_t) Y_t + S_{H,t}\left(\delta_t Y_t + z^S_{H,t}\right) + S_{F,t}\left(\delta^*_t p_{F,t} Y^*_t + z^S_{F,t}\right) + \widetilde{P}_t B_{H,t} + \widetilde{P}^*_t B_{F,t}. \end{aligned} \tag{5.7}$$

Similarly, the budget constraint faced by Foreign's households is

$$\begin{aligned} & P^*_t C^*_t + z^S_{H,t} S^*_{H,t+1} + z^S_{F,t} S^*_{F,t+1} + z^B_{H,t} B^*_{H,t+1} + z^B_{F,t} B^*_{F,t+1} \\ &= (1-\delta^*_t) p_{F,t} Y^*_t + S^*_{H,t}\left(\delta_t Y_t + z^S_{H,t}\right) + S^*_{F,t}\left(\delta^*_t p_{F,t} Y^*_t + z^S_{F,t}\right) + \widetilde{P}_t B^*_{H,t} + \widetilde{P}^*_t B^*_{F,t}. \end{aligned} \tag{5.8}$$

Stochastic processes. There are six independent sources of risk that makes financial markets incomplete. Home and Foreign income follow log-normally distributed AR(1) processes:

$$\ln Y_{t+1} = \rho_y \ln Y_t + \sigma \eta_y u_{y,t+1} \tag{5.9}$$

$$\ln Y^*_{t+1} = \rho_y \ln Y^*_t + \sigma \eta_y u^*_{y,t+1} \tag{5.10}$$

where $\left(u_{t+1}, u^*_{t+1}\right)$ are standard normal innovations scaled by the perturbation parameter σ. The model of interest corresponds to the case $\sigma = 1$.

In turn, redistributive shocks $\widetilde{\delta}_t = \ln\left(\delta_t / (1-\delta_t)\right)$ and $\widetilde{\delta}^*_t = \ln\left(\delta^*_t / (1-\delta^*_t)\right)$ follow AR(1) processes of the form:

$$\widetilde{\delta}_{t+1} = (1-\rho_\delta)\,\widetilde{\delta} + \rho_\delta \widetilde{\delta}_t + \sigma \eta_\delta u_{\delta,t+1} \tag{5.11}$$

$$\widetilde{\delta}^*_{t+1} = (1-\rho_\delta)\,\widetilde{\delta} + \rho_\delta \widetilde{\delta}^*_t + \sigma \eta_\delta u^*_{\delta,t+1} \tag{5.12}$$

These specifications ensure that the shares δ_t and δ^*_t are bounded within their natural domain. In particular, $\widetilde{\delta} = \ln\left(\overline{\delta}/(1-\overline{\delta})\right)$ ensures that the DSS of both δ_t and δ^*_t equals $\overline{\delta} \in (0,1)$.

Finally, preference shocks q_t and q^*_t follow log-normally distributed AR(1) processes:

$$\ln q_{t+1} = \rho_q \ln q_t + \sigma \eta_q u_{q,t+1} \tag{5.13}$$

$$\ln q^*_{t+1} = \rho_q \ln q^*_t + \sigma \eta_q u^*_{q,t+1} \tag{5.14}$$

I assume that all innovations are serially uncorrelated, but will allow for intra-period correlation to match observed long-term correlations of output across countries, and between output and financial shares within countries.

Market-clearing conditions. At each period, equilibrium in the two-good markets requires

$$\begin{aligned} c_{H,t} + c^*_{H,t} &= Y_t, \\ c_{F,t} + c^*_{F,t} &= Y^*_t. \end{aligned} \tag{5.15}$$

Similarly, equilibrium in equity markets requires

$$
\begin{aligned}
S_{H,t} + S^*_{H,t} &= 1, && (5.16)\\
S_{F,t} + S^*_{F,t} &= 1, && (5.17)
\end{aligned}
$$

and equilibrium in bond markets requires

$$
\begin{aligned}
B_{H,t} + B^*_{H,t} &= 0, && (5.18)\\
B_{F,t} + B^*_{F,t} &= 0. && (5.19)
\end{aligned}
$$

5.2 Equilibrium

5.2.1 Auxiliary model

The model of interest just described cannot be solved by standard perturbation for the usual indeterminacy reasons. To solve the model by two-parameter perturbation, the auxiliary model introduces Portfolio Adjustment Costs (PAC) in the budget constraints. Specifically, agents in both countries face quadratic costs of holding stocks and bonds in a quantity different from some long-run levels. These costs are paid to a national agency that simply returns them to the households as lump-sum transfers T_t and T^*_t. The new budget constraints replacing (5.7) and (5.8) are:

$$
\begin{aligned}
& P_t C_t + z^S_{H,t} S_{H,t+1} + z^S_{F,t} S_{F,t+1} + z^B_{H,t} B_{H,t+1} + z^B_{F,t} B_{F,t+1} \\
&= (1-\delta_t) Y_t + S_{H,t}\left(\delta_t Y_t + z^S_{H,t}\right) + S_{F,t}\left(\delta^*_t p_{F,t} Y^*_t + z^S_{F,t}\right) + \widetilde{P}_t B_{H,t} + \widetilde{P}^*_t B_{F,t} \\
& -\frac{\psi}{2}(1-\varepsilon)\left[z^S_{H,t}\left(S_{H,t+1}-\overline{a}\right)^2 + z^S_{F,t}\left(S_{F,t+1}-\overline{a}_F\right)^2\right] \\
& -\frac{\psi}{2}(1-\varepsilon)\left[z^B_{H,t}\left(B_{H,t+1}-\overline{b}\right)^2 + z^B_{F,t}\left(B_{F,t+1}-\overline{b}_F\right)^2\right] + T_t \qquad (5.20)
\end{aligned}
$$

for Home households, and

$$
\begin{aligned}
& P^*_t C^*_t + z^S_{H,t} S^*_{H,t+1} + z^S_{F,t} S^*_{F,t+1} + z^B_{H,t} B^*_{H,t+1} + z^B_{F,t} B^*_{F,t+1} \\
&= (1-\delta^*_t) p_{F,t} Y^*_t + S^*_{H,t}\left(\delta_t Y_t + z^S_{H,t}\right) + S^*_{F,t}\left(\delta^*_t p_{F,t} Y^*_t + z^S_{F,t}\right) + \widetilde{P}_t B^*_{H,t} + \widetilde{P}^*_t B^*_{F,t} \\
& -\frac{\psi}{2}(1-\varepsilon)\left[z^S_{H,t}\left(S^*_{H,t+1}-(1-\overline{a})\right)^2 + z^S_{F,t}\left(S^*_{F,t+1}-(1-\overline{a}_F)\right)^2\right] \\
& -\frac{\psi}{2}(1-\varepsilon)\left[z^B_{H,t}\left(B^*_{H,t+1}+\overline{b}\right)^2 + z^B_{F,t}\left(B^*_{F,t+1}+\overline{b}_F\right)^2\right] + T^*_t. \qquad (5.21)
\end{aligned}
$$

for Foreign households. In both equations, $\psi > 0$ is the auxiliary parameter controlling quadratic costs, $\left(\overline{a}, \overline{a}_F, \overline{b}, \overline{b}_F\right)$ is the vector of parameters that will pin down the DSS of asset holdings, and ε is the new perturbation parameter. As always, we recover the model of interest by evaluating both equations at $\varepsilon = 1$.

5.2.2 Equilibrium conditions

Model with bonds and equities. Appendix C lists the 21 equilibrium conditions of the auxiliary model, including the 8 Euler equations (two for each asset and country) that characterize inter-temporal optimization. These are also the equilibrium conditions of the model of interest once we evaluate them at $(\varepsilon, \sigma) = 1$. Exploiting the asset market-clearing conditions (5.16)-(5.19), we can reduce the endogenous state-space in terms of Home states only. Hence, the 4 endogenous state variables are bond holdings $B_{H,t}$ and $B_{F,t}$, and stock holdings $S_{H,t}$ and $S_{F,t}$. The 6 exogenous states are Home and Foreign income Y_t and Y_t^*, Home and Foreign shares δ_t and δ_t^*, and preference shocks q_t and q_t^*. The 11 control variables are: consumptions C_t and C_t^*, equity prices $p_{H,t}^S$ and $p_{F,t}^S$, bond prices $p_{H,t}^B$ and $p_{F,t}^B$, price indices P_t, P_t^*, $\widetilde{P}_t$ and $\widetilde{P}_t^*$, and terms of trade $p_{F,t}$.

Appendix C also pins down the DSS of the auxiliary model $(\varepsilon, \sigma) = (0, 0)$. In the baseline calibration, perfect symmetry across countries leads to the parameter restriction $\overline{a}_F = 1 - \overline{a}$, and $\overline{b}_F = -\overline{b}$. It follows that the pair $\left(\overline{a}, \overline{b}\right)$ uniquely pins down the DSS of the endogenous states: $B_{H,d} = \overline{b}$, $B_{F,d} = -\overline{b}$, $S_{H,d} = \overline{a}$, and $S_{F,d} = 1 - \overline{a}$.

Model with equities only. I will compare the ability of the full model to account for the international diversification puzzle with a simplified equities-only version where $B_{H,t} = B_{F,t} = 0$ at all times. This reduces the model to 15 equilibrium conditions with 2 endogenous states ($S_{H,t}$ and $S_{F,t}$), 6 exogenous states (same as in the full model), and 7 control variables (C_t, C_t^*, $p_{H,t}^S$, $p_{F,t}^S$, P_t, P_t^*, and $p_{F,t}$). Under perfect symmetry, the DSS of the endogenous states is $S_{H,d} = \overline{a}$, and $S_{F,d} = 1 - \overline{a}$.

5.2.3 Solution

The equilibrium conditions of this model are a particular case of the Two-parameter perturbation model described in Section 2. Thus, we can solve this application with available solution packages by treating ε as a exogenous state that is constant (or extremely persistent) over time: $\varepsilon_{t+1} = \varepsilon_t$. Since the pair $\left(\overline{a}, \overline{b}\right)$ uniquely pins down the DSS of the endogenous states, we can use the algorithm of Section 3.4 and solve for values of $\left(\overline{a}, \overline{b}\right)$ that approximate the SSS of the endogenous states: $B_{H,s} = \overline{b}$, $B_{F,s} = -\overline{b}$, $S_{H,s} = \overline{a}$, and $S_{F,s} = 1 - \overline{a}$. As the DSS of the state variables will equal their SSS in the model of interest, in what follows I omit s and d sub-indices for these variables when refering to their steady-state.

I use a second-order approximation to the policy rules to approximate the SSS, and build third-order approximations around this point to study the equilibrium dynamics. As shown in the results section, this suffices to produce highly accurate solutions. Moreover, I have verified that the contribution of higher orders in this model (to both SSS and dynamics) is virtually nil.

5.2.4 Additional variables of interest

The model also deliver implications for a set of variables of interest that can be expressed as functions of the equilibrium ones.

Domestic wealth, portfolio shares, and asset returns. End-of-period domestic wealth W_t is the sum of all asset gross purchases:

$$W_t = z^S_{H,t} S_{H,t+1} + z^S_{F,t} S_{F,t+1} + z^B_{H,t} B_{H,t+1} + z^B_{F,t} B_{F,t+1}. \tag{5.22}$$

The portfolio shares are: $w^S_{H,t} = z^S_{H,t} S_{H,t+1}/W_t$ for Home equity, $w^S_{F,t} = z^S_{F,t} S_{F,t+1}/W_t$ for Foreign equity, $w^B_{H,t} = z^B_{H,t} B_{H,t+1}/W_t$ for Home bonds, and $w^B_{F,t} = z^B_{F,t} B_{F,t+1}/W_t$ for Foreign bonds.[27] The returns of each asset are: $R^S_{H,t} = \left(\delta_t Y_t + z^S_{H,t}\right)/z^S_{H,t-1}$ for Home equity, $R^S_{F,t} = \left(\delta^*_t p_{F,t} Y^*_t + z^S_{F,t}\right)/z^S_{F,t-1}$ for Foreign equity, $R^B_{H,t} = \widetilde{P}_t / z^B_{H,t-1}$ for Home bonds, and $R^B_{F,t} = \widetilde{P}^*_t / z^B_{F,t-1}$ for Foreign bonds.

Using the previous definitions, the endogenous return to wealth (specific to each country) is the weighted sum of asset returns, where the weights are the portfolio shares chosen the previous period. For Home, we have:

$$R^W_{H,t} = w^S_{H,t-1} R^S_{H,t} + w^S_{F,t-1} R^S_{F,t} + w^B_{H,t-1} R^B_{H,t} + w^B_{F,t-1} R^B_{F,t}. \tag{5.23}$$

Combining (5.22) and (5.23), Home's budget constraint (5.7) can be rewritten as a standard wealth accumulation equation:

$$W_t = W_{t-1} R^W_{H,t} + (1-\delta_t) Y_t - P_t C_t.$$

Its associated wealth Euler equation in the model of interest is:

$$\beta \left(\frac{C_t}{\overline{\overline{C}}}\right)^{-\kappa} \mathbb{E}_t \left[\left(\frac{P_t}{P_{t+1}}\right) \frac{R^W_{H,t+1}}{C^\gamma_{t+1}}\right] = \frac{1}{C^\gamma_t}, \tag{5.24}$$

which I will use to compute Euler equation errors.[28]

External assets positions and Equity home bias. Following the conventions of Lane and Milesi-Ferretti (2018), Home's external equity assets are its gross purchases of Foreign equity $z^S_{F,t} S_{F,t+1}$, and Home's external equity liabilities are Foreign's gross purchases of Home equity $z^S_{H,t} S^*_{H,t+1}$. In equilibrium, each country will be long in the bond denominated in its own price index, and short in the other. From Home's perspective, this means that $z^B_{H,t} B_{H,t+1}$ are its external debt assets, and $z^B_{F,t} B^*_{F,t+1}$ its external debt liabilities. Thus, total external assets TEA_t and

[27] Note that, by construction, $w^S_{H,t} + w^S_{F,t} + w^B_{H,t} + w^B_{F,t} = 1$.

[28] This equation can also be obtained as the weighted sum of the Euler equations (C.1)-(C.8) evaluated at $\varepsilon = 1$.

liabilities TEL_t are:

$$\begin{aligned}\text{TEA}_t &= z^S_{F,t} S_{F,t+1} + z^B_{H,t} B_{H,t+1}, \\ \text{TEL}_t &= z^S_{H,t} S^*_{H,t+1} + z^B_{F,t} B^*_{F,t+1}.\end{aligned}$$

In turn, net foreign assets NFA_t is the difference between the two. Combining (5.22) with (5.17) and (5.19), it can be shown that $\text{NFA}_t = W_t - z^S_{H,t}$.

The share of foreign equities in Home's equity holdings is $z^S_{F,t} S_{F,t+1} / \left(z^S_{H,t} S_{H,t+1} + z^S_{F,t} S_{F,t+1} \right)$, and the share of foreign equities in the World market portfolio is $z^S_{F,t} / \left(z^S_{H,t} + z^S_{F,t} \right)$. Following Coeurdacier and Rey (2013), the measure of equity home bias is:

$$\text{EHB}_t = 1 - \frac{\text{Share of foreign equities in Home's equity holdings}}{\text{Share of foreign equities in the World portfolio}}$$

Provided that countries are symmetric, it is easy to verify that the SSS of this equation is:

$$\text{EHB} = 2S_H - 1 = 1 - 2S_F. \tag{5.25}$$

Real exchange rate, trade balance, and current account. The real exchange rate is the ratio between Home and Foreign price indices: $\text{RER}_t = P_t / P^*_t$. In the stochastic simulations, I will also consider an alternative measure not adjusted for preference shocks ($\text{RER}_t = \widetilde{P}_t / \widetilde{P}^*_t$), which might be relevant given the difficulty of quantifying these shocks in the data. Home's trade balance (or net exports) is the difference between Home's output and consumption expenditures:

$$\text{TB}_t = Y_t - P_t C_t. \tag{5.26}$$

The current account is defined as the sum of portfolio reallocations, excluding valuation effects:

$$\text{CA}_t = z^S_{H,t} \Delta S_{H,t+1} + z^S_{F,t} \Delta S_{F,t+1} + z^B_{H,t} \Delta B_{H,t+1} + z^B_{F,t} \Delta B_{F,t+1},$$

where the operator Δ denotes first differences: $\Delta X_{t+1} = X_{t+1} - X_t$.

Capital inflows and outflows. Following Broner et al. (2013), end-of-period capital inflows by Foreign CIF_t is equal to the net purchases of domestic assets by Foreign, and capital outflows by domestic (Home) agents COD_t is equal to the net purchases of foreign assets by domestic agents. Given that in equilibrium each country will be long in the bond denominated in its own price index, we can write:

$$\begin{aligned}\text{CIF}_t &= z^S_{H,t} \Delta S^*_{H,t+1} + z^B_{F,t} \Delta B^*_{F,t+1}, \\ \text{COD}_t &= z^S_{F,t} \Delta S_{F,t+1} + z^B_{H,t} \Delta B_{H,t+1},\end{aligned}$$

where, consistent with the dataset of Broner et al. (2013), the measures exclude valuation effects. Then, total gross flows is the sum $\text{CIF}_t + \text{COD}_t$, and net capital flows are equal to the

Table 1: Calibration of the two-country DSGE model

Parameter		Value
	Model parameters	
β	Discount factor	0.96
γ	Risk aversion	2
α	Share of domestic consumption	0.85
ϕ	Trade elasticity	2
κ	Uzawa parameter	0.007
	Stochastic processes	
$\bar{\delta}$	DSS dividend share	0.036
ρ_y	Persistence income shocks	0.51
ρ_δ	Persistence redistributive shocks	0.42
ρ_q	Persistence preference shocks	0.46
η_y	Std. income shocks	0.018
η_δ	Std. redistributive shocks	0.059
η_q	Std. preference shocks	0.007
ρ_{yy^*}	$\text{Corr}\left(u_{y,t+1}, u^*_{y,t+1}\right)$	0.68
$\rho_{y\delta}$	$\text{Corr}\left(u_{y,t+1}, u_{\delta,t+1}\right)$	0.12

difference $\text{CIF}_t - \text{COD}_t$. It is easy to verify that, in equilibrium, net capital flows are the negative of the current account.

5.3 Calibration

Table 1 reports the baseline calibration of the two-country model, chosen to match key moments of annual data for advanced economies (mostly the United States). The discount factor $\beta = 0.96$ delivers a DSS net return of 4% for all assets.[29] The risk aversion parameter $\gamma = 2$ is large enough to ensure that households want to increase their income when their consumption goods are more expensive (Coeurdacier and Rey, 2013). The home bias in consumption $\alpha = 0.85$ is set to match the average U.S. import ratio, and the trade elasticity is set to $\phi = 2$, which falls within the conservative range of empirical estimates using sectoral trade data (Imbs and Mejean (2015), Boehm et al. (2023)). In the results section, I will explore how changes to these two parameters affects the equilibrium.

Regarding the stochastic processes (5.9)-(5.12), the paramaters are set to match estimates from AR(1) regressions using U.S. data. Specifically, I collect annual data from the FRED database for U.S. real GDP and net corporate dividends (as a share of GDP) for the time period

[29] Since this model lacks features such as time-varying volatility or disaster risk, the SSS returns will be close to 4% for all assets too.

1947-2022. The average dividend share of 3.6% pins down the DSS share $\overline{\delta} = 0.036$. Then, I apply the Hodrick-Prescott filter to detrend the variables $\ln Y_t$ and $\widetilde{\delta}_t$, and run AR(1) regressions using their cyclical components. The procedure yields auto-correlations of $\rho_y = 0.51$ and $\rho_\delta = 0.42$, and conditional standard deviations of $\eta_y = 0.018$ and $\eta_\delta = 0.059$. Moreover, the correlation between the two time-series pins down the correlation of income and redistributive shocks to $\rho_{y\delta} = 0.12$.

The remaining parameters are calibrated as follows. The correlation between Home and Foreign income shocks $\rho_{yy^*} = 0.68$ helps to match the observed cross-country correlation of GDP between the U.S. and a set of OECD countries reported by Corsetti, Dedola and Leduc (2008, Table 3). Since preference shocks are unobserved, I set the auto-correlation $\rho_q = 0.42$ such that $\rho_\delta < \rho_q < \rho_y$, and in the next subsection I will calibrate a standard deviation $\eta_q = 0.0067$ such that the model with bonds and equities matches a degree of equity home bias equal to 0.66 (the value for the U.S. in 2008 reported by Coeurdacier and Rey (2013, Table 1)). The Uzawa parameter $\kappa = 0.007$ helps to match the auto-correlation of net exports over GDP ratio for the U.S., equal to 0.62 using (HP detrended) FRED data. Finally, I set the PAC auxiliary parameter to $\psi = 0.001$ to optimize dynamic accuracy as measured by Euler equation errors.

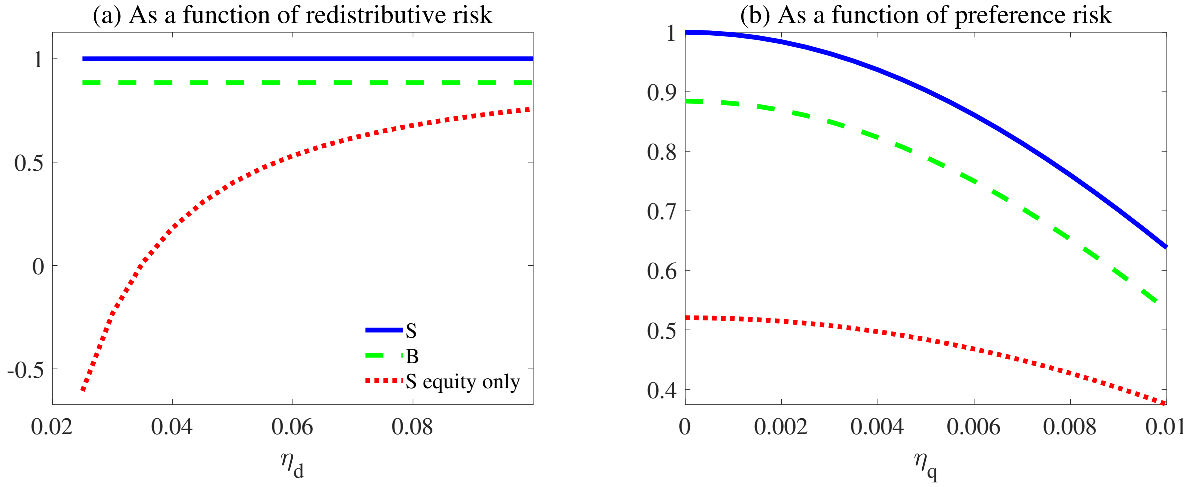
5.4 Results

5.4.1 Comparative statics at the stochastic steady-state

Bonds still matter. This numerical exercise compares the SSS properties of the model with bonds and equities against the equities-only model by performing the same comparative statics studied by CG in their Online Appendix. The first experiment shuts down preference risk by setting a very small value of $\eta_q = 0.0001$. I then let η_δ vary between 0.025 and 0.1, compute the associated SSS asset values for both models, and plot the results in Figure 1, panel (a). The second experiment sets redistributive risk at its calibrated value $\eta_\delta = 0.059$. I then let η_q vary between 0 and 0.01, compute the associated SSS asset values, and report them in Figure 1, panel (b). In both panels, the red dotted line is the value of S_H in the equities-only model, and the blue solid and green dashed lines correspond to S_H and B_H in the model with bonds and equities.

The qualitative findings of Figure 1 are identical to those reported by CG in Figure 1 of their Online Appendix (top and bottom panels with trade elasticity above unity). In that sense, their key theoretical results survive in a quantitative model. The intuition behind these findings is therefore the same as in their two-period model. Consider first the case with only equities. In panel (a), sufficiently small values of η_δ deliver a worse-than-you-think-puzzle scenario (Baxter and Jermann, 1997): holding a portfolio strongly biased towards Foreign ($S_H < 0.5$) is optimal because Home equity returns $R^S_{H,t}$ are strongly correlated with Home's non-financial income $(1 - \delta_t)\, Y_t$. However, this correlation decreases as η_δ increases because a positive redistributive shock increases Home's dividends $d_t Y_t$ at the expense of Home's non-financial income. For enough large redistributive shocks the equities-only model can generate a significant Home bias ($S_H > 0.5$), just as CG found in their two-period model.

Figure 1: SSS Portfolios



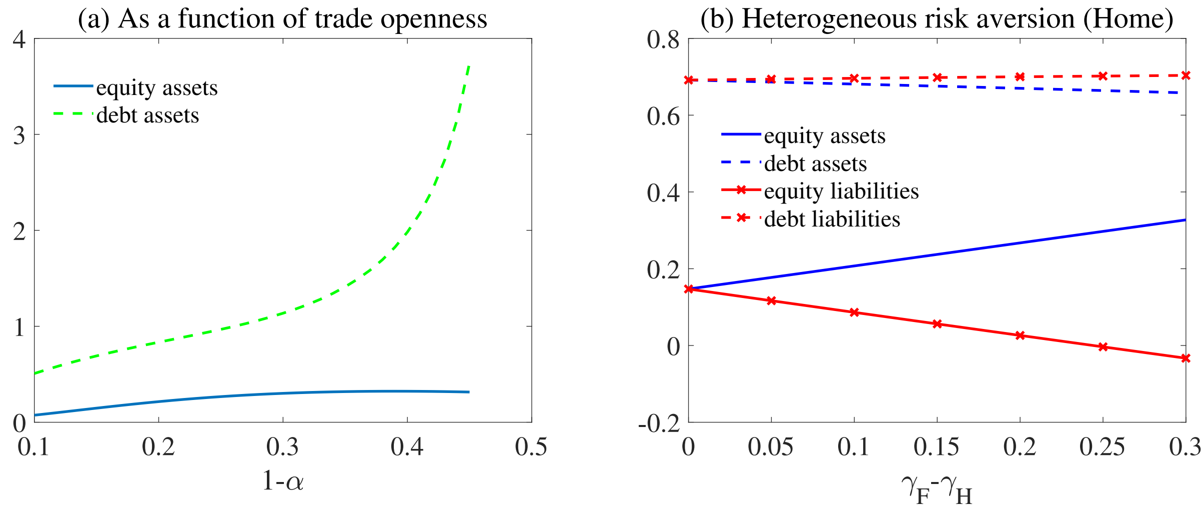
Therefore, whether an incomplete-markets structure with only equities can account for the International diversification puzzle is a quantitative question that only this model can address. Panels (a) and (b) together provide a clear negative answer: the calibrated value $\eta_\delta = 0.059$ delivers a S_H barely above 0.5 (zero Home bias), and increasing preference risk only pushes the model into worse-than-you-think-puzzle territory. Since S_H should be above 0.80 to match the high levels of equity home bias observed in the data (0.66 for the U.S. and 0.62 for the Eurozone in 2008), the equities-only model is strongly rejected by the data.[30]

Next, consider the model with bonds and equities. Since panel (a) effectively shuts down preference risk, financial markets are complete, leading to the perfect risk-sharing scenario that CG cover in the main text of their paper. In this case, a large external debt position that is long in the Home bond and short in the Foreign one provides an optimal hedge against relative income shocks. Intuitively, shocks that increase Home's relative income lead to a RER depreciation, which lowers the bond return differential $\ln\left(R^B_{H,t}/R^B_{F,t}\right)$ and generates a transfer from Home to Foreign (see also the IRFs of Figure 3). As CG emphasize, all what is left for equities is to provide a hedge against redistributive risk, which residents in each country can achieve by holding all their domestic equity. The result is therefore full equity home bias ($S_H = 1$).

While this extreme case is also rejected by the data, panel (b) shows that as η_q increases both B_H and S_H decrease. Intuitively, a positive shock to the preference for Home goods leads to a (welfare based) RER depreciation and to a simultaneous positive bond return differential (see also the IRFs of Figure 5). Thus, a bond portfolio long in Home and short in Foreign is a bad hedge against this shock because it generates a positive transfer from Foreign to Home just when Home's goods are relatively cheaper. This gives countries an incentive to reduce their gross external debt positions and partially replace them with foreign equity. In particular, calibrating the standard deviation of (unobserved) preference shocks to $\eta_q = 0.0067$ delivers

[30] The numbers come from Coeurdacier and Rey (2013). Note that even a massive value $\eta_\delta = 0.1$ that nearly doubles redistributive risk is unable to match the observed level of home bias. More generally, the equities-only model cannot generate a large equity home bias unless one imposes strongly counterfactual calibrations.

Figure 2: SSS External wealth



$S_H = 0.83$, which matches the U.S. equity home bias of 0.66 (see equation (5.25)). Since this calibration also delivers a external gross debt position comparable to those held by advanced economies in the Lane and Milesi-Ferretti dataset, I conclude that, from the point of view of SSS portfolios, the incomplete-markets model with bonds and equities is consistent with the data.

Trade and financial linkages. In the steady-state of the baseline model, Trade openness (exports plus imports over GDP) is given by $2(1-\alpha)$, where $\alpha > 0.5$ controls home bias in goods. To explore the effects of trade openness on external asset positions, I let $1-\alpha$ vary between 0.1 and 0.45, compute the associated steady states, and use them to construct measures of external equity and debt assets as indicated in Section 5.2.4. Panel (a) of Figure 2 shows the results, where the blue solid line is external equity, and the dashed green line is external debt. Since countries are symmetric, equity and debt liabilities equal their asset counterparts, total external assets equals external liabilities, and financial openness (the sum of the two measures) equals twice the vertical sum of the blue and green lines.

The results of Panel (a) are consistent with the data. First, the model predicts a strong positive relationship between trade and financial openness, as documented by Heathcote and Perry (2013).[31] Furthermore, this positive relationship is almost entirely driven by a large increase in the external debt position of countries, whereas the change in the external equity position is much more modest and can even decrease for sufficiently large levels of trade openness (Khalil, 2019). Since these patterns hold empirically both across countries and over time, the findings of Panel (a) help to rationalize the slow rate of change of Equity home bias despite the last decades of intense trade globalization and large increases in external gross positions (Coeurdacier and Rey, 2013; Gourinchas and Rey, 2014). Last but not least, the magnitudes of

[31] Indeed, the fast rate of increase in this model does a better job that theirs in rationalizing Figure 2 of their paper, where they plot a sample of advanced economies in the trade-financial space.

external asset positions delivered by the model are comparable to their empirical counterparts for advanced economies (see the dataset of Lane and Milesi-Ferretti, 2018).

What is the intuition behind these predictions? Holding a leveraged bond position (long in Home and short in Foreign) delivers stabilizing transfers because the cash-flows of Home and Foreign bonds change differently in response to the same income shocks. But since the degree of this asymmetry is a direct function of Home bias in goods, lower home bias (higher trade openness) implies smaller transfers other things equal. To compensate for this effect, countries that are more trade integrated must leverage up their bond positions just to maintain the same transfer size. As α approaches 0.5, the asymmetry between cash-flows almost entirely vanishes, and a extremely large gross debt position is needed for the transfer scheme to work at all.

As for equities, the equilibrium portfolio is determined by two opposing forces: redistributive shocks incentivize a less diversified portfolio (smaller holdings of foreign equity), and preference shocks encourage a more diversified one. In this model, the relative strength of these two forces changes with α in a non-monotonic way. Specifically, the net incentive towards diversification is strong for low levels of trade openness, but it weakens as trade openness becomes large enough.[32] As such, the pattern of external equities in Panel (a) is the logical answer to these changing incentives.

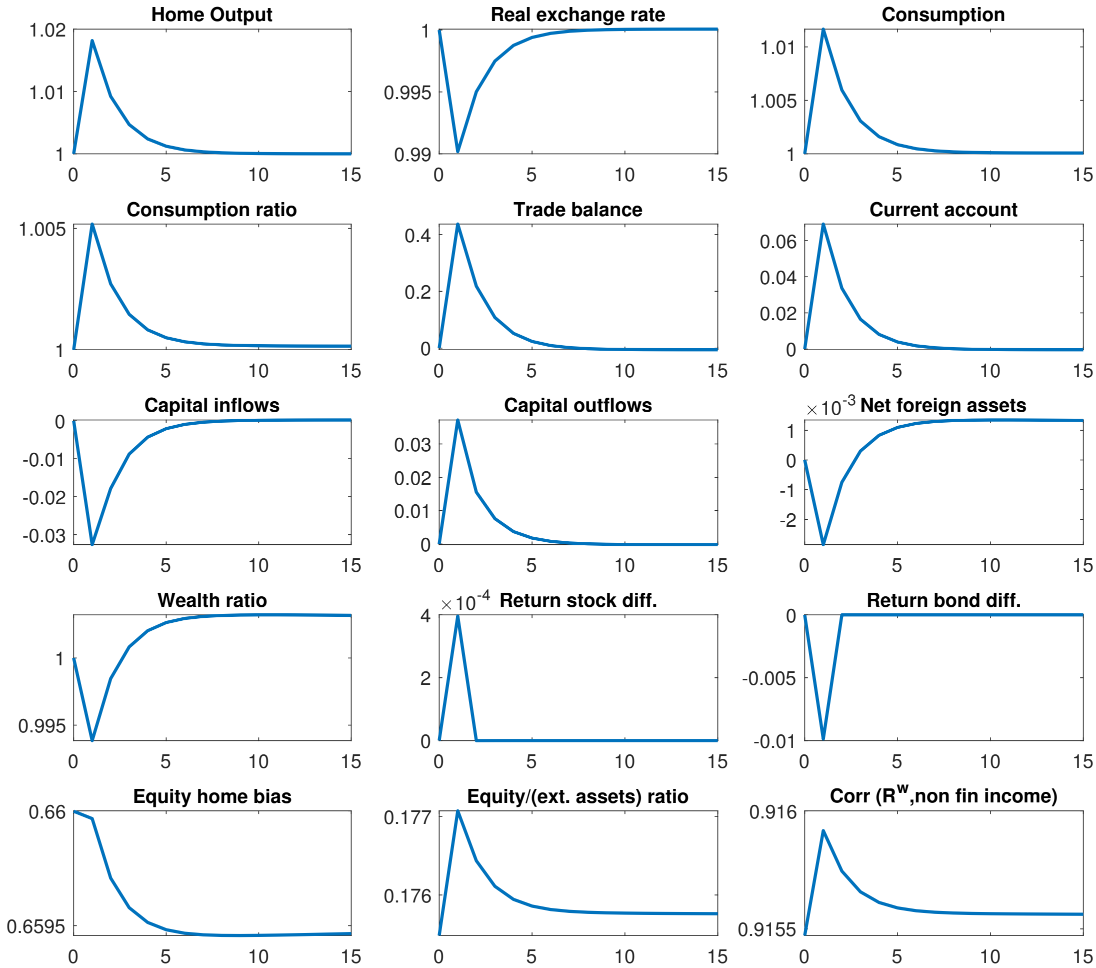
Long-run global imbalances. As a final experiment, I explore the implications of introducing a parameter asymmetry between countries, wherein domestic households become less risk-averse than foreigners. The goal of this exercise is twofold. Firstly, Rabitsch, Stepanchuk, and Tsyrennikov (2015) document that the solution method proposed by Devereux and Sutherland (2010) struggles to generate accurate portfolio dynamics in asymmetric settings induced by risk. This issue arises because the approximation point of the solution method is the symmetric DSS. In contrast, the SSS naturally incorporates any asymmetry related to risk, making Two-parameter perturbation a more suitable tool for studying portfolio-choice models with long-run global imbalances.[33] Secondly, recent papers, such as Gourinchas, Rey, and Govillot (2017), Sauzet (2022a), and Stepanchuk and Tsyrennikov (2015) have considered this type of asymmetry to rationalize the international role of the United States as a global banker. It is then natural to examine the predictions of this model and compare them with the data.

To introduce the asymmetry, the new risk-aversion parameters are $\gamma_H = \overline{\gamma} - \tau$ for Home, and $\gamma_F = \overline{\gamma} + \tau$ for Foreign, where $\overline{\gamma}$ is the level of risk aversion in the baseline model, and $\tau \geq 0$ represents the gap between countries' risk aversion. I then vary τ between 0 and 0.15, compute the associated steady states, and use them to construct measures of external assets and liabilities. Panel (b) of Figure 2 presents the results. Here, red lines with markers represent external liabilities, and blue lines without them correspond to external assets. Similarly, solid lines indicate equity positions, and dashed lines indicate debt positions.

[32] Consistent with this argument, the size of impulse responses to the return differential of stocks changes with α in a non-monotonic way.

[33] Indeed, Table 2 shows that the stochastic simulation with the parameter asymmetry is as accurate as the baseline scenario.

Figure 3: Impulse responses to a Home income shock



The success of the model in replicating the long-run patterns of the international financial system is mixed. On the one hand, the model correctly predicts that the country with lower risk aversion (akin to the U.S. in the data) significantly increases its exposure to foreign equity assets. On the other hand, the model predicts that the same country should be a net creditor with a long-run positive NFA position, while the data shows the opposite. Indeed, the trade-off generated by the model, as highlighted in Table 2 ("Different γ's" column), is that Home enjoys a larger long-run consumption in exchange for an increase in its volatility relative to Foreign.[34]

5.4.2 Impulse responses

Income shock. Figure 3 reports impulse responses of selected variables following a surprise shock to Home output Y_t of one standard deviation at $t = 1$, starting from the SSS. All variables are expressed in their natural units and, unless otherwise specified, refer to Home measures.

The economic effects can be summarized as follows. The shock to domestic income is persistent but mean-reverting, and it dies out after 10 years. Home goods are relatively more abundant during this period, which worsens Home's terms of trade (larger $p_{F,t}$) and causes a RER depreciation that mirrors the income shock. Due to a substantial trade elasticity, domestic non-financial income increases compared to Foreign.

The Return bond differential is negative on impact but zero afterwards, resulting in a one-time transfer from Home to Foreign (since the SSS bond position is long in Home and short in Foreign). However, this transfer does not fully offset the relative increase of Home's non-financial income. Exploiting this situation, domestic households accumulate both domestic and foreign assets, leading to Home's current account surpluses driven by a simultaneous decline of capital inflows and a surge of capital outflows.

Net Foreign Assets (NFA) decline on impact due to valuation effects, as the RER depreciation implies cheaper Home bonds compared to Foreign ones. Consequently, the wealth ratio falls on impact too. However, as the RER reverts to unity, both variables converge to medium-run values above their SSS. This allows domestic households to enjoy a consumption stream larger than foreigners in both the short and medium-run, as reflected in the impulse response of the Home-to-Foreign consumption ratio that outlasts the income shock.

The last row of Figure 3 shows the effects of the income shock on portfolio reallocation. Equity home bias gradually decreases until convergence to a medium-run level below its SSS.[35] We observe a similar pattern with the share of equity over total external assets: following the initial jump, there is convergence to a medium-run level above the SSS. What explains this extremely persistent portfolio reallocation? The answer is that, as Home becomes richer than Foreign, it also becomes less risk averse relative to Foreign. As a result, Home is inclined to embrace a riskier external position.[36] The increased (ex-ante) correlation between Home's endogenous wealth returns $R^W_{H,t}$ and non-financial income confirms the willingness of domestic households to bear a riskier portfolio than foreigners, in both the short and medium-run.

Redistributive shock. Figure 4 reports impulse responses of selected variables following a surprise shock to the Home dividend share of one standard deviation at $t = 1$, starting from the SSS. Since this shock implies an income transfer from domestic households to shareholders of Home's equity (via a positive return stock differential), and domestic households are the main owners of the asset, the resulting effect is a small transfer from domestic to foreign households.

All the impulse responses can therefore be interpreted under this light. The RER slightly de-

[34] The positive NFA position might follow because the risk premia generated by the model is too low. Thus, a promising avenue for future research is to incorporate disaster risk, as in Gourinchas, Rey, and Govillot (2017).

[35] In the long-run there is full convergence back to the initial SSS.

[36] See Sauzet (2022a) for a similar result linking relative wealth to portfolio reallocations.

Figure 4: Impulse responses to a Home redistributive shock

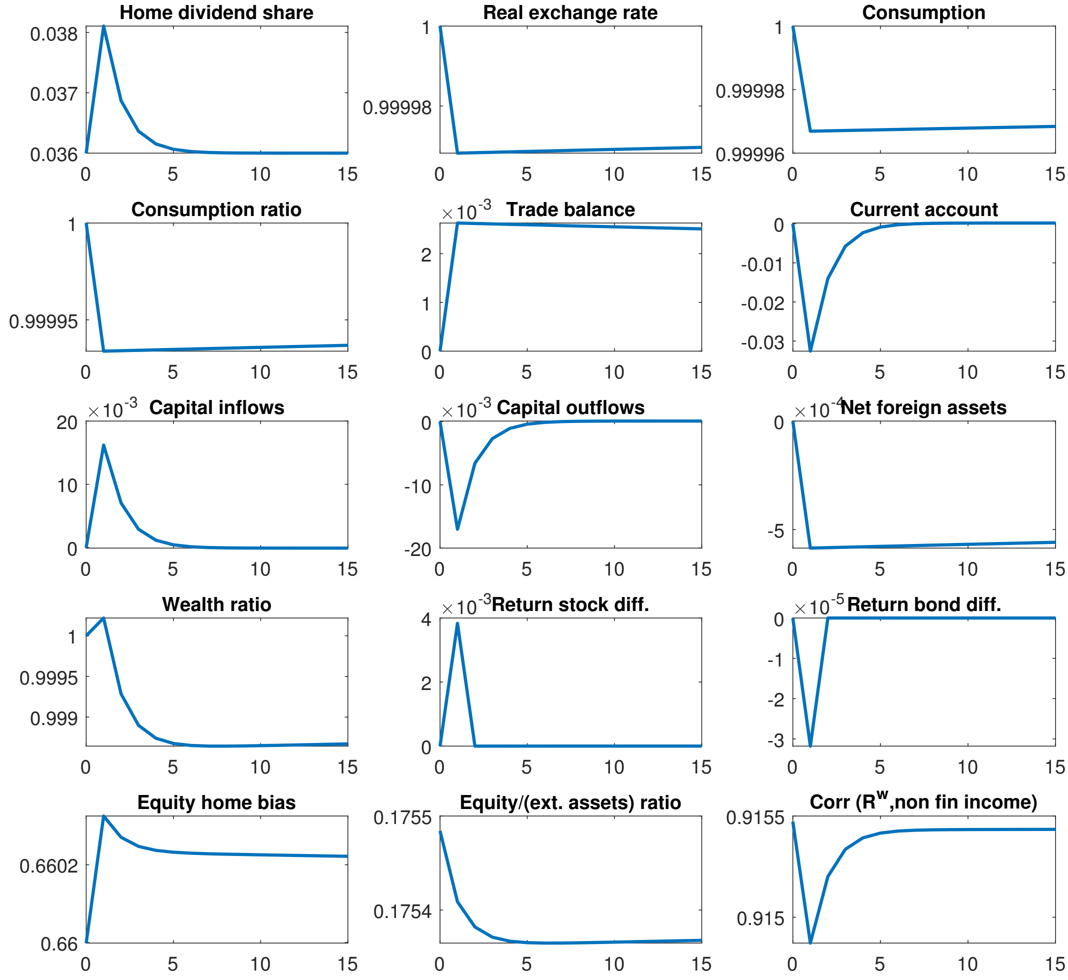
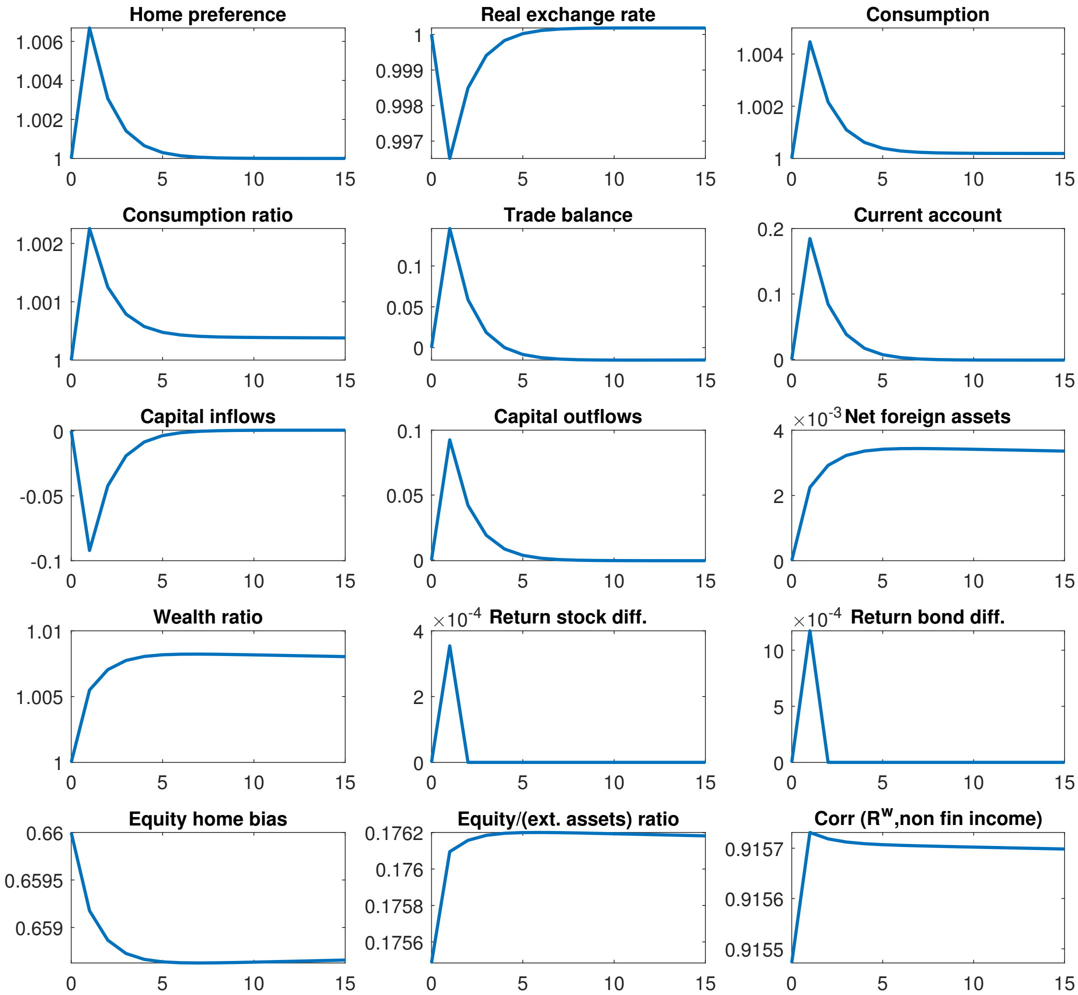


Figure 5: Impulse responses to a Home preference shock



preciates (due to Foreign bias towards their goods), which leads to a (one-time) very small negative return bond differential. To smooth out the income transfer over time, foreign households accumulate both Home and Foreign assets, as reflected by domestic current account deficits driven by an influx of capital inflows, and a decline of capital outflows. Consequently, there is a very persistent (albeit small) decrease of Home's NFA and consumption, and a gradual decline of relative wealth converging to a medium-run level slightly below the steady-state. Consistent with these patterns, Home opts for a relatively safer portfolio, as illustrated by the increase in equity home bias, the decline of the share of equity over total external assets, and the reduced correlation between $R^W_{H,t}$ and non-financial income.

Preference shock. Figure 5 reports impulse responses of selected variables following a surprise shock to the preference for the Home good of one standard deviation at $t = 1$, starting from the SSS. This shock implies a welfare-based RER depreciation coupled with a simultaneous improvement of Home's terms of trade (lower $p_{F,t}$). Since the return differentials of bonds and stocks are positive on impact, domestic households receive an income transfer just when their non-financial income is relatively larger, and their consumption expenditures are relatively cheaper. To smooth out these positive effects over time, domestic households accumulate both Home and Foreign assets, as reflected by domestic current account surpluses driven by a decline of capital inflows, and a rise of capital outflows. As a result, there is a very persistent increase of Home's NFA and consumption, and a gradual rise of relative wealth converging to a medium-run level slightly above the steady-state. Consistent with these patterns, Home chooses a relatively riskier portfolio, as shown by the decrease in equity home bias, the increase of the share of equity over total external assets, and the larger correlation between $R^W_{H,t}$ and non-financial income.

5.4.3 Stochastic Simulations

Table 2 reports moments from stochastic simulations of the model. The "Baseline" column corresponds to a simulation using the calibration specified in Table 1. Relative to this scenario, the "High ϕ" column doubles the trade elasticity ($\phi = 4$), the "Low α" column considers a World with lower Home bias in goods ($\alpha = 0.75$), and the "Different γ's" column introduces a parameter asymmetry between countries by making domestic households less risk-averse than foreigners: $\gamma = 1.75$ for Home, and $\gamma = 2.25$ for Foreign (the moments reported are from the Home economy). For reference, the "Data" column reports the empirical counterparts for the United States. In all cases, I simulate the model $100,000$ periods (with a burn-in of $10,000$ additional periods) feeding the decision rules with pseudo-random innovations.

Despite the absence of a pruning scheme, these simulations produce well-behaved unimodal distributions for all model variables. In the first three columns, I calibrate the volatility of preference shocks to match a SSS of Equity Home bias equal to 0.66 as in the data. Thus, the fact that the averages of this variable also equal 0.66 shows that, at least in this model, the SSS is very close to the ergodic mean. It is then no surprise that the model performs reasonably well in delivering average external asset positions (all measured as ratios to GDP) comparable to the data, with external equity assets of 0.15 in the model versus 0.19 in the data, and total external assets of 0.84 in the model versus 1.04 in the data.[37]

The model is less successful in matching second moments of the data. On the positive side, the model is able to match the volatility of the trade balance (std. of 0.56%), and it correctly predicts that consumption is less volatile than GDP. But compared to the data, the std. of the RER (0.54%) is about seven times lower, the std. of the current account (0.31%) is about two times lower, and the std. of the change in NFA (0.56%) is about five times lower.

[37] Since in the first three columns countries are perfectly symmetric, there are no long-run global imbalances: on average, total external liabilities exactly match total external assets.

Table 2: Simulated Moments of the two-country DSGE model

	Baseline	High ϕ	Low α	Different γ's	Data
Averages:					
Equity home bias	0.66	0.66	0.66	0.49	0.66
External equity assets	0.15	0.15	0.15	0.30	0.19
Total external assets	0.84	1.97	1.43	0.96	1.04
Total external liabilities	0.84	1.97	1.43	0.70	1.38
Euler errors (in log10)	-6.01	-6.25	-6.10	-6.02	
Standard deviations (relative to GDP):					
Consumption	0.98	0.93	0.95	1.07	0.94
RER	0.54	0.26	0.27	0.54	3.90
Absolute standard deviations:					
Trade balance	0.54	0.65	0.54	0.56	0.56
Current account	0.31	0.29	0.30	0.31	0.66
Δ(Net foreign assets)	0.50	0.61	0.50	1.12	2.36
Gross capital flows	0.01	0.01	0.01	0.01	
Equity home bias	0.03	0.02	0.02	0.01	
Serial correlations:					
Trade balance	0.62	0.58	0.61	0.61	0.62
Current account	0.46	0.46	0.46	0.46	0.67
Δ(Net foreign assets)	0.03	-0.08	0.02	-0.19	0.03
Equity home bias	0.99	0.99	0.99	0.99	
Cross-Correlations:					
RER and relative consumption	-0.37	-0.20	-0.29	-0.27	0.71
Home and Foreign consumption	0.89	0.94	0.93	0.89	0.60
Capital inflows and outflows	-0.99	-0.99	-1	-0.99	0.78
Current account and GDP	0.07	0.11	0.10	0.04	-0.52
Δ(Net foreign assets) and GDP	-0.10	-0.11	-0.08	0.40	-0.27
Conditional hedge ratios:					
Relative bond returns	0.81	0.82	0.76	0.82	0.94
Relative equity returns	-0.18	-0.07	-0.09	-0.17	-0.01

Notes. The table reports simulated moments of the two-country DSGE model, using a third-order approximation to the policy rules around the SSS. Each simulation contains $100,000$ observations to generate an ergodic set. RER is the real exchange rate. The column "Baseline" reports results using the baseline calibration. The column "High ϕ" doubles the elasticity of substitution between Home and Foreign goods. The column "Low α" sets $\alpha = 0.75$ to match the World's import share. The column "Different γ's" makes Home and Foreign heterogeneous by setting $\gamma_H = 1.75$ and $\gamma_F = 2.25$, and reports results from Home's perspective. The sources and periods of the data are as follows: Coeurdacier and Rey (2013) for the 2008 U.S. home bias; the dataset by Lane and Milesi-Ferretti (2018) for 2008 U.S. external positions; Corsetti, Dedola and Leduc (2008) for U.S. relative standard deviations, and cross-correlations of RER and consumption; Coeurdacier, Kollman and Martin (2010) for data on U.S. net foreign assets, Broner et al. (2013) for the correlation between inflows and outflows for advanced economies; Coeurdacier and Gourinchas (2016) for U.S. hedge ratios, and the author's calculations using U.S. data from the FRED for the remaining variables.

More troubling are the results for gross capital flows. Broner et al. (2013) show that, empirically, gross capital flows are much more volatile than net capital flows (the negative of the current account). In Table 2, the opposite is true. To understand this result, note that in all figures depicting impulse responses, capital inflows (CIF) and capital outflows (COD) are the opposite mirror of each other: if CIF rises, COD falls by almost the same magnitude. It follows that CIF and COD are almost perfectly negatively correlated in the model (whereas in the data the correlation is 0.78), and thus they cancel each other as a sum, which is the definition of gross capital flows.[38]

The model is able to match the empirical serial correlations of the trade balance (0.62) and of the change in NFA (0.03), but it underestimates the one of the current account (0.46 in the model versus 0.67 in the data). In turn, and consistent with the impulse responses, the equity home bias follows an extremely persistent process with a serial correlation of 0.99.

The model also struggles reproducing international co-movements. The simulations deliver a negative correlation between the RER and relative consumption (-0.37 in the baseline scenario), while in the data this correlation is strongly positive, at least for the U.S. (0.71). Still, this result is an improvement relative to complete-markets models where full risk-sharing leads to perfectly negative correlations (Backus and Smith, 1993).[39] Moreover, if one uses instead the measure of RER not adjusted for preference shocks, the correlation raises significantly (-0.11). As an additional symptom of excessive risk-sharing, Home and Foreign consumptions are too synchronized, both relative to the empirical counterpart (correlation of 0.89 or above in the model, versus 0.60 in the data), and relative to the cross-correlation between Home and Foreign GDP (0.68), leading to what is commonly referred to as the Consumption Correlation Puzzle (Backus, Kehoe, and Kydland, 1992).

Another problem, shared with many open-economy models, is that the simulations fail to reproduce the empirical negative co-movement between the current account and GDP (0.07 of above in the model, versus -0.52 in the data). On a positive note, the model correctly predicts a small and negative correlation between the NFA and GDP (about -0.10 in the model, -0.27 in the data).

The model is relatively successful in reproducing observed conditional hedge ratios between changes in the RER and relative asset returns. Similar to CG, these ratios correspond to estimates from OLS regressions, where the dependent variable is the change in the RER, and the regressors are (the logs of) the return differentials of equities and bonds. The model simulations successfully reproduce a large hedge ratio for relative bond returns (0.81 in the model, compared to 0.94 in the empirical application of CG), and a small, negative hedge ratio for relative equity returns (-0.18 in the model, -0.01 in the empirical application of CG).[40] Again,

[38]See Tille and van Wincoop (2010) for a model with a positive correlation between CIF and COD, driven by time-varying second moments of the return differentials. While in this paper these moments are also time-varying, their impact on portfolio reallocation is almost nil.

[39]Figure 5 shows that preference shocks are the main driver behind this departure from perfect risk-sharing, since they trigger positive transfers from Foreign when Home is doing fine. Indeed, the correlation between RER and relative consumption converges to -1 as I shut down preference risk.

[40]See van Wincoop and Warnock (2010) for a similar result concerning relative equity returns.

using a measure of RER not adjusted for preference shocks improves the results substantially, yielding hedge ratios of 1 for bond returns, and 0 for equity returns. Overall, this outcome simply reflects a point already made in the previous subsections: that bonds provide a much effective hedge against income shocks compared to equities.

Finally, Table 2 reveals that changes to the trade elasticity or the Home bias in goods make little difference in shaping the ergodic distributions of the simulations, as all moments remain very similar. However, the same is not true when introducing an asymmetry that makes domestic households less risk-averse than foreigners. On the one hand, average total external assets (0.96) are larger than average total external liabilities (0.70), indicating a long-run global imbalance where Home, on average, serves as a net creditor. On the other hand, domestic consumption becomes more volatile than GDP (and foreign consumption, not reported, becomes substantially less volatile). That is, the price paid by Home for a larger average consumption is an increase in its volatility.

5.4.4 Accuracy

To test the accuracy of the third-order solution, I feed the wealth Euler equation (5.24) with the policy rules to compute average unit-free Euler errors (in log10 scale) of the simulated time-series (Aruoba et al, 2006), discretizing the innovations with monomials to approximate expectations (Judd, Maliar and Maliar, 2011).[41] The last row of "Averages" in Table 2 shows that the simulations are, indeed, highly accurate. In all scenarios, the mean errors are slightly below -6, and the max. errors (not reported) are well below -4. To put this result in context, Aruoba et al. (2006, Table 5) find that a fifth-order approximation of the stochastic growth model delivers a mean error of -5.43, and a max. error of -3.33.

6 Conclusion

This paper has proposed a small generalization to the standard perturbation approach by introducing a new perturbation parameter, in addition to the standard one scaling future shocks, that interacts with a subset of the model parameters. The resulting two-parameter perturbation model is well suited for solving open-economy models with incomplete markets, as it allows to solve for the dynamics of a nearby auxiliary model, and then use perturbation to reach the model of interest. Exploiting that two-parameter perturbation can approximate the model of interest around a different point than its implied deterministic steady-state, the paper has developed a simple algorithm that, backed by theoretical results, approximates the DSGE model around the stochastic steady-state. Since the method is fully compatible with popular solution toolboxes, now researchers can solve DSGE models with portfolio-choice just like any other standard model, and small open-economy models free of *ad hoc* modifications to induce stationary.

[41] The log10 scale allows for an intuitive interpretation: a value of -3 means $1 mistake for each $1,000, a value of -4 a $1 mistake for each $10,000, and so on.

As a main application, the paper has extended the two-period, multi-asset model of Coeurdacier and Gourinchas (2016) to an infinite-horizon setting, and has performed a rigorous quantitative analysis that includes the study of portfolio dynamics. Among other results, the paper has successfully demonstrated that an incomplete-market structure with bonds and equities is essential to generate long-run external gross positions comparable to the data, and has uncovered a new natural link between trade and financial integration consistent with the main patterns of globalization during the last decades.

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Appendix

A Convergence to the true stochastic steady-state

Let $\overline{\mathbf{x}}^k$ be the vector that solves the system (3.12) for a given order k. The following assumptions are sufficient to prove that $\overline{\mathbf{x}}^k$ converges to $\mathbf{x_{1,s}}$ as k goes to infinity:

Assumption 1. *There exists a vector $\overline{\boldsymbol{\psi}}$ of values of the auxiliary parameters $\boldsymbol{\psi}$ such that $\mathbf{x_s}$ satisfies the DSS condition (3.3).*

Assumption 2. *The function $\mathbf{h}(\mathbf{x}, \varepsilon, \sigma)$ is analytic in a neighborhood around $(\mathbf{x_s}, 0, 0)$ that includes $(\mathbf{x_s}, 1, 1)$.*

The first assumption ensures that the true SSS can be the DSS of the auxiliary model. The second assumption ensures that the Taylor series of $\mathbf{h}$ about $(\mathbf{x_s}, 0, 0)$ converges to $\mathbf{h}(\mathbf{x}, 1, 1)$ when evaluated at the true SSS. In practice, this will be satisfied provided that both $\overline{\psi}$ and the matrix η scaling innovations are small enough in magnitude.

Under these regularity assumptions, the following proposition holds:

PROPOSITION. *If Assumptions 1 and 2 hold, $\overline{\mathbf{x}}^k$ converges to $\mathbf{x_{1,s}}$ as $k \to \infty$.*

PROOF. Let $\mathbf{h^T}_1(\mathbf{x})$ be the Taylor series of $\mathbf{h}_1$ about $(\mathbf{x}, 0, 0)$ evaluated at $(\mathbf{x}, 1, 1)$, given by:

$$\mathbf{h^T}_1(\mathbf{x}) = H_0 + \sum_{i=1}^{k} \frac{1}{i!} H_i \begin{pmatrix} 0_{n_x \times 1} \\ 1 \\ 1 \end{pmatrix}^{\otimes i} + \zeta^k,$$

where $H_0 = \mathbf{h}_1(\mathbf{x}, 0, 0)$ is a $n_{x1} \times 1$ vector, and

$$\zeta^k = \sum_{i=k+1}^{\infty} \frac{1}{i!} H_i \begin{pmatrix} 0_{n_x \times 1} \\ 1 \\ 1 \end{pmatrix}^{\otimes i}$$

is the error term of the k-order Taylor series evaluated at $(\mathbf{x}, 1, 1)$. Fix $\boldsymbol{\psi} = \overline{\boldsymbol{\psi}}$ and evaluate $\mathbf{h^T}_1$ at $\mathbf{x} = \mathbf{x_s}$. Since Assumption 1 holds, $\mathbf{x_s}$ is the fixed-point of $\mathbf{h}(\mathbf{x}, 0, 0)$, and therefore $H_0 = \mathbf{x_{1,s}}$. Since Assumption 2 holds, convergence of the Taylor series gives $\mathbf{h^T}_1(\mathbf{x_s}) = \mathbf{h}_1(\mathbf{x_s}, 1, 1)$, and by definition (3.8) we have $\mathbf{x_{1,s}} = \mathbf{h_1}(\mathbf{x_s}, \mathbf{1}, \mathbf{1})$. Combining these two equalities gives $\mathbf{h^T}_1(\mathbf{x_s}) = \mathbf{x_{1,s}}$, which is equivalent to

$$\sum_{i=1}^{k} \frac{1}{i!} H_i \begin{pmatrix} 0_{n_x \times 1} \\ 1 \\ 1 \end{pmatrix}^{\otimes i} + \zeta^k = 0. \tag{A.1}$$

Note that each H_i is a continuous function of $\mathbf{x_{1,s}}$. If $\mathbf{h^T}_1(\mathbf{x_s})$ is an exact k-order Taylor series, then $\zeta^k = 0$ and the system (A.1) is identical to (3.12) in the main text, which automatically gives $\overline{\mathbf{x}}^k = \mathbf{x_{1,s}}$. Otherwise Assumption 2 implies that $\lim_{k\to\infty}\zeta^k = 0$ holds, and the systems (A.1) and (3.12) become identical as $k \to \infty$, leading to $\overline{\mathbf{x}}^k \to \mathbf{x_{1,s}}$. ■

B Examples

B.1 Neoclassical growth model

The equilibrium equations of the model of interest are standard:

$$\begin{aligned} y_t &= c_t + i_t, && \text{(B.1)}\\ y_t &= e^{z_t}k_t^{\alpha}, && \text{(B.2)}\\ k_{t+1} &= (1-\delta)k_t + i_t, && \text{(B.3)}\\ \frac{1}{c_t^{\gamma}} &= \beta\mathbb{E}_t\left[\left(\alpha\frac{y_{t+1}}{k_{t+1}} + 1 - \delta\right)\frac{1}{c_{t+1}^{\gamma}}\right], && \text{(B.4)}\\ z_{t+1} &= \rho z_t + \eta u_{t+1}, && \text{(B.5)} \end{aligned}$$

where y_t is output, c_t consumption, i_t investment, z_t the log of TFP, k_t current capital, and u_{t+1} the iid standard normal innovation.

The auxiliary model includes the perturbation parameters ε and σ, and introduces an auxiliary parameter ψ such that the effective discount factor is $\beta[1+\psi(1-\varepsilon)]^{-1}$. This way we allow for a small change to the discount factor such that the DSS of the auxiliary model is different from the one implied by the model of interest.[42] Perturbation around $(\varepsilon, \sigma) = (0, 0)$ then corrects for this deviation and risk simultaneously. The new equilibrium conditions replacing (B.4) and (B.5) are:

$$\begin{aligned} \frac{1}{c_t^{\gamma}} &= \beta[1+\psi(1-\varepsilon)]^{-1}\mathbb{E}_t\left[\left(\alpha\frac{y_{t+1}}{k_{t+1}} + 1 - \delta\right)\frac{1}{c_{t+1}^{\gamma}}\right],\\ z_{t+1} &= \rho z_t + \sigma\eta u_{t+1}. \end{aligned}$$

As always, we recover the model of interest by setting $(\varepsilon, \sigma) = (1, 1)$.

At $(\varepsilon, \sigma) = (0, 0)$, the deterministic steady-state (DSS) of capital in the auxiliary model is:

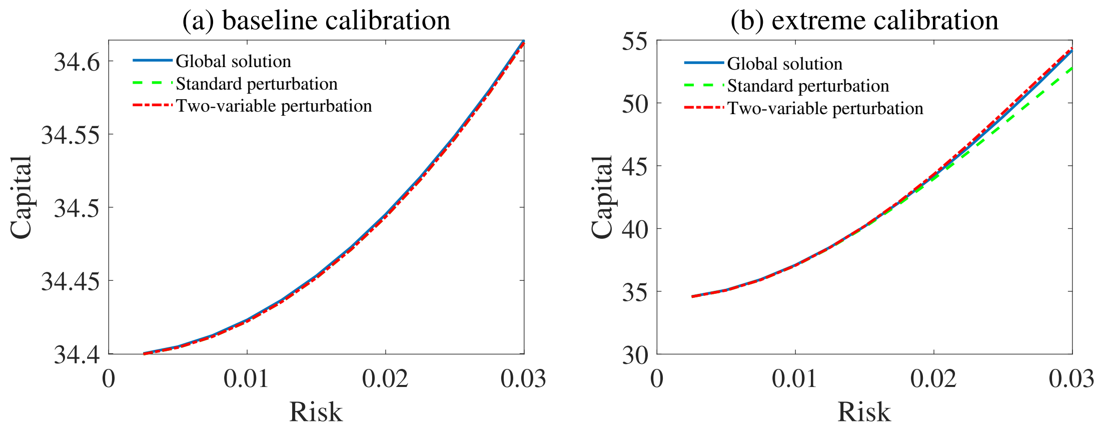
$$k_d = \left[\frac{\alpha}{([1+\psi]/\beta - (1-\delta))}\right]^{1/(1-\alpha)}.$$

It follows that by setting

$$\psi\left(\bar{k}\right) = \beta\left[\frac{\alpha}{\left(\bar{k}\right)^{1-\alpha}} + (1-\delta)\right] - 1$$

[42] Another natural candidate for perturbation is δ. Perturbing α is also feasible but less efficient because one cannot solve analytically for ψ as a function of the target DSS of capital .

Figure 6: Comparative statics of the Growth model across methods



we can choose a wide range of values for $\bar{k}$ such that $k^d = \bar{k}$. Since $\bar{k}$ uniquely pins down k_d, we can use the algorithm of Section 3.4 to build a local solution around the stochastic steady-state (SSS) of the model of interest such that $k_s = k_d = \bar{k}$.

B.1.1 Numerical performance

The baseline calibration of the model of interest is standard: $\beta = 0.99$, $\gamma = 2$, $\alpha = 0.35$, $\delta = 0.025$, $\rho = 0.95$, and $\eta = 0.01$. In addition, I consider an extreme calibration combining high levels of risk and risk aversion: $\eta = 0.025$ and $\gamma = 20$. I compute third-order approximations to the policy rule for capital using both standard perturbation and the two-parameter perturbation. To evaluate accuracy, I compare their performance against a global solution computed with the endogenous grid method (Carroll, 2006) using a grid of 400 grid-points for capital and 15 grid-points for productivity, discretizing the AR(1) process for z_t with the Rouwenhorst method.

First, I perform a comparative statics exercise by looking at the implied SSS of the Growth model across methods and calibrations. Specifically, for each method I solve the model for different values of risk (with the conditional standard deviation of productivity shocks η ranging from 0.0025 to 0.03), and then calculate the implied SSS of each policy rule.[43] Figure 6 reports the results. In Panel (a) the risk-aversion parameter is set to $\gamma = 2$, and the key finding is that the three solutions deliver virtually the same SSS, which increases with η due to the precautionary-saving motive.

In contrast, in panel (b) I set a high value of $\gamma = 20$, and find that the SSS of standard perturbation (green dashed line) deviates from the global solution (blue solid line) for enough large risk, whereas the one computed with two-parameter perturbation (red dash-dotted line) manages to follow the global solution closely. This happens because a high-risk and risk-aversion scenario boosts the precautionary-saving effect, leading to a true SSS far away from the deter-

[43]For standard perturbation, I find the SSS by iterating the state vector (k_t, z_t) over time until convergence using the third-order policy rule. For the global solution, the SSS is the point where the policy rule of capital (conditional on $z = 0$) crosses the 45° line.

Table 3: Accuracy check for the Growth model

Approximation	Baseline		Extreme	
	Mean	Max	Mean	Max
Standard perturbation	0.016	0.017	1.74	6.68
Two-parameter perturbation	0.016	0.017	0.40	1.32

Notes. The table reports key statistics of the time-series for absolute percent errors of two local approximations (relative to the global solution) under two different calibrations.

ministic one. Since a large value of $\gamma = 20$ greatly increases the non-linearity of the policy rule, the region of the state-space where the true SSS belongs falls beyond the radius of convergence of a local approximation around the DSS, thus worsening the accuracy.

Next, I fix $\gamma = 2$ and $\eta = 0.01$ in the baseline calibration, $\gamma = 20$ and $\eta = 0.025$ in the extreme calibration, and run a stochastic simulation of the model with a length of $10,000$ periods (burn-in of $1,000$ periods) for the three solution methods. To make a fair comparison, in all three cases productivity z follows the same Markov chain as in the global solution. Hence different time-paths of capital across models are due to differences in the policy rules only. Then, for the two local approximations I compute the (absolute) percent errors of the time-series of capital with respect to the time-series of the global solution, which I treat as the "true" solution.

Table 3 reports mean and maximum values for each simulation. Under the baseline calibration, these statistic are very small and virtually identical across local approximations. However, under the extreme calibration the simulation of two-parameter perturbation clearly outperforms standard perturbation: its mean absolute error of 0.4% is about four times smaller, and its maximum of 1.32% is about six times smaller. The intuition behind this result is the same as in the comparative statics exercise: under the extreme calibration, the dynamics of capital gravitate around a true SSS far above the deterministic one. By changing the point of approximation, the two-parameter perturbation corrects for this effect, leading to more accurate time-series.

B.2 Small open economy model

A representative agent in a small open economy (SOE) seeks to maximize a standard CRRA utility function:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma} - 1}{1-\gamma} \right].$$

At each period the agent receives an endowment income y_t and can save or borrow net foreign assets b_t with a (gross) world interest rate R_t. The budget constraint is:

$$c_t = Y_t + b_t - \frac{b_{t+1}}{R_t}. \tag{B.6}$$

The exogenous states $y_t = \ln Y_t$ and $z_t = \ln\left(\frac{R_t}{\overline{R}}\right)$ (with $\overline{R}$ as the DSS level of R_t) follow AR(1) processes:

$$y_{t+1} = \rho_y y_t + \sigma \eta_y u_{y,t+1}, \tag{B.7}$$

$$z_{t+1} = \rho_r z_t + \sigma \eta_r u_{r,t+1}, \tag{B.8}$$

where and $u_{y,t+1}$ and $u_{r,t+1}$ are serially uncorrelated standard normal disturbances scaled by the perturbation parameter σ.

The first-order conditions of this problem, together with a no-Ponzi-game condition, lead to the following Euler Equation for bonds:

$$c_t^{-\gamma} = R_t \beta \mathbb{E}_t\left[c_{t+1}^{-\gamma}\right]. \tag{B.9}$$

Thus, the equilibrium conditions of the model of interest are given by equations (B.6)-(B.9).

B.2.1 Auxiliary models

I consider two well-known modifications of the standard SOE model that have no other purpose than to induce stationarity of the equilibrium dynamics (Schmitt-Grohé and Uribe, 2003). In both cases the control variable is c_t, and the state variables are b_t, y_t, and z_t.

(i) Uzawa preferences. The new lifetime utility function is:

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty} \theta_t \frac{c_t^{1-\gamma}-1}{1-\gamma}\right],$$

where the discount factor θ_t depends on the perturbation parameter ε, and evolves over time according to

$$\theta_0 = 1,$$

$$\theta_{t+1} = \theta_t \overline{\beta}\left(1+\psi_2 \varepsilon^2\right)\left(\kappa \widetilde{c}_t\right)^{-\psi_1(1-\varepsilon^2)}.$$

Here $\widetilde{c}_t$ denotes average consumption, $\psi_1 > 0$ and $\psi_2 < 0$ are two auxiliary parameters, κ is a calibration parameter discussed below, and $\overline{\beta} = 1/\overline{R}$. When $\varepsilon = 1$ the discount factor becomes exogenous and equal to $\beta = \overline{\beta}\,(1+\psi_2)$, which allows the model of interest to satisfy the well-known stationarity condition $\overline{R}\beta < 1$ arising from precautionary savings.[44] The new Euler equation replacing (B.9) is:

$$c_t^{-\gamma} = R_t \overline{\beta}\left(1+\psi_2\varepsilon^2\right)\left(\kappa c_t\right)^{-\psi_1(1-\varepsilon^2)} \mathbb{E}_t\left[c_{t+1}^{-\gamma}\right] \tag{B.10}$$

where I have already imposed $\widetilde{c}_t = c_t$. Thus, the equilibrium conditions of the new model are given by equations (B.6), (B.7), (B.8), and (B.10). The deterministic auxiliary model corresponds

[44] See Aiyagari (1994), and Chamberlain and Wilson (2000).

to the case $(\varepsilon, \sigma) = (0, 0)$, and the model of interest is $(\varepsilon, \sigma) = (1, 1)$.

Finally, imposing stationarity to (B.10) gives $c_d = 1/\kappa$, which we can substitute into (B.6) to pin down $b_d = (1/\kappa - 1)/(1 - 1/\overline{R})$. Since this expression is invertible in κ, we can set $\kappa = \left[1 + \overline{b}\left(1 - 1/\overline{R}\right)\right]^{-1}$, and target a wide range of DSS values for external assets such that $b_d = \overline{b}$.[45]

(ii) Portfolio adjustment costs (PAC). The new budget constraint takes into account that agents face convex costs of holding external assets in a quantity different from some long-run level $\overline{b}$:

$$c_t = y_t + b_t - \frac{b_{t+1}}{R_t} - \frac{\psi_1}{2}\left(1 - \varepsilon^2\right)\left(b_{t+1} - \overline{b}\right)^2 \tag{B.11}$$

where $\psi_1 > 0$ is an auxiliary parameter. The exogenous discount factor is $\beta = \overline{\beta}\left(1 + \psi_2 \varepsilon^2\right)$, with $\psi_2 < 0$ and $\overline{\beta} = 1/\overline{R}$. It follows that the new Euler equation replacing (B.9) is:

$$\left(\frac{1}{R_t} + \psi_1\left(1 - \varepsilon^2\right)\left(b_{t+1} - \overline{b}\right)\right) c_t^{-\gamma} = \overline{\beta}\left(1 + \psi_2 \varepsilon^2\right) \mathbb{E}_t\left[c_{t+1}^{-\gamma}\right] \tag{B.12}$$

The equilibrium conditions of the new model are given by equations (B.7), (B.8), (B.11), and (B.12). Again, we recover the model of interest by setting $(\varepsilon, \sigma) = (1, 1)$. Finally, imposing stationarity to (B.12) immediately yields $b_d = \overline{b}$.

B.2.2 Comparative statics

Consider the special case where there are only income shocks ($\eta_r = 0$), so that $R_t = \overline{R}$ at all times. This allows me to compare the performance of the approximations with a global solution to the income fluctuation problem based on the endogenous grid method. The baseline calibration is $\overline{R} = 1.04$, $\gamma = 4$, $\rho_y = 0.85$, and $\eta_y = 0.02$. The auxiliary parameters are set to $\psi_1 = 10^{-5}$ and $\psi_2 = -10^{-4}$. These low values helps perturbation to reach the model of interest starting from the auxiliary ones.

To implement the global algorithm, I introduce the borrowing constraint $b_{t+1} \geq \underline{b}$ (as opposed to a no-Ponzi-game condition), and set a value of $\underline{b}$ low enough such that its effects on the dynamics are negligible. To maximize accuracy, I use a dense grid of $1,000$ nodes for b_t and 25 nodes for income Y, discretizing its AR(1) process with the Rouwenhorst method. The solution delivers a SSS of $b_s = -2.0582$, which corresponds to the point where the policy rule of b_{t+1} (conditional on $Y = 1$) crosses the 45° line.

When I approximate the SSS with the algorithm of Section 3.4 I obtain the following values: $b_s = -2.0510$ and $b_s = -2.0583$ using second and fourth-order approximations to the auxiliary PAC model, and $b_s = -2.0855$ and $b_s = -2.0584$ using second and fourth-order approximations to the auxiliary Uzawa model. Thus, regardless of the auxiliary model used, a second-order perturbation already provides a good approximation to the true SSS, which becomes a near-perfect fit as the order of perturbation increases.

[45]The restriction is that DSS consumption must be positive: $\overline{b} > -\left(1 - 1/\overline{R}\right)^{-1}$.

Figure 7: Comparative statics of the SOE model across methods

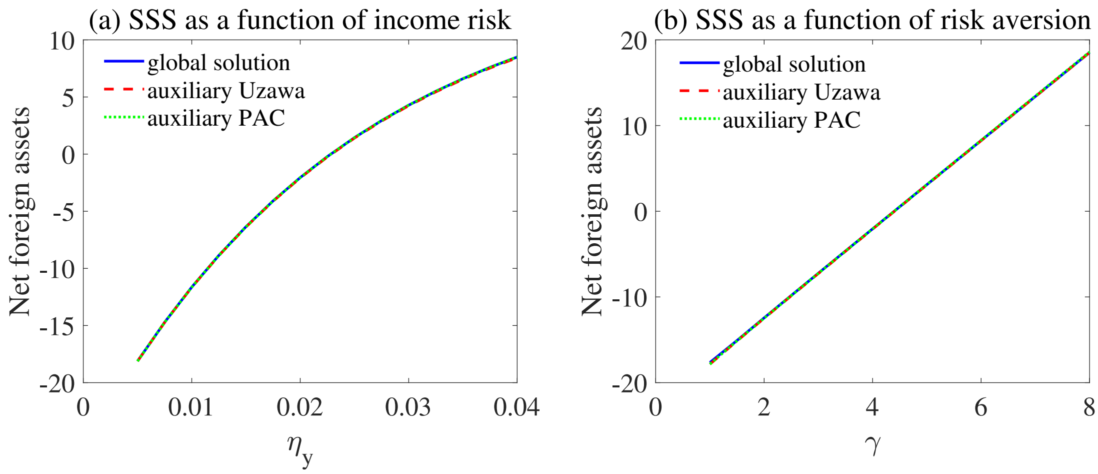


Table 4: Calibration of the SOE model

Parameter		Value
$\overline{R}$	DSS gross interest rate	1.0144
β	Discount factor	0.9851
γ	Risk aversion	2
ρ_y	Auto-correlation of income	0.749
ρ_r	Auto-correlation of interest rate	0.572
η_y	Std. of income shocks	0.0180
η_r	Std. of interest rate shocks	0.0161
$\rho_{y,r}$	Correlation of innovations	-0.62

Next, I perform a comparative statics exercise by computing the SSS for different values of income risk (η_y ranging from 0.005 to 0.04), and risk-aversion (γ ranging from 1 to 8). Figure 7 reports the results. Panel (a) plots the SSS values as a function of income risk, and Panel (b) plots the SSS values as a function of risk-aversion. In both panels the blue solid line corresponds to the global solution, the green dotted line to the second-order perturbation using the auxiliary PAC model, and the red dashed line to the second-order perturbation using the auxiliary Uzawa model. Since the three lines are virtually identical in both panels, the main result here is that two-parameter perturbation does an excellent job in approximating the true SSS of the model, regardless of the auxiliary model involved.

B.2.3 Simulations with a realistic calibration

Table 4 reports the calibration used to simulate the SOE economy. Here I follow Mendoza (2010) and set values to the structural parameters that help to match key moments from Mexican data for the period 1993:I-2005:II. The only parameters different from this study are $\overline{R}$ (set to match

Figure 8: Ergodic distribution of the SOE model

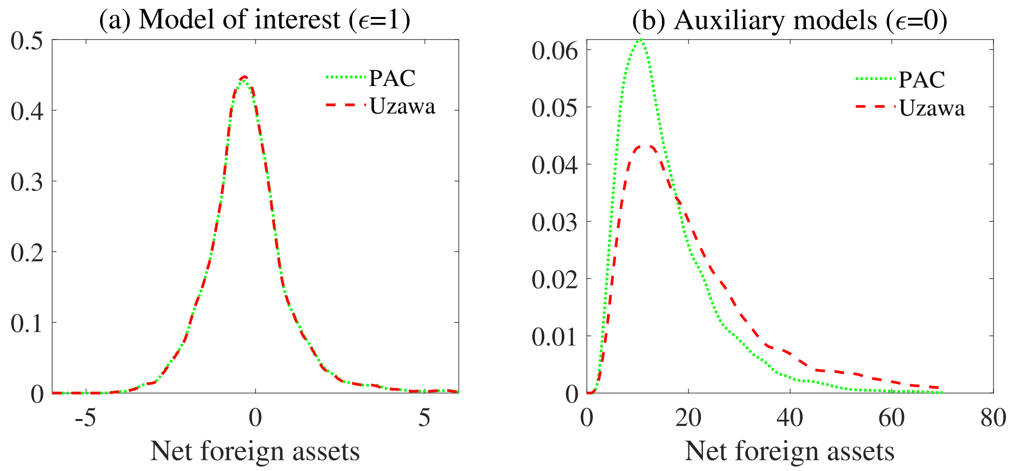


Table 5: Euler errors of the SOE model

Approximation	Auxiliary Uzawa		Auxiliary PAC	
	Mean	Max	Mean	Max
Third order	-5.19	-3.42	-5.28	-3.50
Fifth order	-6.92	-4.32	-6.82	-4.25

Notes. The table reports the mean and maximum values of the Euler errors (in log10 scale) across the ergodic set of each approximation.

the average (annual equivalent) of 5.9 percent from Uribe and Yue (2006)), and β, which I calibrate to deliver a SSS ratio of net foreign assets to GDP of -44 percent (Mexico's average ratio over the period 1985–2004 in the Lane and Milesi-Ferretti (2018) database).

Using a third-order perturbation to the model of interest for each auxiliary model, I simulate the economy for $100,000$ periods (burn-in of $1,000$ periods), feeding the decision rules with the same pseudo-random innovations. The resulting time-series for net foreign assets allows me to compute kernel distributions (using the Epanechnikov method) that I plot in Figure 8, Panel (a). The first result is that the approximations deliver well-defined ergodic distributions of net foreign assets (dotted green line for PAC, and red dashed line for Uzawa) , despite the use of a third-order approximation without pruning the state-space. The second result is that the two distributions are extremely close to each other, indicating near-identical long-run dynamics. To show that this is not an artifact implied by similar underlying auxiliary models, Panel (b) repeats the same exercise evaluating the approximations without correcting for the stationary-inducing modifications (that is, $(\varepsilon, \sigma) = (0, 1)$). Now the two kernel distributions are widely different, both from each other and from the ones implied by the model of interest.

Finally, I compute the mean and maximum of unit-free Euler errors (in log10 scale) of the simulated time-series. Table 5 reports the results for third and fifth-order approximations to the policy rules. The overall message is that, regardless of the auxiliary model used, the two-variable perturbation method delivers approximations with good global properties, with mean Euler errors below -5 using a third-order approximation, and well below -6 using a fifth-order one.

C Derivations of the two-country DSGE model

C.1 Equilibrium conditions

The Euler equations of the Home agent for Home and Foreign stocks are:

$$\beta\left(\frac{C_t}{\overline{C}}\right)^{-\kappa}\mathbb{E}_t\left[\frac{\delta_{t+1}Y_{t+1}+z^S_{H,t+1}}{P_{t+1}C^{\gamma}_{t+1}}\right]=\frac{z^S_{H,t}}{P_tC^{\gamma}_t}\left[1+\psi\left(1-\varepsilon\right)\left(S_{H,t+1}-\overline{a}\right)\right], \tag{C.1}$$

$$\beta\left(\frac{C_t}{\overline{C}}\right)^{-\kappa}\mathbb{E}_t\left[\frac{\delta^*_{t+1}p_{F,t+1}Y^*_{t+1}+z^S_{F,t+1}}{P_{t+1}C^{\gamma}_{t+1}}\right]=\frac{z^S_{F,t}}{P_tC^{\gamma}_t}\left[1+\psi\left(1-\varepsilon\right)\left(S_{F,t+1}-\overline{a}_F\right)\right]. \tag{C.2}$$

where I have already imposed $\widetilde{C}_t=C_t$. The Euler equations of the Home agent for Home and Foreign bonds are:

$$\beta\left(\frac{C_t}{\overline{C}}\right)^{-\kappa}\mathbb{E}_t\left[\frac{\widetilde{P}_t}{P_{t+1}C^{\gamma}_{t+1}}\right]=\frac{z^B_{H,t}}{P_tC^{\gamma}_t}\left[1+\psi\left(1-\varepsilon\right)\left(B_{H,t+1}-\overline{b}\right)\right], \tag{C.3}$$

$$\beta\left(\frac{C_t}{\overline{C}}\right)^{-\kappa}\mathbb{E}_t\left[\frac{\widetilde{P}^*_t}{P_{t+1}C^{\gamma}_{t+1}}\right]=\frac{z^B_{F,t}}{P_tC^{\gamma}_t}\left[1+\psi\left(1-\varepsilon\right)\left(B_{F,t+1}-\overline{b}_F\right)\right]. \tag{C.4}$$

In turn, the Euler equations of the Foreign agent for Home and Foreign stocks are:

$$\beta\left(\frac{C^*_t}{\overline{C}^*}\right)^{-\kappa}\mathbb{E}_t\left[\frac{\delta_{t+1}Y_{t+1}+z^S_{H,t+1}}{P^*_{t+1}\left(C^*_{t+1}\right)^{\gamma}}\right]=\frac{z^S_{H,t}}{P^*_t\left(C^*_t\right)^{\gamma}}\left[1-\psi\left(1-\varepsilon\right)\left(S_{H,t+1}-\overline{a}\right)\right], \tag{C.5}$$

$$\beta\left(\frac{C^*_t}{\overline{C}^*}\right)^{-\kappa}\mathbb{E}_t\left[\frac{\delta^*_{t+1}p_{F,t+1}Y^*_{t+1}+z^S_{F,t+1}}{P^*_{t+1}\left(C^*_{t+1}\right)^{\gamma}}\right]=\frac{z^S_{F,t}}{P^*_t\left(C^*_t\right)^{\gamma}}\left[1-\psi\left(1-\varepsilon\right)\left(S_{F,t+1}-\overline{a}_F\right)\right], \tag{C.6}$$

where I have used the market conditions (5.16) and (5.17) to impose $S^*_{H,t+1}=1-S_{H,t+1}$, and $S^*_{F,t+1}=1-S_{F,t+1}$. Also, $\overline{C}^*$ is the DSS of C^*_t in the auxiliary model. The Euler equations of the Foreign agent for Home and Foreign bonds are:

$$\beta\left(\frac{C^*_t}{\overline{C}^*}\right)^{-\kappa}\mathbb{E}_t\left[\frac{\widetilde{P}_t}{P^*_{t+1}\left(C^*_{t+1}\right)^{\gamma}}\right]=\frac{z^B_{H,t}}{P^*_t\left(C^*_t\right)^{\gamma}}\left[1-\psi\left(1-\varepsilon\right)\left(B_{H,t+1}-\overline{b}\right)\right], \tag{C.7}$$

$$\beta\left(\frac{C^*_t}{\overline{C}^*}\right)^{-\kappa}\mathbb{E}_t\left[\frac{\widetilde{P}^*_t}{P^*_{t+1}\left(C^*_{t+1}\right)^{\gamma}}\right]=\frac{z^B_{F,t}}{P^*_t\left(C^*_t\right)^{\gamma}}\left[1-\psi\left(1-\varepsilon\right)\left(B_{F,t+1}-\overline{b}_F\right)\right], \tag{C.8}$$

where I have used the market-clearing conditions (5.18) and (5.19) to impose $B^*_{H,t}=-B_{H,t}$, and $B^*_{F,t}=-B_{F,t}$.

In the auxiliary equilibrium, the PAC of each country cancels out with their respective lump-sum transfer (and in the model of interest there is no PAC in the first place). It follows that the equilibrium budget constraints are (5.7) for Home, and

$$\begin{aligned} & P_t^* C_t^* + z_{H,t}^S \left(1 - S_{H,t+1}\right) + z_{F,t}^S \left(1 - S_{F,t+1}\right) - z_{H,t}^B B_{H,t+1} - z_{F,t}^B B_{F,t+1} \\ = & \left(1 - \delta_t^*\right) p_{F,t} Y_t^* + \left(1 - S_{H,t}\right) \left(\delta_t Y_t + z_{H,t}^S\right) + \left(1 - S_{F,t}\right) \left(\delta_t^* p_{F,t} Y_t^* + z_{F,t}^S\right) - \widetilde{P}_t B_{H,t}^* - \widetilde{P}_t^* B_{F,t}^*. \end{aligned} \tag{C.9}$$

for Foreign, where I have already imposed the market-clearing substitutions. Finally, substituting the relative demands (5.3) and (5.4) into the market-clearing condition (5.15) gives

$$\alpha P_t^{\phi} \left(q_t\right)^{\phi - 1} C_t + (1 - \alpha) \left(P_t^*\right)^{\phi} \left(q_t\right)^{\phi - 1} C_t^* = Y_t. \tag{C.10}$$

Bonds and equities model. The 21 equilibrium conditions of the full model are: the Euler equations (C.1)-(C.8), budget constraints (5.7) and (C.9), price-index equations (5.1), (5.2), (5.5) and (5.6), market-clearing condition (C.10), and AR(1) processes (5.9)-(5.14).

Equities-only model. I also consider an equities-only version of the model, where $B_{H,t} = B_{F,t} = 0$ at all times. In this case, the 15 equilibrium conditions are the Euler equations (C.1), (C.2), (C.5) and (C.6), the budget constraints (5.7) and (C.9), price-index equations (5.1) and (5.2), market-clearing condition (C.10), and AR(1) processes (5.9)-(5.14).

C.2 Deterministic steady-state

The first step is to pin down the DSS of asset holdings. Let $\overline{C} = C$ and $\overline{C}^* = C^*$ be the DSS of Home and Foreign consumption. Evaluating at $(\varepsilon, \sigma) = (0, 0)$ and imposing stationarity, Euler equations (C.1) and (C.5) become:

$$\begin{aligned} \beta \left[\frac{\overline{\delta} + z_H^S}{z_H^S} \right] &= 1 + \psi \left(S_H - \overline{a}\right), \\ \beta \left[\frac{\overline{\delta} + z_H^S}{z_H^S} \right] &= 1 - \psi \left(S_H - \overline{a}\right). \end{aligned}$$

Equating the left hand-sides of the equations delivers $S_H = \overline{a}$, which implies $z_H^S = \frac{\beta}{1-\beta}\overline{\delta}$. An identical procedure with the pair (C.2) and (C.6) gives $S_F = \overline{a}_F$, and $z_F^S = \frac{\beta}{1-\beta} p_F \overline{\delta}$. In turn, combining the pairs (C.3)-(C.7) and (C.4)-(C.8) yields DSS bond holdings $B_H = \overline{b}$ and $B_F = \overline{b}_F$, and DSS bond prices $z_H^B = \beta P$ and $z_F^B = \beta P^*$.[46] Thus, it follows that the vector $\left(\overline{a}, \overline{a}_F, \overline{b}, \overline{b}_F\right)$ uniquely pins down the DSS of the endogenous states.

Next, impose stationarity in the budget constraints (5.7) and (C.9), and use the previous

[46]Note that the steady-state values of P_t and its non-adjusted counterpart $\widetilde{P}_t$ are identical: $P^d = \widetilde{P}^d = P$. Similarly, $(P^*)^d = \left(\widetilde{P}^*\right)^d = P^*$.

results to obtain:

$$C = \left(\frac{1}{P}\right)\left[(1-\delta)+\delta\left(a+p_F a_F\right)+(1-\beta)\left(Pb+P^* b_F\right)\right] \quad (C.11)$$

$$C^* = \left(\frac{1}{P^*}\right)\left[(1-\delta)\,p_F+\delta\left((1-a)+p_F\left(1-a_F\right)\right)-(1-\beta)\left(Pb+P^* b_F\right)\right] \quad (C.12)$$

where I have omitted hat symbols to ease notation. Likewise, the stationary versions of the price-index equations (5.1) and (5.2) are:

$$P = \left[\alpha+(1-\alpha)\,p_F^{1-\phi}\right]^{1/(1-\phi)}, \quad (C.13)$$

$$P^* = \left[(1-\alpha)+\alpha p_F^{1-\phi}\right]^{1/(1-\phi)}, \quad (C.14)$$

and the stationary version of the market-clearing equation (C.10) is:

$$\alpha P^{\phi} C+(1-\alpha)\left(P_t^*\right)^{\phi} C_t^* = 1. \quad (C.15)$$

Thus, we can substitute (C.13) and (C.14) into (C.11) and (C.12), and in turn these into (C.15) to obtain an equation with p_F as the only unknown. Once this variable is solved, we can immediately compute the remaining DSS values of price indices (P, P^*), consumptions (C, C^*), and asset prices z_F^S, z_H^B, and z_F^B. Finally, the calculations for the equities-only version of the model are identical after setting $b = b_F = 0$.

Symmetrical case. The previous solution is general enough to allow for approximations to the policy rules around arbitrary values of $\left(\bar{a}, \bar{a}_F, \bar{b}, \bar{b}_F\right)$. But if one is working with perfectly symmetric countries, exploiting this symmetry simplifies calculations. In particular, perfect symmetry imposes a strong parameter restriction: $\bar{a}_F = 1-\bar{a}$, and $\bar{b}_F = -\bar{b}$. To see this, note that under perfect symmetry it must be the case that $p_F = 1$, and $P = P^* = 1$. Plugging these results into (C.11) and (C.12), and substituting the resulting expressions into (C.15) gives

$$(1-\delta)+2(1-\alpha)\,\delta+(2\alpha-1)\left(\delta A+(1-\beta)\,B\right) = 1.$$

where $A = a + a_F$, and $B = b + b_F$. For this equation to hold good for arbitrary values of δ and α, we require $A = 1$ and $B = 0$, which delivers the result. Moreover, since one obtains an identical expression by imposing stationarity in the model of interest $(\varepsilon, \sigma) = (1, 1)$, the same restrictions apply to the SSS of asset holdings. The consequences are that, when solving for the SSS with the algorithm of Section 3.4, (i) we just have to solve for $\bar{a}$ and $\bar{b}$, and (ii) recalculating the DSS starting from arbitrary values of $\bar{a}$ and $\bar{b}$ does not require any intermediate numerical procedure.