We are not in a Gaussian world anymore: Implications for the composition of official foreign assets

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Abstract

In the aftermath of the 1997-98 Asian crises, many emerging markets self-insured by accumulating international reserves (i.e., non-contingent assets) in excess of what many models predicted, while relying relatively little on state-contingent assets. This apparent over-reliance on self-insurance has been viewed as a puzzle in search of an explanation. A related, and still outstanding, puzzle is why the benefits of financial liberalization appear to be so small and, yet, financial globalization has been unprecedented in recent decades. We show that these two puzzles can be solved by incorporating rare macroeconomic disasters in income risk. To this effect, we first fit a fat-tailed distribution to long output time series for 156 countries. We then develop a theoretical framework to quantify (i) the increase in welfare gains of financial integration and (ii) how the composition of official reserves (non-contingent and contingent) responds to bigger shocks. Our results show that fat tails lead to a sharp increase in both the gains of financial integration and self-insurance for standard values of the coefficient of risk aversion.

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"True wisdom comes to each of us when we realize how little we understand about life, ourselves, and the world around us"
— Socrates

1 Introduction

How do emerging markets (EM) protect themselves against risk? The evidence seems to suggest that they accumulate abundant international reserves and other risk-free assets as self-insurance, but little state-contingent assets. In fact, the "over-accumulation" of international reserves in the aftermath of the 1997-98 Asian crises has been viewed as a puzzle in search of an explanation because it typically exceeded the predictions/prescriptions of standard models of demand for international reserves.[1] More generally, Caballero and Panageas (2004, 2005) have argued that EM hold too many non-contingent assets that are costly to maintain and inefficient in providing a cushion against financial crises. They argue that EM would be better off by holding contingent assets linked to, for instance, the volatility index (VIX) so as to be able to access more liquidity in bad times.

A related puzzle – but typically studied separately from international reserve accumulation – is the size of welfare gains from financial integration. In a seminal contribution, Cole and Obstfeld (1991) show, in the context of a standard two-country model with output uncertainty, that the gains from financial integration are surprisingly small (in the order of 0.20 percent of GDP per year). In a growth model, Gourinchas and Jeanne (2006) conclude that the welfare gains from switching from financial autarky to perfect capital mobility are about 1 percent of GDP. Small welfare gains are also reported in Obstfeld (1992), Tesar (1995), and Mendoza (1995).[2] Martin (2010) revisits Cole and Obstfeld (1991) and shows that the wealth costs of shutting down trade in financial assets across countries increase by 3 to 20 percent when allowing for rare disasters in the income process. Martin (2010), however, needs to assume very large risk aversion coefficients (up to 8) to obtain substantive

[1] See, among others, Aizenman and Marion (2002), Jeanne and Ranciere (2006), Jeanne (2007), and Bianchi *et al.* (2018). To explain "over-accumulation," these papers had to resort to non-income risk (such as substantial sudden stops and roll-over risk). In other words, income risk by itself cannot explain the data. As will become clear below, we will focus on fat-tailed income risk to explain the data.

[2] Van Wincoop (1998) argues that small welfare gains are mainly due to the assumption of stationary output processes and, hence, that such results should be viewed as the welfare gains from eliminating business cycles. This interpretation is consistent with Lucas' (1987) conclusion that the costs of business cycles are negligible.

gains from financial integration.

The starting point of our paper is the observation that these two puzzles are the two sides of the same coin. As illustrated in Figure 1, we view financial integration as consisting of two stages (or segments): (i) from financial autarky to incomplete markets (i.e., a risk-free bond) and (ii) from incomplete markets to complete markets (in the Arrow-Debreu sense). If the welfare gains from the first stage are small, for example, there is little incentive for authorities to accumulate risk-free assets. If the gains from the second stage are large, authorities will want to acquire a lot of state-contingent assets. We show quantitatively that, under a normal distribution, the welfare gains from the first stage are very small. In fact, to fix ideas, notice that welfare gains from the first stage would be zero for a quadratic consumer. Hence, almost all welfare gains follow from the second stage. Under a normal distribution, we should thus see almost no risk-free assets compared to state-contingent assets. This is the opposite of what we see in practice, as argued by Caballero and Panageas (2004, 2005). In other words, we are re-stating the first puzzle as referring to the *composition* of welfare gains and hence risk-free versus state-contingent assets. The second puzzle remains but refers to the *level* of welfare gains (as opposed to the composition).

The key to solving both puzzles is the introduction of macroeconomic disasters governed by a power-law distribution.[3] In fact, following Taleb (2007), Vegh *et al.* (2018) show that it is hard to find any relevant macroeconomic time series that follows a normal (or log-normal) distribution, as traditionally assumed. We do not live in a Gaussian world anymore! Figure 2 illustrates this phenomenon by showing a histogram of Dow Jones returns. The visual message is clear: a normal distribution is not a good fit for the data but a q-Gaussian distribution is. The q-Gaussian distribution is a particular case of a power-law distribution and allows for fatter tails than the normal distribution.[4] To compare the predictive power of the normal and q-Gaussian distributions, notice that the normal distribution would predict that an event such as the 1987 stock market crash would take place every 57,408 years compared to 34 years for the q-Gaussian.

Based on a sample of 156 countries (20 industrial and 136 developing) for the period 1900-2018, we first estimate the parameters of a normal distribution for GDP growth. Plugging these parameters into a stochastic model of a small open economy,

[3] The idea that rare macroeconomic disasters can explain a variety of asset return puzzles (particularly the equity-premium puzzle) goes back to Rietz (1988) and includes Barro (2006, 2009), Barro and Jin (2011), and Martin (2011). Barro and Jin (2011) resort to a power-law distribution.

[4] The Jarque-Bera test clearly rejects the null hypothesis that the data follows a normal distribution.

we compute the welfare gains from financial autarky to complete markets to be in the range 0.18-0.44 percent of steady-state consumption for coefficients of risk aversion in the interval 2 to 5. Hence, as in Cole and Obstfeld (1991) and most of the literature, the *level* of welfare gains is tiny. Interestingly though, welfare gains from incomplete markets to complete markets are in the range 0.18-0.43 percent. Hence, essentially all of the welfare gains occur in the second segment in Figure 1. In other words, the model cannot explain large holdings of non-contingent assets relative to state-contingent assets.

We then fit a power-law distribution to rare events. The estimation indicates that a cumulative drop in GDP of 16 percent is needed to qualify as a disaster (i.e., a rare event). Further, the probability that such a disaster occurs is 2.7 percent. On average, then, we should see less than three major output contractions per century. Under the power-law distribution, welfare gains from financial integration increase dramatically and reach 4.30 percent of steady-state consumption for a coefficient of risk aversion equal to 3.5 and 20.5 percent for a coefficient of risk aversion of 4.5. Compared to the normal distribution case, the overall *level* of welfare gains is thus around six times bigger for a coefficient of risk aversion equal to 3.5 and 53 times bigger for a coefficient of risk aversion equal to 4.5. Further, the *composition* of welfare gains changes drastically relative to the normal distribution case. Welfare gains from IM to CM are about half of the total welfare gains compared to essentially 100 percent under a normal distribution. The implication is that the authorities will now choose to self-insure substantially in light of possible macroeconomic disasters.

In sum, the size of potential output shocks is the key to understanding the composition of official assets (non-contingent versus contingent). When shocks are unusually large, both types of insurance increase (self-insurance and contingent assets) but non-contingent increases more. Intuitively, under incomplete markets, prudent authorities grow increasingly concerned about very large negative shocks and raise self-insurance accordingly. In contrast, under complete markets, full insurance renders individual outcomes less risky.

The paper proceeds as follows. Section 2 provides the empirical estimates from fitting a power-law distribution to our sample. Section 3 uses the simplest 2-period stochastic model (with a binomial distribution) to introduce the key messages from our analysis. Section 4 works out a more sophisticated stochastic model for two types of risk: normal and power-law distributions. Section 5 closes the paper with some policy implications. Detailed technical derivations have been relegated to appendices.

2 Evidence on macroeconomic disasters

This section tries to gain a better understanding of the underlying sample distribution of output growth. To this end, we combine information from the Maddison Project Database of Bolt *et al.* (2018) with recent estimates from the World Economic Outlook (IMF) to generate long annual series of real GDP per capita for 156 countries (20 industrial and 136 developing) covering the period 1900 to 2018.[5]

To identify candidate episodes in the disaster regime, we follow Barro and Jin (2011) and assume that an episode begins when real GDP per capita falls and ends when the initial (i.e., pre-episode) level is exceeded. We then compute peak-to-trough declines as our measure of the size of the GDP contraction. For each episode, x is calculated as the minimum real GDP per capita relative to the pre-episode level (i.e., the ratio reflects the maximum cumulative drop). We then use the reciprocal of the contraction size, $z = 1/x$, to fit our power-law distribution. This strategy enables us to capture a broad set of candidate episodes (922 in total) with different contraction intensities and origins (e.g., economic mismanagement, natural disasters, and wars). As in Barro (2006), Barro and Urzua (2008), and Barro and Jin (2011), disaster events are allowed to be correlated across countries (as in world wars, the Great Depression, and the global financial crisis of 2007-08).[6]

For continuous random variables like z, fitting a power-law distribution is relatively simple. The maximum likelihood estimator (MLE) of the scaling parameter ($\hat{\alpha}$) *conditional* on a threshold value (z_{min}), which determines the inclusion of data points in the estimation sample, is given by

$$\hat{\alpha} = 1 + n \left[\sum_{i=1}^{n} ln \left(\frac{z_i}{z_{min}} \right) \right]^{-1}, \tag{1}$$

where n is the number of observations. Equation 1 makes clear that, in principle, we can estimate $\hat{\alpha}$ for *any* value of z_{min}. Nonetheless, it must be emphasized that the choice of z_{min} is critical for the consistency of the estimator. After all, in most empirical data, power-law behavior is expected to occur only at the extremes of the distributions – in our case, for values of z greater than or equal to z_{min}. As stated by Clauset *et al.* (2009, p. 9), "[i]f we choose too low a value for $\hat{z}_{min}$, we will get

[5] Appendix A provides starting years for each country.

[6] Given that the purpose of the peak-to-trough measurements of GDP decline is to provide a baseline approximation to the model's jump contractions, and following Barro and Jin (2011), we are not concerned about the uneven length of disaster events, their correlations across space and time, and the possibly temporary nature of shocks (since, in practice, a number of disasters might be offset by high recovery rates).

a biased estimate of the scaling parameter $\hat{\alpha}$ since we will be attempting to fit a power-law model to non-power-law data. On the other hand, if we choose too high a value for $\hat{z}_{min}$, we are effectively throwing away legitimate data points $z_i < \hat{z}_{min}$, which increases both the statistical error of the scaling parameter and the bias from finite size effects." As a sensible solution to this issue, Clauset *et al.* (2009) propose a strategy based on objective criteria to select suitable values of z_{min} in empirical settings. This is the procedure we follow in the rest of our calibration exercise.

Specifically, the approach consists in finding the value of z_{min} that minimizes the distance between the cumulative density function (CDF) of the data and that of the fitted model. Using the Kolmogorov-Smirnov statistic as a measure of distance (D), the optimal z_{min} minimizes:

$$D = \max_{z \geq z_{min}} |S(x) - P(x)|, \tag{2}$$

where $S(x)$ is the CDF of the data that satisfies $z \geq z_{min}$, and $P(x)$ is the CDF of the power-law model fitted through maximum likelihood also in the region $z \geq z_{min}$. That is, our algorithm calls for (i) sequentially running a set of plausible z_{min} values through the MLE procedure, represented in equation (1), to estimate a set of $\hat{\alpha}$; (ii) using each $\hat{\alpha}$ to fit a cumulative distribution function $P(x)$; and (iii) find the minimum distance between the fitted CDFs and the ones found in the data given each z_{min} used. Once optimal values of the threshold and scaling parameter ($\hat{z}_{min}$ and $\hat{\alpha}$) are found for the disaster observations, we turn to the rest of the observations. Specifically, we put together a panel dataset that combines (i) the annual observations of candidate disaster episodes not part of the power-law distribution (i.e., with $z < \hat{z}_{min}$) and (ii) the annual observations of the episodes of purely positive growth in real GDP per capita (so far excluded from the analysis). We use this "tranquil-times" sample to fit the log-normal distribution with parameters μ (mean) and σ (standard deviation) of the relative income, y, through maximum likelihood, with y defined as the ratio of real GDP per capita in each year with respect to its level in the preceding year. Finally, as in Barro and Jin (2011), we compute the disaster probability, p, as the ratio of the number of disasters to the number of non-disaster years. Hence, the gross growth rate of real GDP per capita follows, with probability p, an inverse power-law distribution and, with probability $1 - p$, a log-normal distribution

Estimates from the application of this procedure are presented in Table 1. For the total sample, we estimate $\hat{z}_{min} = 1.1899$, meaning that an event must register a cumulative drop in real GDP per capita of at least 15.96 percent to be considered a disaster.[7] We estimate $\hat{\alpha} = 4.5980$ and a disaster probability $p = 0.0274$ (i.e., mac-

[7] Note that a drop of 15.96 percent in GDP per capita implies that $z = 1/0.8404 = 1.1899$.

roeconomic contractions of 15.96% or more occur around three times per century), which is consistent with the estimates of Barro and Jin (2011).

Figures 3 and 4 provide a visual assessment of the goodness-of-fit of our power-law model. Figure 3 compares the empirical density with the estimated density, while Figure 4 compares the empirical CDF with the estimated CDF. As is well-known – and as illustrated in Figure 4 – the CDF of a power-law distribution in a doubly logarithmic plot is linear. A visual inspection of these figures makes clear that the power-law distribution provides a good fit for the data.

Lastly – and also reported in Table 1 – using the tranquil-times sample, we obtain estimates for the mean, $\hat{\mu} = 0.0262$, and standard deviation, $\hat{\sigma} = 0.0450$, of the corresponding log-normal distribution. We will use this information on tranquil-times in Section 4 to think about risk characterized by a normal distribution in good times and a power-law distribution in bad (i.e., rare) times.

3 Basic ideas

This section introduces the basic ideas behind our analysis in the simplest possible setup: a two-period model with a binomial distribution. The emphasis is on qualitative messages. Based on our quantitative analysis in Section 2, Section 4 will then derive our main results in a two-period model with both normal and power-law distributions. Finally, Appendix C extends the results to an infinite horizon.

3.1 Prudent consumer

Consider a two-period small open economy.[8] There is a single (tradable and non-storable) good. The endowment in period 1, y_1, is exogenous. The endowment in period 2, y_2, is stochastic and follows a binomial distribution:

$$y_2 = \begin{cases} y_2^H \text{ with probability } p, \\ y_2^L \text{ with probability } 1-p, \end{cases}$$

where $y_2^H \geq y_1 \geq y_2^L$. For simplicity, we will assume that the expected value of y_2 ($E(y_2) = py_2^H + (1-p)y_2^L$) is equal to y_1 and that $p = 1/2$.

Lifetime expected utility (welfare) is given by

$$W = u(C_1) + \beta\mathbb{E}\{u(C_2)\},$$

[8] For all analytical derivations, the reader is referred to Appendix B.

where W denotes welfare, $0 < \beta < 1$ is the discount factor, and $u(.)$ is given by a standard constant-relative-risk-aversion utility function:

$$\begin{array}{ll} u(C) = \frac{C^{1-\gamma}-1}{1-\gamma}, & \gamma \neq 1, \\ u(C) = \log(C), & \gamma = 1, \end{array} \tag{3}$$

where $\gamma > 0$ is the coefficient of risk aversion. Except in the case of the quadratic consumer below, we assume $u'''(c) > 0$ (i.e., prudent consumers). We assume $\beta(1+r) = 1$, where r is the world risk-free interest rate. Initial net assets are assumed to be zero.

We solve this model under three different asset market configurations: (i) financial autarky (FA), (ii) incomplete markets (IM) (i.e., a single risk-free bond), and (iii) complete markets (CM) in the Arrow-Debreu sense. Figure 1 offers a schematic representation. We compute lifetime utility (i.e., welfare) in each case and compare the welfare achieved in each of the three cases.

As shown in Appendix B, welfare in the case of complete markets is given by

$$W_{CM} = \frac{y_1^{1-\gamma} - 1}{1-\gamma}(1+\beta). \tag{4}$$

Welfare in the financial autarky case is given by

$$W_{FA}(\phi_1) = \frac{[y_1(1+\phi_1)]^{1-\gamma} - 1}{1-\gamma} + \beta\left[0.5\frac{[y_2^H(1+\phi_1)]^{1-\gamma} - 1}{1-\gamma} + 0.5\frac{[y_2^L(1+\phi_1)]^{1-\gamma} - 1}{1-\gamma}\right], \tag{5}$$

where, for further analysis, we have added a potential compensating variation to consumption equal to the factor $1+\phi_1$.[9]

The welfare gains of having complete markets relative to the case of financial autarky are the value of ϕ_1 that satisfies

$$W_{FA}(\phi_1) = W_{CM}.$$

Using (4) and (5), we can obtain a reduced form for ϕ_1:

$$\phi_1 = \left\{\frac{1+\beta}{1+\beta\,(0.5)\left[\left(\frac{y_2^H}{y_1}\right)^{1-\gamma} + \left(\frac{y_2^L}{y_1}\right)^{1-\gamma}\right]}\right\}^{\frac{1}{1-\gamma}} - 1. \tag{6}$$

[9] The compensating variation can be thought of as a flat tax (across time and across states of nature).

As a particular case, notice that if $y_2^H = y_2^L = y_1$, then $\phi_1 = 0$. In other words, since there is no uncertainty and a flat output path allows consumers to smooth consumption over time, access to financial markets is redundant.

In the same vein, welfare for the incomplete markets case is given by

$$W_{IM}(\phi_2) = \frac{\left[y_1(1+\phi_2)\left(1-\widetilde{b}_1\right)\right]^{1-\gamma}-1}{1-\gamma} \tag{7}$$
$$+0.5\beta\left[\frac{\left(y_1(1+\phi_2)\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]\right)^{1-\gamma}-1}{1-\gamma}+\frac{\left(y_1(1+\phi_2)\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]\right)^{1-\gamma}-1}{1-\gamma}\right],$$

where $\widetilde{b}_1$ is the current account as a percentage of GDP and $1+\phi_2$ is a compensating variation that captures the welfare gains from moving from incomplete to complete markets (see Figure 1). Formally,

$$W_{CM} = W_{IM}(\phi_2).$$

As shown in Appendix B,

$$\phi_2 = \left\{\frac{1+\beta}{\left[\left(1-\widetilde{b}_1\right)\right]^{1-\gamma}+0.5\beta\left[\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]^{1-\gamma}+\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]^{1-\gamma}\right]}\right\}^{\frac{1}{1-\gamma}}-1. \tag{8}$$

Notice that we cannot obtain a reduced form for ϕ_2 ($\widetilde{b}_1$ is, of course, endogenous). Hence, we need to solve for ϕ_2 numerically (see Appendix B).[10]

The computation of ϕ_1 and ϕ_2 is central to analyzing the welfare gains from financial autarky (FA) to complete markets (CM) and from incomplete markets (IM) to complete markets (CM), respectively, as suggested in Figure 1. In fact, the comparison between ϕ_1 and ϕ_2 also provides a quantification of the welfare gains from FA to IM, as formally shown in Subsection 3.3.

Table 2 reports the values of saving, ϕ_2 and ϕ_1, for values of γ between 2 and 5.[11] Two points follow from this table. The first, well-known, point is that the gains

[10] Note, again, that in the case $y_2^H = y_2^L = y_1$ (which implies $\widetilde{b}_1 = 0$), then $\phi_2 = 0$ because financial autarky is not binding.

[11] We assume $\beta = 0.98$ (as in Cole and Obstfeld), $y_2^H = 1.0601$, and $y_2^L = 0.9399$. The output

from financial integration – as measured by ϕ_1 in this case – are quite small. We know this since Cole and Obstfeld (1991), who estimated this gain to be around 0.20 percent of GDP per year. In our case, for values of γ between 2 and 5, the upper bound for the welfare gains is 0.45 percent. The second point, much less known, has to do with the *composition* of total welfare gains: essentially all the gains from financial integration occur in the second segment in Figure 1 (from IM to CM) and almost none in the first segment (FA to IM). Indeed, even for $\gamma = 5$, the difference between ϕ_1 and ϕ_2 is only 0.008 percent.

3.2 Quadratic consumer

It should be intuitively clear from the above discussion that quadratic preferences will imply that $\phi_1 = \phi_2 > 0$. In other words, the welfare gains of moving from FA to IM will be zero and all the gains will accrue going from IM to CM. To check this, consider the following quadratic preferences:

$$u(C) = -\frac{1}{2}(\bar{C} - C)^2, \quad C \leq \bar{C},$$

where $\bar{c}$ is the bliss point.

Proceeding as in the prudent case above, it is straightforward to check (see Appendix B) that ϕ_1 is given by the same expression; that is, equation (6). The same is true for ϕ_2: the expression for the quadratic consumer continues to be given by equation (8). But, in the quadratic case and assuming that $\mathbb{E}(y_2) = y_1$, we can further simplify expression (8) by taking into account that $\widetilde{b_1} = 0$ (i.e., the trade balance is zero). In fact, when we impose $\widetilde{b_1} = 0$, we conclude that $\phi_2 = \phi_1$, as expected. This means that there are no welfare gains from moving from FA to IM. All welfare gains will accrue when moving from IM to CM.

Intuitively, under quadratic utility, certainty equivalence holds. This implies that when $\mathbb{E}(y_2) = y_1$, the consumer chooses $C_1 = C_2 = y_1$ and the trade balance is zero (i.e., there are no precautionary savings). In other words, the lack of access to international capital markets is not binding, and having a risk-free bond is thus redundant. Incomplete markets do not increase welfare relative to financial autarky. In contrast, complete markets allow consumers to equate consumption across states of nature (i.e., $C_2^H = C_2^L$), which is not feasible under IM.

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values are based on the estimate for our sample of the standard deviation of growth (see Table 4). We assume $\gamma = 3$, a standard value. Again, we solve for ϕ_2 numerically, as detailed in Appendix B.

3.3 Relationship between welfare gains

Let $\widetilde{\phi}_1$ denote the welfare gains from FA to IM (see Figure 1). Before proceeding further, it will be helpful to derive a basic relationship between $\widetilde{\phi}_1$, ϕ_1, and ϕ_2.

By definition,

$$W_{FA}(\widetilde{\phi}_1) = W_{IM}.$$

Substituting into this last equation expression (5), evaluated at $\widetilde{\phi}_1$, and expression (7), evaluated at $\phi_2 = 0$, and taking into account (6) and (8), we can derive the following relationship (see Appendix B):

$$\underbrace{\log(1+\phi_1)}_{\text{total gains}} = \underbrace{\log(1+\widetilde{\phi}_1)}_{\text{gains FA-IM}} + \underbrace{\log(1+\phi_2)}_{\text{gains IM-CM}}. \tag{9}$$

There is thus a log-linear relationship between $1+\phi_1$, $1+\widetilde{\phi}_1$, and $1+\phi_2$. Therefore, given ϕ_1 and ϕ_2, we can compute $\widetilde{\phi}_1$. Intuitively, as expected from Figure 1, total welfare gains can be broken down into gains from FA to IM plus gains from IM to CM. As expected, in the quadratic case, $\phi_1 = \phi_2$, and hence, by equation (9), $\widetilde{\phi}_1 = 0$.

3.4 Composition of welfare gains

We have shown above (Table 2) that for log-normal shocks (which can be thought of as shocks at the business cycle frequency), almost all of the welfare gains occur from IM to CM and practically none from FA to IM. But how does the composition of welfare gains change with the size of the shock?[12]

To gain insights into this issue, Table 3 computes ϕ_1, ϕ_2, and $\widetilde{\phi}_1$ as a function of the output distribution (keeping $\mathbb{E}(y_2) = y_1$).[13] The main message that follows from this table is that the larger the shock, the larger the benefits of non-contingent assets (the risk-free bond in this case) relative to contingent assets. Specifically, for $y^H = 1.1$ and $y^L = 0.9$, we have the result obtained above that essentially all the welfare gains occur from IM to CM. In fact, the ratio of ϕ_2/ϕ_1 (which measures the welfare gains from IM to CM relative to total gains) is 98 percent. At the other extreme ($y^H = 1.6$ and $y^L = 0.4$), the ratio ϕ_2/ϕ_1 is just 51.7 percent. In other words, about

[12] Of course, if we take our model literally, our small open economy will hold either no foreign assets (under financial autarky), contingent bonds (under incomplete markets), or state-contingent bonds (under complete markets).

[13] We are not trying to be "realistic" here in terms of the size of the shocks. We will calibrate output shocks using the normal and power-law distributions in the next section.

half of the total gains are now due to incomplete markets. Intuitively, as the output shock becomes larger (and thus the fatter the tails become), consumers' marginal utility in the bad state of nature rises sharply, thus increasing the demand for non-contingent assets.[14] This effect dominates the increase in demand for contingent claims in which case consumers are already smoothing consumption across states of nature.

Finally, an important caveat in terms of how to use our model to think about the composition of official international reserves. If we take our model literally, our small open economy will hold either no foreign assets (under financial autarky), or contingent bonds (under incomplete markets) or state-contingent bonds (under complete markets). We, therefore, want to think of a world with a large number of small open economies characterized by some idiosyncratic financial friction (for example, accessing complete markets involves a fixed cost while accessing bond markets is costless).[15] Further, the size of the fixed cost varies across countries. In this world, economies with high fixed costs will hold only non-contingent assets and economies with small fixed costs will hold only contingent assets. The "average" economy will thus hold some of its reserves in non-contingent assets and some in contingent assets. Further, in response to common endowment shocks, the response will vary across economies and thus the "average" economy will change the proportion of non-contingent to contingent assets.[16]

4 A small open economy model under normal and power-law distributions

To show our theoretical results in a much richer stochastic structure, we will stick to our two-period model. We first compute welfare results under a normal distribution. We then analyze how these outcomes vary with the addition of rare disasters to the income process. We show that the presence of a power-law distribution for the size of macroeconomic disasters substantially affects welfare gains.

[14] Notice that, since the probabilities do not change as the mean-preserving spread increases, tails are becoming fatter. Hence, we can say that the fatter the tails, the smaller ϕ_2/ϕ_1. This way of thinking will remain valid in Section 4 when risk is modeled using normal and power-law distributions.

[15] There is a large literature on segmented asset markets. See, for example, Alvarez *et al.* (2002).

[16] Naturally, a formal development of this model would require a different paper. For the purposes of this paper, however, all we need is to interpret our discussion of the composition of official reserves as applying to the "average" economy.

Consider a small open economy with a single (and non-storable) good. Output is deterministic in period 1, given by $Y_1 = Y$, and stochastic in period 2, given by

$$Y_2(x) = Yx, \tag{10}$$

where x is a continuous random variable with a probability density function given by $f_X(x)$. We assume that $\mathbb{E}[x] = exp(g_x)$. Thus, $\mathbb{E}[Y_2(x)] = Y exp(g_x)$.

As in section 3, preferences of the representative household are given by

$$\frac{(C_1)^{1-\gamma} - 1}{1-\gamma} + \beta \mathbb{E}\left[\frac{(C_2(x))^{1-\gamma} - 1}{1-\gamma}\right]. \tag{11}$$

The rest of the world consists of a continuum of identical economies with the same discount factor β. We will assume that there is full risk sharing among the economies in the rest of the world. Consumption for the rest of the world is thus given by

$$\begin{aligned}
&C_1^* = Y^* = Y, \\
&C_2(x^*) = \mathbb{E}[Y_2^*(x^*)] = \mathbb{E}[Y_2(x)] = Y exp(g_x).
\end{aligned}$$

Full risk sharing in the rest of the world implies that our analysis will focus on the degree of risk sharing of income fluctuations around its average value. Hence, the gains from risk sharing provided by the international financial markets will correspond to the welfare benefits in terms of smoothing consumption around its average path. In the two-period model developed here, this distinction is not essential but it is important for the generalization of our results in Appendix C in an infinite horizon set-up. We now present the resulting allocations for the small open economy under the three market arrangements already considered in the previous section: complete markets, financial autarky, and incomplete markets.

4.1 Complete markets

Suppose that there is a full set of state-contingent bonds. A contingent bond that pays out one unit of consumption in period 2 in state x has a price $p(x)$ in period 1. The budget constraint can thus be written as

$$C_1 + \int p(x) C_2(x) dx = Y + \int p(x) Y_2(x) dx. \tag{12}$$

The optimality conditions for the consumption stream are

$$(C_1)^{-\gamma} = \lambda, \tag{13}$$

$$\beta f_X(x)(C_2(x))^{-\gamma} = \lambda p(x), \tag{14}$$

where λ is the marginal utility of income (i.e., the Lagrange multiplier corresponding to budget constraint (12)). The assumption of full risk sharing in the rest of the world implies

$$p(x) = \beta f_X(x)\left(\frac{Y}{Y exp(g_x)}\right)^{\gamma} = \beta f_X(x)exp(-\gamma g_x), \tag{15}$$

where $f_X(\cdot)$ is the probability density function of x. Combined with (15), condition (14) yields

$$\beta f_X(x)(C_2(x))^{-\gamma} = \lambda\beta f_X(x)exp(-\gamma g_x) = (C_1)^{-\gamma}\beta f_X(x)exp(-\gamma g_x). \tag{16}$$

This last condition implies $C_1 = C_2(x)exp(-g_x) = \bar{C}\ \forall x$. Combining this with (12), we get:

$$\bar{C}(1+\beta exp((1-\gamma)g_x)) = Y+\beta Y exp(-\gamma g_x)\mathbb{E}\left[Y_2(x)\right] = Y(1+\beta\exp((1-\gamma)g_x)), \tag{17}$$

which implies $\bar{C} = Y$. Hence, welfare under complete markets is given by:

$$W_{CM} = \frac{(Y)^{1-\gamma}-1}{1-\gamma} + \beta\frac{(Y)^{1-\gamma}exp((1-\gamma)g_x)-1}{1-\gamma}. \tag{18}$$

4.2 Financial autarky

Under financial autarky, this small open economy is not integrated at all with the rest of the world. Hence, it cannot diversify any income risk in period 2. Therefore, $C_2(x) = Yx\ \forall x$. Welfare under financial autarky, adding a potential compensation to consumption of a factor of $1+\phi_1$, is thus given by

$$W_{FA}(\phi_1) = \frac{(Y(1+\phi_1))^{1-\gamma}-1}{1-\gamma} + \beta\mathbb{E}\left[\frac{(Yx(1+\phi_1))^{1-\gamma}-1}{1-\gamma}\right]. \tag{19}$$

The welfare gain of having complete markets relative to the case of financial autarky is the value of ϕ_1 that satisfies $W_{FA}(\phi_1) = W_{CM}$. Using (18) and (19), it follows that

$$(1+\phi_1)^{1-\gamma} + (1+\phi_1)^{1-\gamma}\beta\mathbb{E}\left[(x)^{1-\gamma}\right] = (1+\beta exp((1-\gamma)g_x)).$$

The last expression implies

$$\phi_1 = \left(\frac{1+\beta exp((1-\gamma)g_x)}{1+\beta\mathbb{E}\left[(x)^{1-\gamma}\right]}\right)^{\frac{1}{1-\gamma}} - 1. \tag{20}$$

The value ϕ_1 captures the benefits in terms of consumption of having a full set of contingent bonds relative to the financial autarky case. Assuming a specific distribution for x will make it possible to obtain an analytical expression for $\mathbb{E}\left[(x)^{1-\gamma}\right]$ and thus a closed-form solution for ϕ_1.

4.3 Incomplete markets

Under incomplete markets, there is a non-state-contingent bond that pays a gross risk-free rate of $R = 1 + r$ in period 2. Let b_1 denote the stock of this risk-free bond held by the representative household. The budget constraints in periods 1 and 2 are thus given by, respectively,

$$\begin{aligned} C_1 &= Y - b_1, \\ C_2 &= Yx + Rb_1. \end{aligned} \tag{21}$$

The optimal choice for b_1 satisfies:

$$(Y - b_1)^{-\gamma} = \beta R \mathbb{E}\left[(Yx + Rb_1)^{-\gamma}\right].$$

Defining $\widetilde{b}_1 \equiv b_1/Y$, then the optimal saving decision can be written as

$$\left(1 - \widetilde{b}_1\right)^{-\gamma} = \beta R \mathbb{E}\left[\left(x + R\widetilde{b}_1\right)^{-\gamma}\right]. \tag{22}$$

Since this non-contingent bond is priced by the rest of the world, R satisfies $1/R = \beta exp(-\gamma g_x)$. Hence, equation (22) becomes:

$$\left(1 - \widetilde{b}_1\right)^{-\gamma} = \exp(\gamma g_x)\mathbb{E}\left[(x + R\widetilde{b}_1)^{-\gamma}\right]. \tag{23}$$

Welfare under incomplete markets, compensating the household by a factor of $1+\phi_2$, is thus given by

$$W_{IM}(\phi_2) = \frac{(Y(1+\phi_2)(1-\widetilde{b}_1))^{1-\gamma} - 1}{1-\gamma} + \beta \mathbb{E}\left[\frac{(Y(1+\phi_2)(x + R\widetilde{b}_1))^{1-\gamma} - 1}{1-\gamma}\right]. \tag{24}$$

Hence, the welfare gains from moving from incomplete to complete markets are the compensating variation for incomplete markets, ϕ_2, such that $W_{IM}(\phi_2) = W_{CM}$. Using (18) and (24), it follows that

$$(1+\phi_2)^{1-\gamma}(1-\widetilde{b}_1)^{1-\gamma} + (1+\phi_2)^{1-\gamma}\beta\mathbb{E}\left[(x + R\widetilde{b}_1)^{1-\gamma}\right] = (1 + \beta exp((1-\gamma)g_x)).$$

Solving for ϕ_2,

$$\phi_2 = \left(\frac{1 + \beta exp((1-\gamma)g_x)}{(1-\widetilde{b}_1)^{1-\gamma} + \beta \mathbb{E}\left[(x + R\widetilde{b}_1)^{1-\gamma} \right]} \right)^{\frac{1}{1-\gamma}} - 1. \tag{25}$$

The value ϕ_2 captures the benefits in terms of consumption of having a full set of state-contingent bonds relative to the case with only a non-state-contingent bond. Unfortunately, equation (22) does not have an analytical solution for $\widetilde{b}_1$ for a general probability density function $f_X(x)$. Therefore, numerical integration is needed to obtain the expression on the right-hand side of (22) for a given value for $\widetilde{b}_1$. The optimal value for $\widetilde{b}_1$ is obtained by numerical computation. Once we have the optimal value for $\widetilde{b}_1$, we can get ϕ_2 with numerical integration of $\mathbb{E}\left[(x + R\widetilde{b}_1)^{1-\gamma} \right]$.

4.4 Types of risk and welfare gains

This subsection computes the welfare gains of financial integration, ϕ_1 and ϕ_2, under two types of risk for income in period 2. The first case assumes a log-normal distribution for income in period 2. The second case considers a distribution with fat tails, combining a log-normal distribution with a power-law distribution, as in Barro and Jin (2011). We will obtain expressions for the gains of complete financial integration (i.e., complete markets) relative to the financial autarky case, ϕ_1, for both types of risks. Likewise, we will also compute the saving rate $\widetilde{b}_1$ and, thus, the welfare gains of complete markets relative to the case of incomplete markets, given by ϕ_2, for the two types of risk.

4.4.1 Income risk type I: Normal distribution

Suppose that the distribution of income in period 2 is given by a log-normal distribution for x:

$$\log(x) \sim N(\mu_x, \sigma_x^2).$$

With this specification, $\mathbb{E}[Y_2(x)] = exp(\mu_x + \frac{1}{2}\sigma_x^2)$, implying that $g_x = \mu_x + \frac{1}{2}\sigma_x^2$. Moreover, in this case, we obtain

$$\mathbb{E}[x^{1-\gamma}] = \exp((1-\gamma)\mu_x + \frac{1}{2}(1-\gamma)^2\sigma_x^2) = \exp\left((1-\gamma)g_x + \frac{1}{2}\gamma(\gamma-1)\sigma_x^2 \right),$$

and, therefore,

$$\phi_1 = \left(\frac{1 + \beta exp((1-\gamma)g_x)}{1 + \beta \exp\left((1-\gamma)g_x + \frac{1}{2}\gamma(\gamma-1)\sigma_x^2\right)} \right)^{\frac{1}{1-\gamma}} - 1.$$

Table 4 shows the estimation of the parameters for type I risk for our sample of 156 countries since 1900. Using all the observations in our sample, we estimate an average growth of 1.65 percent with a standard deviation of 6.01 percentage points.[17] Table 5 shows the gains of financial integration measured by ϕ_1. We use $\beta = 0.98$ and a range for the coefficient of risk aversion, γ, from 2 to 5. Table 5 indicates that, with type I risk, the gains of financial integration are very small (i.e., below one percent of average consumption). The small magnitude of the welfare gains is consistent with Cole and Obstfeld (1991) and Martin (2010), among others, when income risk is normally distributed.

In the case of incomplete markets, we need to solve first for the optimal value of $\widetilde{b}_1$ that satisfies (22) and then numerically integrate $\mathbb{E}[(x + R\widetilde{b}_1)^{1-\gamma}]$. Consider the case where $\gamma > 1$. Since $\log(x)$ is normally distributed, $\mathbb{E}[(x + R\widetilde{b}_1)^{1-\gamma}] < \infty$ if $\widetilde{b}_1$ is finite. The computation of the welfare gains from moving from the non-contingent bond equilibrium to the case of complete markets requires a numerical approximation to obtain the bond holdings, $\widetilde{b}_1$.

Table 6 shows bond holdings ($\widetilde{b}_1$) in the case of type I risk for different values of the coefficient of risk aversion, γ. We conclude that, under normally-distributed income risk, saving for self-insurance is very low and below one percent of income for a range of the risk aversion coefficient between 2 and 5. Given how low savings are, we can already guess that the welfare gains of moving from financial autarky to incomplete markets will be rather low. Put differently, foregoing the possibility of such a small amount of precautionary saving cannot be too costly.

Table 7 reports the welfare gains, ϕ_2, under type I risk and for different values of γ. Further, comparing ϕ_1 in Table 5 with ϕ_2 in Table 7, we can see that the difference $(\phi_1 - \phi_2)$ is negligible. In fact, even for $\gamma = 5$, welfare gains from incomplete to complete markets (ϕ_2) are 0.43 percent, while the welfare gains from financial autarky to complete markets (ϕ_1) are 0.44 percent. In other words, just 0.01 percentage points of the total welfare gains come from self-insurance.

[17] Comparing these estimates with those for tranquil times in Table 1 indicates that, as expected, growth in tranquil times is greater than for the whole sample (0.0262 compared to 0.0165) while volatility is lower (0.0450 compared to 0.0601).

4.4.2 Income risk type II: Combination of normal and power-law distributions

Suppose now that the income distribution is a combination of two random variables: (i) a normal distribution conditional on a non-disaster state and (ii) a fat-tailed distribution conditional on a disaster event. Formally, we follow Barro and Jin (2011) and assume that x has the following distribution:

$$\begin{array}{ll} \text{with probability } 1-p: & \log(x) = \log(\tilde{x}) \sim N(\tilde{\mu}_x, \tilde{\sigma}_x^2), \\ \text{with probability } p: & x = 1/z, \end{array} \tag{26}$$

where z, the reciprocal of the contraction size, follows a power-law distribution with probability density function given by

$$f_Z(z) = (\alpha - 1)(z_{min})^{\alpha-1} z^{-\alpha},$$

where $\alpha > 1$. Thus, conditional on a disaster event, income follows an inverse power-law distribution.

A moment's reflection should make clear that, for the purposes of estimating the combined distribution (26), we can simply rely on the estimates shown in Table 1. These estimates are based on the iterative procedure detailed in Section 2, which assigns every growth point in our total sample of 156 countries to either a disaster year or tranquil times. The probability of each assignment reflects the parameter p, computed as the ratio of disaster years to non-disaster years. The estimate of z_{min} is the value such that all observations above it (i.e., $z \geq z_{min}$) are disaster events. The remaining observations belong to tranquil times. Conditional on z_{min}, the estimate of α provides us with the critical power-law parameter. For convenience, then, Table 8 reports the point estimates of z_{min}, α, p, $\tilde{\mu}_x$, and $\tilde{\sigma}_x$ and the confidence interval for α, all obtained from Table 1.[18] The estimates $\tilde{\mu}_x$, and $\tilde{\sigma}_x$ parameterize the log-normal distribution followed by all observations in tranquil times. All these estimates now become parameters that we use to calibrate the combined distribution (26) and thus compute welfare gains for type II risk.

In order to compute welfare under financial autarky, we need to obtain (recall equation (20))

$$\mathbb{E}[(x)^{1-\gamma}] = p\mathbb{E}[(z)^{\gamma-1}] + (1-p)\mathbb{E}[(\tilde{x})^{1-\gamma}].$$

Hence, we need to compute the expectation of income growth conditional on the disaster event and conditional on the non-disaster state. The second term on the

[18]This range allows us to check how the main results change as tails become fatter (as captured by α).

right-hand side can be computed as previously done since $\tilde{x}$ is log-normally distributed. Thus, we can show that

$$\mathbb{E}[(\tilde{x})^{1-\gamma}] = \exp((1-\gamma)\tilde{\mu}_x + \frac{1}{2}(1-\gamma)^2\tilde{\sigma}_x^2).$$

For the power-law distribution component of the expectation above, we can obtain:

$$\begin{aligned}\mathbb{E}[(z)^{\gamma-1}] &= (\alpha-1)(z_{min})^{\alpha-1}\int_{z_{min}}^{\infty} z^{\gamma-1-\alpha}dz = (\alpha-1)(z_{min})^{\alpha-1}\int_{z_{min}}^{\infty}\frac{d}{dz}(\frac{z^{\gamma-\alpha}}{\gamma-\alpha})dz \\ &= (\alpha-1)(z_{min})^{\alpha-1}\int_{z_{min}}^{\infty} d\left(\frac{z^{\gamma-\alpha}}{\gamma-\alpha}\right)\end{aligned}$$

This integral is well defined if $\alpha > \gamma$, implying in that case that

$$\mathbb{E}[(z)^{\gamma-1}] = \frac{\alpha-1}{\alpha-\gamma}(z_{min})^{\gamma-1}. \tag{27}$$

Therefore, when $\alpha > \gamma$, it follows that

$$\phi_1 = \left(\frac{1+\beta exp((1-\gamma)g_x)}{1+\beta(1-p)\exp((1-\gamma)\tilde{\mu}_x + \frac{1}{2}(1-\gamma)^2\tilde{\sigma}_x^2) + \beta p\frac{\alpha-1}{\alpha-\gamma}(z_{min})^{\gamma-1}}\right)^{\frac{1}{1-\gamma}} - 1.$$

It should be noted that if $\alpha \leq \gamma$, then $\mathbb{E}[(z)^{\gamma-1}] \to \infty$. Moreover, if $\gamma > 1$ we get $\phi_1 \to \infty$ as well. Hence, when $\alpha \leq \gamma$, welfare gains (ϕ_1) are unbounded. Table 9 shows the gains of financial integration for different values of γ based on the parameter values for type II risk presented in Table 8. As in Table 2, we use $\beta = 0.98$. The table indicates that for $\alpha = 3.9846$, welfare gains are unbounded for $\gamma \geq 4$. Intuitively, recall that the lower is α, the fatter the tails, and the bigger will be rare events.

In the case of incomplete markets, we proceed as follows. First, we can prove that $\mathbb{E}[(x+R\widetilde{b}_1)^{1-\gamma}] < \infty$ if $\widetilde{b}_1$ is finite, using the fact that conditional on the disaster event, $1/x$ has a power-law distribution. Hence, we know that $\mathbb{E}[(x+R\widetilde{b}_1)^{1-\gamma}] < \mathbb{E}[(R\widetilde{b}_1)^{1-\gamma}] = (R\widetilde{b}_1)^{1-\gamma}$. Second, as with type I risk, we perform a numerical integration for each possible value of $\widetilde{b}_1 \in [0,1]$ to solve the fixed point value that satisfies (22).

Table 10 shows the bond holdings ($\widetilde{b}_1$) in the case of type II risk for different values of γ and α. We can see that self-insurance can account for several percentage points of income. Specifically, for γ between 3 and 5 and α in the lower range of the estimated values (3.9-4.5), precautionary savings are in the range 2.2-9.5 percent of income, substantially higher than in the case of only normally-distributed income risk (where welfare gains were below one percent, as indicated by Table 5).

Welfare gains (ϕ_2) from the non-contingent bond equilibrium to complete markets with type II risk are shown in Table 11 for different values of γ and α. Depending on the size of the tail distribution of disasters, the welfare gains from incomplete markets to complete markets lie in the range 0.39 to 1.48 percent for risk aversion coefficients around 2-3. For a risk aversion coefficient between 4 and 5, the welfare gains turn out to be as high as 5.59 percent for the lower bound of the estimated α (3.98-4.59).

To capture how much welfare gains change when we allow for rare disasters risk, we compute the factor increase in ϕ_1 and ϕ_2 in the case of type II risk (reported in Tables 9 and 11, respectively), relative to type I risk (reported in Tables 5 and 7, respectively). A factor increase of, say, 3 means that welfare gains triple. The factor increase for different risk aversion coefficients is presented in Table 12. We can see that even for low values of γ, such as $\gamma = 2$, welfare gains may double for both ϕ_1 and ϕ_2. For higher values of γ, such as $\gamma = 4$, welfare gains may increase by a factor of 8 for ϕ_2 and become unbounded for ϕ_1 . Welfare gains thus increase dramatically when we allow for tail or rare events captured by the power-law distribution, even if the occurrence of the tail or rare event has a small probability.

Appendix C shows that our main results on the welfare gains from financial integration with macroeconomic disasters remain valid in an infinite-horizon model. Adding the risk of the size distribution of macroeconomic disasters as it is observed in long-span data increases the welfare gains of financial integration by factors of 3 to 7 for values of $\gamma \leq 2.5$ (relative to standard computations based only on normally-distributed income fluctuations). Welfare gains may become unbounded for higher values of γ.

4.5 The role of the distribution of the size of disasters

Our framework above focuses on incorporating disaster risk into the analysis. We characterize empirically rare disasters as events with a low probability of occurrence and where the fall in income is distributed with fat tails conditional on the occurrence of rare disasters. While the distribution of falls in income conditional on rare disasters is well described in long-span data with a power-law distribution, other types of distributions can be used to capture rare disasters.

In particular, Barro (2006) models rare disasters with a deterministic fall in income in order to quantitatively explain the size of the equity premium. In the same vein, Martin (2013) shows that incorporating rare disasters into the analysis makes it easier to match observed asset pricing behavior without requiring implausibly large risk aversion coefficients. Martin (2013) assumes that the fall in dividends follows a normal distribution conditional on the occurrence of a rare disaster. He also stresses that the gains from financial integration are much larger when adding rare disasters in the Cole-Obstfeld two-country model. In this case, Martin (2010) also assumes a normal distribution for the fall in income conditional on the rare disaster event.

In this context, this subsection assesses how important is the size distribution of income in the disaster event. We will compute the welfare gains with rare disasters in the income process, but without imposing a power-law distribution for the fall in income conditional on the rare event. Specifically, we will analyze two cases for the size distribution of income under rare disasters. First, we assume that, in the case of a rare event, the fall in income is deterministic. Second, we assume a normal distribution for the size of income in the rare event.

Formally, the first case will assume the following process for income growth in period 2, given by x:

$$\begin{array}{ll} \text{with probability } 1-p & \log(x) = \log(\tilde{x}) \sim N(\tilde{\mu}_x, \tilde{\sigma}_x^2), \\ \text{with probability } p & x = b, \end{array}$$

where b is a deterministic value. To keep the same expected value for x, we set b such that $b = \mathbb{E}[1/z]$, where z follows the same power-law distribution described above. The second case assumes that x follows the distribution given by:

$$\begin{array}{ll} \text{with probability } 1-p & \log(x) = \log(\tilde{x}) \sim N(\tilde{\mu}_x, \tilde{\sigma}_x^2), \\ \text{with probability } p & \log(x) = \log(\tilde{z}) \sim N(\tilde{\mu}_z, \tilde{\sigma}_z^2). \end{array}$$

In this case, rare events also follow a normal distribution. We choose the value of parameters $\tilde{\mu}_z$ and $\tilde{\sigma}_z$ in order to match the first and second moments of $1/z$ in the power-law distribution case; that is, $\mathbb{E}[1/z] = \exp(\tilde{\mu}_z + \frac{1}{2}\tilde{\sigma}_z^2)$ and $\mathbb{E}[1/z^2] = \exp(2\tilde{\mu}_z + 2\tilde{\sigma}_z^2)$.

Hence, in the first case, the expected value for $\mathbb{E}[x^{1-\gamma}]$ is given by

$$\mathbb{E}[x^{1-\gamma}] = (1-p)\exp((1-\gamma)\tilde{\mu}_x + \frac{(1-\gamma)^2}{2}\tilde{\sigma}_x^2) + pb^{1-\gamma}. \tag{28}$$

In the second case, we obtain:

$$\mathbb{E}[x^{1-\gamma}] = (1-p)\exp((1-\gamma)\tilde{\mu}_x + \frac{(1-\gamma)^2}{2}\tilde{\sigma}_x^2) + p\exp((1-\gamma)\tilde{\mu}_z + \frac{(1-\gamma)^2}{2}\tilde{\sigma}_z^2). \tag{29}$$

We substitute equations (28) and (29) into equation (20) to compute the welfare gains measured by ϕ_1. Table 13 shows the gains under these two cases. Since different values for α imply different moments for $1/z$, we have a range for the implied values for b, $\tilde{\mu}_z$ and $\tilde{\sigma}_z$.

Two observations follow from Table 13. First, welfare gains (ϕ_1) are higher in the normal distribution case relative to the deterministic case by about a factor of around 2 for high values of γ and values of b, $\tilde{\mu}_z$ and $\tilde{\sigma}_z$ consistent with $\alpha = 3.98$ (compare, for instance, 3.50 percent to 1.60 percent). Second – and very important for the message of our paper on the critical role of fat-tailed distributions – the welfare gains for the two cases captured in Table 13 are much smaller than under the power-law distribution illustrated in Table 9. To see this, Table 14 shows the factors by which welfare gains increase in the deterministic and normal distribution cases relative to our benchmark case captured by Table 13. Comparing this with the equivalent table for the case of the power-law distribution (Table 12) makes this point abundantly clear.

We thus conclude that the way of modeling disaster risk is critical in generating large welfare gains due to financial integration. Specifically, assuming a power-law distribution for disaster risk compared to either a deterministic distribution or normal distribution implies much larger welfare gains. Hence, incorporating disaster risk is necessary, but not sufficient, to generate large welfare gains. In addition, we need to assume a power-law distribution for disaster risk.

4.6 Composition of welfare gains

Based on a binomial distribution and back-of-the-envelope calculations, Section 3 showed that overall gains from financial integration (i.e., from financial autarky to complete markets) are very small (less than one percent), as suggested by Cole and Obstfeld (1991). In terms of the composition of welfare gains, we showed that most of the welfare gains accrue when moving from incomplete markets to complete markets. Welfare gains from financial autarky to incomplete markets are close to zero (and literally zero in the quadratic case). We also showed that, as the shock becomes larger, welfare gains from financial autarky to incomplete markets become much more important.

How do these results hold in the much more refined stochastic model of this section? Before answering this question, we can use equation (9) to compute $\widetilde{\phi}_1$ (welfare gains from financial autarky to incomplete markets) in the normal distribution case, for given ϕ_1 and ϕ_2. To this effect, we use the results summarized in Tables 5 and 7 to compute $\widetilde{\phi}_1$. We then compute $\widetilde{\phi}_1$ for the power-law case using Tables 9 and 11.

Figure 5 presents the results of these calculations. Each bar represents the welfare gains from financial integration, with the blue segment capturing welfare gains from financial autarky to incomplete markets and the orange one from incomplete to complete markets. Several important observations follow. First, as in Section 3, the welfare gains under a normal distribution (first bar) are around 0.5 percent and essentially all accruing from incomplete to complete markets. Second, for the power-law distribution (the remaining three bars), the fatter the tail (i.e., the lower the α), the larger the welfare gains. For the last bar ($\alpha = 4$), welfare gains are about 4 percent (i.e., eight times larger than under the normal distribution). Third, the ratio ϕ_2/ϕ_1 varies from close to one (first bar) to around 50 percent (last bar). In other words, for big shocks, the government holds the same amount of non-contingent and contingent assets.[19] In this light, the large holdings of non-contingent assets held by governments in emerging markets (in the form of international reserves and risk-free assets such as U.S. treasury bills) would be justified.

5 Policy conclusions

As a benchmark case, we have shown that, in a Gaussian world, overall welfare gains of financial integration are quite small (i.e., less than 1 percent of GDP), which is consistent with most of the existing literature. In terms of the *composition* of welfare gains (a novel dimension), almost all gains accrue from incomplete to complete markets and essentially none from financial autarky to incomplete markets. The policy implication in this Gaussian world would thus be to hold few non-contingent assets given that the pay-off is almost nil. This, however, clearly contradicts the empirical evidence that points to relatively large holdings of non-contingent assets (i.e., international reserves).

What are we, as a profession, missing? The key, to paraphrase Dorothy in The Wizard of Oz, is that we are not in a Gaussian world anymore! Truth be told, we never were, but macroeconomic theory assumed that we did! However, it is almost impossible to find a relevant macroeconomic time series (output, stock markets, etc.) that follows our convenient, but false, assumption of normality. As we showed for output growth for a sample of 156 countries for the period 1900-2018, all such macro-series are very well described by a fat-tailed distribution.

In the presence of a power-law distribution, welfare gains from financial integration increase by, at least, a factor of 6. Further, in terms of the composition, our preferred parameterization suggests that welfare gains are about the same in the first

[19]Note that, in both the normal and power-law cases, as we move to the left of the distribution, the probability of disaster falls. But it falls much faster in the normal than in the power-law case. So, the same intuition discussed in Section 3 holds,: the fatter the tails, the smaller ϕ_2/ϕ_1 is.

segment (financial autarky to incomplete markets) as in the second (incomplete to complete markets), at about 2 percent of GDP each, for a total of 4 percent of GDP. Intuitively, when countries are, on average, facing the probability of falls in GDP of 16 percent around 3 times in a century – and completing the markets is not feasible and/or too expensive – precautionary savings are bound to become quite large even for low levels of prudence. This scenario fits much better what we see in the real world. Our policy conclusion is thus that, in a world of fat tails, there is a clear theoretical case for self-insurance.

Eventually, of course, the first-best equilibrium is for emerging and developing economies to have access to full state-contingent assets such as "catastrophe bonds." In the meantime, however, it makes sense for emerging and developing economies to invest in non-contingent assets (including sovereign wealth funds and safe assets) when facing large uncertain risks.

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6 Appendices

6.1 Appendix A: Start years of the sample of countries

This appendix provides the year in which the GDP sample starts for each of our 156 countries (20 developed and 136 developing); see Table A.1.

6.2 Appendix B (online): Two-period model with binomial distribution

This appendix shows the formal derivation of expressions used in Section 3.

Consider a small open economy perfectly integrated into world good markets. There is a single, non-storable tradable good. We will consider three different scenarios in terms of financial integration with the rest of the world: (i) complete markets (i.e., the economy can fully insure against risk in the second period); (ii) financial autarky (i.e., no borrowing/lending from/to the rest of the world); and (iii) incomplete markets (i.e., there is a single, risk-free bond in the world).

The endowment in the first period is known with certainty (and equal to y_1). Period 2-endowment is given by

$$y_2 = \begin{cases} y_2^H \text{ with probability } p, \\ y_2^L \text{ with probability } 1-p, \end{cases}$$

where $y_2^H \geq y_1 \geq y_2^L$. For simplicity, we will assume that $\mathbb{E}(y_2) = py_2^H + (1-p)y_2^L$ is equal to y_1 and that $p = 1/2$.

Lifetime expected utility (welfare) is given by

$$W = u(C_1) + \beta \mathbb{E}\{u(C_2)\}, \tag{30}$$

where W denotes welfare, $0 < \beta < 1$ is the discount factor, and $u(.)$ is given by a standard constant-relative-risk-aversion (CRRA) utility function:

$$\begin{array}{ll} u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}, & \gamma \neq 1, \\ u(c) = \log(c), & \gamma = 1. \end{array} \tag{31}$$

where $\gamma > 0$ is the coefficient of risk aversion. We assume $\beta(1+r) = 1$, where r is the world risk-free interest rate. Initial net assets are assumed to be zero.

6.2.1 Complete markets

Under complete markets, consumers can buy contingent claims that promise to deliver one unit of output in the good (bad) state of nature at a second-period price of q^H (q^L). The intertemporal budget constraint thus takes the form:

$$y_1 + \frac{q^H y_2^H + q^L y_2^L}{1+r} = C_1 + \frac{q^H C_2^H + q^L C_2^L}{1+r}. \tag{32}$$

It is easy to show that, if prices are actuarially fair (i.e., $q^H/q^L = p/(1-p)$), then $C_1 = C_2^H = C_2^L$. Consumers smooth consumption across states of nature and across time (recall that $\beta(1+r) = 1$). Since $q^H = p = 0.5$ and $q^L = 1 - p = 0.5$ (and, by assumption, $\mathbb{E}(y_2) = y_1$), we can use (32) to solve for C_1:

$$\begin{aligned}
y_1 + \frac{q^H y_2^H + q^L y_2^L}{1+r} &= C_1 + \frac{q^H C_1 + q^L C_1}{1+r}, \\
y_1 + \frac{\mathbb{E}(y_2)}{1+r} &= C_1\left(1 + \frac{q^H + q^L}{1+r}\right), \\
y_1\left(1 + \frac{1}{1+r}\right) &= C_1\left(1 + \frac{q^H + q^L}{1+r}\right), \\
y_1 &= C_1.
\end{aligned}$$

Hence,

$$C_1 = C_2^H = C_2^L = y_1.$$

Welfare under complete markets is thus given by

$$\begin{aligned}
W_{CM} &= \frac{y_1^{1-\gamma} - 1}{1-\gamma} + \beta\left[0.5\frac{y_1^{1-\gamma} - 1}{1-\gamma} + 0.5\frac{y_1^{1-\gamma} - 1}{1-\gamma}\right], \\
W_{CM} &= \frac{y_1^{1-\gamma} - 1}{1-\gamma}(1+\beta). \qquad (33)
\end{aligned}$$

6.2.2 Financial autarky

Under financial autarky, the following equilibrium conditions must hold:

$$\begin{aligned}
C_1 &= y_1, \\
C_2^H &= y_2^H, \\
C_2^L &= y_2^L.
\end{aligned}$$

Welfare under financial autarky (FA) is thus given by

$$W_{FA}(\phi_1) = \frac{[y_1(1+\phi_1)]^{1-\gamma}-1}{1-\gamma} + \beta\left[0.5\frac{[y_2^H(1+\phi_1)]^{1-\gamma}-1}{1-\gamma} + 0.5\frac{[y_2^L(1+\phi_1)]^{1-\gamma}-1}{1-\gamma}\right], \tag{34}$$

where, for further reference, we have added a potential compensating variation to consumption of a factor $1+\phi_1$.

The welfare gains of having complete markets relative to the case of financial autarky is the value of ϕ_1 that satisfies

$$W_{FA}(\phi_1) = W_{CM}.$$

Using (33) and (34), we can solve for ϕ_1:

$$\phi_1 = \left\{\frac{1+\beta}{1+\beta\,(0.5)\left[\left(\frac{y_2^H}{y_1}\right)^{1-\gamma} + \left(\frac{y_2^L}{y_1}\right)^{1-\gamma}\right]}\right\}^{\frac{1}{1-\gamma}} - 1. \tag{35}$$

As a particular case, notice that if $y_1 = y_2^H = y_2^L$, then $\phi_1 = 0$ because consumption is already fully smoothed (across states and time) in the financial autarky case.

6.2.3 Incomplete markets

The flow budget constraints are given by:

$$b_1 = y_1 - C_1, \tag{36}$$

$$0 = (1+r)b_1 + y_2^H - C_2^H, \tag{37}$$

$$0 = (1+r)b_1 + y_2^L - C_2^L, \tag{38}$$

where C_2^H $\left(C_2^L\right)$ is consumption in the high-output (low-output) state of nature and b_1 is the trade balance (i.e., saving) in the first period.

For further reference, rewrite the budget constraints as:

$$C_1 = y_1\left(1-\widetilde{b}_1\right), \tag{39}$$

$$C_2^H = y_1\left[(1+r)\widetilde{b}_1 + \frac{y_2^H}{y_1}\right], \tag{40}$$

$$C_2^L = y_1\left[(1+r)\widetilde{b}_1 + \frac{y_2^L}{y_1}\right], \tag{41}$$

where $\widetilde{b}_1 = b_1/y_1$ is saving as a proportion of output (GDP).

Combining the flow constraints, we obtain the intertemporal constraint for each state of nature in period 2:

$$y_1 + \frac{1}{1+r} y_2^H = C_1 + \frac{1}{1+r} C_2^H, \tag{42}$$

$$y_1 + \frac{1}{1+r} y_2^L = C_1 + \frac{1}{1+r} C_2^L. \tag{43}$$

The consumer maximizes expected utility, given by (30), subject to (42) and (43). Combining first-order conditions, we obtain the familiar stochastic Euler equation:

$$u'(C_1) = \mathbb{E}\{u'(C_2)\}. \tag{44}$$

Solution of the model for $u'''(c) > 0$ In this case (the more natural case), consumers engage in precautionary saving. Since the CRRA preferences given by (31) imply that $u'''(C) > 0$, $u'(C)$ is a convex function. Hence, $\mathbb{E}\{u'(C_2)\} > u'(\mathbb{E}\{C_2\})$. The Euler equation (44) can be rewritten as

$$u'(C_1) > u'[\mathbb{E}\{C_2\}].$$

It follows that

$$C_1 < \mathbb{E}\{C_2\}. \tag{45}$$

Consumers do not smooth consumption (in an expected value sense). In fact, their period-1 consumption is less than their period-2 expected consumption because they wish to save more (relative to the certainty case or the quadratic case) in case the bad state of nature materializes in period 2.

To see the implication of (45), first multiply (42) by p and (43) by $1-p$ to obtain

$$C_1 + \frac{\mathbb{E}\{C_2\}}{1+r} = y_1 + \frac{\mathbb{E}\{y_2\}}{1+r}. \tag{46}$$

Suppose again that $y_1 = \mathbb{E}\{y_2\}$ and rewrite this last equation as

$$C_1 + \frac{\mathbb{E}\{C_2\}}{1+r} = \left(\frac{2+r}{1+r}\right) y_1. \tag{47}$$

Since $\mathbb{E}\{C_2\} > C_1$, it follows that

$$C_1 + \frac{\mathbb{E}\{C_2\}}{1+r} > \left(\frac{2+r}{1+r}\right) C_1.$$

Using this last inequality, it follows from (47) that

$$\begin{aligned}\left(\frac{2+r}{1+r}\right)y_1 &> \left(\frac{2+r}{1+r}\right)C_1\\ y_1 &> C_1.\end{aligned}$$

This means that $TB_1(=S_1) > 0$. Consumers thus engage in precautionary saving. Since they are concerned about the second period's uncertainty, they save in period 1 so that they can consume a little bit more in case the low state of nature materializes.

Derivation of welfare gains Using our CRRA specification, we can compute the welfare gains from financial integration (FA to IM). First, rewrite the Euler equation (44) as

$$C_1^{-1/\gamma} = p(C_2^H)^{-1/\gamma} + (1-p)(C_2^L)^{-1/\gamma}.$$

Recall that $p = 1/2$. Then:

$$\frac{2}{C_1^\gamma} = \frac{1}{(C_2^H)^\gamma} + \frac{1}{(C_2^L)^\gamma}, \tag{48}$$

$$0 = (1+r)(y_1 - C_1) + y_2^H - C_2^H, \tag{49}$$

$$0 = (1+r)(y_1 - C_1) + y_2^L - C_2^L. \tag{50}$$

This is a system of 3 equations in 3 unknowns (C_1, C_2^H, and C_2^L) with the log case ($\gamma = 1$) as a particular case.

By definition, welfare is given by

$$W_{IM} = \frac{C_1^{1-\gamma} - 1}{1-\gamma} + \beta\left[0.5\frac{\left(C_2^H\right)^{1-\gamma} - 1}{1-\gamma} + 0.5\frac{\left(C_2^L\right)^{1-\gamma} - 1}{1-\gamma}\right].$$

Substituting (39), (40), and (41) into the last equation, we obtain:

$$W_{IM} = \frac{\left[y_1\left(1-\widetilde{b}_1\right)\right]^{1-\gamma} - 1}{1-\gamma} + 0.5\beta\left[\frac{\left(y_1\left[(1+r)\widetilde{b}_1 + \frac{y_2^H}{y_1}\right]\right)^{1-\gamma} - 1}{1-\gamma} + \frac{\left(y_1\left[(1+r)\widetilde{b}_1 + \frac{y_2^L}{y_1}\right]\right)^{1-\gamma} - 1}{1-\gamma}\right] \tag{51}$$

Adding a compensating variation, ϕ_2, to capture how much we would need to give to the consumer under incomplete markets for him/her to attain the complete-markets welfare:

$$
\begin{aligned}
W_{IM}(\phi_2) &= \frac{\left[y_1(1+\phi_2)\left(1-\widetilde{b}_1\right)\right]^{1-\gamma}-1}{1-\gamma} \\
&+0.5\beta\left[\begin{array}{c}\dfrac{\left(y_1(1+\phi_2)\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]\right)^{1-\gamma}-1}{1-\gamma} \\ +\dfrac{\left(y_1(1+\phi_2)\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]\right)^{1-\gamma}-1}{1-\gamma}\end{array}\right]
\end{aligned}
$$

Using (33) and (51), the welfare gains of moving from incomplete to complete markets are given by

$$
\begin{aligned}
W_{CM} &= W_{IM}(\phi_2) \\
\frac{y_1^{1-\gamma}-1}{1-\gamma}(1+\beta) &= \frac{\left[y_1(1+\phi_2)\left(1-\widetilde{b}_1\right)\right]^{1-\gamma}-1}{1-\gamma} \\
&+0.5\beta\left[\begin{array}{c}\dfrac{\left(y_1(1+\phi_2)\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]\right)^{1-\gamma}-1}{1-\gamma} \\ +\dfrac{\left(y_1(1+\phi_2)\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]\right)^{1-\gamma}-1}{1-\gamma}\end{array}\right] \\
\frac{(1+\beta)y_1^{1-\gamma}}{1-\gamma}-\frac{1+\beta}{1-\gamma} &= \frac{(1+\phi_2)^{1-\gamma}\left[y_1\left(1-\widetilde{b}_1\right)\right]^{1-\gamma}}{1-\gamma}-\frac{1}{1-\gamma} \\
&+0.5\beta\left[\begin{array}{c}\dfrac{(1+\phi_2)^{1-\gamma}\left(y_1\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]\right)^{1-\gamma}}{1-\gamma}-\dfrac{1}{1-\gamma} \\ +\dfrac{(1+\phi_2)^{1-\gamma}\left(y_1\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]\right)^{1-\gamma}}{1-\gamma}-\dfrac{1}{1-\gamma}\end{array}\right]
\end{aligned}
$$

Notice that all the terms that do not have output cancel each other out. Hence,

$$
\begin{aligned}
\frac{(1+\beta)y_1^{1-\gamma}}{1-\gamma} &= \frac{(1+\phi_2)^{1-\gamma}\left[y_1\left(1-\widetilde{b}_1\right)\right]^{1-\gamma}}{1-\gamma} \\
&\quad +0.5\beta\left[\begin{array}{c} \frac{(1+\phi_2)^{1-\gamma}\left(y_1\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]\right)^{1-\gamma}}{1-\gamma} \\ +\frac{(1+\phi_2)^{1-\gamma}\left(y_1\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]\right)^{1-\gamma}}{1-\gamma} \end{array}\right] \\
(1+\beta)y_1^{1-\gamma} &= (1+\phi_2)^{1-\gamma}\left\{\begin{array}{l} \left[y_1\left(1-\widetilde{b}_1\right)\right]^{1-\gamma} \\ +0.5\beta\left(y_1\left[(1+r)\widetilde{b}_1+\dfrac{y_2^H}{y_1}\right]\right)^{1-\gamma} \\ +0.5\beta\left(y_1\left[(1+r)\widetilde{b}_1+\dfrac{y_2^L}{y_1}\right]\right)^{1-\gamma} \end{array}\right\}
\end{aligned}
$$

Solving for ϕ_2:

$$
\phi_2 = \left\{\frac{(1+\beta)y_1^{1-\gamma}}{\left[y_1\left(1-\widetilde{b}_1\right)\right]^{1-\gamma}+0.5\beta\left[\left(y_1\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]\right)^{1-\gamma}+\left(y_1\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]\right)^{1-\gamma}\right]}\right\}^{\frac{1}{1-\gamma}} -1.
$$

Simplifying, we get our final expression for ϕ_2:

$$
\phi_2 = \left\{\frac{1+\beta}{\left[\left(1-\widetilde{b}_1\right)\right]^{1-\gamma}+0.5\beta\left[\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]^{1-\gamma}+\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]^{1-\gamma}\right]}\right\}^{\frac{1}{1-\gamma}} - 1. \tag{52}
$$

A check that we can do here is that, under no uncertainty (i.e., $y_2^H = y_2^L$), ϕ_2 should be 0. To show this, notice that, in this equilibrium, it must be the case that $y_2^H = y_2^L = y_1$ (to make cases comparable) because under complete markets we assume that $\mathbb{E}(y_2) = y_1$. Hence, $\widetilde{b}_1 = 0$. Substituting this into (52) yields

$$
\begin{aligned}
\phi_2|_{\widetilde{b}_1=0} &= \left\{ \frac{1+\beta}{\left[\left(1-\widetilde{b}_1\right)\right]^{1-\gamma} + \beta\left[\left[(1+r)\widetilde{b}_1+1\right]^{1-\gamma}\right]} \right\}^{\frac{1}{1-\gamma}} - 1, \\
&= \left\{\frac{1+\beta}{1+\beta}\right\}^{\frac{1}{1-\gamma}} - 1 = 0.
\end{aligned}
$$

For the purposes of Tables 2 and 3, we computed ϕ_2 by solving the system (48)-(50) numerically in Mathematica with the FindRoot command, then using $c1$ to obtain $\widetilde{b}_1$, and finally substituting $\widetilde{b}_1$ into (52).

6.2.4 Quadratic consumer

Quadratic preferences are given by $u(C) = -\frac{1}{2}(\bar{C} - C)^2$ for $C \leq \bar{C}$ and where $\bar{C}$ is the "bliss" level of consumption. In this case, $u'(C) = \bar{C} - C \geq 0$, $u''(C) = -1 < 0$, and $u'''(C) = 0$.

Notice that if $u'''(C) = 0$, then $u'(C)$ is linear and thus:

$$pu'(C_2^H) + (1-p)u'(C_2^L) = u'[pC_2^H + (1-p)C_2^L].$$

We can thus rewrite the Euler equation (44) as

$$
\begin{aligned}
u'(C_1) &= u'[pC_2^H + (1-p)C_2^L], \\
u'(C_1) &= u'[\mathbb{E}\{C_2\}],
\end{aligned}
$$

which implies that

$$C_1 = \mathbb{E}\{C_2\}. \tag{53}$$

Substitute this last equation into (46) to obtain:

$$C_1 = \frac{1+r}{2+r}\left[y_1 + \frac{\mathbb{E}\{y_2\}}{1+r}\right].$$

The trade balance will be given by:

$$
\begin{aligned}
TB_1 &= y_1 - C_1, \qquad (54)\\
TB_1 &= \frac{1}{2+r}\left[y_1 - \mathbb{E}\{y_2\}\right]. \qquad (55)
\end{aligned}
$$

The reduced forms for C_2^H and C_2^L are given by:

$$
\begin{aligned}
C_2^H &= y_2^H + \frac{1+r}{2+r}\left[y_1 - \mathbb{E}\{y_2\}\right], \\
C_2^L &= y_2^L + \frac{1+r}{2+r}\left[y_1 - \mathbb{E}\{y_2\}\right].
\end{aligned}
$$

We can rewrite C_1 as (note that $TB_1 = b_1$):

$$
\begin{aligned}
\frac{b_1}{y_1} &= 1 - \frac{C_1}{y_1}, \\
\frac{C_1}{y_1} &= 1 - \widetilde{b_1}, \\
C_1 &= y_1\left(1 - \widetilde{b_1}\right).
\end{aligned}
$$

We rewrite C_2^H and C_2^L as

$$
\begin{aligned}
C_2^H &= y_2^H + \frac{1+r}{2+r}\left[y_1 - \mathbb{E}\{y_2\}\right], \\
C_2^H &= y_2^H + (1+r)\, b_1, \\
\frac{C_2^H}{y_1} &= \frac{y_2^H}{y_1} + (1+r)\, \widetilde{b_1}, \\
C_2^H &= y_1\left(\frac{y_2^H}{y_1} + (1+r)\, \widetilde{b_1}\right), \\
C_2^L &= y_1\left(\frac{y_2^L}{y_1} + (1+r)\, \widetilde{b_1}\right).
\end{aligned}
$$

6.2.5 Computation of welfare

Recall that welfare for the IM case is given by

$$
\begin{aligned}
W_{IM} &= \frac{C_1^{1-\gamma}-1}{1-\gamma}+\beta\left[0.5\frac{\left(C_2^H\right)^{1-\gamma}-1}{1-\gamma}+0.5\frac{\left(C_2^L\right)^{1-\gamma}-1}{1-\gamma}\right] \\
&= \frac{\left[y_1\left(1-\widetilde{b}_1\right)\right]^{1-\gamma}-1}{1-\gamma}+(0.5)\,\beta\left[\begin{array}{c}\dfrac{\left(y_1\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]\right)^{1-\gamma}-1}{1-\gamma} \\ +\dfrac{\left(y_1\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]\right)^{1-\gamma}-1}{1-\gamma}\end{array}\right].
\end{aligned}
$$

Adding the compensating variation ϕ_2:

$$
\begin{aligned}
W_{IM} &= \frac{\left[y_1(1+\phi_2)\left(1-\widetilde{b}_1\right)\right]^{1-\gamma}-1}{1-\gamma}+(0.5)\,\beta\left[\begin{array}{c}\dfrac{\left(y_1(1+\phi_2)\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]\right)^{1-\gamma}-1}{1-\gamma} \\ +\dfrac{\left(y_1(1+\phi_2)\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]\right)^{1-\gamma}-1}{1-\gamma}\end{array}\right] \\
W_{IM} &= \frac{\left[y_1(1+\phi_2)\left(1-\widetilde{b}_1\right)\right]^{1-\gamma}}{1-\gamma}-\frac{1}{1-\gamma} \\
&\quad +(0.5)\,\beta\left[\begin{array}{c}\dfrac{\left(y_1(1+\phi_2)\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]\right)^{1-\gamma}}{1-\gamma}-\dfrac{1}{1-\gamma} \\ +\dfrac{\left(y_1(1+\phi_2)\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]\right)^{1-\gamma}}{1-\gamma}-\dfrac{1}{1-\gamma}\end{array}\right]
\end{aligned}
$$

The welfare gains from moving from IM to CM are given by

$$
\begin{aligned}
W_{CM} &= W_{IM}(\phi_2) \\
\frac{y_1^{1-\gamma}-1}{1-\gamma}(1+\beta) &= \frac{\left[y_1(1+\phi_2)\left(1-\widetilde{b}_1\right)\right]^{1-\gamma}-1}{1-\gamma} \\
&\quad + (0.5)\,\beta \left[\begin{array}{c} \dfrac{\left(y_1(1+\phi_2)\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]\right)^{1-\gamma}-1}{1-\gamma} \\ +\dfrac{\left(y_1(1+\phi_2)\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]\right)^{1-\gamma}-1}{1-\gamma} \end{array} \right] \\
\frac{(1+\beta)y_1^{1-\gamma}}{1-\gamma} - \frac{1+\beta}{1-\gamma} &= \frac{\left[y_1(1+\phi_2)\left(1-\widetilde{b}_1\right)\right]^{1-\gamma}}{1-\gamma} - \frac{1}{1-\gamma} \\
&\quad + (0.5)\,\beta \left[\begin{array}{c} \dfrac{\left(y_1(1+\phi_2)\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]\right)^{1-\gamma}}{1-\gamma} - \dfrac{1}{1-\gamma} \\ +\dfrac{\left(y_1(1+\phi_2)\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]\right)^{1-\gamma}}{1-\gamma} - \dfrac{1}{1-\gamma} \end{array} \right] \\
\frac{(1+\beta)y_1^{1-\gamma}}{1-\gamma} &= \frac{\left[y_1(1+\phi_2)\left(1-\widetilde{b}_1\right)\right]^{1-\gamma}}{1-\gamma} \\
&\quad + (0.5)\,\beta \left[\begin{array}{c} \dfrac{\left(y_1(1+\phi_2)\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]\right)^{1-\gamma}}{1-\gamma} \\ +\dfrac{\left(y_1(1+\phi_2)\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]\right)^{1-\gamma}}{1-\gamma} \end{array} \right] \\
(1+\beta) &= (1+\phi_2)^{1-\gamma} \left\{ \begin{array}{l} \left[\left(1-\widetilde{b}_1\right)\right]^{1-\gamma} \\ +(0.5)\,\beta\left[(1+r)\widetilde{b}_1+\dfrac{y_2^H}{y_1}\right]^{1-\gamma} \\ +(0.5)\,\beta\left[(1+r)\widetilde{b}_1+\dfrac{y_2^L}{y_1}\right]^{1-\gamma} \end{array} \right\} \\
(1+\phi_2)^{1-\gamma} &= \frac{1+\beta}{\left[\left(1-\widetilde{b}_1\right)\right]^{1-\gamma} + (0.5)\,\beta\left[\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]^{1-\gamma} + \left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]^{1-\gamma}\right]}
\end{aligned}
$$

$$\phi_2 = \left\{ \frac{1+\beta}{\left[\left(1-\widetilde{b}_1\right)\right]^{1-\gamma} + (0.5)\,\beta\left[\left[(1+r)\widetilde{b}_1 + \frac{y_2^H}{y_1}\right]^{1-\gamma} + \left[(1+r)\widetilde{b}_1 + \frac{y_2^L}{y_1}\right]^{1-\gamma}\right]} \right\}^{\frac{1}{1-\gamma}} -1,$$

which is the same expression that we had obtained for incomplete markets (see equation (51)).

Note that the expression for ϕ_1 will be the same as for the CRRA computation above (see equation (35)), replicated here for convenience:

$$\phi_1 = \left\{ \frac{1+\beta}{1+\beta\,(0.5)\left[\left(\frac{y_2^H}{y_1}\right)^{1-\gamma} + \left(\frac{y_2^L}{y_1}\right)^{1-\gamma}\right]} \right\}^{\frac{1}{1-\gamma}} - 1.$$

If $\widetilde{b}_1 = 0$, these two expressions are the same (i.e., $\phi_1 = \phi_2$). And that will be the case under quadratic preferences and $y_1 = \mathbb{E}(y_2)$. Intuitively, the quadratic consumer does not need to run a trade imbalance, so the lack of a risk-free bond is not binding. If y_1 is not equal to $\mathbb{E}(y_2)$, then the lack of a risk-free bond is binding and $\phi_1 > \phi_2$.

In contrast, there is no substitute for CM in the sense that only CM can equate consumption across markets So CM will always be binding. The risk-free bond may or may not be binding. It depends on preferences and shocks.

6.2.6 Welfare gains from FA to IM ($\widetilde{\phi}_1$)

To compute the welfare gains from moving from FA to IM, denoted by $\widetilde{\phi}_1$,consider

$$W_{FA}(\widetilde{\phi}_1) = W_{IM}.$$

$$\left(\begin{array}{c} \dfrac{\left(y_1(1+\widetilde{\phi}_1)\right)^{1-\gamma}-1}{1-\gamma} \\ +\beta\left[\begin{array}{c} 0.5\dfrac{\left(y_2^H(1+\widetilde{\phi}_1)\right)^{1-\gamma}-1}{1-\gamma} \\ +0.5\dfrac{\left(y_2^L(1+\widetilde{\phi}_1)\right)^{1-\gamma}-1}{1-\gamma} \end{array}\right] \end{array} \right) = \left(\begin{array}{c} \dfrac{\left[y_1\left(1-\widetilde{b}_1\right)\right]^{1-\gamma}-1}{1-\gamma} \\ +0.5\beta\left[\begin{array}{c} \dfrac{\left(y_1\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]\right)^{1-\gamma}-1}{1-\gamma} \\ +\dfrac{\left(y_1\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]\right)^{1-\gamma}-1}{1-\gamma} \end{array}\right] \end{array} \right)$$

$$\left(\begin{array}{c} \frac{\left(y_1(1+\widetilde{\phi}_1\right))^{1-\gamma}}{1-\gamma} - \frac{1}{1-\gamma} \\ +\beta \left[\begin{array}{c} 0.5\frac{\left(y_2^H(1+\widetilde{\phi}_1\right))^{1-\gamma}-1}{1-\gamma} \\ +0.5\frac{\left(y_2^L(1+\widetilde{\phi}_1\right))^{1-\gamma}-1}{1-\gamma} \end{array} \right] \end{array} \right) = \left(\begin{array}{c} \frac{\left[y_1\left(1-\widetilde{b}_1\right)\right]^{1-\gamma}}{1-\gamma} - \frac{1}{1-\gamma} \\ +0.5\beta \left[\begin{array}{c} \frac{\left(y_1\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]\right)^{1-\gamma}}{1-\gamma} - \frac{1}{1-\gamma} \\ +\frac{\left(y_1\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]\right)^{1-\gamma}}{1-\gamma} - \frac{1}{1-\gamma} \end{array} \right] \end{array} \right)$$

$$\left[(1+\widetilde{\phi}_1)^{1-\gamma}\right] \left(\begin{array}{c} \frac{(y_1)^{1-\gamma}}{1-\gamma} \\ +\beta \left[\begin{array}{c} 0.5\frac{(y_2^H)^{1-\gamma}}{1-\gamma} \\ +0.5\frac{(y_2^L)^{1-\gamma}}{1-\gamma} \end{array} \right] \end{array} \right) = \left(\begin{array}{c} \frac{\left[y_1\left(1-\widetilde{b}_1\right)\right]^{1-\gamma}}{1-\gamma} \\ +0.5\beta \left[\begin{array}{c} \frac{\left(y_1\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]\right)^{1-\gamma}}{1-\gamma} \\ +\frac{\left(y_1\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]\right)^{1-\gamma}}{1-\gamma} \end{array} \right] \end{array} \right)$$

$$(1+\widetilde{\phi}_1)^{1-\gamma} = \frac{\left[\left(1-\widetilde{b}_1\right)\right]^{1-\gamma} + 0.5\beta\left[\left(\left[(1+r)\widetilde{b}_1+\frac{y_2^H}{y_1}\right]\right)^{1-\gamma} + \left(\left[(1+r)\widetilde{b}_1+\frac{y_2^L}{y_1}\right]\right)^{1-\gamma}\right]}{1+\beta\,(0.5)\left[(\frac{y_2^H}{y_1})^{1-\gamma}+(\frac{y_2^L}{y_1})^{1-\gamma}\right]} \frac{1+\beta}{1+\beta}$$

$$(1+\widetilde{\phi}_1)^{1-\gamma} = \frac{(1+\phi_1)^{1-\gamma}}{(1+\phi_2)^{1-\gamma}}$$

It follows that

$$\begin{aligned} (1+\widetilde{\phi}_1)^{1-\gamma} &= \frac{(1+\phi_1)^{1-\gamma}}{(1+\phi_2)^{1-\gamma}}, \\ \underbrace{\log(1+\phi_1)}_{\text{total gains}} &= \underbrace{\log(1+\widetilde{\phi}_1)}_{\text{gains FA-IM}} + \underbrace{\log(1+\phi_2)}_{\text{gains IM-CM}}. \end{aligned}$$

Given ϕ_1 and ϕ_2, we can compute $\widetilde{\phi}_1$.

6.3 Appendix C (online): Welfare gains in an infinite horizon model

This appendix extends our two-period model in Section 4 to an infinite horizon. We will conclude that the key results from our two-period model remain valid in an infinite horizon context.

Let time be denoted by $t = 0, 1, 2, 3, \ldots$. The rate of growth of income from period $t-1$ to t is given by x_t; that is, $Y_t = x_t Y_{t-1}$. Income in period 0 is exogenously given and equal to Y_0. The uncertainty of income growth in period t is given by the realization $s_t \in \mathbb{S}$: $x_t = x(s_t)$. We define the state of the world in period t as the history of events from 1 to t, $h_t = (s_1, s_2, \ldots, s_t)$. Thus, $Y_t(h_t)$ is the income in period t as a function of $(s_1, s_2, \ldots, s_t)$. Note that $Y_t(h_t) = Y_t((h_{t-1}, s_t)) = Y_{t-1}(h_{t-1})x(s_t)$. From the perspective of period 0, $\pi_t(h_t)$ is the probability of the sequence of events $(s_1, s_2, \ldots, s_t)$.

Households' consumption is described by the sequence $\{C_t(h_t)\}_{t=0}^{\infty}$, where $C_t(h_t)$ is consumption in period t as a function of $(s_1, s_2, \ldots, s_t)$. Households' expected utility is given by

$$\sum_{t=0}^{\infty} \sum_{h_t} \beta^t u(C_t(h_t)) \pi_t(h_t), \tag{56}$$

where

$$u(C) = \frac{C^{1-\gamma} - 1}{1 - \gamma}.$$

We will analyze this model under two different asset market configurations: (i) complete markets, and (ii) financial autarky. We will derive the consumption path and welfare under these two market arrangements and compute the welfare gains of complete markets relative to financial autarky.

6.3.1 Complete markets

When markets are complete, there is a full set of contingent bonds for all states of nature and for each period. We assume that these contingent bonds are traded in period 0. Let $q_t(h_t)$ denote the price at period 0 of a security that gives one unit of consumption in period t in the sequence of events h_t. With a complete set of these securities, the household's intertemporal budget constraint is given by

$$\sum_{t=0}^{\infty} \sum_{h_t} q_t(h_t) C_t(h_t) \le \sum_{t=0}^{\infty} \sum_{h_t} q_t(h_t) Y_t(h_t). \tag{57}$$

Maximizing (56) subject to (57) yields the following condition:

$$\beta^t (C_t(h_t))^{-\gamma} \pi_t(h_t) = \mu q_t(h_t), \qquad \forall t, h_t, \tag{58}$$

where μ is the Lagrange multiplier associated with budget constraint (57). As in our two-period model, we assume that the rest of the world consists of a continuum of identical economies with the same income process. These economies can diversify all income fluctuations along the average path. Hence, the contingent-security prices satisfy:

$$\beta^t \pi_t(h_t) \left(\frac{\mathbb{E}[Y_t]}{Y_0} \right)^{-\gamma} = q_t(h_t), \qquad \forall t, h_t. \tag{59}$$

Combining (58) and (59), we obtain:

$$(C_t(h_t))^{-\gamma} = \mu \left(\frac{\mathbb{E}[Y_t]}{Y_0} \right)^{-\gamma}, \qquad \forall t, h_t. \tag{60}$$

This last expression implies $C_t(h_t) = (\mathbb{E}[Y_t]/Y_0)\, C_0$. Using the household's budget constraint, we can write:

$$\frac{C_0}{Y_0} \sum_{t=0}^{\infty} \beta^t \left(\frac{\mathbb{E}[Y_t]}{Y_0} \right)^{-\gamma} \mathbb{E}[Y_t] = \sum_{t=0}^{\infty} \beta^t \left(\frac{\mathbb{E}[Y_t]}{Y_0} \right)^{-\gamma} \mathbb{E}[Y_t].$$

If $\sum_{t=0}^{\infty} \beta^t \left(\frac{\mathbb{E}[Y_t]}{Y_0} \right)^{-\gamma} \mathbb{E}[Y_t] < \infty$, we obtain $C_0 = Y_0$, implying that

$$C_t(h_t) = \mathbb{E}[Y_t], \qquad \forall t, h_t.$$

Therefore, welfare under complete markets is given by

$$\hat{W}_0 = \sum_{t=0}^{\infty} \beta^t \left(\frac{(\mathbb{E}[Y_t])^{1-\gamma} - 1}{1-\gamma} \right).$$

6.3.2 Financial autarky

When the economy cannot trade assets with the rest of the world, the consumption path is equal to the income path: $C_t(h_t) = Y_t(h_t)$. Welfare under financial autarky is thus given by

$$\hat{W}_1 = \sum_{t=1}^{\infty} \sum_{h_t} \beta^t \pi_t(h_t) \frac{(Y_t(h_t))^{1-\gamma} - 1}{1-\gamma} = \sum_{t=1}^{\infty} \beta^t \frac{(\mathbb{E}[(Y_t)^{1-\gamma}] - 1)}{1-\gamma}. \tag{61}$$

6.3.3 Welfare gains from financial autarky to complete markets

Welfare gains from financial autarky to complete markets can be obtained as the consumption compensation factor, $\hat{\phi}_1$, such that

$$\sum_{t=0}^{\infty} \beta^t \frac{\left((1+\hat{\phi}_1)^{1-\gamma} \mathbb{E}[(Y_t)^{1-\gamma}] - 1 \right)}{1-\gamma} = \hat{W}_0. \tag{62}$$

Solving for $\hat{\phi}_1$, we obtain

$$\hat{\phi}_1 = \left(\frac{\sum_{t=0}^{\infty} \beta^t \left(\mathbb{E}[Y_t] \right)^{1-\gamma}}{\sum_{t=0}^{\infty} \beta^t \mathbb{E}[(Y_t)^{1-\gamma}]} \right)^{\frac{1}{1-\gamma}} - 1. \tag{63}$$

In this multi-period setting, we will consider the same income process used in Section 4 for the second period of our two-period model. Before computing this welfare gain, however, we will need to specify whether the income path has a deterministic or stochastic trend. Welfare costs will be sensitive to this assumption since a deterministic trend implies no increase over time in the risk of income fluctuations because the welfare costs will be of a similar magnitude as the one considered in the text when the income risk is normally distributed. In contrast, when income has a stochastic trend, the income risk rises over time increasing the value of insurance against income fluctuations. For this reason, for each type of risk analyzed in the text for the two-period model, we will analyze two cases here (one with a deterministic trend and the other with a stochastic trend).

Type I risk and deterministic trend Assume that the income path fluctuates around a deterministic trend. The fluctuations around this trend are normally distributed:

$$\log(Y_t) \sim N(\log(Y_0) + t\mu_x, \sigma_x^2), \qquad \forall t \geq 1.$$

This specification implies that

$$\begin{aligned} \mathbb{E}[Y_t] &= Y_0 \exp\left(t\mu_x + \frac{1}{2}\sigma_x^2 \right), \\ \mathbb{E}[(Y_t)^{1-\gamma}] &= (Y_0)^{1-\gamma} \exp\left((1-\gamma)t\mu_x + \frac{(1-\gamma)^2}{2}\sigma_x^2 \right). \end{aligned}$$

Substituting the last two expressions in (63), we obtain

$$\begin{aligned} \hat{\phi}_1 &= \left(\frac{\sum_{t=0}^{\infty} \beta^t (Y_0)^{1-\gamma} \exp((1-\gamma)(t\mu_x + \frac{1}{2}\sigma_x^2))}{\sum_{t=0}^{\infty} \beta^t (Y_0)^{1-\gamma} \exp(()t\mu_x + \frac{(1-\gamma)^2}{2}\sigma_x^2)} \right)^{\frac{1}{1-\gamma}} - 1, \\ &= \left(\frac{(Y_0)^{1-\gamma} \exp(\frac{(1-\gamma)}{2}\sigma_x^2)}{1 - \beta \exp((1-\gamma)\mu_x)} \frac{1 - \beta \exp((1-\gamma)\mu_x)}{(Y_0)^{1-\gamma} \exp(\frac{(1-\gamma)^2}{2}\sigma_x^2)} \right)^{\frac{1}{1-\gamma}} - 1, \\ &= \left(\exp((1-\gamma)(\gamma)\frac{1}{2}\sigma_x^2) \right)^{\frac{1}{1-\gamma}} - 1, \\ &= \exp(\frac{\gamma}{2}\sigma_x^2) - 1. \end{aligned}$$

Type I risk and stochastic trend In this case, income growth follows an independent and identically distributed (i.i.d.) log-normal distribution:

$$\log(Y_t/Y_{t-1}) = x \sim iidN(\mu_x, \sigma_x^2), \tag{64}$$

which implies that conditional on Y_0, Y_t also has a log-normal distribution, but with a variance that grows over time:

$$\log(Y_t) \sim N(\log(Y_0) + t\mu_x, t\sigma_x^2). \tag{65}$$

It follows that

$$\begin{aligned} \mathbb{E}[Y_t] &= Y_0 \exp(t\mu_x + \frac{1}{2}\sigma_x^2 t) = Y_0 \left[\exp(\mu_x + \frac{1}{2}\sigma_x^2)\right]^t, \\ \mathbb{E}[(Y_t)^{1-\gamma}] &= (Y_0)^{1-\gamma} \exp(t(1-\gamma)\mu_x + \frac{1}{2}(1-\gamma)^2\sigma_x^2 t) = Y_0 \left[\exp((1-\gamma)\mu_x + \frac{(1-\gamma)^2}{2}\sigma_x^2)\right]^t. \end{aligned}$$

Using the last two expressions and imposing that

$$\beta \exp(\mu_x + \frac{1}{2}\sigma_x^2) < 1 \text{ and } \beta \exp((1-\gamma)\mu_x + \frac{(1-\gamma)^2}{2}\sigma_x^2) < 1, \tag{66}$$

we obtain:

$$\sum_{t=0}^{\infty} \beta^t \left(\mathbb{E}[Y_t]\right)^{1-\gamma} = \frac{Y_0}{1 - \beta \exp\left((1-\gamma)(\mu_x + \frac{1}{2}\sigma_x^2)\right)}, \tag{67}$$

$$\sum_{t=0}^{\infty} \beta^t \mathbb{E}[(Y_t)^{1-\gamma}] = \frac{(Y_0)^{1-\gamma}}{1 - \beta \exp\left((1-\gamma)\mu_x + \frac{(1-\gamma)^2}{2}\sigma_x^2\right)}, \tag{68}$$

which, using equation (63), implies

$$\hat{\phi}_1 = \left(\frac{1 - \beta \exp\left((1-\gamma)\mu_x + \frac{(1-\gamma)^2}{2}\sigma_x^2\right)}{1 - \beta \exp\left((1-\gamma)(\mu_x + \frac{1}{2}\sigma_x^2)\right)}\right)^{\frac{1}{1-\gamma}} - 1. \tag{69}$$

Table 15 shows the welfare gains measured by $\hat{\phi}_1$ under type I risk. We use the same value of the parameters considered in the two period model of Section 4 with type I risk. The first row of Table 15 corresponds to the deterministic trend, whereas the second row is the computation with a stochastic trend. As we did in the text, the welfare gains are computed for a range of 2 to 5 of the risk aversion coefficient. Welfare gains with the deterministic trend are quite small (in a range of 0.36 to 0.91 percent), although about twice as much as in the two-period model (recall Table 5). When the income process has a stochastic trend, welfare gains are much larger and in the range of 10-12 percent.

Type II risk and deterministic trend As in the case of the normal distribution, we will assume a deterministic path for the average income over time, keeping the variance constant and independent of time. In contrast to the normal distribution, however, we will assume that fluctuations around this deterministic path are a combination of a normal and power-law distributions. Hence, for $t \geq 1$, the process for income in period t is given by

$$\log(Y_t) = \log(Y_0) + (t-1)\mu_x + x,$$

where x is independent and identically distributed as follows:

$$x = \begin{cases} \tilde{x} \sim N(\tilde{\mu}_x, \tilde{\sigma}_x^2), & \text{with probability } 1-p, \\ 1/z \sim \text{inverse power-law distribution}, & \text{with probability } p. \end{cases} \tag{70}$$

As indicated in the text, the probability density function for z is:

$$f_Z(z) = (\alpha - 1)(z_{min})^{\alpha-1}(z)^{-\alpha}, \ \alpha > 1, z \geq z_{min}.$$

We can compute ($\forall t \geq 1$):

$$\begin{aligned} \mathbb{E}[Y_t] &= Y_0 \exp((t-1)\mu_x) \left[(1-p)\exp(\tilde{\mu}_x + \frac{1}{2}\tilde{\sigma}_x^2) + p\mathbb{E}[1/z]\right], \\ \mathbb{E}[(Y_t)^{1-\gamma}] &= (Y_0)^{1-\gamma} \exp((1-\gamma)(t-1)\mu_x) \left[(1-p)\exp((1-\gamma)\tilde{\mu}_x + \frac{(1-\gamma)^2}{2}\tilde{\sigma}_x^2) + p\mathbb{E}[1/(z^{1-\gamma})]\right]. \end{aligned}$$

Substituting these two last expressions into (63), and rearranging terms as before:

$$\begin{aligned} &\hat{\phi}_1 \\ &= \left(\frac{Y_0^{1-\gamma} + \sum_{t=1}^{\infty} \beta^t (Y_0)^{1-\gamma} \exp((t-1)(1-\gamma)\mu_x) \left[(1-p)\exp(\tilde{\mu}_x + \frac{1}{2}\tilde{\sigma}_x^2) + p\mathbb{E}[1/z]\right]^{1-\gamma}}{Y_0^{1-\gamma} + \sum_{t=1}^{\infty} \beta^t (Y_0)^{1-\gamma} \exp((t-1)(1-\gamma)\mu_x) \left[(1-p)\exp((1-\gamma)\tilde{\mu}_x + \frac{(1-\gamma)^2}{2}\tilde{\sigma}_x^2) + p\mathbb{E}[1/(z^{1-\gamma})]\right]} \right)^{\frac{1}{1-\gamma}} - 1 \\ &= \left(\frac{\left[(1-p)\exp(\tilde{\mu}_x + \frac{1}{2}\tilde{\sigma}_x^2) + p\mathbb{E}[1/z]\right]^{1-\gamma}}{\left[(1-p)\exp((1-\gamma)\tilde{\mu}_x + \frac{(1-\gamma)^2}{2}\tilde{\sigma}_x^2) + p\mathbb{E}[1/(z^{1-\gamma})]\right]} \right)^{\frac{1}{1-\gamma}} - 1 \end{aligned} \tag{71}$$

Type II risk and stochastic trend In this case, the log income growth in each period also follows an i.i.d. process:

$$\log(Y_t/Y_{t-1}) = x = \begin{cases} \log(\tilde{x}) \sim N(\tilde{\mu}, \tilde{\sigma}_x^2), & \text{with probability } 1-p, \\ 1/z \sim \text{inverse power-law distribution}, & \text{with probability } p, \end{cases} \tag{72}$$

where again the density probability function for z is given by:

$$f_Z(z) = (\alpha - 1)(z_{min})^{\alpha-1}(z)^{-\alpha}, \ \alpha > 1, z \geq z_{min}.$$

Hence,

$$\mathbb{E}[Y_t] = Y_0 \left((1-p)\exp(\tilde{\mu}_x + \frac{\tilde{\sigma}_x^2}{2}) + p\frac{\alpha-1}{\alpha}(z_{min})^{-1} \right)^t.$$

If $\alpha > \gamma$, we have

$$\mathbb{E}[(Y_t)^{1-\gamma}] = (Y_0)^{1-\gamma} \left((1-p)\exp((1-\gamma)\tilde{\mu}_x + \frac{(1-\gamma)^2\tilde{\sigma}_x^2}{2}) + p\frac{\alpha-1}{\alpha-\gamma}(z_{min})^{\gamma-1} \right)^t.$$

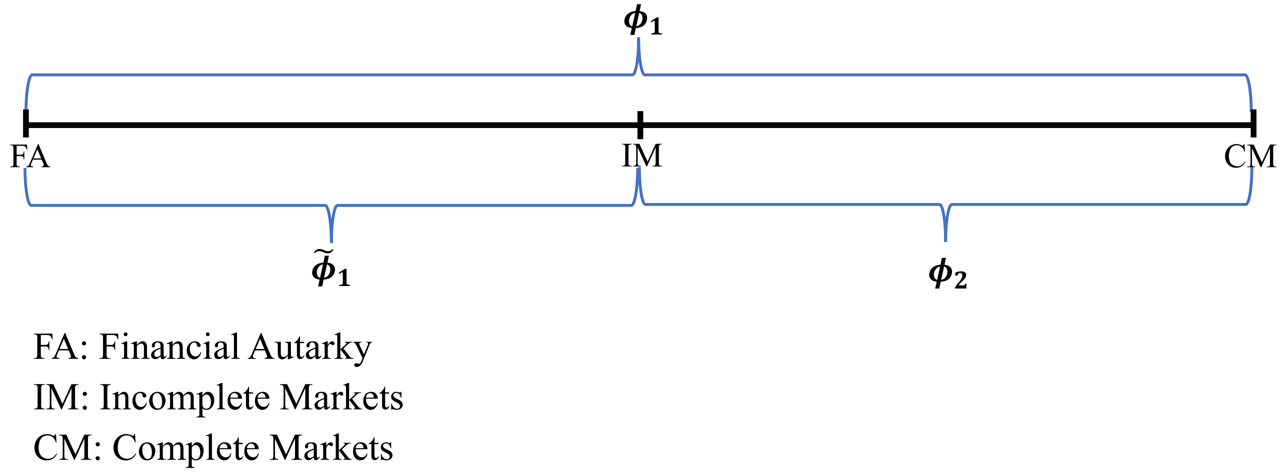
If $\alpha \leq \gamma$, $\mathbb{E}[(Y_t)^{1-\gamma}]$ is unbounded and not well-defined. Therefore, as long as $\alpha > \gamma$ and $\beta\left((1-p)\exp((1-\gamma)\tilde{\mu}_x + \frac{(1-\gamma)^2\tilde{\sigma}_x^2}{2}) + p\frac{\alpha-1}{\alpha-\gamma}(z_{min})^{\gamma-1}\right) < 1$, we obtain:

$$\hat{\phi}_1 = \left(\frac{1-\beta\left((1-p)\exp((1-\gamma)\tilde{\mu}_x + \frac{(1-\gamma)^2\tilde{\sigma}_x^2}{2}) + p\frac{\alpha-1}{\alpha-\gamma}(z_{min})^{\gamma-1}\right)}{1-\beta\left((1-p)\exp(\tilde{\mu}_x + \frac{\tilde{\sigma}_x^2}{2}) + p\frac{\alpha-1}{\alpha}(z_{min})^{-1}\right)^{1-\gamma}} \right)^{\frac{1}{1-\gamma}} - 1. \quad (73)$$

If $\alpha > \gamma$, but $\beta\left((1-p)\exp((1-\gamma)\tilde{\mu}_x + \frac{(1-\gamma)^2\tilde{\sigma}_x^2}{2}) + p\frac{\alpha-1}{\alpha-\gamma}(z_{min})^{\gamma-1}\right) \geq 1$, then $\hat{\phi}_1 \to \infty$. When $\alpha \leq \gamma$, we also get $\hat{\phi}_1 \to \infty$.

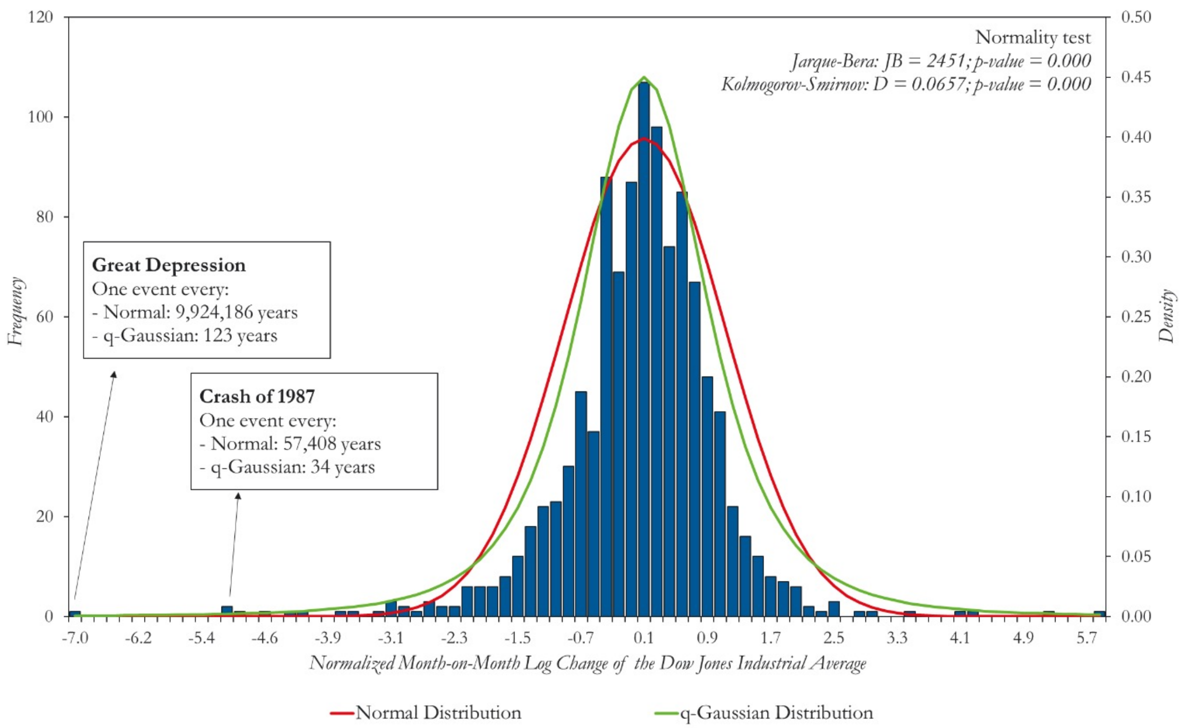
Again, we consider the same parameter values used for type II risk in the two-period model. The welfare computation is presented in Table 16. The top panel shows the welfare gains with a deterministic trend and the bottom panel with a stochastic trend. In contrast to type I risk, welfare gains of financial integration increase substantially and are very sensitive to a rise in the coefficient of risk aversion. To quantify the effect of adding rare disasters, Table 17 shows the factor increase in the welfare computation of $\hat{\phi}_1$ between values in Table 16 and Table 15. This factor increase in $\hat{\phi}_1$ can be compared to the one reported in the first row of Table 12 in the text for the case of ϕ_1. This comparison highlights that adding the risk of macroeconomic disasters increases the welfare gains of financial integration by factors of 2 to 14 for values of $\gamma \leq 2.5$ (relative to standard computations based only on normally-distributed income fluctuations). Welfare gains may become unbounded for higher values of γ. We conclude that the main results of a substantial increase in the welfare gains of financial integration when adding rare disasters in income risk and with reasonable values for the risk aversion coefficient (i.e., between 2 an 5) still hold in an infinite horizon model.

Figure 1: Welfare gains from financial integration



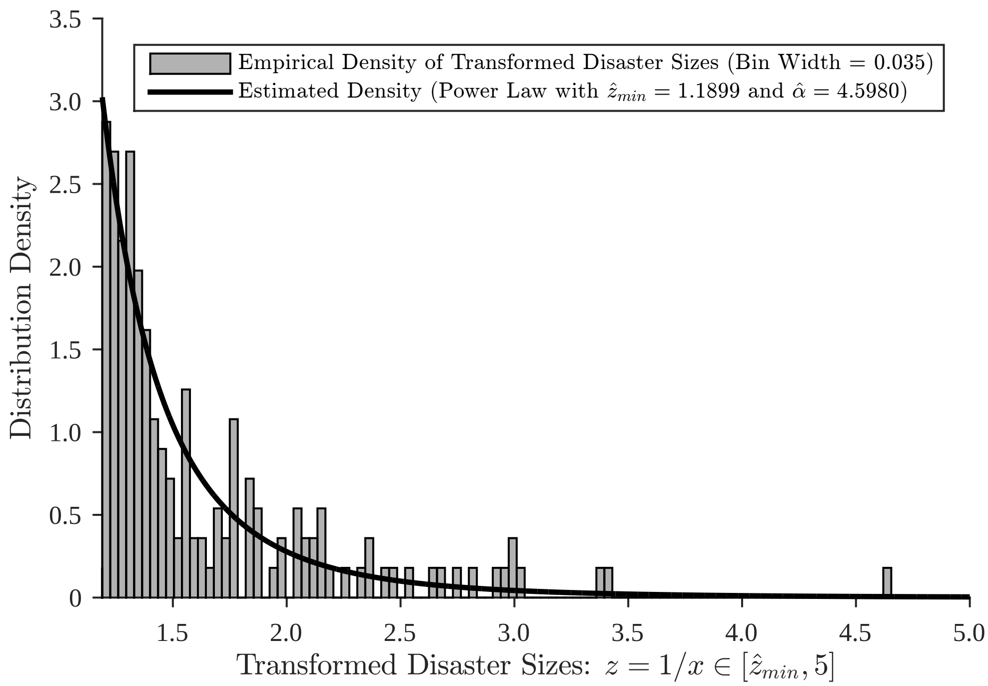
Notes: ϕ_1 is welfare gains from FA to CM; ϕ_2 is welfare gains from IM to CM; and $\widetilde{\phi}_1$ is welfare gains from FA to IM.

Figure 2: Distribution of the Dow Jones Returns



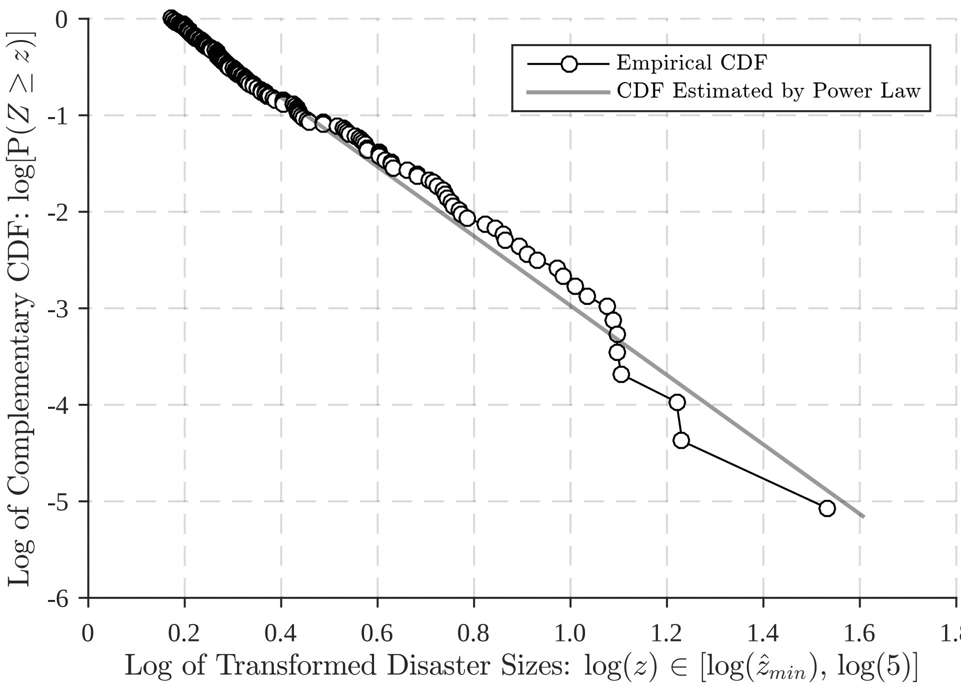
Note: The value of q is 1.67.

Figure 3: Density comparison between the data and the fitted model



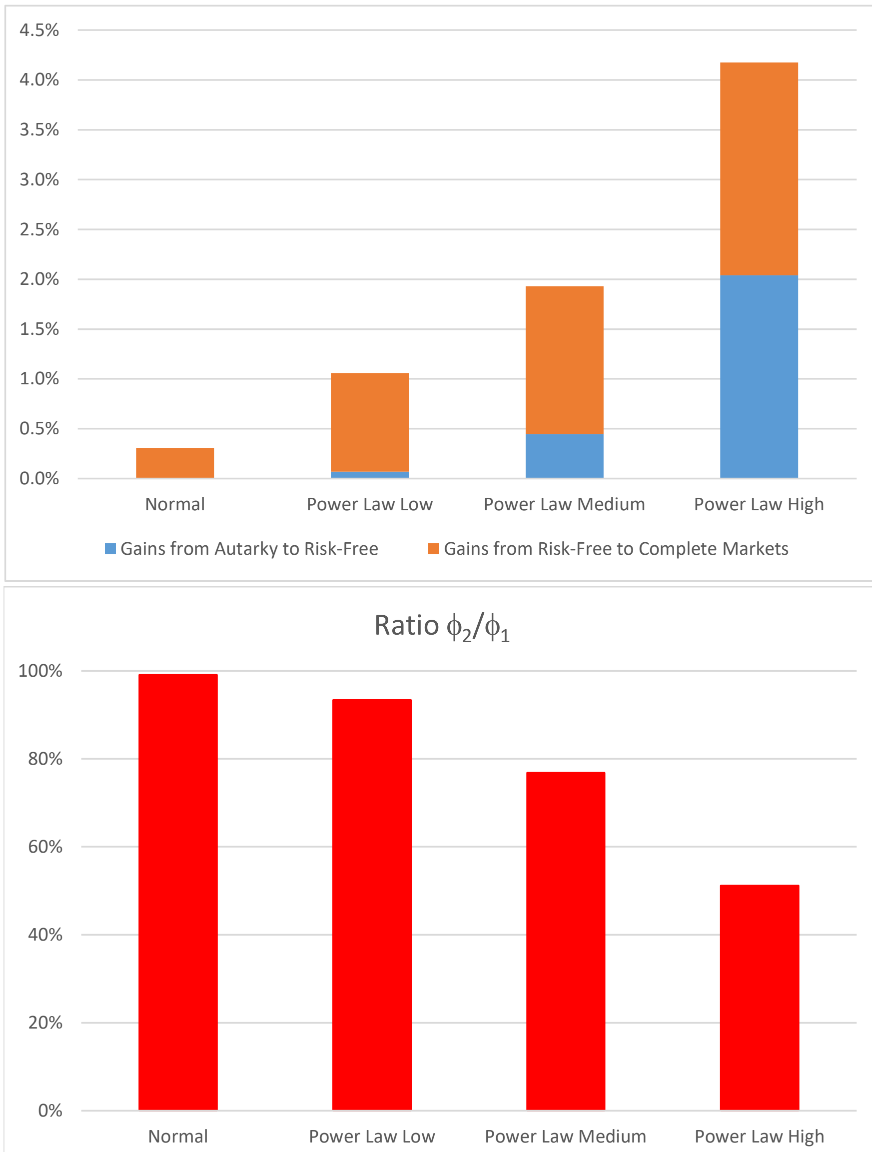
Notes: Following Barro and Jin (2011), a visual assessment on the goodness-of-fit of the model can be made by comparing a histogram of the empirical density of transformed disaster sizes, z, with realizations of the probability density function (pdf) of the estimated power-law distribution ($\hat{z}_{min} = 1.1899$ and $\hat{\alpha} = 4.5980$).

Figure 4: Cumulative distribution function comparison between data and fitted model



Notes: Following Clause *et al.* (2009), the optimal power-law model is characterized by the threshold value, $\hat{z}_{min}$, that allows the estimated model's CDF to be as close as possible to the data CDF in the region $z \geq \hat{z}_{min}$. In the plot, the straight line corresponds to the optimal $\hat{z}_{min} = 1.1899$ and $\hat{\alpha} = 4.5980$.

Figure 5: Decomposition of Welfare Gains from Financial Integration



Notes: In this computation, we use a coefficient of risk aversion of 3.5 (γ), and values of 4.0 (power law high), 4.6 (power law medium), and 5.6 (power law low) for the exponent of the power law distribution (α).

Table 1: Estimated parameters of the income distribution

Disaster regime (Power-law dist.)					*Tranquil times (Log-normal dist.)*	
$\hat{z}_{min}$		$\hat{\alpha}$		p	$\hat{\mu}$	$\hat{\sigma}$
Est.	95% Confidence Interval	Est.	95% Confidence Interval			
1.1899	(1.1158, 1.3917)	4.5980	(3.9846, 5.6343)	0.0274	0.0262	0.0450

Notes: The power-law and log-normal distributions are fitted through R implementation of Gillespie (2015) and Delignette-Muller and Dutang (2015), respectively. 95% confidence intervals of $\hat{z}_{min}$ and $\hat{\alpha}$ from 5,000 bootstrap simulations, following methods developed in Clauset *et al.* (2009). The point estimates for $\hat{\mu}$ and $\hat{\sigma}$ and the value of p are computed as explained in the text.

Table 2: Welfare gains

γ	2	3	4	5
Savings	0.27%	0.35%	0.44%	0.53%
ϕ_1	0.179%	0.269%	0.360%	0.450%
ϕ_2	0.179%	0.267%	0.355%	0.442%

Table 3: Welfare gains as function of the size of the shocks

y^H	y^L	ϕ_1 (total)	ϕ_2 (segment 2)	ϕ_2/ϕ_1	$\tilde{\phi}_1$ (segment 1)
1.0	1.0	0	0	n/a	0
1.1	0.9	0.75	0.74	98.0%	0.01%
1.2	0.8	3.13	2.90	92.5%	0.2%
1.3	0.7	7.54	6.36	84.3%	1.11%
1.4	0.6	14.8	11.0	74.4%	3.41%
1.5	0.5	26.7	16.9	63.4%	8.35%
1.6	0.4	46.6	24.1	51.7%	18.2%

Note: Calculations assume $\gamma = 3$.

Table 4: Estimates of type I risks

Parameter	Value
μ_x	0.0165
σ_x	0.0601

Table 5: Type I risks. Welfare gains from financial autarky to complete markets (ϕ_1)

	γ						
	2.0	2.5	3.0	3.5	4.0	4.5	5.0
	0.177%	0.221%	0.264%	0.307%	0.350%	0.393%	0.435%

Table 6: Non-contingent bond holdings with type I risks ($\widetilde{b}_1$)

	γ						
	2.0	2.5	3.0	3.5	4.0	4.5	5.0
	0.26%	0.31%	0.35%	0.39%	0.43%	0.47%	0.51%

Table 7: Type I risk. Welfare gains from non-contingent bond to complete markets (ϕ_2)

	γ						
	2.0	2.5	3.0	3.5	4.0	4.5	5.0
	0.177%	0.220%	0.263%	0.305%	0.347%	0.387%	0.429%

Table 8: Estimates of type II risks

Parameter	Value
z_{min}	1.1899
p	0.0274
Point estimate for α	4.5980
Range for α	[3.9846; 5.6343]
$\tilde{\mu}_x$	0.0262
$\tilde{\sigma}_x$	0.0450

Table 9: Type II risks. Welfare gains from financial autarky to complete markets (ϕ_1)

	γ						
α	2.0	2.5	3.0	3.5	4.0	4.5	5.0
3.9846	0.69%	1.11%	1.91%	4.30%	∞	∞	∞
4.5980	0.53%	0.80%	1.20%	1.95%	3.80%	20.50%	∞
5.6343	0.40%	0.56%	0.77%	1.07%	1.52%	2.35%	4.34%

Table 10: Non-contingent bond holdings with type II risks ($\widetilde{b}_1$)

	γ						
α	2.0	2.5	3.0	3.5	4.0	4.5	5.0
3.9846	1.58%	2.42%	3.46%	4.91%	6.06%	7.81%	9.52%
4.5980	1.10%	1.53%	2.24%	3.12%	4.62%	5.72%	7.59%
5.6343	0.74%	0.99%	1.32%	1.84%	2.39%	3.19%	4.33%

Table 11: Type II risks. Welfare gains from non-contingent bond to complete markets (ϕ_2)

	γ						
α	2.0	2.5	3.0	3.5	4.0	4.5	5.0
3.9846	0.66%	1.01%	1.48%	2.16%	2.98%	4.16%	5.59%
4.5980	0.52%	0.74%	1.06%	1.49%	2.19%	2.95%	4.14%
5.6343	0.39%	0.54%	0.73%	1.00%	1.29%	1.74%	2.34%

Table 12: Factor of increase in ϕ_1 and ϕ_2 from adding disasters with power-law distribution

	γ						
	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Factor of increase in ϕ_1	2 to 4	3 to 5	3 to 7	3 to 14	4 to ∞	6 to ∞	10 to ∞
Factor of increase in ϕ_2	2 to 4	2 to 5	3 to 6	3 to 7	4 to 9	4 to 11	5 to 13

Table 13: Welfare gains ϕ_1 under alternative assumptions for the size distribution of disasters

	γ						
Deterministic	2.0	2.5	3.0	3.5	4.0	4.5	5.0
	Deterministic income fall under disaster						
b consistent with $\alpha = 3.9846$	0.42%	0.55%	0.71%	0.89%	1.10%	1.33%	1.60%
b consistent with $\alpha = 4.5980$	0.36%	0.48%	0.60%	0.75%	0.91%	1.09%	1.30%
b consistent with $\alpha = 5.6343$	0.30%	0.40%	0.50%	0.61%	0.73%	0.87%	1.01%
	γ						
Normal distribution	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$\tilde{\mu}_z, \tilde{\sigma}_z$ consistent with $\alpha = 3.9846$	0.56%	0.79%	1.08%	1.46%	1.96%	2.62%	3.50%
$\tilde{\mu}_z, \tilde{\sigma}_z$ consistent with $\alpha = 4.5980$	0.46%	0.64%	0.85%	1.12%	1.45%	1.87%	2.40%
$\tilde{\mu}_z, \tilde{\sigma}_z$ consistent with $\alpha = 5.6343$	0.37%	0.49%	0.64%	0.82%	1.02%	1.27%	1.57%

Table 14: Factor of increase in ϕ_1 under under alternative assumptions for the size distribution of disasters

	γ						
	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Deterministic	2	2 to 3	2 to 3	2 to 3	2 to 3	2 to 3	2 to 4
Normal distribution	2 to 3	2 to 4	2 to 4	3 to 5	3 to 6	3 to 7	4 to 8

Table 15: Type I risk. Welfare gains in the infinite horizon model ($\hat{\phi}_1$)

	γ						
	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Deterministic trend	0.36%	0.45%	0.54%	0.63%	0.73%	0.82%	0.91%
Stochastic trend	10.15%	10.51%	10.86%	11.23%	11.61%	12.03%	12.49%

Table 16: Type II risks. Welfare gains in the infinite horizon model ($\hat{\phi}_1$)

	γ						
α	2.0	2.5	3.0	3.5	4.0	4.5	5.0
	Deterministic trend						
3.9846	1.41%	2.26%	3.89%	8.5%	∞	∞	∞
4.5980	1.09%	1.63%	2.46%	3.96%	7.7%	35.8%	∞
5.6343	0.82%	1.15%	1.58%	2.19%	3.12%	4.76%	8.56%
	Stochastic trend						
3.9846	60.28%	145.7%	∞	∞	∞	∞	∞
4.5980	39.90%	62.75%	192.4%	∞	∞	∞	∞
5.6343	26.37%	33.85%	48.50%	97.7%	∞	∞	∞

Table 17: Factor of increase in $\hat{\phi}_1$ when adding rare disasters in the infinite horizon model

	γ						
	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Deterministic trend	2 to 4	3 to 5	3 to 7	3 to 13	4 to ∞	6 to ∞	9 to ∞
Stochastic trend	3 to 6	3 to 14	4 to ∞	9 to ∞	∞	∞	∞

Table A1: Start year of real GDP per capita series, by country

Industrial Countries							
Australia	1900	Finland	1900	Japan	1900	Spain	1900
Austria	1900	France	1900	Netherlands	1900	Sweden	1900
Belgium	1900	Germany	1900	New Zealand	1900	Switzerland	1900
Canada	1900	Greece	1900	Norway	1900	United Kingdom	1900
Denmark	1900	Italy	1900	Portugal	1900	United States	1900
Developing Countries							
Afghanistan	1950	Croatia	1991	Latvia	1990	Rwanda	1950
Algeria	1970	Cyprus	1950	Lesotho	1950	Samoa	1998
Angola	1975	Czech Republic	1993	Lithuania	1990	Sao Tome and Principe	1950
Antigua and Barbuda	1980	Djibouti	1950	Luxembourg	1950	Senegal	1950
Argentina	1900	Dominica	1950	Madagascar	1950	Serbia	2006
Armenia	1990	Dominican Republic	1950	Malawi	1950	Seychelles	1950
Azerbaijan	1990	East Timor	2000	Maldives	1980	Sierra Leone	1950
Bahamas	1980	Ecuador	1900	Mali	1950	Slovak Republic	1993
Bahrain	1970	El Salvador	1920	Malta	1950	Slovenia	1991
Bangladesh	1950	Equatorial Guinea	1950	Marshall Islands	1997	Solomon Islands	1980
Barbados	1950	Eritrea	1992	Mauritania	1950	Somalia	2011
Belarus	1990	Estonia	1990	Mauritius	1950	South Sudan	2011
Belize	1980	Eswatini	1950	Mexico	1900	Sri Lanka	1900
Benin	1950	Ethiopia	1950	Micronesia	1995	St. Kitts and Nevis	1980
Bhutan	1980	Fiji	1980	Moldova	1990	St. Lucia	1950
Bolivia	1900	Gabon	1950	Mongolia	1950	St. Vincent & Grenadines	1980
Bosnia and Herzegovina	1991	Gambia	1950	Montenegro	2006	Sudan	1950
Botswana	1950	Georgia	1990	Mozambique	1950	Suriname	1990
Brazil	1900	Grenada	1980	Namibia	1950	Tajikistan	1990
Brunei	1985	Guatemala	1920	Nauru	2004	Tanzania	1950
Burkina Faso	1950	Guinea	1950	Nicaragua	1920	Togo	1950
Burundi	1950	Guinea-Bissau	1950	Niger	1950	Tonga	1980
Cambodia	1950	Guyana	1980	Nigeria	1950	Trinidad and Tobago	1950
Cameroon	1950	Haiti	1945	North Macedonia	1991	Turkmenistan	1990
Cape Verde	1950	Honduras	1920	Oman	1950	Tuvalu	2002
Central African Republic	1950	Iceland	1950	Pakistan	1950	Uganda	1950
Chad	1950	India	1900	Palau	2000	Ukraine	1990
Chile	1900	Israel	1950	Panama	1906	Uruguay	1900
Colombia	1900	Kazakhstan	1990	Papua New Guinea	1980	Uzbekistan	1990
Comoros	1950	Kenya	1950	Paraguay	1939	Vanuatu	1980
Congo, Dem.	1950	Kiribati	1980	Peru	1900	Venezuela	1900
Congo, Rep.	1950	Kuwait	1974	Puerto Rico	1950	Yemen	1950
Costa Rica	1920	Kyrgyz Republic	1990	Qatar	1974	Zambia	1950
Cote d'Ivoire	1950	Laos	1950	Russia	1990	Zimbabwe	1950

Notes: In our exercise, we consider that historical "industrial" nations are those that achieved OECD membership by 1980—with a population greater than one million—and classify all remaining countries as "developing."