Generalized Method of Moments with Grouped Heterogeneous Validity of Moment Conditions in Panel Data Models*

Zhan Gao†

Department of Economics

University of Southern California

Abstract

This paper provides a unified framework for the selection of valid moment conditions and detection of latent group structure based on the moment condition validity in general nonlinear generalized method of moments (GMM) panel data models, which can accommodate a diverging number of moment conditions and group-specific heterogeneous validity of moment conditions across agents. The proposed method integrates the pairwise adaptive fused Lasso and the adaptive Lasso regularization into the GMM estimation. The estimator is shown to be consistent and achieve classification and moment selection consistency simultaneously. The asymptotic distribution of a post-regularization estimator is derived, and its oracle properties are established. The finite-sample performance of the proposed method is evaluated through a Monte Carlo simulation experiment.

Key words: Generalized method of moments, pairwise adaptive fused Lasso, classification, adaptive Lasso, moment selection, IV

JEL code: C13, C33, C36, C38, C52

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†Correspondence: zhangao@usc.edu.

1 Introduction

Panel data models are extensively employed in empirical economics research, utilizing data that encompass various units—such as workers, firms, or countries—observed over time. These models offer the distinct advantage of enabling the analysis of critical economic relationships while controlling for unobserved heterogeneity. However, this unobserved heterogeneity also presents statistical challenges for accurate estimation and inference, leading researchers to navigate trade-offs in the degree of unobserved heterogeneity accommodated by the model.

Recent years have seen a surge in interest towards developing panel data models that incorporate latent group structures, a concept pioneered by Hahn and Moon (2010); Bonhomme and Manresa (2015); Su et al. (2016). These models assume that units are categorized into a finite number of groups with homogeneous parameters within each group but heterogeneity across groups. This approach strikes a balance between model flexibility, statistical efficiency, and empirical interpretability.

Panel data models are often characterized by sets of moment conditions, such as endogenous regression models with instrumental variables and nonlinear structural models. In many cases, the available moment conditions, which are possibly misspecified, significantly outnumber the parameters of interest. It is, therefore, crucial to selectively use valid moment conditions in the estimation process to prevent biased and misleading outcomes. For instance, approaches for shrinkage GMM estimation (Liao, 2013; Cheng and Liao, 2015) have been proposed to address these challenges by penalizing slackness parameters that signify the validity of moment conditions.

Given the prevalence of unobserved heterogeneity in panel data, assuming universal validity of moment conditions across all units is overly restrictive. It is more pragmatic to acknowledge the heterogeneity in the validity of moment conditions across units. For example, in instrumental variable (IV) models, the exogeneity of instruments typically hinges on foundational model assumptions or empirical evidence. Given the inherent heterogeneity across units, it is unrealistic to assume a universal applicability of any single IV. Recognizing this, it becomes essential to accommodate the notion that varied units may necessitate distinct sets of IVs for accurately identifying causal relationships of interest. Similarly, in the context of moment conditions derived from economic theory, assuming a one-size-fits-all model that uniformly represents all facets of the data-generating process for every unit is overly optimistic. A more nuanced approach involves tailoring the model to accurately reflect specific subsets of features for different units, as defined by corresponding sets of moment conditions, thereby ensuring a more accurate and nuanced representation of the underlying economic realities.

This paper introduces a unified framework for selecting valid moment conditions and detecting latent group structures in nonlinear GMM panel data models, accommodating an expanding number of moment conditions and group-specific heterogeneity in the validity of these conditions. By integrating the pairwise adaptive fused Lasso with the adaptive Lasso regularization for moment selection, our method offers a comprehensive solution for achieving consistent and efficient parameter estimation, accurate classification of latent groups, and the selection of valid moment conditions for each group simultaneously. This is the inaugural work, to the best of my knowledge, that integrates the moment selection and group classification in the nonlinear GMM panel data models. This integration bridges the gap between moment selection methodologies and panel data models featuring latent group structures and leverages the strengths of both approaches.

This work contributes to various strands of the literature. Firstly, it is related to the vast literature on the generalized method of moments (GMM) since Hansen (1982), with a particular emphasis on overidentifying moment conditions with possible misspecification. In early works, Andrews (1999); Andrews and Lu (2001) develop moment selection criteria based on J overidentification test statistic. Han and Phillips (2006) derive the asymptotic properties of the GMM estimator with many moment conditions. Liao (2013); Cheng and Liao (2015) propose shrinkage GMM estimation with adaptive Lasso regularization of the slackness parameters for moment selection. Another strand of the literature focuses on the empirical likelihood approach to moment models. For example, Moon and Schorfheide (2009) study the properties of the empirical likelihood estimator with overidentifying moment inequalities. Shi (2016); Chang et al. (2018); Ando and Sueishi (2019); Chang et al. (2022) investigate the penalized empirical likelihood estimation in high-dimensional environments. As an important class of moment condition models, linear IV models with high-dimensional instruments have attracted wide attention. Selection of IVs via regularization has been extensively studied, see Okui (2011); Belloni et al. (2012); Fan and Liao (2014); Luo (2014); Windmeijer et al. (2019); Gautier and Rose (2021); Liang et al. (2022); Lin et al. (2022) and the references therein.

In this work, the wisdom from these works, particularly the shrinkage GMM estimation approach in Cheng and Liao (2015), is brought to the nonlinear panel data settings with latent group structures to account for heterogeneity in the validity of moment conditions.

Panel data models with latent group structure become increasingly popular in the literature during the past decade. Several prominent approaches have been developed. The k-means algorithm has been introduced to the panel data literature for estimation of latent group structures and group-specific parameters by Lin and Ng (2012); Bonhomme and

Manresa (2015) and this line of research has been flourishing, see Bonhomme et al. (2019, 2022); Liu et al. (2020, 2023); Miao et al. (2020); Okui and Wang (2021); Wang et al. (2023); Lumsdaine et al. (2023); Cytrynbaum (2020); Cheng et al. (2023). Some works borrow the sequential binary segmentation algorithm initally applied in the structural break detection literature, for examples Ke et al. (2016); Wang and Su (2021); Su et al. (2023), and recent works focus on spectral clustering algorithms, see Ma et al. (2022); Chetverikov and Manresa (2022); Yu et al. (2022). The Bayesian approach has been proposed in Zhang (2023); Huang (2023). Last but not least, a major strand of the literature focuses on identifying group structures in panel data models via dedicated penalization schemes. Su et al. (2016) propose the classifier-Lasso (C-Lasso) penalty to identify group structure in a nonlinear profile likelihood framework and the linear IV models. Following their work, the method has been extended to a variety of settings, for example, Su and Lu (2017); Su and Ju (2018); Su et al. (2019); Huang et al. (2020, 2021), and the computational issues of C-Lasso is discussed in Gao and Shi (2021); Huang et al. (2023).

Many of the existing works on grouped panel models focus on linear or nonlinear regression models or linear IV models, with Cheng et al. (2023) as an exception in which the authors study a nonlinear GMM model with an emphasis on multi-dimensional grouped heterogeneity. This work contributes to the nonlinear GMM panel data models with latent group structure with a focus on heterogeneity in the validity of moment conditions. In addition, the model in this paper deals with parameters with increasing dimensions by allowing for a diverging number of moment conditions and facilitates variable selection in the estimation.

Our work adopts the pairwise adaptive fused Lasso (PAFL) penalization regularization to identify the grouped structure in the validity of moment conditions. The PAFL penalty originates from of idea of the adaptive Lasso (Zou, 2006), group Lasso (Yuan and Lin, 2006), fused Lasso (Tibshirani et al., 2005) and the group fused Lasso (Qian and Su, 2016; Okui and Wang, 2021; Lumsdaine et al., 2023) and is proposed for clustering problems as in Hocking et al. (2011); Radchenko and Mukherjee (2017). The method is applied for the detection of group structures in panel data models in Gu and Volgushev (2019) and Mehrabani (2023). This technique offers computational benefits over alternatives like the Classifier-Lasso, k-means clustering, and the sequential binary segmentation algorithm, thanks to its convex nature, and simplifies the training process with a single tuning parameter and provides a spectrum of tuning parameters to accurately determine the number of groups. Discussion and comparison of the PAFL penalty with other methods will be delineated in Remark 2.

Gu and Volgushev (2019) adopts a scaler version of the L_1-norm-baed PAFL penalization to study the quantile regression models with grouped fixed effects, while our model deals with increasing dimensional parameters. The PAFL penalty in our work has the same form as in Mehrabani (2023). However, the proof in Mehrabani (2023) was found to only support individual classification consistency with a few technical gaps in the development of the asymptotic theory. This paper provides a novel proof to first rigorously establish uniform classification consistency with the pairwise adaptive fused Lasso, which can also serve as a remedy for Mehrabani (2023) in the linear panel data models with latent group structures.

1.1 Notations

For any positive integer N, $\boldsymbol{I}_N \in \mathbb{R}^{N\times N}$, $\mathbf{0}_N \in \mathbb{R}^{N\times 1}$ and $\boldsymbol{\iota}_N \in \mathbb{R}^{N\times 1}$ denotes the $N \times N$ identity matrix, $N \times 1$ zero vector and $N \times 1$ vector of ones, respectively. Denote $[N] = \{1, 2, \cdots, N\}$. For generic vectors $\boldsymbol{a} \in \mathbb{R}^N$ and matrices $\boldsymbol{A} \in \mathbb{R}^{N\times K}$, $\|\cdot\|$ denotes Frobenius norm; $\boldsymbol{A}'$ is the transpose of $\boldsymbol{A}$; $\sigma_{\max}(\boldsymbol{A})$ and $\sigma_{\min}(\boldsymbol{A})$ denote the largest and smallest singular values of $\boldsymbol{A}$, respectively. Let a_i and a_{ik} denote the i-th element of $\boldsymbol{a}$ and the (i, k)-th element of $\boldsymbol{A}$, respectively. For a vector-valued function $f(\boldsymbol{a}) : \mathbb{R}^N \to \mathbb{R}^K$, $\frac{\partial f(\boldsymbol{a})}{\partial \boldsymbol{a}'} \in \mathbb{R}^{K\times N}$ denotes the Jacobian matrix with (i, k)-th element as $\frac{\partial f_i(x)}{\partial a_k}$ where $f_i(\cdot)$ is the i-th component of $f(\cdot)$. Let $\mathcal{S} = \left\{i_{(1)}, i_{(2)}, \cdots, i_{(S)}\right\} \subset [N]$, denote $\boldsymbol{a}_{\mathcal{S}} = \left(a_{i_{(1)}}, a_{i_{(2)}}, \cdots, a_{i_{(S)}}\right)'$ and $f_{\mathcal{S}}(\cdot) = \left(f_{i_{(1)}}, f_{i_{(2)}}, \cdots, f_{i_{(S)}}\right)'$ as the subvectors of $\boldsymbol{a}$ and $f(\cdot)$, respectively. For a generic set $\mathcal{A}$, $|\mathcal{A}|$ is the cardinality of $\mathcal{A}$. We use C (c) to denote generic large (small) positive constants. Define $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$. $\Delta x_{it} := x_{it} - x_{i,t-1}$ denotes the first difference operator. "$\xrightarrow{p}$" and "$\xrightarrow{d}$" denote convergence in probability and convergence in distribution, respectively; "w.p.a.1" abbreviates "with probability approach 1". Denote $a_N = o_P(b_N)$ if for any $\varepsilon > 0$, $\Pr(|a_N/b_N| > \varepsilon) \to 0$ as $N \to \infty$; $a_N = \mathcal{O}_p(b_N)$ if for any $\varepsilon > 0$, there exist finite $C_\varepsilon > 0$ such that $\lim_{N\to\infty} \Pr(|a_N/b_N| \geq C_\varepsilon) < \varepsilon$. We use $(N, T) \to \infty$ to signify that N and T jointly go to infinity.

The rest of the paper is organized as follows. Section 2 presents the model setup and motivating examples and proposes the penalized GMM estimator. Section 3 investigates the asymptotic properties of the proposed method. The finite sample performance of the proposed method is evaluated through a Monte Carlo simulation experiment in Section 4. Section 5 concludes. Proofs are relegated to the Appendix A.1.

2 Framework

This section lays the foundation for our investigation of moment models with grouped heterogeneous validity of moment conditions with panel data. Firstly, we delineate the model's framework and examine illustrative econometric examples that motivate our study in Section 2.1 and 2.2. Subsequently, in Section 2.3 we introduce a penalized GMM estimator, specifically designed for parameter estimation, classification, and moment selection.

2.1 Model Setup

We examine an observed dataset wherein each observation $\boldsymbol{z}_{it} \in \mathbb{R}^{p_z}$, indexed by $i \in [N]$ and $t \in [T]$. This dataset may be conceptualized as either classical panel data or as clustered data. Within the framework of panel data, the index i represents the individual dimension, while t pertains to the time dimension. In the case of clustered data, on the other hand, i indicates the cluster dimension, with t identifying units within these clusters.

2.1.1 Moment Conditions

Consider $g(\boldsymbol{z}, \boldsymbol{\theta}) \in \mathbb{R}^L$, which represents a vector of moment functions associated with the parameter $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^{p_\theta}$, where p_θ is a predetermined positive integer. Let $\boldsymbol{\theta}^0$ denote the true parameter values of interest for $\boldsymbol{\theta}$.

There exists a subset $\mathcal{S} \subset [L]$ of the moment conditions, satisfying $p_\theta \leq L_{\mathcal{S}} = |\mathcal{S}| < L$, such that

$$\mathbb{E}\left[g_{\mathcal{S}}\left(\boldsymbol{z}_{it}, \boldsymbol{\theta}\right)\right] = 0 \tag{2.1}$$

is fulfilled if and only if $\boldsymbol{\theta} = \boldsymbol{\theta}^0$, thus identifying the structural parameter $\boldsymbol{\theta}$ in accordance with (2.1). We assume that both p_θ and L_S are constant, while $L \equiv L_{NT}$ may increase with the sample sizes N and T.

The set $\mathcal{S}$ of *sure* moment conditions, which are correctly specified as per (2.1) and are sufficient for the identification of $\boldsymbol{\theta}$, is presumed to be known to the researchers based on economic theory or prior empirical evidence. Let $\mathcal{D} = [L] \setminus \mathcal{S}$, with $L_{\mathcal{D}} = L - L_{\mathcal{S}}$, denote the set of *doubtful* moment conditions, which may be potentially misspecified. Without loss of generality, let $\mathcal{D} = [L_{\mathcal{D}}]$. To model the pattern of misspecification, we introduce slackness parameters $\boldsymbol{\delta}_i \in \Theta_\delta \subset \mathbb{R}^{L_{\mathcal{D}}}$, $i \in [N]$, such that

$$\mathbb{E}\left[g_{\mathcal{D}}\left(\boldsymbol{z}_{it}, \boldsymbol{\theta}^0\right)\right] = \boldsymbol{\delta}_i, \tag{2.2}$$

thereby allowing for heterogeneous validity of moment conditions across i. Specifically, $\delta_{il} = 0$ indicates that moment condition $l \in \mathcal{D}$ is correctly specified for i, whereas $\delta_{il} \neq 0$ signifies that moment condition $l \in \mathcal{D}$ is misspecified for i. For each individual i, the set of doubtful moment conditions, $\mathcal{D}$, is partitioned into $\mathcal{V}_i$ and $\mathcal{I}_i$, where $\mathcal{V}_i = \{l \in [L_{\mathcal{D}}] : \delta_{il} = 0\}$ represents the subset of conditions correctly specified, and $\mathcal{I}_i = \{l \in [L_{\mathcal{D}}] : \delta_{il} \neq 0\}$ denotes those misspecified. The parameter

$$\zeta_{NT} = \min_{1 \leq i \leq N} \min_{l \in \mathcal{I}_i} |\delta_{il}^0|$$

quantifies the minimum degree of misspecification across individuals and conditions, which is permitted to approach zero.

2.1.2 Grouped Structure

Consider that $\mathcal{G} = \{\mathcal{G}_k\}_{k=1}^{K^0}$ constitutes a partition of the set $[N]$, such that $\bigcup_{k=1}^{K^0} \mathcal{G}_k = [N]$ and $\mathcal{G}_k \cap \mathcal{G}_j = \emptyset$ for all $j \neq k$. This partitioning organizes the sample into $K_0 \geq 1$ distinct groups. Define $N_k = \sum_{i=1}^{N} \mathbb{1}\{i \in \mathcal{G}_k\}$ as the number of observations within group k, for $k \in [K^0]$. Furthermore, the group membership function $k(i) = \sum_{k=1}^{K^0} k \mathbb{1}\{i \in \mathcal{G}_k\}$ is introduced, alongside the indicator function $\lambda_{ik} = \mathbb{1}\{i \in \mathcal{G}_k\}$, for each $i \in [N]$ and $k \in [K^0]$.

Rather than assuming arbitrary patterns of heterogeneity in $\boldsymbol{\delta}_i$, we introduce a latent group structure by positing that

$$\boldsymbol{\delta}_i^0 = \sum_{k=1}^{K^0} \boldsymbol{\alpha}_k^0 \mathbb{1}\{i \in \mathcal{G}_k\} = \boldsymbol{\alpha}_{k(i)}^0, \tag{2.3}$$

where $\boldsymbol{\alpha}_k^0 \neq \boldsymbol{\alpha}_j^0$ for all $j \neq k$, signifying that the validity of moment conditions is homogeneous within groups but varies across different groups.[1] Accordingly, we define $\mathcal{V}_k = \{l \in \mathcal{D} : \alpha_{kl} = 0\}$ and $\mathcal{I}_k = \{l \in \mathcal{D} : \alpha_{kl} \neq 0\}$ for each group $k \in [K^0]$. Define

$$\begin{aligned} \mathcal{Z}_0 &= \{(i,j) \in [N] \times [N] : k(i) = k(j) \text{ and } i < j\}, \\ \mathcal{Z}_1 &= \{(i,j) \in [N] \times [N] : k(i) \neq k(j) \text{ and } i < j\}, \end{aligned} \tag{2.4}$$

to represent the sets of observation pairs within the same group and across different groups, according to their true latent group memberships, respectively. The notation of $\mathcal{Z}_0$ and $\mathcal{Z}_1$ is particularly helpful for rigorous analysis of the asymptotic properties of the estimator that will be proposed in Section 2.3 based on the adaptive fused Lasso penalty. Let

$$\rho_{NT} = \min_{1 \leq k < k' \leq K^0} \left\| \boldsymbol{\alpha}_k^0 - \boldsymbol{\alpha}_{k'}^0 \right\| = \min_{(i,j) \in \mathcal{Z}_1} \left\| \boldsymbol{\delta}_i^0 - \boldsymbol{\delta}_j^0 \right\|,$$

[1] It is a direct extension to allow the structural parameters $\boldsymbol{\theta}_i$ to be heterogeneous and share the latent group structure with the slackness parameters $\boldsymbol{\delta}_i$. In this case, (2.1) and (2.2) are modified to $\mathbb{E}[g_{\mathcal{S}}(\boldsymbol{z}_{it}, \boldsymbol{\theta}_i)] = 0$, which is fulfilled if and only if $\boldsymbol{\theta}_i = \boldsymbol{\theta}_i^0$, and $\mathbb{E}\left[g_{\mathcal{D}}\left(\boldsymbol{z}_{it}, \boldsymbol{\theta}_i^0\right)\right] = \boldsymbol{\delta}_i$. Denote $\boldsymbol{\beta}_i = (\boldsymbol{\theta}_i', \boldsymbol{\delta}_i')'$. We impose the latent group structure on $\boldsymbol{\beta}_i$ by modifying (2.3), $\boldsymbol{\beta}_i^0 = \sum_{k=1}^{K^0} \boldsymbol{\alpha}_k^0 \mathbb{1}\{i \in \mathcal{G}_k\} = \boldsymbol{\alpha}_{k(i)}^0$, where $\boldsymbol{\alpha}_k^0 \neq \boldsymbol{\alpha}_j^0$ for all $j \neq k$. With minimal modification, we can propose the penalized GMM estimator for $\boldsymbol{\beta}$ and develop the same set of asymptotic properties as in Section 2.3 and 3.

which delineates the minimum degree of separation between groups, a measure that is permitted to approach zero as the sample sizes N and T increase.

Remark 1. Define the matrix $\underset{N\times L_{\mathcal{D}}}{\boldsymbol{D}} = (\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \ldots, \boldsymbol{\delta}_N)'$, the group membership matrix $\underset{N\times K}{\boldsymbol{\Lambda}} = (\lambda_{ik})$, and the loading matrix $\underset{L_{\mathcal{D}}\times K}{\boldsymbol{A}} = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_K)$. Consequently, in matrix notation, we have $\boldsymbol{D} = \boldsymbol{\Lambda}\boldsymbol{A}'$, which reveals a low-rank factor structure, as highlighted in the works of Ma et al. (2022); Chetverikov and Manresa (2022); Bonhomme et al. (2022). Within this framework, $\boldsymbol{\lambda}_i$ represents the factor loading, while $\boldsymbol{\alpha}_k$ denotes the latent factor. The interconnection between these two strands of literature and their implications for directions of extensions are briefly discussed in Remark 9.

The primary objective of this study is to facilitate the efficient estimation and inference of $\boldsymbol{\theta}^0$, alongside the detection of the group structure $\mathcal{G}$ and the validity parameters $\boldsymbol{\alpha}_k$, for $k \in [K^0]$.

2.2 Motivating Examples

Example 1 (Measurement Errors and Linear IV). Consider a first-differenced linear panel model, articulated as

$$\Delta y_{it} = \theta^0 \Delta x_{it} + \Delta\varepsilon_{it},$$

where i indexes individuals in the set $[N]$ and t denotes time periods within $[T]$. The researcher is concerned about the endogeneity of Δx_{it} caused by possible measurement errors, and identifies θ^0 by an exogenous IV, $z_{\mathcal{S},it}$, satisfying

$$\mathbb{E}\left[z_{\mathcal{S},it}\left(\Delta y_{it} - \theta^0 \Delta x_{it}\right)\right] = 0. \tag{2.5}$$

In addition to $z_{\mathcal{S},it}$, suppose there exists an alternative measure to x_{it}, $\widetilde{x}_{it}$, such that

$$x_{it} = x_{it}^{\star} + u_{it} + \gamma_i v_{it},\ \gamma_i = \mathbb{1}\left\{i \in \mathcal{N}_1\right\},$$
$$\widetilde{x}_{it} = x_{it}^{\star} + \widetilde{u}_{it} + \widetilde{\gamma}_i \widetilde{v}_{it},\ \widetilde{\gamma}_i = \mathbb{1}\left\{i \in \mathcal{N}_2\right\},$$

where $x_{it}^{\star}$ is the latent explanatory variable, with $\mathbb{E}\left(\Delta\varepsilon_{it} \mid \Delta x_{it}^{\star}\right) = 0$, measured by observed proxy variables x_{it} and $\widetilde{x}_{it}$, u_{it} and $\widetilde{u}_{it}$ are classical measurement errors such that $\mathbb{E}\left(\Delta\varepsilon_{it} \mid \Delta u_{it}\right) = \mathbb{E}\left(\Delta\varepsilon_{it} \mid \Delta\widetilde{u}_{it}\right) = 0$ while v_{it} and $\widetilde{v}_{it}$ are nonclassical measurement errors that are correlated with the structural shocks $\Delta\varepsilon_{it}$ and consequently the source of endogeneity. $\mathcal{N}_1, \mathcal{N}_2 \subset [N]$ denote the set of individuals for which the proxy variables, x_{it} and $\widetilde{x}_{it}$, are endogenous due to the entering of v_{it} and $\widetilde{v}_{it}$, respectively. Let $\boldsymbol{z}_{\mathcal{D},it} = (\Delta x_{it}, \Delta\widetilde{x}_{it})$ and we then have the set of doubtful moment conditions,[2]

$$\mathbb{E}\left[\boldsymbol{z}_{\mathcal{D},it}\left(\Delta y_{it}-\theta^{0}\Delta x_{it}\right)\right]=\boldsymbol{\delta}_{i},$$

where

$$\boldsymbol{\delta}_{i}=\begin{cases}(\delta_1,\delta_2)' & \text{if } i\in\mathcal{N}_1\cap\mathcal{N}_2,\\ (\delta_1,0)' & \text{if } i\in\mathcal{N}_1\setminus\mathcal{N}_2,\\ (0,\delta_2)' & \text{if } i\in\mathcal{N}_2\setminus\mathcal{N}_1,\\ (0,0)' & \text{otherwise,}\end{cases}$$

and $\delta_1=\mathbb{E}\left(\Delta v_{it}\Delta\varepsilon_{it}\right)\neq 0$ and $\delta_2=\mathbb{E}\left(\Delta\widetilde{v}_{it}\Delta\varepsilon_{it}\right)\neq 0$. $\boldsymbol{\delta}_i$ exhibits a group structure as in (2.3) capturing the heterogeneity of the validity of moment conditions across individuals. In a concrete empirical context, consider a simplified labor supply model, as in Liao (2013), that is specified as follows[3]

$$\Delta\log\left(y_{it}\right)=\theta^{0}\Delta\log\left(x_{it}\right)+\Delta\varepsilon_{it},$$

where y_{it} denotes the annual hours worked, x_{it} represents the hourly wage rate and the parameter θ^0 captures the inter-temporal substitution elasticity of labor supply in response to evolutionary changes in wages. As discussed in MaCurdy (1981); Altonji (1986), researchers are concerned about measurement errors in $\Delta\log\left(x_{it}\right)$, which cause the OLS estimator to be inconsistent. MaCurdy (1981) suggest employing a set of family background variables (parents' education; parents' economic status when the individual is young; education, age and interaction between education and age) as IVs, whereas Altonji (1986) advocate alternative wage measures $\widetilde{x}_{i,t}$ to construct IVs.[4] The set $\mathcal{S}$ can be formed by including IVs for which there is a higher degree of confidence in their exogeneity, such as the economic status of the parents as in Liao (2013), and we investigate the potential group structure of the validity of moment conditions by including $\Delta\log\left(x_{it}\right)$, $\Delta\log\left(\widetilde{x}_{it}\right)$ and other IVs proposed in MaCurdy (1981) in $\boldsymbol{z}_{\mathcal{D},it}$.

[2]If Δx_{it} is exogenous, the generalized least squares (GLS) estimator, which can be constructed as GMM estimator based on (2.5) together with $\mathbb{E}\left[\Delta x_{it}\left(\Delta y_{it}-\theta^{0}\Delta x_{it}\right)\right]=0$, is consistent and efficient. We can include Δx_{it} in $\boldsymbol{z}_{\mathcal{D},it}$ to check the exogeneity of Δx_{it}.

[3]We abstract the time-varying constant term from the model presented in Liao (2013) for illustration purposes.

[4]As also summarized in Liao (2013, Note 11), x_{it} is constructed by dividing the annual labor income of individual i by the annual labor supply and gross national product (GNP) price deflator in MaCurdy (1981); Altonji (1986). $\widetilde{x}_{it}$ is the hourly wage rate of individual i if the person is paid on hourly basis in Altonji (1986).

Example 2 (**Dynamic Panel).** Consider the dynamic panel model

$$\Delta y_{it} = \theta^0 \Delta y_{i,t-1} + \Delta\varepsilon_{it},$$

$i \in [N]$ and $t \in [T]$. $\Delta y_{i,t-1}$ and $\Delta\varepsilon_{it}$ are naturally correlated, which leads OLS estimator to be inconsistent. Suppose ε_{it} is distributed independent across i and has no serial correlation, then $y_{i,t-2-j}$, $j = 0, 1, \cdots, t-2$ are valid instruments for $\Delta y_{i,t-1}$ (Arellano and Bond, 1991) satisfying $\mathbb{E}\left(y_{i,t-2-j}\Delta y_{i,t-1}\right) \neq 0$ and

$$\mathbb{E}\left(y_{i,t-2-j}\Delta\varepsilon_{it}\right) = 0,$$

which can be used to construct the $\mathcal{S}$ set, i.e.

$$g_{\mathcal{S}}\left(z_{it}, \theta\right) = \left[y_{i,t-2-j}\left(\Delta y_{it} - \theta\Delta y_{i,t-1}\right)\right]_{j=0,1,\cdots,t-2}.$$

We can derive additional moment conditions to form $\mathcal{D}$ by imposing more restrictions on the data-generating process. For example,

$$\begin{aligned} \mathbb{E}\left(y_{i,t}\Delta\varepsilon_{i,t+1} - y_{i,t+1}\Delta\varepsilon_{i,t+2}\right) &= 0,\ t = 1, 2, \ldots, T-2 \\ \mathbb{E}\left(\bar{\varepsilon}_i \Delta\varepsilon_{i,t+1}\right) &= 0,\ t = 1, 2, \cdots, T-1, \end{aligned} \tag{2.6}$$

where $\bar{\varepsilon}_i = T^{-1}\sum_{t=1}^{T}\varepsilon_{it}$, hold under homoskedasticity across time $\mathbb{E}\left(\varepsilon_{it}^2\right) = \sigma_i^2$ (Ahn and Schmidt, 1995). However, the homoskedasticity assumption may not hold uniformly across unit i, say $\mathbb{E}\left(\varepsilon_{it}^2\right) = \sigma_i^2 + \gamma_i\omega_{it}$, where $\gamma_i = \mathbb{1}\left(i \in \mathcal{N}\right)$ for some subset of individuals $\mathcal{N} \subset [N]$ and ω_{it} captures the heteroskedasticity across t. In this case, we have at least two groups based on the validity of the homoskedasticity assumption and in turn the moment conditions in (2.6). Additional linear or nonlinear moment conditions can be derived under homoskedasticity or initial condition restrictions (Arellano and Bover, 1995; Ahn and Schmidt, 1997; Blundell and Bond, 1998). See Hsiao (2022, Chapter 3) or Pesaran (2015, Chapter 37) for a textbook treatment.

Example 3 (**Shift-share (Bartik) IV).** The shift-share IV has become increasingly popular in empirical studies. Recent developments including Borusyak et al. (2022), Goldsmith-Pinkham et al. (2020) and Adao et al. (2019) provide theoretical justification of the shift-share IV. Consider the model with N locations, T time periods, and J industries,

$$y_{it} = \theta^0 x_{it} + \varepsilon_{it},$$

where the endogenous variable employs the accounting identity $x_{it} = \sum_{j=1}^{J} s_{ijt} v_{ijt}$, and the location-industry-time shift can be decomposed as $v_{ijt} = v_{jt} + \widetilde{v}_{ijt}$, where v_{jt} is the industry-time shift and $\widetilde{v}_{ijt}$ is idiosyncratic shock. The shift-share IV is constructed by

$$z_{it} = \sum_{j=1}^{J} s_{ij0} v_{jt}, \tag{2.7}$$

which is the inner product of initial shares and aggregated level industry-time shifts. A notable example is provided by Autor et al. (2013), who investigate the causal impact of increased import penetration from China on local labor markets within the United States. In their analysis, the endogenous variable x_{it} quantifies the local exposure to the surge in imports from China, s_{ij0} represents the employment share of the manufacturing industry j within location i, measured a decade prior to each period t, and v_{jt} denotes the growth of imports of products in industry j from China into the eight comparable economies over the period t.

The validity of the shift-share instrumental variable (IV) hinges on the exogeneity of either v_{jt} or s_{ij0}, with the other variable being treated as fixed. However, the exogeneity assumption may not hold for each location-industry pair (i, j) since locations have different industrial structures and local amenities. In this case, the IV constructed in (2.7) by combining all industries for all i can be invalid. For instance, considering a scenario where the geographical location is Silicon Valley and the analysis incorporates the technology sector as part of (2.7), it is not plausible that the exogeneity condition holds. Consequently, it is crucial to select suitable subsets of industries for different groups of locations to construct valid shift-share IVs. In practice, we can construct the set of IVs as

$$z_{\mathcal{S},it} = \sum_{j \in \mathcal{J}_S} s_{ij0} v_{jt} \text{ and } z_{l,it} = \sum_{j \in \mathcal{J}_S \cup \widetilde{\mathcal{J}}_l} s_{ij0} v_{jt},$$

where $\mathcal{J}_S$ is a subset of industries for which the exogeneity condition holds based on prior knowledge or empirical evidence, and we add more industries in $\widetilde{\mathcal{J}}_l$, $l \in [L_{\mathcal{D}}]$, to construct additional IVs whose validity is subject to detection. Denote $\boldsymbol{z}_{\mathcal{D},it} = [z_{l,it}]_{l=1}^{L_{\mathcal{D}}}$. It is expected that the validity of constructed IVs is heterogeneous across locations.

2.3 Penalized GMM Estimation

To facilitate understanding, we introduce the following notations. Let

$$m(\boldsymbol{z},\boldsymbol{\theta})=\begin{bmatrix} g_{\mathcal{S}}(\boldsymbol{z},\boldsymbol{\theta}) \\ g_{\mathcal{D}}(\boldsymbol{z},\boldsymbol{\theta}) \end{bmatrix},\ g(\boldsymbol{z},\boldsymbol{\theta},\boldsymbol{\delta})=\begin{bmatrix} g_{\mathcal{S}}(\boldsymbol{z},\boldsymbol{\theta}) \\ g_{\mathcal{D}}(\boldsymbol{z},\boldsymbol{\theta})-\boldsymbol{\delta} \end{bmatrix}$$

represent the moment functions with and without slackness parameters, respectively. Correspondingly,

$$\overline{m}_i(\boldsymbol{\theta})=\mathbb{E}\left[m(\boldsymbol{z}_{it},\boldsymbol{\theta})\right],\ \overline{g}_i(\boldsymbol{\theta},\boldsymbol{\delta})=\mathbb{E}\left[g(\boldsymbol{z}_{it},\boldsymbol{\theta},\boldsymbol{\delta})\right],$$
$$\overline{m}_{\mathcal{S},i}(\theta)=\mathbb{E}\left[g_{\mathcal{S}}(\boldsymbol{z}_{it},\boldsymbol{\theta})\right],\ \overline{m}_{\mathcal{D},i}(\theta)=\mathbb{E}\left[g_{\mathcal{D}}(\boldsymbol{z}_{it},\boldsymbol{\theta})\right].$$

The empirical counterparts are denoted as

$$\widehat{m}_{i,T}(\boldsymbol{\theta})=\frac{1}{T}\sum_{t=1}^{T}m(\boldsymbol{z}_{it},\boldsymbol{\theta}),\ \widehat{g}_{i,T}(\boldsymbol{\theta},\boldsymbol{\delta}_i)=\frac{1}{T}\sum_{t=1}^{T}g(\boldsymbol{z}_{it},\boldsymbol{\theta},\boldsymbol{\delta}_i),$$
$$\widehat{m}_{\mathcal{S},i,T}(\boldsymbol{\theta})=\frac{1}{T}\sum_{t=1}^{T}g_{\mathcal{S}}(\boldsymbol{z}_{it},\boldsymbol{\theta}),\ \widehat{m}_{\mathcal{D},i,T}(\boldsymbol{\theta})=\frac{1}{T}\sum_{t=1}^{T}g_{\mathcal{D}}(\boldsymbol{z}_{it},\boldsymbol{\theta}).$$

We also define the Jacobian matrices as

$$\Gamma_{\mathcal{S},i}(\boldsymbol{\theta})=\frac{\partial\overline{m}_{\mathcal{S},i}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}'},\ \Gamma_{\mathcal{D},i}(\boldsymbol{\theta})=\frac{\partial\overline{m}_{\mathcal{D},i}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}'},\ \Gamma_i(\boldsymbol{\theta})=\begin{bmatrix}\Gamma_{\mathcal{S}}(\boldsymbol{\theta}) & \mathbf{0}_{L_{\mathcal{S}}\times L_{\mathcal{D}}} \\ \Gamma_{\mathcal{D}}(\boldsymbol{\theta}) & -\boldsymbol{I}_{L_{\mathcal{D}}\times L_{\mathcal{D}}}\end{bmatrix}.$$

We propose the penalized GMM estimator,

$$\left(\widehat{\boldsymbol{\theta}},\widehat{\boldsymbol{D}}\right)=\underset{\boldsymbol{\theta}\in\Theta,\,\boldsymbol{D}\in{\Theta_\delta}^N}{\arg\min}\ \widehat{Q}_{NT}(\boldsymbol{\theta},\boldsymbol{D})+P_{\psi_1,\psi_f}(\boldsymbol{D}),\tag{2.8}$$

where

$$\widehat{Q}_{NT}(\boldsymbol{\theta},\boldsymbol{D})=\frac{1}{N}\sum_{i=1}^{N}\widehat{Q}_{i,NT}(\boldsymbol{\theta},\boldsymbol{\delta}_i),$$
$$\widehat{Q}_{i,NT}(\boldsymbol{\theta},\boldsymbol{\delta}_i)=\widehat{g}_{i,T}(\boldsymbol{\theta},\boldsymbol{\delta}_i)'\boldsymbol{W}_{i,NT}\widehat{g}_{i,T}(\boldsymbol{\theta},\boldsymbol{\delta}_i),$$
$$P_{\psi_1,\psi_c}(\boldsymbol{D})=\frac{\psi_1}{N}\sum_{i=1}^{N}\sum_{l=1}^{L_{\mathcal{D}}}\dot{w}_{il}\left|\delta_{il}\right|+\frac{\psi_f}{N^2}\sum_{1\le i<j\le N}\dot{\mu}_{ij}\left\|\boldsymbol{\delta}_i-\boldsymbol{\delta}_j\right\|,\tag{2.9}$$

and $\boldsymbol{W}_{i,T}$ is a positive definite weighting matrix.

The penalty scheme $P_{\psi_1,\psi_c}(\boldsymbol{D})$ integrates a variant of the adaptive Lasso penalty (Zou, 2006; Cheng and Liao, 2015) and the pairwise adaptive fused Lasso (PAFL) penalty (Mehrabani, 2023) with the adaptive weights $\dot{w}_{il} = \left|\dot{\delta}_{il}\right|^{-\kappa_1}$ and $\dot{\mu}_{ij} = \left\|\dot{\boldsymbol{\delta}}_i - \dot{\boldsymbol{\delta}}_j\right\|^{-\kappa_f}$ based on a preliminary consistent estimator $\dot{\boldsymbol{D}}$, where $\kappa_1, \kappa_f \geq 2$.

The adaptive Lasso penalty is designed to differentiate the valid and invalid moment conditions by shrinking the slackness parameter δ_{il} associated with valid moment conditions, $l \in \mathcal{V}_i$, to zero while leaving δ_{il} stay away from zero for $l \in \mathcal{I}_i$. In the case of valid moment conditions, i.e. $l \in \mathcal{V}_i$, the adaptive weights $\dot{w}_{il}$ tends to be large since $\dot{\delta}_{il}$ is close to zero by its consistency, which imposes heavy penalties to ensure $\widehat{\delta}_{il} = 0$ for $l \in \mathcal{V}_i$. In contrast, invalid moment conditions are subject to small penalties so that they are asymptotically associated with nonzero estimated slackness parameters.

We employ the PAFL penalty to achieve the classification of individuals based on the validity of moment conditions. The penalty encourages $\boldsymbol{\delta}_i = \boldsymbol{\delta}_j$ if i and j belong to the same group, i.e. $\boldsymbol{\delta}_i^0 = \boldsymbol{\delta}_j^0$. The adaptive weights $\dot{\mu}_{ij}$ works similarly as $\dot{w}_{il}$ following the intuition of the adaptive Lasso (Zou, 2006).

Remark 2. It is also possible to utilize other methodologies to handle the latent group structure. For example, we can consider the classifier-Lasso (Su et al., 2016),

$$P_{\psi_c}(\boldsymbol{D}, \boldsymbol{A}) = \frac{\psi_c}{N} \sum_{i=1}^{N} \prod_{k=1}^{K} \|\boldsymbol{\delta}_i - \boldsymbol{\alpha}_k\|.$$

The estimation with the classifier-Lasso penalty, even associated with convex moment conditions, is a non-convex optimization problem. The numerical solution is approximated by solving a sequence of convex optimization problems (Gao and Shi, 2021), which is computationally expensive, and the convergence is not guaranteed. Furthermore, the number of groups K is a tuning parameter, in addition to the regularization parameter ψ_c, that needs to be selected in advance. On the contrary, the PAFL penalty is convex and relies on a single tuning parameter ψ_f to control the strength of the penalty.

2.3.1 Preliminary Estimator

Consider the GMM estimator using moment conditions in $\mathcal{S}$ for initial estimation of $\boldsymbol{\theta}^0$ and the plug-in estimator for $\boldsymbol{\delta}_i$,

$$\dot{\boldsymbol{\theta}}_i = \underset{\boldsymbol{\theta} \in \Theta}{\arg\min}\, \widehat{m}_{\mathcal{S},i,T}(\boldsymbol{\theta})' \boldsymbol{W}_{i,T} \widehat{m}_{\mathcal{S},i,T}(\boldsymbol{\theta}) \text{ and } \dot{\boldsymbol{\delta}}_i = \widehat{m}_{\mathcal{D},i,T}\left(\dot{\boldsymbol{\theta}}\right), \tag{2.10}$$

for $i \in [N]$, where $\boldsymbol{W}_{i,T}$ is a positive definite weighting matrix. Denote $\dot{\boldsymbol{D}} = \left(\dot{\boldsymbol{\delta}}_1, \dot{\boldsymbol{\delta}}_2, \cdots, \dot{\boldsymbol{\delta}}_N\right)'$. The asymptotic properties of $\dot{\boldsymbol{D}}$ will be developed in Lemma A.1 in the appendix. Theoretically, any preliminary estimator satisfying the properties in Lemma A.1 can be employed.

3 Asymptotic Properties

This section is devoted to investigating the asymptotic properties of the penalized GMM estimator introduced in Section 2.3. Initially, we outline the assumptions necessary for the estimator's consistency and proceed to derive the convergence rates of $\widehat{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\delta}}_i$. Subsequently, we establish the consistency of classification and moment selection, in the sense that the estimated group structure and the selected set of moment conditions for each group coincide with the true underlying sets with probability approaching 1. In the final part of this section, we study the asymptotic distribution of the penalized GMM estimator and its oracle properties. Detailed proofs are relegated in Appendix A.1.

3.1 Consistency and Preliminary Rate of Convergnce

To begin with, we introduce the following regularity conditions.

Assumption 1. *(i)* $\boldsymbol{z}_i = \{\boldsymbol{z}_{it} : t \in [T]\}$ *are independently distributed across* i*. For each* i*,* $\boldsymbol{z}_i$ *is stationary strong mixing with mixing coefficients* $\boldsymbol{a}_i(\cdot)$*, where*

$$\boldsymbol{a}(\cdot) = \sup_N \max_{1\le i\le N} \boldsymbol{a}_i(\cdot)$$

satisfies $\boldsymbol{a}(s) \le c_a r^s$ *for some* $c_a > 0$ *and* $r \in (0,1)$*.*

(ii) $\boldsymbol{\theta}^0$ *lies in the interior of a compact set* Θ*.*

(iii) There exists $f(\boldsymbol{z}_{it})$ *s.t.* $\sup_{\boldsymbol{\theta}\in\Theta} \|m(\boldsymbol{z}_{it},\boldsymbol{\theta})\| \le f(\boldsymbol{z}_{it})$ *and*

$$\left\|m(\boldsymbol{z}_{it},\boldsymbol{\theta}) - m\left(\boldsymbol{z}_{it},\overline{\boldsymbol{\theta}}\right)\right\| \le f(\boldsymbol{z}_{it})\left\|\boldsymbol{\theta}-\overline{\boldsymbol{\theta}}\right\|,$$

for all $\boldsymbol{\theta},\overline{\boldsymbol{\theta}} \in \Theta$*.* $\mathbb{E}\left|f(\boldsymbol{z}_{it})\right|^q < \infty$ *for some* $q \ge 6$*.*

(iv) $N = \mathcal{O}(T^{\frac{q}{2}-1})$ *where* $q \ge 6$ *is the constant in (iii).*

Remark 3. Assumption 1 is comparable to Assumption A1 in Su et al. (2016), which is essential to guarantee the convergence of the sample moments to the population moments uniformly over the parameter space and i at a desired rate, which is formally shown in Lemma A.1. The observations are assumed to be cross-sectionally independent and the dependence across t within each i is controlled by a mixing condition as in Assumption 1(i). Assumption

1(ii) regulates the parameter space. Assumption 1(iii) imposes a Lipschitz bound on the moment functions. Alternatively, one can follow Bonhomme and Manresa (2015); Liu et al. (2020) to impose tail conditions on $f(\boldsymbol{z}_{it})$ directly. Finally, Assumption 1(iv) specifies the relative growth rate of N and T depending on $q \geq 6$.

Assumption 2. *(i)* $\overline{m}_{\mathcal{S},i}(\boldsymbol{\theta})$ *is continuous in* $\boldsymbol{\theta}$ *for all* i*, and for any* $\epsilon > 0$*,*

$$\inf_N \min_{1\leq i\leq N} \inf_{\|\boldsymbol{\theta}-\boldsymbol{\theta}^0\|>\epsilon} \|\overline{m}_{\mathcal{S},i}(\boldsymbol{\theta})\| > 0.$$

(ii) $|\delta_{il}^0| \leq C$ *for* $i \in [N]$ *and* $l \in [L_{\mathcal{D}}]$*.*

(iii) There exists a sequence of constants $\tau_T \to 0$ *with* $\tau_T^{-1} = \mathcal{O}\left(T^{\frac{1}{2}}\right)$ *and a fixed constant* η *such that*

$$\sup_{\|\boldsymbol{\theta}-\boldsymbol{\theta}^0\|\leq\eta} \|\widehat{m}_{i,T}(\boldsymbol{\theta}) - \overline{m}_i(\boldsymbol{\theta})\| = \mathcal{O}_p(\tau_T),$$

for $i \in [N]$*.*

(iv) There exist nonrandom matrices $\boldsymbol{W}_{i,NT}$ *and some constant* $c_w, C_w > 0$ *such that*

$$\Pr\left(\max_{1\leq i\leq N} \|\boldsymbol{W}_{i,NT} - \boldsymbol{W}_i\| \geq \epsilon\right) = o(1)$$

for any $\epsilon > 0$*, and*

$$c_w < \inf_N \min_{1\leq i\leq N} \sigma_{\min}(\boldsymbol{W}_i) \leq \sup_N \max_{1\leq i\leq N} \sigma_{\max}(\boldsymbol{W}_i) < C_w.$$

(v) $\overline{m}_i(\boldsymbol{\theta})$ *is continuously differentiable for any* θ *in the local neighborhood of* $\boldsymbol{\theta}^0 \in \Theta$ *for all* i*, and there exists a constant* $c_\Gamma, C_\Gamma > 0$ *such that for some* $\eta > 0$

$$c_\Gamma < \inf_N \min_{1\leq i\leq N} \inf_{\|\boldsymbol{\theta}-\boldsymbol{\theta}^0\|\leq\eta} \sigma_{\min}\left(\Gamma_i(\boldsymbol{\theta})' \Gamma_i(\boldsymbol{\theta})\right) \leq \sup_N \max_{1\leq i\leq N} \sup_{\|\boldsymbol{\theta}-\boldsymbol{\theta}^0\|\leq\eta} \sigma_{\max}\left(\Gamma_i(\boldsymbol{\theta})' \Gamma_i(\boldsymbol{\theta})\right) < C_\Gamma.$$

Remark 4. Assumption 2(i) ensures the structural parameter $\boldsymbol{\theta}$ is identifiable by the moment conditions in $\mathcal{S}$ as in (2.1). Assumption 2(ii) imposes a compactness condition on the true parameter $\boldsymbol{\delta}^0$, which is essentially assuming the existence of moments for moment conditions in $\mathcal{D}$. Assumption 2(iii) is a high-level condition on the convergence rate of the empirical process indexed by the moment function $m(\boldsymbol{z}_{it}, \boldsymbol{\theta})$. When the number of moment conditions, L is fixed, standard empirical process theory for dependent data implies $\tau_T = T^{-\frac{1}{2}}$ (Dehling and Philipp, 2002). The τ_T is introduced for an increasing number of moment conditions. Cheng and Liao (2015, Lemma D.1) provides sufficient conditions for

$\tau_T = \sqrt{L/T}$ to hold. Assumption 2(iv) imposes regularity conditions on the weighting matrix as in Su et al. (2016, Assumption B1(iv)) and (v) regulates the first-order derivatives of the moment conditions.

Assumption 3. *Let* $\widetilde{\varkappa}_{NT} = \left(\frac{L_{\mathcal{D}}}{T}\right)^{\frac{1}{2}} (\log T)^3$ *and* $\varkappa_{NT} = \left(\frac{L_{\mathcal{D}}}{T}\right)^{\frac{1}{2}} (\log T)^{3+\upsilon}$ *for some* $\upsilon > 0$.

(i) $\left(\rho_{NT}^{-1} \vee \zeta_{NT}^{-1}\right) \varkappa_{NT} = o(1)$.

(ii) $\psi_f = \mathcal{O}\left(L_{\mathcal{D}}^{-\frac{1}{2}} \rho_{NT}^{\kappa_f} \tau_T\right)$ *and* $\psi_1 = \mathcal{O}\left(L_{\mathcal{D}}{}^{-\frac{1}{2}} \zeta_{NT}^{\kappa_1} \tau_T\right)$.

Remark 5. Assumption 3 is a set of rate conditions. As shown in Lemma A.1 and Theorem 2(i), the rates $\widetilde{\varkappa}_{NT}$ and $\varkappa_{NT}$ controls the uniform convergence of the preliminary estimator defined in (2.10) and the penalized GMM estimator, respectively. Assumption 3(i) ensures that the estimators converge to the true parameters faster than the minimal degree of group separation ρ_{NT} and the minimal degree of misspecification ζ_{NT} so that we can still correctly separate different groups and identify the invalid moment conditions based on $\widehat{\boldsymbol{\delta}}_i$ even when $\rho_{NT}, \zeta_{NT} \to 0$. Assumption 3(ii) specifies upper bounds on the tuning parameters ψ_1 and ψ_f to ensure that the penalty scheme cannot dominate the GMM objective so that the consistency of the penalized GMM estimator pertains.

With the assumptions outlined above, we can derive the rate of convergence of the penalized GMM estimator $\widehat{\boldsymbol{\theta}} \in \mathbb{R}^{p_\theta}$ and $\widehat{\boldsymbol{D}} \in \mathbb{R}^{L_{\mathcal{D}} \times N}$ in the following Theorem.

Theorem 1. *Suppose Assumption 1 - 3 holds, then*

(i) $\left\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0\right\| = \mathcal{O}_p(\tau_T)$ *and* $\left\|\widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0\right\| = \mathcal{O}_p(\tau_T)$ *for* $i \in [N]$.

(ii) $N^{-1} \sum_{i=1}^N \left\|\widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}^0\right\|^2 = \mathcal{O}_p(\tau_T^2)$.

Remark 6. Theorem 1 establishes the pointwise and mean square convergence $\widehat{\boldsymbol{\delta}}_i$. As shown in (A.16) in the proof in Appendix A.1, the rate of convergence depends on $a_{NT} = \psi_f \left(\max_{(i,j)\in\mathcal{Z}_1} \dot{\mu}_{ij}\right)$ and $b_{NT} = \psi_1 \max_{1\leq i\leq N} \|\dot{\boldsymbol{w}}_{i,\mathcal{I}_i}\|$, where $\dot{\boldsymbol{w}}_{i,\mathcal{I}_i}$ is the subvector of $\dot{\boldsymbol{w}}_i$ with element w_{il}, for $l \in \mathcal{I}_i = \{l \in \mathcal{D} : \delta_{il}^0 \neq 0\}$, in addition to τ_T, which is comparable to the results in Cheng and Liao (2015, Theorem 3.2) and Su et al. (2016, Theorem 2.1). With Assumption 3(ii), we can simplify the results as in Theoreom 1 by showing $a_{NT} = \mathcal{O}_p(\tau_T)$ and $b_{NT} = \mathcal{O}_p(\tau_T)$.

3.2 Classification and Moment Selection Consistency

In addition to Assumptions in Section 3.1, we further introduce the following assumption.

Assumption 4. *(i)* $\psi_f = O\left(\rho_{NT}^{\kappa_f} L_{\mathcal{D}}^{-\frac{1}{2}} \widetilde{\varkappa}_{NT}^2\right)$ *and* $\psi_1 = O\left(\zeta_{NT}^{\kappa_1} L_{\mathcal{D}}^{-\frac{1}{2}} \widetilde{\varkappa}_{NT}^2\right)$.

(ii) $\psi_f^{-1} = o\left(\left(\tau_T \sqrt{L_{\mathcal{D}}} \widetilde{\varkappa}_{NT}^{\kappa_f}\right)^{-1}\right)$ *and* $\psi_1^{-1} = o\left(\left(\tau_T \widetilde{\varkappa}_{NT}^{\kappa_1}\right)^{-1}\right)$.

(iii) $\lim_{N\to\infty} \min_{1\le k\le K^0} N_k/N \to \pi_{\min} \in (0,1)$.

Remark 7. As we will see in Theorem 2, the classification and moment selection consistency relies on the uniform consistency of the penalized GMM estimator across i, which is a stronger result than Theorem 1(i). Consequently, it necessitates a more restrictive upper bound on the tuning parameters ψ_f and ψ_1 in Assumption 4(i) than that in Assumption 3(ii). Assumption 4(ii) delineates lower bounds for ψ_f and ψ_1. This ensures that for any $(i,j) \in \mathcal{Z}_0$ and $l \in \mathcal{V}_i$, ψ_f and ψ_1, in conjunction with the adaptive weights $\dot{\mu}_{ij}$ and $\dot{w}_{il}$, levy sufficiently heavy penalties on $\|\boldsymbol{\delta}_i - \boldsymbol{\delta}_j\|$ and $|\delta_{il}|$, respectively, which is imperative for the consistency of the classification and moment selection. The upper and lower bounds for the tuning parameters also hinge on the range we can allow for ζ_{NT}, ρ_{NT} and the number of moments $L_{\mathcal{D}}$, and guide the choice of κ_1 and κ_f. Consider the case where $\tau_T = \sqrt{\frac{L_{\mathcal{D}}}{N}}$. Let $\zeta_{NT} = T^{-\phi_\zeta}$, $\rho_{NT} = T^{-\phi_\rho}$ and $L_{\mathcal{D}} = T^{\phi_L}$ take polynomial rates of T for some $\phi_\zeta, \phi_\rho, \phi_L > 0$. It is required that $\frac{\tau_T \widetilde{\varkappa}_{NT}^{\kappa_1}}{\zeta_{NT}^{\kappa_1} L_{\mathcal{D}}^{-\frac{1}{2}} \widetilde{\varkappa}_{NT}^2} = o_p(1)$, which leads to $\frac{1}{2} - \frac{1}{2}(1 - 2\phi_\zeta - \phi_L)\kappa_1 < 0$. With $\kappa_1 = 2$, we can allow

$$2\phi_\zeta + \phi_L < \frac{1}{2}. \tag{3.1}$$

Similarly, we can derive conditions for κ_f, ϕ_ρ and ϕ_L that $\left(\frac{1}{2} + \frac{\phi_L}{2}\right) - \frac{1}{2}(1 - 2\phi_\rho - \phi_L)\kappa_f < 0$, which call for a large choice of κ_f. With $\kappa_f = 3$, we can allow

$$3\phi_\rho + 2\phi_L < 1. \tag{3.2}$$

With (3.1) and (3.2), Assumption 3(i), which requires $\phi_L + 2(\phi_\zeta \vee \phi_\rho) < 1$, is automatically satisfied under the current setting.

Remark 8. In instances where the degree of misspecification of moment conditions is minimal, as indicated by a large value of ϕ_ζ, or when groups are not well separated, i.e. ϕ_ρ takes a large value, our methodology may not consistently identify invalid moment conditions or accurately discern the underlying group structure. Moreover, as we shall show in Theorem 2, the penalized GMM estimator can detect invalid moment conditions up to the rate of convergence $\varkappa_{NT}$ while it fails to achieve consistent moment selection when $\zeta_{NT} \asymp \frac{1}{\sqrt{T}}$, which is the case when the moment conditions are locally misspecified for at least one group.

These intricacies prompt the exploration of robust bias-aware inference techniques in future research. Armstrong and Kolesár (2021) propose bias-aware confidence intervals, in the presence of local misspecification at $\sqrt{T}$-rate, that can be constructed by taking the GMM estimator and adding and subtracting the standard error times a critical value that takes into account the potential bias from misspecification of the moment conditions. It is a fruitful direction to consider the post-regularization estimation as in (3.7) and constructing bias-aware confidence intervals for the structural parameters following Armstrong and Kolesár (2021).

Remark 9. As noticed in Remark 1, the group structure admits a factor structure $\boldsymbol{D} = \boldsymbol{\Lambda}\boldsymbol{A}'$. The approximate moment conditions with $\alpha_{kl} = \frac{c}{\sqrt{T}}$ for a constant $c \neq 0$ for some $l \in [L_{\mathcal{D}}]$ corresponds to weak factor issues in the interactive fixed effects models, for which the robust inference method is investigated in the recent work by Armstrong et al. (2023).

Remark 10. Assumption 4(iii) imposes the condition that no group has an asymptotically trivial size, which is a convenient handle for the technical derivation. At the cost of cumbersomeness in notations and derivations, we can relax this condition to allow for $\pi_{\min} = 0$.

Now we are readily to establish the following theorem, which directly implies classification and moment selection consistency which will be elaborated in Remark 13.

Theorem 2. *Suppose Assumption 1 - 4 hold, then as* $(N,T) \to \infty$,

(i) $\max_{1\leq i\leq N} \left\|\widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0\right\| = O_p\left(\varkappa_{NT}\right).$

(ii) $\Pr\left(\max_{(i,j)\in\mathcal{Z}_0} \left\|\widehat{\boldsymbol{\delta}}_i - \widehat{\boldsymbol{\delta}}_j\right\| > 0\right) \to 0$ *and* $\Pr\left(\min_{(i,j)\in\mathcal{Z}_1} \left\|\widehat{\boldsymbol{\delta}}_i - \widehat{\boldsymbol{\delta}}_j\right\| > 0\right) \to 1.$

(iii) $\Pr\left(\max_{1\leq i\leq N} \max_{l\in\mathcal{V}_i} \left|\widehat{\delta}_{il}\right| > 0\right) \to 0$ *and* $\Pr\left(\min_{1\leq i\leq N} \min_{l\in\mathcal{I}_i} \left|\widehat{\delta}_{il}\right| > 0\right) \to 1.$

Remark 11. As in Appendix A.1, we show the first statement of Theorem 2(ii) in the first place by investigating the Karush-Kuhn-Tucker (KKT) conditions of (2.8) and making use of Assumption 4(ii). With this result, for any $i \in [N]$ such that $\left\|\widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0\right\| > \varkappa_{NT}$, we have $\left\|\widehat{\boldsymbol{\delta}}_j - \boldsymbol{\delta}_i^0\right\| > \varkappa_{NT}$ for all $j \in \mathcal{G}_{k(i)}$, which is essential to prove the uniform consistency in Theorem 2(i). It is then guaranteed that we can separate different groups and detect invalid moment conditions up to the rate $\varkappa_{NT}$. Together with Assumption 3(i), we can show the second part of both Theorem 2(ii) and (iii).

Remark 12. Mehrabani (2023, Theorem 3.2) claim a similar results to the first statement in Theorem 2(ii). However, the proof in Mehrabani (2023) actually attempts to show

$$\Pr\left(\min_{j\in\mathcal{G}_{k(i)}} \left\|\widehat{\boldsymbol{\delta}}_i - \widehat{\boldsymbol{\delta}}_j\right\| > 0\right) \to 0$$

as $(N,T) \to \infty$ for each individual $i \in [N]$, which is neither a proper notion of classification consistency nor a property uniform over i. Since the uniform consistency is not established in Mehrabani (2023), the second statement in Theorem 2(ii) is not guaranteed and hence the classification consistency is not established as claimed therein. The proof in A.1 can serve as a remedy in their setting.

Remark 13. Denote

$$\widehat{\mathcal{Z}}_0 = \left\{(i,j) \in [N] \times [N] : \widehat{\boldsymbol{\delta}}_i = \widehat{\boldsymbol{\delta}}_j, i < j\right\} \text{ and } \widehat{\mathcal{Z}}_1 = \left\{(i,j) \in [N] \times [N] : \widehat{\boldsymbol{\delta}}_i \neq \widehat{\boldsymbol{\delta}}_j, i < j\right\}.$$

Theorem 2(ii) shows that

$$\Pr\left(\mathcal{Z}_0 \subset \widehat{\mathcal{Z}}_0\right) \to 1,\ \Pr\left(\mathcal{Z}_1 \subset \widehat{\mathcal{Z}}_1\right) \to 1, \tag{3.3}$$

as $(N,T) \to \infty$. By triangle inequality, Theorem 2(i) and the rate condition Assumption 3(ii), we have

$$\begin{aligned}\Pr\left(\max_{(i,j)\in\widehat{\mathcal{Z}}_0} \|\boldsymbol{\delta}_i^0 - \boldsymbol{\delta}_j^0\| \geq \rho_{NT}\right) &\leq \Pr\left(\max_{(i,j)\in\widehat{\mathcal{Z}}_0} \left\|\widehat{\boldsymbol{\delta}}_i - \widehat{\boldsymbol{\delta}}_j\right\| + \left\|\widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0\right\| + \left\|\widehat{\boldsymbol{\delta}}_j - \boldsymbol{\delta}_j^0\right\| \geq \rho_{NT}\right) \\ &\leq \Pr\left(\max_{1\leq i\leq N} \left\|\widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0\right\| \geq \frac{\rho_{NT}}{2}\right) \to 0,\end{aligned}$$

as $(N,T) \to \infty$, which implies that $\Pr\left(\widehat{\mathcal{Z}}_0 \subset \mathcal{Z}_0\right) \to 1$ and together with (3.3), we have

$$\Pr\left(\widehat{\mathcal{Z}}_0 = \mathcal{Z}_0,\ \widehat{\mathcal{Z}}_1 = \mathcal{Z}_1\right) \to 1 \text{ as } (N,T) \to \infty. \tag{3.4}$$

Similarly, Theorem 2(iii) implies

$$\Pr\left(\bigcap_{i=1}^{N} \left\{\widehat{\mathcal{V}}_i = \mathcal{V}_i\right\}\right) \to 1 \text{ as } (N,T) \to \infty, \tag{3.5}$$

where $\widehat{\mathcal{V}}_i = \{l \in [L_{\mathcal{D}}] : \delta_{il} = 0\}$, for $i \in [N]$.

Let $\{\widehat{\boldsymbol{\alpha}}_1, \widehat{\boldsymbol{\alpha}}_2, \cdots, \widehat{\boldsymbol{\alpha}}_{\widehat{K}}\}$ be the distinct values of $\left\{\widehat{\boldsymbol{\delta}}_1, \widehat{\boldsymbol{\delta}}_2, \cdots, \widehat{\boldsymbol{\delta}}_N\right\}$. For $k = 1, 2, \cdots, \widehat{K}$, let $\widehat{\mathcal{G}}_k = \left\{i \in [N] : \widehat{\boldsymbol{\delta}}_i = \widehat{\boldsymbol{\alpha}}_k\right\}$ and $\widehat{\mathcal{V}}_k = \{l \in \mathcal{D} : \widehat{\boldsymbol{\alpha}}_{kl} = 0\}$ denote the estimated group membership and the set of selected moment conditions for group k, respectively. We formalized the classification and moment selection consistency in the following corollary.

Corollary 3. *Suppose Assumption 1 - 4 hold, then*

$$\Pr\left(\left\{\widehat{K}=K^0\right\}\right)\to 1,\ \Pr\left(\bigcap_{k=1}^{K^0}\left\{\widehat{\mathcal{G}}_k=\mathcal{G}_{(k)}\right\}\right)\to 1 \ \textit{and}\ \Pr\left(\bigcap_{k=1}^{K^0}\left\{\widehat{\mathcal{V}}_k=\mathcal{V}_{(k)}\right\}\right)\to 1, \quad (3.6)$$

as $(N,T)\to\infty$*, where* $\{(1),(2),\cdots,(K^0)\}$ *is a suitable permutation of* $[K^0]$.

Corollary 3 can be directly implied from Theorem 2 and Remark 13. Note that $\widehat{\mathcal{Z}}_0\subset\mathcal{Z}_0$ implies $K^0\leq\widehat{K}$ and $\mathcal{Z}_0\subset\widehat{\mathcal{Z}}_0$ implies $\widehat{K}\leq K^0$. By (3.4), we have the consistency of the estimated number of groups. With $\widehat{K}=K^0$, $\widehat{\mathcal{Z}}_0\subset\mathcal{Z}_0$ implies $\widehat{\mathcal{G}}_k\subset\mathcal{G}_{(k)}$, for $k\in[K^0]$ and $\{(1),(2),\cdots,(K^0)\}$ is a permutation of $[K^0]$; conversely $\mathcal{Z}_0\subset\widehat{\mathcal{Z}}_0$ implies $\mathcal{G}_{(k)}\subset\widehat{\mathcal{G}}_k$, for $k\in[K^0]$, which implies classification consistency. Together with (3.5), we have the moment selection consistency.

3.3 Asymptotic Distribution

Under Theorem 2 and Corollary 3, $\widehat{\boldsymbol{\alpha}}_{k,\mathcal{V}_k}=\mathbf{0}$ w.p.a.1 for $k\in[K^0]$. It remains to develop the asymptotic distribution for $\boldsymbol{\theta}$ and $\boldsymbol{\alpha}_{k,\mathcal{I}_k}$. Denote $\boldsymbol{\beta}_k^0=\left(\boldsymbol{\theta}^{0\prime},\boldsymbol{\alpha}_{k,\mathcal{I}_k}^{0}{}'\right)'$. For $\boldsymbol{\alpha}_{k,\widehat{\mathcal{I}}_k}\in\mathbb{R}^{|\widehat{\mathcal{I}}_k|}$, denote $\widetilde{\boldsymbol{\alpha}}_k\in\mathbb{R}^{L_\mathcal{D}}$ with $\widetilde{\boldsymbol{\alpha}}_{k,\mathcal{I}_k}=\boldsymbol{\alpha}_{k,\mathcal{I}_k}$ and $\widehat{\boldsymbol{\alpha}}_{k,\mathcal{V}_k}=\mathbf{0}$. Let

$$\widetilde{g}_{i,T}^{(k)}\left(\boldsymbol{\theta},\boldsymbol{\alpha}_{k,\widehat{\mathcal{I}}_k}\right)=\frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix} g_S\left(\boldsymbol{z}_{it},\boldsymbol{\theta}\right)\\ g_\mathcal{D}\left(\boldsymbol{z}_{it},\boldsymbol{\theta}\right)-\widetilde{\boldsymbol{\alpha}}_k\end{bmatrix}.$$

We define the post-regularization estimator $\widehat{\boldsymbol{\beta}}_k^{\text{post}}$, $k\in\left[\widehat{K}\right]$, as

$$\left(\widehat{\boldsymbol{\theta}}^{\text{post}\prime},\widehat{\boldsymbol{\alpha}}_{k,\widehat{\mathcal{I}}_k}^{\text{post}\prime}\right)'=\underset{\boldsymbol{\theta}\in\Theta,\boldsymbol{\alpha}_{k,\widehat{\mathcal{I}}_k}\in\Theta_\delta^{|\widehat{\mathcal{I}}_k|}}{\arg\min}\left(\frac{1}{\widehat{N}_k}\sum_{i\in\widehat{\mathcal{G}}_k}\widetilde{g}_{i,T}^{(k)}\left(\boldsymbol{\theta},\boldsymbol{\alpha}_{k,\widehat{\mathcal{I}}_k}\right)\right)'\boldsymbol{W}_{k,NT}\left(\frac{1}{\widehat{N}_k}\sum_{i\in\widehat{\mathcal{G}}_k}\widetilde{g}_{i,T}^{(k)}\left(\boldsymbol{\theta},\boldsymbol{\alpha}_{k,\widehat{\mathcal{I}}_k}\right)\right), \quad (3.7)$$

and we let $\widehat{\boldsymbol{\beta}}_k^{\text{post}}=\left(\widehat{\boldsymbol{\theta}}^{\text{post}\prime},\widehat{\boldsymbol{\alpha}}_{k,\widehat{\mathcal{I}}_k}^{\text{post}\prime}\right)'$

Assumption 5. *(i) Let* $\widehat{\nu}_{i,T}=\widehat{m}_{i,T}\left(\boldsymbol{\theta}\right)-\overline{m}_{i,T}\left(\boldsymbol{\theta}\right)$*. There exists a sequence of constants* $\varsigma_T\to 0$ *such that*

$$\sup_{\boldsymbol{\theta}_1,\boldsymbol{\theta}_2\in\{\boldsymbol{\theta}\in\Theta:\|\boldsymbol{\theta}-\boldsymbol{\theta}^0\|\leq\eta_T\}}\frac{\|\widehat{\nu}_{i,T}(\boldsymbol{\theta}_1)-\widehat{\nu}_{i,T}(\boldsymbol{\theta}_2)\|}{T^{-\frac{1}{2}}+\|\boldsymbol{\theta}_1-\boldsymbol{\theta}_2\|}=\mathcal{O}_p(\varsigma_T), \quad (3.8)$$

for some sequence $\eta_T\to 0$ *with* $\eta_T^{-1}\tau_T=o\left(1\right)$*,* $\forall i\in[N]$.

(ii) For $k \in [K^0]$, define the variance of the moment conditions as

$$\boldsymbol{\Omega}_{i,T} = \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathbb{E}\left[g\left(\boldsymbol{z}_{it}, \boldsymbol{\theta}^0, \boldsymbol{\delta}_i^0\right) g\left(\boldsymbol{z}_{is}, \boldsymbol{\theta}^0, \boldsymbol{\delta}_i^0\right)'\right] \text{ and } \boldsymbol{\Omega}_k = \lim_{(N,T)\to\infty} N_k^{-1}\sum_{i\in\mathcal{G}_k}\boldsymbol{\Omega}_{i,T},$$

and the Jacobian matrix is

$$\boldsymbol{\Gamma}_k = \lim_{N\to\infty}\sum_{i=1}^{N}\begin{bmatrix}\Gamma_{\mathcal{S}}\left(\boldsymbol{\theta}^0\right) & \boldsymbol{0}_{L_{\mathcal{S}}\times L_{\mathcal{D}}}\\ \Gamma_{\mathcal{D}}\left(\boldsymbol{\theta}^0\right) & -\Upsilon_k\end{bmatrix}$$

where $\Upsilon = \operatorname{diag}(\boldsymbol{\upsilon})$ in which $\boldsymbol{\upsilon}_{\mathcal{I}_k} = -\boldsymbol{\iota}_{|\mathcal{I}_k|}$ and $\upsilon_{\mathcal{V}_k} = \boldsymbol{0}_{|\mathcal{V}_k|}$. Assume that $\boldsymbol{\Omega}_k$ and Γ_k exists and

$$c < \sigma_{\min}\left(\boldsymbol{\Omega}_k\right) \leq \sigma_{\max}\left(\boldsymbol{\Omega}_k\right) < C,$$
$$c < \sigma_{\min}\left(\boldsymbol{\Gamma}_k'\boldsymbol{\Gamma}_k\right) \leq \sigma_{\max}\left(\boldsymbol{\Gamma}_k'\boldsymbol{\Gamma}_k\right) < C,$$

for constant $c, C > 0$, for $k \in K^0$.

(iii) Assumption 2(iv) holds with $\boldsymbol{W}_{i,NT}$ and $\boldsymbol{W}_i$ replaced by $\boldsymbol{W}_{k,NT}$ and $\boldsymbol{W}_k$, respectively, for $k \in [K^0]$.

(iv) For any $\boldsymbol{\gamma} \in \mathbb{R}^{L_{\mathcal{D}}}$ with $\|\gamma\| = 1$, $\boldsymbol{\gamma}'\left(\frac{1}{\sqrt{NT}}\sum_{i\in\mathcal{G}_k}\sum_{t=1}^{T} g\left(\boldsymbol{z}_{it}, \theta^0, \alpha_k^0\right)\right) \xrightarrow{d} \mathcal{N}\left(0, \boldsymbol{\gamma}'\boldsymbol{\Omega}_k\boldsymbol{\gamma}\right)$.

(v) $\sqrt{L_{\mathcal{D}}}\tau_T^2 = o\left(T^{-\frac{1}{2}}\right)$ and $\varsigma_T\tau_T = o\left(T^{-\frac{1}{2}}\right)$.

Remark 14. Assumption 5(i) restates the stochastic equicontintuity condition in Cheng and Liao (2015, Assumption 3.5). The rate of convergence ς_T is introduced to accommodate a diverging number of moments $L_{\mathcal{D}}$. If $L_{\mathcal{D}}$ is fixed, we can replace the right-hand side of (3.8) with $o_p(1)$. Cheng and Liao (2015, Lemma D.2) provides primitive low-level conditions under which (3.8) holds with $\varsigma_T = \sqrt{L_{\mathcal{D}}/T}$. Assumption 5(ii) and (iii) regulate the covariance of the moment conditions, the Jacobian matrix and the weight matrix. Assumption 5(iv) assumes the Lindeberg-Feller central limit theorem holds for the moment conditions, for which Su et al. (2016, Lemma S1.12) provides verification details based on the same set of Assumptions.

Theorem 4. *Let $\boldsymbol{\Sigma}_k = \left(\boldsymbol{\Gamma}_k'\boldsymbol{W}_k\boldsymbol{\Gamma}_k\right)^{-1}\left(\boldsymbol{\Gamma}_k'\boldsymbol{W}_k\boldsymbol{\Omega}_k\boldsymbol{W}\boldsymbol{\Gamma}_k\right)\left(\boldsymbol{\Gamma}_k'\boldsymbol{W}_k\boldsymbol{\Gamma}_k\right)^{-1}$. For any $\boldsymbol{\gamma} \in \mathbb{R}^{p_\theta+|\mathcal{I}_k|}$ with $\|\boldsymbol{\gamma}\| = 1$,*

$$\sqrt{N_k T}\boldsymbol{\gamma}'\left(\widehat{\boldsymbol{\beta}}_k^{\text{post}} - \boldsymbol{\beta}_{(k)}^0\right) \xrightarrow{d} \mathcal{N}\left(0, \boldsymbol{\gamma}'\boldsymbol{\Sigma}_k\boldsymbol{\gamma}\right),$$

as $(N,T) \to \infty$, for all $k \in [K^0]$, where $\{(1), (2), \cdots, (K^0)\}$ is a suitable permutation of $[K^0]$.

Remark 15. The post-regularization estimator has the same asymptotic distribution as the infeasible estimator based on the true group membership and the set of invalid moment conditions being known, i.e. it possesses the oracle property.

4 Monte Carlo Simulation

In this section, we present a simulation experiment with a simple linear IV model to verify that our proposed method is feasible and effective. In later updates, a comprehensive simulation study will be conducted to evaluate the finite-sample performance of the proposed method in a variety of settings.

4.1 Simulation Design

The structural equation is

$$y_{it} = \theta x_{it} + \varepsilon_{it}^{(0)},\ i \in [N],$$

where $\varepsilon_{it}^{(0)}$ is the structural error and x_{it} is an endogenous regressor. The reduced-form equation for the endogenous variable is

$$x_{it} = \boldsymbol{z}_{it}'\boldsymbol{\gamma} + \varepsilon_{it}^{(1)},$$

where $\varepsilon_{it}^{(1)}$ is the reduced-form error. $\boldsymbol{z}_{it} \in \mathbb{R}^L$ is a vector of instruments. Let

$$\begin{pmatrix} \varepsilon_{it}^{(0)} \\ \varepsilon_{it}^{(1)} \end{pmatrix} \sim N\left(\mathbf{0}_{2\times 1}, \begin{pmatrix} 1 & \rho_\varepsilon \\ \rho_\varepsilon & 1 \end{pmatrix}\right),$$

and

$$z_{it,l} = \delta_{il}\varepsilon_{it}^{(0)} + \sqrt{1-\delta_{il}^2}\tilde{z}_{il},$$

where $\tilde{\boldsymbol{z}}_{it} \sim N\left(\mathbf{0}_{L\times 1}, \boldsymbol{\Sigma}_z\right)$ and $\boldsymbol{\Sigma}_z = \begin{pmatrix} 1 & \rho_z & \rho_z & \dots & \rho_z \\ \rho_z & 1 & \rho_z & \dots & \rho_z \\ \rho_z & \rho_z & 1 & \dots & \rho_z \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_z & \rho_z & \rho_z & \dots & 1 \end{pmatrix}$. The non-zero correlation between $\varepsilon_{it}^{(0)}$ and $\varepsilon_{it}^{(1)}$, ρ_ε, is the source of endogeneity. $|\rho_z| > 0$ ensures the relevance conditions hold for each instrument z_{il}. In the experiment, $\theta = 1$, $\boldsymbol{\gamma} = (1, 1, 0, \cdots, 0)$, $\rho_\varepsilon = 0.5$, $\rho_z = 0.5$. We consider sample sizes $N = 250, 500$, $T = 50, 100$ and fix $L = 8$. There are 3 groups, i.e. $K^0 = 3$. Denote N_k as the number of observations in each group and let $N_1 : N_2 : N_3 = 0.3 : 0.3 : 0.4$. The validity of instruments are characterized by $\boldsymbol{D} = (\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \cdots, \boldsymbol{\delta}_N)' \in \mathbb{R}^{N \times L}$, where $\boldsymbol{D} = \boldsymbol{\Lambda} \boldsymbol{A}'$, where $\lambda_{ik} = 1$ if $i \in \mathcal{G}_k$ and $\lambda_{ik} = 0$ otherwise, for $i \in [N]$ and $k \in [K^0]$. $\boldsymbol{A} = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3)$, where $\alpha_{k1} = 0$ for all $k = 1, 2, 3$, $\alpha_{12} = \alpha_{13} = \alpha_{23} = \alpha_{24} = \alpha_{34} = \alpha_{45} = 0$, and we generate $\alpha_{kl} \sim \text{unif}\,(0.5, 0.9)$ elsewhere and fix it across replications. This means that the sure set $\mathcal{S} = \{1\}$ and $\mathcal{V}_1 = \{2, 3\}$, $\mathcal{V}_2 = \{3, 4\}$ and $\mathcal{V}_3 = \{4, 5\}$.

4.2 Tuning Parameter Selection and Optimization Routine

In the experiment, we choose $\kappa_1 = 2$ and $\kappa_f = 3$. Following the discussion in Remark 7, the tuning parameters are set as $\psi_1 = c_1 \operatorname{Var}(\boldsymbol{y})\, T^{-\frac{5}{6}}$ and $\psi_f = c_f \operatorname{Var}(\boldsymbol{y})\, T^{-\frac{3}{2}}$ based on $\frac{\log L}{\log T} \approx \frac{1}{3}$ according to the simulation design. In future updates of the paper, we will develop a data-driven guidance based on Cheng and Liao (2015) and the information criteria as in Mehrabani (2023) and provide supporting asymptotic results.

For a linear IV model, the optimization problem (2.8) is nicely convex. Leveraging on modern convex optimization solver, `MOSEK` (MOSEK ApS, 2024) for example, and the modeling techniques summarized in Gao and Shi (2021), the estimation problem can be efficiently solved. If the (2.8) is nonconvex, we can use the Gauss-Newton algorithm which has been justified for nonconvex GMM problems by Forneron and Zhong (2023).

4.3 Results

We report the finite sample performance across $1,000$ replications in Table 1. As expected, the penalized GMM estimator achieves a high classification correct ratio and moment selection accuracy. The bias and RMSE of the estimator are small.

5 Conclusion

In this paper, we provide a unified framework for the selection of valid moment conditions and detection of latent group structure based on the moment condition validity in general nonlinear generalized method of moments (GMM) panel data models, which can accommodate a diverging number of moment conditions and group-specific heterogeneous validity of moment conditions across agents. The proposed penalized GMM estimator is shown to be consistent and achieve classification and moment selection consistency simultaneously. The asymptotic distribution of a post-regularization estimator is derived, and its oracle properties are established. The extension to incorporate locally misspecified moment conditions

Table 1: Finite Sample Performance of the Penalized GMM Estimator for Linear IV Model

		$\theta = 1$		Classification	Moment Selection	
N	T	Bias	RMSE	% $\widehat{k(i)} = k(i)$	% Invalid IV	$\%\widehat{K} = K^0$
250	50	0.005	0.073	94.4	1.1	99.2
	100	-0.004	0.025	97.2	0.5	100
500	50	0.004	0.054	96.2	0.3	99.5
	100	0.003	0.019	98.8	0.1	100

Notes: Generically, bias and RMSE are calculated by $R^{-1}\sum_{r=1}^{R}\left(\hat{\theta}^{(r)} - \theta_0\right)$ and $\sqrt{R^{-1}\sum_{r=1}^{R}\left(\hat{\theta}^{(r)} - \theta_0\right)^2}$, respectively, for true parameter θ_0 and its estimate $\hat{\theta}^{(r)}$, across $R = 1,000$ replications. We relabel the estimated group with $\widehat{\boldsymbol{\alpha}}_k$ to the closest $\boldsymbol{\alpha}_k^0$ and compute the percentage of correct classification by $R^{-1}\sum_{r=1}^{R} N^{-1}\sum_{i=1}^{N} \mathbb{1}\left(i \in \widehat{\mathcal{G}}_{k(i)}\right)$. The percentage of invalid IV is calculated by $R^{-1}\sum_{r=1}^{R}\sum_{k=1}^{K} |\mathcal{I}_k|^{-1}\sum_{l\in\mathcal{I}_k} \mathbb{1}\left(\widehat{\alpha}_{kl} \neq 0\right)$. The percentage of correct number of group is calculated by $R^{-1}\sum_{r=1}^{R} \mathbb{1}\left(\widehat{K} = K^0\right)$.

and robust inference is discussed. The finite-sample performance of the proposed method is evaluated through a Monte Carlo simulation experiment.

The paper is at a preliminary stage and subject to active updates. We plan to complete the data-driven parameter tuning methods and conduct a comprehensive simulation study to evaluate the finite-sample performance of the proposed method in a variety of settings. The method will also be applied to empirical data to illustrate its practical relevance.

Appendix

A.1 Proofs of Main Results

We first introduce the following notations for simplicity of exposition in the proofs. Denote $\widehat{\nu}_{i,T}(\boldsymbol{\theta}) := \widehat{m}_{i,T}(\boldsymbol{\theta}) - \overline{m}_i(\boldsymbol{\theta})$ as in Assumption 5(i) and $\widehat{R}_{i,T}(\boldsymbol{\theta}) := \widehat{\nu}_{i,T}(\boldsymbol{\theta})' \boldsymbol{W}_i \widehat{\nu}_{i,T}(\boldsymbol{\theta})$. In addition, denote $\overline{Q}_i(\boldsymbol{\theta}, \boldsymbol{\delta}) = \overline{g}_i(\boldsymbol{\theta}, \boldsymbol{\delta}) \boldsymbol{W}_i \overline{g}_i(\boldsymbol{\theta}, \boldsymbol{\delta})$. For any two sequences a_N and b_N, let $a_N \lesssim b_N$ denote $Ca_N \leq b_N$ where $C > 0$ is some fixed finite constant, and $a_N \gtrsim b_N$ denote $b_N \lesssim a_N$. With abuse of notation, we reuse the notation Ξ to denote algebraic terms for convenience of exposition.

Lemma A.1. *Let* $\widetilde{\varkappa}_{NT} = \sqrt{\frac{L_{\mathcal{D}}}{T}} (\log T)^3$. $\dot{\boldsymbol{\delta}}_i$ *is the preliminary estimator defined in* (2.10). $\dot{w}_{il} = \left|\dot{\delta}_{il}\right|^{-\kappa_1}$ *and* $\dot{\mu}_{ij} = \left\|\dot{\boldsymbol{\delta}}_i - \dot{\boldsymbol{\delta}}_j\right\|^{-\kappa_f}$ *are the adaptive weights introduced in Section 2.3. Under Assumption 1 - 3,*

(i) $\max_{1 \leq i \leq N} \|\widehat{m}_{i,T}(\boldsymbol{\theta}) - \overline{m}_i(\boldsymbol{\theta})\| = \mathcal{O}_p(\widetilde{\varkappa}_{NT})$

(ii) $\max_{1 \leq i \leq N} \left\|\dot{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0\right\| = \mathcal{O}_p(\widetilde{\varkappa}_{NT})$

(iii) $\max_{(i,j) \in \mathcal{Z}_1} \dot{\mu}_{ij} = \mathcal{O}_p\left(\rho_{NT}^{-\kappa_f}\right)$.

(iv) $\max_{1 \leq i \leq N} \max_{l \in \mathcal{I}_i} \dot{w}_{il} = \mathcal{O}_p\left(\zeta_{NT}^{-\kappa_1}\right)$ *and* $\max_{1 \leq i \leq N} \|\dot{\boldsymbol{w}}_{i,\mathcal{I}_i}\| = \mathcal{O}_p\left(\sqrt{L_{\mathcal{D}}}\zeta_{NT}^{-\kappa_1}\right)$ *where* $\dot{\boldsymbol{w}}_{i,\mathcal{I}_i}$ *is the subvector of* $\dot{\boldsymbol{w}}_i$ *with element* w_{il}*, for* $l \in \mathcal{I}_i = \{l \in [L_{\mathcal{D}}] : \delta_{il}^0 \neq 0\}$.

Proof of Lemma A.1. Part (i) restates Lemma S1.2(i) in Su et al. (2016) with the dimension of the moment function, $L_{\mathcal{D}}$, diverging. By applying the proof arguments element-by-element to $|\widehat{m}_{il,T}(\boldsymbol{\theta}) - \overline{m}_{il}(\boldsymbol{\theta})|^2$, we have the modified convergence rate $\varkappa_{NT}$ with the new term $\sqrt{L_{\mathcal{D}}}$ showing up compared to the original rate in Su et al. (2016).

Part (ii). Note that we leverage on the identification of $\boldsymbol{\theta}$ with fixed dimensional moment conditions $\overline{m}_{\mathcal{S},i}(\boldsymbol{\theta}) = 0$ to construct the initial GMM estimator for $\boldsymbol{\theta}^0$, standard asymptotic theory as in Newey and McFadden (1994) yield the $\sqrt{T}$ rate of convergence, i.e. $\left\|\dot{\boldsymbol{\theta}} - \boldsymbol{\theta}^0\right\| = \mathcal{O}_p\left(T^{-\frac{1}{2}}\right)$. By mean value theorem,

$$\begin{aligned}
\dot{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0 &= \widehat{m}_{\mathcal{D},i,T}\left(\dot{\boldsymbol{\theta}}\right) - \overline{m}_{\mathcal{D},i}\left(\dot{\boldsymbol{\theta}}\right) + \overline{m}_{\mathcal{D},i}\left(\dot{\boldsymbol{\theta}}\right) - \overline{m}_{\mathcal{D},i}\left(\boldsymbol{\theta}^0\right) \\
&= \widehat{m}_{\mathcal{D},i,T}\left(\dot{\boldsymbol{\theta}}\right) - \overline{m}_{\mathcal{D},i}\left(\dot{\boldsymbol{\theta}}\right) + \Gamma_{\mathcal{D},i}\left(\widetilde{\boldsymbol{\theta}}\right)\left(\dot{\boldsymbol{\theta}} - \boldsymbol{\theta}^0\right),
\end{aligned}$$

where $\widetilde{\boldsymbol{\delta}}$ is between $\boldsymbol{\theta}^0$ and $\dot{\boldsymbol{\theta}}$. Then

$$\begin{aligned}\max_{1\le i\le N}\left\|\dot{\boldsymbol{\delta}}_i-\boldsymbol{\delta}_i^0\right\| &\le \max_{1\le i\le N}\left\|\widehat{m}_{\mathcal{D},i,T}\left(\dot{\boldsymbol{\theta}}\right)-\overline{m}_{\mathcal{D},i}\left(\dot{\boldsymbol{\theta}}\right)\right\|+\left(\max_{1\le i\le N}\left\|\Gamma_{\mathcal{D},i}\left(\widetilde{\boldsymbol{\theta}}\right)\right\|\right)\left\|\dot{\boldsymbol{\theta}}-\boldsymbol{\theta}^0\right\|\\ &\le \mathcal{O}_p\left(\widetilde{\varkappa}_{NT}\right)+\mathcal{O}_p\left(T^{-\frac{1}{2}}\right)=\mathcal{O}_p\left(\widetilde{\varkappa}_{NT}\right),\end{aligned}$$

where the first line follows from triangle inequality and Cauchy-Schwartz inequality, and the second inequality holds under part (i) and Assumption 2(v).

Part (iii). Note that $\max_{(i,j)\in\mathcal{Z}_1}\dot{\mu}_{ij}=\left(\min_{(i,j)\in\mathcal{Z}_1}\left\|\dot{\boldsymbol{\delta}}_i-\dot{\boldsymbol{\delta}}_j\right\|\right)^{-\kappa_f}$. For $(i,j)\in\mathcal{Z}_1$ and sufficiently large (N,T),

$$\min_{(i,j)\in\mathcal{Z}_1}\left\|\dot{\boldsymbol{\delta}}_i-\dot{\boldsymbol{\delta}}_j\right\|\ge\min_{(i,j)\in\mathcal{Z}_1}\left|\|\boldsymbol{\delta}_i^0-\boldsymbol{\delta}_j^0\|-\left\|\left(\dot{\boldsymbol{\delta}}_i-\boldsymbol{\delta}^0\right)-\left(\dot{\boldsymbol{\delta}}_j-\boldsymbol{\delta}^0\right)\right\|\right|\ge\rho_{NT}-2\max_{1\le i\le N}\left\|\dot{\boldsymbol{\delta}}_i-\boldsymbol{\delta}_i^0\right\|. \tag{A.1}$$

by triangle inequality. By the uniform convergence in part (ii) and the rate condition in Assumption 3(i), $\max_{(i,j)\in\mathcal{Z}_1}\dot{\mu}_{ij}=O_p\left(\rho_{NT}^{-\kappa_f}\right)$.

Part (iv). By parallel arguments as in part (iii), we have $\max_{1\le i\le N}\max_{l\in\mathcal{I}_i}\dot{w}_{il}=\mathcal{O}_p\left(\zeta_{NT}^{-\kappa_1}\right)$,

$$\max_{1\le i\le N}\|\dot{\boldsymbol{w}}_{i,\mathcal{I}_i}\|=\mathcal{O}_p\left(\sqrt{L_{\mathcal{D}}}\zeta_{NT}^{-\kappa_1}\right)$$

holds by noting that $\|\dot{\boldsymbol{w}}_{i,\mathcal{I}_i}\|\le\sqrt{L_{\mathcal{D}}}\max_{l\in\mathcal{I}_i}\dot{w}_{il}$.

□

Lemma A.2. *Under Assumption 1 - 3,* $\widehat{\boldsymbol{\theta}}\xrightarrow{p}\boldsymbol{\theta}^0$ *as* $(N,T)\to\infty$.

Proof of Lemma A.2. By the optimality in (2.8),

$$\begin{aligned}\widehat{Q}_{NT}\left(\widehat{\boldsymbol{\theta}},\widehat{\boldsymbol{D}}\right)\le\widehat{Q}_{NT}\left(\boldsymbol{\theta}^0,\boldsymbol{D}^0\right)&+\frac{\psi_f}{N^2}\sum_{1\le i<j\le N}\dot{\mu}_{ij}\left(\|\boldsymbol{\delta}_i^0-\boldsymbol{\delta}_j^0\|-\left\|\widehat{\boldsymbol{\delta}}_i-\widehat{\boldsymbol{\delta}}_j\right\|\right)\\ &+\frac{\psi_1}{N}\sum_{i=1}^{N}\sum_{l=1}^{L_{\mathcal{D}}}\dot{w}_{il}\left(\left|\delta_{il}^0\right|-\left|\widehat{\delta}_{il}\right|\right)\end{aligned} \tag{A.2}$$

Note that, $\widehat{g}_{i,T}\left(\boldsymbol{\theta}^0,\boldsymbol{\delta}_i^0\right)=\widehat{\nu}_{i,T}\left(\boldsymbol{\theta}^0\right)=\widehat{m}_{i,T}\left(\boldsymbol{\theta}^0\right)-\overline{m}_i\left(\boldsymbol{\theta}^0\right)$, then by Assumption 2(iv) and Lemma A.1(i),

$$\begin{aligned}\widehat{Q}_{NT}\left(\boldsymbol{\theta}^0,\boldsymbol{D}^0\right)&=\frac{1}{N}\sum_{i=1}^{N}\widehat{\nu}_{i,T}\left(\boldsymbol{\theta}^0\right)'\boldsymbol{W}_i\widehat{\nu}_{i,T}\left(\boldsymbol{\theta}^0\right)+\widehat{\nu}_{i,T}\left(\boldsymbol{\theta}^0\right)'\left(\boldsymbol{W}_{i,NT}-\boldsymbol{W}_i\right)\widehat{\nu}_{i,T}\left(\boldsymbol{\theta}^0\right)\\ &\le\frac{1}{N}\sum_{i=1}^{N}\left(\sigma_{\max}\left(\boldsymbol{W}_i\right)+\|\boldsymbol{W}_{i,NT}-\boldsymbol{W}_i\|\right)\left\|\widehat{\nu}_{i,T}\left(\boldsymbol{\theta}^0\right)\right\|^2\end{aligned}$$

$$
\begin{aligned}
&\leq \left(C_w + \max_{1\leq i\leq N} \|\boldsymbol{W}_{i,NT} - \boldsymbol{W}_i\|\right)\left(\max_{1\leq i\leq N} \left\|\widehat{\nu}_{i,T}\left(\boldsymbol{\theta}^0\right)\right\|\right)^2 \\
&= \left(C_w + o_p\left(1\right)\right) o_p\left(1\right) = o_p\left(1\right). \qquad \text{(A.3)}
\end{aligned}
$$

The second term on R.H.S. of (A.2) due to the fused Lasso penalty is bounded by

$$
\begin{aligned}
&\frac{\psi_f}{N^2}\sum_{1\leq i<j\leq N} \dot{\mu}_{ij}\left(\left\|\boldsymbol{\delta}_i^0 - \boldsymbol{\delta}_j^0\right\| - \left\|\widehat{\boldsymbol{\delta}}_i - \widehat{\boldsymbol{\delta}}_j\right\|\right) \\
=&\frac{\psi_f}{N^2}\left[-\sum_{(i,j)\in\mathcal{Z}_0} \dot{\mu}_{ij}\left\|\widehat{\boldsymbol{\delta}}_i - \widehat{\boldsymbol{\delta}}_j\right\| + \sum_{(i,j)\in\mathcal{Z}_1} \dot{\mu}_{ij}\left(\left\|\boldsymbol{\delta}_i^0 - \boldsymbol{\delta}_j^0\right\| - \left\|\widehat{\boldsymbol{\delta}}_i - \widehat{\boldsymbol{\delta}}_j\right\|\right)\right] \\
\leq&\frac{\psi_f}{N^2}\sum_{(i,j)\in\mathcal{Z}_1} \dot{\mu}_{ij}\left(\left\|\widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0\right\| + \left\|\widehat{\boldsymbol{\delta}}_j - \boldsymbol{\delta}_j^0\right\|\right) \\
\leq&\psi_f\left(\max_{(i,j)\in\mathcal{Z}_1} \dot{\mu}_{ij}\right)\left(\frac{1}{N}\sum_{i=1}^N \left\|\widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0\right\|\right) \qquad \text{(A.4)} \\
\leq&\mathcal{O}_p\left(\psi_f \rho_{NT}^{-\kappa_f}\sqrt{L_{\mathcal{D}}}\right) = o_p(1)
\end{aligned}
$$

where the third line follows from (reverse) triangle inequality, and the last line is due to Lemma A.1(iii) and Assumption 2(ii) and Assumption 3(ii).

It follows from triangle inequality, Assumption 2(ii) and 3(ii) and $\delta_{il}^0 = 0$ for $l \in \mathcal{I}_i$ that

$$
\begin{aligned}
\psi_1 \max_i \sum_{l=1}^{L_{\mathcal{D}}} \dot{w}_{il}\left(\left|\delta_{il}^0\right| - \left|\widehat{\delta}_{il}\right|\right) \leq&\psi_1 \max_i \left\{\sum_{l\in\mathcal{I}_i} \dot{w}_{il}\left(\left|\delta_{il}^0\right| - \left|\widehat{\delta}_{il}\right|\right) - \psi_1 \sum_{l\in\mathcal{V}_i} \dot{w}_{il}\left|\widehat{\delta}_{il}\right|\right\} \\
\leq&\psi_1 \max_i \sum_{l\in\mathcal{I}_i} \dot{w}_{il}\left|\delta_{il}^0 - \widehat{\delta}_{il}\right| \lesssim \psi_1 \max_i \sum_{l\in\mathcal{I}_i} \dot{w}_{il} \\
\leq&\psi_1 \max_i \left\|\dot{w}_{i,\mathcal{I}_i}\right\| = o_p(1), \qquad \text{(A.5)}
\end{aligned}
$$

by Lemma A.1(iv) and Assumption 3(ii), which implies the third term of R.H.S. of (A.2) is of order $o_p(1)$.

Therefore,

$$
\widehat{Q}_{NT}\left(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{D}}\right) = \frac{1}{N}\sum_{i=1}^N \widehat{g}_{i,T}\left(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\delta}}_i\right)' \boldsymbol{W}_{i,T}\widehat{g}_{i,T}\left(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\delta}}_i\right) = o_p(1).
$$

On the other side,

$$\widehat{Q}_{NT}\left(\widehat{\boldsymbol{\theta}},\widehat{\boldsymbol{D}}\right)\geq\left(\min_i\sigma_{\min}\left(\boldsymbol{W}_i\right)\right)\left(\frac{1}{N}\sum_{i=1}^{N}\left\|\widehat{g}_{i,T}\left(\widehat{\boldsymbol{\theta}},\widehat{\boldsymbol{\delta}}_i\right)\right\|^2\right)+o_p(1)$$
$$\geq c_w\left(\frac{1}{N}\sum_{i=1}^{N}\left\|\widehat{g}_{i,T}\left(\widehat{\boldsymbol{\theta}},\widehat{\boldsymbol{\delta}}_i\right)\right\|^2\right)+o_p(1),$$

which implies

$$\frac{1}{N}\sum_{i=1}^{N}\left\|\widehat{g}_{i,T}\left(\widehat{\boldsymbol{\theta}},\widehat{\boldsymbol{\delta}}_i\right)\right\|^2=o_p(1). \tag{A.6}$$

Note that

$$\left\|\widehat{g}_{i,T}\left(\widehat{\boldsymbol{\theta}},\widehat{\boldsymbol{\delta}}_i\right)\right\|^2=\left\|\widehat{m}_{\mathcal{S},i,T}\left(\widehat{\boldsymbol{\theta}}\right)\right\|^2+\left\|\widehat{m}_{\mathcal{D},i,T}\left(\widehat{\boldsymbol{\theta}}\right)-\widehat{\boldsymbol{\delta}}_i\right\|^2\geq\left\|\widehat{m}_{\mathcal{S},i,T}\left(\widehat{\boldsymbol{\theta}}\right)\right\|^2, \tag{A.7}$$

and by triangle inequality,

$$\left\|\widehat{m}_{\mathcal{S},i,T}\left(\widehat{\boldsymbol{\theta}}\right)\right\|\geq\left|\left\|\overline{m}_{\mathcal{S},i}\left(\widehat{\boldsymbol{\theta}}\right)\right\|-\left\|\widehat{m}_{\mathcal{S},i,T}\left(\widehat{\boldsymbol{\theta}}\right)-\overline{m}_{\mathcal{S},i}\left(\widehat{\boldsymbol{\theta}}\right)\right\|\right|. \tag{A.8}$$

Combine (A.6), (A.7) and (A.8), we have

$$\begin{aligned}o_p(1)=&\frac{1}{N}\sum_{i=1}^{N}\left\|\widehat{g}_{i,T}\left(\widehat{\boldsymbol{\theta}},\widehat{\boldsymbol{\delta}}_i\right)\right\|^2\\ \geq&\min_i\left\|\overline{m}_{\mathcal{S},i}\left(\widehat{\boldsymbol{\theta}}\right)\right\|^2-2\left(\max_i\left\|\overline{m}_{\mathcal{S},i}\left(\widehat{\boldsymbol{\theta}}\right)\right\|\right)\left(\max_i\left\|\widehat{m}_{\mathcal{S},i,T}\left(\widehat{\boldsymbol{\theta}}\right)-\overline{m}_{\mathcal{S},i}\left(\widehat{\boldsymbol{\theta}}\right)\right\|\right)\\ =&\min_i\left\|\overline{m}_{\mathcal{S},i}\left(\widehat{\boldsymbol{\theta}}\right)\right\|^2-2O_p\left(1\right)o_p(1)\\ =&\min_i\left\|\overline{m}_{\mathcal{S},i}\left(\widehat{\boldsymbol{\theta}}\right)\right\|^2+o_p(1),\end{aligned}$$

which follows from Assumption 1(ii) and Lemma A.1(i). Then, $\min_i\left\|\overline{m}_{\mathcal{S},i}\left(\widehat{\boldsymbol{\theta}}\right)\right\|=o_p(1)$. By identification of $\boldsymbol{\theta}^0$ imposed by Assumption 2(i), we reach the desired result $\widehat{\boldsymbol{\theta}}\xrightarrow{p}\boldsymbol{\theta}^0$. □

Lemma A.3 (Lemma B.1 in Su et al. (2016))**.** *Suppose Assumption 2(iv) holds, then*

$$\Pr\left(c_Q\left(\frac{1}{2}\overline{Q}_i\left(\boldsymbol{\theta},\boldsymbol{\delta}_i\right)-\widehat{R}_{i,T}\left(\boldsymbol{\theta}\right)\right)\leq\widehat{Q}_{i,T}\left(\boldsymbol{\theta},\boldsymbol{\delta}_i\right)\leq C_Q\left(2\overline{Q}_i\left(\boldsymbol{\theta},\boldsymbol{\delta}_i\right)+2\widehat{R}_{i,T}\left(\boldsymbol{\theta}\right)\right)\right)=1-o\left(1\right)$$

for $\boldsymbol{\theta}\in\Theta$, $\boldsymbol{\theta}_i\in\Theta_\delta$, where c_Q and C_Q are positive constants with $0<c_Q<1<C_Q<\infty$.

Proof of Theorem 1. For simplicity, we denote

$$a_{NT} = \psi_f \left(\max_{(i,j)\in\mathcal{Z}_1} \dot{\mu}_{ij} \right) \text{ and } b_{NT} = \psi_1 \max_{1\le i\le N} \|\dot{\boldsymbol{w}}_{i,\mathcal{I}_i}\|,$$

where $\dot{\boldsymbol{w}}_{i,\mathcal{I}_i}$ is the subvector of $\dot{\boldsymbol{w}}_i$ with element w_{il}, for $l \in \mathcal{I}_i = \{l \in [L_\mathcal{D}] : \delta_{il}^0 \neq 0\}$. By Lemma A.1(iii) - (iv) and the rate condition Assumption 3(ii), we have $a_{NT} = \mathcal{O}_p(\tau_T)$ and $b_{NT} = \mathcal{O}_p(\tau_T)$.

Part (i). The proof is again starting with (A.2). From (A.5) and Cauchy Schwartz inequality, we have

$$\psi_1 \sum_{l=1}^{L_\mathcal{D}} \dot{w}_{il} \left(|\delta_{il}^0| - \left|\widehat{\delta}_{il}\right| \right) \le \psi_1 \|\dot{\boldsymbol{w}}_{i,\mathcal{I}_i}\| \left\| \widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0 \right\| \le b_{NT} \left\| \widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0 \right\|. \tag{A.9}$$

From (A.2), (A.4) and (A.9), we have

$$\frac{1}{N} \sum_{i=1}^{N} \left\{ \widehat{Q}_{i,NT}\left(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\delta}}_i\right) - \widehat{Q}_{i,NT}\left(\widehat{\boldsymbol{\theta}}, \boldsymbol{\delta}_i^0\right) - (a_{NT} + b_{NT}) \left\| \widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0 \right\| \right\} \le 0 \tag{A.10}$$

By Lemma A.3, w.p.a.1,

$$\begin{aligned} &\widehat{Q}_{i,NT}\left(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\delta}}_i\right) - \widehat{Q}_{i,NT}\left(\widehat{\boldsymbol{\theta}}, \boldsymbol{\delta}_i^0\right) \\ &\qquad \ge \frac{c_Q}{2} \overline{Q}_i\left(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\delta}}_i\right) - 2C_Q \overline{Q}_i\left(\widehat{\boldsymbol{\theta}}, \boldsymbol{\delta}_i^0\right) - (c_Q + 2C_Q) \widehat{R}_{i,T}\left(\widehat{\boldsymbol{\theta}}, \boldsymbol{\delta}_i^0\right). \end{aligned} \tag{A.11}$$

With sufficiently large (N, T),

$$\widehat{R}_{i,T}\left(\widehat{\boldsymbol{\theta}}\right) = \widehat{\nu}_{i,T}\left(\widehat{\boldsymbol{\theta}}\right)' W_i \widehat{\nu}_{i,T}\left(\widehat{\boldsymbol{\theta}}\right) \le C_w \left\| \widehat{\nu}_{i,T}\left(\widehat{\boldsymbol{\theta}}\right) \right\|^2 = \mathcal{O}_p\left(\tau_T^2\right), \tag{A.12}$$

where the first inequality follows from Assumption 2(iv) and the last equality is due to Assumption 2(iii) and consistency of $\widehat{\boldsymbol{\theta}}$ shown in Lemma A.2.

Similarly, with sufficiently large (N, T), by first-order Taylor expansion, Assumption 2(iv) and (v) and Lemma A.2,

$$\overline{Q}_i\left(\widehat{\boldsymbol{\theta}}, \boldsymbol{\delta}_i^0\right) \le C_w \left\| \overline{g}_i\left(\widehat{\boldsymbol{\theta}}, \boldsymbol{\delta}_i^0\right) \right\|^2 = C_w \left\| \Gamma_i\left(\widetilde{\boldsymbol{\theta}}\right) \begin{bmatrix} \widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0 \\ \mathbf{0}_{L_\mathcal{D}} \end{bmatrix} \right\|^2 \le C_w C_\Gamma \left\| \widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0 \right\|^2, \tag{A.13}$$

$$\overline{Q}_i\left(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\delta}}_i\right) \ge c_w \left\| \overline{g}_i\left(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\delta}}_i\right) \right\|^2 = c_w \left\| \Gamma_i\left(\overline{\boldsymbol{\theta}}\right) \begin{bmatrix} \widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0 \\ \widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0 \end{bmatrix} \right\|^2 \ge c_w c_\Gamma \left(\left\| \widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0 \right\|^2 + \left\| \widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0 \right\|^2 \right), \tag{A.14}$$

where $\widetilde{\boldsymbol{\theta}}$, $\overline{\boldsymbol{\theta}}$ are between $\boldsymbol{\theta}^0$ and $\widehat{\boldsymbol{\theta}}$.

Combine (A.10), (A.11), (A.4), (A.9), (A.12), (A.13) and (A.14), we have

$$\frac{1}{N}\sum_{i=1}^{N}\left\{\left\|\widehat{\boldsymbol{\delta}}_i-\boldsymbol{\delta}_i^0\right\|^2-(a_{NT}+b_{NT})\left\|\widehat{\boldsymbol{\delta}}_i-\boldsymbol{\delta}_i^0\right\|-\mathcal{O}_p\left(\tau_T^2\right)-\left\|\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^0\right\|^2\right\}\lesssim 0, \tag{A.15}$$

which, together with Lemma A.1(ii) and Assumption 3(ii), implies

$$\left\|\widehat{\boldsymbol{\delta}}_i-\boldsymbol{\delta}_i^0\right\|\lesssim\mathcal{O}_p\left(\tau_T+a_{NT}+b_{NT}\right)=\mathcal{O}_p\left(\tau_T\right). \tag{A.16}$$

Note that with sufficiently large (N,T) and consistency of $\widehat{\boldsymbol{\theta}}$, (A.13) implies

$$\overline{Q}_i\left(\widehat{\boldsymbol{\theta}},\boldsymbol{\delta}^0\right)\lesssim\left\|\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^0\right\|,$$

which further implies a twist of (A.15),

$$\frac{1}{N}\sum_{i=1}^{N}\left\{\left\|\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^0\right\|^2-\left\|\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^0\right\|+\left\|\widehat{\boldsymbol{\delta}}_i-\boldsymbol{\delta}_i^0\right\|^2-(a_{NT}+b_{NT})\left\|\widehat{\boldsymbol{\delta}}_i-\boldsymbol{\delta}_i^0\right\|-\mathcal{O}_p\left(\tau_T^2\right)\right\}\lesssim 0.$$

Plug in the rate of $\widehat{\boldsymbol{\delta}}_i$ in (A.16), we have

$$\left\|\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^0\right\|=\mathcal{O}_p\left(\tau_T\right) \tag{A.17}$$

which completes the proof of part (i).

Part (ii). Let $\widehat{\boldsymbol{D}}=\boldsymbol{D}^0+\tau_T\widehat{\boldsymbol{V}}$ where $\widehat{\boldsymbol{V}}=(\widehat{\boldsymbol{v}}_1,\ldots,\widehat{\boldsymbol{v}}_N)\in\mathbb{R}^{L_{\mathcal{D}}\times N}$. Note that

$$\frac{1}{N}\sum_{i=1}^{N}\left\|\widehat{\boldsymbol{\delta}}_i-\boldsymbol{\delta}^0\right\|^2=\tau_T^2\left(\frac{1}{N}\sum_{i=1}^{N}\left\|\widehat{\boldsymbol{v}}_i\right\|^2\right),$$

and we want to show that

$$\frac{1}{N}\sum_{i=1}^{N}\left\|\widehat{\boldsymbol{v}}_i\right\|^2=\mathcal{O}_p\left(1\right). \tag{A.18}$$

For any $\widetilde{\boldsymbol{D}}=\boldsymbol{D}^0+\tau_T\boldsymbol{V}$ with $\left(N^{-1}\sum_{i=1}^{N}\left\|\boldsymbol{v}_i\right\|^2\right)^{-1}=o_p\left(1\right)$,

$$\begin{aligned}&\tau_T^{-2}\left[\left(\widehat{Q}_{NT}\left(\widehat{\boldsymbol{\theta}},\widetilde{\boldsymbol{D}}\right)+P_{\psi_1,\psi_f}\left(\widetilde{\boldsymbol{D}}\right)\right)-\left(\widehat{Q}_{NT}\left(\boldsymbol{\theta}^0,\boldsymbol{D}^0\right)+P_{\psi_1,\psi_f}\left(\boldsymbol{D}^0\right)\right)\right]\\ =&\tau_T^{-2}\left(\widehat{Q}_{NT}\left(\widehat{\boldsymbol{\theta}},\widetilde{\boldsymbol{D}}\right)-\widehat{Q}_{NT}\left(\boldsymbol{\theta}^0,\boldsymbol{D}^0\right)\right)+\tau_T^{-2}\left[\frac{\psi_f}{N^2}\sum_{1\le i<j\le N}\dot{\mu}_{ij}\left(\left\|\boldsymbol{\delta}_i^0+\tau_T\boldsymbol{v}_i-\boldsymbol{\delta}_j^0-\tau_T\boldsymbol{v}_j\right\|-\left\|\boldsymbol{\delta}_i^0-\boldsymbol{\delta}_j^0\right\|\right)\right]\end{aligned}$$

$$
+ \tau_T^{-2} \left[\frac{\psi_1}{N} \sum_{i=1}^{N} \sum_{l=1}^{L_{\mathcal{D}}} \dot{w}_{il} \left(\left| \delta_{il}^0 + v_{il} \right| - \left| \delta_{il}^0 \right| \right) \right]
$$

$$
:= \Xi_{1,NT} + \Xi_{2,NT} + \Xi_{3,NT}. \tag{A.19}
$$

The first term in (A.19) can be bounded w.p.a.1

$$
\begin{aligned}
\Xi_{1,NT} \geq & \tau_T^{-2} \left(\frac{c_Q}{2N} \sum_{i=1}^{N} \overline{Q}_i \left(\widehat{\boldsymbol{\theta}}, \boldsymbol{\delta}_i^0 + \tau_T \boldsymbol{v}_i \right) - \frac{1}{N} \sum_{i=1}^{N} \left(\widehat{R}_{i,T} \left(\widehat{\boldsymbol{\theta}} \right) + \widehat{R}_{i,T} \left(\boldsymbol{\theta}^0 \right) \right) \right) \\
\geq & \frac{c_Q c_w c_\Gamma}{2N} \sum_{i=1}^{N} \left(\tau_T^{-2} \left\| \widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0 \right\|^2 + \left\| \boldsymbol{v}_i \right\|^2 - \mathcal{O}_p(1) \right) \\
\gtrsim & \frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{v}_i \right\|^2 + \mathcal{O}_p(1),
\end{aligned} \tag{A.20}
$$

where the first inequality follows from Lemma A.3 and $\widehat{Q}_{NT} \left(\boldsymbol{\theta}^0, \boldsymbol{D}^0 \right) = N^{-1} \sum_{i=1}^{N} \widehat{R}_{i,T} \left(\boldsymbol{\theta}^0 \right)$, the second inequality is due to (A.12) and (A.14) and the last line holds with the result in part (i).

By (A.4), Assumption 3(ii) and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\Xi_{2,NT} \geq & - \tau_T^{-2} \left[\frac{\psi_f}{N^2} \sum_{1 \leq i < j \leq N} \dot{\mu}_{ij} \left(\left\| \boldsymbol{\delta}_i^0 - \boldsymbol{\delta}_j^0 \right\| - \left\| \boldsymbol{\delta}_i^0 + \tau_T \boldsymbol{v}_i - \boldsymbol{\delta}_j^0 - \tau_T \boldsymbol{v}_j \right\| \right) \right] \\
\gtrsim & - \tau_T^{-1} \psi_f \left(\max_{(i,j) \in \mathcal{Z}_1} \dot{\mu}_{ij} \right) \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{v}_i \right\| \right) \gtrsim -\mathcal{O}\left(1\right) \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{v}_i \right\|^2 \right)^{\frac{1}{2}}.
\end{aligned} \tag{A.21}
$$

By (A.9), Assumption 3(ii) and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\Xi_{3,NT} = & - \tau_T^{-2} \frac{\psi_1}{N} \sum_{i=1}^{N} \sum_{l=1}^{L_{\mathcal{D}}} \dot{w}_{il} \left(\left| \delta_{il}^0 \right| - \left| \widetilde{\delta}_{il} \right| \right) \\
\gtrsim & - \tau_T^{-1} b_{NT} \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{v}_i \right\| \right) \gtrsim -\mathcal{O}\left(1\right) \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{v}_i \right\|^2 \right)^{\frac{1}{2}}.
\end{aligned} \tag{A.22}
$$

Combine (A.19), (A.20) and (A.21), for sufficiently large (N, T),

$$
\begin{aligned}
& \tau_T^{-2} \left[\left(\widehat{Q}_{NT} \left(\widehat{\boldsymbol{\theta}}, \widetilde{\boldsymbol{D}} \right) + P_{\psi_1, \psi_f} \left(\widetilde{\boldsymbol{D}} \right) \right) - \left(\widehat{Q}_{NT} \left(\boldsymbol{\theta}^0, \boldsymbol{D}^0 \right) + P_{\psi_1, \psi_f} \left(\boldsymbol{D}^0 \right) \right) \right] \\
\gtrsim & \frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{v}_i \right\|^2 - \mathcal{O}\left(1\right) \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{v}_i \right\|^2 \right)^{\frac{1}{2}} + \mathcal{O}_p(1)
\end{aligned}
$$

$$>0 \tag{A.23}$$

w.p.a.1 since $N^{-1}\sum_{i=1}^{N}\|\boldsymbol{v}_i\|^2$ is diverging, which implies $\left\{\widehat{\boldsymbol{\theta}}, \widetilde{\boldsymbol{D}}\right\}$ does not minimize (2.8) and hence (A.18) holds, which completes the proof of part (ii). □

Lemma A.4. *Suppose the conditions in Lemma A.1 and Assumption 4(ii) hold, then*

(i) $\psi_f^{-1}\tau_T\sqrt{L_{\mathcal{D}}}\max_{(i,j)\in\mathcal{Z}_0}\dot{\mu}_{ij}^{-1} = o_p(1)$.

(ii) $\psi_1^{-1}\tau_T\max_i\max_{l\in\mathcal{V}_i}\dot{w}_{il}^{-1} = o_p(1)$.

Proof of Lemma A.4.

$$\begin{aligned}
\psi_f^{-1}\tau_T\sqrt{L_{\mathcal{D}}}\max_{(i,j)\in\mathcal{Z}_0}\dot{\mu}_{ij}^{-1} &= \psi_f^{-1}\tau_T\sqrt{L_{\mathcal{D}}}\max_{(i,j)\in\mathcal{Z}_0}\left\|\dot{\boldsymbol{\delta}}_i - \dot{\boldsymbol{\delta}}_j\right\|^{\kappa_f}\\
&\leq \psi_f^{-1}\tau_T\sqrt{L_{\mathcal{D}}}\max_{(i,j)\in\mathcal{Z}_0}\left\{\left\|\dot{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0\right\| + \left\|\dot{\boldsymbol{\delta}}_j - \boldsymbol{\delta}_j^0\right\|\right\}^{\kappa_f}\\
&\lesssim \psi_f^{-1}\tau_T\sqrt{L_{\mathcal{D}}}\max_i\left\|\dot{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0\right\|^{\kappa_f}\\
&\leq \psi_f^{-1}\tau_T\sqrt{L_{\mathcal{D}}}\mathcal{O}_p\left(\widetilde{\varkappa}_{NT}^{\kappa_f}\right)\\
&= o_p(1),
\end{aligned} \tag{A.24}$$

where we apply triangle inequality in the second line, the fourth line invokes Lemma A.1(ii) and the last line follows from Assumption 4(ii), and (A.24) implies (i).

$$\begin{aligned}
\psi_1^{-2}\tau_T^2\max_i\max_{l\in\mathcal{V}_i}\dot{w}_{il}^{-2} = \psi_1^{-2}\tau_T^2\max_i\max_{l\in\mathcal{V}_i}\dot{\delta}_{il}^{2\kappa_1} &\leq \psi_1^{-2}\tau_T^2\max_i\left\|\dot{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0\right\|^{2\kappa_1}\\
&\leq \psi_1^{-2}\tau_T^2\mathcal{O}_p\left(\widetilde{\varkappa}_{NT}^{2\kappa_1}\right)\\
&= o_p(1),
\end{aligned} \tag{A.25}$$

where the second line invokes Lemma A.1(ii) and the last line follows from Assumption 4(ii), and (A.25) implies (ii). □

Proof of Theorem 2. In the proof, we first show the first statement in (ii) and (iii). Then we can leverage the results to show the uniform consistency of $\widehat{\boldsymbol{\delta}}_i$ in part (i), and the second statement in (ii) and (iii) directly follow.

Rewrite the objective function (2.8), with notation $\widehat{\Psi}_{NT}$, as

$$\widehat{\Psi}_{NT}(\boldsymbol{\theta},\boldsymbol{D}) = \frac{1}{N}\sum_{i=1}^{N}\left\{\widehat{g}_{i,T}(\boldsymbol{\theta},\boldsymbol{\delta}_i)'\boldsymbol{W}_{i,NT}\widehat{g}_{i,T}(\boldsymbol{\theta},\boldsymbol{\delta}_i) + \frac{\psi_f}{2N}\sum_{j=1}^{N}\dot{\mu}_{ij}\|\boldsymbol{\delta}_i-\boldsymbol{\delta}_j\| + \psi_1\sum_{l=1}^{L_{\mathcal{D}}}\dot{w}_{il}|\delta_{il}|\right\}. \tag{A.26}$$

The Karush-Kuhn-Tucker (KKT) condition (with respect to δ_{il}) evaluated at $\left(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{D}}\right)$, scaled up by τ_T^{-1}, is

$$\begin{aligned} 0 &= 2\tau_T^{-1}\boldsymbol{\gamma}_{L_{\mathcal{S}}+l}' \boldsymbol{W}_{i,NT}\widehat{g}_{i,T}\left(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\delta}}_i\right) + \frac{\psi_f \tau_T^{-1}}{2N}\sum_{j\notin\mathcal{G}_{k(i)}} \dot{\mu}_{ij}\widehat{e}_{ij,l} + \frac{\psi_f \tau_T^{-1}}{2N}\sum_{j\in\mathcal{G}_{k(i)}} \dot{\mu}_{ij}\widehat{e}_{ij,l} + \psi_1\tau_T^{-1}\dot{w}_{il}\widehat{s}_{il} \\ &:= \Xi_{il,m} + \Xi_{il,\mathcal{Z}_1} + \Xi_{il,\mathcal{Z}_0} + \Xi_{il,1}, \qquad \text{(A.27)} \end{aligned}$$

where $\widehat{\boldsymbol{e}}_{ij} = \frac{\widehat{\boldsymbol{\delta}}_i - \widehat{\boldsymbol{\delta}}_j}{\|\widehat{\boldsymbol{\delta}}_i - \widehat{\boldsymbol{\delta}}_j\|}$ if $\left\|\widehat{\boldsymbol{\delta}}_i - \widehat{\boldsymbol{\delta}}_j\right\| \neq 0$ and $\|\widehat{\boldsymbol{e}}_{ij}\| \leq 1$ otherwise; $\widehat{s}_{il} = \operatorname{sgn}\left(\widehat{\delta}_{il}\right)$ if $\widehat{\delta}_{il} \neq 0$ and $\widehat{s}_{il} \in [-1, 1]$ if $\widehat{\delta}_{il} = 0$; $\boldsymbol{\gamma}_{L_{\mathcal{S}}+l} \in \mathbb{R}^L$ is the vector with $(L_{\mathcal{S}} + l)$-th element equal to 1 and others being 0.

In the KKT condition (A.27):

$$\begin{aligned} |\Xi_{il,m}| &\leq 2\tau_T^{-1}\left(\|\boldsymbol{W}_i\| + \|\boldsymbol{W}_{i,NT} - \boldsymbol{W}_i\|\right)\left\|\widehat{g}_{i,T}\left(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\delta}}_i\right)\right\| \\ &\leq 2\tau_T^{-1}\left(C_w + o_p(1)\right)\left(C_\Gamma\left(\left\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0\right\|^2 + \left\|\widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0\right\|^2\right)\right)^{\frac{1}{2}} \\ &\leq 2\tau_T^{-1}\left(C_w + o_p(1)\right)\mathcal{O}_p(\tau_T) \\ &= \mathcal{O}_p(1), \qquad \text{(A.28)} \end{aligned}$$

where we apply Cauchy-Schwartz inequality in the first inequality, Assumption 2(iv) and (v) as in (A.14) for the second inequality, and make use of the convergence rate derived in Theorem 1(i) to reach the result that $\Xi_{il,m}$ is stochastically bounded.

$$|\Xi_{il,\mathcal{Z}_1}| \leq \frac{\psi_f \tau_T^{-1}}{2N}\left(\max_{(i,j)\in\mathcal{Z}_1}\dot{\mu}_{ij}\right)\left|\sum_{j\notin\mathcal{G}_{k(i)}}\widehat{e}_{ij,l}\right| \leq \psi_f\tau_T^{-1}\left(\max_{(i,j)\in\mathcal{Z}_1}\dot{\mu}_{ij}\right)\frac{N - N_{k(i)}}{2N} \leq \mathcal{O}_p(1), \qquad \text{(A.29)}$$

which is due to Assumption 3(ii) and Lemma A.1(iii).

To facilitate the analysis of $\Xi_{il,\mathcal{Z}_0}$ and $\Xi_{il,1}$, we introduce the following notations. Let $c_e \in (0, \frac{1}{3})$ be a constant. Denote

$$\widehat{\mathcal{Z}}_{i,0} = \left\{j \in \mathcal{G}_{k(i)} : \left\|\widehat{\boldsymbol{\delta}}_i - \widehat{\boldsymbol{\delta}}_j\right\| = 0\right\} \text{ and } \widehat{\mathcal{Z}}_{i,1} = \left\{j \in \mathcal{G}_{k(i)} : \left\|\widehat{\boldsymbol{\delta}}_i - \widehat{\boldsymbol{\delta}}_j\right\| > 0\right\}.$$

Define the events $\mathcal{E}_{G,i} = \left\{\frac{|\widehat{\mathcal{Z}}_{i,1}|}{N_{K(i)}} > c_e\right\}$ and $\mathcal{E}_{S,i} = \left\{\max_{l\in\mathcal{V}_i}\left|\widehat{\delta}_{il}\right| > 0\right\}$.

Conditioning on $\mathcal{E}_{G,i}$, we have $\left|\left\{j \in \mathcal{G}_{k(i)} : \left|\widehat{\delta}_{il} - \widehat{\delta}_{jl}\right| > 0\right\}\right| > c_e \frac{N_{k(i)}}{\sqrt{L_{\mathcal{D}}}}$, and

$$\begin{aligned}|\Xi_{il,\mathcal{Z}_0}| =& \psi_f \tau_T^{-1} \left(\min_{(i,j)\in\mathcal{Z}_0} \dot{\mu}_{ij}\right) \left|\frac{1}{2N} \sum_{j\in\mathcal{G}_{k(i)}} \widetilde{\mu}_{ij} \widehat{e}_{ij,l}\right| \\ \gtrsim& \psi_f \tau_T^{-1} L_{\mathcal{D}}^{-\frac{1}{2}} \left(\min_{(i,j)\in\mathcal{Z}_0} \dot{\mu}_{ij}\right) \frac{c_e N_{k(i)}}{N} \xrightarrow{p} \infty, \end{aligned} \tag{A.30}$$

where $\widetilde{\mu}_{ij} := \frac{\dot{\mu}_{ij}}{\min_{(i,j)\in\mathcal{Z}_0} \dot{\mu}_{ij}} \geq 1$ for $(i,j) \in \mathcal{Z}_0$, the inequality holds since the system $\sum_{j\in\mathcal{G}_{k(i)}} \widetilde{\mu}_{ij}\widehat{e}_{ij,l} = 0$, with $\|\widehat{\boldsymbol{e}}_{ij}\| = 1$ if $\left\|\widehat{\boldsymbol{\delta}}_i - \widehat{\boldsymbol{\delta}}_j\right\| > 0$, for $i \in \mathcal{G}_k$ for some $k \in [K^0]$ and $l \in [L_{\mathcal{D}}]$, is over-determined in terms of $\widehat{e}_{ij,l}$ and does not vanish, and the probability limit follows from Assumption 4(iii) that $\lim_{N\to\infty} N_{k(i)}/N > \pi_{\min}$ and Lemma A.4(i). As a result, $|\Xi_{il,\mathcal{Z}_1}|$ is asymptotically explosive.

Conditional on $\mathcal{E}_{S,i}$, we have

$$|\Xi_{il,1}| \geq \tau_T^{-1}\psi_1 \min_i \min_{l\in\mathcal{V}_i} \dot{w}_{il} \xrightarrow{p} \infty \text{ if } l \in \mathcal{V}_i \text{ and } \left|\widehat{\delta}_{il}\right| > 0, \tag{A.31}$$

by Lemma A.4(ii).

Combine the KKT condition (A.27) and (A.28), (A.29), (A.30), (A.31), and triangle inequality, for each $i \in [N]$,

$$\Pr\left(\mathcal{E}_{G,i} \bigcup \mathcal{E}_{S,i}\right) \leq \Pr\left(||\Xi_{il,\mathcal{Z}_0}| - |\Xi_{il,1}|| \leq |\Xi_{il,\mathcal{Z}_1}| + |\Xi_{il,m}|, \mathcal{E}_{G,i} \bigcup \mathcal{E}_{S,i}\right) \to 0 \tag{A.32}$$

as $(N,T) \to \infty$, since $|\Xi_{il,m}| + |\Xi_{il,\mathcal{Z}_1}| = \mathcal{O}_p(1)$ while $||\Xi_{il,\mathcal{Z}_0}| - |\Xi_{il,1}|| \xrightarrow{p} \infty$ with suitable choice of κ_1 and κ_f that guarantees $\Xi_{il,\mathcal{Z}_0}$ and $\Xi_{il,1}$ do not coincide conditional on $\mathcal{E}_{G,i} \bigcap \mathcal{E}_{S,i}$.

Then we turn to the desired uniform results. Denote the event

$$\mathcal{E}_G = \left\{\max_{(i,j)\in\mathcal{Z}_0} \left\|\widehat{\boldsymbol{\delta}}_i - \widehat{\boldsymbol{\delta}}_j\right\| > 0\right\} = \left\{\exists\,(i^\star, j^\star) \in \mathcal{Z}_0 \text{ s.t. } \left\|\widehat{\boldsymbol{\delta}}_{i^\star} - \widehat{\boldsymbol{\delta}}_{j^\star}\right\| > 0\right\}.$$

By (A.32), we have

$$\begin{aligned}\Pr(\mathcal{E}_G) <& \Pr\left(\mathcal{E}_G \bigcap \left\{\frac{\left|\widehat{\mathcal{Z}}_{i^\star,0}\right|}{N_{k(i^\star)}} > c_e\right\}\right) + \Pr\left(\mathcal{E}_G \bigcap \left\{\frac{\left|\widehat{\mathcal{Z}}_{j^\star,0}\right|}{N_{k(j^\star)}} > c_e\right\}\right) \\ &+ \Pr\left(\mathcal{E}_G \bigcap \left\{\frac{\left|\mathcal{G}_k \setminus \left(\widehat{\mathcal{Z}}_{i^\star,0} \bigcup \widehat{\mathcal{Z}}_{j^\star,0}\right)\right|}{N_{k(i^\star)}} \geq 1 - 2c_e\right\}\right) \\ \leq& \Pr(\mathcal{E}_{G,j^\star}) + 2\Pr(\mathcal{E}_{G,i^\star}) \to 0, \end{aligned} \tag{A.33}$$

as $(N,T)\to\infty$, which completes the proof of the first statement in part (ii).

Conditioning on $\left\{\bigcup_{i=1}^{N}\mathcal{E}_{S,i}\right\}$, there exist $i^{\star}\in[N]$ such that $\max_{l\in\mathcal{V}_{i^\star}}\left|\widehat{\delta}_{i^\star l}\right|>0$. Note that (A.31) holds uniformly across i, and the above arguments can go through for $i^\star$, which leads

$$\Pr\left(\max_i\max_{l\in\mathcal{V}_i}\left|\widehat{\delta}_{il}\right|>0\right)\to 0$$

as $(N,T)\to\infty$ and the first statement in part (iii) is established.

Next, we turn to the proof of the uniform convergence result in part (i). Note that it suffices to show

$$\Pr\left(\max_i\left\|\widehat{\boldsymbol{\delta}}_i-\boldsymbol{\delta}_i^0\right\|>\varkappa_{NT}\,\middle|\,\max_{(i,j)\in\mathcal{Z}_0}\left\|\widehat{\boldsymbol{\delta}}_i-\widehat{\boldsymbol{\delta}}_j\right\|=0\right)=o\left(1\right), \tag{A.34}$$

given that $\Pr\left(\max_{(i,j)\in\mathcal{Z}_0}\left\|\widehat{\boldsymbol{\delta}}_i-\widehat{\boldsymbol{\delta}}_j\right\|=0\right)\to 1$ as $(N,T)\to\infty$. Rewrite that the penalized GMM objective in (A.26) as

$$\widehat{\Psi}_{NT}\left(\boldsymbol{\theta},\boldsymbol{D}\right)=\widehat{Q}_{NT}\left(\boldsymbol{\theta},\boldsymbol{D}\right)+\frac{1}{N}\sum_{i=1}^{N}\frac{\psi_f}{2N}\sum_{j=1}^{N}\dot{\mu}_{ij}\left\|\boldsymbol{\delta}_i-\boldsymbol{\delta}_j\right\|+\frac{\psi_1}{N}\sum_{i=1}^{N}\sum_{l=1}^{L_{\mathcal{D}}}\dot{w}_{il}\left|\delta_{il}\right|, \tag{A.35}$$

and denote

$$\Pi_{NT}=\left\{\boldsymbol{D}\in\Theta_\delta^N:\max_{1\le i\le N}\left\|\boldsymbol{\delta}_i-\boldsymbol{\delta}_i^0\right\|>\varkappa_{NT},\text{ and }\max_{(i,j)\in\mathcal{Z}_0}\left\|\boldsymbol{\delta}_i-\boldsymbol{\delta}_j\right\|=0\right\}.$$

It is then desired to show that, w.p.a.1,

$$\inf_{\boldsymbol{D}\in\Pi_{NT}}\widehat{\Psi}_{NT}\left(\widehat{\boldsymbol{\theta}},\boldsymbol{D}\right)>\widehat{\Psi}_{NT}\left(\boldsymbol{\theta}^0,\boldsymbol{D}^0\right), \tag{A.36}$$

which implies that the minimizer $\widehat{\boldsymbol{D}}\notin\Pi_{NT}$ w.p.a.1.

We first establish an upper bound for $\widehat{\Psi}_{NT}\left(\boldsymbol{\theta}^0,\boldsymbol{D}^0\right)$. (A.3) together with Lemma A.1(i) implies

$$\widehat{Q}_{NT}\left(\boldsymbol{\theta}^0,\boldsymbol{D}^0\right)=\mathcal{O}_p\left(\widetilde{\varkappa}_{NT}^2\right). \tag{A.37}$$

Following the similar arguments as in (A.4) and (A.5), together with Lemma A.1(iii)-(iv) and Assumption 4(i), we have

$$\frac{1}{N}\sum_{i=1}^{N}\frac{\psi_f}{2N}\sum_{j=1}^{N}\dot{\mu}_{ij}\left\|\boldsymbol{\delta}_i^0-\boldsymbol{\delta}_j^0\right\| \lesssim \max_{1\le i\le N}\frac{\psi_f}{N}\sum_{j\notin\mathcal{G}_{k(i)}}\dot{\mu}_{ij}\left\|\boldsymbol{\delta}_i^0-\boldsymbol{\delta}_j^0\right\|$$

$$\le \psi_f\sqrt{L_{\mathcal{D}}}\left(\max_{(i,j)\in Z_1}\dot{\mu}_{ij}\right) = \psi_f\sqrt{L_{\mathcal{D}}}\mathcal{O}_p\left(\rho_{NT}^{-\kappa_f}\right) = \mathcal{O}_p\left(\widetilde{\varkappa}_{NT}^2\right), \tag{A.38}$$

$$\frac{\psi_1}{N}\sum_{i=1}^{N}\sum_{l=1}^{L_{\mathcal{D}}}\dot{w}_{il}\left|\delta_{il}^0\right| \le \psi_1\max_{1\le i\le N}\left\|\dot{\boldsymbol{w}}_{i,\mathcal{I}_i}\right\| = \psi_1\sqrt{L_{\mathcal{D}}}\mathcal{O}_p\left(\zeta_{NT}^{-\kappa_1}\right) = \mathcal{O}_p\left(\widetilde{\varkappa}_{NT}^2\right). \tag{A.39}$$

Combine (A.37) - (A.39), we establish the upper bound for $\widehat{\Psi}_{NT}\left(\boldsymbol{\theta}^0,\boldsymbol{D}^0\right)$ as

$$\widehat{\Psi}_{NT}\left(\boldsymbol{\theta}^0,\boldsymbol{D}^0\right) = \mathcal{O}_p\left(\widetilde{\varkappa}_{NT}^2\right). \tag{A.40}$$

Next, we investigate the L.H.S. of (A.36). Denote $\Pi_{k,NT} = \left\{\boldsymbol{D}\in\Theta_\delta^N : \left\|\boldsymbol{\delta}_i-\boldsymbol{\delta}_i^0\right\| > \varkappa_{NT},\ \forall i\in\mathcal{G}_k\right\}$.

$$\inf_{\boldsymbol{D}\in\Pi_{NT}}\widehat{\Psi}_{NT}\left(\widehat{\boldsymbol{\theta}},\boldsymbol{D}\right) \ge \inf_{1\le k\le K^0}\inf_{\boldsymbol{D}\in\Pi_{k,NT}}\widehat{Q}_{i,NT}\left(\widehat{\boldsymbol{\theta}},\boldsymbol{D}\right) \ge \inf_{1\le k\le K^0}\inf_{\boldsymbol{D}\in\Pi_{k,NT}}\frac{1}{N}\sum_{i\in\mathcal{G}_k}\widehat{Q}_{i,NT}\left(\widehat{\boldsymbol{\theta}},\boldsymbol{\delta}_i\right)$$

$$\gtrsim \inf_{1\le k\le K^0}\inf_{\boldsymbol{D}\in\Pi_{k,NT}}\frac{1}{N}\sum_{i\in\mathcal{G}_k}\left\|\widehat{m}_{\mathcal{D},i,T}\left(\widehat{\boldsymbol{\theta}}\right)-\boldsymbol{\delta}_i\right\|^2$$

$$\ge \inf_{1\le k\le K^0}\inf_{\boldsymbol{D}\in\Pi_{k,NT}}\frac{1}{N}\sum_{i\in\mathcal{G}_{k(i^\star)}}\left|\left\|\widehat{m}_{\mathcal{D},i,T}\left(\widehat{\boldsymbol{\theta}}\right)-\boldsymbol{\delta}_i^0\right\|-\left\|\boldsymbol{\delta}_i-\boldsymbol{\delta}_i^0\right\|\right|^2$$

where the second line follows from Assumption 2(iv) and the last line follows from the triangle inequality. Note that

$$\max_{1\le i\le N}\left\|\widehat{m}_{\mathcal{D},i,T}\left(\widehat{\boldsymbol{\theta}}\right)-\boldsymbol{\delta}_i^0\right\| \le \max_{1\le i\le N}\left\|\widehat{m}_{\mathcal{D},i,T}\left(\widehat{\boldsymbol{\theta}}\right)-\overline{m}_{\mathcal{D},i}\left(\widehat{\boldsymbol{\theta}}\right)\right\| + \left(\max_{1\le i\le N}\left\|\Gamma_{\mathcal{D},i}\left(\widetilde{\boldsymbol{\theta}}\right)\right\|\right)\left\|\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^0\right\|$$

$$\le \mathcal{O}_p\left(\widetilde{\varkappa}_{NT}+\tau_T\right) = \mathcal{O}_p\left(\widetilde{\varkappa}_{NT}\right),$$

where $\widetilde{\boldsymbol{\theta}}$ is between $\widehat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}^0$, the first inequality follows from the triangle inequality and mean value theorem, the second inequality is due to Assumption 2(v), Theorem 1(i) and Lemma A.1(i), and the last equality follows from Lemma A.1(i) because the uniform rate of convergence must be at least as slow as the rate of convergence of the sample moments for each i. Meanwhile, for $\boldsymbol{D}\in\Pi_{k,NT}$ and $i\in\mathcal{G}_k$, $\left\|\boldsymbol{\delta}_i-\boldsymbol{\delta}_i^0\right\| > \varkappa_{NT}$. Then for sufficiently large

(N, T), we have

$$\inf_{\boldsymbol{D}\in\Pi_{NT}} \widehat{\Psi}_{NT}\left(\widehat{\boldsymbol{\theta}}, \boldsymbol{D}\right) > \left(\inf_{1\leq k\leq K^0} \frac{N_k}{N}\right)\left(\varkappa_{NT} - \max_{1\leq i\leq N}\left\|\widehat{m}_{\mathcal{D},i,T}\left(\widehat{\boldsymbol{\theta}}\right) - \boldsymbol{\delta}_i^0\right\|\right)^2 \gtrsim \varkappa_{NT}^2, \quad \text{(A.41)}$$

where we apply Assumption 4(iii). Combine (A.40) and (A.41), we reach (A.36) and complete the proof part (i).

The second statement in both parts (ii) and (iii) are directly implied by (i). For sufficiently large (N, T),

$$\begin{aligned}
\Pr\left(\min_{(i,j)\in\mathcal{Z}_1}\left\|\widehat{\boldsymbol{\delta}}_i - \widehat{\boldsymbol{\delta}}_j\right\| > 0\right) &\geq \Pr\left(\rho_{NT} - \max_{(i,j)\in\mathcal{Z}_1}\left\{\left\|\widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0\right\| + \left\|\widehat{\boldsymbol{\delta}}_j - \boldsymbol{\delta}_j^0\right\|\right\} > 0\right) \\
&\geq \Pr\left(\rho_{NT} - 2\max_i\left\|\widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0\right\| > 0\right) \\
&\geq \Pr\left(\max_i\left\|\widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0\right\| < \varkappa_{NT}\right) \\
&= 1 - o(1) \qquad \text{(A.42)}
\end{aligned}$$

by the uniform convergence result in (i) and the rate condition Assumption 3(i).

For sufficiently large (N, T), we have

$$\begin{aligned}
\Pr\left(\min_{1\leq i\leq N}\min_{l\in\mathcal{I}_i}\left|\widehat{\delta}_{il}\right| > 0\right) &\geq \Pr\left(\min_i\min_{l\in\mathcal{I}_i}\left[\left|\delta_{il}^0\right| - \left|\widehat{\delta}_{il} - \delta_{il}^0\right|\right] > 0\right) \\
&\geq \Pr\left(\zeta_{NT} - \max_i\max_{l\in\mathcal{I}_i}\left|\widehat{\delta}_{il} - \delta_{il}^0\right| > 0\right) \\
&\geq \Pr\left(\zeta_{NT} - \max_i\left\|\widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0\right\| > 0\right) \geq \Pr\left(\max_i\left\|\widehat{\boldsymbol{\delta}}_i - \boldsymbol{\delta}_i^0\right\| < \varkappa_{NT}\right) \\
&= 1 - o(1) \qquad \text{(A.43)}
\end{aligned}$$

by triangle inequality, the uniform convergence result in (i) and the rate condition Assumption 3(i).

□

Proof of Theorem 4. In the proof, we drop the superscript "post" for notational simplicity. By Corollary 3, we have $\widehat{\mathcal{G}}_k = \mathcal{G}_{(k)}$ and $\widehat{\mathcal{I}}_k = \mathcal{I}_{(k)}$ w.p.a.1. for $k \in [K^0]$, so $\widehat{\boldsymbol{\beta}}_k$ has the same asymptotic distribution as the oracle estimator $\overline{\boldsymbol{\beta}}$, assuming the group structure and the set of invalid moment conditions are known, defined as

$$\left(\overline{\boldsymbol{\theta}}', \overline{\boldsymbol{\alpha}}'_{k,\mathcal{I}_k}\right)' = \underset{\boldsymbol{\theta}\in\Theta, \boldsymbol{\alpha}_{k,\mathcal{I}_k}\in\Theta_\delta^{|\mathcal{I}_k|}}{\arg\min} \left(\frac{1}{N_k}\sum_{i\in\mathcal{G}_k} \widetilde{g}_{i,T}^{(k)}\left(\boldsymbol{\theta}, \boldsymbol{\alpha}_{k,\mathcal{I}_k}\right)\right)' \boldsymbol{W}_{k,NT} \left(\frac{1}{N_k}\sum_{i\in\mathcal{G}_k} \widetilde{g}_{i,T}^{(k)}\left(\boldsymbol{\theta}, \boldsymbol{\alpha}_{k,\mathcal{I}_k}\right)\right), \tag{A.44}$$

and we let $\overline{\boldsymbol{\beta}}_k = \left(\overline{\boldsymbol{\theta}}', \overline{\boldsymbol{\alpha}}'_{k,\widehat{\mathcal{I}}_k}\right)'$. The proof directly follows from Theorem 3.3 in Cheng and Liao (2015). □

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