Learning Dynamics in a Network of Cournot Economies

Sebastian Kreuzmair*, [a]

[a]CeNDEF, Amsterdam School of Economics, University of Amsterdam, The Netherlands

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Abstract

In this paper, firms in a network of Cournot economies learn about the intercept of the demand curve using past sales history. I decompose the learning process into economically interpretable components and find that learning aggregate quantities happens at a faster rate than individual quantities both within markets and within firms. This speed depends on the network topology and the slope of the demand function. The slowest learning component is the distribution of a correct aggregate amount between markets, which drives the slow convergence of individual quantities. The convergence rate of individual quantities is the same for all sufficiently connected networks and is independent of the slope of the demand function. Increasing the density of a random network has a non-monotonic effect on the convergence speed of aggregate quantities. Convergence speeds first decrease relative to isolated market-firm pairs, increasing again after the graph becomes sufficiently connected.

Keywords: Cournot competition, least squares learning, limited information, networks

1 Introduction

The market as an information aggregation device has been a central theme of modern economic theory since its early days (Smith, 1776; Hayek, 1945). Markets, suppos-

*s.t.kreuzmair@uva.nl

edly, allow economic actors using decentralized information to arrive at (constrained) optimal decisions even though none of the actors have all the necessary information. How markets fulfill this role exactly is not entirely clear (Bowles et al., 2017). In this paper, I use a Cournot oligopoly model with limited information and learning firms to decompose the learning dynamics into economically meaningful components and characterize how information aggregated through the market leads to faster convergence of aggregate quantities.

Often we focus on analyzing market equilibria because we assume that agents learn these equilibria quickly. This is however not straightforward, as markets can allow for (relatively) fast learning of aggregate quantities but individual quantities may still be learned slowly. Hence, the plausibility of analyzing equilibrium outcomes may depend on the aggregation level being analyzed. This has also implications for welfare: in this market, aggregate quantities affect consumer surplus directly while individual quantities only affect firm revenue. Thus, if aggregate quantities are learned faster than individual quantities, welfare losses (relative to the full-information equilibrium) due to wrong aggregate quantities decrease faster than losses due to a wrong distribution of individual quantities. So in settings where information is limited but informative signals on an aggregate level are available and consumer utility is directly affected by aggregate quantities only, out of equilibrium welfare losses may disappear relatively quickly.

Expectations play an important role in many economic models and a key benchmark are rational expectations. Rational expectations were first introduced by Muth (1961) and the formulation was further strengthened by Lucas and Prescott (1971)[1]. Rational expectations as defined by Lucas and Prescott (1971) have been the primary formulation used in economic modeling. They assume that when forming expectations all participants know the true distribution of future variables and form the correct expectations, which also implies that all agents have the same expectations. This can be motivated by an eductive argument (Guesnerie, 1992), i.e. agents are able to derive the rational expectations equilibrium as a Nash equilibrium of the economic model based on their understanding of the model, which however requires a high computational ability. Another motivation is by an evolutive learning argument, whereby agents update their expectations over time and may converge to the rational expectations equilibrium (Bray & Savin, 1986; Marcet & Sargent, 1989). While the possibility of convergence has been studied extensively, the speed at which this convergence happens has received less attention.

[1]See Wagener (2014) for a discussion.

The central question of this paper is how long the convergence to an equilibrium takes. To the best of my knowledge, this is the first paper to systematically analyze convergence speeds theoretically in a network of markets and provide new insights into how the Cournot market mechanism disseminates information. Convergence speed is important for multiple reasons: first, if it is slow the rational expectations assumption and focusing on equilibrium comparative statics may not be justified as the economy may never reach this equilibrium before shocks or structural shifts change it. It is also interesting from a market design perspective; if convergence of individual quantities is an important goal a market mechanism with aggregate signals may not be the best choice because individual quantities will converge relatively slowly.

Vives (1993) analyzes convergence speeds in a 'rational' learning environment, i.e. where agents have a correctly specified model, whereas in this paper agents are boundedly rational in the sense that they have a misspecified model. Vives (1993) analyzes convergence speeds in terms of the rate that the (stochastic) estimate of the parameter under consideration converges in probability to the (deterministic) true value. The focus in this paper is on the convergence rate of the expected value of the estimator to the true value. The estimator is still a random variable, but as the agents base their decisions on the mean of this random variable the remaining randomness is not important for the economic outcomes.

Wagener (2014) analyzes data from learning-to-forecast experiments and finds slow convergence speeds. If learning in real life is slow, the transient period before convergence is of interest. In addition, if we want to understand how markets disseminate information it is important to analyze this transient period as once the rational expectations equilibrium is reached the information aggregation and distribution role of the market is limited. Agents will have beliefs consistent with the equilibrium and thus no longer need to update their beliefs.

In this paper, I reduce the informational requirements in a one-good, many-markets oligopoly model to study how firm decisions evolve when firms are learning about their environment. In particular, I analyze how fast convergence to equilibria – if any exist – takes place. Firms are active in multiple markets determined by an exogenous network and know the slope of the demand function and their sales, i.e. price–quantity pairs. Each firm believes demand is a linear function of their own quantity and as new information arrives, firms update their estimate of the market size, i.e. the intercept of the demand function using a recursive least squares learning algorithm. Since firms are connected to multiple markets they simultaneously try to learn about all the markets in their neighborhood. Firms' production costs are a quadratic function of total production, which gives rise to tradeoffs between producing for one market or another. This tradeoff implies that if a firm's belief about any market in its network changes, it has an effect on its production for all markets. The linearity of perceived demand and the aggregate price signal in the Cournot market imply that for strongly connected market networks, individual quantities converge at a constant rate, which is independent of the network topology, to the full information Cournot-Nash equilibrium. A sufficient condition for a market network to be strongly connected is that each firm is active in at least two markets and each market has two firms active in it. This guarantees that every firm faces tradeoffs and that no firm is the only active firm in a market. So as long as this condition is satisfied we know that all individual quantities will converge at a constant rate, which is slower than for aggregate quantities, regardless of what the network looks like.

The recursive least squares algorithm describing how firms update their beliefs about market size is a stochastic algorithm, which takes the form of a nonautonomous stochastic difference equation. I approximate this with a system of linear ordinary differential equations using results from the stochastic approximation literature (Benveniste et al., 1990; Evans & Honkapohja, 2001). This approximation allows us to analyze the convergence speeds of the system using the eigenvalues of the matrix that represents the dynamics. In addition, we can link the convergence speeds to different learning tasks via the eigenspaces of this matrix. This analysis is independent of the network topology and allows me to make statements holding for any network.

Next, I numerically look at how aggregate convergence speeds vary between different standard network types and within types depending on fundamentals of the economy. Convergence speeds of aggregate quantities depend on the network topology and the slope of the demand function. I find that for networks with a constant number of links per node, e.g. the circle or the tree network, aggregate convergence speeds decrease as the size of the network increases[2] and approaches the slower rate of individual quantities. For the star network, the convergence speed is constant in the size of the network, as the convergence speed is limited by the leaf nodes whose number of connections do not change as the network size increases. Convergence speed in the complete network is increasing in the size of the network. These results indicate that if information and trade flows are limited the market is less efficient in disseminating information as shown by the slower learning speeds. This could be an argument for facilitating trade and information flows between markets and in favor of increasing globalization as the total consumer surplus increases with the number of firms. The

[2] Note that in the tree and circle network, the number of firms and markets increases together, i.e. we cannot vary the number of markets independently of the number of firms and vice versa.

slope of demand has a heterogeneous effect that depends on the network topology. In the complete network, convergence speeds increase as inverse demand reacts more strongly to price changes (see discussion below). In the circle and star network, this effect goes in the opposite direction and convergence speed decreases as inverse demand becomes more sensitive to quantity changes.

The topology of these standard networks is by definition fixed and only covers a fraction of all possible networks. To gain some more insight into how the network topology affects the convergence speed I look at draws of an Erdős-Renyi graph with a fixed number of markets and firms but a varying number of connections. An Erdős-Renyi graph is a random graph that is uniformly drawn from the set of all graphs with a given number of nodes and connections. By varying the number of connections we gain insights into how the connectedness of the graph affects convergence speeds and by looking at many draws we also gain insights into how much the network topology changes the convergence speeds for a given number of connections. We find that for a given demand slope, the convergence rate of aggregate demand is hump-shaped and decreases as the network goes from single market-firm pairs, where there is no information exchange, to a sparsely connected network, where information can travel between markets. The increase in connections slows down aggregate convergence because most markets are only indirectly connected to each other so information takes longer to disseminate across the network and changes in a market that a firm does not observe may affect the conditions in the market the firm does observe. As the network becomes more dense convergence speed increases again as information can now travel more directly between markets. Once the network becomes a complete network there is a sharp increase in aggregate convergence speed as now all firms have access to all information. The decrease in convergence speed is more pronounced and takes longer to reverse the steeper the inverse demand function is. This can be explained by the increase in price volatility as the sensitivity of the inverse demand function to quantity changes increases. With a limited number of connections, and thus a limited information flow, more pronounced price changes make it harder for a firm to distinguish what caused prices to change and thus it takes longer for aggregate quantities to converge. Interestingly, this has the opposite effect in the complete network; convergence speeds increase sharply as the slope of the inverse demand function increases. The reasoning for this is similar, as every firm is connected to every market there are only direct information flows. Then, with a steeper inverse demand curve fewer observations reveal more information about the equilibrium beliefs as a small deviation of quantities from equilibrium will have a bigger effect on the difference between observed and expected intercepts, which in turn affects how strongly beliefs are updated, than with a less sensitive inverse demand.

There is an extensive literature on learning and expectations in (macro-) economic models (Evans & Honkapohja, 2001). Paramount among these is the cobweb model (Ezekiel, 1938; Nerlove, 1958; Brock & Hommes, 1997; Guesnerie, 1992; Hommes et al., 2022), which is characterized by a simple market where suppliers have a one (or more) period production lag and thus base their production decisions on expected prices instead of actual prices.

The literature on learning in Cournot oligopoly models evolved largely in parallel to the literature on learning in cobweb models but is seldom talked about together. In cobweb models, typically a continuum of firms is trying to learn about how an exogenous variable drives demand and firms do not behave strategically. In contrast, in Cournot oligopoly models the strategic interaction between firms is a key component in the learning process.

Cournot (1838) was one of the first to model learning in an economic context. He introduced the best reply dynamics as a justification for the Cournot-Nash equilibrium (Cox & Walker, 1998). In the Cournot best-reply dynamics firms know perfectly the demand they are facing but are unaware of what the other firms are supplying before making the production decision. Each period they choose their quantities by optimally reacting to the previous period's aggregate supply. This learning process converges to the Nash equilibrium but it also requires a lot of information.

Kirman (1975) analyzes a linear duopoly model with imperfect substitutes where firms compete in prices in a single market. Firms use Bayesian updating and recursive least squares to learn about the demand they are facing. When using least squares learning firms arrive at an equilibrium that is not the Nash equilibrium.

Tuinstra (2004) generalizes the demand function in a similar framework to Kirman (1975). Firms are assumed to know the true slope of the demand curve they are facing at the most recent price. Based on the price-quantity pair and the slope the firms update the intercept of the linear approximation of the true demand curve at the previous price. Tuinstra finds that the adjustment process is stable for demand functions which exhibit little cross-price effects and are not too non-linear. Otherwise, endogenous fluctuations may arise.

Bischi et al. (2007) is most similar to the present set-up: they analyze a repeated Cournot oligopoly game in a single market where firms have more limited information on the demand function. In particular, they only know the partial derivative of the demand function with respect to their own quantity at the current aggregate production level. I am using the same information structure and learning process but while they focus on the local stability of the steady states and how the stability compares to learning under best reply dynamics, I focus on characterizing the convergence dynamics, their relation to information aggregation, and how these two relate to the network structure.

Gode and Sunder (1993) analyze the allocative efficiency of market institutions with zero-intelligence traders, i.e. trades with only very simple behavioral rules. They find that even in these circumstances some market institutions may achieve in aggregate efficient outcomes. This result does, however, not extend to individual outcomes. The setup differs from the present one as there is no learning by agents, but the results share similarities in terms of how the efficiency of the market structure in achieving equilibrium outcomes differs when comparing aggregation levels.

The paper is structured as follows. Section 2 presents the model structure and the learning framework. Section 3 discusses learning dynamics in a general network and characterizes learning speeds of individual and aggregate quantities. Sections 4.1 and 4.2 analyze learning dynamics for the particular cases of a single market and a complete network, respectively, and Section 5 provides some comparative statics for the economic fundamentals. Section 6 concludes.

2 Model

There is a finite set of markets $\mathcal{M} \equiv \{1, \dots, M\}$ indexed by m and a finite set of firms $\mathcal{J} \equiv \{1, \dots, J\}$ indexed by j. A firm is connected to some subset $\mathcal{M}_j \subseteq \mathcal{M}$ of markets and markets are connected to some subset of firms, $\mathcal{J}_m \subseteq \mathcal{J}$. These connections are not directed, so $m \in \mathcal{M}_j \iff j \in \mathcal{J}_m$. A firm-market connection means that the firm is aware of the market and is able to sell its good on the market if the firm finds it profitable. The connections are assumed to be exogenous and heterogeneous across firms. This network can be described by a bipartite graph $\mathcal{G} = (\mathcal{M}, \mathcal{J}, E)$ where E is the set of paired vertices, i.e. the connections between markets and firms. Let $G \in \{0,1\}^{M \times J}$ denote the biadjacency matrix of the graph $\mathcal{G}$, with elements, g_{mj}, equal to 1 if there is an edge connecting m, j. The good is assumed to be homogeneous so consumers are indifferent between consuming output from different firms. The demand in each market is identical and denoted by $D_m(p_t^m)$, and is assumed to be a linear function of the price, p_t^m. Each firm chooses the amount to supply by maximizing expected profits.

In the example network of Figure 1, firm two can, for example, sell its goods on markets one and two, whereas firm one can only sell its goods on market one.

In each period t firm j is choosing quantities $q_t^{m,j}$ to sell on market m in its network

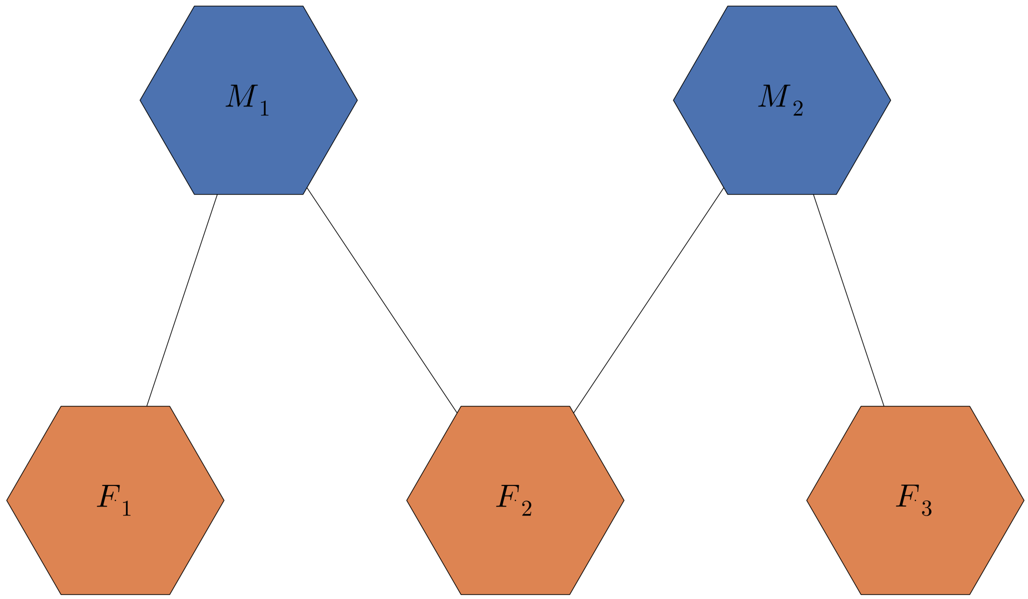

Figure 1: An example of a Cournot network with 2 markets and 3 firms.

$\mathcal{M}_j$ to maximize expected profits. Firms are endowed with a production technology that is summarized by a cost function, $c_j\left(q_t^j\right) : \mathbb{R}_+^{|\mathcal{M}_j|} \to \mathbb{R}_+$. I assume these costs to be quadratic in total production $Q_t^j \equiv \sum_{m \in \mathcal{M}_j} q_t^{m,j}$. I make the quadratic cost assumption as then the optimal quantity choice for a particular market depends on the beliefs about all markets via the total quantity in the cost function. This implies a trade-off between producing for different markets, which is not present if costs were linear. With linear costs, each firm could make the production decision for each market independently as the optimal choice only depends on the beliefs about the market in question, which would make the network structure irrelevant[3]. This also depends on the assumption that demand is isolated in the sense that consumers in one market only have access to the good in that market.

Firms are assumed to make profit-maximizing decisions but face a restricted information set, i.e. they can only base their decisions on information about the market size that they have gathered through previous market interaction. Firms believe that

[3]Interestingly, the speed of convergence of individual quantities is the same in the linear case as in the quadratic case, but the speed of convergence of aggregate quantities is independent of the network structure and only depends on the firms active on a market, see Appendix A.10.

demand is linear, which matches the true demand function. An avenue for future work is to extend this by an additional layer of misspecification and see how nonlinear demand functions affect the learning dynamics. The perceived demand function is given by

$$p_t^{m,j} = a^{m,j} - \frac{\beta}{2} q_t^{m,j} + \upsilon_t^{m,j} \,, \tag{1}$$

where $a > 0$ is an unknown parameter that the firm tries to learn, $\beta > 0$ is the true slope and $\upsilon_t^{m,j}$ is an error term containing variables that affect the price but that the firm has no information on. Firms only take into account their own quantity because of their restricted information set. The only quantity that they observe is their own as they have no way of knowing how much the other firms are producing. Therefore, their best estimate of the price is as a function of their own supply. In an equilibrium, this would also be the correct relationship as then other firms would keep their supply constant at the equilibrium level (at least for one period). I assume here, in line with some of the previous literature (Tuinstra, 2004; Bischi et al., 2007), that the slope is known to the firms. Given that the demand function is linear, the unobserved quantities only affect the perceived intercept and thus learning about the slope would be unaffected by the other firms.

Now, as noted above, firms are not aware of the market size, i.e. they do not know the parameter a of their perceived demand function. Through market exchange, the firms can collect data and extend their information set. Using the data they collect firms estimate these parameters using least squares learning, i.e. they estimate the model in (1) yielding estimates $a_{t-1}^{m,j}$ of the parameter a. Based on this the firm forms a perceived inverse demand function

$$\hat{p}_t^{m,j}(q) = a_{t-1}^{m,j} - \frac{\beta}{2} q \,. \tag{2}$$

The firm problem can thus be described by,

$$\mathbf{q}_t^j = \underset{\{q^{m,j}\}_{m \in \mathcal{M}_j}}{\arg\max} \left[\left(\sum_{m \in \mathcal{M}_j} \hat{p}_t^{m,j}(q^{m,j}) q^{m,j} \right) - \frac{c}{2} \left(Q^j \right)^2 \right] . \tag{3}$$

Aggregate demand is described by a linear inverse demand function

$$P_m \left(Q_t^m \right) = \alpha - \frac{\beta}{2} Q_t^m + \varepsilon_t^m \,, \tag{4}$$

where $\alpha, \beta > 0$, Q_t^m is total production for market m, and, $\varepsilon_t^m \sim \text{iid}\,(0, \sigma_\varepsilon^2)$ is a shock to demand. The demand shock is assumed to be bounded by some constant C.

Total production in a market is given by

$$S_m(a_{t-1}) = \sum_{j \in \mathcal{J}_m} q_t^{m,j} \,, \tag{5}$$

where the quantities depend on firm beliefs and a_{t-1} is a vector containing the parameter estimates of all markets of all firms active in market m (because the quantity supplied to market m of firm j depends not only on the parameter estimate for market m but on the estimates for all markets in which firm j is active, due to the quadratic cost function).

As usual in Cournot oligopoly models, prices are assumed to adjust such that markets clear, i.e.

$$p_t^m = \max\left\{P_m\left(S_m\left(a_{t-1}\right)\right), 0\right\} \,. \tag{6}$$

3 General Network

I will first discuss the general network described by the graph $\mathcal{G} = (\mathcal{M}, \mathcal{J}, E)$, which is given by the biadjacency matrix, G. The whole adjacency matrix is given by

$$H = \begin{pmatrix} \mathbb{0}_M & G \\ G^T & \mathbb{0}_J \end{pmatrix} \,,$$

where G has dimensions $M \times J$ and the m, j-th element g_{mj} is one if there is an edge between the m-th market and the j-th firm. The matrix $\mathbb{0}_i$ is the $i \times i$ matrix of zeros. The adjacency matrix H characterizes the connections between the M markets and J firms. As this is a bipartite graph there are no connections between nodes of the same type, i.e. no connections among markets and no connections among firms.

The graph can be summarized by the matrix G alone. The graph in Figure 1 is described by the following biadjacency matrix

$$G = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \,. \tag{7}$$

For the analytical results, I will assume that all markets are symmetric, i.e. $\alpha^m = \alpha$ and $\beta^m = \beta$ for all m, and all firms have the same quadratic cost function with coefficient $c = 1$.

Firm decisions

The first-order conditions of the firm problem are

$$a^{i,j} - \beta q^{i,j} - \sum_{m \in \mathcal{M}_j} q^{m,j} = 0 \quad \forall i \in \mathcal{M}_j \,. \tag{8}$$

Note that $i \in \mathcal{M}_j \iff g_{ij} = 1$. So we can rewrite the above equation as

$$a^{i,j} - \beta q^{i,j} - \sum_{m=1}^{M} g_{mj} q^{m,j} = 0 \,. \tag{9}$$

This assumes that the firm finds it profitable to produce for all markets it is connected to. If this is not the case then the firm will not produce for some markets and the first-order conditions will only hold for markets it is producing for. This is discussed in Appendix A.1. We can sum over all first-order conditions

$$\sum_{i \in \mathcal{M}_j} \left[a^{i,j} - \beta q^{i,j} - \sum_{m \in \mathcal{M}_j} q^{m,j} \right] = 0$$

$$\sum_{i \in \mathcal{M}_j} a^{i,j} - \beta Q^j - M_j Q^j = 0$$

$$Q^j = \frac{\sum_{i \in \mathcal{M}_j} a^{i,j}}{M_j + \beta} = \frac{\sum_i g_{ij} a^{i,j}}{M_j + \beta} \,, \tag{10}$$

where $M_j = |\mathcal{M}_j|$ is the degree of firm j, i.e. the number of markets firm j is connected to and, similarly, J_m is the degree of market m, the number of firms active on market m.

Plugging aggregate production (10) back in the first-order condition (8) and assuming it holds with equality yields

$$a^{m,j} - \beta q^{m,j} - \frac{\sum_i g_{ij} a^{i,j}}{M_j + \beta} = 0 \,, \tag{11}$$

which we can solve for individual quantities

$$q_t^{m,j} = \frac{2}{\beta} \left(\frac{(M_j + \beta - 1)\, a_{t-1}^{m,j}}{2\,(M_j + \beta)} - \frac{\sum_i (g_{ij} - \delta_{im})\, a_{t-1}^{i,j}}{2\,(M_j + \beta)} \right) , \tag{12}$$

where δ_{im} is Kronecker's delta, defined as

$$\delta_{im} = \begin{cases} 1 & \text{if } i = m \\ 0 & \text{if } i \neq m \end{cases} .$$

Note that we have derived this under the assumption that the firm produces positive quantities for all markets in $\mathcal{M}_j$. This is not necessarily true and the firm may produce only for a subset $\mathcal{M}_j^* \subseteq \mathcal{M}_j$ of markets. This subset can be found iteratively (see Appendix A.1), but for the derivations, I will assume that $\mathcal{M}_j^* = \mathcal{M}_j$.

Quantities are a linear function of firm beliefs and can be split into two components

$$q_t^{m,j} = \frac{2}{\beta} g_{mj} \left(s_j a_{t-1}^{m,j} + t_j \sum_i \left(g_{ij} - \delta_{im} \right) a_{t-1}^{i,j} \right) , \tag{13}$$

where s_j and t_j are defined as follows:

$$s_j = \frac{\left(M_j + \beta - 1 \right)}{2 \left(M_j + \beta \right)} > 0 \tag{14}$$

$$t_j = -\frac{1}{2 \left(M_j + \beta \right)} < 0 \, . \tag{15}$$

The belief about the market under consideration is multiplied by the positive factor s_j, which implies the bigger the firm believes the market to be the more it will produce for it, ceteris paribus. It can be interpreted as the benefit of producing for market m. The sum of beliefs of all other markets, i.e. the total market size apart from market m, is multiplied by t_j. This can be interpreted as the implicit cost of not producing for one of the other markets, which is a result of the quadratic cost function. Also note that $s_j = t_j + \frac{1}{2}$, which we will use to keep notation more concise.

Increasing production for market m will increase the marginal costs for all markets. If marginal revenue equaled marginal costs for the remaining markets before the production increase this implies that now marginal costs are higher and the production for the other markets has to be decreased. Adding a link to another market cannot decrease total production of a firm and cannot increase production for already connected markets. If production for the new market is positive, marginal costs have increased and thus the production for the other markets has to decrease in comparison to before. The decrease has to be no bigger than the increase in production for the new market, as adding a new market allows the firm to spread its total production over more markets, which increases the marginal revenue on each market, assuming demand is downward sloping, and thus can sustain a higher equilibrium aggregate production. Aggregate consumer surplus then increases as the network becomes more connected as the total production increases. The distribution of the aggregate consumer surplus changes.

We are going to proceed in steps: first starting with the quantity choice of a fully connected firm, i.e. a firm connected to all markets, and then adjusting it to arrive at the general formulation. If the firm is not connected to all markets then beliefs about markets it is not connected to do not influence its decision and the quantity it produces for a market it is not connected to is zero. For this adjustment, we need to define the projection matrix, $\operatorname{diag} G_j$, which is a diagonal matrix with the j-th column of G on the diagonal. This matrix allows us to define base matrices that represent a fully connected firm and simply "turn off" the markets that the firm is not connected to.

Quantities of fully connected firms can be written as

$$q_t^{m,j} = \frac{2}{\beta} g_{mj} \left(\sum_i g_{ij} t_j a_{t-1}^{i,j} + \frac{1}{2} a_{t-1}^{m,j} \right) , \tag{16}$$

where $g_{ij} = 1 \; \forall i$ as the firm is connected to all markets. In vector notation we can write

$$q_t^{m,j} = \frac{2}{\beta} \left(t_j \mathbf{1} + \frac{1}{2} \mathbf{e}_m \right)^\top a_{t-1}^j , \tag{17}$$

where $\mathbf{1}$ is the vector of ones, usually assumed to be in $\mathbb{R}^M$ but on occasion it may be adapted to the size of the matrix being multiplied, and $\mathbf{e}_m \in \mathbb{R}^M$ the standard unit vector in the m-th dimension.

The sum in Equation (16) is over all markets the firm is connected to. In the vectorized version in Equation (17) the sum $t_j \mathbf{1}^\top a_{t-1}$ is over all markets because for a fully connected firm $\mathcal{M}_j = \mathcal{M}$. For the general formulation, we will multiply the above expression with the projection matrix $\operatorname{diag} G_j$, which will turn off the markets that the firm is not connected to. So, $t_j \left(\operatorname{diag} G_j \mathbf{1} \right)^\top a_{t-1} = t_j G_j^\top a_{t-1}$ only sums over beliefs of markets the firm is connected to.

The last thing to consider is that if firm j is not connected to market m then its production for market m is zero. This is why we will multiply the above expression with the $m, j-$th element of the biadjacency matrix, g_{mj}, which is one if the firm is connected to market m and zero otherwise. Then, we arrive at the final expression for the quantity choice of a firm

$$q_t^{m,j} = \frac{2}{\beta} g_{mj} \left(\operatorname{diag} G_j \left(t_j \mathbf{1} + \frac{1}{2} \mathbf{e}_m \right) \right)^\top a_{t-1}^j . \tag{18}$$

We will call

$$u^{m,j} = g_{mj}\left(\operatorname{diag} G_j \left(t_j \mathbf{1} + \frac{1}{2}\mathbf{e}_m\right)\right)^T$$

the aggregation mapping which aggregates the beliefs of firm j to arrive at production for market m, $q_t^{m,j} = \frac{2}{\beta} u^{m,j} a_{t-1}^j$.

Equation (18) represents the quantity choice of a particular market-firm combination. We can further vectorize this problem by stacking all quantities of a firm on top of each other

$$q_t^j = \begin{pmatrix} q_t^{1,j} \\ \vdots \\ q_t^{M,j} \end{pmatrix} = \frac{2}{\beta} \begin{pmatrix} u^{1,j} \\ \vdots \\ u^{M,j} \end{pmatrix} a_{t-1}^j \,. \tag{19}$$

We can write this in matrix form as

$$q_t^j = \frac{2}{\beta} L_j a_{t-1}^j \,, \tag{20}$$

where

$$L_j = \operatorname{diag} G_j K_j \operatorname{diag} G_j \,, \tag{21}$$

and

$$K_j = \left(t_j \mathbb{1} + \frac{1}{2} I\right) \,, \tag{22}$$

with $\mathbb{1}$ the matrix of all ones. Here K_j is again the default matrix for a fully connected firm. Multiplying it with $\operatorname{diag} G_j$ from the right turns off the beliefs of markets the firm is not connected to and multiplying it with $\operatorname{diag} G_j$ from the left sets production for markets the firm is not connected to zero.

The matrix L_j aggregates the beliefs of firm j to arrive at production for all markets. By combining the L_j matrices in a block diagonal matrix we can write the whole system as

$$q_t = \frac{2}{\beta} \begin{pmatrix} L_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & L_J \end{pmatrix} a_{t-1} \,. \tag{23}$$

Dynamics of firm beliefs

Next, we describe the price formation process. Firms believe[4] prices are determined (if nonnegative) according to

$$\hat{p}_t^m = a_{t-1}^{m,j} - \frac{\beta}{2} q_t^{m,j} \quad m \in \mathcal{M}_j \,, \tag{24}$$

while price are actually determined by

$$p_t^m = \alpha - \frac{\beta}{2} \left(\sum_{j \in \mathcal{J}_m} q_t^j \right) + \varepsilon_t^m \,. \tag{25}$$

From this, it can be seen that there is misspecification in the firms' model of the economy as they only have information on their quantities and base their price predictions on this information whereas the actual price depends on the quantities of all firms, i.e. they only use the subset of the explanatory variables that they have access to.

Firms update their beliefs recursively using recursive least squares estimation,

$$a_t^{m,j} = a_{t-1}^{m,j} + \frac{1}{t} \left(p_t^m + \frac{\beta}{2} q_t^{m,j} - a_{t-1}^{m,j} \right) \,. \tag{26}$$

Based on the firm's perceived price formation process and an observation of the actual price p_t^m and quantity $q_t^{m,j}$ the firm infers the realized market size $\tilde{a}_t^{m,j}$ as

$$\tilde{a}_t^{m,j} = p_t^m + \frac{\beta}{2} q_t^{m,j} \,. \tag{27}$$

This is the market size belief that would be consistent with the realized price and chosen quantity. The expected market size is given by $a_{t-1}^{m,j}$. The firm thus increases its estimate of $a^{m,j}$ if the realized price is higher than expected and decreases it if the realized price is lower than expected.

Using the actual price formation process we can rewrite the above equation as

$$a_t^{m,j} = a_{t-1}^{m,j} + \frac{1}{t} \left(\alpha - \frac{\beta}{2} \left(\sum_{i \in \mathcal{J}_m \setminus j} q_t^{m,i} \right) - a_{t-1}^{m,j} + \varepsilon_t^m \right)$$

[4]We can interpret this as firms believing that prices are determined this way but we can also equally well interpret it as firms being aware that this equation is misspecified but given that their information set does not include more they use this approximation.

$$= a_{t-1}^{m,j} + \frac{1}{t}\left(\alpha - \sum_{i\in\mathcal{J}_m\setminus j}\left(\operatorname{diag} G_j\left(t_j\mathbf{1} + \frac{1}{2}\mathbf{e}_m\right)\right)^\top a_{t-1}^i - a_{t-1}^{m,j} + \varepsilon_t^m\right). \tag{28}$$

This assumes that quantity choices and shock realizations are such that prices are not at the zero bound. For reasonable initial values a_0 and support of the shocks this is the case.

Equation (28) is the updating rule for one market-firm combination. We will next vectorize it and write the whole learning process in matrix form.

To do so we introduce the operator $\operatorname{vec}(C)$ which transforms a matrix to a vector by stacking the columns on top of each other, and $B \circ C$ is the Hadamard, or componentwise, product which for two matrices is defined as

$$(B \circ C)_{ij} = B_{ij}C_{ij}\,. \tag{29}$$

Lastly, for two vectors $u \in \mathbb{R}^K$ and $v \in \mathbb{R}^L$, $\otimes$ is the Kronecker product, defined as

$$u \otimes v = \begin{pmatrix} u_1 v \\ \vdots \\ u_K v \end{pmatrix} \in \mathbb{R}^{KL}\,. \tag{30}$$

Let us first consider the learning process for firm one, which corresponds to the first M rows in the matrix form of the learning process. The learning process for firm one is given by

$$a_t^1 = \begin{pmatrix} a_t^{1,1} \\ \vdots \\ a_t^{M,1} \end{pmatrix} = a_{t-1}^1 + \frac{1}{t}\left(\alpha G_1 - \sum_{j\in\mathcal{J}\setminus\{1\}} \operatorname{diag} G_1 L_j a_{t-1}^j - a_{t-1}^1 + \varepsilon_t \circ G_1\right). \tag{31}$$

This system of equations follows directly from Equation (28) by stacking the equations for all markets on top of each other, apart from multiplying the terms with G_1, the first column of G or $\operatorname{diag} G_1$, the diagonal matrix with G_1 on the diagonal. The reason for this is that if firm one is not connected to a particular market n its belief about the market size of that market is constant at zero. By multiplying the quantities of the other firms with $\operatorname{diag} G_1$ we ensure that the production of the other firms who may be connected to market n and thus produce something for market n does not affect the beliefs of firm one who is not connected to this market.

By stacking the difference equations for all firms we can write the learning process in matrix form as

$$a_t = a_{t-1} + \frac{1}{t}\left(\alpha \operatorname{vec} G - A a_{t-1} + \mathcal{E}_t\right) , \tag{32}$$

where

$$A = \operatorname{diag} G \,(L + I) \in \mathbb{R}^{JM \times JM} , \tag{33}$$

$$L = \begin{pmatrix} 0 & L_2 & \dots & L_J \\ L_1 & 0 & \dots & L_J \\ \vdots & \vdots & \ddots & \vdots \\ L_1 & L_2 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{J \times J} , \tag{34}$$

and,

$$\mathcal{E}_t = (\mathbf{1} \otimes \varepsilon_t) \circ \operatorname{vec} G . \tag{35}$$

Again multiplying with $\operatorname{diag} G \equiv \operatorname{diag} \operatorname{vec} G$ ensures that if firm i is not connected to a market the dynamics of firm i's belief for this market are constant at 0. The last term in Equation (32) is the noise component and is zero in expectations.

Equation (32) is a stochastic difference equation and thus has no steady state. However, by taking expectations we can find the steady state of the deterministic part of the system. The steady state is implicitly determined by $\alpha \operatorname{vec} G - A\bar{a} = 0$. If the network is not complete then A is not invertible as there are some rows and columns that are all zeros. These rows and columns correspond to the missing market-firm connections, i.e. where $g_{mj} = 0$. Again, the corresponding rows of A are zero because $a_t^{m,j} = 0 \; \forall t$ if firm j is not connected to market m and the corresponding columns are zero because a firm's belief about a market it is not connected to does not affect other firms' beliefs who are connected to it. As the beliefs corresponding to the missing connections are constant at zero we can simply remove the corresponding rows and columns from A and $\alpha \operatorname{vec} G$ and solve the system for the remaining variables. See Appendix A.2 for a proof of the invertibility of the reduced matrix A^{-}.

It can be shown that the steady-state beliefs solve the same first-order condition as the quantities in a Cournot-Nash equilibrium and thus, quantities at the steady-state beliefs are equal to the Cournot-Nash equilibrium quantities (see Appendix A.3).

We can rewrite the system dynamics in terms of deviations from the steady state – $\hat{a}_t = a_t - \bar{a}$ – as

$$\hat{a}_t = \hat{a}_{t-1} - \frac{1}{t}\left(A\hat{a}_{t-1} - \mathcal{E}_t\right) . \qquad (36)$$

To keep the notation simple I will drop the hat from now on and assume that all variables are in deviations from the steady state.

Stochastic approximation and analysis

Proposition 3.1. *The system dynamics in Equation* (36) *can be approximated by the ordinary differential equation*

$$\dot{a} = -Aa \, .$$

In particular, if $\bar{a}$ is a globally stable steady state of the ordinary differential equation then a_t will converge to $\bar{a}$ almost surely from any starting point. In addition if $a(\tau)$ is the solution to the ordinary differential equation then

$$a_t \approx a(\tau) \ ,$$

where the continuous time τ is related to the step-size $\frac{1}{t}$ by

$$\tau = \sum_{k=1}^{t} \frac{1}{k} \approx \int_1^k \frac{1}{k} \, dk = \log t \, .$$

Proof. This follows from results in the stochastic approximation literature. The stochastic algorithm in Equation (36) satisfies conditions A.1-A.5 in Benveniste et al. (1990) and thus the system dynamics can be approximated by the ordinary differential equation in Equation (37).

□

Thus, we can analyze the dynamics of the system by studying the eigenvalues and eigenspaces of the matrix A.

The solution to the differential equation is given by

$$a(\tau) = \sum_i^N c_i e^{-\lambda_i \tau} v^i \, ,$$

where the c_i are constants depending on the initial condition and N is the number of eigenvalues. This assumes that the matrix A is diagonalizable and thus has MJ linearly independent eigenvectors. I do not show that this is the case and therefore the solution of the differential equation may require generalized eigenvectors. This has no impact on the results.

In addition, the dynamics of each eigenvector component are governed by the corresponding eigenvalue, i.e.

$$v^i(\tau) = c_i e^{-\lambda_i \tau} v^i \,,$$

or in discrete time

$$v_t^i = c_i t^{-\lambda_i} v^i \,.$$

Let us consider the example in Figure 1. Then $A \in \mathbb{R}^{6\times 6}$ and $L_j \in \mathbb{R}^{2\times 2}$ for all j. The belief vector is structured as

$$a_t = \begin{pmatrix} a_t^1 \\ a_t^2 \\ a_t^3 \end{pmatrix} \,,$$

where $a_t^j \in \mathbb{R}^2$ is the belief vector of firm j. Firm one is not connected to market two and firm three is not connected to market one. Therefore, the corresponding beliefs are zero for all periods.

The L_j are given by

$$L_1 = \begin{pmatrix} s_1 & 0 \\ 0 & 0 \end{pmatrix} \,, \quad L_2 = \begin{pmatrix} s_2 & t_2 \\ t_2 & s_2 \end{pmatrix} \,, \quad L_3 = \begin{pmatrix} 0 & 0 \\ 0 & s_3 \end{pmatrix} \,, \tag{37}$$

and A is given by

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{\operatorname{diag} G} \underbrace{\begin{pmatrix} 1 & 0 & s_2 & t_2 & 0 & 0 \\ 0 & 1 & t_2 & s_2 & 0 & 0 \\ s_1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & s_3 \\ 0 & 0 & s_1 & t_1 & 1 & 0 \\ 0 & 0 & t_1 & s_1 & 0 & 1 \end{pmatrix}}_{L+I}$$

$$= \begin{pmatrix} 1 & 0 & s_2 & t_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ s_1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & s_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_1 & s_1 & 0 & 1 \end{pmatrix} . \tag{38}$$

The second row determines the dynamics of $a_t^{2,1}$. As firm one is not connected to market two this belief is zero. By considering only the matrix $L+I$ we can see that $a_t^{2,1}$ would not remain constant at zero but would be driven by the dynamics of $a_{t_1}^{1,2}$ and $a_{t_1}^{2,2}$. However, as firm one is not connected to market two we multiply from the left with $\operatorname{diag} G$ ensuring that the corresponding row in A is zero and thus $a_t^{2,1}$ remains constant at zero.

Having sufficiently defined the problem we can start with the analysis. We will proceed in steps: as can be seen from Equation (33), the learning process is strongly tied to the L matrix. Therefore, we will first analyze the properties of L and then move on to the properties of A. The analysis of L will be split into two parts: first, we will analyze the properties of the L_j and then see what these properties imply for L itself. Our goal is to characterize the eigenspaces of A as the eigenvalues determine the stability of the system and the eigenvectors tell us what kind of information is learned at what speed.

It will be useful to define a measure of the density of the network. In particular, I will consider the total number of connections in the network, which is given by

$$D = \mathbf{1}^\top G \mathbf{1} = \sum_{j=1}^{J} M_j = \sum_{m=1}^{M} J_m \, .$$

For the following results we will need two definitions:

Definition 3.1 (Weak connectivity)**.** A network of markets is weakly connected if $D > M + J - 1$

which is a lower bound on the density of the network, i.e. the number of connections, and

Definition 3.2 (Strong connectivity)**.** A network of markets is strongly connected if $M_j \geq 2 \quad \forall j$ and $J_m \geq 2 \quad \forall m$, which means that each firm is active in at least two markets and each market has at least two firms active in it.

We show in Appendix A.4 that the eigenvalues of A are nonnegative. There is an eigenvalue $\lambda_1 = 0$ associated with the eigenvectors in the kernel of $\operatorname{diag} G$. These eigenvectors correspond to learning of firms about markets they are not connected to and thus remain constant at zero.

Proposition 3.2. *For any weakly connected market network, the smallest nonzero eigenvalue of A is $\lambda_2 = \frac{1}{2}$. The associated eigenspace has dimension $D-(M+J-1)$.*

Proof. Proof in Appendix A.4. □

Knowing this eigenvalue allows us to conclude that at least some firm beliefs converge at a rate equal to $-\frac{1}{2}$ for any weakly connected network.

The eigenspace E_{λ_2} consists of vectors

$$v = \begin{pmatrix} v^1 \\ \vdots \\ v^J \end{pmatrix} \in \mathbb{R}^{MJ}$$

with the property that if we were to reshape v by taking the J components v^j of size M corresponding to each firm and stack them side-by-side in a matrix

$$V = \begin{pmatrix} v^1 & \dots & v^J \end{pmatrix} \in \mathbb{R}^{M \times J},$$

then v needs to be in two subspaces

$$\begin{aligned} W_1 &= \{v \mid V\mathbf{1} = 0\} \\ W_2 &= \left\{v \mid V^\top \mathbf{1} = 0\right\}. \end{aligned}$$

So, the vector of ones is both in the left- and right-nullspace of V. This means that the rows and columns of V sum to zero. The columns of V correspond to the beliefs of a firm about all markets and the rows correspond to the beliefs of all firms about a single market. So, the rows of V summing to zero mean that in this eigenvector component of a aggregate beliefs of firms are correct and the columns summing to zero mean that in this eigenvector the aggregate beliefs of markets are correct. Economically, this means that the hardest learning task is for firms to distribute an, in aggregate, correct amount between markets.

Combining the result of Proposition 3.2 with the learning problem leads us to the following result.

Theorem 1 (Individual Learning)**.** *For any strongly connected network quantities converge polynomially at a rate of* $-\frac{1}{2}$ *to the steady state values.*

Proof. Proof in Appendix A.4. □

So, for any strongly connected network, the individual quantities converge at the same rate regardless of the topology of the network. This includes among others the complete network, the star network, the circle network, and the tree network.

Theorem 2 (Informational Efficiency)**.** *The Cournot market mechanism aggregates information such that aggregate production converges at a faster rate than individual production both within markets and within firms. Prices are determined by aggregate production and are thus also learned at the faster rate.*

Proof. Proof in Appendix A.4. □

The results of Theorems 1 and 2 can be interpreted as follows: in a market that sends out signals that are informative about the aggregate state, in this case through the price, aggregate quantities are learned relatively faster, while individual quantities are learned slowly. One could thus conclude that markets are learning fast and market outcomes converge fast, but given that the Cournot market is restricted to aggregate signals it is less efficient in ensuring that also individual quantities are learned at the same rate.

These results are valid for strongly connected networks. For weakly connected networks some quantities will converge at the same rate as aggregate quantities and some will converge at the rate $-\frac{1}{2}$. If the network is also not weakly connected then it is rather sparse with few connections. In this case, the convergence rate of all individual quantities will be faster than $-\frac{1}{2}$.

A common learning process with more informational requirements is the best-response dynamics. Here, the demand function as well as the previously chosen quantities of the other firms are assumed to be known. In the simplest case, firms have naive expectations, i.e. they assume that the quantities of the other firms in the next period are equal to the previously chosen quantities. Then, firms choose their quantities as a best response to the expected quantities of the other firms. It is known that the steady state under best-response dynamics can be unstable (Theocharis, 1960; Fisher, 1961; al-Nowaihi & Levine, 1985) if speeds of adjustment are too large and there are too many firms.

If the system is stable, the increased information environment is also reflected in the convergence properties. Quantities under the best-response dynamics converge exponentially fast to the steady state (see Appendix A.7 for details). Numerical results confirm the intution that the aggregate price signal is, in combination with the previous quantities of the other firms, informative of the direction of change of individual quantities. Thus, the convergence rate of individual quantities is the same as the convergence rate of aggregate quantities. This is in contrast to the previously discussed case where the aggregate price signal is informative of the aggregate quantities but not of the individual quantities.

We can look at some particularly regular networks to gain some more analytical insights into the learning dynamics.

4 Networks with a special structure

4.1 Single Cournot market

Let us first consider the case of a single market oligopoly for exposition, i.e. $|\mathcal{M}| = 1$, $|\mathcal{J}| = J$, and $G = \mathbf{1}^\top \in \mathbb{R}^{1\times J}$. Note that $D = \sum_j M_j = J$ so that this network is neither strongly nor weakly connected and thus Theorem 1 does not apply.

Assuming homogenous[5] quadratic costs with the multiplicative coefficient equal to one, profit-maximizing quantities are given by

$$\tilde{q}_t^j = \frac{a_{t-1}^j}{1+\beta}\,. \tag{39}$$

Assuming $J \geq 2$, A now takes the simple form

$$A = \begin{pmatrix} 1 & \frac{\beta}{2(1+\beta)} & \cdots & \frac{\beta}{2(1+\beta)} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\beta}{2(1+\beta)} & \frac{\beta}{2(1+\beta)} & \cdots & 1 \end{pmatrix}, \tag{40}$$

and we can characterize its eigenspaces analytically.

Proposition 4.1. *In a single market oligopoly with at least two firms A has two eigenvalues, $\lambda_1 = 1 - \frac{\beta}{2(1+\beta)}$ and $\lambda_2 = 1 + (J-1)\frac{\beta}{2(1+\beta)}$ with corresponding eigenspaces,*

[5] Allowing for heterogeneous quadratic costs does not change the result but makes it hard to characterize the convergence dynamics analytically, see Appendix A.11.

$E_{\lambda_1} = \mathbf{1}^\perp$, the distributive learning component, and $E_{\lambda_2} = \operatorname{span}\mathbf{1}$, the aggregate learning component. The distributive learning component preserves the aggregate belief but moves some of it from one firm to the other. The aggregate learning component changes the aggregate belief by uniformly changing individual beliefs.

Let us consider the case $J = 2$ for intuition. Then $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \mathbf{1}$. v_2 moves the beliefs in the same direction, either increasing or decreasing them. It can be interpreted as the aggregate learning component, i.e. should firms produce more or less in aggregate. In contrast, v_1 keeps aggregate beliefs constant and only moves some from one firm to the other. It corresponds to the distribution of aggregate production between firms.

In the single market case, $u = \frac{1}{1+\beta}\mathbf{1}^\top$ which is obviously in the kernel of $E_{\lambda_1} = \mathbf{1}^\perp$ and so aggregate quantities do indeed converge faster.

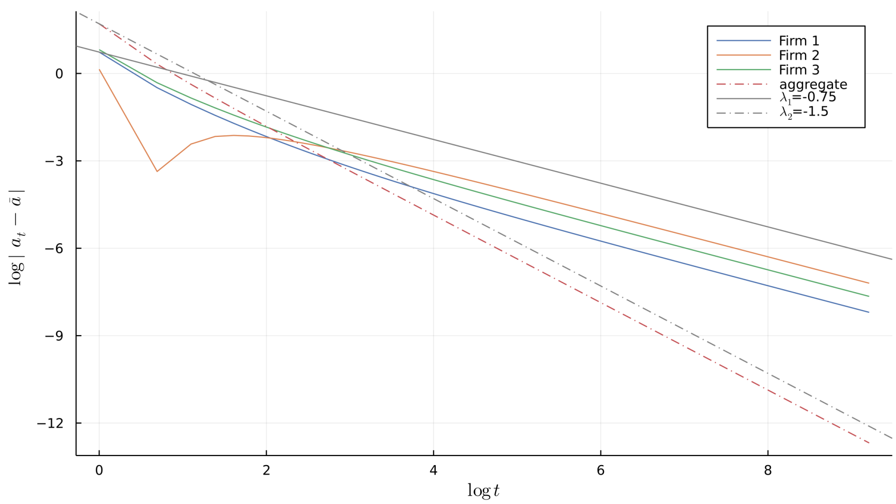

Figure 2: Log-deviations of aggregate and individual beliefs from the equilibrium values for homogenous cost functions.

As can be seen in Figure 3 if shocks are added to the model, aggregate beliefs converge at the rate predicted by the smaller eigenvalue until the magnitude of the shocks becomes larger than the deterministic reduction in the deviations from the steady-state. This is intuitive as markets, via prices, provide signals to firms allowing for faster convergence. If this signal becomes more noisy, efficiency decreases.

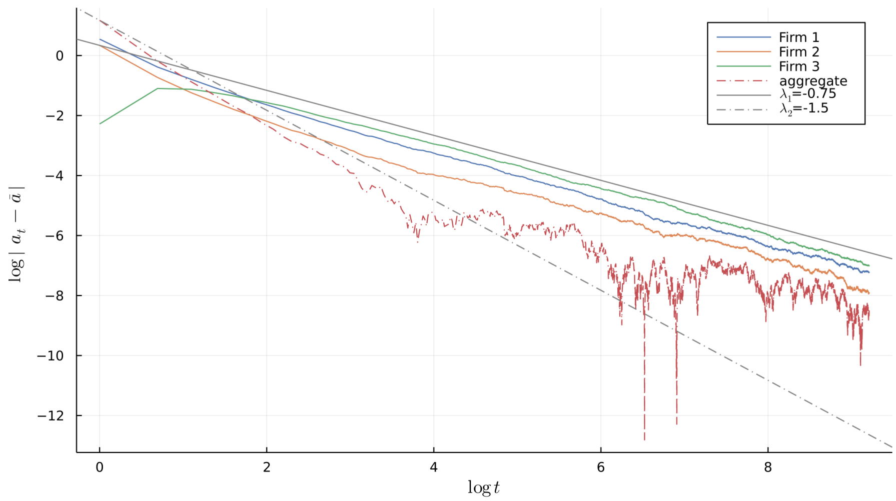

Figure 3: Log-deviations of aggregate and individual beliefs from the equilibrium values for homogenous cost functions with shocks.

4.2 Complete Network

Let us consider the complete network as the second special case: the learning process is the same as in section 3 but with $G = \mathbb{1}$ and $\operatorname{diag} G = I$, which simplifies the expressions. Now, $L_j = K_j = K \ \forall j$. Note also that the complete network is a strongly connected network.

In matrix notation, we have (in expectation)

$$a_t = a_{t-1} + \frac{1}{t}\left(\alpha \mathbf{1} - A a_{t-1}\right) , \tag{41}$$

where

$$A = \begin{pmatrix} I & K & \dots & K \\ K & I & \dots & K \\ \vdots & \vdots & \ddots & \vdots \\ K & K & \dots & I \end{pmatrix} , \tag{42}$$

and

$$K = \begin{pmatrix} s & t & \dots & t \\ t & s & \dots & t \\ \vdots & \vdots & \ddots & \vdots \\ t & t & \dots & s \end{pmatrix} , \tag{43}$$

where

$$s = \frac{(M-1)+\beta}{2\,(M+\beta)} \tag{44}$$

$$t = -\frac{1}{2\,(M+\beta)}\,. \tag{45}$$

We can solve for the steady-state belief, which is the same for each firm,

$$\bar{a} = \frac{2\alpha\,(M+\beta)}{2M+(J+1)\,\beta}\,,$$

and at this belief quantities correspond to the Cournot-Nash equilibrium.

Because of the symmetry of the matrices in this special case, we can say a lot about the eigenspaces of A.

Proposition 4.2. *In a complete network, there are four eigenspaces, which are distinguished by which entities – markets or firms – hold the correct beliefs in aggregate, e.g. a firm has the correct sum of all market sizes but the distribution between markets is incorrect. We can distinguish between the following cases:*

1. *Neither firms nor markets have the correct aggregate beliefs:* $\lambda_2 = \frac{2M+(J+1)\beta}{2(M+\beta)}$.
2. *Firms have the correct aggregate beliefs but markets do not:* $\lambda_1 = \frac{J+1}{2}$.
3. *Firms do not have the correct aggregate beliefs but markets do:* $\lambda_4 = \frac{2M+\beta}{2(M+\beta)}$.
4. *Both firms and markets have the correct aggregate beliefs:* $\lambda_3 = \frac{1}{2}$.

Proof. The proof is given in Appendix A.6. □

Using the same arguments as in the previous sections we can show that the aggregation vector is orthogonal to the component where both firms and markets have the correct aggregate beliefs and the component where markets have the correct aggregate beliefs but firms do not. Surprisingly, the component with the fastest convergence rate is the one where firms have the correct aggregate beliefs but markets do not, and not the component where nothing is correct.

5 Numerical comperative statics for standard networks

In this section, we will see how the convergence speed of aggregate beliefs, as determined by the largest eigenvalue not orthogonal to the aggregation vector, varies from network to network as well as how it depends on the network size and the slope parameter.

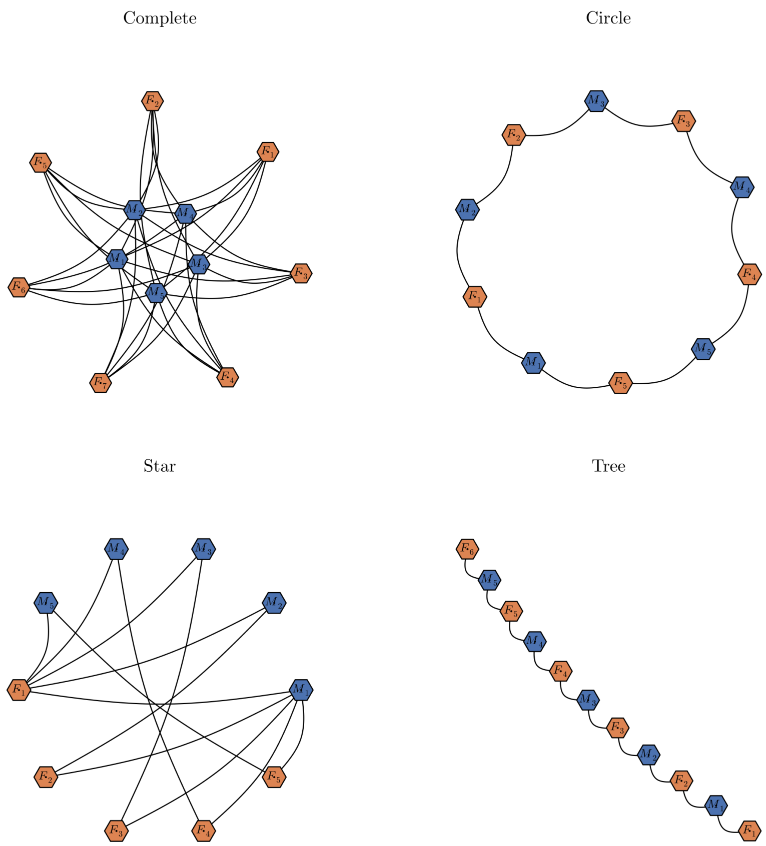

Figure 4: Example graphs.

We can compare the convergence speeds of aggregate quantities for some standard networks.

In Figure 5, we can see that convergence is fastest for the complete network (here $J = M$) and slowest for the tree and circle network (here every firm is connected to

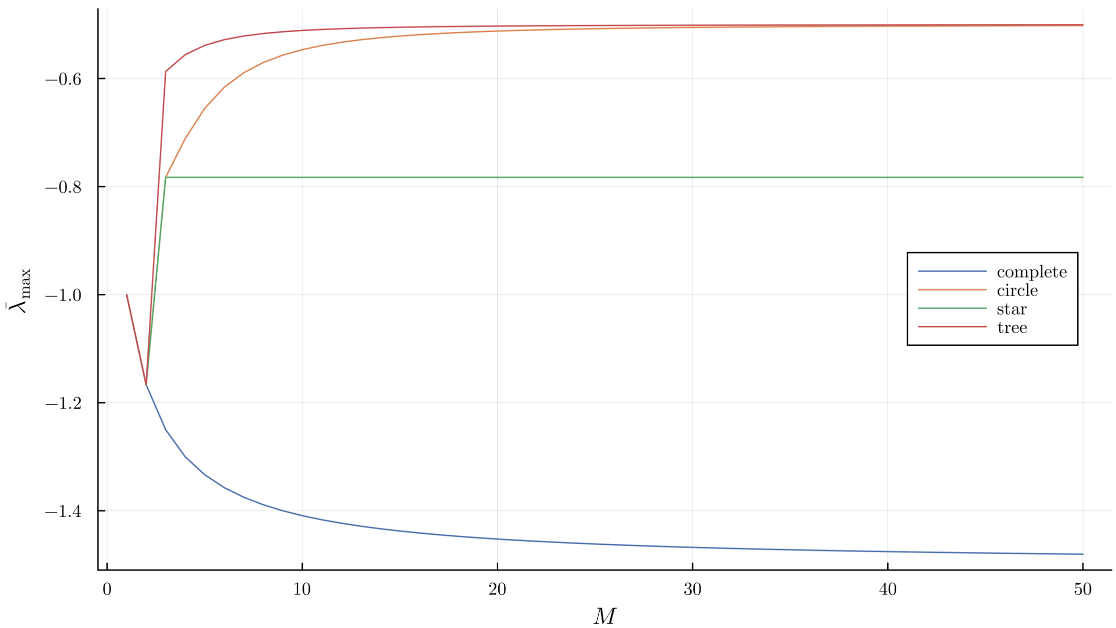

Figure 5: Comparison of convergence speeds for specific networks as a function of network size.

two markets). The star network is in between with a constant convergence speed, which is due to the leaf markets converging at a slower rate than the center market and the number of edges of the leaf markets being constant and thus independent of the network size.

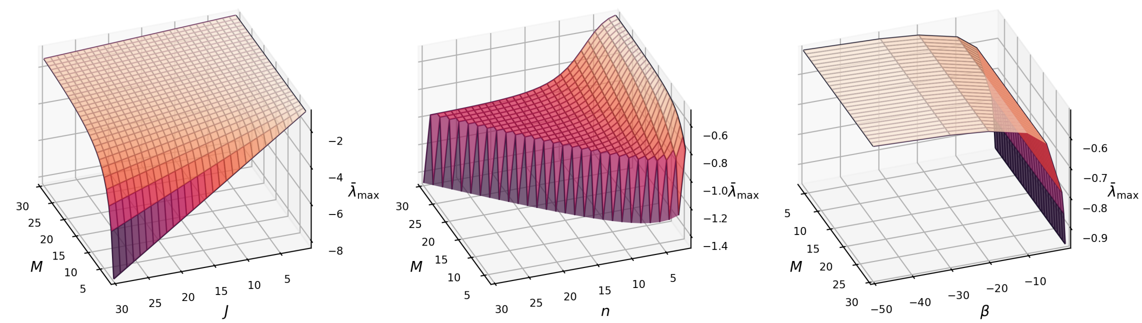

Figure 6: Convergence speeds as a function of network size, number of connections, and slope parameters in the complete, circle, and star network, respectively.

In the left panel of Figure 6 we can see how the convergence speed depends on the number of firms and markets in a complete network. Convergence is fastest for a small number of markets and a large number of firms. For a fixed large number of firms convergence speed initially decreases sharply as the number of markets increases, but then increases rather slowly thereafter. For a smaller number of firms, this effect is similar but less pronounced as convergence speed starts at a higher level. For a fixed number of markets convergence speed decreases linearly as the number of firms increases. This decrease is stronger for a small number of markets.

In the center panel of Figure 6 we can see how the convergence speed depends on the number of markets/firms in a circular network. Convergence is slowest in the standard circle network where each firm is connected to two markets and each firm shares each market with exactly one firm. As we increase the number of connections each firm has convergence speed initially increases sharply and then tapers off before a sharp drop when the network becomes fully connected.

In the right panel of Figure 6 we see again that in the star network, the network size does not affect convergence speed but convergence speed decreases with the steepness of the demand function.

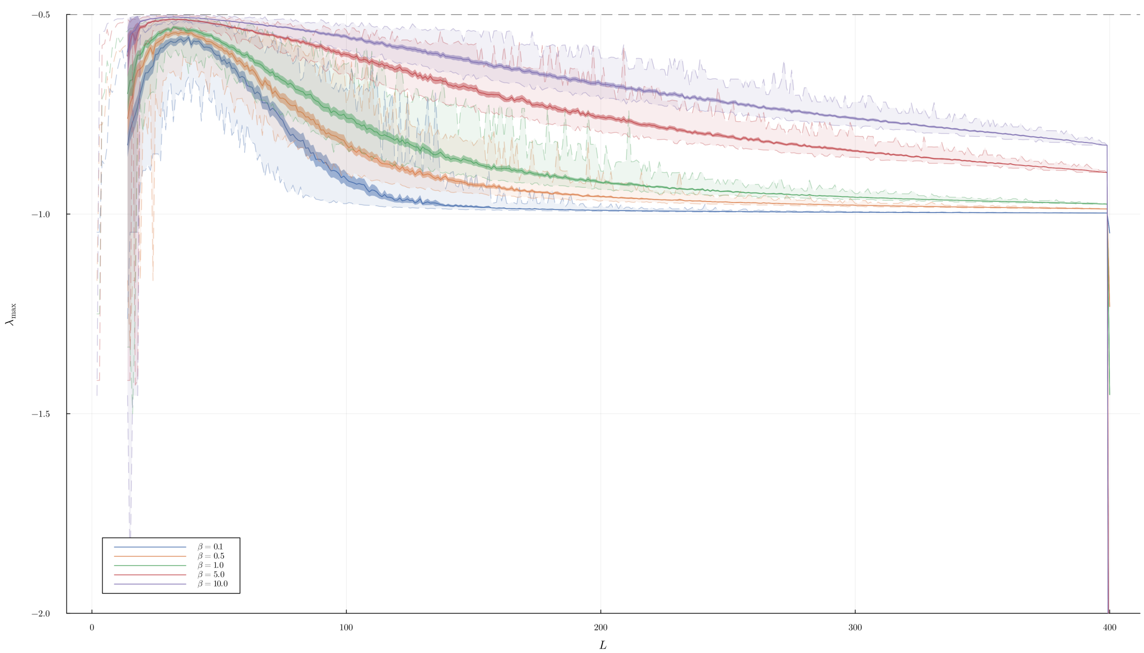

Figure 7: Average convergence speeds of an Erdős-Rényi graph with 20 markets and firms for a varying number of connections. Darker shades indicate the 99% confidence interval and lighter shades indicate the maximum and minimum values.

Figure 7 shows the average slowest[6] aggregate convergence speed, the 99% confidence interval, and the minimal and maximal values of 100 draws of an Erdős-Rényi graph where each graph $G(N, L)$, i.e. each graph with N nodes and L edges is chosen with equal probability[7]. This allows us to get an idea of the effect the number of edges have on convergence speed without imposing any structure on the graph as is the case with more standard, symmetric networks. We have seen in Figure 5 that the complete network converges faster than the other standard networks, which have fewer edges. We also observe faster convergence in a circle network when increasing the number of connections as in Figure 6. In Figure 7 we can see that this relationship is nonlinear: for a small number of edges convergence speed is relatively high as markets are disconnected from each other and the network consists mainly of single market-firm pairs. As the number of edges increases, convergence speed decreases as the network becomes more connected and information needs to be transmitted between markets. However, much of this information is indirect effects of markets that are not directly connected. As the number of edges increases further, convergence speed increases again as the network becomes more connected and information can be transmitted more directly between markets. Also here we observe that convergence speed is slower for a steeper slope parameter β, except for the special case of the complete network where the opposite is true.

6 Conclusion

In this paper, I analyze how agents learn about the market size in a limited information Cournot oligopoly model. I find that steady-state beliefs are stable for all networks. I characterize convergence speeds for any network topology and decompose the learning process into economically interpretable components. In particular, I show that aggregate quantities converge at a faster rate which depends on the network structure, and that individual quantities take longer to converge. For a subclass of networks that have a sufficient number of connections, this rate is polynomial with coefficient $-\frac{1}{2}$, which is independent of the network structure or the slope of demand. I decompose the learning process into different regimes corresponding to whether aggregate beliefs are correct or not. The distribution of a correct aggregate amount, both within firms and across markets, is learned the slowest and drives the slow convergence of individual quantities. A reason for this could be that the Cournot market mechanism gives signals related to aggregate quantities via the prices. Therefore, it

[6] Aggregate convergence speeds may differ across the network if the graph is disconnected.

[7] The y-values for $\beta = -5$ and $\beta = -10$ for the complete network case extend outside the graph.

may be expected that the convergence speed of aggregate quantities is faster than that of individual quantities as an individual firm does not receive a direct signal related to its individual quantity. It may be interesting to see how this result changes if the market mechanism is changed. The rate of convergence of aggregate quantities is nonlinearly related to the number of connections in the network. As markets are isolated with few connections, convergence speed is relatively fast. Convergence speeds then decrease as firms become connected but many of these connections are indirect which means that information takes longer to spread through the network. Convergence speeds increase again as the number of firms directly connected to many markets increases, with the complete network having the fastest convergence rate. For not fully connected networks a larger slope of the inverse demand function leads to slower aggregate convergence as prices are more sensitive to small changes in quantities. As firms are not connected to all markets they may find it difficult to distinguish what caused the price change and thus take longer to learn the correct quantities. This effect is not present in fully connected networks as all firms are connected to all markets and thus have a better picture of what caused the price changes. A more sensitive inverse demand function then leads to faster convergence as for a given deviation of quantities from the steady state, price expectations and realized prices are further apart, which provides a stronger signal for firms to adjust their beliefs.

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A Appendix

A.1 Proof of how to find the set of active markets

Proof. This is an optimization problem with nonnegativity constraints so we can look at the Karush-Kuhn-Tucker conditions. The conditions are

$$\frac{\partial \pi\left(q_t^*, a_t\right)}{\partial q_t^i} \leq 0 \quad q_t^i \geq 0 \quad \frac{\partial \pi\left(q_t^*, a_t\right)}{\partial q_t^i} q_t^i = 0 \quad \forall i \in \mathcal{M}_j \,.$$

The first condition is equivalent to

$$a^i - \beta q^i - Q \leq 0 \quad \forall i \in M_j \,.$$

Let us fix Q at the unconstrained optimal value. Not taking the nonnegativity constraints into account, the smaller a^i is the smaller q^i is. Without loss of generality let the a^m be ordered from largest to smallest. If the constraints are not satisfied we can set all $q^i < 0$ equal to zero and remove them from $\mathcal{M}_j$ and repeat the procedure. Why can we remove all of them? Let us assume we only remove one at a time. Then removing $q^M < 0$ increases Q and q^{M-1} will be even more negative (assuming it was negative before), so we can remove it as well. However, removing the first set of markets is not sufficient. Assuming q^N was the quantity with the smallest a^N that was still positive then by the same argument removing the $N+1$-th to M-th markets increases Q, so Q^N which may have been positive when the other Q were still included may no longer be positive. So we optimize again but only using markets $1, \ldots, N$. If the constraints are not satisfied we remove the next set of markets and repeat the procedure. We continue this until all constraints are satisfied. We know that the constraints are satisfied at least for one market as if all $q^i = 0 \quad i \neq 1$, then $Q^{-1} = 0$ and we can choose $q^1 = \frac{a^1}{2(1+\beta)}$. Can there be positive production for more than one market? Let us consider two markets, then

$$\begin{aligned} a^1 - (1+\beta)\, q^1 - q^2 &= 0 \\ a^2 - (1+\beta)\, q^2 - q^1 &= 0 \,. \end{aligned}$$

Solving this yields

$$q^1 = \frac{a^1(1+\beta) - a^2}{\beta\,(2+\beta)} \quad q^2 = \frac{a^2(1+\beta) - a^1}{\beta\,(2+\beta)} \,.$$

So as long as $(1+\beta) \geq \frac{a^1}{a^2}$ and $(1+\beta) \geq \frac{a^2}{a^1}$ we can have positive production for more than one market. Thus, this algorithm finds the optimal solution in a finite amount of steps. □

A.2 Proof of invertibility of A^-

Proof. A is a $MJ \times MJ$ matrix that is noninvertible if the network is not complete. If it is complete we have characterized the eigenspaces in Section 4.2 and shown that A is invertible. Let us remember that the matrix A takes in the vector of beliefs and outputs the vector of beliefs in the next period. If the network is not complete then some $a^{m,j}$ are zero and do not affect the dynamics. The corresponding $m \times j$ rows and columns of A are zero. The columns of A are zero because this belief does not affect any other firm's beliefs. The rows are zero because the firm has no belief about this market so it needs to stay at zero. If $N = |E|$ is the number of edges then we can permute the rows and columns of A to arrive at a new matrix

$$A = \begin{pmatrix} A^- & 0 \\ 0 & 0 \end{pmatrix}, \tag{46}$$

where $A^- \in \mathbb{R}^{N \times N}$. We know that A^- has no all-zero rows or columns by construction. In addition, no row is a linear combination of the other rows as how a firm's next period's beliefs are affected by the belief vector is unique. Thus, A^- is invertible. □

A.3 Proof of steady-state beliefs inducing Cournot-Nash equilibrium

Proof. If we consider one row of the system defining steady state beliefs $\alpha \operatorname{vec} G - A\bar{a} = 0$,

$$\alpha - \sum_{i \in \mathcal{J}_m \setminus \{j\}} L_i \bar{a}^i - \bar{a}^{m,j} = \alpha - \frac{\beta}{2} \sum_{i \in \mathcal{J}_m \setminus \{j\}} \bar{q}^{m,j} - \bar{a}^{m,j} = 0\,. \tag{47}$$

In the full information Cournot-Nash equilibrium the first-order condition is given by

$$\alpha - \frac{\beta}{2} \sum_{i \in \mathcal{J}_m} \bar{q}^{m,i} - \beta q^{m,j} - \sum_{n \in \mathcal{M}_j} q^{n,j} = 0\,. \tag{48}$$

If $\bar{a}^{m,j} = \beta q^{m,j} + \sum_{n \in \mathcal{M}_j} q^{n,j}$, the steady state beliefs imply the Cournot-Nash equilibrium quantities. Using the expression for quantities as a function of beliefs we can show that this is indeed the case:

$$\bar{a}^{m,j} = \beta q^{m,j} + \sum_{n \in \mathcal{M}_j} q^{n,j} \tag{49}$$

$$\bar{a}^{m,j} = 2t_j M_j \bar{a}^j + \bar{a}^{m,j} + 2t_j \frac{M_j^2}{\beta} \bar{a}^j + \frac{M_j}{\beta} \bar{a}^j \tag{50}$$

$$\bar{a}^{m,j} = 2t_j \frac{M_j}{\beta} (M_j + \beta) + \frac{M_j}{\beta} \bar{a}^j + \bar{a}_{m,j} \tag{51}$$

$$\bar{a}^{m,j} = -2 \frac{1}{2(M_j + \beta)} \frac{M_j}{\beta} (M_j + \beta) + \frac{M_j}{\beta} \bar{a}^j + \bar{a}_{m,j} \tag{52}$$

$$\bar{a}^{m,j} = \bar{a}_{m,j} \,. \tag{53}$$

□

A.4 Full proof of main results for the general network

Let us first consider the eigenvalues of L_j.

Lemma A.1. *L_j is positive semi-definite and all eigenvalues are in $\left[0, \frac{1}{2}\right]$ $\forall j$. If the degree $M_j \geq 2$ then there is at least one eigenvalue $\lambda = \frac{1}{2}$.*

Proof. Recall, $L_j = \operatorname{diag} G_j K_j \operatorname{diag} G_j$ and $K_j = t_j \mathbb{1} + \frac{1}{2} I$ is a symmetric matrix with positive elements on the diagonal and negative elements on the off-diagonal

$$K_j = \begin{pmatrix} t_j + \frac{1}{2} & t_j & \dots & t_j \\ t_j & t_j + \frac{1}{2} & \dots & t_j \\ \vdots & \vdots & \ddots & \vdots \\ t_j & t_j & \dots & t_j + \frac{1}{2} \end{pmatrix} .$$

Let us first consider the eigenspaces of K_j. Take $\lambda_1 = s_j - t_j = \frac{1}{2}$. Then, $K_j - \frac{1}{2} I = t_j \mathbb{1}$, which has rank 1 and is thus not invertible. So, λ_1 is an eigenvalue of K_j with geometric multiplicity $M - 1$. As $t_j \mathbb{1}$ is a matrix with every element equal to t_j the associated eigenvectors are given by the orthogonal complement to the vector of ones, $\mathbf{1}^\perp$. Next consider $\lambda_2 = s_j + (M-1)\, t_j = M t_j + \frac{1}{2} = \frac{\beta}{2(M+\beta)}$. Then, $K_j - \left(M t_j + \frac{1}{2}\right) I = t_j \mathbb{1} - M t_j I$ which has rank $M - 1$ and is thus not invertible. So, λ_2 is an eigenvalue of K_j with geometric multiplicity 1. It can be seen that each row of $K_j - \lambda_2 I$ sums to zero, so $\mathbf{1}$ is the eigenvector associated with λ_2. Note that $0 < \lambda_2 < \frac{1}{2}$. Thus, K_j is positive definite.

Since $\operatorname{diag} G_j = (\operatorname{diag} G_j)^\top$, we have for any vector v

$$v^\top L_j v = (\operatorname{diag} G_j v)^\top K_j \operatorname{diag} G_j v = w^\top K_j w \geq 0 \,, \tag{54}$$

because K_j is positive definite. This inequality is strict if $v \notin \ker \operatorname{diag} G_j$. Thus, L_j is positive semi-definite as well. We may distinguish three cases: if the degree M_j of firm j is equal to the number of markets M then $\operatorname{diag} G_j = I$ and $L_j = K_j$ and the eigenvalues are as described above. If $1 < M_j < M$ the eigenvalues of L_j are the same as the eigenvalues of K_j except the M in the expressions for the eigenvalues is replaced by M_j as some of the rows and columns are of K_j are set to zero. In addition, there is a third eigenvalue, corresponding to these zero columns, $\lambda_3 = 0$ with geometric multiplicity $\dim E_{\lambda_3} = M - M_j$, reducing the geometric multiplicity of λ_1 to $M_j - 1$. If $M_j = 1$ then L_j has only one nonzero element, $s_j = \frac{\beta}{2(1+\beta)} \in \left(0, \frac{1}{2}\right)$ which is on the diagonal and the eigenvalues are $\lambda_1 = s_j$ with geometric multiplicity 1 and $\lambda_2 = 0$ with geometric multiplicity $M - 1$. Thus, in general L_j is positive semi-definite and all eigenvalues are in $\left[0, \frac{1}{2}\right]$.

□

Let me define for future use the block matrix $\mathbf{I} \in \mathbb{R}^{MJ \times MJ}$ with all blocks being the $M \times M$ identity matrix.

Based on the eigenvalues of L_j we can find an upper bound on the eigenvalues of L, which in turn will give us an upper bound on the eigenvalues of the learning matrix A.

Let me introduce two concepts from linear algebra we will need. The spectrum of a matrix A is the set of its eigenvalues, $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$. The spectral radius of a matrix A is the maximum absolute value of its eigenvalues, $\rho(A) = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$.

Proposition A.1. *The smallest eigenvalue $\lambda_{\min}$ of L satisfies $\lambda_{\min} \geq -\frac{1}{2}$.*

Proof. Let[8] $\lambda_{\min}(A) = \min \sigma(A)$ be the smallest eigenvalue of a matrix A and $\lambda_{\max}(A) = \max \sigma(A)$ the largest eigenvalue of a matrix A.

The dynamics of the system are given by $-A = -\operatorname{diag} G (L + I)$. So if we can find a lower bound on the eigenvalues of L this will also allow us to find a lower bound on the eigenvalues of A. We will show that the smallest eigenvalue of L is greater or equal to $-\frac{1}{2}$.

If $\sigma(L) = \{\lambda_1, \ldots, \lambda_N\}$ with the eigenvalues ordered from smallest to largest, then $\sigma(-L) = \{-\lambda_N, \ldots, -\lambda_1\}$. So by showing that the most positive eigenvalue of $-L$

[8] I thank user1551 (2023) for help with this proof.

satisfies $\lambda_{\max}(-L) = -\lambda_1 \in \left(0, \frac{1}{2}\right]$, we can also conclude that $\lambda_{\min}(L) \geq -\frac{1}{2}$.

Let us first assume that there is at least one firm that is connected to more than one market. Then, by Lemma A.1 $\rho(L_j) \leq \frac{1}{2}$ with strict equality for at least one j.

$$2\,\mathrm{diag}\,L = \mathrm{diag}\left(2L_1, 2L_2, \dots, 2L_M\right)\,, \tag{55}$$

is a symmetric positive semi-definite matrix. Its square root is,

$$P = \mathrm{diag}\left(\sqrt{2L_1}, \sqrt{2L_2}, \dots, \sqrt{2L_M}\right)\,,$$

which is also positive semi-definite and

$$\sigma(P) = \cup_j \sigma\left(\sqrt{2L_j}\right) = \cup_j \sqrt{2\sigma(L_j)}\,,$$

as $2\,\mathrm{diag}\,L$ is a block-diagonal matrix.

Let $\|A\|_2$ be the spectral norm of the matrix A, which is defined as $\|A\|_2 = \sqrt{\lambda_{\max}(A^\top A)}$.

Since $\max_j \rho(L_j) = \frac{1}{2}$, $\|P\|_2 = 1$.

Note that

$$L = \begin{pmatrix} 0 & L_2 & \dots & L_M \\ L_1 & 0 & \dots & L_M \\ \vdots & \vdots & \ddots & \vdots \\ L_1 & L_2 & \dots & 0 \end{pmatrix} = (\mathbf{I} - I)\begin{pmatrix} L_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & L_M \end{pmatrix} = \frac{1}{2}(\mathbf{I} - I)\,P^2\,, \tag{56}$$

and

$$-L = \frac{1}{2}(I - \mathbf{I})\,P^2\,. \tag{57}$$

Since for square matrices $\sigma(XY) = \sigma(YX)$ (see e.g. Horn & Johnson, 2013), we have

$$\lambda_{\max}(-L) = \frac{1}{2}\lambda_{\max}\left((I - \mathbf{I})\,P^2\right) = \frac{1}{2}\lambda_{\max}\left(P\,(I - \mathbf{I})\,P\right)\,. \tag{58}$$

From there it follows that

$$\lambda_{\min}(L) = -\lambda_{\max}(-L) = -\frac{1}{2}\lambda_{\max}\left(P\,(I - \mathbf{I})\,P\right)\,. \tag{59}$$

$P\,(I - \mathbf{I})\,P$ is a real, symmetric matrix, so it has real eigenvalues as does L. In addition, $\operatorname{tr} L = 0$ so its maximum eigenvalue is nonnegative.

Let v be a unit eigenvector corresponding to the maximum eigenvalue of $P\,(I - \mathbf{I})\,P$. Then

$$
\begin{aligned}
0 \leq \frac{1}{2}\lambda_{\max}\left(P\,(I - \mathbf{I})\,P\right) &= \frac{1}{2} v^\top P\,(I - \mathbf{I})\,Pv && (60)\\
&= \frac{1}{2}\|Pv\|_2^2 \left(\frac{Pv}{\|Pv\|_2}\right)^\top (I - \mathbf{I}) \left(\frac{Pv}{\|Pv\|_2}\right) && (61)\\
&\leq \frac{1}{2} \left(\frac{Pv}{\|Pv\|_2}\right)^\top (I - \mathbf{I}) \left(\frac{Pv}{\|Pv\|_2}\right) && (62)\\
&\leq \frac{1}{2} \max_{\|y\|_2 = 1} y^\top (I - \mathbf{I})\, y && (63)\\
&= \frac{1}{2}\lambda_{\max}\,(I - \mathbf{I}) && (64)\\
&= \frac{1}{2}\,. && (65)
\end{aligned}
$$

The first equality comes from

$$v^\top P\,(I - \mathbf{I})\,Pv = v^\top \lambda_{\max} v = \lambda_{\max}\,. \qquad (66)$$

The second inequality stems from the fact that

$$\|Pv\|_2 \leq \|P\|_2 \|v\|_2 = 1\,, \qquad (67)$$

and the third follows from seeing that if we multiply any matrix with a unit vector from the left and right $x^\top M x \in \mathbb{R}$ the result needs to be less or equal to the product with the vector maximizing the same expression, $\max_{\|y\|_2=1} y^\top M y$.

The last equality follows from $\lambda_{\max}\,(I - \mathbf{I}) = 1$.

Thus

$$\frac{1}{2}\lambda_{\max}\left(P\,(I - \mathbf{I})\,P\right) \leq \frac{1}{2}\,. \qquad (68)$$

Using Equation (59) we can conclude that

$$\lambda_{\min}\,(L) = -\frac{1}{2}\lambda_{\max}\left(P\,(I - \mathbf{I})\,P\right) \geq -\frac{1}{2}\,. \qquad (69)$$

We can thus conclude that the most negative eigenvalue of L is greater or equal to $-\frac{1}{2}$.

If there is no firm that is connected to more than two markets, $s_j = s \, \forall j$ and $\max|\sigma(L_j)| = |s| \, \forall j$. Thus, by replacing 2 with $\frac{1}{s}$ in Equation (55) and $\frac{1}{2}$ with s in Equation (56), respectively, we obtain the desired result, with the tighter bound $\lambda_{min}(L) \geq -s \geq -\frac{1}{2}$.

□

Adding the identity matrix to L simply shifts the spectrum by 1 but multiplying by $\operatorname{diag} G$ is less trivial. However, after some linear algebra, we can show that the eigenvectors of A are the same as the eigenvectors of $L+I$. The same holds for the eigenvalues except that the eigenvalues $\lambda = 1$ of $L+I$, which are associated with the eigenvectors in the kernel of $\operatorname{diag} G$, are replaced with $\lambda = 0$ for A.

Lemma A.2. *The eigenvectors of* $A = \operatorname{diag} G(L+I)$ *are the same as the eigenvectors of* $L+I$*. In addition, also the eigenvalues are the same except the eigenvalues* $\lambda = 1$*, which are associated with the eigenvectors in the kernel of* $\operatorname{diag} G$*, are replaced with* $\lambda = 0$.

Proof. Note that $\ker \operatorname{diag} G \subseteq \ker L$, as $L \operatorname{diag} G = L$, so if $\operatorname{diag} Gx = 0$, $Lx = 0$. We thus know that L has at least $M^2 - \operatorname{tr} \operatorname{diag} G$ standard unit vectors, e_j, as eigenvectors, where j corresponds to the zeros of $\operatorname{diag} G$ with eigenvalue $\lambda = 0$.

We also know that $\sigma(L+I) = \sigma(L) + 1$. Note, however, that the eigenvectors of $L+I$ are the same as the eigenvectors of L.

Next, we want to show that the e_j are still eigenvectors of $\operatorname{diag} G(L+I)$ but now no longer with eigenvalue $\lambda = 1$ but with $\lambda = 0$. As the e_j are in the kernel of $\operatorname{diag} G$, i.e. $\operatorname{diag} Ge_j = \mathbf{0}$,

$$\operatorname{diag} G(L+I)e_j = \operatorname{diag} G\lambda' e_j = \lambda' \mathbf{0} = \mathbf{0},$$

so again the eigenvalues of $\operatorname{diag} G(L+I)$ corresponding to the eigenvectors e_j are $\lambda = 0$.

All other eigenvalues of $L+I$ remain unchanged by the multiplication with the projection matrix $\operatorname{diag} G$ as eigenvectors corresponding to different eigenvalues are linearly independent. Then, the other eigenvectors are not in the kernel of $\operatorname{diag} G$, so we can choose vectors v such that $\operatorname{diag} Gv = v$, and as $\operatorname{diag} G$ is a projection

$\sigma\left(\operatorname{diag} G\,(L+I)\right)=\sigma\left(\operatorname{diag} G\,(L+I)\operatorname{diag} G\right)$. So,

$$\operatorname{diag} G\,(L+I)\operatorname{diag} G v=\operatorname{diag} G\,(L+I)\,v=\lambda\operatorname{diag} G v=\lambda v\quad\forall v\notin\ker\left(\operatorname{diag} G\right).$$

Thus, if v is an eigenvector of $L+I$ not in the kernel of $\operatorname{diag} G$ with eigenvalue λ then $\operatorname{diag} G v$ is an eigenvector of A with eigenvalue λ. □

So far we have shown that all eigenvalues of A are nonnegative and that the zero eigenvalues correspond to the beliefs of firms about markets they are not connected to and which thus remain constant at zero. For the remaining eigenvalues, we have found a lower bound of $\frac{1}{2}$. With this, we can conclude that the learning process is stable as then the relevant eigenvalues of $-A$ are negative.

Next, we will show that Proposition 3.2 is indeed true.

Proof of Proposition 3.2

Proof. Combining Proposition A.1 and Lemma A.2 we can conclude that the largest nonzero eigenvalue of A is greater or equal to $\frac{1}{2}$. Now we will show that $\lambda_2=\frac{1}{2}$ is indeed an eigenvalue for weakly connected networks.

We can further decompose L as

$$L=(\mathbf{I}-I)\operatorname{diag} G\begin{pmatrix}K_1 & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & K_M\end{pmatrix}\operatorname{diag} G\,. \tag{70}$$

So

$$A=\operatorname{diag} G\left((\mathbf{I}-I)\operatorname{diag} G\begin{pmatrix}K_1 & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & K_M\end{pmatrix}\operatorname{diag} G+I\right)= \tag{71}$$

$$=\operatorname{diag} G B\operatorname{diag} G\,, \tag{72}$$

where

$$B=(\mathbf{I}-I)\operatorname{diag} G\begin{pmatrix}K_1 & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & K_M\end{pmatrix}+I \tag{73}$$

$$= (\mathbf{I} - I) \operatorname{diag} G \left(\begin{pmatrix} t_1 \mathbb{1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & t_M \mathbb{1} \end{pmatrix} + \frac{1}{2} I \right) + I \,. \tag{74}$$

Let us consider the kernel of the matrix $(B - \lambda I)$. We assume that $v = (v_1, \dots, v_M)^\top \in \mathbb{R}^{M \times J}$ is such that $\operatorname{diag} G v = v$ and that if we ordered the vectors v_i in a matrix $V \in \mathbb{R}^{M \times J}$ where the j-th column is v_j such that $V\mathbf{1} = \mathbf{0}$ and $V^\top \mathbf{1} = \mathbf{0}$. Note that if a firm is only connected to one market the only vector v_j satisfying these conditions is the zero vector. For the existence of such vectors, it is necessary that the condition in Definition 3.1 holds, as can be seen by the dimension of the eigenspace.

Economically this can be interpreted as aggregate beliefs across firms and within firms being correct. Then

$$(B - \lambda I) v = \left((\mathbf{I} - I) \operatorname{diag} G \left(\begin{pmatrix} t_1 \mathbb{1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & t_M \mathbb{1} \end{pmatrix} + \frac{1}{2} I \right) + (1 - \lambda) I \right) v \tag{75}$$

$$= \left((\mathbf{I} - I) \operatorname{diag} G \begin{pmatrix} t_1 \mathbb{1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & t_M \mathbb{1} \end{pmatrix} + \frac{1}{2} (\mathbf{I} - I) \operatorname{diag} G + (1 - \lambda) I \right) v \tag{76}$$

$$= (\mathbf{I} - I) \operatorname{diag} G \begin{pmatrix} t_1 \mathbb{1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & t_M \mathbb{1} \end{pmatrix} v + \frac{1}{2} (\mathbf{I} - I) v + (1 - \lambda) v \tag{77}$$

$$= \left(\frac{1}{2} - \lambda \right) v \,, \tag{78}$$

which is equal to zero for $\lambda_2 = \frac{1}{2}$. Here we have made use of the fact that $\mathbf{I} v = \begin{pmatrix} \sum_j v_{j,1} & \dots & \sum_j v_{j,M} \end{pmatrix}^\top = \mathbf{0}$ because $V\mathbf{1} = \mathbf{0}$ and $\mathbb{1} v_j = \mathbf{0}$ because $V^\top = \mathbf{0}$.

We may notice that the above restrictions on v define three subspaces of which v needs to be in the intersection. First, all eigenvectors with nonzero eigenvalue will be in the subspace

$$U = \left\{ v \in \mathbb{R}^{MJ} \mid v = \operatorname{diag} G v \right\} ,$$

with $\dim U = D$.

Let us denote by W_1 the subspace such that the aggregate production across markets is correct, i.e.

$$W_1 = \{v \mid V\mathbf{1} = 0\} \subset U\,. \tag{79}$$

This imposes M conditions on W_1 so that $\dim W_1 = D - M$.

The second subspace, W_2, is given by vectors such that aggregate production within firms is correct, i.e.

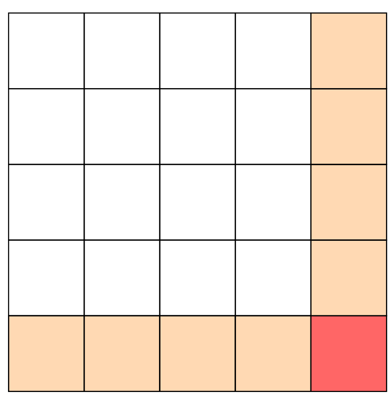
$$W_2 = \left\{v \mid V^\top \mathbf{1} = 0\right\} \subset U\,. \tag{80}$$

This imposes J conditions on W_2 so that $\dim W_2 = D - J$.

Then,

$$E_{\lambda_2}(A) = W_1 \cap W_2\,.$$

We know that $\operatorname{codim} W_1 = M$ and $\operatorname{codim} W_2 = J$ and $\max(M, J) \leq \operatorname{codim}(W_1 \cap W_2) \leq M + J$. Now, we know that for each row we have M restrictions from W_1 and we know that for each column we would have J restrictions from W_2 leading to a total of $M + J$ restrictions. This, however, counts one restriction twice since if we restrict the M rows there are only $J - 1$ columns left to restrict. Thus, $\operatorname{codim}(W_1 \cap W_2) = M + J - 1$ and $\dim E_{\lambda_2}(A) = D - (M + J - 1)$. Assuming, without loss of generality, that we can use the elements in the last row and column to satisfy the restriction we can represent this visually as

,

which shows that if we simply added the codimensions we would count one restriction twice. □

Now it only remains to prove the remaining two results.

Proof of Theorem 1

Proof. Combining the result of Proposition 3.2 with the learning problem leads us to this theorem.

If the network is weakly connected we have shown in Proposition 3.2 that the smallest nonzero eigenvalue of A is $\frac{1}{2}$ and thus $-\frac{1}{2}$ is the largest nonzero eigenvalue of $-A$. This eigenvalue will determine the long-run convergence rate of the belief system. In addition, if the network is strongly connected then each belief is influenced by the eigenvector associated with the eigenvalue $-\frac{1}{2}$.

Note that the component of the eigenvectors associated with the eigenvalue $-\frac{1}{2}$ of firms who are only connected to one market is the zero vector as shown in the proof of Proposition 3.2. Since there are no eigenvalues in $\left(-\frac{1}{2}, 0\right)$ individual quantities of firms connected to only one market converge faster than the quantities of firms who need to learn about more markets.

This is also true for the case where two firms are active on one market one of which is also active on other markets, then the quantity of the multi-market firm for this particular market will converge at a faster rate.

Take for example the network,

$$G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} .$$

This network is weakly but not strongly connected. There is one eigenvector with eigenvalue $-\frac{1}{2}$ and the component of this eigenvector associated with firm one is of the shape, $v_1 = \begin{pmatrix} b & -b & 0 \end{pmatrix}^\top$ because $v_3 = \mathbf{0}$. Firm one's beliefs can be aggregated to quantities for market three with the vector, $u^{3,1} = \begin{pmatrix} t & t & s \end{pmatrix}^\top$. Now, it is clear that $u^{3,1} \perp v_1$ so that the dynamics of $q_t^{3,1}$ is not influenced by this eigenspace. □

Proof of Theorem 2

Proof. Aggregate production is given by

$$q_t^m = u^m a_{t-1} , \tag{81}$$

with u^m the aggregation mapping transforming beliefs into aggregate quantities for a particular market, given by

$$u^m = \frac{2}{\beta}\left(\begin{pmatrix} g_{m1}t_1G_1 \\ \vdots \\ g_{mJ}t_JG_J \end{pmatrix} + \frac{1}{2}\begin{pmatrix} g_{m1}e_m \\ \vdots \\ g_{mJ}e_m \end{pmatrix}\right)^\top . \tag{82}$$

As we assume $a_t^{m,j} = 0$ for all m, j such that $g_{mj} = 0$ we know that the eigenvector associated with the eigenvalue $\lambda_1 = 0$ is not relevant for the dynamics and thus v^1 in the solution of the differential equation is the zero vector. This naturally implies

$$u^m v = 0 \text{ if } v \in E_{\lambda_1} . \tag{83}$$

Next, take a vector $v \in E_{-\frac{1}{2}}(A)$. Then

$$u^m v = \frac{2}{\beta}\sum_k g_{mk}t_k \sum_n v_{n,k} + \frac{1}{\beta}\sum_j v_{m,j} = 0 , \tag{84}$$

where the last equality follows from the fact that $v \in E_{-\frac{1}{2}} \subset W_1 \cap W_2$.

Then

$$\hat{q}_t^m = q_t^m - \bar{q} = (u^m)^\top \left(\sum_{i\in\{3,\ldots,N\}} \left(v_{t-1}^i - \bar{v}^i\right)\right) . \tag{85}$$

We know that the eigenvector with the eigenvalue closest to zero that is not perpendicular to u will dominate the dynamics. We have not characterized the other eigenspaces so we will assume this to be λ_3, but it may also be an eigenvalue even further from zero. We can approximate the dynamics of aggregate quantities as

$$q_t^m - \bar{q} \approx u^\top \left(v_{t-1}^3 - \bar{v}^3\right) . \tag{86}$$

Thus,

$$q_t^m - \bar{q} \sim t^{-\lambda_3} . \tag{87}$$

Therefore, the convergence rate of aggregate quantities is given by the eigenvalue λ_3 in contrast to individual quantities which are given by the eigenvalue $-\frac{1}{2}$.

The aggregation mapping transforming beliefs into quantities for a particular firm follows straightforwardly from the L_j matrix

$$q_t = \mathbf{1}^\top q_t^j = \mathbf{1}^\top \frac{2}{\beta} L_j a_{t-1}^j \,. \tag{88}$$

Take a vector $v \in E_{-\frac{1}{2}}(A)$, but only consider the component v_j that corresponds to firm j, i.e. rows $[(j-1)M+1, jM]$. Then

$$u^j v_j = \frac{1}{\beta}\mathbf{1}^\top v_j = 0 \,, \tag{89}$$

which follows from the fact that $v_j \in W_2$.

We, thus, have shown that both market aggregate quantities and firm aggregate quantities converge faster than individual quantities. □

A.5 Proof of Proposition 4.1

Proof. $A - \lambda_1$ is a square matrix with $\frac{\beta}{2(1+\beta)}$ in every position. It has rank one and therefore $\lambda_1 = 1 - \frac{\beta}{2(1+\beta)}$ is indeed an eigenvalue with geometric multiplicity $J-1$ and corresponding eigenspace

$$E_{\lambda_1} = \mathbf{1}^\perp \,.$$

$A - \lambda_2$ is a $J \times J$ square matrix with $\frac{\beta}{2(1+\beta)}$ on the offdiagonal and $-(J-1)\frac{\beta}{2(1+\beta)}$ on the diagonal. It is, thus, of rank $J-1$ and $\lambda_2 = 1 + (J-1)\frac{\beta}{2(1+\beta)}$ is indeed an eigenvalue with geometric multiplicity 1 and corresponding eigenspace

$$E_{\lambda_2} = \operatorname{span} \mathbf{1} \,.$$

□

A.6 Proof of Proposition 4.2

Proof. The matrix A is a square, symmetric matrix with dimensions $MJ \times MJ$ so it will have MJ eigenvalues and MJ linearly independent eigenvectors.

A and K share a similar structure so let us first characterize the eigenspaces of K. As before, we can decompose K as

$$K = t\mathbb{1} + \frac{1}{2} I \,. \tag{90}$$

Let us consider $\mu_1 = s - t = \frac{1}{2}$. Then the the matrix $K - \mu_1 I = t\mathbb{1}$ has rank 1 and μ_1 is an eigenvalue with multiplicity $M-1$ and has eigenspace

$$E_{\mu_1} = \operatorname{span} \mathbf{1}^{\perp} ,$$

with dimension $M-1$.

Another eigenvalue is $\mu_2 = s + (M-1)t = Mt + \frac{1}{2}$. Then, the matrix $K - \mu_2 I = t\mathbb{1} - (Mt)\, I$ has rank $M-1$ and μ_2 is indeed an eigenvalue. Its associated eigenspace is

$$E_{\mu_2} = \operatorname{span} \mathbf{1} ,$$

where $\dim \left(E_{\mu_2}\right) = 1$.

We noted that the structure of A mimics the structure of K, so an educated guess is that the eigenvectors of the two will also mimic each other. We will confirm this in the following paragraphs.

Take the potential eigenvector $w = (v, \ldots, v)$. Then

$$Aw = \begin{pmatrix} v + (J-1)Kv \\ \vdots \\ v + (J-1)Kv \end{pmatrix} = \left(1 + (J-1)\,\mu\right) \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix} = \left(1 + (J-1)\,\mu\right) w ,$$

and w is indeed an eigenvalue of A. We know there are $M-1$ linearly independent eigenvectors associated with the eigenvalue $\mu_1 = s - t = \frac{1}{2}$, so

$$\lambda_1 = 1 + (J-1)\mu_1 = 1 + (J-1)\frac{1}{2}$$

is an eigenvalue of A with eigenspace E_{λ_1} with $\dim \left(E_{\lambda_1}\right) = M-1$. It can be characterized by

$$E_{\lambda_1} = \oplus_{v \in \mathbf{1}^{\perp}} \operatorname{span} \left(v, \ldots, v\right) .$$

In words, the eigenspace of the matrix A associated with the eigenvalue $\lambda_1 = 1 + (J-1)\mu_1$ is the direct sum of the span of all vectors of the shape $(v, \ldots, v)$ with v the eigenvectors of the matrix K associated with the eigenvalue μ_1. This corresponds to the case of each firm producing the correct amount in aggregate but markets do not have the correct aggregate amount yet.

Another eigenvalue of K is $\mu_2 = s + (M-1)t$ with associated eigenvector $v = (1, \ldots, 1)^\top$. Then $\lambda_2 = 1 + (J-1)\mu_2 = 1 + (J-1)s + (J-1)(M-1)t$ is also an eigenvalue of A with associated eigenspace

$$E_{\lambda_2} = \operatorname{span}(\mathbf{1}, \ldots, \mathbf{1})$$

with $\dim(E_{\lambda_2}) = 1$. This corresponds to the case of neither firms producing the correct amount in aggregate nor markets having the correct aggregate amount yet.

These eigenvectors with shape $w = (v, \ldots, v)$ mimic the eigenvector $v = (1, \ldots, 1)$ of K. The other eigenspaces of K are given by the orthogonal complement to the eigenspace of this first eigenvector. We will thus further postulate that the other eigenspaces of A are given by the orthogonal complement to the eigenspace associated with the eigenvector $w = (v, \ldots, v)$. The eigenvectors associated with this other eigenspace should form a basis for $\mathbb{R}^{M(M-1)}$. We will confirm this for the case of a complete network of size three.

One basis for the orthogonal complement of (v, v, v) is given by

$$\{(v, -v, 0), (v, 0, -v)\} .$$

Take the first potential eigenvector $w = (v, -v, 0)$. Then

$$Aw = \begin{pmatrix} v - Kv \\ Kv - v \\ 0 \end{pmatrix} = (1-\mu) \begin{pmatrix} v \\ -v \\ 0 \end{pmatrix} = (1-\mu)\, w\,.$$

So w is indeed an eigenvector of A with associated eigenvalues $\lambda_3 = 1 - (s + (M-1)t)$ and $\lambda_4 = 1 - (s - t) = \frac{1}{2}$.

Then let us take the second potential eigenvector $w = (v, 0, -v)$. Then

$$Aw = \begin{pmatrix} v - Kv \\ 0 \\ Kv - v \end{pmatrix} = (1-\mu) \begin{pmatrix} v \\ 0 \\ -v \end{pmatrix} = (1-\mu)\, w\,.$$

So w is indeed also an eigenvector of A also with associated eigenvalues λ_3 and λ_4.

The eigenspace E_{λ_3} is characterized by

$$E_{\lambda_3} = \oplus_{v \in \operatorname{span} \mathbf{1}} \operatorname{span}(v, \ldots, v)^\perp ,$$

which has $\dim(E_{\lambda_3}) = M - 1$. This corresponds to the case of firms not having the correct aggregate amount but markets do.

The eigenspace E_{λ_4} is characterized by

$$E_{\lambda_4} = \oplus_{v \in \mathbf{1}^\perp} \operatorname{span}(v, \ldots, v)^\perp \ ,$$

which has $\dim(E_{\lambda_4}) = (M-1)^2$. Here both firms and markets have the correct aggregate amount. Together, these eigenspaces cover all of $\mathbb{R}^{M^2}$.

The eigenvalues of A at the steady state then are from largest to smallest,

$$(M-1) : \lambda_1 = \frac{J+1}{2} \tag{91}$$

$$1 : \lambda_2 = \frac{1}{2(M+\beta)}(2M + (J+1)\beta) \tag{92}$$

$$(M-1) : \lambda_3 = \frac{1}{2(M+\beta)}(2M+\beta) \tag{93}$$

$$(M-1)^2 : \lambda_4 = \frac{1}{2} \tag{94}$$

which are positive for $\beta > 0$.

Note that the dynamics of the system are given by $-A$. The relevant eigenvalues are then $\lambda = \{-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4\}$, ordered from most negative to least negative.

□

A.7 Best response dynamics

Another common learning framework is the best response dynamics, which requires more information than least squares learning. It is interesting to compare how the two learning dynamics differ in terms of convergence speed. The firm problem under best response dynamics is

$$\max_{\{q^{m,j}\}_{m \in \mathcal{M}_j}} \sum_{n \in \mathcal{M}_j} \left[\left(\alpha - \frac{\beta}{2} \sum_{i \in \mathcal{J}_n \setminus \{j\}} q_{t-1}^{n,i} - \frac{\beta}{2} q^{n,j} \right) q^{n,j} \right] - \frac{1}{2} \left(\sum_{k \in \mathcal{M}_j} q^{k,j} \right)^2 , \tag{95}$$

with first-oder conditions

$$\alpha - \frac{\beta}{2} \sum_{i \in \mathcal{J}_m \setminus \{j\}} q_{t-1}^{m,i} - \beta q^{m,j} - \sum_{k \in \mathcal{M}_j} q^{k,j} = 0 \tag{96}$$

$$\alpha - \frac{\beta}{2} \sum_{i \in \mathcal{J}_m \setminus \{j\}} q_{t-1}^{m,i} - \sum_{k \in \mathcal{M}_j} q^{k,j} = \beta q^{m,j} \,. \tag{97}$$

Summing over markets yields

$$\beta Q^j = M_j \alpha - \frac{\beta}{2} \sum_{m \in \mathcal{M}_j} \sum_{i \in \mathcal{J}_m \setminus \{j\}} q_{t-1}^{m,i} - M_j Q^j \tag{98}$$

$$Q^j = \frac{M_j \alpha}{M_j + \beta} - \frac{\beta}{2\left(M_j + \beta\right)} \sum_{m \in \mathcal{M}_j} \sum_{i \in \mathcal{J}_m \setminus \{j\}} q_{t-1}^{m,i} \,. \tag{99}$$

Plugging back in

$$\beta q^{m,j} = \alpha - \frac{\beta}{2} \sum_{i \in \mathcal{J}_m \setminus \{j\}} q_{t-1}^{m,i} - \frac{M_j \alpha}{M_j + \beta} + \frac{\beta}{2\left(M_j + \beta\right)} \sum_{n \in \mathcal{M}_j} \sum_{i \in \mathcal{J}_n \setminus \{j\}} q_{t-1}^{n,i} \tag{100}$$

$$q^{m,j} = \frac{\alpha}{M_j + \beta} - \frac{1}{2} \sum_{i \in \mathcal{J}_m \setminus \{j\}} q_{t-1}^{m,i} + \frac{1}{2\left(M_j + \beta\right)} \sum_{n \in \mathcal{M}_j} \sum_{i \in \mathcal{J}_n \setminus \{j\}} q_{t-1}^{n,i} \tag{101}$$

We can vectorize this as

$$q_t = \max\left\{\tilde{\alpha} \operatorname{vec} G - \operatorname{diag} G L \operatorname{diag} G q_{t-1}, 0\right\} \,, \tag{102}$$

where L is given by

$$L = \frac{1}{2}\left(\mathbf{I} - I\right) - \begin{pmatrix} t_1 \mathbb{0} & t_1 \mathbb{1} & \dots & t_1 \mathbb{1} \\ t_2 \mathbb{1} & t_2 \mathbb{0} & \dots & t_2 \mathbb{1} \\ \vdots & \vdots & \ddots & \vdots \\ t_J \mathbb{1} & t_J \mathbb{1} & \dots & t_J \mathbb{0} \end{pmatrix} \circ \begin{pmatrix} \operatorname{diag} G_1 & \operatorname{diag} G_1 & \dots & \operatorname{diag} G_1 \\ \operatorname{diag} G_2 & \operatorname{diag} G_2 & \dots & \operatorname{diag} G_2 \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{diag} G_M & \operatorname{diag} G_M & \dots & \operatorname{diag} G_M \end{pmatrix} , \tag{103}$$

$t_j = -\frac{1}{2(M_j + \beta)}$, and $\tilde{\alpha} = \frac{\alpha}{M_j + \beta}$.

Let $A = \operatorname{diag} G L \operatorname{diag} G$, then we can rewrite the dynamics in deviations from the steady state

$$\hat{q}_t = -A \hat{q}_{t-1} \,, \tag{104}$$

with solution

$$\hat{q}_t = \left(-A\right)^t \hat{q}_0 \,. \tag{105}$$

We can equally write this as a sum over the eigenvectors of A (assuming A is diagonalizable)

$$\hat{q}_t = \sum_{i=1}^{MJ} (-\lambda_i)^t \, c_i v_i \,, \tag{106}$$

where

$$\sum_{i=1}^{MJ} c_i v_i = \hat{q}_0 \,. \tag{107}$$

As t increases the dynamics of the system are dominated by the largest eigenvalue in absolute terms, $\lambda_{\max} = \max_{\lambda_i} \{|\lambda_i|\}$. The dynamics are thus approximately proportional to

$$|\hat{q}_t| \approx \left|(-\lambda_{\max})^t\right| \hat{q}_0 \,, \tag{108}$$

or in logarithms

$$\log|\hat{q}_t| \approx t \log|\lambda_{\max}| + \log \hat{q}_0 \,. \tag{109}$$

Thus, if a stable steady state exists, quantities will converge exponentially fast with rate $\log|\lambda_{\max}|$ under the best response dynamics. Therefore, learning is faster than under least-squares learning because under least-squares learning convergence only happens polynomially fast at the rate $-\frac{1}{2}$.

Numerical results indicate that aggregate quantities converge at the same rate as individual quantities. I show this analytically for the single market case.

A.8 Single market

In the single market case

$$A = \frac{\beta}{2(1+\beta)} \left(\mathbb{1} - I\right) \,, \tag{110}$$

and

$$q_t = \max\left\{\frac{\alpha}{\beta}\mathbf{1} - A q_{t-1}, 0\right\} \,. \tag{111}$$

The eigenvalues of A are $\lambda_1 = \frac{\beta}{2(1+\beta)}$, with eigenspace $E_{\lambda_1} = \mathbf{1}^\perp$, and $\lambda_2 = \frac{\beta}{2(1+\beta)}(J-1)$ with eigenspace $E_{\lambda_2} = \mathbf{1}$.

As $|\lambda_2| \geq 1$ for $J \geq \frac{2(1+\beta)}{\beta} + 1$ the dynamics can become unstable if the number of firms is sufficiently large. The nonnegativity constraint leads to oscillations between the monopoly solution and 0 in case of instability.

Also, note that aggregate quantities converge at the same rate as individual quantities as in the single market case the aggregation vector is $\mathbf{1}$, which is clearly not orthogonal to E_{λ_2}.

A.9 Complete Network

In the complete network case

$$A = \frac{1}{2}(\mathbf{I} - I) - \begin{pmatrix} t_1\mathbb{0} & t_1\mathbb{1} & \dots & t_1\mathbb{1} \\ t_2\mathbb{1} & t_2\mathbb{0} & \dots & t_2\mathbb{1} \\ \vdots & \vdots & \ddots & \vdots \\ t_J\mathbb{1} & t_J\mathbb{1} & \dots & t_J\mathbb{0} \end{pmatrix} \tag{112}$$

$$= \begin{pmatrix} t_1\mathbb{0} & t_1\mathbb{1} + \frac{1}{2}I & \dots & t_1\mathbb{1} + \frac{1}{2}I \\ t_2\mathbb{1} + \frac{1}{2}I & t_2\mathbb{0} & \dots & t_2\mathbb{1} + \frac{1}{2}I \\ \vdots & \vdots & \ddots & \vdots \\ t_J\mathbb{1} + \frac{1}{2}I & t_J\mathbb{1} + \frac{1}{2}I & \dots & t_J\mathbb{0} \end{pmatrix} \tag{113}$$

and $t_j = t$.

Let $v_j \in \mathbb{R}^M$ be an eigenvector component corresponding to firm j. Consider a vector

$$w = \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix},$$

so $v_j = v$ for all j. In addition, if we rearrange w in the familiar matrix with the firm eigenvector components as columns then v needs to be such that $W^\top \mathbf{1} = 0$. Then w is an eigenvector of A. Let us consider a block of A,

$$t\mathbb{1}v + \frac{1}{2}v = \frac{1}{2}v\,. \tag{114}$$

There are $J-1$ such blocks per row and thus $Aw = \frac{J-1}{2}w$ and w is an eigenvector with eigenvalue $\lambda = \frac{J-1}{2}$. For $J \geq 3$ this eigenvalue is greater than 1 in absolute terms. Thus, the steady state is unstable and the system will not converge to the steady state.

A.10 Linear costs

If costs are linear, quantities are $q_t^{m,j} = \frac{a_{t-1}^{m,j} - c^j}{\beta}$ if $a_{t-1}^j > c^j$. Then

$$a_t = a_{t-1} + \frac{1}{t}\left(\alpha \operatorname{vec} G - \frac{1}{2}\left(A a_{t-1} - C\left(c \otimes \mathbf{1}\right)\right) + \left(\mathbf{1} \otimes \varepsilon_t\right) \circ \operatorname{vec} G\right), \tag{115}$$

where

$$A = \operatorname{diag} G \left(\mathbf{I} + I\right) \operatorname{diag} G \tag{116}$$

$$C = \operatorname{diag} G \left(\mathbf{I} - I\right) \operatorname{diag} G, \tag{117}$$

and

$$c = \begin{pmatrix} c^1 & \dots & c^J \end{pmatrix}^\top . \tag{118}$$

This system is derived under the assumption that initial beliefs and costs are such that all firms produce for all markets they are connected to and that prices are never zero. It can well be that some firms do not find it profitable to produce for a market in equilibrium or the transient phase. In that case, the corresponding columns of A and C are all zeroes and we would analyze a smaller system. This has no effect on the convergence properties and without loss of generality we can assume that all firms produce for all markets they are connected to.

Note that we analyze the eigenvalues of A but the dynamics of a_t are governed by the eigenvalues of $-\frac{1}{2}A$.

Same as before there is an eigenvalue $\lambda_1 = 0$ with eigenspace $E_{\lambda_1} = \ker \operatorname{diag} G$ and eigenvectors $\mathbf{e}^i$ if $\operatorname{diag}(G)_{i,i} = 0$. The dimension of this eigenspace is $MJ - D$, i.e. the total possible number of connections minus the actual number of connections.

There are two further types of eigenvectors of A. For the first set we construct a vector in $\mathbb{R}^{MJ}$ using the standard unit vectors $\mathbf{e}^m$ for $m \in 1, \dots, M$ as

$$v_m = \operatorname{diag} G \left(\mathbf{1} \otimes \mathbf{e}^m\right) = \operatorname{diag} G \begin{pmatrix} \mathbf{e}^m \\ \vdots \\ \mathbf{e}^m \end{pmatrix}, \tag{119}$$

i.e. the vector v_m consists of a combination of standard unit vectors if the corresponding firm is connected to market m, and zero vectors if the firm is not connected to market m. With some abuse of notation, we denote the corresponding eigenvalues by $\lambda_2 = J_m + 1$ for $m \in 1, \ldots, M$, where J_m is the number of firms in market m. These M eigenvalues may be distinct or not, but the geometric multiplicity of each is equal to the algebraic multiplicity.

Let us denote the collection of these eigenvectors by

$$V = \{\operatorname{span} v_m\}_{m \in \mathcal{M}} \,, \tag{120}$$

and v_m as described above.

Note that $v \in V$ implies $v \in \operatorname{im} \operatorname{diag} G$. Then,

$$\operatorname{diag} G \left(\mathbf{I} + I\right) \operatorname{diag} G v_m = \operatorname{diag} G \left(J_m(\mathbf{1} \otimes \mathbf{e}^m) + v_m\right) = \left(J_m + 1\right) v_m \,. \tag{121}$$

Thus, $v_m \in V$ is indeed an eigenvector with corresponding eigenvalue λ_2.

Similar to the proof of Proposition 3.2 in Appendix A.4, we can construct two subspaces to define the last eigenspace.

Let us for a vector

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_J \end{pmatrix} \in \mathbb{R}^{MJ}$$

again define the corresponding $M \times J$ matrix

$$W = \begin{pmatrix} w_1 & \ldots & w_J \end{pmatrix} \,.$$

Let

$$U = \left\{ w \in \mathbb{R}^{MJ} \mid w = \operatorname{diag} G w \right\} \tag{122}$$

and

$$S = \{ w \mid W\mathbf{1} = \mathbf{0} \} \,. \tag{123}$$

Take a vector $w \in S \cap U$, then

$$\operatorname{diag} G \left(\mathbf{I} + I\right) \operatorname{diag} G w = \operatorname{diag} G \left(\mathbf{I} + I\right) w = \operatorname{diag} G w = w \,, \tag{124}$$

where $\mathbf{I}w = \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix}$ because $w \in S$. Thus, w is an eigenvector with eigenvalue $\lambda_3 = 1$ and the eigenspace $E_{\lambda_3} = S \cap U$ has dimension $D - M$ as shown in the proof of Proposition 3.2 in Appendix A.4.

Aggregate production for market m is given by

$$q_t^m = \sum_{j \in \mathcal{J}_m} \frac{a_{t-1}^{m,j} - c^j}{\beta} = \frac{1}{\beta} \left(\operatorname{diag} G \begin{pmatrix} \mathbf{e}^m \\ \vdots \\ \mathbf{e}^m \end{pmatrix} \right)^\top (a_{t-1} - c \otimes \mathbf{1}) \, . \tag{125}$$

Take a vector $v \in E_{\lambda_3}$, then

$$\left(\operatorname{diag} G \begin{pmatrix} \mathbf{e}^m \\ \vdots \\ \mathbf{e}^m \end{pmatrix} \right)^\top v = 0 \quad \forall m \in \mathcal{M} \, , \tag{126}$$

because $v \in S \cap U$.

Thus, also in the linear cost case, we see that vectors $v \in E_{\lambda_3}$ are orthogonal to the aggregation mapping and thus aggregate quantities converge faster than individual quantities. The speed of convergence of individual quantities and beliefs is the same as in the quadratic cost case.

A.11 Heterogeneous quadratic costs

If we allow cost functions to be heterogeneous then, $q_t^j = \frac{a_{t-1}^j}{(c^j + \beta)}$, and

$$a_t^j = a_{t-1}^j + \frac{1}{t} \left(\alpha - \frac{\beta}{2} \sum_{i \neq j} \frac{a_{t-1}^i}{(c^i + \beta)} - a_{t-1}^j + \varepsilon_t \right) \, , \tag{127}$$

Rewriting leads to a system of linear difference equations,

$$a_t = a_{t-1} + \frac{1}{t} (\alpha \mathbf{1} - A a_{t-1}) + \frac{1}{t} (\varepsilon_t \mathbf{1}) \, , \tag{128}$$

where

$$A = \begin{pmatrix} 1 & \frac{\beta}{2(c^2+\beta)} & \cdots & \frac{\beta}{2(c^J+\beta)} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\beta}{2(c^1+\beta)} & \frac{\beta}{2(c^2\beta+)} & \cdots & 1 \end{pmatrix} . \tag{129}$$

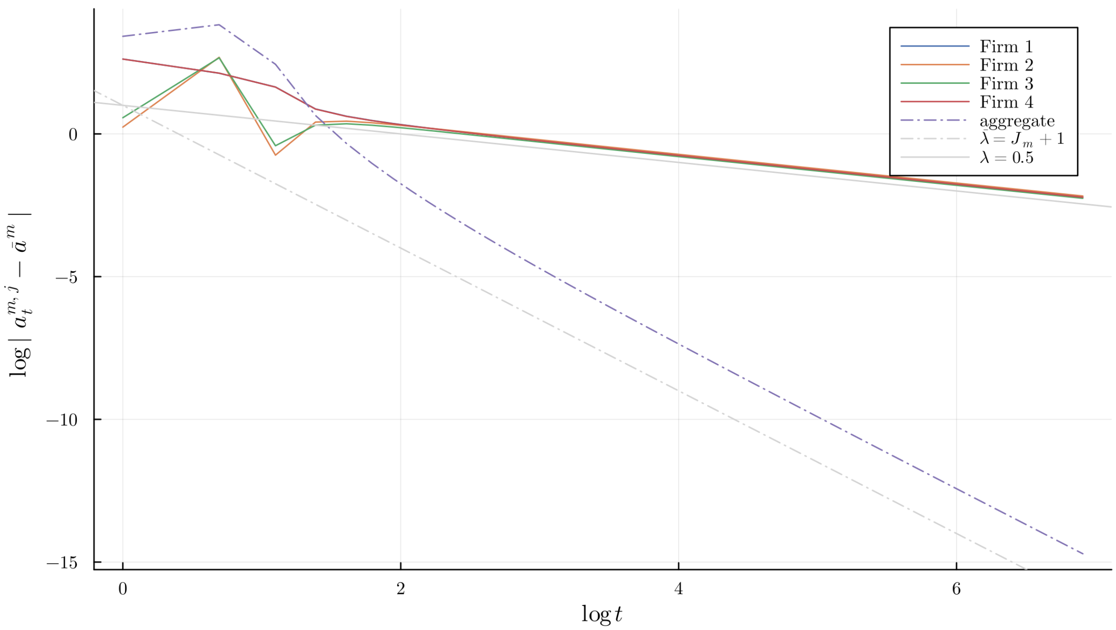


Figure 8: Log-deviations of aggregate and individual beliefs from the equilibrium values for linear cost functions.

This matrix is no longer symmetric making it more difficult to deduce the exact eigenvalues. However, we can bound the eigenvalues of A and show that they are negative such that also with heterogeneous costs the steady state is stable.

Proposition A.2. *Let* $s_{\max} = \max_j \frac{\beta}{2(c^j+\beta)} \in \left(0, \frac{1}{2}\right)$. *Then, all eigenvalues of* $-A$ *are negative and bounded above by* $\lambda \leq s_{\max} - 1 \leq -\frac{1}{2}$.

Proof. This is a slightly modified proof of Theorem A.1. Let $s_j = \frac{\beta}{2(c^j+\beta)} \in \left(0, \frac{1}{2}\right)$. We can rewrite A as,

$$A = (\mathbb{1} - I)\,\text{diag}\,(s_1, \ldots, s_J) + I\,.$$

Furthermore, denote by $s_{\max} = \max_j s_j$ and let $P = \text{diag}\left(\sqrt{\frac{s_1}{s_{max}}}, \ldots, \sqrt{\frac{s_J}{s_{max}}}\right)$. Then, $\text{diag}\,(s_1, \ldots, s_J) = s_{\max} P^2$ and P is positive definite and $\|P\|_2 = 1$.

Then,

$$A = s_{\max}\,(\mathbb{1} - I)\,P^2 + I\,,$$

and,

$$-A = s_{\max}\,(I - \mathbb{1})\,P^2 - I\,.$$

Thus,

$$\sigma\,(-A + I) = \sigma\left(s_{\max}\,(I - \mathbb{1})\,P^2\right) = s_{\max}\sigma\,(P\,(I - \mathbb{1})\,P)\,.$$

A matrix A is congruent to a matrix B if there exists an invertible matrix P such that $A = PBP^T$. Here P is a square diagonal matrix with positive entries on the diagonal so it is invertible. Thus, $P\,(I - \mathbb{1})\,P$ is congruent to the matrix $I - \mathbb{1}$. Sylvester's law of inertia states that congruent matrices have the same number of positive, negative and zero eigenvalues.

The matrix $I - \mathbb{1}$ has $J - 1$ positive eigenvalues of 1 and one negative eigenvalue of $-(J - 1)$.

Since, $P\,(I - \mathbb{1})\,P$ is congruent to the matrix $I - \mathbb{1}$ by Sylvester's law of inertia it also has $(J - 1)$ positive eigenvalue. Let v be the unit eigenvector associated with the largest positive eigenvalue of $P\,(I - \mathbb{1})\,P$, $\lambda_{\max}$. Then,

$$\begin{aligned} 0 &< s_{\max}\lambda_{\max}\,(P\,(I - \mathbb{1})\,P) \\ &= s_{\max} v^T P\,(I - \mathbb{1})\,Pv \end{aligned}$$

$$
\begin{aligned}
&= s_{\max} \|Pv\|_2^2 \left(\frac{Pv}{\|Pv\|_2} \right)^T (I - \mathbb{1}) \left(\frac{Pv}{\|Pv\|_2} \right) \\
&\leq s_{\max} \left(\frac{Pv}{\|Pv\|_2} \right)^T (I - \mathbb{1}) \left(\frac{Pv}{\|Pv\|_2} \right) \\
&\leq s_{\max} \max_{\|y\|_2 = 1} y^T (I - \mathbb{1})\, y \\
&= s_{\max} \lambda_{\max} (I - \mathbb{1}) \\
&= s_{\max}.
\end{aligned}
$$

It follows that $0 < \lambda_{\max}(-A + I) \leq s_{\max}$ and $\lambda_{\max}(A) \leq s_{\max} - 1 \leq -\frac{1}{2}$. □