Risk-taking with Financing Constraints*

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February 2024

Abstract

We study the impact of liquidity constraints on firms' risk-taking that depends on leveraged returns and leveraged volatility. We show that an easier liquidity condition may or may not encourage risk-taking. Following an interest-rate cut, fewer firms take risks if their leverage is low, but more firms take risks if their leverage is high. Therefore, in the aggregate, we can observe a non-monotone relationship between the interest rate and firm-return volatility. A calibrated version of the model shows that an optimal real (deposit) interest rate should depend on the correlation of firms' risky project returns. The model also implies an optimal mix of interest rate and prudential policies that are non-monotone concerning risk spillover.

Key Words: financing constraint; leveraged returns; leveraged volatility; risk-taking

JEL code: G32; E44

*We thank useful discussions with Antonio Guarino, Leo Kaas, Rong Li, Albert Marcet, Franck Portier, Morten Ravn, Wei Xiong, and Shengxing Zhang. We are also grateful for comments from participants in various seminars and workshops. This research is supported by the ESRC grant #ES/R007667/1. Renbin Zhang thanks the National Natural Science Foundation of China for financial support through Grant 72203130.

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1 Introduction

For the society as a whole, great opportunities often come from risk-taking. The development of new ideas and paths also relies on risk-taking. On the individual level, however, the incentive to take risks depends on many factors, including the risk profile of a project, the risk tolerance of the decision maker, financing constraints, and the relevant interest rate. What are the effects of financing constraints on firm risk-taking behaviors in a low interest rate environment? Does an interest-rate cut encourage or discourage risk-taking? Should interest rate policy and leverage policy coordinate when risk-taking is concerned?

This paper answers these questions by examining jointly interest rate and leverage limit on firms' risk-taking behaviors. We consider an environment with three types of investment, risk-free savings, a risk-free, costly project with a higher return, and a risky project with the highest expected return. In the model, firms can borrow when implementing a risky or risk-free project. The firm incurs idiosyncratic costs when implementing a risky project. For example, testing and prototyping costs in research and development can vary widely depending on the nature of the projects and the industries. The existence of the risk-free project provides an alternative option to the risky project. Firms may or may not implement the risky project depending on the idiosyncratic cost and the borrowing capacity. Therefore, the model features an "extensive margin" of risk-taking.

We emphasize two effects. First, a rise in liquidity available to firms, either by increasing the leverage or reducing the interest rate, mitigates the financing constraint and raises the firm profit or firm value. This "valuation effect" *reduces* the incentive to take risks since firms would like to secure the benefits from the rise in liquidity and accumulate more capital with certainty. Nevertheless, more liquidity still brings the second, more conventional "searching for yield" effect that makes the risky project more attractive, or a "substitution effect". We show that there can be a non-monotone relationship between liquidity conditions and firms' risk-taking, which highlights the complex nature of liquidity policy. This complex nature is especially important for the interaction between interest rate and macro-prudential policy, which directly influences the tightness of financing constraints.

Specifically, the model generates a threshold cost of implementing the risky project, below which firms opt for the risky project. We study how leverage constraints and interest rate affect this threshold. The effects could be captured by two terms: 1). an impact on leverage return, summarized as the return ratio between the risky and the risk-free projects; 2). an impact on the leveraged risk, summarized as an adjusted Sharpe ratio of the leveraged return.

We highlight this non-monotone result through the above value and substitution effects. When firm leverage is already low, a further cut in interest rate R mainly increases the wealth, and fewer firms take risks. This logic explains why a low-interest-rate policy may not stimulate firms to take risky but socially productive projects. When a firm can take a high leverage, the (leveraged) valuation effect is dominated by the substitution effect arising from the decline in the adjusted Sharpe ratio of running a risky project (for the same amount of interest rate cut). Therefore, in this case, an interest rate cut encourages more risk-taking. This case also implies that "searching-for-yield" (e.g. Rajan (2006)) type of behaviors in a low-interest rate environment is more likely when a firm /financial institution is highly leveraged.

Using various proxy measures for risk-taking, we compare the effect of an interest-rate cut by the Federal Reserve in different periods characterized by high and low leverages. The evidence is consistent with our theory. Our model may show why cutting interest rates may not stimulate firms to take risky but socially productive projects, leaving marginal products of capital and/or interest rates low in equilibrium. Our calibration suggests that an optimal interest rate level can exist for leverage limit. Importantly, the spillover of risks across projects moves the optimal interest rate. It turns out to be even better to have an optimal mix of interest rate and leverage limit. As project outcomes become more positively correlated, interest rates should be lowered to better move resources towards firms with projects; however, to avoid too much or too little risk-taking (as compared to the socially desired level), the leverage limit should fall first and then quickly rise with the correlation.

Related literature. Our theory of option value with financial conditions are closest to the literature on entrepreneurship with financial development, e.g. Buera and Shin (2013) and Buera et al. (2015). There, agents can choose between an entrepreneur or a worker, and financial frictions affect the choice. The agent's optimization problem exhibits non-convexity. The risk-taking channel is added on top of career choice in Vereshchagina and Hopenhayn (2009) where agents make occupational choices and risky project choices under no-borrowing constraint. Agents who choose to be entrepreneurs but who have an intermediate range of wealth may become risk-seeking, since this "lottery" can smooth the value functions. Compared to this literature, we emphasize how changes in the financing constraints affect agents' risk-taking and the effect of interest rate policy and welfare implications when projects become correlated. To illustrate the mechanism at work, we choose a highly tractable framework to deliver closed-form risk-taking policy functions. This technical aspect may be of independent interest.

Our framework connects naturally to financial frictions[1] and risks in general. Building on Bernanke et al. (1999), Christiano et al. (2014) show that risk shocks in a quantitative business-cycle model with financing constraints generate counter-cyclical spreads and account for a large fraction of macroeconomic fluctuations. Miao and Wang (2010) include long-term defaultable debt in a macroeconomic model with financial shocks to the recovery rate. Recently, Cui and Kaas (2020) develop a framework to analyze belief risks in credit contracts. We add to this literature by examining the effect of financing constraint/interest rates on endogenous risk-taking behaviors.

Our work also adds to the strand of literature on active risk management: the seminal paper by Froot et al. (1993) in the static case and Bolton et al. (2011) in the dynamic case show that financing constraints generate the rationale for active risk management. DELL'ARICCIA et al. (2017) illustrates the channel of risk shifting of financial intermediaries when the interest rate changes. We illustrate the potential non-monotone effect between the project choice and financing constraint via a project selection problem. This problem becomes important even at the aggregate level since the liquidity policy may generate unintended volatilities and welfare effects.

The rest of the paper is organized as follows: We first present stylized facts about the non-monotone firm volatility to variations in interest rates. Section 3 presents a simple two-period model, which illustrates the key effects of interest rate and leverage on risk-taking. Section 4 extends the two-period model into an infinite-horizon economy in which the correlation between project returns is considered. Section 5 shows quantitatively how the interest rate and leverage limit shape the macroeconomic outcomes, followed by the optimal mix of interest rate policy and leverage policy and particularly how correlated projects affect the result. Section 6 concludes.

[1] The literature on financial frictions is vast, and the seminal contribution at least includes Kiyotaki and Moore (1997), Bernanke et al. (1999), Mendoza (2010), and Brunnermeier and Sannikov (2014).

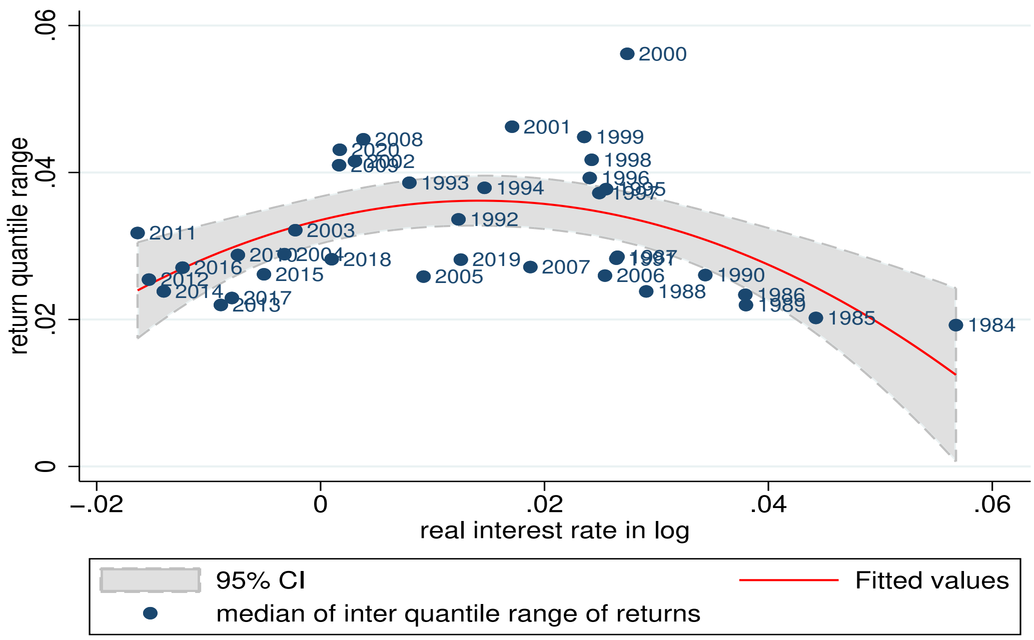
2 Motivating Evidence

We start with some evidence about the relationship between firms' risk-taking behavior and interest rate/leverage.

For the risk-taking behavior of a firm in each year, as a proxy, we first use a variation measure of its stock return (in log) in that year. When firms implement riskier projects, the stock return volatility should be higher (keeping everything else constant) as there is a lot more uncertainty involved. We draw the daily stock return for non-financial firms from CRSP (Appendix A contains the data description) between 1984 and 2020. The daily return's inter-quartile range (IQR) is then used to approximate the degree of firms' risk-taking. IQR, compared to standard deviation, has the advantage of not being influenced by extreme volatility. The result is robust if we use standard deviation. Further, for an alternative proxy for "risk-taking" behavior, the result is also robust if we replace stock return variation by risky investment-spending intensity measured by the return on assets (ROA) variation or ratio of firms' R&D spending to total assets (from COMPUSTAT).

Figure 1 plots the median IQR of all individual stock returns against the log level of the Cleveland rate (a popular measure of real interest rate provided by the Federal Reserve Bank of Cleveland) across years. A quadratic fitting between these two variables with a 95% confidence interval is also plotted. When the real interest rate is relatively high, we observe a negative relationship between the interest rate and stock return variability. As the interest rate drops to a lower level, the mentioned relationship turns positive. In addition, we find that this non-monotone relationship holds within each sector classified by the one-digit SIC code (see Appendix B).

Figure 1: volatility and interest rate



To further quantify the effects of interest rates on a risk-taking proxy, we consider the following empirical specification:

$$y_{ijt} = \beta_0 + \beta_1(R_t)\Delta R_t + X_{ijt-1} + f_i + h_j + \Gamma_t, \tag{1}$$

where y_{ijt} is the inter-quantile range of the stock return of firm i in sector j at year t in the baseline regression, R_t is the level of interest rate, ΔR_t is interest-rate shock, Γ_t is an aggregate control which includes the lagged interest rate, the lagged and current terms of inflation, unemployment rate and real GDP growth rate, f_i controls the firm fixed effects and h_j controls for the sector effect. We also control for a number of factors (X_{ijt-1}) as the interest rate might influence risk-taking activities through these variables. The vector of controls X_{ijt-1} includes the firm leverage ratio measured by long-term debt over fixed assets (property, plant, and equipment), size measured by total assets in log, market to book ratio, return on assets (ROA), return on equity (ROE), return on investment (ROI), Tobin's Q, cash holdings measured by cash, short-term investment over total assets the lagged dependent variable y_{ijt-1}. All controls in X_{ijt-1} are in lagged terms.[2] We cluster standard errors in firm ID and year to account for correlation within firms and year.

Our main coefficient of interest is β_1, showing how the interest rate level affects firms' risk-taking behavior measured by y_{ijt}. Notice that $\beta_1(R_t)$ is allowed to depend on the interest rate level. Motivating by Figure 1, we divide the sample periods into two, a period of low interest rate levels and another period of high interest rate levels. To determine whether a particular period is characterized by low or high interest rates. We use different measures for real interest rate: the log level of the federal fund rate deflated by the past inflation ("ffr" in Table 1) and deflated by the expected inflation ("ffr2"); the log level of the Cleveland rate; and the log level of the real prime rate (prime rate in Table 1).

We use the median of the interest as the cutoff. The period with an interest rate higher (lower) than the median is classified as the high (low) rate period. For (exogenous) interest-rate variations, we use the Jarociński and Karadi (2020) monetary policy shock (mpshock) as the exogenous interest rate changes.

Table 1 reports the results from estimating the specification in equation (1) using the firm inter-quantile range to measure the risk-taking behaviors and β_1 is significantly positive when the interest rate is low while and negative when the interest rate is high. An unexpected unit of positive interest rate shock[3] will increase the firm inter-quantile range by roughly 100-200 bps in a low-interest-rate environment while reducing it by 20-60 bps in a high-interest-rate environment.

Following JOHN et al. (2008) and Boubakri et al. (2013), we use variation in ROA to measure risk-taking behaviours. More specifically, "risk1" is constructed by using the standard deviation of the leading five years (t from 0 to 4) ROA[4] of the firm. Results are included in Table 2. An unexpected unit of positive interest rate shock will increase the standard deviation of ROA in a low-interest-rate environment, while reducing it in a high-interest-rate environment.

For robustness, in Appendix B, Table 5 and Table 6 consider alternative measures for risk-taking: "risk2" is the difference between the maximum and the minimum of the leading five years (t from 0 to 4) ROA(similarly constructed as in Table 2); and research intensity($R\&D$ expenses over total assets). The results are similar.

[2]We draw these firm level-financial data from US COMPUSTAT, which covers the period from 1984-2016.
[3]3-6 bps according to Jarociński and Karadi (2020).
[4]subtracts the industry mean level ROA

Table 1: Variation in IQR with respect to Changes in Interest Rate.

VARIABLES	(1) return iqr	(2) return iqr	(3) return iqr	(4) return iqr	(5) return iqr	(6) return iqr	(7) return iqr	(8) return iqr
mpshock	0.019***	-0.006	0.015**	-0.002	0.009*	-0.003	0.014***	-0.005
	(0.005)	(0.006)	(0.005)	(0.012)	(0.005)	(0.008)	(0.004)	(0.007)
Constant	0.031***	-0.015	0.034***	-0.029	0.042***	-0.022	0.018*	-0.007
	(0.007)	(0.018)	(0.006)	(0.022)	(0.007)	(0.018)	(0.009)	(0.019)
Observations	45,344	48,562	45,054	49,149	43,501	49,554	45,344	48,562
R-squared	0.701	0.676	0.698	0.681	0.702	0.679	0.699	0.675
Firm FE	YES	YES	YES	YES	YES	YES	YES	YES
Interest use	ffr	ffr	ffr2	ff2	Cleveland rate	Cleveland rate	prime rate	prime rate
Interest level	low	high	low	high	low	high	low	high

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Note: Results from estimating equation (1). Alternative measures of the real interest rate are employed to classify the interest rate regime: the log level of the real federal fund rate deflated by the past inflation (ffr columns 1 and 2) and deflated by the expected inflation (ffr2 columns 3 and 4); the Cleveland rate(columns 5 and 6); the log level of the real prime rate (columns 7 and 8).

Table 2: Variation in ROA with respect to Changes in Interest Rate.

VARIABLES	(1) risk1 in log	(2) risk1 in log	(3) risk1 in log	(4) risk1 in log	(5) risk1 in log	(6) risk1 in log	(7) risk1 in log	(8) risk1 in log
mpshock	1.021**	0.189	1.597***	-1.166***	1.532**	-0.683**	1.216***	0.208
	(0.394)	(0.291)	(0.493)	(0.269)	(0.539)	(0.273)	(0.265)	(0.302)
Constant	0.867*	1.520	0.912*	-0.093	0.907**	1.242	1.758***	1.071
	(0.449)	(0.909)	(0.502)	(0.804)	(0.363)	(0.798)	(0.505)	(0.883)
Observations	43,195	45,250	42,934	45,720	41,423	46,311	43,195	45,250
R-squared	0.817	0.888	0.802	0.892	0.708	0.889	0.818	0.888
Firm FE	YES	YES	YES	YES	YES	YES	YES	YES
Interest use	ffr	ffr	ffr2	ff2	Cleveland rate	Cleveland rate	prime rate	prime rate
Interest level	low	high	low	high	low	high	low	high

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Note: Results from estimating equation (1). Alternative measures of the real interest rate are employed to classify the interest rate regime: the log level of the real federal fund rate deflated by the past inflation (ffr columns 1 and 2) and deflated by the expected inflation (ffr2 column 3 and 4); the Cleveland rate(columns 5 and 6); the log level of the real prime rate (columns 7 and 8).

3 A Simple Leveraged Risk-taking Model

In this section, we examine the effect of financing constraints on risk-taking behaviors in a simple two-period model. When the payoff of running a firm is bounded below by risk-free projects, entrepreneurs have incentives to select riskier projects because of an option-value consideration. However, leverage constraints change the option value significantly, and we explicitly model this effect. We use an environment similar to Cui and Kaas (2020) to achieve a closed-form solution, but different in the crucial risk-taking aspect.

3.1 The Environment

There are two periods and two types of projects in the economy. A continuum of entrepreneurs, with measure one indexed by $i \in [0, 1]$, pick one of the two types of projects. The risky project has two possible returns: with probability p, the return is Π^h; and with probability $1 - p$, the return is Π^l. The expected return of the risky project is denoted by

$$\Pi = p\Pi^h + (1-p)\Pi^l.$$

An entrepreneur may choose a safe/risk-free project, with return Π^f. When implementing a project, the entrepreneur can borrow at a gross interest rate of R.

For exposition reasons, we assume that

$$\Pi^h > \Pi > \Pi^f > R > \Pi^l. \tag{A1}$$

It is natural to have the expected return Π of the risk project to be higher than the return Π^f of a safe project. Unlike Vereshchagina and Hopenhayn (2009), who study choices of projects with the same expected return,[5] we impose $\Pi > \Pi^f$ so that socially it may be optimal to take risks while it may not be the case at individual levels. The return of the risky project is higher than Π^f when the project turns out to be successful (i.e., with the return Π^h), while the return is lower than Π^f if the project is unsuccessful (i.e., with the return Π^l). In this simple model, we assume that $\Pi^f > R$ so the safe project has a higher return than the risk-free saving rate. This might be natural as running a firm generally should have a higher return than that of borrowing. Finally, our analysis abstracts from saving in risk-free assets (instead of projects) since they are dominated by running safe projects. In the macroeconomic model, we will consider a case later in which an entrepreneur could be indifferent between a safe project and using the risk-free rate to save.

In addition, throughout this paper, we assume that

$$\Pi^h\Pi^l > \Pi^f\hat{\Pi} \text{ where } \hat{\Pi} \equiv p\Pi^l + (1-p)\Pi^h \tag{A2}$$

so risk-free project return is not too high, which captures reasonable features in practice and which turns out to be useful in simplifying the analysis.

Let ω denote the wealth level of the entrepreneur each period after repaying previous debt, and let s be the amount invested in a project which can be used as collateral. We assume the utility function of the entrepreneur as

$$u(c) = \log c,$$

where consumption/dividends $c = \omega - s$. It may initially seem that the entrepreneur owns the firm, but the entrepreneur can also be its manager, so c should be interpreted as a dividend payout. Besides obtaining closed-form solutions, there are several advantages of assuming $u(c) = \log c$. The curvature in $u(.)$ captures the manager's preference for dividend smoothing. Lintner (1956) first showed that managers consider dividend smoothing over time, a fact further confirmed by subsequent studies. In addition, increasing c can be interpreted as equity share repurchases while reducing c can be thought of as sales of new shares. Putting curvature in $u(.)$ is thus a simple way of modeling the speed with which firms can vary the funding source when financial conditions change.[6]

There is a (utility) running cost from implementing the risky project, denoted by η which is drawn from an i.i.d. distribution with a support $[\underline{\eta}, +\infty)$ (note: the lower bound does not have to be positive).

[5] Their main goal is to answer why entrepreneurs take risks even when expected returns are the same as the safe return.
[6] Jermann and Quadrini (2012) also assume dividend adjustment costs, supported by empirical evidence cited therein.

The value of an entrepreneur with running (utility) cost η and wealth ω can be written as

$$V(\omega,\eta) = \max\{V^r(\omega,\eta), V^f(\omega)\}.$$

Here, we use the realization of η to control for whether the firm chooses a risky or a risk-free project in period t. Using utility cost as in the firm default problem of Cui and Kaas (2020) permits closed-form solutions, and it is equivalent to costs in terms of goods proportional to entrepreneurs' wealth (given the utility and technology in this environment). We abstract from the default issue to focus on the project choices, and firms are required to repay all their debt in both cases.[7]

Let b denote the level of borrowing. The value of an entrepreneur, who implements the risky project, $V^r(\omega,\eta)$ can be written as

$$\begin{aligned} V^r(\omega,\eta) = \max_{s,b}\{&\log(\omega-s) - \eta + \beta p \log(\Pi^h(s+b) - Rb) \\ &+ \beta(1-p)\log(\Pi^l(s+b) - Rb)\} \\ \text{s.t. } & b \leq \bar{\theta}s, \end{aligned}$$

where $\bar{\theta}$ is a parameter that governs the tightness of the borrowing constraint. The entrepreneur can put in internal savings s, together with the borrowing b; next period, the entrepreneur earns either $\Pi^h(s+b)$ if the project turns out to be productive, or $\Pi^l(s+b)$ if the project turns out to be unproductive. The interest payment is naturally Rb. The amount of borrowing, b, is limited due to financial frictions. We use a collateral constraint specification, i.e., the entrepreneur can borrow up to $\bar{\theta}$ fraction of its capital, s at the time of borrowing.

The value function of an entrepreneur who implements the safe project $V^f(\omega)$ can be written as

$$\begin{aligned} V^f(\omega) = \max_{s,b} &\log(\omega-s) + \beta\log(\Pi^f(s+b) - Rb) \\ \text{s.t. } & b \leq \bar{\theta}s. \end{aligned}$$

Compared to the value function $V^r(\omega,\eta)$, the cost of implementing the risk-free project is normalized to be 0, and the return next period is certain.

The optimization shows that s is linear in wealth level ω. That is, $s = \varphi\omega$, with $\varphi = \frac{\beta}{1+\beta}$. When entrepreneurs choose the risk-free project, their firms will borrow up to the limit $\theta = \bar{\theta}$ since $\Pi^f > R$. When entrepreneurs choose the risky project, depending on the interest rate R, θ can be one of the three, 0, $\bar{\theta}$, and θ^* as the level of the leverage a firm will choose (by ignoring the credit limit) which satisfies:

$$\theta^* \equiv -\frac{\Pi^h\Pi^l - R\hat{\Pi}}{(\Pi^l - R)(\Pi^h - R)} = -\left[p\frac{\Pi^l}{\Pi^l - R} + (1-p)\frac{\Pi^h}{\Pi^h - R}\right]. \tag{2}$$

Assumptions A1 and A2 imply that $\theta^* > 0$. The following proposition determines the threshold cost of taking the risky project, $\tilde{\eta}$.

Proposition 1. *Suppose A1 and A2 hold. The leverage of a firm that implements the risky project satisfies*

$$\theta = \min\{\theta^*, \bar{\theta}\},$$

[7] As in Cui and Kaas (2020), default can introduce multiple equilibria, which is left for future research.

where θ^* is defined in (2) and θ^* decreases in the interest R. Those entrepreneurs with $\eta \le \max\{\underline{\eta}, \tilde{\eta}\}$ choose the risky project and the threshold level $\tilde{\eta}$ satisfies

$$\tilde{\eta} = \beta p \log\left(\frac{\Pi^h + (\Pi^h - R)\theta}{\Pi^f + (\Pi^f - R)\bar{\theta}}\right) + \beta(1-p)\log\left(\frac{\Pi^l + (\Pi^l - R)\theta}{\Pi^f + (\Pi^f - R)\bar{\theta}}\right). \tag{3}$$

Proof. See Appendix C. □

3.2 Non-monotone Risk-takings

What is the effect of interest rate (or, more generally, financial conditions) on risk-taking decisions? Specifically, we investigate how borrowing interest rates affect the risk-taking behavior characterized by $\tilde{\eta}$, respectively. First, we show that risk-taking may not be monotone to the interest rate (borrowing cost). A falling borrowing cost does not necessarily induce more risk-taking. Then, we provide a heuristic approach to understand the results.

To simplify the discussion, let us first focus on the case in which the financing constraint is binding if a risky project is implemented. We will relax the assumption later. Proposition 1 implies as interest increases, θ^* falls so that the financing constraints for entrepreneurs who choose the risky project might change from binding to non-binding.Notice that when $\theta = \bar{\theta}$

$$\frac{\partial \tilde{\eta}}{\partial R} = \beta\left[\frac{\bar{\theta}}{\Pi^f + (\Pi^f - R)\bar{\theta}} - p\frac{\bar{\theta}}{\Pi^h + (\Pi^h - R)\bar{\theta}} - (1-p)\frac{\bar{\theta}}{\Pi^l + (\Pi^l - R)\bar{\theta}}\right], \tag{4}$$

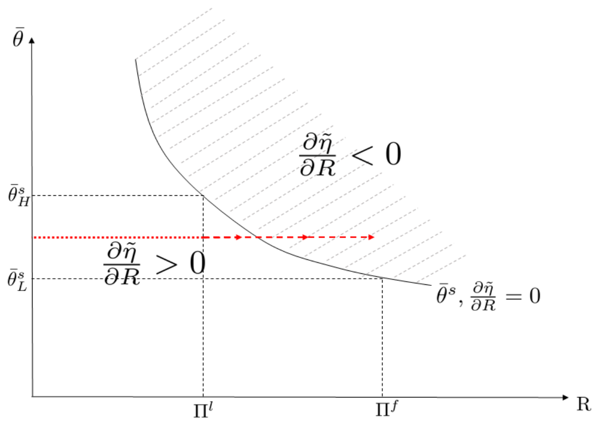
and as will become clear the sign of $\partial\tilde{\eta}/\partial R$ depends on the leverage upper bound $\bar{\theta}$. A higher interest rate will reduce the leveraged return of both projects; however, whether the safe project or the risky project is more attractive, depends on the maximum leverage $\bar{\theta}$. If $\bar{\theta}$ is low (high) enough, the threshold $\tilde{\eta}$ increases (decreases) in the interest rate R, and the risky (safe) project is thus preferred. Overall, $\tilde{\eta}$ has a hump-shaped relationship with R.

Proposition 2. *Suppose A1 and A2 hold. Assuming the financing constraint is binding when the risky project is implemented (i.e., $\bar{\theta} < \theta^*$). There exist cutoff levels $\bar{\theta}^s_L < \bar{\theta}^s_H$ such that the threshold level $\tilde{\eta}$ increases in R when $0 < \bar{\theta} \le \bar{\theta}^s_L$, decreases in R when $\bar{\theta}^s_H \le \bar{\theta} < \theta^*$, and is hump-shaped in R when $\bar{\theta}^s_L < \bar{\theta} < \bar{\theta}^s_H$.*

Proof. See Appendix C. □

The proposition can be illustrated by Figure 2. The horizontal axis is the interest rate R, and the vertical axis is the leverage upper limit $\bar{\theta}$. The downward-sloping curve is $R^s(\bar{\theta})$, which corresponds to the interest rate such that $\partial\tilde{\eta}/\partial R = 0$ (for a given leverage $\bar{\theta}$) and separates the two regions with different signs of the derivative $\partial\tilde{\eta}/\partial R$. Alternatively, the downward-sloping curve also represents $\bar{\theta}^s(R)$,which corresponds to the leverage upper bound such that $\partial\tilde{\eta}/\partial R = 0$ for a given interest rate R. The shaded region above the curve has the feature of $\partial\tilde{\eta}/\partial R < 0$, and the blank area under the curve has the feature of $\partial\tilde{\eta}/\partial R > 0$. When $\bar{\theta} < \bar{\theta}^s_L$, as R increases from the low return realization Π^l to the safe return Π^f, $\partial\tilde{\eta}/\partial R$ is always positive. When $\bar{\theta} > \bar{\theta}^s_H$, as R increases from the low return realization Π^l to the safe return Π^f, $\partial\tilde{\eta}/\partial R$ is always negative. In these two cases, the change of the interest rate R thus has a monotone effect on the threshold $\tilde{\eta}$.

Figure 2: The Hyperplane of $\frac{\partial \tilde{\eta}}{\partial R} = 0$ in $(\bar{\theta}, R)$ Space



However, if $\bar{\theta}$ is middle-ranged i.e. $\bar{\theta} \in (\bar{\theta}^s_L, \bar{\theta}^s_H)$, the effect of interest rate on risk-taking is non-monotone. To see this key result, for the points $(R, \bar{\theta})$ on the red dashed line below the separating curve, $\partial\tilde{\eta}/\partial R$ is positive since $\Pi^l < R < R^s(\bar{\theta})$; for the points $(R, \bar{\theta})$ on the red dashed line above the separating curve, $\partial\tilde{\eta}/\partial R$ turns negative since $R^s(\bar{\theta}) < R < \Pi^f$. In this middle range case, an increase of the interest rate R first encourages more risk-taking behavior and then discourages risk-taking once the interest rate R exceeds $R^s(\bar{\theta})$. This is because an increase in the interest rate will not only increase the cost of borrowing but also affect the trade-off between leveraged return and volatility. The non-monotonicity effect is not unique to varying interest rate; we also find the non-monotonicity effect of leverage as in Appendix C.2.

To better understand the key result, we approximate the threshold $\tilde{\eta}$ to the second order around a point $\tilde{\eta}_s$ (see below) so we can obtain further analytical insight. First, if we ignore the risk involved and imagine there is another hypothetical safe project with the return Π (the same as the expected return of the risky project), the entrepreneur borrows up to the credit limit with this hypothetical safe project (i.e., $\theta = \bar{\theta}$) because the return $\Pi > \Pi^f > R$. Then, the threshold below which an entrepreneur chooses this hypothetical project can be defined as $\tilde{\eta}_s$:

$$
\begin{aligned}
\tilde{\eta}_s &\equiv \beta \log(\Pi(1+\theta) - R\theta) - \beta \log(\Pi^f(1+\theta) - R\theta) \\
&= \beta \log\left(\Pi - R\frac{\bar{\theta}}{1+\bar{\theta}}\right) - \beta \log\left(\Pi^f - R\frac{\bar{\theta}}{1+\bar{\theta}}\right) \\
&\approx \beta(\Pi - \Pi^f)\frac{1}{\Pi^f - R\frac{\bar{\theta}}{1+\bar{\theta}}},
\end{aligned}
$$

after we use (3). Now, we expand $\tilde{\eta}$ to the second order around $\tilde{\eta}_s$ by using the cut-off expression (3):

$$\begin{aligned}\tilde{\eta} \approx \quad & \tilde{\eta}_s \quad +\beta p\left[\frac{1+\bar{\theta}}{\Pi(1+\bar{\theta})-R\bar{\theta}}(\Pi^h-\Pi)-\frac{1}{2}\frac{(1+\bar{\theta})^2}{[\Pi(1+\bar{\theta})-R\bar{\theta}]^2}(\Pi^h-\Pi)^2\right] \\ & \quad +\beta(1-p)\left[\frac{1+\bar{\theta}}{\Pi(1+\bar{\theta})-R\bar{\theta}}(\Pi^l-\Pi)-\frac{1}{2}\frac{(1+\bar{\theta})^2}{[\Pi(1+\bar{\theta})-R\bar{\theta}]^2}(\Pi^l-\Pi)^2\right] \\ = \quad & \tilde{\eta}_s \quad \underbrace{-\frac{1}{2}\frac{\beta\sigma^2}{\left[\Pi-R\frac{\bar{\theta}}{1+\bar{\theta}}\right]^2}}_{\tilde{\eta}_\sigma}, \end{aligned} \tag{5}$$

where $\sigma^2 \equiv p(\Pi^h-\Pi)^2+(1-p)(\Pi^l-\Pi)^2$ is the variance of the risky project itself. $\tilde{\eta}_s$ is associated with the ratio between the leveraged return from the risky and the risk-free project; this term represents the "valuation effect" of leverage or interest rate. $\tilde{\eta}_\sigma$ is a negative half of the inverse of the leveraged Sharpe ratio after adjusting for discounting; this term reflects the disutility related to the volatility of the leveraged return. It represents the "searching for yield effect" or substitution effect.

When the interest rate R increases, financing costs increase, and the entrepreneurs will be less wealthy. The valuation effect makes the risky project more attractive. Taking risks is a gamble entrepreneurs want to engage in when they are poorer since taking chances may raise wealth quickly. Meanwhile, the adjusted Sharpe ratio falls, and the searching for yield effect makes the risky project less attractive so that

$$\frac{\partial\tilde{\eta}_s}{\partial R}=\frac{\beta\bar{\theta}(1+\bar{\theta})(\Pi-\Pi^f)}{[\Pi^f(1+\bar{\theta})-R\bar{\theta}]^2}>0 \text{ and } \frac{\partial\tilde{\eta}_\sigma}{\partial R}=\frac{-\beta\bar{\theta}(1+\bar{\theta})^2}{[\Pi(1+\bar{\theta})-R\bar{\theta}]^3}\sigma^2<0.$$

Depending on parameters, we thus know that a rise in the interest rate R thus can have a non-monotone effect on $\tilde{\eta}$ because the valuation effect and the substitution effect work in the opposite directions, which explains Proposition 2. In fact, for the interest rate $R \in (\Pi^l, \Pi^h)$, a trade-off between the two effects arises when $\bar{\theta}$ is middle ranged.

Notice that the effect of interest rate on risk-taking depends on the leverage constraint measured by $\bar{\theta}$. Rearranging the terms in (5) and $\tilde{\eta}_s$, we obtain

$$\Pi-\Pi^f=\left(\Pi^f-R\frac{\bar{\theta}}{1+\bar{\theta}}\right)\left[\frac{\tilde{\eta}}{\beta}+\frac{1}{2}\left(\frac{\sigma}{\Pi-R\frac{\bar{\theta}}{1+\bar{\theta}}}\right)^2\right],$$

where $\Pi-\Pi^f$ is the expected excess return of the risky project. Therefore, we have a natural asset pricing interpretation of risk management. For entrepreneurs who are indifferent between a safe project and a risky project, the excess (leveraged) return compensates for the risk and the cost of implementing the risky project. That is, on top of the safe leveraged return, the compensation for the risk is $0.5\sigma^2/\left[\Pi-R\bar{\theta}/(1+\bar{\theta})\right]^2$, while the compensation for the cost is simply $\tilde{\eta}/\beta$.

Therefore, an increase in the interest rate makes the portfolio with longing the risky project and shorting bonds riskier than the risky project itself without leverage. Keeping all other parameters Π, Π^f, σ, and $\bar{\theta}$ the same, this increase will reduce the incentive for taking the risk, i.e., putting downward pressure on $\tilde{\eta}$. Nevertheless, an increase of R also makes the safe leveraged return smaller, and to justify the excess return $\Pi - \Pi^f$ for the *marginal* entrepreneur, it must be that the cost of implementing the risky project, $\tilde{\eta}$, for an indifferent entrepreneur goes up.

So far, we have found the non-monotonicity of risk-taking when the liquidity constraint is binding. That is, $\bar{\theta} < \theta^*_{min}$, where $\theta^*_{min} \equiv \frac{\Pi^h \Pi^l - \Pi^f \hat{\Pi}}{(\Pi^h - \Pi^f)(\Pi^f - \Pi^l)}$ is minimal leverage when entrepreneurs are not constrained, which is obtained by assuming $R = \Pi^f$ in (2). We also generalize that the non-monotonicity can persist when the liquidity constraint is not binding (i.e. $\bar{\theta} \geq \theta^*_{min}$).

Proposition 3. *Let $\bar{R}(\bar{\theta})$ be the interest rate level such that $-\left[p\frac{\Pi^l}{\Pi^l - \bar{R}} + (1-p)\frac{\Pi^h}{\Pi^h - \bar{R}}\right] = \bar{\theta}$ when entrepreneurs just become unconstrained. When $\bar{\theta} \geq \theta^*_{min}$, there exist an interest rate level R^u such that the risk-taking for unconstrained entrepreneurs is decreasing in R when $\bar{R}(\bar{\theta}) \leq R < R^u(\bar{\theta})$ and increases in R when $R^u(\bar{\theta}) \leq R < \Pi^f$.*

Proof. See Appendix C. □

3.3 Numerical Illustrations

Now, we illustrate the above discussion by showing numerical examples. An entrepreneur may or may not be financially constrained when implementing the risky project because of the risk profile of the project and the entrepreneur's risk preference.

Figure 3 is a numerical illustration where the parameterization for the exercise is as follows: $\Pi^f = 1.1$, $\Pi^h = 1.3$, $\Pi^l = 0.94$, $p = 0.65$, and $\beta = 0.96$.[8] We do not have to specify $F(\cdot)$ for individual decision problems given the interest rate R. However, it will be used in the analysis later with an equilibrium in the credit market.

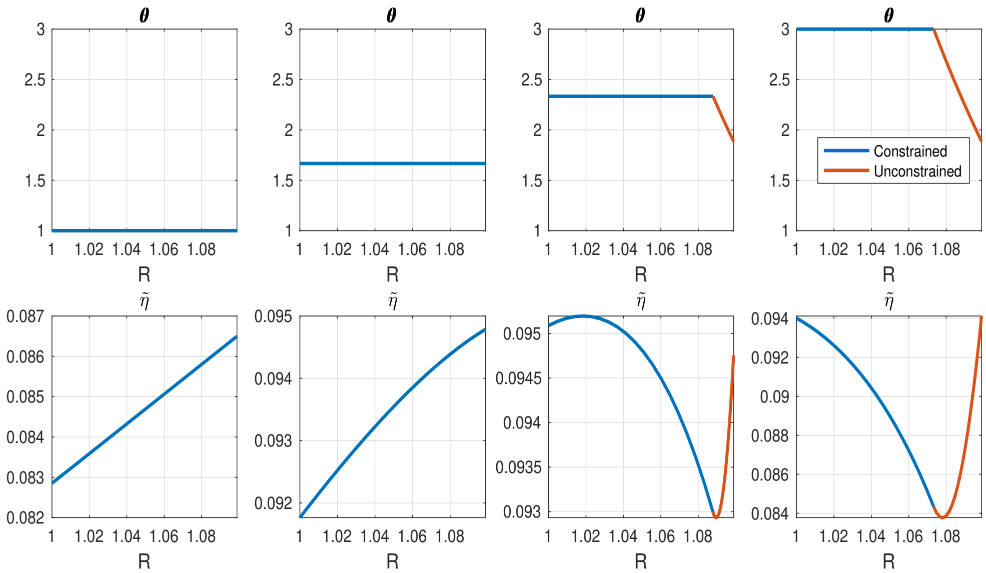
The figure delivers a crucial message of the model: a cut in the interest rate reduces risk-taking when the leverage limit $\bar{\theta}$ is small, while it encourages risk-taking when $\bar{\theta}$ is large. When the leverage is low, a cut in interest rate R mainly works through the "leveraged valuation effect" since entrepreneurs pay less interest; entrepreneurs prefer taking less risk. Notice that the usual valuation effect may encourage risk-taking for an intermediate level of initial wealth and eventually discourage risk-taking when wealth is high enough, also demonstrated in Vereshchagina and Hopenhayn (2009). In our model, the valuation effect is leveraged valuation effect and always discourages risk-taking, keeping everything else constant.

When the leverage is high, however, the leveraged valuation effect is dominated by the substitution effect with the same amount of interest rate cut since the high leverage amplifies the rise in the Sharpe ratio of in the portfolio with the leveraged risky project. This generates a testable implication: "searching for yield" behaviors in a low interest rate environment are more likely when a firm is highly leveraged. In other words, the ease of credit condition encourages risk-taking by reducing the funding cost per unit of capital and increases the Sharpe ratio of the leveraged risky project. The motivating evidence supports this observation.

Note that the financing constraint might become slack for high leverage limits and interest rates, as shown by the orange curve in the third and fourth columns. Consistent with Proposition 3, the risk-taking first decreases and then increases as R rises in this case. Overall, these columns exhibit a complex pattern of risk-taking incentives.

[8] This specification satisfies $\frac{1}{\Pi^f} > \frac{p}{\Pi^h} + \frac{1-p}{\Pi^l}$.

Figure 3: Effect of R on $\tilde{\eta}$ for Alternative $\bar{\theta}$



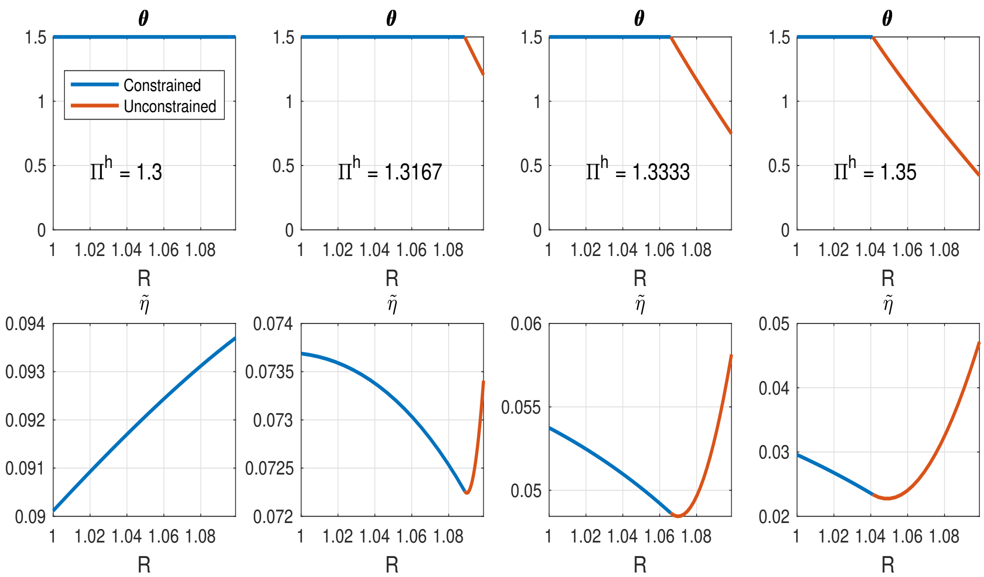
Note: The top panel of each column shows the leverage associated with the risky project; the bottom panel of each column shows the threshold cost for risk-taking. Each column corresponds to a different leverage upper bound (from the left to the right: $\bar{\theta} = 1.00, 1.67, 2.33,$ and 3.00).

Finally, Figure 4 further shows that the effect of borrowing cost R on the risk-taking behavior also depends on the profile of the risky project return Π^l and Π^h. In this experiment, we fix $\bar{\theta} = 1.5$. The first column uses the value of Π^h and Π^l previously specified and shows that a rise in R makes the risky project more appealing as a result of the dominating valuation effect.

Now, we increase Π^h and decrease Π^l simultaneously to keep the mean return of the risky project Π unchanged. For simplicity, let's first look at the case when the financing constraint is binding. If the project becomes riskier, as the interest rate R increases, the adjusted Sharpe ratio falls more, and the risky project becomes less attractive from the volatility concern. In other words, the substitution effect dominates when the risky project becomes riskier. This result is intuitive because when risk goes up, the leveraged volatility is sensitive to interest rate changes.

We also note that as the project becomes riskier, the leverage constraint can cease binding for a high interest rate R. Then, the substitution effect will eventually be dominated again by the valuation effect as R increases further.

Figure 4: Effect of R on $\tilde{\eta}$ for different risk profiles



Note: The top panel of each column shows the leverage associated with the risky project; the bottom panel of each column shows the threshold cost for risk-taking. Each column corresponds to a different risk profile (from the left to the right: $\Pi^h = 1.3, 1.32, 1.33,$ and 1.35), while the expected return $\Pi = p\Pi^h + (1-p)\Pi^l$ is kept the same across all panels.

4 An Extended Infinite-horizon Model with Risk Spillover

Now we extend the previous two-period model into an infinite-horizon one and introduce credit market equilibrium. We also introduce the correlation of project success/failure without changing the key ingredients of the model. The next section will use the equilibrium model with the potential spillover effect of risk-taking for policy discussion.

4.1 The Environment

Time is discrete and infinite. We use recursive notation, i.e., let variable x denote x_t and let x' to denote x_{t+1}. The economy is populated by a unit measure of entrepreneurs who may run firms. These entrepreneurs have the production technologies mentioned above. They can choose among the risk-free project, the risky project, and risk-free savings. Furthermore, similar to η, we assume that there is a fixed (utility) cost η^f associated with running the risk-free project. The economy also has a household sector and a government which runs flat-rate value-added tax policy[9] and credit policy.

Entrepreneurs Entrepreneurs may run firms. The firm technology is represented by

$$y = (zk)^{\alpha} (\ell)^{1-\alpha},$$

[9]VAT tax policy is equivalent to tax labor income and capital income at the same rate. We also experimented with a lump-sum tax policy, and the qualitative results are very similar.

where k is capital input, ℓ is labor input, z is risky if an entrepreneur chooses a risky project, and z is a constant number if the entrepreneur chooses a safe project. The depreciation rate of capital is δ. Consider a firm with capital k. In the labor market, the firm hires workers at the competitive wage rate w, which leads to labor demand proportional to the firm's effective capital input. The firm maximizes its capital return by solving

$$\max_{\ell}\{(1-\tau^y)(zk)^{\alpha}(\ell)^{1-\alpha}-w\ell\}.$$

where τ^y is the VAT tax rate set by the government. The optimal labor choice is:

$$\ell = zk\left(\frac{(1-\alpha)}{w/(1-\tau^y)}\right)^{\frac{1}{\alpha}}. \tag{6}$$

Therefore, the firm's capital return (but before debt repayment) is rzk, where r can be shown as

$$r \equiv (1-\tau^y)\alpha\left[\frac{(1-\alpha)}{w/(1-\tau^y)}\right]^{\frac{1-\alpha}{\alpha}}. \tag{7}$$

The$\{z^f, z^h, z^l\}$ are specified such that

$$\Pi^f = rz^f + 1 - \delta, \quad \Pi^h = rz^h + 1 - \delta, \text{ and } \Pi^l = rz^l + 1 - \delta. \tag{8}$$

Then, the decision problem introduced below will be similar to the problems discussed in the simple model.

Let $V(\omega)$ be the expected value of an entrepreneur with wealth ω before knowing the cost of implementing the risky project. Conditioning on having good production technology and the implementing cost, we let $V^r(\omega,\eta)$ and $V^f(\omega)$ (as before) denote the value for an entrepreneur who chooses the risky and risk-free projects, respectively. Let $V^d(\omega)$ be the value of an entrepreneur who chooses to save in deposits with a return R^d. We have the following relationship:

$$V(\omega) = \mathbb{E}\left[\max\{V^r(\omega,\eta), V^f(\omega), V^d(\omega)\}\right],$$

where the expectation is taken over the distribution of η. The value function of an entrepreneur who takes the risky project can be rewritten as

$$\begin{aligned} V^r(\omega,\eta) = &\max_{s,b}\Big\{(1-\beta)\log(\omega - s) - \eta + \beta p V(\Pi^h(s+b) - Rb) \\ &\quad + \beta(1-p)V(\Pi^l(s+b) - Rb)\Big\} \\ &\text{s.t. } b \le \bar{\theta}s, \end{aligned}$$

where $(1-\beta)$ serves as a normalization. As before, the entrepreneur can invest with internal saving s, together with the borrowing b; next period, the entrepreneur earns either $\Pi^h(s+b)$ (if the project turns out to be productive) or $\Pi^l(s+b)$ (if the project turns out to be less productive). The interest payment is naturally Rb. The value function of an entrepreneur who takes a safe project or saves in deposits can be rewritten as

$$V^f(\omega) = \max_{s,b} \Big\{(1-\beta)\log(\omega - s) - \eta^f + \beta V(\Pi^f(s+b) - Rb)\Big\}$$
$$\text{s.t. } b \le \bar{\theta} s;$$

$$V^d(\omega) = \max_{s} \Big\{(1-\beta)\log(\omega - s) + \beta V(R^d s)\Big\},$$

respectively. When entrepreneurs choose the risk-free project, they will borrow up to the limit $\theta = \bar{\theta}$ if $\Pi^f > R$. Notice that the deposit rate R^d may differ from the lending rate R. We allow a constant τ to be the difference between the deposit rate R^d and the lending rate R:

$$R = R^d (1 + \tau) \tag{9}$$

The parameter τ stands for the intermediation costs or interest rate markup.

Appendix 4 shows that the value functions satisfy the following forms $V^r(\omega, \eta) = \log(\omega) - \eta + v^r$, $V^f(\omega) = \log(\omega) - \eta^f + v^f$, and $V^d(\omega) = \log(\omega) + v^d$ for some endogenous v^r, v^f, and v^d. As in the two period model, the saving function is linear in wealth $s = \beta\omega$.

Proposition 4. *Define $l^d \equiv \log R^d$ and $l^f \equiv \log\left(\Pi^f + \bar{\theta}\left(\Pi^f - R\right)\right) - \beta^{-1}\eta^f$. Those entrepreneurs with $\eta \le \max\{\underline{\eta}, \tilde{\eta}\}$ choose the risky project, where $\tilde{\eta}$ satisfies,*

$$\tilde{\eta} = \beta p \log\left(\Pi^h + (\Pi^h - R)\theta\right) + \beta(1-p)\log\left(\Pi^l + (\Pi^l - R)\theta\right)$$
$$- \beta \max\{l^f, l^d\}, \tag{10}$$

and the leverage $\theta = \min\{\theta^*, \bar{\theta}\}$. Additionally, let ϕ be the fraction of entrepreneurs who choose to save in deposits, we have the following condition

$$\begin{cases} \phi = 1 & l^d > l^f \\ \phi \in (0,1) & l^d = l^f \\ \phi = 0 & l^d < l^f \end{cases} . \tag{11}$$

Proof. See Appendix C. □

In equilibrium with $\phi \in (0, 1)$, those entrepreneurs who choose the safe project should be indifferent between saving via the risk-free deposits and their firm technology. That is, $l^d = l^f$ and

$$\Pi^f = \frac{1}{1+\bar{\theta}}\left(\frac{e^{\eta^f/\beta}}{1+\tau} + \bar{\theta}\right) R,$$

for $\Pi^f > R$. Therefore, $\eta^f > \beta \log(1+\tau)$ has to be true in equilibrium with $\phi \in (0, 1)$ so that some entrepreneurs choose to save in risk-free deposits.

Households Households are hand-to-mouth consumers and they supply labors to firms.[10] To focus on the firm side, we assume a Greenwood-Hercowitz-Huffman log utility function, and the household maximizes

$$(1-\beta)\sum_{t=0}^{\infty}\beta^t \log\left(C_t - \frac{\kappa L_t^{1+\gamma}}{1+\gamma}\right),$$

where C_t is households' consumption level, L_t is the labor supply, $\kappa > 0$ is disutility parameter, and γ is the inverse of the Frisch labor elasticity. The household consumption and labor supply decision satisfies (where again we ignore the time subscript)::

$$C = wL; \tag{12}$$

$$w = \kappa L^{\gamma}. \tag{13}$$

Spillovers In the previous section, we consider there are infinitely many i.i.d. risky projects. Now, we relax the independent assumption and assume the project realizations are correlated. Once the risky project is chosen, each agent will be randomly assigned to one of them. At the aggregate level, how many risky projects will achieve high realization may exhibit correlations. To characterize the dependence structure in Bernoulli distribution, we follow the approach of Conway-Maxwell Binomial/Poisson (CMB/CMP) distribution as in Shmueli et al. (2005) and Kadane (2016). The specification does not change the individual problems above and allows a straightforward aggregation. To the best of our knowledge, this specification is new in a macroeconomic setup, and it modifies the baseline model with minimal departure because the individual problem stays the same.

Specifically, we characterize the overall dependence structure through an exogenous parameter ν. For CMB distribution:

$$Pr(m=k) = \frac{\binom{n}{k}^{\nu} p^k(1-p)^{n-k}}{\sum_{j=0}^{n}\binom{n}{j}^{\nu} p^j(1-p)^{n-j}} \equiv \frac{\binom{n}{k}^{\nu} p^k(1-p)^{n-k}}{D(\nu,p,n)},$$

where m is the total number of successes in n trials, $\binom{n}{k} \equiv \frac{n!}{k!(n-k)!}$, and we have used

$$D(\nu,p,n) \equiv \sum_{j=0}^{n}\binom{n}{j}^{\nu} p^j(1-p)^{n-j}.$$

The expression stands for the probability of k realization of high returns among n risky projects where the probability of realizing high return Π^h for each project is p. If $\nu = 1$, it becomes the standard binomial distribution. When $0 < \nu < 1$, risky projects' returns are positively correlated, and when $\nu > 1$,risky projects' returns are negatively correlated.[11] To see this, denote X_i as the random variable of result at i^{th} position. Consider $P(X_2 = z^l | X_1 = z^l)$, the conditional probability of the second project

[10] The assumption is less strong than it seems. In incomplete-market models, the interest rate is lower than the time-preference rate. Households would like to borrow, but (unlike entrepreneurs) households do not have collaterals, so they will not borrow in equilibrium.

[11] See Kadane (2016) for more detailed discussions.

achieving a low return conditional on the low return of the first project, which is

$$\begin{aligned}\frac{P(X_2=z^l,X_1=z^l)}{P(X_1=z^l)} &= \frac{P(X_2=z^l,X_1=z^l)}{P(X_2=z^l,X_1=z^l)+P(X_2=z^h,X_1=z^l)} \\ &= \frac{\frac{1}{D(\nu,p,2)}(1-p)^2}{\frac{1}{D(\nu,p,2)}(1-p)^2+\frac{1}{2}\frac{1}{D(\nu,p,2)}\binom{2}{1}^{\nu}p(1-p)} \\ &= \frac{(1-p)}{(1-p)+2^{\nu-1}p},\end{aligned}$$

where we have used the property of the exchangability: $P(X_2 = z^h, X_1 = z^l) = P(X_2 = z^l, X_1 = z^h)$. When $\nu < 1$, we have $2^{\nu-1} < 1$, then $P(X_2 = z^l|X_1 = z^l) > 1-p$ which means a positive correlation (positive spillover). If $\nu > 1$, we have $2^{\nu-1} > 1$ and then $P(X_2 = z^l|X_1 = z^l) < 1-p$ which implies a negative correlation (negative spillover). Similarly, $P(X_2 = z^h|X_1 = z^h) > p$ if $\nu < 1$ and $P(X_2 = z^h|X_1 = z^h) < p$ if $\nu > 1$.

Denote the overall proportion of success to be $p^s(\nu)$ when the number of projects n goes to infinity. The aggregate expectation of risky projects $\Pi(\nu)$ becomes:

$$\Pi(\nu) = p^s(\nu)\Pi^h + (1-p^s(\nu))\Pi^l.$$

This will matter for the social welfare as we aggregate the economy. The key feature of CMB is that for individuals, when they choose the project at given interest R, the probability of Π^h is always p. This preserves the solution of the individual optimization problem. For the social planner, this correlation tends to be crucial in determining social welfare. This modeling captures a kind of pecuniary externality where an individual's choice ignores aggregate correlations which influence the aggregate liquidity demand and thus the interest rate that clears the credit market.

To understand how this correlation affects the overall success probability $p^s(\nu)$ and the aggregate expectation of risky projects $\Pi(\nu)$, let's consider a special case which we can trace analytically. When n goes to infinity, and p is small enough, we know that the CMB distribution converges to the CMP distribution (see Shmueli et al. (2005) and Daly and Gaunt (2016)):

$$Pr(m=k) = \frac{\frac{(\lambda)^k}{(k!)^{\nu}}}{\sum_{j=0}^{j=n}\frac{(\lambda)^j}{(j!)^{\nu}}},$$

where $\lambda = n^{\nu}p$.[12] And the overall success probability $p^s(\nu)$ and the average return $\Pi(\nu)$ satisfies: the following:[13]

$$p^s(\nu) = \begin{cases} p^{\frac{1}{\nu}} & \text{if } p < p^* \\ (1-(1-p)^{\frac{1}{\nu}}) & \text{if } p \geq p^{**}, \end{cases} \tag{14}$$

for some small $p^* \in (0,1)$ and large $p^{**} \in (0,1)$, respectively. When p is small, high returns are relatively more scarce. The number of high return realization converges to Conway-Maxwell Poisson distribution. Taking into account the correlation, the social planner realizes that the overall probability of Π^h is $p^{1/\nu}$. When p is large, $1-p$ is small and counting the low realizations Π^l will converge to

[12] $\lambda = n^{\nu}(1-p)$ when p is large.

[13] see Appendix D.1 for more details,

the Conway-Maxwell Poisson distribution with $\lambda = n^{\nu}(1-p)$. The social planner realizes that the overall probability of Π^l is $(1-p)^{1/\nu}$. Finding p^* and p^{**} is a numerical question, but one may not even need to find the p^* and p^{**} for the numerical analysis below, where we obtain $p^s(\nu)$ and hence $\Pi(\nu)$ via simulating CMB samples taking a large n.

Meanwhie the aggregate expectation of the risky projects $\Pi(\nu)$ satisfies:

$$\Pi(\nu) = \begin{cases} p^{\frac{1}{\nu}}\Pi^h + (1-p^{\frac{1}{\nu}})\Pi^l & \text{if } p < p^* \\ (1-(1-p)^{\frac{1}{\nu}})\Pi^h + (1-p)^{\frac{1}{\nu}}\Pi^l & \text{if } p \geq p^{**}. \end{cases} \tag{15}$$

To understand whether the spillover effect lowers or raises the overall return compared to $\Pi = \Pi(1)$ (i.e., without the spillover effect), consider the case of positive correlation, i.e., $\nu < 1$: when $p > p^{**}$, $p^s(\nu) = 1-(1-p)^{\frac{1}{\nu}} > p$ and $\Pi(\nu) > \Pi$. Intuitively, when p is large, high returns are relatively more abundant, the positive spillover strengthens the dominance of high returns and the overall success probability viewed by the social planner becomes higher, which raises the overall aggregate expectation of risky projects; when $p < p^*$, $\Pi(\nu) < \Pi$ holds similarly.

The (consolidated) government agency We consider a consolidated government which conducts joint monetary and fiscal policies.

$$G + R_d B_{-1} = T + B \tag{16}$$

The expenditure side includes government spending and repaying debt $R_d B_{-1}$ accumulated previously. For simplicity, we assume that the government can borrow without the intermediation cost so that repaying interest rate is R_d because of no-arbitrage. The revenue side includes tax $T = \tau^y Y$ (where Y is output to be more specifically defined below) and the newly issued debt.

To understand how B represents liquidity policy, one can consider the government agency as a consolidated identity that includes a monetary authority, a fiscal authority, and financial intermediaries, which lend to the firm sector. Therefore, B is the net debt position of the joint agency; when $B < 0$, the consolidated identity lend to the firm sector. Alternatively, if we interpret the consolidated agency with only monetary and fiscal authorities, then B can be considered as an outcome of quantitative easing (QE) policy. Notice that we simplify the institutional details of liquidity policy to focus on the equilibrium effect of interest rate on entrepreneurs' risk-taking, as well as how the correlation of projects affects the optimal interest rate.

We specify that the government agency determines (G, R^d) every period. Then, B will be determined in the credit market, while tax rate τ^y will be the residual of the government budget constraint.

4.2 Equilibrium

Denote the total wealth of private agents as Ω. The dynamics of Ω can be derived from the wealth accumulation:

$$\Omega' = \beta\Omega\left\{\phi\left[1-F(\tilde{\eta})\right]R^d + (1-\phi)\left[1-F(\tilde{\eta})\right]\left[\Pi^f(1+\bar{\theta}) - R\bar{\theta}\right] + F(\tilde{\eta})\left[\Pi(\nu)(1+\theta) - R\theta\right]\right\}, \tag{17}$$

where again ϕ is the probability of saving entrepreneurs among those who do *not* run risky projects (note: they save with risk-less saving rate $R^d \equiv R/(1+\tau)$ with τ denoting the intermediary markup). The next period's wealth Ω' will come from three sources. All entrepreneurs put a β fraction of their wealth (which explains $\beta\Omega$) aside. The group of entrepreneurs that do not take risks have ϕ fraction of savers (who has a return R^d) and $(1-\phi)$ fraction implementing the safe project with a return $\Pi^f(1+\bar{\theta}) - R\bar{\theta}$. The remaining entrepreneurs take risks, and the project return after leverage is $\Pi(\nu)(1+\theta) - R\theta$.

In the aggregate, capital productivity is endogenous because of credit-induced investment in different technologies. That is, capital reallocation linked with credit conditions determines the endogenous productivity of capital. Define Z as the endogenous productivity that takes into account the technology choices of entrepreneurs:

$$Z = \left[z^f(1-F)(1-\phi)(1+\bar{\theta}) + \bar{z}F(1+\theta)\right],$$

where

$$\bar{z} = p^s(\nu)z^h + (1-p^s(\nu))z^l.$$

Then, the aggregate output can be written as

$$Y = \int (z_i k_i)^\alpha \ell_i^{1-\alpha} di = (Z\beta\Omega)^\alpha L^{1-\alpha}. \tag{18}$$

Notice that each firm produces $z_i k_i/\alpha$ in which $z_i k_i$ is retained. Additionally, $Z\beta\Omega$ can be regarded as the effective capital stock used in production,

To define equilibrium, we look at market clearing conditions for labor and for credit. For the labor market, hours of work L should be the labor supply from households, so that the "rental rate" r can be written in a conventional way, which is the marginal product of effective capital stock (using L in the product)

$$r = \alpha (Z\beta\Omega)^{\alpha-1} L^{1-\alpha}. \tag{19}$$

For the credit market, the consolidated agency conducts the liquidity policy that determines the market-clearing interest rate. As mentioned before, we can think of the consolidated policy maker conducting an interest rate policy by choosing R^d. An interest rate R^d corresponds to a particular liquidity provision choice B according to the market clearing condition. The total liquidity provided by savers at period t is $(1-F(\tilde{\eta}))\phi\beta\Omega$. The total credit demanded by entrepreneurs who borrow is $\left[(1-F(\tilde{\eta}))(1-\phi)\bar{\theta} + F(\tilde{\eta})\theta\right]\beta\Omega$, a fraction $[1-F(\tilde{\eta})](1-\phi)$ of whom use the safe technology and borrow to the credit limit implied by $\bar{\theta}$ and a fraction $F(\tilde{\eta})$ of whom use the risky technology and leverage to $\theta \leq \bar{\theta}$. Therefore, the clearing of credit market implies

$$\left[(1-F(\tilde{\eta}))(1-\phi)\bar{\theta} + F(\tilde{\eta})\theta\right]\beta\Omega = [1-F(\tilde{\eta})]\,\phi\beta\Omega - B, \tag{20}$$

which then implies the level of B to implement the deposit interest rate R^d.

Definition. Given states including the entrepreneurs' wealth Ω, a government spending G, and an interest-rate policy target R^d, a recursive competitive equilibrium is a collection of variables {L, C, $\tilde{\eta}$, θ, ϕ, Π^h, Π^l, Π^f, r, w, τ^y, B, R, Ω'} such that

- Households supply labor according to (13) ; their budget constraint (12) holds;
- $\tilde{\eta}$ and θ solve the entrepreneurs' problem with risk-taking choice and leverage choice as shown in Proposition 4;
- project returns (8) are satisfied, with r given by (7);
- The wealth dynamics (17) holds;
- The government budget constraint is satisfied, i.e., (16) holds with tax revenue $T = \tau^y Y$ where output is defined in (18);
- The credit market clears, i.e., (20) holds, with (11) satisfied, and the lending rate R is determined by (9);
- The labor market clears, i.e., (19) holds.

Importantly, one can immediately see that the aggregate correlation, measured by ν, matters. It affects the market demand for liquidity and therefore the tax revenue T. When $\upsilon < 1$ and p is small, $\Pi(\nu) < \Pi$ and the required tax revenue T is lower. We will also show that the spillover will affect the equilibrium level of the share of the savers ϕ as well. Both households and entrepreneurs welfare may change. We are interested in how the optimal interest rate R^d is related to this degree of correlation υ. In the next quantitative section, we give the social welfare function and examine the optimal interest rate.

5 Macroeconomic Effects of Interest-rate Policy

Using the macroeconomic environment introduced above, we now analyze the aggregate long-run effects of interest rate policy, including whether risk-taking should be incentivized by monetary policy and how the optimal policy is affected by the leverage level and the degree of spillover ν.

5.1 Parameterization

It becomes too complex to study the optimal interest rate fully analytically. We choose to calibrate the model first and assess the policy effects.

Table 3 reports the calibrated parameters. Some parameters are exogenous. The capital share and the discount factor rare set to conventional values $\alpha = 0.33$ and $\beta = 0.95$. The inverse of the Frisch elasticity of labor supply γ is set to 1, also a conventional number for macroeconomic models. The discussion below illustrates the key steps in calibrating other parameters.

Table 3: Calibration

	Value	Explanation/Target		Value	Explanation/Target
β	0.95	Discount factor	z^f	1	Normalization
γ	1	Inverse Frisch elasticity	z^h	1.8152	S&P index return top 90th per.
κ	2.3611	Hours 0.33	z^l	0.1270	S&P index return bottom 10th per.
α	0.33	Capital share	p	0.5953	Annual stock volatility
δ	0.0884	Investment-to-output 16%	μ	-3.0483	Debt-to-output ratio: 65%
τ	0.0208	Interest-rate spread	σ	0.0161	Elasticity of stock volatility: 2.98
$\bar{\theta}$	0.5600	Median leverage ratio	G	0.0843	Gov-spending-to-output 17%
η^f	0.0360	Median S&P index return	R	1.0330	Prime rate

κ, which governs the disutility of labor, is calibrated to hit the labor hours such that $L = 0.33$ after normalizing hours to unity. The model targets government spending to GDP ratio G/Y as 18%, which pins down G. The capital share α is calibrated to the investment-to-output ratio 16%.[14]

We choose the prime rate to be the status quo interest rate R, since there is no default in the model and since the prime rate is a benchmark interest rate used for high-quality borrowers. We use the data covering 1954-2018 to calibrate the model (see Appendix A for data description). The average annualized gross prime rate during this period is 1.033, while the average annualized gross federal funds rate during this period is 1.012. Thus, the interest-rate markup τ is set to 2.08%.

The baseline sets $\nu = 1$ (no correlation among projects). We use the top 90^{th} percentile return and the bottom 10^{th} percentile return of the S&P 500's (Π^{90th} and Π^{10th}) to represent the net leveraged return for high realization and low realization, i.e., $\Pi^{90th} \equiv (1+\bar{\theta})\Pi^h - R\bar{\theta}$ and $\Pi^{10th} \equiv (1+\bar{\theta})\Pi^l - R\bar{\theta}$. We map the S&P 500's median return to the net leveraged risk-free return $\Pi^m \equiv (1+\bar{\theta})\Pi^f - R\bar{\theta}$. The top 90^{th}percentile, bottom 10^{th} percentile, and median net return for the period of 1954-2018 are 22%, -13%, and 5%, respectively, so that $\Pi^{90} = 1.22$, $\Pi^{10} = 0.87$, and $\Pi^m = 1.05$. The benchmark exercise normalizes $z^f = 1$ and assumes that some entrepreneurs save. Then, we obtain η^f from the indifference condition $\log R^d = \log(\Pi^m) - \eta^f/\beta$, and we obtain r as a function of $\bar{\theta}$:

$$r \quad = \quad \frac{\Pi^m + R\bar{\theta}}{1+\bar{\theta}} - (1-\delta).$$

Additionally, z^h and z^l are also functions of $\bar{\theta}$ by substituting this expression for r to the expression for Π^{90th} and Π^{10th}:

$$z^h = \frac{1}{r}\left[\frac{\Pi^{90th} + R\bar{\theta}}{1+\bar{\theta}} - (1-\delta)\right]; \quad z^l = \frac{1}{r}\left[\frac{\Pi^{10th} + R\bar{\theta}}{1+\bar{\theta}} - (1-\delta)\right].$$

The leverage parameter $\bar{\theta}$ is set according to the firm leverage ratio used in Section 2. We drop the observations of firms whose return standard deviation is higher than the 95 percentile or lower than the 95 percentile at the industry level (at the one-digit SIC level), and we only consider firms whose illiquid

[14]In the model, we derive the investment output ratio as

$$\frac{I}{Y} = \frac{\delta\left(1 + \left[(1-F(\tilde{\eta}))(1-\phi)\bar{\theta} + F(\tilde{\eta})\bar{\theta}\right] - (1-F(\tilde{\eta}))\phi\right)}{\frac{r}{\alpha}\left[z^f(1-F)(1-\phi)(1+\bar{\theta}) + \bar{z}F(1+\bar{\theta})\right]}$$

ratio is positive. The median of the leverage ratio is 0.56. Then, r, z^h, and z^l are identified immediately as shown above.

We assume the cost distribution follows a log-normal distribution with the mean as μ and variance σ^2 after log transformation.[15] Therefore, the remaining key parameters are: $\{\mu, \sigma, p\}$. They are chosen to match the following three moments (17):

(1). We target the debt-output ratio[16] to 0.65 as in the data[17] after we use the total debt of the non-financial business.

(2). Assuming risk-taking firms are financially constrained in the status quo, we have the standard deviation of the overall return in the model

$$\sigma_I = \sqrt{F(\tilde{\eta})}(1+\bar{\theta})\sqrt{p(1-p)}\left(\Pi^h - \Pi^l\right) = \sqrt{F(\tilde{\eta})}\sqrt{p(1-p)}\left(\Pi^{90th} - \Pi^{10th}\right),$$

corresponding to the standard deviation of S&P 500's annual return of 0.132 in the data.

(3). Most importantly, its sensitivity to the interest rate becomes

$$\frac{\partial \sigma_I}{\partial R}\frac{R}{\sigma_I} = \left[\frac{1}{2}\frac{f(\tilde{\eta})}{F(\tilde{\eta})}\frac{\partial \tilde{\eta}}{\partial R} + \frac{1}{\Pi^h - \Pi^l}\left(\frac{\partial \Pi^h}{\partial R} - \frac{\partial \Pi^l}{\partial R}\right)\right] R \tag{21}$$

corresponding to 2.98 in the data. Note that under financing-constrained case the derivative $\partial\tilde{\eta}/\partial R$ can be computed as

$$\frac{\partial \tilde{\eta}}{\partial R} = \begin{cases} \beta p \frac{1}{\Pi^{90th}}\left(\frac{\partial \Pi^h}{\partial R} + \left(\frac{\partial \Pi^h}{\partial R} - 1\right)\bar{\theta}\right) \\ +\beta(1-p)\frac{1}{\Pi^{10th}}\left(\frac{\partial \Pi^l}{\partial R} + \left(\frac{\partial \Pi^l}{\partial R} - 1\right)\bar{\theta}\right) \\ -\beta \frac{1}{\Pi^m}\left(\frac{\partial \Pi^f}{\partial R} + \left(\frac{\partial \Pi^f}{\partial R} - 1\right)\bar{\theta}\right) \end{cases}.$$

Since this outcome is unique to our model, we elaborate the procedure. First, the calibration assumes that those entrepreneurs who do not take risks are indifferent between saving in safe assets and implementing the safe project. That is, we have the second branch of ((11)) so that

$$\Pi^f = \frac{\frac{e^{\beta^{-1}\eta^f}}{1+\tau} + \bar{\theta}}{1+\bar{\theta}} R \rightarrow \frac{\partial \Pi^f}{\partial R} = \frac{\frac{e^{\beta^{-1}\eta^f}}{1+\tau} + \bar{\theta}}{1+\bar{\theta}},$$

and $\partial\Pi^h/\partial R$ and $\partial\Pi^l/\partial R$ follow immediately. Otherwise, one needs to solve the whole equilibrium to examine the effect of interest rate R on the risk-taking threshold $\tilde{\eta}$ and eventually the volatility measure σ_I, which brings the calibration overly relying on the general equilibrium effect. Second, we also examine the case when risk-taking entrepreneurs may not be financing constrained, i.e., replacing $\bar{\theta}$ by θ above, but the final result points to the constrained scenario.

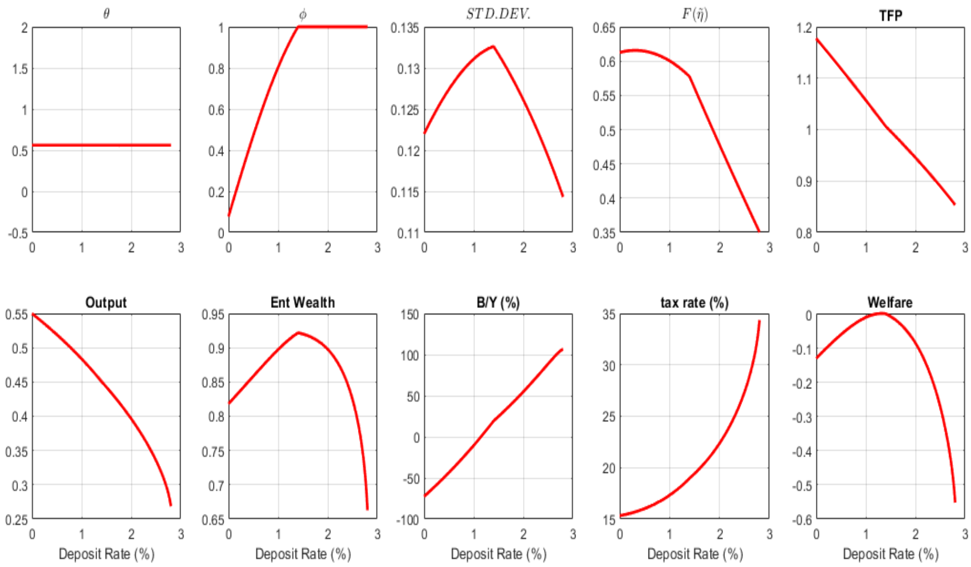
[15] we tried other distributions, which are not crucial for the qualitative conclusion shown below.

[16] $\tilde{\eta}$ is solved by (10) and ϕ is solved by (17). After simplification

$$\frac{D}{Y} = \frac{\left[(1-F(\tilde{\eta}))(1-\phi)\bar{\theta} + F(\tilde{\eta})\bar{\theta}\right]}{\frac{r}{\alpha}\left[z^f(1-F)(1-\phi)(1+\bar{\theta}) + \bar{z}F(1+\bar{\theta})\right]}$$

[17] Total debt of the non-financial businesses over the total output of the non-financial businesses.

Figure 5: Numerical illustration



Note: This figure plots equilibrium variables as functions of the (net) deposit rate $R^d - 1$ at the calibrated parameter values. The variables shown are the optimal leverage θ, the share of saving firms among non-risk-taking firms ϕ, standard deviation of project returns, the share of risk-taking firms $F(\tilde{\eta})$, TFP, aggregate output, aggregate entrepreneurs' wealth (Ent Wealth), and the consumption-equivalent (in the calibrated economy) welfare.

5.2 The Effects of Interest-rate Policy

Suppose the government implements different levels of interest rate, achieved by more borrowing by the joint government agency. Figure 5 shows how key variables change with the safe interest rate R^d.

This numerical example shows a weak hump-shaped relationship between the risk-taking threshold $\tilde{\eta}$ and the interest rate. However, the model shows a strong hump-shaped standard deviation of investment return as the interest rate varies. When the interest rate rises, saving in safe deposit becomes more attractive. The share of the savers among non-risk-takers who provide liquidity, ϕ, increases in the deposit rate R^d and becomes 100% at the upper limit if R^d is high enough. Notice that the total factor productivity (TFP), calculated as

$$TFP \equiv Z^\alpha = \left[z^f(1 - F(\tilde{\eta}))(1 - \phi)(1 + \bar{\theta}) + \bar{z}F(\tilde{\eta})(1 + \theta)\right]^\alpha,$$

falls with the interest rate because a higher interest rate mostly discourages risk-taking in the range considered. Therefore, less risk-taking puts downward pressure on TFP.

Entrepreneurs' wealth first increases in R^d and then declines in R^d. When R^d is low, risk-taking behavior is not sensitive to the increase in interest rate (note: as the interest rate goes up, F increases slightly and falls slightly). Therefore, the effect on the wealth of those who run firms is not sensitive to the rise of the interest rate. The increase in entrepreneurs' wealth is driven by the higher deposit rate, which benefits those who save in deposits. When R^d is high, equilibrium risk-taking falls quickly. At the aggregate level, since risk-taking generates, on average, a higher return than those on safe projects and safe deposits, the entrepreneur's wealth thus falls.

Notice that the output measured by

$$Y = \frac{r\beta\Omega Z}{\alpha} = \left(\frac{r}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} L$$

decreases in r and increases in L. Given that r naturally rises with the deposit rate R^d while labor does the opposite, output thus drops when the government runs a higher interest rate. We thus observe falls in investment and consumption when R^d is higher (not plotted), a downward sloping long-run "IS" relationship arising from the risk-taking channel.

5.3 Optimal Interest Rate

Suppose the planner can choose the deposit rate R^d to maximize the joint social welfare of entrepreneurs V and households W. Given the value functional form and the nature of the steady state economy, the social welfare measure, which is related to present value of utility of consumption from all groups, thus becomes (see Appendix D for details) :

$$V(\Omega, \phi) + W = \log(\Omega) + \tilde{V}(\phi) + \log\left(C^h - \frac{\kappa L^{1+\gamma}}{1+\gamma}\right) + \text{constants},$$

and $\tilde{V}(\phi)$ is the value of the entrepreneurs depending on the specific value of ϕ:

$$\tilde{V}(\phi) = (1-\beta)^{-1}\left(\beta \max\{l^d, l^f\} + \int^{\tilde{\eta}} F(\eta) d\eta\right).$$

The social welfare depends on the aggregate wealth Ω, safe project return and taxes. The direct effects of risk-taking are contained in V^{ϕ}. The social welfare includes the base return (either risk-free return or safe deposit return, i.e., the first term in $\tilde{V}(\phi)$) and the relative value gain from choosing the risky project (the rest of the terms in $\tilde{V}(\phi)$).

When assessing the policy effect on the social welfare of a target economy, we compute the corresponding consumption equivalence measure ψ of the baseline calibrated economy. Given that entrepreneurs' consumption is $(1-\beta)$ fraction of their wealth, we calculate the gain/loss of wealth of entrepreneurs (Ω_{base}) and households (C^h_{base}) such that the baseline economy with $(1+\psi)\Omega_{base}$ and $(1+\psi)C^h_{base}$ (and everything else stays as their calibrated level) has the same welfare measure as in the target economy.

Notice that a natural upper bound of R^d is β^{-1} as otherwise the household would save.[18] However, the effective upper bound may be lower than this level because the implied consequence on the tax rate is not feasible. We also assume a lower bound of R^d. This could be because of liquidity traps and/or adverse expectations when the interest rate becomes ultra low. The numerical examples below set the lower bound to be 1, but having 0.98 or 0.97 (i.e., -2% or -3% net interest rate) does not change the result. Additionally, implementing persistent -2% or -3% interest rate in the model requires the government to hold privately-issued assets close to 100% of aggregate output, which is unrealistic. For all these reasons, we set the lower bound to be unity.

[18] We check this type of equilibrium in which households save. The welfare is dominated by cases in which the deposit rate is lower, i.e., $R^d < \beta^{-1}$. The reason is that the planner is incentivised to lower the interest rate and redistribute resources towards entrepreneurs with productive projects.

Figure 5 shows entrepreneurs' welfare closely tracks their wealth. When R^d rises, the government authority borrows more (or saves less, as can be seen by the debt-to-output ratio B/Y) since firms save more. As explained above, initially, risk-taking behavior increases with the interest rate, but the government has to raise more taxes from the private agents to satisfy the government budget constraint. Risk-taking behavior also falls significantly when R^d becomes high enough. Therefore, the welfare of everyone eventually falls, and the social welfare displays a hump shape, suggesting the optimal deposit interest rate being around 1.3%.

Leverage limit and optimal interest rate. As shown in the simple model, risk-taking behavior depends on leverage, which in turn affects the aggregate. How does leverage limit influence the optimal level of interest rate?

When $\bar{\theta}$ falls by 50% from the calibrated level $\bar{\theta} = 0.56$ to $\bar{\theta} = 0.28$ (the blue dash-dotted lines in Figure 6), the share of risk taking entrepreneurs $F(\tilde{\eta})$, turns sensitive to the variation in interest rate. It increases with the interest rate when the rate is low, while it decreases when the rate is high. This result confirms the previous finding in which the income effect of interest rate dominates when leverage is low. The equilibrium share of savers ϕ mostly shifts down since the demand for liquidity is lower due to the lower external financing limit $\bar{\theta}$. Meanwhile, TFP falls uniformly as the leverage falls, and output thus shifts down. However, entreprenuers' wealth shifts up for low levels of interest rate. The reason is the stronger saving incentive when entrepreneurs face lower borrowing capacity. But as the interest rate goes up thanks to government borrowing, the higher taxation need reduces the returns on projects and eventually entrepreneurs' wealth.

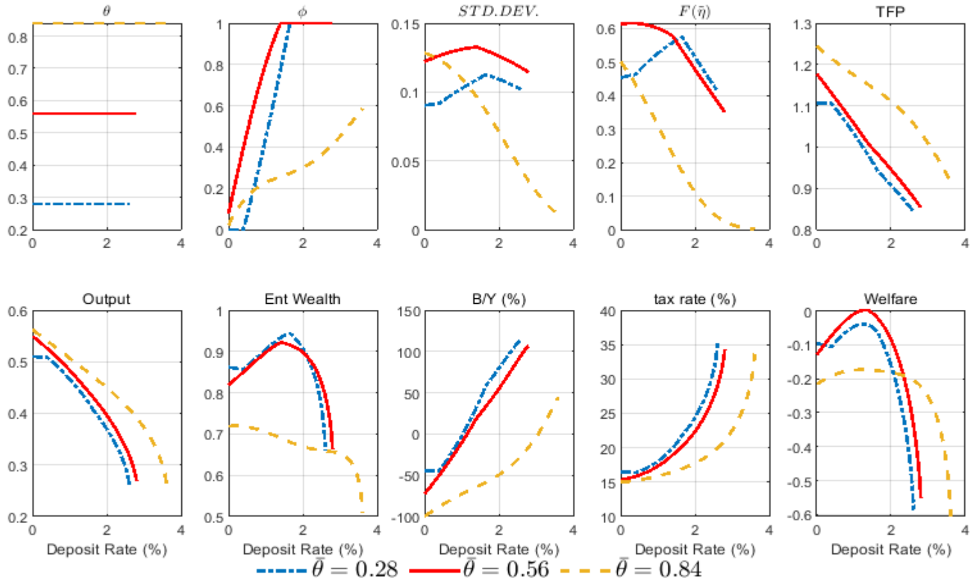
Note that the optimal interest rate falls slightly even if $\bar{\theta}$ is halved. There are two competing forces behind this insensitivity of optimal interest rate. On the one hand, a lower interest rate can compensate the adverse effect of tougher financial conditions when $\bar{\theta}$ falls; on the other hand, given that the income effect of risk-taking is dominating, raising the interest rate encourages risk-taking when leverage is low.

As a comparison, we let $\bar{\theta}$ increase by 50% to 0.84 (i.e., shown by the yellow dash lines). $F(\tilde{\eta})$ now is downward sloping because the substitution effect dominates when leverage is high. That is, when $\bar{\theta}$ is high (note: as θ panel shows, the entrepreneurs may not be financing constrained), a rise in interest rate discourages risk-taking, consistent with Proposition 2. At the same time, for any given interest rate, raising $\bar{\theta}$ up to unity pushes down risk-taking incentives (as shown by the shift-down of the $F(\tilde{\eta})$ curve), thanks to the valuation effect of raising $\bar{\theta}$ while keeping the interest rate constant. This result is consistent with Corollary C.2.

Overall, TFP rises as the leverage increases and more resources can be allocated to entrepreneurs who implement projects (risky or safe ones). As a result, there is less need for entrepreneurs to save, and thus, the wealth Ω of entrepreneurs falls. The optimal interest rate level still exists to balance the incentives of entrepreneurs who are borrowers to implement projects and who are savers. The level is slightly lower than that of the calibrated level (44% lower than unity).

Spillover and optimal interest rate We experiment with different scenarios by having positive correlation (i.e., $\nu < 1$) and negative correlation (i.e., $\nu > 1$). Notice that when projects exhibit positive correlation, as discussed in Section 4, we know that $\Pi^{\nu} > \Pi$. The reason is that the positive spillover of the higher return dominates the positive spillover effect of the low return. The opposite is true if project

Figure 6: Alternative financing constraints

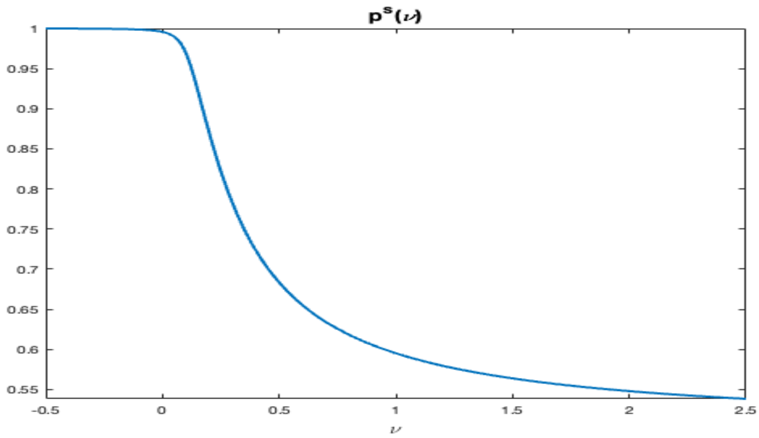


Note: This figure shows the equilibrium outcomes under three possible leverage limits of $\bar{\theta}$. Variables are the same as in Figure 5.

outcomes are negatively correlated. The "social" probability of success, $p^s(\nu)$ is thus a decreasing function of ν, as shown by the simulation result in Figure 7

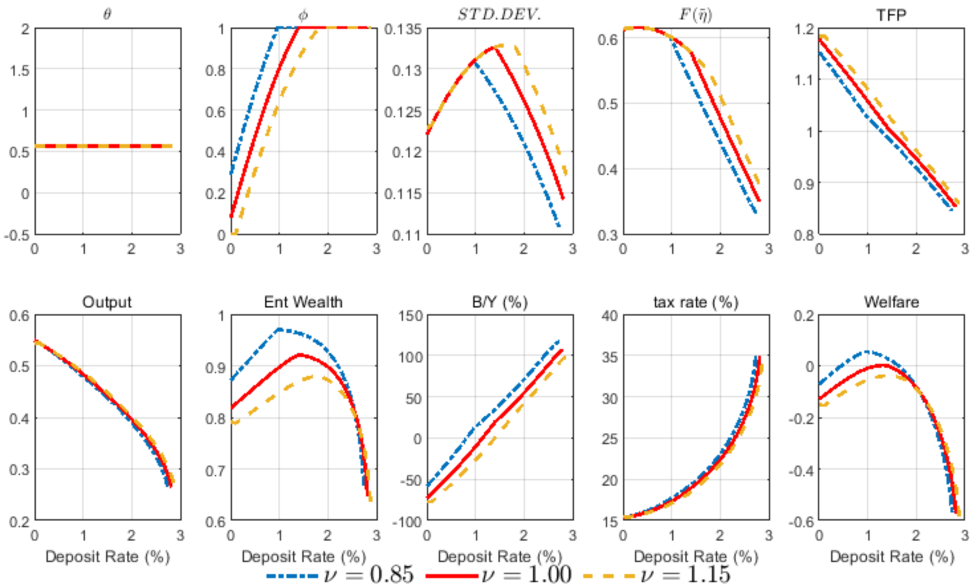
When project outcomes become positively correlated (i.e. when ν falls below unity), the social return assigns a higher probability $p^s(\nu)$ to Π^h, although individuals still believe in the probability p. As a result, capital accumulation goes up (so does entrepreneurs' wealth) for most interest-rate levels, and the net return of effective wealth r thus falls for a given deposit rate. The gross return of risky project Π^h and Π^l and consequently $\tilde{\eta}$, the risk-taking measure, decrease. With lower r, the gross return of safe project Π^f is also lower, making the safe project less attractive. We thus see a weakly higher ϕ for any interest-rate levels. With lower $\tilde{\eta}$ and higher ϕ, TFP falls, leading to more precautionary

Figure 7: Welfare changes under different ν.



Note: The simulated probability of success $p^s(\nu)$ from CMB as a function of ν.

Figure 8: Welfare changes under different ν.



Note: This figure plots the equilibrium outcomes for different levels of project correlation. $\nu = 1$ is the benchmark case (no correlation) as shown in Figure 5. $\nu < 1$ means project outcomes are positively correlated; $\nu > 1$ means project outcomes are negatively correlated.

savings and, again, higher entreprenuers' wealth. Accordingly, the positive spillover leads to higher entrepreneurs' welfare than the baseline. With more firms choosing to save in safe deposits, more government borrowing is needed to clear the credit market. This leads to higher taxation to balance the government budget constraint, so the welfare becomes lower for a high enough interest rate when compared to the case of no-correlation (the red line).

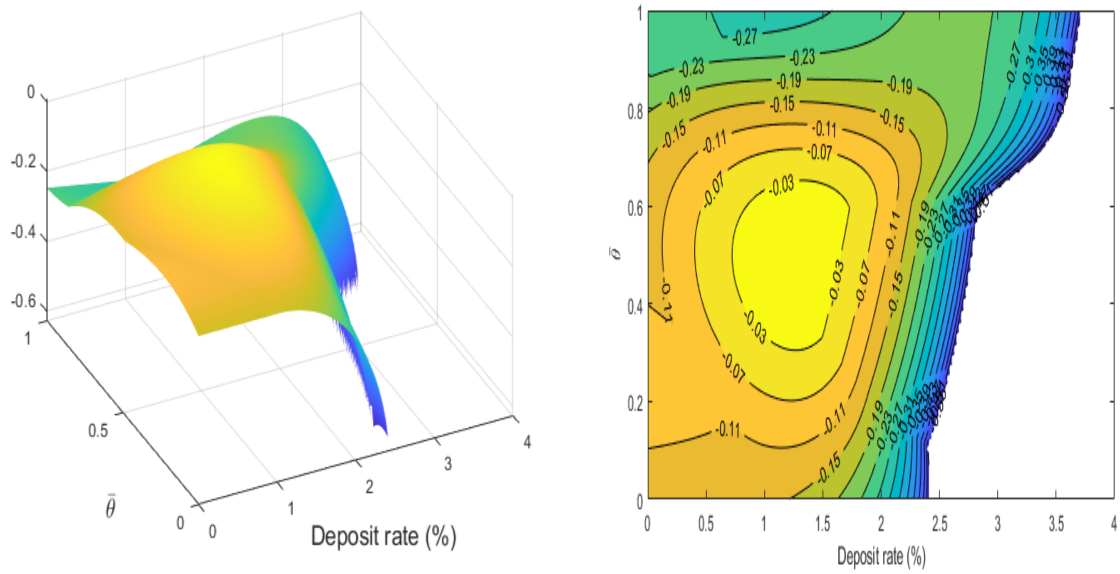
As for the optimal interest rate, the level is lower compared to that of the baseline. The optimal level should always balance the the welfare of entrepreneurs who implement projects and that of those who save in deposits; however, the risk-taking behavior becomes more sensitive to interest rate when ν falls (see the difference between the blue and the red lines of the $F(\tilde{\eta})$ panel). Therefore, a cut of interest rate is needed to utilize the sensitive increase of socially beneficial risk-taking behaviors.

When project outcomes become negatively correlated, i.e., $\nu > 1$, the results are reversed. The central bank should increase the interest rate to deal with the negative correlation, which is consistent with the previous analysis of positive correlation (see the peaks of curves in the welfare panel).

One common feature of the optimal interest rate (whether there is a correlation) can be summarized as follows. The optimal interest rate should be generally low to encourage resources used by productive entrepreneurs. However, the optimal interest rate should be set such that the economy is at the steep branch of the $F(\tilde{\eta})$ curve (typically when the interest rate is high). The steep branch of the curve means that the risk-taking behavior is sensitive to interest rate variations.

The spillover effect on the optimal interest rate depends on the leverage limit. Table 4 shows the optimal interest rate for different levels of $\bar{\theta}$ and spillovers. When risky project returns are positively correlated, the optimal deposit interest rate appears robustly lower. A positive correlation implies a higher Π^{ν} than Π, so the planner should take into account the overall positive spillover externality and maintain a lower interest rate.

Figure 9: The welfare consequence of interest rate and leverage limit



Note: The left panel shows the consumption-equivalent loss as a function of deposit rate $R^d - 1$ and leverage limit $\bar{\theta}$. The right panel is the corresponding contour plot of the 3-dimensional plot on the left.

Table 4: Optimal interest rate $100(R^d - 1)$

	ν		
$\bar{\theta}$	0.85	1.00	1.15
0.28	1.20	1.26	1.30
0.56	0.97	1.32	1.42
0.84	0.77	1.22	1.79

Table 4 highlights the importance of a proper mix of interest-rate and macro-prudential policies (which typically directly affects $\bar{\theta}$). In fact, the optimal deposit interest rate decreases (increases) in $\bar{\theta}$ when when risky projects returns are positively (negatively) correlated, further suggesting the need for a proper mix. The discussion below analyzes the optimal mix.

5.4 Optimal Mix of Interest Rate and Leverage Limit

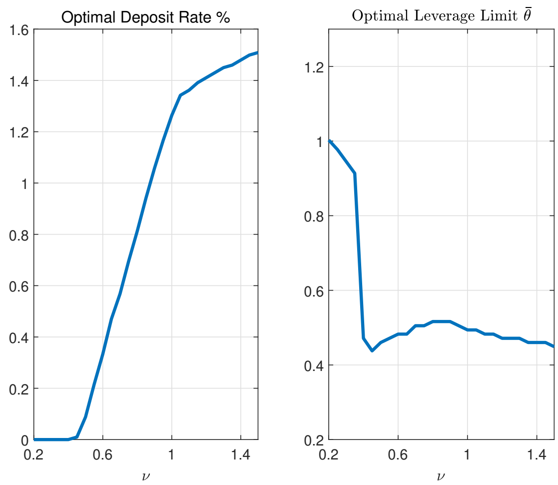
Given a leverage limit $\bar{\theta}$, we have shown that there can be an optimal interest rate level. We further explore the possibility of a joint pair of $(R, \bar{\theta})$ that maximizes social welfare. In this regard, we are searching for an optimal mix of policies, i.e., interest-rate and (macro-)prudential policies.

We start by examining the economy with calibrated parameters. In particular, with $\nu = 1$, projects exhibit no correlation. The optimal interest rate and leverage pair are found to be (1.25%, 0.4972) as shown by Figure 9. The optimal rate is only five basis points above the calibrated level of 1.20%, while the leverage limit is below the calibrated level of 0.56 by about 6.3 percentage points.

We can draw a few comparisons from the exercise in the previous section. When the deposit rate is above 2% (and thus the lending rate is above 4.1% thanks to the banking markup), the tax rate rises quickly, distorting the rates of project returns. The optimal policy should avoid such a scenario. When the leverage limit reaches 0.7-0.8, the social welfare drops quickly if leverage increases further. This result can be explained by the valuation effect illustrated in the theory part; for any given interest rate, entrepreneurs want to take less risk if leverage is already high and keeps going up. With the observation that a drop in leverage limits resource allocation and that a cut in interest rate hurts the savers, the social planner may want an interior optimal mix of interest rate and leverage limit.

Finally, we assess how spillover affects the optimal pair of interest rates and leverage limits. Figure 10 illustrates the effects under varying spillover conditions measured by different levels of ν. We show how optimal interest rate and leverage limit pairs respond to variations in the correlation of project outcomes.

Figure 10: Optimal policy



As outcomes of projects shift from negatively correlated to positively correlated, the economy calls for a monotone fall in interest rates because more resources should be given to the firms. With more and more positively correlated outcomes, at the aggregate level, the firms are more productive. However, the private economy may have too little or too much risk-taking, thanks to the externality not being considered by individual entrepreneurs. To address the externality issue, the leverage limit arrives at the center stage. When project outcomes are negatively correlated, the leverage limit decreases; however, when project outcomes become more positively correlated, the optimal leverage shape becomes non-monotonic, initially rising gradually, then falling, and finally rising sharply when the nominal interest rate is binding.

6 Conclusion

In this paper, we illustrate the impact of liquidity constraints on firms risk-taking behavior. When firm leverage is low, an increase in the leverage encourages risk-taking. When the leverage is high, an increase in leverage actually encourages firms to choose risk-free projects instead. The effect of interest rate on risk-taking could also be non-monotone as well. In particular, an interest rate cut only encourages risk-taking when the leverage is sufficiently high. Our analysis may explain why earlier studies found mixed results between liquidity and firm volatility. Our study suggests that in a low interest rate environment, firms may not implement risky but socially desirable projects.

Apart from the leverage conditions,we also show whether an interest rate incentivizes risk taking behavior or not may depend crucially on the spillover effects among the risk takers. We highlight the need for careful mix of interest-rate policy and macro prudential policy. In future research, we expect to find that taxations that alter projects' risk profiles should also be carefully mixed with interest rate policy.

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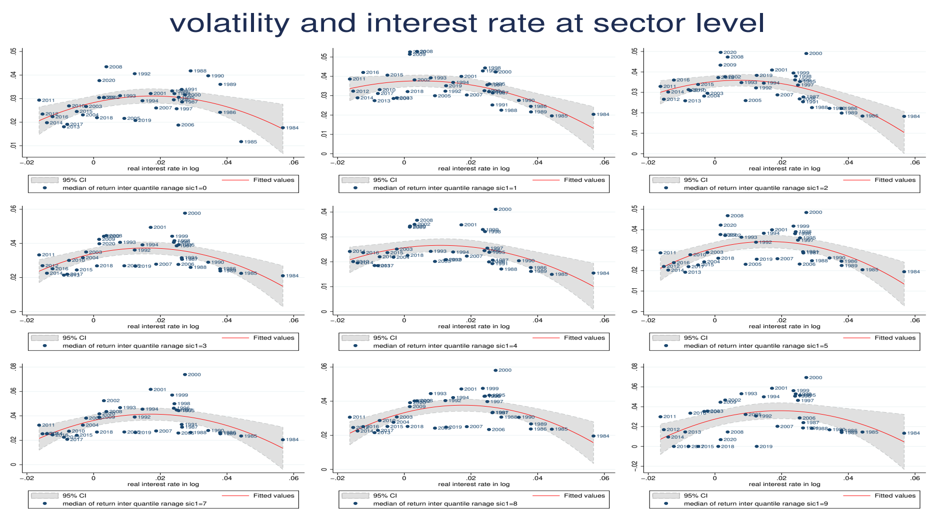
Figure 11: volatility and interest rate



A Data Source

We use the daily stock return from CRSP to calculate the inter quantile range of the daily return and the annualized stock return standard deviation. CRSP is a database that maintains the most comprehensive collection of security price, return, and volume data for the NYSE, AMEX and NASDAQ stock markets. Other financial data is from US COMPUSTAT from 1984-2016, and firms from financial sector are excluded. We also excluded the observations where the return standard deviation is larger than the 95th percentile or smaller than the 5th percentile of the industry level.

We use the real federal fund rate as the benchmark to measure the borrowing rate (results are similar when switched to the prime rate). It is retrieved from FRED maintained by the Federal Reserve Bank of St. Louis. The federal fund rate and inflation are from Federal Reserve Bank of St. Louis. Both the real interest rate and expected inflation are from Federal Reserve Bank of Cleveland and retrieved from FRED.

B Further Empirical Evidence

Table 5: Variation in ROA Changes with respect to the Interest Rate II

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
VARIABLES	risk2 in log	risk2 in log	risk2 in log	risk2 in log	risk2 in log	risk2 in log	risk2 in log	risk2 in log
mpshock	1.011**	0.095	1.584***	-1.257***	1.517**	-0.795**	1.186***	0.104
	(0.372)	(0.291)	(0.484)	(0.263)	(0.535)	(0.285)	(0.269)	(0.309)
Constant	0.771	1.223	0.936	-0.563	0.890*	0.842	1.588**	0.680
	(0.497)	(0.899)	(0.529)	(0.827)	(0.441)	(0.799)	(0.532)	(0.897)
Observations	43,195	45,250	42,934	45,720	41,423	46,311	43,195	45,250
R-squared	0.807	0.882	0.793	0.886	0.700	0.882	0.808	0.882
Firm FE	YES	YES	YES	YES	YES	YES	YES	YES
Interest use	ffr	ffr	ffr2	ff2	Cleveland rate	Cleveland rate	prime rate	prime rate
Interest level	low	high	low	high	low	high	low	high

Robust standard errors in parentheses

*** p<0.01, ** p<0.05, * p<0.1

Note: Results from estimating equation (1).Risk2 is the difference between the maximum and the minimum of the leading 5 years(t from 0 to 4)ROA(similarly constructed as in Table 2).Alternative measures of the real interest rate are employed to classify the interest rate regime: the log level the real federal fund rate deflated by the past inflation (ffr collumn 1 and 2) and deflated by the expected inflation (ffr2 collumn 3 and 4); the Cleveland rate(collumn 5 and 6) ;the log level of real prime rate (collumn 7 and 8).

Table 6: Variation in Research Intensity with respect to Changes in Interest Rate.

VARIABLES	(1) research intensity	(2) research intensity	(3) research intensity	(4) research intensity	(5) research intensity	(6) research intensity	(7) research intensity	(8) research intensity
mpshock	-0.017	-0.024***	0.000	-0.020*	-0.015	-0.020	-0.014	-0.027***
	(0.021)	(0.007)	(0.013)	(0.010)	(0.028)	(0.014)	(0.018)	(0.007)
Constant	0.251***	0.370***	0.251***	0.401***	0.259***	0.378***	0.244***	0.416***
	(0.036)	(0.092)	(0.039)	(0.098)	(0.046)	(0.096)	(0.045)	(0.099)
Observations	27,768	28,432	27,504	28,875	27,089	28,739	27,768	28,432
R-squared	0.602	0.575	0.596	0.583	0.591	0.590	0.602	0.576
Firm FE	YES	YES	YES	YES	YES	YES	YES	YES
Interest use	ffr	ffr	ffr2	ff2	Cleveland rate	Cleveland rate	prime rate	prime rate
Interest level	low	high	low	high	low	high	low	high

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Note: Results from estimating equation (1).Research intensity is defined as $R\&D$ expenses over total assets. Alternative measures of the real interest rate are employed to classify the interest rate regime: the log level the real federal fund rate deflated by the past inflation (ffr collumn 1 and 2) and deflated by the expected inflation (ffr2 collumn 3 and 4); the Cleveland rate(collumn 5 and 6) ;the log level of real prime rate (collumn 7 and 8).

C Derivations and Proofs

C.1 Proof of Proposition 1

If the entrepreneur chooses a risky project, the value function becomes:

$$\begin{aligned} V^r(\omega,\eta) &= \max_{s,b}\Big\{\log(\omega-s)-\eta+\beta p\log(\Pi^h(s+b)-Rb)\} \\ &\quad+\beta(1-p)\log(\Pi^h(s+b)-Rb)\Big\} \\ &= \max_{s,\theta\le\bar\theta}\Big\{\log(\omega-s)+\beta\log s-\eta+\beta p\log\left(\Pi^h(1+\theta)-R\theta\right) \\ &\quad+\beta(1-p)\log\left(\Pi^l(1+\theta)-R\theta\right)\Big\}. \end{aligned}$$

Through first-order conditions, the optimal saving rate is $s=\varphi\omega$ with $\varphi=\frac{\beta}{1+\beta}$. Denote θ^* as the level of the unconstrained optimal leverage a firm will choose were they not facing the financing constraint and the optimal $\theta>0$, then θ^* solves the first-order condition:

$$p\frac{\Pi^h-R}{\Pi^h(1+\theta)-R}+(1-p)\frac{\Pi^l-R}{\Pi^l(1+\theta)-R}=0$$

Therefore, the cutoff interest rate level $\Pi^h\Pi^l/\hat\Pi$ is obtained by setting $\theta=0$ above and it is straightforward to verify that $\Pi^h\Pi^l/\hat\Pi\in\left(\Pi^l,\Pi^h\right)$. If $R>\Pi^h\Pi^l/\hat\Pi$, then the interest rate is too high to justify borrowing. If $R\le\Pi^h\Pi^l/\hat\Pi$, then we can express θ^* as

$$\theta^*\equiv-\frac{\Pi^h\Pi^l-R(p\Pi^l+(1-p)\Pi^h)}{(\Pi^l-R)(\Pi^h-R)}=-\left[p\frac{\Pi^l}{\Pi^l-R}+(1-p)\frac{\Pi^h}{\Pi^h-R}\right]. \tag{22}$$

Notice that $\theta^*>0$ because $R<\Pi^f\le\Pi^h\Pi^l/\hat\Pi$ under assumption A2. The optimal leverage $\theta=\min\{\theta^*,\bar\theta\}$. We also show that θ^* decreases in R. Notice that

$$
\begin{aligned}
\frac{\partial \theta^*}{\partial R} &= \frac{\hat{\Pi}(\Pi^l - R)(\Pi^h - R) + (-\Pi^h\Pi^l + R\hat{\Pi})(+\Pi^h + \Pi^l - 2R)}{(\Pi^l - R)^2(\Pi^h - R)^2} \\
&= \frac{\Pi^l\Pi^h(R - \Pi) + (\Pi^h\Pi^l - \hat{\Pi}R)R}{(\Pi^l - R)^2(\Pi^h - R)^2}
\end{aligned}
$$

The numerator is quadratic and concave in R. With some algebra, the maximum value of the numerator is $\frac{\Pi^h\Pi^l(\Pi^h\Pi^l - \hat{\Pi}\Pi)}{\hat{\Pi}}$, which is negative. To see this, we can use the convexity, $\frac{p}{\Pi^h} + \frac{1-p}{\Pi^l} > \frac{1}{p\Pi^h + (1-p)\Pi^l} = \frac{1}{\Pi}$ which implies $\Pi^h\Pi^l < \hat{\Pi}\Pi$. Then the numerator is less than 0 and $\frac{\partial \theta^*}{\partial R} < 0$.

If the entrepreneur chooses a safe project, we have:

$$
\begin{aligned}
V^f(\omega) &= \max_{s,b}\{\log(\omega - s) + \beta \log(\Pi^f(s + b) - Rb) \\
&= \max_{s,\theta \le \bar{\theta}}\{\log(\omega - s) + \beta \log s + \beta \log(\Pi^f(1 + \theta) - R\theta)\}.
\end{aligned}
$$

This means that the optimal saving rate is $s = \varphi\omega$ again. Since $\Pi^f > R$, a firm will borrow up to the limit $\theta = \bar{\theta}$.

Finally, we determine the threshold of taking risky project $\tilde{\eta}$, take the difference between $V^r(\omega, \eta)$ and $V^f(\omega)$, we obtain

$$
\tilde{\eta} = v^r - v^f = \beta p \log\left(\frac{\Pi^h + (\Pi^h - R)\theta}{\Pi^f + (\Pi^f - R)\bar{\theta}}\right) + \beta(1 - p)\log\left(\frac{\Pi^l + (\Pi^l - R)\theta}{\Pi^f + (\Pi^f - R)\bar{\theta}}\right).
$$

C.2 Proof of Proposition 2

Assuming $\theta^* > \bar{\theta}$, then $\theta = \bar{\theta}$ when an entrepreneur chooses the risky project. Define $x \equiv R\frac{\bar{\theta}}{1+\bar{\theta}}$ as the debt servicing cost per unit of capital used in production, then $\tilde{\eta}$ can be rewritten as

$$
\tilde{\eta} = \beta p \log\left(\frac{\Pi^h - x}{\Pi^f - x}\right) + \beta(1 - p)\log\left(\frac{\Pi^l - x}{\Pi^f - x}\right).
$$

Then, we have

$$
\frac{\partial \tilde{\eta}}{\partial x} = \beta\frac{\Pi^h\Pi^l - \Pi^f\hat{\Pi} + (\Pi^f - \Pi)x}{(\Pi^h - x)(\Pi^l - x)}.
$$

where $\hat{\Pi} \equiv (1 - p)\Pi^h + p\Pi^l$. Under Assumptions A1 and A2, one can verify that

$$
\Pi^h > \Pi^l > R\frac{\theta^*}{1 + \theta^*} > R\frac{\bar{\theta}}{1 + \bar{\theta}}
$$

Then the denominator is positive. Also, $(\Pi^f - \Pi)$ is negative. We can thus reach $\partial\tilde{\eta}/\partial x = 0$ if $x = x^s$, where

$$
x^s \equiv \frac{\Pi^h\Pi^l - \Pi^f\hat{\Pi}}{\Pi - \Pi^f}
$$

$\partial\tilde{\eta}/\partial x > 0$ if $x < x^s$, and $\partial\tilde{\eta}/\partial x < 0$ if $x > x^s$.

Now, let's look at

$$
\frac{\partial \tilde{\eta}}{\partial \bar{\theta}} = \frac{\partial \tilde{\eta}}{\partial x}\frac{\partial x}{\partial \bar{\theta}}.
$$

We know that $\partial x/\partial\bar{\theta} > 0$. Therefore the sign of $\partial\tilde{\eta}/\partial\bar{\theta}$ depends on the sign of $\partial\tilde{\eta}/\partial x$. Let $\bar{\theta}^s$ be the leverage upper bound such that $x = x^s$, i.e.,$\bar{\theta}^s \equiv \bar{\theta}^s(R) = \frac{x^s}{R - x^s}$. One can show that $0 < \bar{\theta}^s < \theta^*$. Therefore $\tilde{\eta}$ is hump-shaped in $\bar{\theta}$. Moreover, it is easy to see that $\bar{\theta}^s$ decreases in R.

Now, let's look at

$$\frac{\partial \tilde{\eta}}{\partial R} \quad = \quad \frac{\partial \tilde{\eta}}{\partial x}\frac{\partial x}{\partial R}.$$

Similarly, $\partial x/\partial R > 0$, and the sign of $\partial \tilde{\eta}/\partial R$ depends on the sign of $\partial \tilde{\eta}/\partial x$. Let R^s be the interest rate such that $x = x^s$, i.e., $R^s = R^s(\bar{\theta}) = x^s(1/\bar{\theta}+1)$. Note that R^s depends on the leverage upper bound $\bar{\theta}$ and the superscript s indicates that the debt servicing cost is x^s (which is an parameter given the project returns). Therefore,

$$\bar{\theta}^s_H \equiv \frac{\Pi^h \Pi^l - \Pi^f \hat{\Pi}}{(\Pi - \Pi^f)\Pi^l - (\Pi^h \Pi^l - \Pi^f \hat{\Pi})}$$

is the bound such that $R^s(\bar{\theta}^s_H) = \Pi^l$ and

$$\bar{\theta}^s_L \equiv \frac{\Pi^h \Pi^l - \Pi^f \hat{\Pi}}{(\Pi - \Pi^f)\Pi^f - (\Pi^h \Pi^l - \Pi^f \hat{\Pi})}$$

is the bound such that $R^s(\bar{\theta}^s_L) = \Pi^f$. Using the two bounds, we can reach the following conclusion.

If $\bar{\theta} > \bar{\theta}^s_H$, then $R^s(\bar{\theta}) < \Pi^l$. For the range of interest rate under assumptions, $R > \Pi^l > R^s(\bar{\theta})$, implying that the debt servicing cost x is above x^s. In this case, $\partial \tilde{\eta}/\partial x < 0$ and thus $\partial \tilde{\eta}/\partial R < 0$.

If $\bar{\theta} < \bar{\theta}^s_L$, then $R^s(\bar{\theta}) > \Pi^f$. For the range of interest rate under assumptions, $R < \Pi^f < R^s(\bar{\theta})$, implying that the debt servicing cost x is below x^s. In this case, $\partial \tilde{\eta}/\partial x > 0$ and thus $\partial \tilde{\eta}/\partial R > 0$.

Otherwise, $\tilde{\eta}$ is hump-shaped in R and $\Pi^l < R^s(\bar{\theta}) < \Pi^f$ when $\bar{\theta}^s_L < \bar{\theta} < \bar{\theta}^s_H$.

Finally, we have the following corollary:

Corollary. *Suppose A1 and A2 hold. Assuming the financing constraint is binding when the risky project is implemented (i.e., $\bar{\theta} < \theta^*$). $\bar{\theta}^s(R) \in (0, \theta^*)$, the leverage upper bound* such that $\partial \tilde{\eta}/\partial R = 0$, *decreases in R. When $0 < \bar{\theta} < \bar{\theta}^s(R)$, the cutoff $\tilde{\eta}$ is increasing in $\bar{\theta}$; when $\bar{\theta}^s(R) < \bar{\theta} < \theta^*$, the cutoff $\tilde{\eta}$ is decreasing in $\bar{\theta}$.*

Intuitively, when leverage $\bar{\theta}$ is low, the risk-free project is not preferred since its (leveraged) return is smaller than the risky project's. As the leverage rises, the ratio between the leveraged return from the risky and the risk-free projects rises while the returns on both risky and risk-free projects increase. However, the (leveraged) volatility associated with the risky asset increases as leverage increases. If the leverage ratio is above some threshold $\hat{\theta}$, the disutility related to volatility dominates, and the risky project becomes less appealing.

C.3 Proof of Proposition 3

Observe that when leverage is not constrained,

$$\tilde{\eta} = \beta p \log\left(\frac{\Pi^h + (\Pi^h - R)\theta^*}{\Pi^f + (\Pi^f - R)\bar{\theta}}\right) + \beta(1-p)\log\left(\frac{\Pi^l + (\Pi^l - R)\theta^*}{\Pi^f + (\Pi^f - R)\bar{\theta}}\right),$$

where $\theta^* = -\left[p\frac{\Pi^l}{\Pi^l - R} + (1-p)\frac{\Pi^h}{\Pi^h - R}\right]$. Then, we have:

$$\begin{aligned}\tilde{\eta} =& \beta[p\log p + (1-p)\log(1-p) + \log R + \log(\Pi^h - \Pi^l) \\ & - p\log(R - \Pi^l) - (1-p)\log(\Pi^h - R) - \log(\Pi^f + (\Pi^f - R)\bar{\theta})]\end{aligned}$$

Take the derivative with respect to R, we have

$$\frac{\partial \tilde{\eta}}{\partial R} = \beta\left[\frac{\bar{\theta}}{\Pi^f + (\Pi^f - R)\bar{\theta}} - \frac{\theta^*}{R}\right]. \tag{23}$$

To prove the non-monotone behavior of the cutoff level for unconstrained entrepreneurs in Proposition 3 we proceed in two steps: 1) we show that $\tilde{\eta}$ is convex in the interest rate R when the financing constrained is slack; 2) if $\bar{\theta}$ is large enough we will see $\tilde{\eta}$ deceases and then increases in the interest R for unconstrained entrepreneurs.

Lemma 5. *For financially unconstrained entrepreneurs, the cutoff level $\tilde{\eta}$ is convex in R.*

Proof. With the expression for θ^* equation (22), we can rewrite $\frac{\partial \theta^*}{\partial R}$ as

$$\frac{\partial \theta^*}{\partial R} = \frac{\theta^*}{R\hat{\Pi} - \Pi^h\Pi^l}\left[\frac{\Pi^h\Pi^l + \theta^*(\Pi^h\Pi^l - R^2)}{R}\right].$$

From Proposition 1, we know $\frac{\partial \theta^*}{\partial R} < 0$. With this inequality and assumption A2 $\Pi^h\Pi^l > \Pi^f\hat{\Pi}$ indicating that $\Pi^h\Pi^l - R\hat{\Pi} > \Pi^h\Pi^l - \Pi^f\hat{\Pi} > 0$, we have $\Pi^h\Pi^l + \theta^*(\Pi^h\Pi^l - R^2) > 0$. Then, it follows that

$$-\frac{\partial \frac{\theta^*}{R}}{\partial R} = \frac{\theta^*\left[\Pi^l\Pi^h + \theta^*(\Pi^h\Pi^l - R^2)\right] + \theta^*(\Pi^h\Pi^l - R\hat{\Pi})}{R^2(\Pi^h\Pi^l - R\hat{\Pi})} > 0$$

Thus, we have the cutoff level $\tilde{\eta}$ is convex in the interest rate R for unconstrained entrepreneurs as

$$\frac{\partial^2 \tilde{\eta}}{\partial R^2} = \frac{\partial \frac{\bar{\theta}}{\Pi^f + (\Pi^f - R)\bar{\theta}}}{\partial R} - \frac{\partial \frac{\theta^*}{R}}{\partial R} > 0.$$

□

Lemma 6. *At the interest rate $\bar{R}$, when the financial constraints changes from binding to non-binding, $\tilde{\eta}$ decreases in the interest rate R, i.e., $\frac{\partial \tilde{\eta}}{\partial R}|_{R=\bar{R}} < 0$.*

Proof. To show this we use the non-monotone result from the binding case in 2 since when interest rate is higher than R^s, which is the interest rate such that $\tilde{\eta}$ takes its maximum when the financial constraints are binding, then $\frac{\partial \tilde{\eta}}{\partial R} < 0$. If $\bar{R} > R^s$ then $\frac{\partial \tilde{\eta}}{\partial R}|_{R=\bar{R}} < 0$ follows.

With some algebra, we can verify that $\theta^*_{min} = \bar{\theta}^s_L$ because $\Pi + \hat{\Pi} = \Pi^h + \Pi^l$. The proof in proposition 2 has shown that when $\bar{\theta} > \bar{\theta}^s_L$, we have $R^s(\bar{\theta}) < \Pi^f$. To show $\bar{R}(\bar{\theta}) > R^s(\bar{\theta})$, it would be more convenient to compare the debt servicing cost, $R\frac{\bar{\theta}}{1+\bar{\theta}}$ in both cases: from the proof of Proposition 2 in Appendix C.2

$$R^s\frac{\bar{\theta}}{1+\bar{\theta}} \quad = \quad x^s = \frac{\Pi^h\Pi^l - \Pi^f\hat{\Pi}}{\Pi - \Pi^f} = g(\Pi^f)$$

where $g(s) \equiv \frac{\Pi^h\Pi^l - s\hat{\Pi}}{\Pi - s}$. Meanwhile

$$\bar{R}\frac{\bar{\theta}}{1+\bar{\theta}} = \frac{-p\Pi^l(\Pi^h - \bar{R}) - (1-p)\Pi^h(\Pi^l - \bar{R})}{-p(\Pi^h - \bar{R}) - (1-p)(\Pi^l - \bar{R})} = \frac{\Pi^h\Pi^l - \bar{R}\hat{\Pi}}{\Pi - \bar{R}} = g(\bar{R}).$$

If we observe entrepreneurs become unconstrained then it must be that $\bar{R}(\bar{\theta}) < \Pi^f$. To see this claim, first we observe that θ^* is a decreasing function of R and $\bar{R}$ is a function of $\bar{\theta}$. By the definition of $\bar{R}$, we have $\theta^*(R = \bar{R}(\bar{\theta})) = \bar{\theta}$. By the definition of θ^*_{min}, we have $\theta^*(R = \Pi^f) = \theta^*_{min}$. Second, if $\bar{R}(\bar{\theta}) \geq \Pi^f$ that is opposite of the claim, then $\bar{\theta} = \theta^*(R = \bar{R}(\bar{\theta})) \leq \theta^*(R = \Pi^f) = \theta^*_{min}$ as θ^* decreases in R, which contradicts with our assumption that $\bar{\theta} > \theta^*_{min}$ and we prove the claim. Finally, by convexity, $\frac{p}{\Pi^h} + \frac{1-p}{\Pi^l} > \frac{1}{p\Pi^h + (1-p)\Pi^l} = \frac{1}{\Pi}$ which implies $\Pi^h\Pi^l < \hat{\Pi}\Pi$. Then, the expression $g(s)$ is decreasing in s.[19] This implies $\bar{R}\frac{\bar{\theta}}{1+\bar{\theta}} > R^s\frac{\bar{\theta}}{1+\bar{\theta}}$ and thus $\bar{R} > R^s$. □

Finally, using lemma 5 and lemma 6, we can finish the proof of Proposition 3, the non-monotonicity of $\tilde{\eta}$ in the interest R for unconstrained case. To show $\tilde{\eta}$ changes from decreasing to increasing in R, we need to further find an interest rate level such that $\frac{\partial \tilde{\eta}}{\partial R} > 0$.

[19]The first order condition of $g(s) = \frac{\Pi^h\Pi^l - s\hat{\Pi}}{\Pi - s}$ with respect to s is $\frac{-\hat{\Pi}(\Pi - s) + \Pi^h\Pi^l - s\hat{\Pi}}{(\Pi - s)^2} = \frac{\Pi^h\Pi^l - \hat{\Pi}\Pi}{(\Pi - s)^2} < 0$.

In the proof of (C.2), the derivative of $\tilde{\eta}$ with respect to the debt servicing cost x is zero when $x = x^s$; we see that $\frac{\partial \tilde{\eta}}{\partial R}|_{R=\Pi^f} = 0$ when $\bar{\theta}$ takes the value of $\bar{\theta}_L^s$, because the debt servicing cost is indeed $x = x^s$ when $\bar{\theta} = \bar{\theta}_L^s$ and $R = \Pi^f$. It is straightforward to verify that $\frac{\partial^2 \tilde{\eta}}{\partial R \partial \bar{\theta}} > 0$ according to (23), which means that $\frac{\partial \tilde{\eta}}{\partial R}$ increases in $\bar{\theta}$. Thus $\frac{\partial \tilde{\eta}}{\partial R}|_{R=\Pi^f} > 0$ when $\bar{\theta} > \theta^*_{min}$ since we already know that $\theta^*_{min} = \bar{\theta}^s_L$. Then by Lemma 5, Lemma 6, and the mean value theorem, when $\bar{\theta} > \theta^*_{min}$ there exists an interest rate $R^u(\bar{\theta})$ where $\bar{R}(\bar{\theta}) < R^u(\bar{\theta}) < \Pi^f$ such that $\frac{\partial \tilde{\eta}}{\partial R}|_{R=R^u} = 0$, as $\frac{\partial \tilde{\eta}}{\partial R}|_{R=\bar{R}} < 0$ (from Lemma 6), $\frac{\partial \tilde{\eta}}{\partial R}|_{R=\Pi^f} > 0$, and $\frac{\partial \frac{\partial \tilde{\eta}}{\partial R}}{\partial R} > 0$ (from Lemma 5). The implied risk taking threshold for unconstrained entrepreneurs decreases with R first and increases with R later, i.e., $\frac{\partial \tilde{\eta}}{\partial R} < 0$ if $\bar{R}(\bar{\theta}) \leq R \leq R^u(\bar{\theta})$and $\frac{\partial \tilde{\eta}}{\partial R} > 0$ if $R^u(\bar{\theta}) < R(\bar{\theta}) \leq \Pi^f$.

C.4 Proof of Proposition 4

Guess that $V^r(\omega, \eta) = \log(\omega) + v^r - \eta$, $V^f(\omega) = \log(\omega) + v^f - \eta^f$, $V^d(\omega) = \log(\omega) + v^d$. Plugging these guessed forms into entrepreneur's Bellman equation, we have the following (define $B \equiv (1-\beta)\log(1-\beta) + \beta \log \beta$)

$$\begin{aligned} v^r = & B + \beta p \left[\log\left(\Pi^h(1+\theta) - R\theta\right)\right] + \beta(1-p)\left[\log\left(\Pi^l(1+\theta) - R\theta\right)\right] \\ & + \beta \mathbb{E} \max\{v^r - \eta', v^f - \eta^f, v^d\}; \end{aligned} \tag{24}$$

$$v^f = B + \beta\left[\log(\Pi^f(1+\bar{\theta}) - R\bar{\theta})\right] + \beta \mathbb{E} \max\{v^r - \eta', v^f - \eta^f, v^d\}; \tag{25}$$

$$v^d = B + \beta \log(R^d) + \beta \mathbb{E} \max\{v^r - \eta', v^f - \eta^f, v^d\}. \tag{26}$$

One can solve for the three unknown v^r, v^f, and v^d from the above three equations, and verify the guess. They are, however, not important for our purposes. The choice between the risk-free project and deposits depends on the maximal value of the two, $v^f - \eta^f$ and v^d. Using v^f and v^d just above, we then have

$$\phi = \begin{cases} 0 & \text{if } \log(\Pi^f(1+\bar{\theta}) - R\bar{\theta}) - \eta^f/\beta > \log R^d \\ (0,1) & \text{if } \log(\Pi^f(1+\bar{\theta}) - R\bar{\theta}) - \eta^f/\beta = \log R^d \\ 1 & \text{if } \log(\Pi^f(1+\bar{\theta}) - R\bar{\theta}) - \eta^f/\beta < \log R^d \end{cases}$$

Notice that when $\phi = 0$, the risk-free project strictly dominates; when $\phi = 1$, the risk-free deposits strictly dominates; when $0 < \phi < 1$, an entrepreneur is indifferent between implementing the risk-free project and the risk-free deposits.

For our analysis, the choice of leverage θ is identical to the two-period model and the proof has been provided before. For the cutoff, there exists a cutoff such that below the cutoff agents becomes entrepreneurs who choose risky projects and above which agents might choose risk-free project or deposit. Similar to the proof in the 2-period model, we have $\tilde{\eta}$ as:

$$\begin{aligned} \tilde{\eta} &= v^r - \max\{v^f - \eta^f, v^d\} \\ &= \beta p \log\left(\Pi^h + (\Pi^h - R)\theta\right) + \beta(1-p)\log\left(\Pi^l + (\Pi^l - R)\theta\right) \\ &\quad - \beta \max\{\log(\Pi^f(1+\bar{\theta}) - R\bar{\theta}) - \eta^f/\beta, \log R^d\}. \end{aligned}$$

where we use the value functions in the beginning of the proof and $\theta = \min\{\theta^*, \bar{\theta}\}$ (again θ^* is the unconstrained solution given by (22)). Of course, if the right-hand side is negative, $\tilde{\eta} = 0$. Therefore, those entrepreneurs with $\eta \leq \max\{\underline{\eta}, \tilde{\eta}\}$ choose the risky project.

D Welfare

D.1 The average return in the society

Let m denote number of realizations of Π^h in the economy, the proportion of success $p^s(\nu) = \mathbb{E}\left[\frac{m}{n}\right]$ as n goes to infinity, which is the share of the risky projects with high return Π^h as n goes to infinity.

It has been shown in Shmueli et al. (2005) that for the Conway-Maxwell Poisson distribution, $\mathbb{E}[m]$ converges to $\lambda^{1/\nu} - \frac{\nu-1}{2\nu}$ when n goes to infinity and p is small i.e. $p < p^*$. This suggests that $p^s(\nu)$ converges to $p^{1/\nu}$ when n goes to infinity once substituting $\lambda = n^\nu p$ as illustrated in Daly and Gaunt (2016). Note that $p^{1/\nu}$ is the probability of high realizations assessed by the social planner, which differs from the individual's assessment p whenever there is correlation in the projects (i.e., $\nu \neq 1, \nu > 0$). Then the average return from the society's view becomes $p^{\frac{1}{\nu}}\Pi^h + (1 - p^{\frac{1}{\nu}})\Pi^h$.

The case when p is large i.e. $p > p^{**}$ is similar if we replace p by $1 - p$ in the argument above.

D.2 Welfare function

We first derive entrepreneurs' welfare, which come from three types: those who take risks, those who implement the risk-free project, and those who save in safe deposits:

$$V_e(\omega) = \int^{\tilde{\eta}} V^r(\omega, \eta)dF(\eta) + (1 - \phi)\int_{\tilde{\eta}} V^f(\omega, \eta^f)dF(\eta) + \phi\int_{\tilde{\eta}} V^d(\omega)dF(\eta) \tag{27}$$

We have three scenarios below, each of which determines the expected value $\mathbb{E}\max\{v^r - \eta', v^f - \eta^f, v^d\}$ differently.

Scenario 1: When $\phi = 0$, no one chooses the safe deposit option. That is

$$\begin{aligned} v^f - \eta^f \quad &> \quad v^d; \\ \eta^f \quad &< \quad \beta\left[\log(\Pi^f(1 + \bar{\theta}) - R\bar{\theta})\right] - \beta\log(R^d). \end{aligned}$$

Then, we can explicitly express $\mathbb{E}\max\{v^r - \eta', v^f - \eta^f, v^d\}$ as follows

$$\begin{aligned} \mathbb{E}\max\{v^r - \eta', v^f - \eta^f, v^d\} \quad &= \quad v^d + \mathbb{E}\max\{v^r - v^d - \eta', v^f - v^d - \eta^f, 0\} \\ &= \quad v^d + v^f - v^d - \eta^f + \mathbb{E}\max\{\tilde{\eta} - \eta', 0\} \\ &= \quad v^f - \eta^f + \int^{\tilde{\eta}} F(\eta)d\eta; \end{aligned}$$

since $v^f - v^d - \eta^f > 0$ and $\tilde{\eta} = v^r - v^f + \eta^f$. This means that we can explicitly solve the values in ((24)) - ((26)). First,

$$\begin{aligned} v^f \quad &= \quad B + \beta\left[\log(\Pi^f(1 + \bar{\theta}) - R\bar{\theta})\right] + \beta\left[v^f - \eta^f + \int^{\tilde{\eta}} F(\eta)d\eta\right] \\ &= \quad \frac{B}{1 - \beta} + \frac{\beta}{1 - \beta}\left[\log(\Pi^f(1 + \bar{\theta}) - R\bar{\theta})\right] + \frac{\beta}{1 - \beta}[\int^{\tilde{\eta}} F(\eta)d\eta - \eta^f]; \end{aligned}$$

where the second equality rearranges the v^f in the first equality on the right-hand side and divides $(1 - \beta)$ on both sides. Second, we have

$$
\begin{aligned}
v^r &= B + \beta p \left[\log\left(\Pi^h(1+\theta) - R\theta\right)\right] \\
&+ \beta(1-p)\left[\log\left(\Pi^l(1+\theta) - R\theta\right)\right] + \beta\left[v^f - \eta^f + \int^{\tilde{\eta}} F(\eta)d\eta\right] \\
&= \frac{B}{1-\beta} + \frac{\beta}{1-\beta}\Big\{p\left[\log\left(\Pi^h(1+\theta) - R\theta\right)\right] \\
&+ (1-p)\left[\log\left(\Pi^l(1+\theta) - R\theta\right)\right]\Big\} + \frac{\beta}{1-\beta}\left[\int^{\tilde{\eta}} F(\eta)d\eta - \tilde{\eta}\right],
\end{aligned}
$$

where the second equality uses $v^f - \eta^f = v^r - \tilde{\eta}$ and rearranges v^r. Finally, we have

$$
\begin{aligned}
v^d &= B + \beta\log(R^d) + \beta\left[v^f - \eta^f + \int^{\tilde{\eta}} F(\eta)d\eta\right] \\
&= \frac{B}{1-\beta} + \frac{\beta}{1-\beta}\left[(1-\beta)\log\left(R^d\right) + \beta\log(\Pi^f(1+\bar{\theta}) - R\bar{\theta})\right] + \frac{\beta}{1-\beta}\left[\int^{\tilde{\eta}} F(\eta)d\eta - \eta^f\right],
\end{aligned}
$$

where the second equality uses $v^f - \eta^f = v^r - \tilde{\eta} = \tilde{\eta}_2 + v^d - \tilde{\eta}$. Therefore, denote $V_e(\omega)$ as the value function of an entrepreneur with wealth level ω, we rewrite ((27)) as

$$
\begin{aligned}
V_e(\omega) &= \int^{\tilde{\eta}} (V^r(\omega,\eta))dF(\eta) + \int_{\tilde{\eta}} V^f(\omega,\eta^f) \\
&= \log(\omega) + v^r F(\tilde{\eta}) - \int^{\tilde{\eta}} \eta dF(\eta) + [1 - F(\tilde{\eta})]\,(v^f - \eta^f) \\
&= \log(\omega) + \tilde{\eta}F(\tilde{\eta}) + (v^f - \eta^f) - F(\tilde{\eta})\tilde{\eta} + \int^{\tilde{\eta}} F(\eta)d\eta \\
&= \log(\omega) + \frac{B}{1-\beta} + \frac{\beta}{1-\beta}\left[\log(\Pi^f(1+\bar{\theta}) - R\bar{\theta})\right] \\
&\quad + \frac{1}{1-\beta}[\int^{\tilde{\eta}} F(\eta)d\eta - \eta^f],
\end{aligned}
$$

where the third equality uses $v^r = v^f - \eta^f + \tilde{\eta}$ and the forth equality uses the expression for v^f above.

Scenario 2: when $0 < \phi < 1$, some entrepreneurs choose the safe deposit option while others choose the safe project option. That is:

$$
\begin{aligned}
v^f - \eta^f &= v^d; \\
\eta^f &= \beta\left[\log(\Pi^f(1+\bar{\theta}) - R\bar{\theta})\right] - \beta\log(R^d).
\end{aligned}
$$

Then, we can explicitly express $\mathbb{E}\max\{v^r - \eta', v^f - \eta^f, v^d\}$ as

$$
\begin{aligned}
\mathbb{E}\max\{v^r - \eta', v^f - \eta^f, v^d\} &= v^f - \eta^f + \mathbb{E}\max\{v^r - v^f + \eta^f - \eta', 0\} \\
&= v^f - \eta^f + \mathbb{E}\max\{\tilde{\eta} - \eta, 0\} \\
&= v^f - \eta^f + \int^{\tilde{\eta}} F(\eta)d\eta;
\end{aligned}
$$

Using the result above and following a similar procedure in Scenario 1, we simplify the values in ((24)) - ((26)):

$$\begin{aligned} v^f &= B + \beta\left[\log(\Pi^f(1+\bar\theta) - R\bar\theta)\right] + \beta(v^f - \eta^f + \int^{\tilde\eta} F(\eta)d\eta) \\ &= \frac{B}{1-\beta} + \frac{\beta}{1-\beta}\left[\log(\Pi^f(1+\bar\theta) - R\bar\theta)\right] + \frac{\beta}{1-\beta}[\int^{\tilde\eta} F(\eta)d\eta - \eta^f] \end{aligned}$$

where the second equality rearranges v^f;

$$\begin{aligned} v^d &= B + \beta\left[\log(R^d)\right] + \beta(v^d + \int^{\tilde\eta} F(\eta)d\eta) \\ &= \frac{B}{1-\beta} + \frac{\beta\log(R^d)}{1-\beta} + \frac{\beta}{1-\beta}\int^{\tilde\eta} F(\eta)d\eta \end{aligned}$$

where we use the fact that $v^f - \eta^f = v^d$ in the first equality and rearrange v^d in the second equality;

$$\begin{aligned} v^r &= B + \beta p\left[\log\left(\Pi^h(1+\theta) - R\theta\right)\right] \\ &\quad +\beta(1-p)\left[\log\left(\Pi^l(1+\theta) - R\theta\right)\right] + \beta(v^f - \eta^f + \int^{\tilde\eta} F(\eta)d\eta) \\ &= \frac{B}{1-\beta} + \frac{\beta}{1-\beta}[p\left[\log\left(\Pi^h(1+\theta) - R\theta\right)\right] \\ &\quad +(1-p)\left[\log\left(\Pi^l(1+\theta) - R\theta\right)\right]] + \frac{\beta}{1-\beta}(\int^{\tilde\eta} F(\eta)d\eta - \tilde\eta) \end{aligned}$$

where we use the fact that $v^f - \eta^f = v^r - \tilde\eta$ and rearrange v^r in the second equality. Therefore, using v^r, v^f, and v^d above, we can rewrite ((27)) as

$$\begin{aligned} V_e(\omega) &= \int^{\tilde\eta}(V^r(\omega,\eta))dF(\eta) + (1-\phi)\int_{\tilde\eta} V^f(\omega,\eta^f) + \phi\int_{\tilde\eta} V^d(\omega) \\ &= \log(\omega) + v^r F(\tilde\eta) - \int^{\tilde\eta}\eta dF(\eta) + (1-\phi)(1-F(\tilde\eta))(v^f - \eta^f) + \phi(1-F(\tilde\eta))v^d \\ &= \log(\omega) + v^r F(\tilde\eta) - \int^{\tilde\eta}\eta dF(\eta) + (1-\phi)(1-F(\tilde\eta))(v^f - v^d - \eta^f) + (1-F(\tilde\eta))(1-\phi)v^d + \phi(1-F(\tilde\eta))v^d \\ &= \log(\omega) + v^d + F(\tilde\eta)(v^r - v^d) - \int^{\tilde\eta}\eta dF(\eta) \\ &= \log(\omega) + v^d + F(\tilde\eta)(v^r - v^f + \eta^f) - \int^{\tilde\eta}\eta dF(\eta) \\ &= \log(\omega) + v^d + F(\tilde\eta)\tilde\eta - F(\tilde\eta)\tilde\eta + \int^{\tilde\eta} F(\eta)d\eta \\ &= \log(\omega) + \frac{B}{1-\beta} + \frac{\beta\log(R^d)}{1-\beta} + \frac{1}{1-\beta}\int^{\tilde\eta} F(\eta)d\eta, \end{aligned}$$

where we used $v^d = v^f - \eta^f$ in the fourth equality, $\tilde\eta = v^r - v^f + \eta^f$ in the sixth equality, and the expression for v^d in the last equality.

Scenario 3: when $\phi = 1$, all entrepreneurs choose the safe deposit option. That is

$$\begin{aligned} v^f - \eta^f &< v^d; \\ \eta^f &> \beta\left[\log(\Pi^f(1+\bar\theta) - R\bar\theta)\right] - \beta\log R^d. \end{aligned}$$

Then, we can again explicitly express $\mathbb{E}\max\{v^r - \eta', v^f - \eta^f, v^d\}$ as

$$
\begin{aligned}
\mathbb{E}\max\{v^r-\eta', v^f-\eta^f, v^d\} &= v^d + E\max\{v^r-v^d-\eta', v^f-\eta^f-v^d, 0\} \\
&= v^d + E\max\{v^r-v^d-\eta', 0\} \\
&= v^d + \int^{\tilde{\eta}} F(\eta)d\eta
\end{aligned}
$$

since $v^f-\eta^f-v^d<0$. Using the result and following a similar procedure in Scenario 1, we simplify the values in ((24)) - ((26)):

$$
\begin{aligned}
v^d &= B+\beta\log(R^d)+\beta\left(v^d+\int^{\tilde{\eta}} F(\eta)d\eta\right) \\
&= \frac{B}{1-\beta}+\frac{\beta\log(R^d)}{1-\beta}+\frac{\beta}{1-\beta}\int^{\tilde{\eta}} F(\eta)d\eta
\end{aligned}
$$

where the second equality rearranges v^d on the right-hand side of the first equality;

$$
\begin{aligned}
v^f &= B+\beta\log(\Pi^f(1+\bar{\theta})-R\bar{\theta})+\beta(v^d+\int^{\tilde{\eta}} F(\eta)d\eta) \\
&= B+\beta\log(\Pi^f(1+\bar{\theta})-R\bar{\theta})+\beta(v^f+v^d-v^f+\int^{\tilde{\eta}} F(\eta)d\eta) \\
&= \frac{B}{1-\beta}+\frac{\beta}{1-\beta}\left[\log(\Pi^f(1+\bar{\theta})-R\bar{\theta})+(v^d-v^f)\right]+\frac{\beta}{1-\beta}\int^{\tilde{\eta}} F(\eta)d\eta \\
&= \frac{B}{1-\beta}+\frac{\beta}{1-\beta}\left[(1-\beta)\log(\Pi^f(1+\bar{\theta})-R\bar{\theta})+\beta\log R^d\right]+\frac{\beta}{1-\beta}\int^{\tilde{\eta}} F(\eta)d\eta
\end{aligned}
$$

where the third equality rearranges the first v^f on the right-hand side of the second equality and $v^d-v^f = \beta\log R^d-\beta\left[\log(\Pi^f(1+\bar{\theta})-R\bar{\theta})\right]$;

$$
\begin{aligned}
v^r &= B+\beta p\left[\log\left(\Pi^h(1+\theta)-R\theta\right)\right]+\beta(1-p)\left[\log\left(\Pi^l(1+\theta)-R\theta\right)\right] \\
&+ \beta(v^d+\int^{\tilde{\eta}} F(\eta)d\eta) \\
&= B+\beta p\left[\log\left(\Pi^h(1+\theta)-R\theta\right)\right]+\beta(1-p)\left[\log\left(\Pi^l(1+\theta)-R\theta\right)\right] \\
&+ \beta\left(v^r+v^d-v^r+\int^{\tilde{\eta}} F(\eta)d\eta\right) \\
&= \frac{\beta}{1-\beta}\left[p\log\left(\Pi^h(1+\theta)-R\theta\right)+(1-p)\log\left(\Pi^l(1+\theta)-R\theta\right)-\tilde{\eta}\right] \\
&+ \frac{\beta}{1-\beta}\int^{\tilde{\eta}} F(\eta)d\eta+\frac{B}{1-\beta} \\
&= \frac{\beta}{1-\beta}\left[(1-\beta)p\log\left(\Pi^h(1+\theta)-R\theta\right)+(1-\beta)(1-p)\log\left(\Pi^l(1+\theta)-R\theta\right)+\beta\log R^d\right] \\
&+ \frac{\beta}{1-\beta}\int^{\tilde{\eta}} F(\eta)d\eta+\frac{B}{1-\beta}
\end{aligned}
$$

where the third equality uses $v^r-v^d=\tilde{\eta}$ (note: in this case $l^d\geq l^f$ and entrepreneurs choose to deposit) and rearranges the first v^r on the right-hand side of the second equality, and the last equality uses again

$$
v^r-v^d=\beta p\left[\log\left(\Pi^h(1+\theta)-R\theta\right)\right]+\beta(1-p)\left[\log\left(\Pi^l(1+\theta)-R\theta\right)\right]-\beta\log R^d
$$

Therefore, using v^r, v^f, and v^d above, we can rewrite ((27)) as

$$\begin{aligned}
V_e(\omega) &= \int^{\tilde{\eta}} (V^r(\omega,\eta))dF(\eta) + \int_{\tilde{\eta}} V^d(\omega) \\
&= \log(\omega) + v^r F(\tilde{\eta}) - \int^{\tilde{\eta}} \eta dF(\eta) + [1 - F(\tilde{\eta})]\, v^d \\
&= \log(\omega) + v^d + F(\tilde{\eta})(v^r - v^d) - F(\tilde{\eta})\tilde{\eta} + \int^{\tilde{\eta}} F(\eta)d\eta \\
&= \log(\omega) + \frac{B}{1-\beta} + \frac{\beta}{1-\beta}\log(R^d) + \frac{1}{1-\beta}\int^{\tilde{\eta}} F(\eta)d\eta.
\end{aligned}$$

where the forth equality uses $v^r - v^d = \tilde{\eta}_2$ and the expression for v^d.

Remark: In a steady-state economy, project choice is independent of the wealth level. The distribution of the wealth level ω is always preserved. Denote V as the value function of the entrepreneurs. Thus, $\omega = \Omega$ and $V = V_e(\Omega)$. We therefore show that entrepreneurs' welfare can be represented by $\log(\Omega) + \tilde{V}(\phi)$ as shown in the main text.