Liquid Equity and Boom-Bust Dynamics*

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Abstract

I develop a monetary model with liquid equity. Equity is a claim on the profits of firms that act as sellers in the search-and-matching market. Buyers in that market devote search to obtain matches with firms, and use equity to relax a liquidity constraint. The dual nature of equity in the search-and-matching market entails a strategic complementary in search effort that operates through buyers' liquidity constraint, and it gives rise to endogenous booms and busts. The economy is stable in an inflation-targeting regime combined with TARP, meaning that the government effectively puts a floor below the value of equity.

Keywords: coordination, equity, liquidity, money-search, sunspots.

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1 Introduction

Many assets have money-like properties and the rapid advance of exchange-traded funds (ETFs) is making it easier to trade listed firms' equity and debt swiftly and cheaply (Lettau and Madhavan, 2018). In essence, this trend allows claims on firms' profits to become an alternative to fiat currency, whilst the use of such assets as liquid wealth is perceived to facilitate financial panics.[1] For exactly this reason, the global financial crisis spurred a hot debate on restricting and regulating money creation by the private sector.[2] More recently, the rise of ETFs has been posed as a threat to financial and macroeconomic stability due to ETFs' perceived liquidity,[3] and central banks have unorthodoxly bought commercial-bond and equity ETFs to stabilize markets, for instance amid the 2020 crash.

In light of the developments above, this paper aims to gain a better theoretic understanding of how liquid equity can be a source of financial and macroeconomic instability, and what policy can do in response, particularly by buying equity to stabilize markets. I develop for this purpose a money-search model à la Lagos and Wright (2005), modified to include liquid equity. Buyers and firms in the model participate in alternating frictional and frictionless markets. They are matched bilaterally in the frictional market according to a constant-returns matching function as in Pissarides (1984), and the matching probabilities depend on buyers' endogenous search.[4] A liquidity constraint entails that buyers need liquid wealth to settle trades with firms. The frictionless market allows the agents to adjust their asset positions in response to past trading opportunities, and quasi-linear preferences entail that buyers choose asset portfolios independently of their trading histories, thus producing a very tractable framework.

The novel feature of the framework lies in the modeling of equity as a liquid claim on firms' profits. It generates, together with buyers' endogenous search, a strategic complementarity that produces endogenous dynamics. The complementarity is reminiscent of that in Diamond (1982) but operates through liquid wealth rather than increasing returns in matching: if other buyers search intensely, firms obtain more matches and earn higher profits, and so the value of equity increases, driving down the liquidity premium

[1] This idea goes back to Fisher (1936) and other proponents of 100% fractional reserve banking.

[2] In 2018 Switzerland held a referendum on a popular initiative to provide the SNB (the Swiss Central Bank) with the sole authority to create money. The initiative was rejected by 76% of the voters.

[3] See, for instance, Pagano, Sánchez Serrano and Zechner (2019).

[4] The frictional market can be thought of a place where buyers purchase tailor-made goods, requiring them to search for firms that have the expertise to produce such goods.

due to a greater supply of liquid wealth. This makes carrying liquid wealth cheaper, in turn making it more attractive for the individual buyer to relax its liquidity constraint, entailing higher expected match surplus and, in turn, greater benefits of intense search.

The core of my contribution is to isolate the joint role of endogenous search and liquid equity as a source of financial and macroeconomy instability. I do so in a framework unifying: search and bilateral matching; a transactions-based demand for assets originating from a liquidity constraint within bilateral matches; an asset resembling the equity of firms which act as sellers in the search-and-matching market; and intrinsically-useless fiat currency. I first analyze a setup in which liquid wealth comprises only currency, supplied at a constant growth rate as commonly assumed in the literature. A well-known assumption entailing that ex-ante demand for liquid wealth is decreasing in the liquidity premium suffices to rule out bounded endogenous dynamics despite endogenous search; only steady states are equilibria and the monetary steady state is generically unique. Adding an asset that pays an exogenous divided as in Lucas (1978) does not change this result, highlighting the difference with equity, whose dividend is inherently endogenous.

I then analyze an environment in which liquid wealth comprises only equity. If search were exogenous, only a wealth channel would be operative; a higher equity value relaxes buyers' liquidity constraint so that firms earn greater profits, in turn feeding back into a higher value for equity. This channel is too weak to generate equilibrium multiplicity, although it can amplify real shocks as in Guerrieri and Lorenzoni (2009).

A search channel arises if endogenous search enters the picture: if the value of equity increases, buyers are more likely to increase their search because they face a looser liquidity constraint, entailing that firms are matched more frequently, leading to an increase in the value of equity. This channel is strong enough to generate equilibrium multiplicity for a set of parameters with positive mass. Particularly, in every time period, buyers can either search lazily, entailing a bust with low equity value and little economic activity; or intensely, entailing a boom with high equity value and much economic activity. This property allows for both deterministic and stochastic boom-bust dynamics.

Importantly, the assumption on liquid-wealth demand that rules out endogenous dynamics if liquid wealth comprises only currency, or both currency and an exogenous-dividend asset, does not conflict with the set of parameters that allows for endogenous dynamics if liquid wealth comprises only equity. This feature thus isolates the joint role of search and liquid equity in producing endogenous dynamics. The result also carries over to an environment in which liquid wealth comprises both intrinsically-useless currency—with supply growing at a constant rate—and equity. Endogenous cycles in that setup exhibit boom-bust dynamics with time-varying inflation.

The finding above begs the question whether stabilizing inflation suffices to stabilize the macroeconomy. I show that if an inflation target is successfully implemented, there can still be endogenous boom-bust dynamics because the strategic complementarity in search remains operative. The economy can be stabilized by combing successful inflation targeting with a policy resembling a troubled-asset relief program (TARP)—the government stands ready to buy equity at a predetermined price with the aim to prevent a self-fulfilling bust—, but this requires fiscal commitment to pass potential losses from TARP on to taxpayers. The latter does not occur on the equilibrium path if the price at which equity is bought is sufficiently high, since the mere fiscal commitment then suffices to stabilize the economy. The economy cannot be stabilized if the TARP price is set too low, entailing that there are contingencies in which TARP is deployed and losses are passed on to taxpayers, i.e., using TARP too conservatively may fiscally backfire.

The TARP results are relevant since major central banks have used TARP policies in response to the global financial crisis and the 2020 COVID-19 crash. While central banks are normally reluctant to buy anything but high-grade government debt, the U.S. Federal Reserve bought about USD 8 billion of commercial bonds amid the 2020 crash. The Bank of Japan started purchasing domestic stocks in 2010 and held about USD 366 billion worth of them mid 2023, amounting to 6% of the Japanese stock market.

Finally, the economy can also be stabilized with inflation targeting when policy adheres to the Friedman rule: a slight deflation to eliminate the opportunity cost of holding currency, thereby eliminating buyers' desire to use equity as liquid wealth. The Friedman rule can also be implemented as a unique monetary steady state if policy targets currency-supply growth rather than inflation, but then there are paths leading to the steady state during which the economy suffers from boom-bust dynamics. This suggests not only that targeting narrow-money growth may be undesirable, but also that broader monetary targets can be unreliable in times of financial innovation which would lead to unpredictable changes in the economic significance of monetary aggregates.[5]

[5] McCallum (1985) mentions this as one of the criticisms against the U.S. Federal Reserve's money-stock targets strategy, being used from 1979 to 1982.

Related literature. Papers with a role for liquid assets other than fiat currency are abundant in the money-search literature (see Lagos, Rocheteau and Wright, 2017, for a review). Some, following Lucas (1978), treat dividends paid by such assets as exogenous (e.g., Geromichalos and Herrenbrueck, 2016,1; Geromichalos, Licari and Suárez-Lledó, 2007; Lagos, 2010; Rocheteau and Wright, 2013). Others let dividends be determined in frictionless markets (e.g., Altermatt, 2022; Andolfatto, Berentsen and Waller, 2016; Lagos and Rocheteau, 2008). Altermatt, Iwasaki and Wright (2021) analyze a rich model to study endogenous asset-price and inflation dynamics if both fiat currency and exogenous-dividend assets comprise liquid wealth. Endogenous dynamics can arise in the aforementioned papers, but only if assets are infinitely lived since the dynamics rely on an infinite chain of asset-price expectations. Further, endogenous dynamics are ruled out by a common assumption in these frameworks, entailing lower ex-ante liquid-wealth demand amid higher liquidity premia.

Rocheteau and Wright (2013) briefly analyze, in an extension, a setup in which the fundamental value of assets is determined in markets in which these assets are used in payment. Their analysis lacks endogenous search though and focuses on firm entry instead, known to generate equilibrium multiplicity regardless of the nature of liquid assets (see, e.g., Berentsen, Menzio and Wright, 2011; Nosal and Rocheteau, 2011; Rocheteau and Wright, 2005). I instead uncover a strategic complementarity in search that arises only if liquid wealth comprises equity. The complementarity is strong enough to entail endogenous dynamics, even if a higher liquidity premium negatively affects ex-ante liquid-wealth demand since the mechanism does not rely on an infinite chain of expectations; the result is derived for an asset that is only one-period lived, elucidating the different nature of the endogenous dynamics and the joint role of search and liquid equity.

I also relate to Guerrieri and Lorenzoni (2009), who study a model in which producers' earning prospects matter for consumers' spending, producing a feedback effect that amplifies shocks. Angeletos and La'O (2013) show how limited communication can produce rational heterogeneous beliefs and endogenous booms and busts in a similar setup. I contribute by showing how a strategic complementarity in search can produce endogenous booms and busts in an environment with homogeneous rational beliefs.

A strand of the labor-search literature studies self-fulfilling prophecies regarding unemployment. Howitt and McAfee (1987) show that if the labor-market matching technology has increasing returns, there are multiple equilibria. Howitt and McAfee (1992) and Kaplan and Menzio (2016) consider constant returns in matching; they instead incorporate a positive demand effect of low unemployment to produce multiplicity. Branch and Silva (2022) study an economy à la Mortensen and Pissarides (1994) with households that use government bonds and the equity of firms as liquid wealth. Their model features a demand channel that works through firm entry as in Berentsen *et al.* (2011). My focus here is on a setup with endogenous search and constant returns in matching, showing that multiplicity can arise if liquid wealth comprises firms' equity.

My analysis of a stable inflation regime contributes to the question whether a central bank should pay attention financial developments over and above the extend to which these affect inflation. Some argue in favor (e.g., Bordo and Jeanne, 2002; Roubini, 2006; Smets, 1997; White and Borio, 2004), while others argue against (e.g., Bernanke and Gertler, 2001; Greenspan, 2007; Schwartz, 2003; Woodford, 2012). I show that inflation stability is insufficient for financial stability; interventions like TARP are also necessary. The analysis of TARP contributes to the literature spurred by Sargent and Wallace (1981), studying the interaction between fiscal and monetary policy. It received renewed attention due to unconventional monetary policies, as losses from them may be inflationary, calling for bailout of the central bank (Reis, 2015; Tanaka, 2021). I contribute by showing that TARP requires fiscal backing, and that such backing can occur on the equilibrium path if the price at which assets are bought in TARP is set too conservatively.

Finally, my work fits a theoretic literature on how various aspects of financial intermediation, e.g., the provision of liquidity insurance (Peck and Shell, 2003), market making (Rubinstein and Wolinsky, 1987), the role of intermediaries' reputation (Gu, Mattesini, Monnet and Wright, 2013), and the creation of information-insensitive liabilities (Gorton and Ordoñez, 2014), generate instability. Gu, Monnet, Nosal and Wright (2020) review many of these aspects analytically. My contribution is to focus on the creation of liquid claims on firms' equity in a framework unifying liquidity constraints and search.

Outline. Section 2 lays out the model and Section 3 revisits the scope for endogenous dynamics if liquid wealth comprises only currency. Section 4 uncovers endogenous dynamics when liquid wealth comprises only equity and Section 5 adds currency. Section 6 studies stabilization policies and Section 7 concludes. Proofs are in Appendix D.

2 Model

Time is discrete and denoted with $t \in \mathbb{N}_0$. The time horizon is infinite. Two markets convene sequentially at time t: first a decentralized market (DM_t) and then a centralized market (CM_t). The DM is a frictional market in which liquid wealth and buyers' search are essential. Appendix C lays out a DM with two-sided search for which the main results derived below hold true. The CM is a frictionless market in which agents re-balance their asset positions. There are two fully perishable and perfectly divisible goods: DM goods and CM goods, which are traded in the DM and the CM, respectively. CM goods are used as the numeraire, so all prices and real values are expressed in CM goods.

The economy is populated by a unit mass of infinitely-lived buyers, overlapping generations of finitely-lived firms, and a government. Buyers' preferences are described by the time t flow-utility function

$$\mathcal{U}(\sigma_t, q_t, x_t) = u(q_t) - s(\sigma_t) + x_t \tag{1}$$

and the buyers discount utility between periods at a rate $\beta \in (0, 1)$. In Equation (1), $q_t \in \mathbb{R}_+$ is consumption of DM_t goods, $x_t \in \mathbb{R}$ is net consumption of CM_t goods, and $\sigma_t \in \Sigma \subseteq [0, 1]$ is DM_t search effort which invokes disutility according to $s : \Sigma \to \mathbb{R}_+$. Search effort will equal the probability of being able to acquire DM goods, as detailed later. Function s is increasing and convex, and u is twice continuously differentiable and satisfies $u(0) = 0$, $u' > 0$, $u'' < 0$, $\lim_{q \to 0} u'(q) = \infty$, and $\lim_{q \to \infty} u'(q) = 0$. For the set of feasible levels of search effort, I assume $\Sigma = \{l, h\}$, with $0 < l < h \leq 1$ and $s(h) - s(l) = k$. This makes the mechanism more transparent and is not critical.[6]

A unit mass of firms is born in CM_t, which are owned by the buyers and live until CM_{t+1}. These firms receive an endowment of y CM_{t+1} goods in DM_{t+1} from which they can produce q DM_{t+1} goods by using $c(q)$ CM_{t+1} goods as an input, where $c(0) = 0$, $c' > 0$, and $c'' \geq 0$. CM_{t+1} goods unused for production in DM_{t+1} are stored until CM_{t+1}.

[6] When facing a liquidity premium associated with carrying assets, increased search makes it more attractive for buyers to also increase their asset holdings. This is because assets can then be spend on DM goods with a higher probability. Taking this complementarity between search and asset holdings into account, marginal benefits of exerting search are increasing in the level of search. Therefore, although optimal search will be generically unique if Σ is a convex set, the set of search levels implementable in equilibrium exhibits gaps when the costs of search are close to linear—search may jump from a high level to a low level for an infinitesimally small change in the liquidity premium. If search cost would be linear, then for convex $\Sigma = [\underline{\sigma}, \overline{\sigma}]$ we get that, depending on asset holdings, buyers either choose $\underline{\sigma}$ or $\overline{\sigma}$.

Two perfectly divisible assets are available in the economy. First, ownership shares of the firms, which are bundled into an ETF-like asset. I normalize the amount of shares issued by each firm to one, and I simply refer to ETF shares as *equity shares*. The second asset is intrinsically useless *currency*, which is issued by the government.

All the aggregate uncertainty in the economy comes from a sunspot—a random variable irrelevant for preferences and technologies. The sunspot generates a realization before markets convene at time t. Agents, in turn, coordinate their behavior based on this realization. I will index all prices, quantities, and values with t rather than with the full history $\mathcal{H}_t$ of the sunspot to simplify the notation. Variables and functions indexed with t are therefore (potentially) stochastic objects.

The results from the model can, in principle, be driven by agents' inability to contract on $\mathcal{H}_t$. This is the case because buyers need to determine already in CM_{t-1} how many assets to carry into DM_t, i.e., before uncertainty about the sunspot is resolved. To eliminate such concerns and to isolate the interaction between search and liquidity, I allow agents to choose the amount of currency and equity shares carried into DM_t contingent on $\mathcal{H}_t$ by means of Arrow securities, as detailed next.

Markets. The CM is a perfectly competitive market in which the incumbent firms pay dividends and subsequently die, ownership shares in the new firms are issued and then traded, and buyers adjust their asset positions by producing or consuming CM goods. The CM_t prices of currency and the newly issued equity shares are Φ_t and Ψ_t, respectively. An Arrow security that delivers one unit of currency in DM_{t+1} contingent on history $\mathcal{H}_{t+1}$, is priced at $\phi(\mathcal{H}_{t+1}|\mathcal{H}_t)$, and likewise, a security that delivers one equity share in DM_{t+1} contingent on history $\mathcal{H}_{t+1}$, is priced at $\psi(\mathcal{H}_{t+1}|\mathcal{H}_t)$. I let $\phi_{t+1} = \phi(\mathcal{H}_{t+1}|\mathcal{H}_t)/\mathcal{P}(\mathcal{H}_{t+1}|\mathcal{H}_t)$ and $\psi_{t+1} = \psi(\mathcal{H}_{t+1}|\mathcal{H}_t)/\mathcal{P}(\mathcal{H}_{t+1}|\mathcal{H}_t)$ be the respective prices adjusted for the contingent probability that history $\mathcal{H}_{t+1}$ indeed realizes. The benefit of this notation is that ϕ_{t+1} and ψ_{t+1} can be interpreted as stochastic variables that represent pricing kernels for currency and equity shares. There should be no arbitrage opportunities, entailing that:

$$\Phi_t = \mathbb{E}_t\{\phi_{t+1}\} \quad \text{and} \quad \Psi_t = \mathbb{E}_t\{\psi_{t+1}\}. \tag{2}$$

The CM_t price of currency thus equals the combined CM_t price of a set of Arrow securities that deliver exactly one unit of currency in DM_{t+1} with certainty. The CM_t price of equity shares is determined analogously.

The newborn firm issues a unit mass of shares, yielding Ψ_t CM goods that are paid to the buyers—the initial owners of the firm. The idiosyncratic risk faced by the firms in DM_{t+1} is diversified away through bundling their shares into the ETF-like asset.

An incumbent firm—born in CM_{t-1}—pays dividend and subsequently dies in CM_t. A firm that holds assets worth z_t^f CM goods and an inventory o_t of CM goods will therefore pay a dividend of $\delta_t = z_t^f + o_t$ CM goods. The incumbent equity shares pay a dividend of Δ_t CM goods, where Δ_t is the aggregated dividend of all the underlying incumbent firms and also the cum-dividend value of the equity share. The shares mature after this dividend payment takes place; the ex-dividend value is zero.

The government is only active in the CM. The supply of currency, measured at the end of CM_t, is denoted with M_t. To close the government's budget, a lump-sum transfer τ_t (tax if negative) accruing to buyers is set according to

$$\tau_t = \Phi_t(M_t - M_{t-1}). \tag{3}$$

Buyers are randomly and bilaterally matched to the firms in the DM and negotiate the terms of trade (q, p), with q the amount of DM goods received by the buyer and p the value of the assets—expressed in CM goods—received by the firm. The quasi-linear preferences imply that the utility surplus for the buyer is $u(q) - p$, whilst the surplus for the firm is $p - c(q)$ (Appendix B provides details). I follow the general approach of Gu and Wright (2016) to determine (q, p), meaning that the underlying negotiation process between the buyer and the firm is summarized by an exogenous payment protocol v, mapping $q \mapsto p$. Utility surplus of the buyer from the transaction is then $L(q) = u(q) - v(q)$ and the firm's profit from the transaction is $\Pi(q) = v(q) - c(q)$. I let q^* solve $u'(q) = c'(q)$ and I assume that v is twice continuously differentiable and such that: (i) $v(0) = 0$, $v' > 0$; (ii) $L(q)$ attains a unique global maximum at $\hat{q} \in (0, q^*]$ and is strictly increasing in q for $q \in (0, \hat{q})$; (iii) $\Pi(q) > 0$ for $q \in (0, \hat{q}]$; and (iv) $\Pi'(q) > 0$ for $q \in (0, \hat{q}]$.[7] These assumptions simply ensure that L and Π are increasing in q, and particularly that $\Pi(q)$ is positive on the relevant domain for q. This generates some meaningful interaction between DM activity

[7] These conditions are satisfied for a broad set of bargaining protocols, including Nash (1950) bargaining, proportional bargaining à la Kalai (1977), and gradual bargaining as in Rocheteau, Hu, Lebeau and In (2021), as well as a payment protocol representing constant-markup pricing.

and the firm's profit.

The buyer's maximization problem. In Appendix B, I show that the quasi-linear preferences imply that the buyer's Bellman equation is

$$V_t(m_t, e_t) = \max_{\sigma_t \in \{l,h\}} \left\{ \sigma_t \max_{q_t \geq 0} \{L(q_t) | \text{ s.t. } v(q_t) \leq z_t(m_t, e_t) \text{ and } c(q_t) \leq y\} - s(\sigma_t) \right\}$$
$$+ \Phi_t m_t + \Delta_t e_t + \tau_t + \Psi_t \tag{4}$$
$$+ \mathbb{E}_t \left\{ \max_{m_{t+1}, e_{t+1} \geq 0} \{\beta V_{t+1}(m_{t+1}, e_{t+1}) - \phi_{t+1} m_{t+1} - \psi_{t+1} e_{t+1}\} \right\},$$

where $z_t(m_t, e_t) = \Phi_t m_t + \chi \Delta_t e_t$ is the buyer's liquid wealth, m_t is currency carried into time t, e_t are equity shares carried into time t, and $\chi \in \{0, 1\}$ indicates whether equity is liquid.

The Bellman equation comprises the following. In DM_t, the buyer first determines search effort σ_t, which equals the probability that the buyer ends up in a match with a firm.[8] If matched to a firm in DM_t, the buyer chooses q_t to maximize its match surplus $L(q_t) = u(q_t) - v(q_t)$ subject to: a liquidity constraint, transpiring that the payment $p = v(q_t)$ must be made with liquid wealth; and the firm's capacity constraint, assumed to be slack. The resulting terms of trade satisfy

$$(q_t, p_t) = \begin{cases} (v^{-1} \circ z_t(m_t, e_t),\ z_t(m_t, e_t)) & \text{if } z_t(m_t, e_t) < v(\hat{q}), \\ (\hat{q},\ v(\hat{q})) & \text{if } z_t(m_t, e_t) \geq v(\hat{q}). \end{cases} \tag{5}$$

The buyer thus ideally consumes $\hat{q}$, but needs liquid wealth $v(\hat{q})$ for that. If it does not command over that amount of liquid wealth, it will spend all liquid wealth on DM_t consumption; $q_t = v^{-1} \circ z_t(m_t, e_t)$ since the liquidity constraint binds. I impose $y \geq c(\hat{q})$ to ensure that the capacity constraint is indeed slack.

In CM_t, the Arrow-like structure of the asset market allows the buyer to choose the time-$t+1$ asset holdings (m_{t+1}, e_{t+1}) contingent on the yet to be realized history $\mathcal{H}_{t+1}$. One can therefore write the maximization problem for asset holdings within the expectations operator. The cost of acquiring the time-$t+1$ portfolio (m_{t+1}, e_{t+1}) is

[8] The setup can be microfounded with a constant-returns-to-scale matching function $\min\{b, f\}$, where f is the mass of firms (equal to one) and b is the effective mass of buyers—the mass of buyers multiplied by their average search $\tilde{\sigma}$. The mass of realized matches is then $\min\{\tilde{\sigma}, 1\}$, the probability that a buyer finds a match is $\sigma \min\{\tilde{\sigma}, 1\}/\tilde{\sigma} = \sigma$, and the probability that a firm finds a match is $\min\{\tilde{\sigma}, 1\} = \tilde{\sigma}$.

$\mathbb{E}_t\{\phi_{t+1}m_{t+1} + \psi_{t+1}e_{t+1}\}$, whilst the value of the time-t portfolio carried from time-$t-1$ is $\Phi_t m_t + \Delta_t e_t$. The quasi-linear preferences entail that the optimal choice for the time-$t+1$ portfolio is independent of the buyer's trading history. Finally, the government transfer and the value of new equity shares (recall newborn firms are owned by the households) entail that the buyer receives $\tau_t + \Psi_t$ CM goods in a lump-sum way.

The recursive nature of the Bellman Equation (4) together with the DM_t terms of trade (5) allow me to summarize the buyer's time-t decisions for assets and search:

$$\max_{\substack{\sigma_t \in \{l,h\},\\ m_t, e_t \geq 0}} \left\{\sigma_t L\left(\min\{v^{-1} \circ z_t(m_t, e_t), \hat{q}\}\right) - s(\sigma_t) + (\Phi_t - \phi_t/\beta)m_t + (\Delta_t - \psi_t/\beta)e_t\right\}. \quad (6)$$

In other words, we can think of the buyer as solving for time-t search and time-t asset holdings simultaneously, stemming from the Arrow-like nature of the asset market.

Firm dividends. Expected dividends that an incumbent firm will pay in CM_t, contingent on the aggregate uncertainty being resolved, i.e., $\mathbb{E}\{\delta_t|\mathcal{H}_t\}$, equal the aggregate dividend payment Δ_t of equity shares. Let $G_t(\sigma, m, e)$ be the likelihood that a randomly drawn buyer in DM_t devotes search effort $\sigma_t \leq \sigma$, and holds currency $m_t \leq m$ and equity shares $e_t \leq e$. Obviously, G_t is an equilibrium object. It follows that:

$$\Delta_t = \iiint \sigma \Pi\left(\min\{v^{-1}(\Phi_t m + \chi\Delta_t e), \hat{q}\}\right) G_t(\mathrm{d}\sigma, \mathrm{d}m, \mathrm{d}e) + y. \quad (7)$$

Equation (7) elucidates that firms receive an endowment of y general goods upon entering DM_t. Each firm then draws a buyer from G_t. The drawn buyer devotes search effort σ and carries liquid wealth $z = \Phi_t m + \chi\Delta_t e$. A match then occurs with probability σ and yields additional profit $\Pi\left(\min\{v^{-1} \circ z_t(m, e), \hat{q}\}\right)$. The firm is thus a one-period-lived asset in the spirit of Lucas (1978), but with an endogenous dividend.

Equilibrium characterization. The equilibrium distribution G_t for search and assets must be in line with the buyers' maximization problem embedded in Equation (6), as well as the transversality condition $\lim_{T\to\infty} \beta^T[\Phi_T m_T + \Delta_T e_T] = 0$ (see Rocheteau and Wright, 2013). Further, it should satisfy market clearance:

$$\iiint m' G_t(\mathrm{d}\sigma', \mathrm{d}m', \mathrm{d}e') = M_{t-1} \quad \text{and} \quad \iiint e' G_t(\mathrm{d}\sigma', \mathrm{d}m', \mathrm{d}e') = 1. \quad (8)$$

This allows me to define

Definition 1. *Given a (stochastic) process* $\{M_{t-1}\}_{t=0}^{\infty}$ *for currency supply, an equilibrium is a (stochastic) process* $\{G_t : \mathbb{R}^3 \to [0,1], \phi_t, \Phi_{t-1}, \psi_t, \Psi_{t-1}, \Delta_t\}_{t=0}^{\infty}$ *such that: (i) the no-arbitrage condition (2) holds; (ii) buyers maximize utility, i.e., any* (σ, m, e) *on the support of* G_t *must solve (6) and satisfy* $\lim_{T\to\infty} \beta^T[\Phi_T m_T + \Delta_T e_T] = 0$*; (iii) the aggregate dividend payment* Δ_t *satisfies (7); and (iv) markets clear, i.e., (8) holds.*

I next characterize equilibrium properties of asset prices, DM outcomes, and liquid wealth that are useful for the remaining analysis.

Equilibrium asset prices. From Equation (6) it follows that m_t and e_t—demand for currency and equity—are bounded only if $\beta\Phi_t \leq \phi_t$ and $\beta\Delta_t \leq \psi_t$ due to quasi-linear utility. The conditions $\beta\Phi_t \leq \phi_t$ and $\beta\Delta_t \leq \psi_t$ must therefore hold true to have an equilibrium. If we then take a buyer's DM$_t$ outcome (σ_t, q_t)—search and, when realized, consumption in a DM$_t$ match—as given and we focus on the interesting case in which asset holdings are positive, the optimality of (m_t, e_t) implies

$$\phi_t = \beta\left[1 + \sigma_t L'(q_t)/v'(q_t)\right]\Phi_t \quad \text{and} \quad \psi_t = \beta\left[1 + \chi\sigma_t L'(q_t)/v'(q_t)\right]\Delta_t, \tag{9}$$

where q_t is determined as a function of the buyer's asset holdings as highlighted in (5). Equation (9) states that the time-discounted benefits of the marginal asset equal the acquisition cost. The benefits comprise two components. First, a savings component, being the CM$_t$ price Φ_t for currency and the CM$_t$ dividend Δ_t for equity. Second, a liquidity component, being $\Phi_t\sigma_t L'(q_t)/v'(q_t)$ for currency and $\Delta_t\chi\sigma_t L'(q_t)/v'(q_t)$ for equity. The liquidity component reflects the marginal value of the respective asset in DM$_t$ stemming from the liquidity constraint. From (9) it is now useful to define

$$\iota_t = \phi_t/\beta\Phi_t - 1, \tag{10}$$

which is essentially a stochastic liquidity premium (SLP) since it equals zero when the aforementioned liquidity components in Equation (9) are absent. The SLP is non-negative because this induces bounded asset demand. Further, the SLP entails

$$\Phi_{t-1} = \beta\mathbb{E}_{t-1}\{(1+\iota_t)\Phi_t\}, \quad \psi_t = (1+\chi\iota_t)\Delta_t, \quad \text{and} \quad \Psi_{t-1} = \beta\mathbb{E}_{t-1}\{(1+\chi\iota_t)\Delta_t\}. \tag{11}$$

Currency in CM_{t-1} is thus priced using stochastic discount factor $\beta(1+\iota_t)$, where only the CM_t price matters since currency pays zero dividend. Equity in CM_{t-1} is priced using stochastic discount factor $\beta(1+\chi\iota_t)$, where only the CM_t dividend matters since the CM_t ex-dividend price is zero.

Equilibrium search and liquid wealth holdings. An individual buyer's search and liquid wealth can be thought of as functions of ι_t. Particularly, (9) and (10) imply

$$\iota_t = \mathcal{L}^\sigma(z_t^\sigma) \equiv \frac{\sigma L'(\min\{v^{-1}(z_t^\sigma), \hat{q}\})}{v'(\min\{v^{-1}(z_t^\sigma), \hat{q}\})}, \tag{12}$$

where z^σ is the liquid wealth held by a buyer that searches at intensity σ. We can let z_t^σ be determined as a function of σ_t and ι_t (unless $\iota_t = 0$) by means of:

Assumption 1. *The payment protocol is such that $L'(q)/v'(q)$ is strictly decreasing in q on the domain $(0, \hat{q})$.*

The marginal value $\mathcal{L}^\sigma$ of liquid wealth is then decreasing.[9] Assumption 1 furthermore implies that z^σ is continuous in ι/σ, decreasing in ι/σ, strictly decreasing in ι/σ for $\iota/\sigma \in (0, I)$, indeterminate up to a lower bound $v(\hat{q})$ for $\iota/\sigma = 0$, and zero for $\iota/\sigma \geq I \equiv \lim_{q\to 0} L'(q)/v'(q)$ (see Gu and Wright, 2016, for a proof).[10]

To determine the buyers' search effort, recall that $k = s(h) - s(l)$. The buyers' maximization in Equation (6) therefore implies that buyers are willing to search at $\sigma_t = h$ ($\sigma_t = l$) if and only if

$$\max_{z\geq 0}\left\{hL(\min\{v^{-1}(z), \hat{q}\}) - \iota_t z\right\} - \max_{z\geq 0}\left\{lL(\min\{v^{-1}(z), \hat{q}\}) - \iota_t z\right\} \geq (\leq) k. \tag{13}$$

Note that $\iota_t z$ is the cost of carrying liquid wealth z. The implication is that buyers intensify search when ι_t is low since the LHS of Equation (13) is decreasing in ι_t. The reason is that search is more attractive when DM_t match surplus is large. This requires, through the liquidity constraint, that the buyer commands of much liquid wealth—search and liquid wealth are complementary. Carrying liquid wealth, in turn, is cheap if ι_t is low. I impose the following to ensure some variation in σ_t:

[9] When terms of trade are determined by proportional bargaining, gradual bargaining, or constant mark-up pricing, this property is always satisfied. When terms of trade are determined by Nash bargaining, this property is satisfied when the bargaining power of the buyer is sufficiently large.

[10] Depending on the negotiation procedure that generates v, we have can that $\lim_{q\to 0} L'(q)/v'(q)$ is either infinity or bounded.

Assumption 2. $\max_{z\geq 0}\{hL\circ v^{-1}(z)-lIz\}<k<(h-l)L(\hat{q})$.

Buyers then choose $\sigma_t=h$ when $\iota_t=0$, but when ι_t becomes sufficiently large, they will, for a uniquely determined threshold $\tilde{\iota}\in(0,lI)$ that depends on k, switch to $\sigma_t=l$ while still holding a positive amount of liquid wealth. I define η_t as the fraction of buyers that search intensely:

$$\eta_t\in\begin{cases}\{1\} & \text{if } \iota_t<\tilde{\iota},\\ [0,1] & \text{if } \iota_t=\tilde{\iota},\\ \{0\} & \text{if } \iota_t>\tilde{\iota}.\end{cases}\tag{14}$$

Liquid wealth in equilibrium. From (12) we know that z_t^σ is determined uniquely as a function of ι_t when $\iota_t>0$, whilst it is indeterminate up to the lower bound $v(\hat{q})$ when $\iota_t=0$. Note that we can assume without loss that all buyers searching at σ hold the same amount of liquid wealth z_t^σ due to quasi-linear preferences. Also note that $\iota_t\leq\tilde{\iota}$—the condition for having $\eta_t>0$—implies $z_t^h\geq\underline{z}^h$, whilst $\iota_t\geq\tilde{\iota}$—the condition for $\eta_t<1$—implies $z_t^l\leq\overline{z}^l$, both with equality if and only if $\iota_t=\tilde{\iota}$, where

$$\underline{z}^h:\quad\tilde{\iota}=\mathcal{L}^h(\underline{z}^h)\equiv\frac{hL'\circ v^{-1}(\underline{z}^h)}{v'\circ v^{-1}(\underline{z}^h)}\quad\text{and}\quad\overline{z}^l:\quad\tilde{\iota}=\mathcal{L}^l(\overline{z}^l)\equiv\frac{lL'\circ v^{-1}(\overline{z}^l)}{v'\circ v^{-1}(\overline{z}^l)}.\tag{15}$$

Equation (15) implies that $0<\overline{z}^l<\underline{z}^h$. This elucidates once more that liquid wealth and search are strategic complements—if a buyer increases search from l to h, it will also hold strictly more liquid wealth.

Buyers' aggregate ex-post demand for liquid wealth—liquid wealth held in DM_t—is

$$z_t^d=\eta_t z_t^h+(1-\eta_t)z_t^l.\tag{16}$$

Ex-post demand is decreasing in ι_t and indeterminate but subject to the lower bound $v(\hat{q})$ when $\iota_t=0$. It is useful for future purposes to note that aggregate ex-post demand can also be mapped into the DM_t marginal value of liquid wealth

$$\iota_t=\mathcal{L}(z_t^d)\equiv\begin{cases}\mathcal{L}^l(z_t^d) & \text{if } z_t^d<\overline{z}^l,\\ \tilde{\iota} & \text{if } \overline{z}^l\leq z_t^d\leq\underline{z}^h,\\ \mathcal{L}^h(z_t^d) & \text{if } z_t^d>\underline{z}^h.\end{cases}\tag{17}$$

Buyers' ex-ante demand for liquid wealth—the cost of acquiring the portfolio of liquid assets in CM_{t-1}—is

$$w_{t-1}^d = \mathbb{E}_t\{\beta(1+\iota_t)z_t^d\}, \tag{18}$$

as follows from the definition of the SLP (10). Ex-ante demand can be increasing or decreasing in ι_t; a higher ι_t on the one hand reduces ex-post demand—a substitution effect—but on the other hand it increase ex-ante demand if ex-post demand were left unaffected—an income effect. Which effect dominates plays a role for the existence of endogenous dynamics, as analyzed further below. Ex-post demand is however key to most of the analysis, so I simply refer to it as *demand* in what follows.

Ex-post liquid-wealth supply z_t^s, which I likewise simply refer to as *supply*, consists of equity (if liquid) and currency:

$$\begin{aligned} z_t^s &= \Phi_t M_{t-1} + \chi\Delta_t \\ &= \Phi_t M_{t-1} + \chi\left[\eta_t h \Pi(\min\{z_t^h, \hat{q}\}) + (1-\eta_t) l \Pi(\min\{z_t^l, \hat{q}\}) + y\right]. \end{aligned} \tag{19}$$

Equations (12), (14), (16), (17) and (19) transpire a key feature of the model—demand and supply of liquid wealth are interwoven if equity is liquid. First, a higher supply reduces ι_t through (17) since demand must equal supply, which in turn increases the search-contingent demands z_t^h and z_t^l through (12). This positively feeds back into supply through firms' dividends. Second, when supply increases so that ι_t drops below the threshold $\tilde{\iota}$, this boosts buyers' search through (14). The search boost, in turn, positively feeds back into supply through: (i) a greater mass of firms that are matched; and (ii) the fact that matches are more profitable when buyers search intensely since $\overline{z}^l < \underline{z}^h$.

3 Liquidity with only currency

It is well-documented in the money-search literature that self-fulfilling dynamics can arise if liquid wealth comprises intrinsically useless currency. This section establishes that the scope for such dynamics does, at least to some extent, not change due to buyers' endogenous search if equity is illiquid. It will thus be the interaction between endogenous search and liquid equity which entails novel results.

Let the supply of currency M_t develop according to

$$M_t = \mu M_{t-1}, \quad \text{with} \quad \mu > \beta, \tag{20}$$

which is a common assumption in the literature. Equation (20) can be used to derive a first-order difference equation that describes the dynamic equilibrium. Define $\mathcal{M}_t \equiv \Phi_t M_{t-1}$ as DM$_t$ real currency balances. Then, using $\chi = 0 \Rightarrow z_t^d = z_t^s = \mathcal{M}_t$, (2), (10), and (17), one can derive

$$\phi_t = \beta \left[1 + \mathcal{L}(\mathcal{M}_t)\right] \mathbb{E}_t\{\phi_{t+1}\}. \tag{21}$$

Equation (21) can be reformulated by defining $x_t \equiv \phi_t M_{t-1}/\mu$ and using $\mathcal{M}_t = \mathbb{E}_t\{x_{t+1}\}$:

$$x_t = f_m(\mathbb{E}_t\{x_{t+1}\}) \equiv \beta \left[1 + \mathcal{L}(\mathbb{E}_t\{x_{t+1}\})\right] \mathbb{E}_t\{x_{t+1}\}/\mu; \tag{22}$$

a simple difference equation in x_t, where the subscript m elucidates that f applies to an economy in which liquid wealth comprises only currency. The focus here is on bounded monetary equilibria, meaning that there exist $\underline{N}, \overline{N} \in \mathbb{R}_{++}$ such that $\mathcal{M}_t \in [\underline{N}, \overline{N}]\ \forall t$.

One bounded monetary equilibrium is the monetary steady state. If features $x_t = x_{t+1} = x_{ss} = \mathcal{M}_{ss} > 0$, which simply implies that

$$\mathcal{L}(x_{ss}) = \iota_{ss} = \mu/\beta - 1 \quad \text{and} \quad \Phi_{t+1} = \Phi_t/\mu. \tag{23}$$

In other words, inflation equals the money growth rate and ι_{ss} is positive (this is why I assume $\mu > \beta$ in (20)).[11] Figure 1 depicts various parameterized examples of f_m, where it has to be noted that f_m will always intersect the 45-degree line from above. The monetary steady state is unique, unless $\mu = \beta(1 + \tilde{\iota})$; for that knife edge case, all $x_{ss} \in [\overline{z}_l, \underline{z}_h]$ are steady states as illustrated in Figure 1e. Buyers are then indifferent between high and low search, so any $\eta \in [0, 1]$ can be part of a steady state.

Equation (21) highlights that not much changes compared to a plain-vanilla model with exogenous search. The only substantial difference lies in the fact that $f_m(x)$ is no longer continuously differentiable at $x = \overline{z}^l$ and $x = \underline{z}^h$, which causes the continuum of steady states for the knife-edge case $\mu = \beta(1 + \tilde{\iota})$. A sufficient condition to have

[11] Existence of the monetary steady state requires $\mu < \beta(1 + lL)$; otherwise, currency balances would be zero.

self-fulfilling bounded dynamics, be it stochastic or deterministic, is

$$-1 > f_m'(x_{ss}) \equiv \beta[1 + \mathcal{L}(x_{ss}) + \mathcal{L}'(x_{ss})x_{ss}]/\mu. \tag{24}$$

This follows from the method of flip-bifurcations—mirroring f_m in the 45-degree line to obtain f_m^{-1} (see Azariadis, 1993). Intersections between f_m and f_m^{-1} that do not lie on the 45-degree line constitute a two cycle. When $(x_{t+1}, x_t) = (x', x'')$ is such a point, it follows that $x'' = f_m(x')$ and $x'' = f_m^{-1}(x')$—the economy can alternate deterministically between $x = x'$ and $x = x''$. From the continuity of f_m it follows that stochastic two cycles then exist, too. Figures 1b, 1f, and 1g illustrate two cycles. We can even have three different two cycles as illustrated in Figure 1g since f_m is not continuously differentiable.

Bounded monetary equilibria other than steady states do not exist if f is monotone increasing. The intuition is depicted in Figure 1h. When $x_t < x_{ss}$, we have $\mathbb{E}_t\{x_{t+1}\} < x_t$, so it must be that there is an equilibrium realization for x_{t+1} such that $x_{t+1} < x_t$. Forward iterating the argument implies that x_t goes to zero with positive probability, so that also $\mathcal{M}_t$ goes to zero with positive probability. Likewise, $x_t > x_{ss}$ implies that $\mathbb{E}_t\{x_{t+1}\} > x_t$, which then implies $\mathcal{M}_t$ will go to infinity with some probability.

A similar argument applies when $\mu = \beta$, commonly know as the *Friedman rule*. We then have $f_m(x) \geq x$ on the domain $\mathbb{R}_{++}$, with equality if and only if $x \geq v(\hat{q})$. It follows that $x_t < v(\hat{q})$ cannot be an equilibrium, as x_t would go to zero with positive probability. On the other hand, all $x_t \geq v(\hat{q})$ are part of an equilibrium, but induce identical real allocations since they all imply that $\iota_t = 0$. In other words, setting $\mu = \beta$ uniquely implements the real allocation $(\sigma_t, q_t) = (h, \hat{q})$ for which the liquidity constraint is slack.

Bounded monetary equilibria other than steady states in the above setup are sustained through a chain of expectations that are rational because currency is an infinitely-lived asset. This result carries over to infinitely-lived assets that pay an exogenous divided as in Lucas (1978), as such an asset is almost the same as currency if the divided is infinitesimal (see Altermatt *et al.*, 2021). To contrast these kind of self-fulfilling equilibria with those that can arise with liquid equity, which pays an endogenous dividend, I impose

Assumption 3. *The parameter specification is such that* $1 + \mathcal{L}(z) + \mathcal{L}'(z)z \geq 0 \; \forall z$.

Assumption 3 rules out bounded monetary equilibria other than steady states if liquid wealth comprises only currency; f_m is monotone increasing. The assumption relates directly to how ex-ante liquid-wealth demand and the SLP move together; it holds true if and only if ex-ante liquid-wealth demand is monotonically decreasing in the SLP since

$$\iota = \mathcal{L}(z) \quad \Rightarrow \quad \mathrm{d}[\beta(1+\iota)z]/\mathrm{d}\iota = \beta(1 + \mathcal{L}(z) + \mathcal{L}'(z)z)/\mathcal{L}'(z). \tag{25}$$

The substitution effect in ex-ante demand thus dominates under Assumption 3 since $\mathcal{L}' < 0$. The point developed further below is that the assumption no longer rules out bounded monetary equilibria other than steady states once *both* currency and equity comprise liquid wealth.

4 Liquidity with only equity

I consider an environment in which liquid wealth comprises only equity before delving into a richer setup in which liquid wealth comprises both currency and equity.

The SLP ι_t is key since it determines z_t^h, z_t^l, and η_t (see Equations (12) and (14)). Demand and supply of liquid wealth (subscript e refers to the current environment) are

$$z^d(\iota_t) = \eta(\iota_t)z^h(\iota_t) + (1 - \eta(\iota_t))z^l(\iota_t), \tag{26}$$

$$z_e^s(\iota_t) = h\eta(\iota_t)\Pi(\min\{z^h(\iota_t), \hat{q}\}) + l(1 - \eta(\iota_t))\Pi(\min\{z^l(\iota_t), \hat{q}\}) + y, \tag{27}$$

where z_t^d is uniquely pinned down unless $\iota_t = 0$; it is then indeterminate up to the lower bound $v(\hat{q})$. An equilibrium occurs when ι_t is such that excess demand for liquid wealth

$$r_e(\iota_t) = z^d(\iota_t) - z_e^s(\iota_t) \tag{28}$$

is zero; there is no need to consider inter-temporal conditions due to the combination of quasi-linear preferences and one-period lived equity, entailing that the economy basically resets every t. Excess demand r_e is uniquely pinned down unless $\iota_t = 0$; it can then take any value $r_e(0) \geq v(\hat{q}) - h\Pi(\hat{q}) - y \equiv \hat{r}_e^h$ since demand is indeterminate up to the lower bound $v(\hat{q})$. One can verify that $\lim_{\iota_t \nearrow \iota^L} r_e(\iota_t) = -y$, whilst $r_e(0)$ can always be strictly positive. An equilibrium therefore exists since $r_e(\iota_t)$ is continuous.

The more relevant question though is whether there are multiple ι_t that are consist with equilibrium. After all, both demand and supply of liquid wealth are decreasing in ι_t through two channels: (i) a *wealth channel* operating through reduced demand when ι_t increases, rendering matches less profitable for firms; and (ii) a *search channel* operating through a reduction in search when ι_t increases beyond $\tilde{\iota}$, entailing fewer matched firms.

We can evaluate the wealth channel by characterizing the derivative of r_e w.r.t. ι:

$$\frac{\partial r_e(\iota_t)}{\partial \iota_t}\bigg|_{\iota_t \neq \tilde{\iota}} = [v'(q) - \sigma \Pi'(q)] \frac{\partial z^\sigma(\iota_t)}{\partial \iota_t}\bigg|_{q = v^{-1} \circ z^\sigma(\iota_t)}, \quad \text{and} \quad \sigma = \begin{cases} h & \text{if } \iota_t < \tilde{\iota}, \\ l & \text{if } \iota_t > \tilde{\iota}. \end{cases} \tag{29}$$

The term in square brackets is positive because $\Pi'(q) < v'(q)$ on the domain $(0, \hat{q}]$; if buyers increase their liquid wealth by a dollar, the firms' profits cannot increase by more than a dollar. The overall effect is therefore negative since the derivative of z^σ w.r.t. ι is negative. There can thus be only one ι that clears the market for liquid wealth if σ is exogenous—the wealth channel is too weak to generate equilibrium multiplicity. On the other hand, the wealth channel can amplify shocks as in Guerrieri and Lorenzoni (2009). Exogenous search implies, for instance

$$\Delta = \sigma \Pi(\min\{v^{-1}(\Delta), \hat{q}\}) + y \quad \Rightarrow \quad \frac{\mathrm{d}\Delta}{\mathrm{d}y} = \frac{v'(q)}{v'(q) - \mathbf{1}_{\{\Delta < v(\hat{q})\}} \sigma \Pi'(q)}\bigg|_{q = \min\{v^{-1}(\Delta), \hat{q}\}}. \tag{30}$$

An increase in the firms' endowment thus leads to a more than one-to-one increase in value of equity if the liquidity constraint binds. A higher endowment namely directly leads to a higher equity value, in turn loosening the buyers' liquidity constraint. This increases the value of equity further, in turn loosening buyers' liquidity constraint further, *etcetera*; a multiplier effect.

The search channel can be evaluated by comparing

$$\tilde{r}_e^- \equiv \lim_{\iota_t \searrow \tilde{\iota}} r_e(\iota_t) = \overline{z}^l - l \Pi \circ v^{-1}(\overline{z}^l) - y \quad \text{and} \quad \tilde{r}_e^+ \equiv \lim_{\iota_t \nearrow \tilde{\iota}} r_e(\iota_t) = \underline{z}^h - h \Pi \circ v^{-1}(\underline{z}^h) - y; \tag{31}$$

the right- and the left-hand limit of $r_e(\iota)$ at $\tilde{\iota}$. There are two opposing forces here. The fact that $\overline{z}^l < \underline{z}^h$ on the one hand drives a positive wedge between $\tilde{r}_e^+$ and $\tilde{r}_e^-$; liquid wealth jumps up if ι_t decreases below $\tilde{\iota}$ because demand for liquid wealth and search are complementary. This generates an upward jump in both firm profit and demand for liquid wealth, where the latter effect dominates since $\Pi'(q) < v'(q)$. But the fact that

$l < h$ drives a negative wedge between $\tilde{r}_e^+$ and $\tilde{r}_e^-$; firms find more matches because search effort jumps up when ι_t moves below $\tilde{\iota}$, increasing the supply of liquid wealth. The latter effect dominates for sure if $\tilde{\iota} \to 0$ since $\overline{z}^l$ and $\underline{z}^h$ are then almost the same (see Equation (15)). This allows for multiplicity when $\tilde{r}_e^+ < 0 < \tilde{r}_e^-$, illustrated in Figure 2.

Proposition 1. *$\tilde{r}_e^+ < 0 < \tilde{r}_e^- \Leftrightarrow (k, y) \in \mathcal{S}_e$, where $\mathcal{S}_e$ has positive mass.*

Three equilibrium levels for ι_t arise in case $(k, y) \in \mathcal{S}_e$: one level $\iota_e^h < \tilde{\iota}$ inducing high search—a *boom*; one level $\iota_e^l > \tilde{\iota}$ inducing low search—a *bust*; and $\tilde{\iota}$ for which some buyers devote high and others devote low search—a *mix* with $\eta_t = \tilde{\eta}_e$. The SLP can freely fluctuate over time between these three levels, entailing endogenous dynamics.

Proposition 1 holds true under Assumption 3, elucidating a qualitatively different scope for equilibrium multiplicity and self-fulfilling dynamics than in Section 3. It also contrasts the common perception that with a finitely-lived asset, there cannot be self-fulfilling dynamics. This perception is based on models in which, following Lucas (1978), an asset earns an exogenous dividend. Since a finitely-lived asset is priced fundamentally when it matures, through backwards induction, a chain of self-fulfilling expectations is ruled out.

Equity in the setup above is the sole means of liquidity, finitely lived, and priced fundamentally when traded in the DM—its value equals the firms' aggregate dividend (see Equation (19)). Yet, the dividend depends on DM trade, and DM trade depends on the dividend through the buyers' liquidity constraint. This intricate relationship entails a strong strategic complementary in search: if other buyers search intensely, the individual buyer wants to search intensely, too; whilst if other buyers search lazily, the individual buyer wants to search lazily, too. The complementarity is reminiscent of that in Diamond (1982) but operates through liquid wealth rather than increasing returns in the matching technology. If other buyers search intensely, liquid-wealth supply increases, driving down the SLP in order to clear the market for liquid wealth. This makes carrying liquid wealth cheaper, in turn making it more attractive for the individual buyer to relax its liquidity constraint, entailing higher match surplus and thus greater benefits of search.

5 Liquidity with currency and equity

I now revisit the scope for self-fulfilling bounded monetary equilibria as in Section 3, but in an environment in which liquid wealth comprises *both* currency and equity. Supply of currency satisfies Equation (20) and the first-order difference equation for x_t is

$$x_t \in f_{me}(\mathbb{E}_t\{x_{t+1}\}) \equiv \beta\left[1 + \mathcal{L}((1+\Delta) \circ \mathbb{E}_t\{x_{t+1}\})\right]\mathbb{E}_t\{x_{t+1}\}/\mu, \tag{32}$$

where the equity dividend $\Delta_t = \Delta(\mathbb{E}_t\{x_{t+1}\})$ depends endogenously on $\mathbb{E}_t\{x_{t+1}\}$:

$$\begin{aligned}\Delta(\mathbb{E}_t\{x_{t+1}\}) = h\eta(\iota)\Pi(\min\{z^h(\iota), \hat{q}\}) + l(1-\eta(\iota))\Pi(\min\{z^l(\iota), \hat{q}\}) + y,\\ \text{where} \quad \iota = \mathcal{L}\left((1+\Delta) \circ \mathbb{E}_t\{x_{t+1}\}\right).\end{aligned} \tag{33}$$

Equation (32) differs from (22) because ι_t now depends also on the value of equity, which, in turn, is a function of ι_t. Equation (33) captures this intricacy and implies that $\Delta(\mathbb{E}_t\{x_{t+1}\})$ can be a correspondence, applying to the difference equation, too. I thus write $x_t \in f_{me}(\mathbb{E}_t\{x_{t+1}\})$, with subscript *me* elucidating that liquid wealth comprises both currency and equity.

Exogenous dividend. I briefly consider the scope for self-fulfilling bounded monetary equilibria when equity would pay an exogenous dividend $\overline{\Delta}$. In that case

$$x_t = \overline{f_{me}}(\mathbb{E}_t\{x_{t+1}\}) \equiv \beta[1 + \mathcal{L}(\mathbb{E}_t\{x_{t+1}\} + \overline{\Delta})]\mathbb{E}_t\{x_{t+1}\}/\mu. \tag{34}$$

Because $\mathcal{L}' < 0$, it follows that

$$\begin{aligned}\mu\overline{f'_{me}}(\mathbb{E}_t\{x_{t+1}\})\beta &= 1 + \mathcal{L}(\mathbb{E}_t\{x_{t+1}\} + \overline{\Delta}) + \mathcal{L}'(\mathbb{E}_t\{x_{t+1}\} + \overline{\Delta})\mathbb{E}_t\{x_{t+1}\}\\ &> 1 + \mathcal{L}(\mathbb{E}_t\{x_{t+1}\} + \overline{\Delta}) + \mathcal{L}'(\mathbb{E}_t\{x_{t+1}\} + \overline{\Delta})(\mathbb{E}_t\{x_{t+1}\} + \overline{\Delta}) \geq 0,\end{aligned} \tag{35}$$

where the fist step uses that $\mathcal{L}' < 0$ and last step uses Assumption 3. The difference equation is monotonically increasing so that bounded monetary equilibria must be steady states.[12] Assumption 3 thus rules out other bounded equilibria if liquid wealth comprises currency and an asset with an exogenous dividend à la Lucas (1978). Likewise, if mone-

[12] Monetary steady states exist if and only of $\beta \leq \mu < \beta(1 + \mathcal{L}(\overline{\Delta}))$.

tary policy sets $\mu = \beta$, the real allocation $(\sigma_t, q_t) = (h, \hat{q})$ prevails uniquely.

Endogenous dividend. Now consider endogenous-dividend equity. It is instructive to first analyze a case with exogenous search. We then have

$$\Delta^\sigma(\mathbb{E}_t\{x_{t+1}\}) = \sigma\Pi\left(\min\{v^{-1}\left(\mathbb{E}_t\{x_{t+1}\} + \Delta^\sigma(\mathbb{E}_t\{x_{t+1}\})\right), \hat{q}\}\right) + y. \tag{36}$$

This equation pins down $\Delta^\sigma(\mathbb{E}_t\{x_{t+1}\})$ uniquely since $\Pi'(q) < v'(q)$. We can then define

$$f^\sigma_{me}(\mathbb{E}_t\{x_{t+1}\}) \equiv \beta\left[1 + \mathcal{L}^\sigma((1 + \Delta^\sigma) \circ \mathbb{E}_t\{x_{t+1}\})\right]\mathbb{E}_t\{x_{t+1}\}/\mu. \tag{37}$$

We can next endogenize search. Equilibrium requires that liquid-wealth demand equals supply:

$$\eta(\iota_t)z^h(\iota_t) + (1 - \eta(\iota_t))z^l(\iota_t) \leq \mathbb{E}_t\{x_{t+1}\} + \eta(\iota_t)\Delta^h(\mathbb{E}_t\{x_{t+1}\}) + (1 - \eta(\iota_t))\Delta^l(\mathbb{E}_t\{x_{t+1}\}), \tag{38}$$

with = if $\iota_t > 0$. From Equation (36) it can be deduced that $\mathbb{E}_t\{x_{t+1}\} = r_e(\iota_t)$, with r_e the excess liquid-wealth demand if liquid wealth comprises only equity as defined in Section 4; given ι_t, real currency balances $\mathcal{M}_t = \mathbb{E}_t\{x_{t+1}\}$ absorb the demand for liquid wealth not provided by equity. We can thus have a monetary equilibrium with $\eta_t = 1$—a boom—if $\mathbb{E}_t\{x_{t+1}\} \geq \max\{\tilde{r}_e^+, \varepsilon\}$, where $\varepsilon > 0$ but infinitesimal as x_t must be strictly positive in monetary equilibrium; and likewise, a monetary equilibrium with $\eta_t = 0$—a bust—if $0 < \mathbb{E}_t\{x_{t+1}\} \leq \tilde{r}_e^-$. The mixed case $\eta_t \in (0, 1)$ requires $\iota_t = \tilde{\iota}$ and exists if there is an $\eta_t \in (0, 1)$ solving $\mathbb{E}_t\{x_{t+1}\} = \eta_t\tilde{r}_e^+ + (1 - \eta_t)\tilde{r}_e^-$ for some $\mathbb{E}_t\{x_{t+1}\} > 0$.

Whether η_t is pinned down by $\mathbb{E}_t\{x_{t+1}\}$ depends on whether $\tilde{r}_e^- < \tilde{r}_e^+$. Proposition 1 clearly indicates we can have both $\tilde{r}_e^- < \tilde{r}_e^+$ and $\tilde{r}_e^- > \tilde{r}_e^+$ because of the search channel identified in Section 4. I distinguish between these two possibilities in what follows.

Real currency balances pin down search. We have $\tilde{r}_e^- < \tilde{r}_e^+$, entailing

$$\eta_t = \eta(\mathbb{E}_t\{x_{t+1}\}) \equiv \begin{cases} 0 & \text{if } \mathbb{E}_t\{x_{t+1}\} \leq \tilde{r}_e^-, \\ \frac{\mathbb{E}_t\{x_{t+1}\} - \tilde{r}_e^-}{\tilde{r}_e^+ - \tilde{r}_e^-} & \text{if } \tilde{r}_e^- < \mathbb{E}_t\{x_{t+1}\} < \tilde{r}_e^+, \\ 1 & \text{if } \mathbb{E}_t\{x_{t+1}\} \geq \tilde{r}_e^+; \end{cases} \tag{39}$$

real currency balances $\mathcal{M}_t = \mathbb{E}_t\{x_{t+1}\}$ pin down η_t uniquely—see Figure 3a—, in turn implying that $f_{me}(\mathbb{E}_t\{x_{t+1}\})$ is a function:

$$x_t = f_{me}(\mathbb{E}_t\{x_{t+1}\}) \equiv \begin{cases} f_{me}^l(\mathbb{E}_t\{x_{t+1}\}) & \text{if } \mathbb{E}_t\{x_{t+1}\} \leq \tilde{r}_e^-, \\ \beta(1+\tilde{\iota})/\mu & \text{if } \tilde{r}_e^- < \mathbb{E}_t\{x_{t+1}\} < \tilde{r}_e^+, \\ f_{me}^h(\mathbb{E}_t\{x_{t+1}\}) & \text{if } \mathbb{E}_t\{x_{t+1}\} \geq \tilde{r}_e^+. \end{cases} \tag{40}$$

Monetary steady states feature $\iota_{ss} = \mu/\beta - 1$ and other bounded equilibria are again ruled out by Assumption 3.[13] To see this, note that for $\mathbb{E}_t\{x_{t+1}\} \in (\tilde{r}_e^-, \tilde{r}_e^+)$, f_{me} is strictly increasing, whilst for other $\mathbb{E}_t\{x_{t+1}\}$, we have

$$\begin{aligned} \mu f_{me}'(\mathbb{E}_t\{x_{t+1}\})/\beta = 1 &+ \mathcal{L}(\mathbb{E}_t\{x_{t+1}\} + \Delta_t) + \mathcal{L}'(\mathbb{E}_t\{x_{t+1}\} + \Delta_t)(\mathbb{E}_t\{x_{t+1}\} + \Delta_t) \\ &- \mathcal{L}'(\mathbb{E}_t\{x_{t+1}\} + \Delta_t)\left(\Delta_t - \Delta'(\mathbb{E}_t\{x_{t+1}\})\mathbb{E}_t\{x_{t+1}\}\right), \end{aligned} \tag{41}$$

where $\mathcal{L}' < 0$. The first line is positive by Assumption 3, whilst the sign of the second line is positive since $\mathcal{L}' < 0$ and

$$\begin{aligned} \Delta_t - \Delta'(\mathbb{E}_t\{x_{t+1}\})\mathbb{E}_t\{x_{t+1}\} &= \left[(\sigma\Pi(q) + y)v'(q) - \sigma\Pi'(q)v(q)\right] / \left[v'(q) - \sigma\Pi'(q)\right] \\ &\geq \left[\sigma v(q)c'(q)\right] / \left[v'(q) - \sigma\Pi'(q)\right], \end{aligned} \tag{42}$$

where the second step uses that $v(q) = \Pi(q) + c(q)$ and $y \geq \sigma c(q)$, the latter being implied by $\sigma \leq 1$ and the firms' slack capacity constraint. Under Assumption 3 only steady states can thus be bounded monetary equilibria. Likewise, $\mu = \beta$ implements the real allocation $(\sigma_t, q_t) = (h, \hat{q})$ for the exact same reason as before.

The analysis above applies to an environment with exogenous search, too. The reason is that η_t is pinned down for a given $\mathbb{E}_t\{x_{t+1}\}$. Comparing with the analysis if liquid wealth comprises currency and an exogenous-dividend asset, the difference is that equity entails a wealth effect. This is evident from Equation (36), elucidating that if the value of currency balances increases, then so does the value of equity. Yet, as in Section 4, the wealth effect is too weak to generate self-fulfilling bounded dynamics if Assumption 3 is imposed or $\mu = \beta$.

[13] Monetary steady states exist if and only if $\beta \leq \mu < \beta(1 + r_e^{-1}(0))$, where $r_e(\iota)$ is invertible since $\tilde{r}_e^- \leq \tilde{r}_e^+$ implies $r_e(\iota)$ is decreasing.

Real currency balances do not pin down search. We have $\tilde{r}_e^- > \tilde{r}_e^+$, entailing[14]

$$\eta_t \in \eta(\mathbb{E}_t\{x_{t+1}\}) \equiv \begin{cases} 0 & \text{if } \mathbb{E}_t\{x_{t+1}\} < \tilde{r}_e^+, \\ \left\{0, \frac{\tilde{r}_e^- - \mathbb{E}_t\{x_{t+1}\}}{\tilde{r}_e^- - \tilde{r}_e^+}, 1\right\} & \text{if } \tilde{r}_e^+ \leq \mathbb{E}_t\{x_{t+1}\} \leq \tilde{r}_e^-, \\ 1 & \text{if } \mathbb{E}_t\{x_{t+1}\} > \tilde{r}_e^-. \end{cases} \tag{43}$$

Thus, $\eta(\mathbb{E}_t\{x_{t+1}\})$ is now a correspondence due the search channel; for real currency balances $\mathcal{M}_t = \mathbb{E}_t\{x_{t+1}\} \in [\tilde{r}_e^+, \tilde{r}_e^-]$, we can have a boom, bust, or mix because of a strong strategic complementary in search—see Figures 3b-3c. Hence, f_{me} is a correspondence, too:

$$x_t \in f_{me}(\mathbb{E}_t\{x_{t+1}\}) \equiv \begin{cases} f_{me}^l(\mathbb{E}_t\{x_{t+1}\}) & \text{if } \mathbb{E}_t\{x_{t+1}\} \leq \tilde{r}_e^+, \\ \left\{f_{me}^l(\mathbb{E}_t\{x_{t+1}\}), \frac{\beta(1+\tilde{\iota})}{\mu}, f_{me}^h(\mathbb{E}_t\{x_{t+1}\})\right\} & \text{if } \tilde{r}_e^+ < \mathbb{E}_t\{x_{t+1}\} < \tilde{r}_e^-, \\ f_{me}^h(\mathbb{E}_t\{x_{t+1}\}) & \text{if } \mathbb{E}_t\{x_{t+1}\} \geq \tilde{r}_e^-. \end{cases} \tag{44}$$

Monetary steady states feature $\iota_{ss} = \mu/\beta - 1$.[15] Yet, monotonicity of f_{me} no longer applies under Assumption 3 if f_{me} is a correspondence on the relevant domain $\mathbb{R}_{++}$. This requires not only $\tilde{r}_e^+ < \tilde{r}_e^-$ but also $0 < \tilde{r}_e^-$, i.e., $\max\{0, \tilde{r}_e^+\} < \tilde{r}_e^-$, which arises for a set of parameters with positive mass.

Proposition 2. $\max\{0, \tilde{r}_e^+\} < \tilde{r}_e^- \Leftrightarrow (k, y) \in \mathcal{S}_{me}$, *with* $\mathcal{S}_e \subseteq \mathcal{S}_{me}$.

The implication is that a weaker condition for the existence of bounded self-fulfilling dynamics arises that does not depend on the properties of $1 + \mathcal{L}(z) + \mathcal{L}'(z)z$ imposed by Assumption 3, but rather on the growth rate of currency supply. For this purpose, it is useful to trace the lowest and highest value for ι_t which can be observed for $\mathbb{E}_t\{x_{t+1}\} \in [\max\{\varepsilon, \tilde{r}_e^+\}, \tilde{r}_e^-]$, where $\varepsilon > 0$ but infinitesimal to account for the fact that $\mathbb{E}_t\{x_{t+1}\} = \mathcal{M}_t > 0$ in a monetary equilibrium. Define $\iota^l(r), \iota^h(r) \geq 0$ as the unique solutions of

$$r = z^\sigma(\iota^\sigma) - \sigma\Pi(\min\{v^{-1} \circ z^\sigma(\iota^\sigma), \hat{q}\}) - y, \quad \sigma \in \{l, h\}, \tag{45}$$

[14] I ignore the knife-edge case $\tilde{r}_e^- = \tilde{r}_e^+$. For that case, η_t is uniquely pinned down unless $\mathbb{E}_t\{x_{t+1}\} = \tilde{r}_e^- = \tilde{r}_e^+$, in which case any value for $\eta_t \in [0, 1]$ goes. However, f_{me} remains a monotonically increasing function, implying that bounded monetary equilibria other than steady states do not exist.

[15] If $\tilde{r}_e^+ \geq 0$, monetary steady states exist if and only if $\beta \leq \mu < \beta(1 + \iota^l(0))$. If $\tilde{r}_e^+ < 0$, monetary steady states exist if and only if $\mu \in [\beta, \beta(1 + \iota_e^h)) \cup [\beta(1 + \tilde{\iota}), \beta(1 + \iota_e^l))$, where $\iota_e^h < \tilde{\iota} < \iota_e^l$ are the solutions to $r_e(0) = \iota$.

and note that, by construction, $f^{\sigma}_{me} = x\beta(1 + \iota^{\sigma}(x))/\mu$. As illustrated by Figures 3b and 3c, the lowest feasible value for ι_t on the aforementioned domain is $\iota^h(\tilde{r}^-_e)$, whilst the highest feasible value is $\iota^l(\max\{\varepsilon, \tilde{r}^+_e\})$. Let $\mathcal{I}$ contain all values in between the extrema:

$$\mathcal{I} \equiv \{\iota \geq 0 : \quad \exists \varepsilon > 0 \quad \text{s.t.} \quad \iota^h(\tilde{r}^-_e) \leq \iota \leq \iota^l(\max\{\varepsilon, \tilde{r}^+_e\})\}. \tag{46}$$

where it has to be noted that $\mathcal{I}$ has positive mass if $(k, y) \in \mathcal{S}_{me}$ and contains $\tilde{\iota}$ in its interior (see the proof of Proposition 3). Whether or not $\mu/\beta - 1 \equiv \iota_{ss} \in \mathcal{I}$ is crucial for the existence of bounded monetary equilibria other than steady states.

Proposition 3. *If $(k, y) \in \mathcal{S}_{me}$, then there exist a two cycle if $\mu/\beta - 1 \in \text{int}(\mathcal{I})$. The cycle represents boom-bust dynamics with counter-cyclical inflation.*

The proof of Proposition 3 is illustrated by Figures 4, 5, and 6, sketching hypothetical f_{me} in the $(\mathbb{E}_t\{x_{t+1}\}, x_t)$-space, where f^l_{me} and f^h_{me} are monotonically increasing due to Assumption 3. There exists an $\hat{x} \in (\max\{0, \tilde{r}^+_e\}, \tilde{r}^-_e)$ such that $f^l_{me}(\hat{x}) > \hat{x} > f^h_{me}(\hat{x})$ if $\mu/\beta - 1 \in \text{int}(\mathcal{I}) \equiv (\iota^h(\tilde{r}^-_e), \iota^l(\max\{0, \tilde{r}^+_e\}))$; this follows from Figures 3b and 3c, and noting that $f^{\sigma}_{me} = x\beta(1 + \iota^{\sigma}(x))/\mu$. Then use $\hat{x}$ to define

$$g(x) \equiv \begin{cases} \{f^l_{me}(x)\} & \text{if } x < \hat{x}, \\ [f^h_{me}(\hat{x}), f^l_{me}(\hat{x})] & \text{if } x < \hat{x}, \\ \{f^h_{me}(x)\} & \text{if } x > \hat{x}. \end{cases} \tag{47}$$

We have that $g(0) = 0$, $g(x) > x \; \forall x \in (0, \hat{x})$, and $g(x) < x \; \forall x > \hat{x}$ by construction (see Figures 4b, 5b, and 6b). In essence, $g'(\hat{x}) = -\infty$ since $f^h_{me}(\hat{x}) < f^l_{me}(\hat{x})$; the graph of g is a vertical line at $\hat{x}$ and intersects 45-degree line there. It follows from the method of flip bifurcations that there exist points x', x'', with $x' \neq x''$, where the graphs of g and g^{-1} intersect offside the 45-degree line. If this intersection does not lie on the vertical part of $\hat{x}$ (see Figure 4b), we have $x' < \hat{x} < x''$ and it follows that we have identified a deterministic two-cycle in which $(\mathcal{M}_t, \eta_t)$ alternates between $(x', 0)$ and $(x'', 1)$, i.e., a boom-bust cycle with counter-cyclical inflation as in Figure 4c. If the intersection between g and g^{-1} lies on the vertical part of g, it turns out that we can construct a stochastic two cycle in which, dependent on whether this intersection lies above or below the 45-degree line (Figure 5b and, resp., 6b), the economy experiences a boom respectively bust for sure for even t with $\mathcal{M}_t > \hat{x}$ respectively $\mathcal{M}_t < \hat{x}$, and a bust with probability ρ and boom with probability $1-\rho$ for odd t with $\mathcal{M}_t = \hat{x}$. The reason is that if $\mathcal{M}_t = \hat{x}$, we can have both $\eta_t = 0$—a bust—and $\eta_t = 1$—a boom—since $\hat{x} \in (\max\{0, \tilde{r}_e^+\}, \tilde{r}_e^-)$. The resulting dynamics again feature counter-cyclical inflation (see Figures 5c and 6c).

Proposition 4. *If $(k, y) \in \mathcal{S}_{me}$ and $\mu/\beta - 1 \in \mathcal{I}/\operatorname{int}(\mathcal{I})$, there exist bounded monetary equilibria that converge to the monetary steady state with a boom-bust cycle.*

Proposition 4 applies to the knife-edge cases $\mu = \beta(1+\iota^h(\tilde{r}_e^-))$; and $\mu = \beta(1+\iota^l(\tilde{r}_e^+))$, where $\tilde{r}_e^+ > 0$. They are characterized by a unique monetary steady state involving a boom respectively bust. However, the steady-state value of currency balances is consistent with having a bust, respectively, boom, too. Figure 7 illustrates how this implies we can transition to the steady state in a boom-bust-boom respectively bust-boom-bust fashion. Interestingly, Proposition 4 can apply at the Friedman rule, i.e. $\mu = \beta$. Particularly, this happens when $\tilde{r}_e^- \geq \lim_{\iota \searrow 0} r(\iota) = \hat{r}_e^h$, i.e., if real currency balances that render the liquidity constraint slack in a boom are consistent with having a bust.

Proposition 5. *If $(k, y) \notin \mathcal{S}_{me}$ and/or $\mu/\beta - 1 \notin \mathcal{I}$, only steady states can be bounded monetary equilibria.*

Intuitively, if $(k, y) \in \mathcal{S}_{me}$ there are values for $\mathbb{E}_t\{x_{t+1}\}$ which can induce both a boom, bust, and mix, but, since $\mu/\beta - 1 \notin \mathcal{I}$, such values are too far away from the monetary steady state. Real currency balances would therefore either grow unbounded or converge to zero if the economy is not in steady state.

Taking stock from the analysis in the current and previous sections, under Assumption 3 only steady states can be bounded monetary equilibria if liquid wealth comprises only currency, or currency and an exogenous-dividend asset; whilst bounded equilibria with self-fulfilling dynamics can exist under Assumption 3 if liquid wealth comprises only equity, or both equity and currency. Comparing the case in which real currency balances pin down search with the case in which they do not indicates the importance of endogenous search if liquid wealth comprises equity and currency. If search were exogenous, Assumption 3 would still rule out self-fulfilling dynamics despite the endogeneity of firms' dividend. But if search is endogenous, a search channel entails that buyers can coordinate on different search intensities for given real currency balances. Existent insights from the literature change given the latter property; endogenous dynamics can arise independently of Assumption 3. Instead, endogenous dynamics arise if and only if the currency-growth rate is contained in a set which can be characterized implicitly. And if the currency-growth rate lies in the interior of this set, two cycles exist.

6 Inflation targeting and stabilization policies

I assumed a constant growth rate for currency supply in the preceding analysis. Many central banks target inflation though. This may help to eliminate the equilibria with endogenous dynamics identified earlier since they feature fluctuating real currency balances and thus fluctuating inflation. I therefore suppose in this section that the government implements gross inflation target π, which, however, turns out to be insufficient to stabilize the economy—the government should react to the value of equity, too.

Inflation targeting. Suppose the government implements gross inflation target π, entailing that currency supply adjusts endogenously to satiate demand arising at the target. The optimal price index is simply the nominal price of CM goods due to quasi-linear preferences, so $1/\Phi_t$ should grow at a gross rate π; $\Phi_{t+1} = \Phi_t/\pi$. From Equation (11) this implies

$$\mathbb{E}_{t-1}\{\iota_t\} = (\pi - \beta)/\beta \equiv i \quad \text{if } \Phi_t > 0. \tag{48}$$

The RHS, i.e., i, is the *Fisher rate*: the nominal interest rate that compensates exactly for inflation and time discounting. In a monetary equilibrium, i pins down only the expected value for ι_t. This preludes that inflation targeting alone may not suffice to stabilize the economy. Note that the non-negativity of ι_t rules out $\pi < \beta$; to have bounded currency demand, deflation may not be too strong. Due to the role of expectations as highlighted in Equation (48), I distinguish between a deterministic and stochastic environment.

Deterministic environment. If all uncertainty about time t, particularly the realization of $\mathcal{H}_t$, is already revealed at time $t-1$, then the inflation target pins down ι_t through the Fisher rate: $\iota_t = i$. It is clear from Equations (12) and (14) that the real allocations at time t are then pinned down uniquely, except for the knife-edge case $i = \tilde{\iota}$.

Stochastic environment. If at time $t-1$ it is still uncertain what the outcomes will be at time t, then buyers can coordinate on the sunspot's history $\mathcal{H}_t$, allowing ι_t to fluctuate. Yet, real currency balances $\mathcal{M}_t \equiv \Phi_t M_{t-1}$ act as a exogenous variable in DM_t since $\Phi_t = \Phi_{t-1}/\pi$ if the inflation target is implemented. Thus, $\mathcal{M}_t = \mathbb{E}_{t-1}\{\mathcal{M}_t\}$; time-$t$ real balances are perfectly predictable at time-$t-1$. I define $\mathbb{M}_{t-1} = \mathbb{E}_{t-1}\{\mathcal{M}_t\}$ to capture this. Recall from Section 4 that demand and supply of liquid wealth are functions of ι_t, where supply now includes $\mathbb{M}_{t-1}$:

$$z^s_{\pi e}(\iota_t) = h\eta(\iota_t)\Pi(\min\{z^h(\iota_t), \hat{q}\}) + l(1-\eta(\iota_t))\Pi(\min\{z^l(\iota_t), \hat{q}\}) + y + \mathbb{M}_{t-1}. \tag{49}$$

Not much changes compared to the analysis in Section 4; market clearance now occurs if $r_e(\iota_t) = \mathbb{M}_{t-1}$ for some positive $\mathbb{M}_{t-1} > 0$. It follows from Proposition 2 that:

Corollary 1. *For $(k, y) \in \mathcal{S}_{me}$ and $\mathbb{M}_{t-1} \in [\max\{0, \tilde{r}_e^+\}, \tilde{r}_e^-]$ we have that $r_e(\iota_t) = \mathbb{M}_{t-1}$ for $\iota^h(\mathbb{M}_{t-1}) < \tilde{\iota}$, $\tilde{\iota}$, and $\iota^l(\mathbb{M}_{t-1}) > \tilde{\iota}$. Otherwise, ι_t is uniquely pinned down by $\mathbb{M}_{t-1}$.*

Defining $\mathbb{P}^h_{t-1} = \mathbb{P}_{t-1}\{\iota_t = \iota^h(\mathbb{M}_{t-1})\}$, and $\mathbb{P}^l_{t-1}$ and $\tilde{\mathbb{P}}_{t-1}$ similarly, it follows from Equation (48) that the corresponding Fisher rate is

$$i = \mathbb{P}^l_{t-1}\iota^l(\mathbb{M}_{t-1}) + \tilde{\mathbb{P}}_{t-1}\tilde{\iota} + \mathbb{P}^h_{t-1}\iota^h(\mathbb{M}_{t-1}). \tag{50}$$

The analysis above takes $\mathbb{M}_{t-1}$ and $(\mathbb{P}^l_{t-1}, \tilde{\mathbb{P}}_{t-1}, \mathbb{P}^h_{t-1})$ as given, but these variables are determined endogenously at time-$t-1$ to satisfy Equation (50) given the inflation target. Recall $\iota^l(\mathbb{M}_{t-1})$ and $\iota^h(\mathbb{M}_{t-1})$ are decreasing in $\mathbb{M}_{t-1}$. From Section 5 we also know that the lowest and highest value for ι_t which can be observed for $\mathbb{M}_{t-1} \in [\max\{0, \tilde{r}_e^+\}, \tilde{r}_e^-]$ are $\iota^h(\tilde{r}_e^-)$ and, respectively, $\iota^l(\max\{0, \tilde{r}_e^+\})$. It follows rather directly that if and only if we have i strictly in between these extrema, there exists an $\mathbb{M}_{t-1} \in [\max\{0, \tilde{r}_e^+\}, \tilde{r}_e^-]$ and probabilities $(\mathbb{P}^l_{t-1}, \tilde{\mathbb{P}}_{t-1}, \mathbb{P}^h_{t-1})$ for which Equation (50) holds and ι_t is non-degenerate.

Proposition 6. *For $(k, y) \in \mathcal{S}_{me}$, we have stochastic equilibrium multiplicity if $i \in \text{int}(\mathcal{I})$. Otherwise, the probability distribution for ι_t is degenerate at i.*

Proposition 6 implies the following if compared to Proposition 3. If two cycles exist for a currency-growth regime entailing steady-state inflation $\pi = \mu$, then if the same inflation rate is implemented successfully in an inflation-targeting regime, there is still scope for stochastic dynamics. The strong complementarity in search thus remains operative in an inflation-targeting environment. This result should not come as a surprise given the role that currency plays under an inflation target; it acts as a risk-free liquid asset that buyers can use as a substitute for equity. But if the Fisher rate is positive, then currency is costly to hold, entailing that buyers also use cheaper, risky equity as a means of liquid wealth. The intricate relationship between liquid-wealth demand and supply uncovered in Section 4 thus remains present.

The flip side of the reasoning above is that if the Fisher rate approaches zero, the scope for stochastic equilibrium multiplicity disappears. Particularly, buyers then have access to a risk-free and costless form of liquid wealth. They thus no longer need to rely on equity, so that the source of equilibrium multiplicity is eliminated:

Corollary 2. $\lim_{i \to 0} \mathbb{P}_{t-1}\{\iota_t = 0\} = 1$.

The results above point towards the desirability of running the Friedman rule in an inflation-targeting environment, i.e., setting $\pi = \beta$ to eliminate the opportunity cost of holding currency. Particularly, over and above the fact that running the Friedman rule is consistent with maximizing economic activity—the liquidity constraint is slack so $q = \hat{q}$ and $e = h$—, it also fosters financial and macroeconomic stability.

Implementing the Friedman rule by directly targeting inflation is also better than implementing it by targeting currency-supply growth. The Friedman rule then requires $\mu = \beta$, implying that the monetary steady state is indeed $\iota_{ss} = 0$; there is no opportunity cost of carrying currency. But Proposition 4 shows that the steady state then need not prevail at all times; the economy may be characterized by transitional dynamics involving a boom-bust-boom pattern. Targeting narrow-money growth, i.e., the growth rate of currency supply, can thus be notoriously unreliable when liquid wealth comprises also assets whose value is closely tied to macroeconomic activity. It is more effective to target the Fisher rate directly by accordingly adjusting currency supply in line with demand, as currency demand endogenously adjusts for the liquid wealth provided by other assets.

Stabilization policy. I close the analysis by considering a stabilization policy that can be combined with achieving the inflation targeting under all contingencies in case policy, for whatever reason, deviates from the Friedman rule. Stabilization implies that the government must intervene in DM_t, as otherwise the DM_t real currency balances $\mathcal{M}_t$ act as a predetermined variable, leading to exactly the same findings as before.

I focus on inflation targets $\pi < \beta(1+\tilde{\imath})$, i.e., the deterministic equilibrium is characterized by intensive search. Stabilization thus requires preventing busts, which can be done with a troubled-asset relief program (TARP) that serves to back equity. Consider that the government stands ready to purchase equity shares at some real price $\underline{\Delta}$ during the DM. Although currency is nominal, guaranteeing a real price is feasible but has fiscal implications that are detailed later. Letting ω_t denote the fraction of equity sold to the government in DM$_t$, we have

$$\omega_t \begin{cases} = 0 & \text{if } \Delta_t > \underline{\Delta}, \\ \in [0,1] & \text{if } \Delta_t = \underline{\Delta}, \\ = 1 & \text{if } \Delta_t < \underline{\Delta}; \end{cases} \tag{51}$$

where $\Delta_t = h\eta(\iota_t)\Pi(\min\{z^h(\iota_t), \hat{q}\}) + l(1-\eta(\iota_t))\Pi(\min\{z^h(\iota_t), \hat{q}\}) + y$ is the actual value of equity. Equation (51) transpires that buyers sell equity shares if their value falls short of the TARP price. The resulting supply of liquidity is

$$z_t^s = \max\{\Delta_t, \underline{\Delta}\} + \mathbb{M}_{t-1}, \tag{52}$$

where $\mathbb{M}_{t-1} = \Phi_t M_{t-1}$ are real currency balances brought into DM$_t$, measured before equity has been sold to the government. Equation (52) elucidates that TARP effectively puts a floor below the value of equity.

Proposition 7. *The lower bound on $\underline{\Delta}$ to rule out stochastic equilibrium multiplicity is*

$$\Delta' \equiv \eta' h\Pi \circ v^{-1}(\underline{z}^h) + (1-\eta')l\Pi \circ v^{-1}(\overline{z}^l), \quad \textit{where} \quad \eta' \equiv \frac{\tilde{r}_e^- - r_e(i)}{\tilde{r}_e^- - \tilde{r}_e^+}. \tag{53}$$

Note that the TARP price can be set below the deterministic-equilibrium value of equity because of the weak wealth effect.[16]

TARP also affects the lump-sum transfer off the equilibrium path, indicating that TARP requires fiscal commitment. With inflation targeting and TARP, the transfer is

$$\tau_t = \Phi_t(M_t - M_{TARP,t-1}), \quad \text{where} \quad M_{TARP,t-1} \equiv M_{t-1} + \omega_t(\underline{\Delta} - \Delta_t)/\Phi_t. \tag{54}$$

[16] It suffices to use as TARP price the value that would prevail in a mixed equilibrium if real currency supply is at the deterministic-equilibrium level $r_e(i)$.

$M_{TARP,t-1}$ is the amount of currency brought into CM_t net of the nominal value of equity shares bought by the government in DM_t. With inflation targeting, the government passively supplies real currency balances $\Phi_t M_t = \Phi_{t+1} M_t/\pi = \mathbb{M}_t$ that buyers carry out of CM_t given the Fisher rate, so that combining with Equation (51), we obtain

$$\tau_t = \mathbb{M}_t/\pi - \mathbb{M}_{t-1} - \max\{\underline{\Delta} - \Delta_t, 0\}, \tag{55}$$

where $\mathbb{M}_t$ is determined by buyers demand for real currency balances in CM_t.

Equation (55) elucidates that TARP is used only when the value Δ_t of equity shares drops below the TARP price $\underline{\Delta}$; TARP entails a loss for the government since equity is bought above fundamental value. This loss has to be passed on to the taxpayer if the inflation target is to be achieved in all contingencies. There would be excess currency supply otherwise since $M_{TARP,t-1} > M_{t-1}$, causing inflationary pressure. The government's commitment to pass on losses to the taxpayer, however, entails that currency injected into the economy by means of TARP has real value. This commitment is sufficiently strong to stabilize the economy if $\underline{\Delta} \geq \Delta'$. TARP is then never deployed on the equilibrium path; there is no reason for the value of equity to drop below $\underline{\Delta}$.

Proposition 8. *If $\underline{\Delta} \in (\Delta'', \Delta')$, where Δ'' solves $\Delta'' = l\Pi \circ v^{-1}(r_e(i) + \Delta'') + y$, then TARP can be deployed with positive probability because it fails to stabilize the economy. If $\underline{\Delta} \leq \Delta''$, then TARP is never deployed but still fails to stabilize the economy.*

Proposition 8 elucidates that applying TARP too conservatively can fiscally backfire and the reason is simple. If the TARP price is set slightly below the threshold $\underline{\Delta}$, then TARP fails to stabilize the economy which allows the value of equity to drop strictly below $\underline{\Delta}$ because of self-fulfilling beliefs. If that happens, buyers sell their equity shares to the government, which runs a loss since it then buys equity at a price above fundamental value. Counter intuitive at first sight, the loss can be avoided by setting the TARP price slightly higher in order to unwind the self-fulfilling beliefs that rationalize the drop in the equity value. If, on the other hand, the TARP price is set very low, TARP still fails to stabilize the economy but, exactly because the price is set very low, the value of equity cannot drop below $\underline{\Delta}$. The economy can thus experience a bust, but the government never actually buys equity shares so it never experiences a loss either.

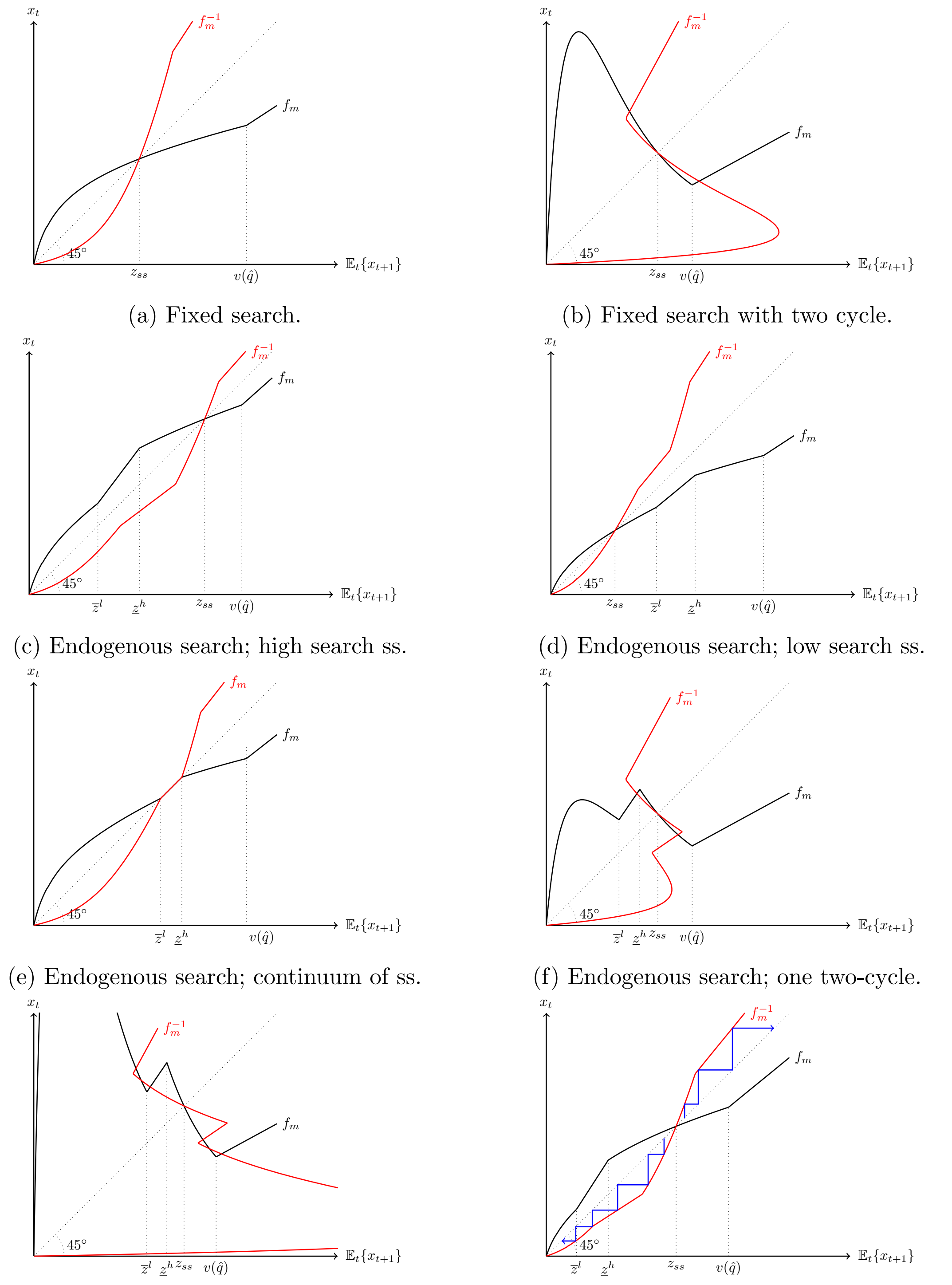
7 Conclusion

This paper introduces liquid equity in a money-search model. Equity is a claim on the profits of firms that sell goods in the search-and-matching market, and simultaneously, equity is used in payment by the buyers in the search-and-matching market. This interwovenness entails a strong strategic complementarity in search, entailing self-fulfilling bounded dynamics. The joint role of liquid equity and search is elucidated by assuming that ex-ante liquid wealth-demand is decreasing in the liquidity premium. Whilst this rules out self-fulfilling bounded dynamics in plain-vanilla models, such dynamics are preserved with liquid equity and endogenous search. The economy is stable at the Friedman rule in an inflation-targeting regime, or, if away from the Friedman rule, if inflation targeting is combined with TARP, which puts a floor below the value of equity.

Directions for future research are twofold. First, the current setup views equity as a one-period lived asset. This is arguable unrealistic, but it also implies that dynamics cannot rely on an infinite chain of self-fulfilling asset-price expectations, which is normally key in money-search models. In that sense, the assumption provides a clean laboratory to analyze the joint role of liquid equity and endogenous search. Relaxing it by modeling equity as a long-lived asset is a useful extension to bring the model to the data.

A second extension would distinguish between direct and indirect liquidity as in Geromichalos and Herrenbrueck (2016,1) and Geromichalos, Jung, Lee and Carlos (2021). The current model has directly-liquid equity; it can be used to purchase goods in the search-and-matching market. In reality, equity is rather indirectly liquid; it must first be sold for directly-liquid assets (currency, deposits, *etcetera*) in a financial market, after which these assets can be used for real transactions. In the current model one can think of these two steps occurring simultaneously; the financial market can be accessed when in a bilateral match. If the steps occur sequentially, indirectly-liquid assets typically inherent the properties of their liquid counterparts. It would be interesting to investigate if indirectly-liquid equity and search interact similarly as directly-liquid equity and search.

A Figures



(a) Fixed search.

(b) Fixed search with two cycle.

(c) Endogenous search; high search ss.

(d) Endogenous search; low search ss.

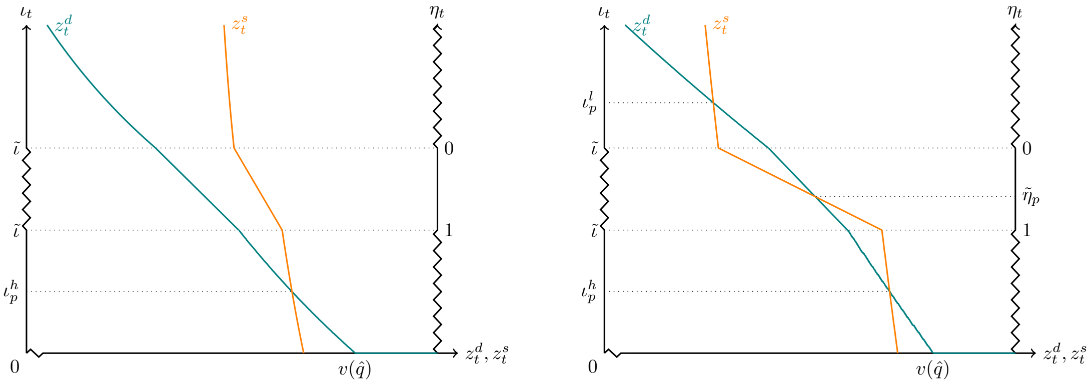
(e) Endogenous search; continuum of ss.

(f) Endogenous search; one two-cycle.

(g) Endogenous search; three two-cycles.

(h) Dynamics with monotone increasing f_e.

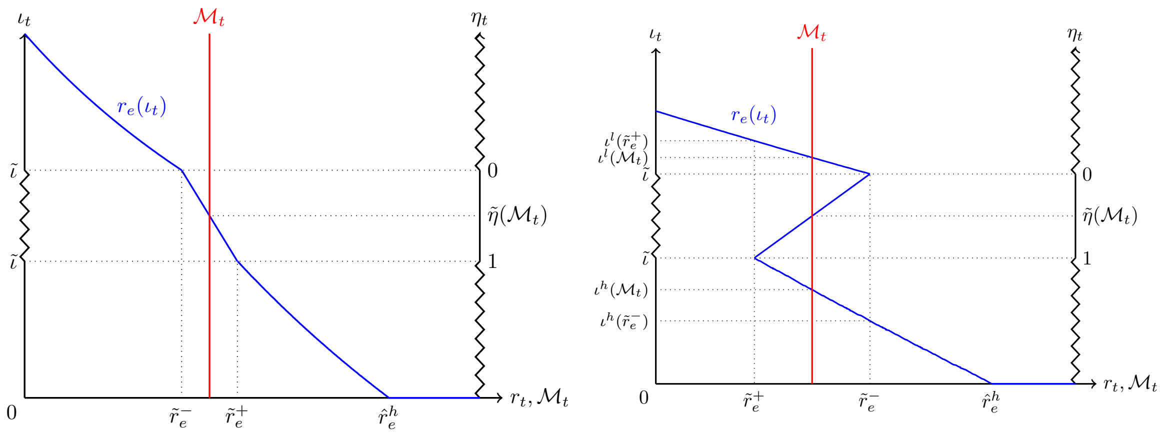
Figure 1: Depiction of f_m and f_m^{-1}.



(a) The case $0 < \tilde{r}_e^- < \tilde{r}_e^+$.

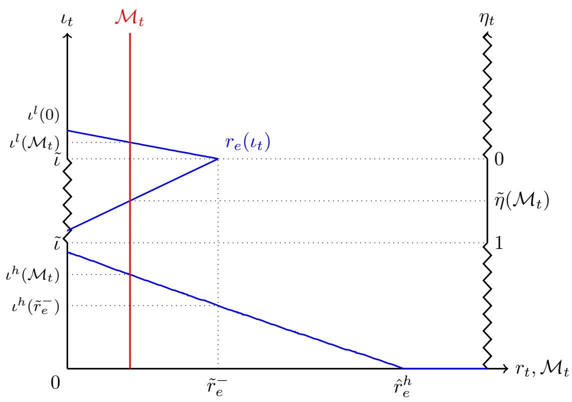
(b) The case $\tilde{r}_e^+ < 0 < \tilde{r}_e^-$.

Figure 2: Depiction of liquid-wealth demand z_t^d and supply z_t^s if liquid wealth comprises only equity.



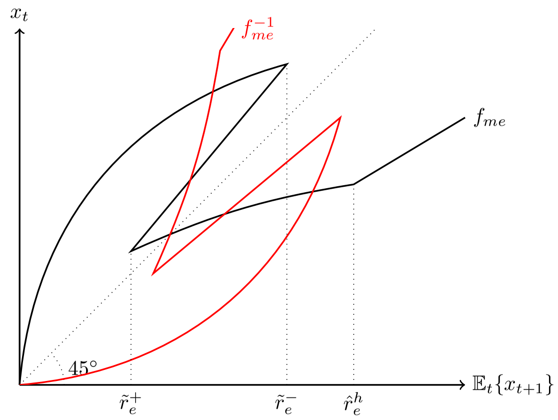
(a) The case $0 < \tilde{r}_e^- < \tilde{r}_e^+$.

(b) The case $0 < \tilde{r}_e^+ < \tilde{r}_e^-$.

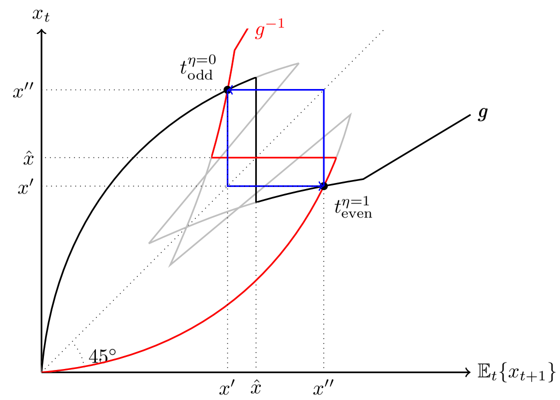


(c) The case $\tilde{r}_e^+ < 0 < \tilde{r}_e^-$.

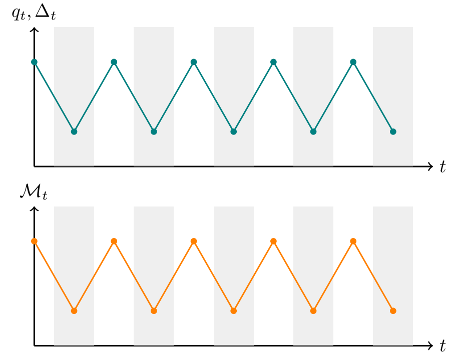
Figure 3: Depiction of excess liquid-wealth demand and currency supply.



(a) Graph of f_{me} and f_{me}^{-1}.

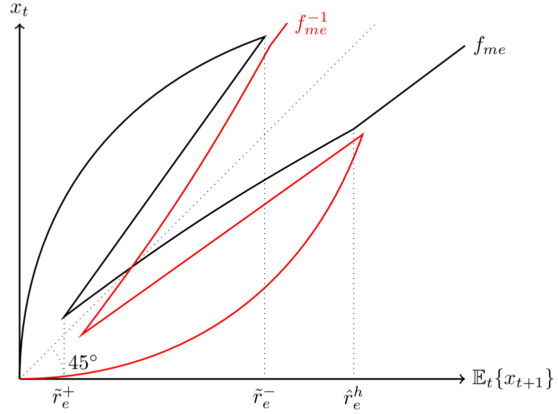


(b) Graph of g and g^{-1}.

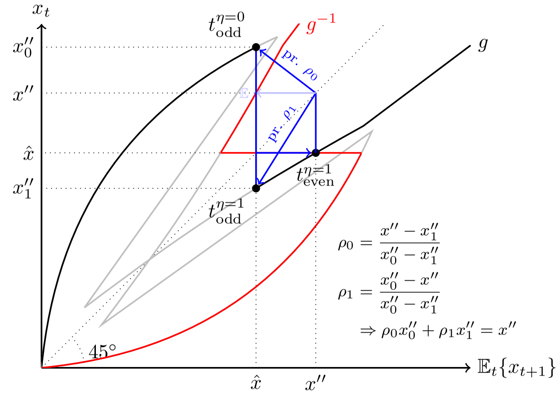


(c) Two-cycle dynamics.

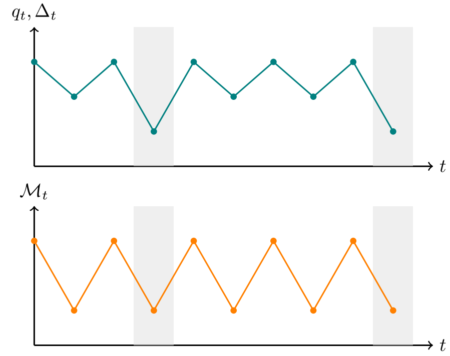
Figure 4: The case for a deterministic two cycle. Gray shaded areas in panel 4c are busts (i.e., $\eta_t = 0$).



(a) Graph of f_{me} and f_{me}^{-1}.

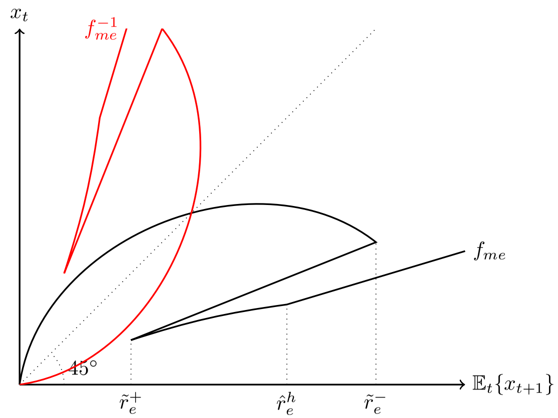


(b) Graph of g and g^{-1}.

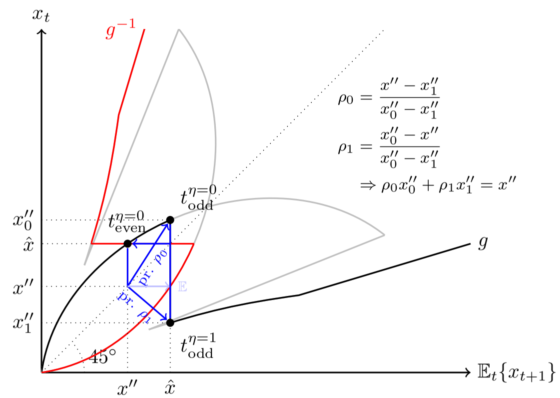


(c) Two-cycle dynamics.

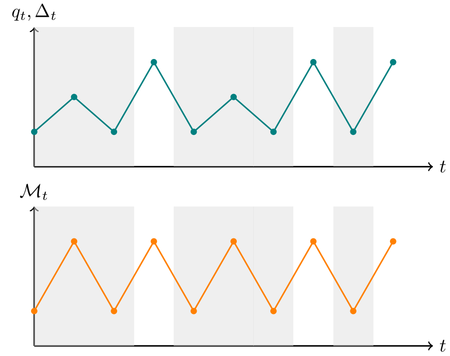
Figure 5: The case for a stochastic two cycle with occasional busts. Gray shaded areas in panel 5c are busts.



(a) Graph of f_{me} and f_{me}^{-1}.

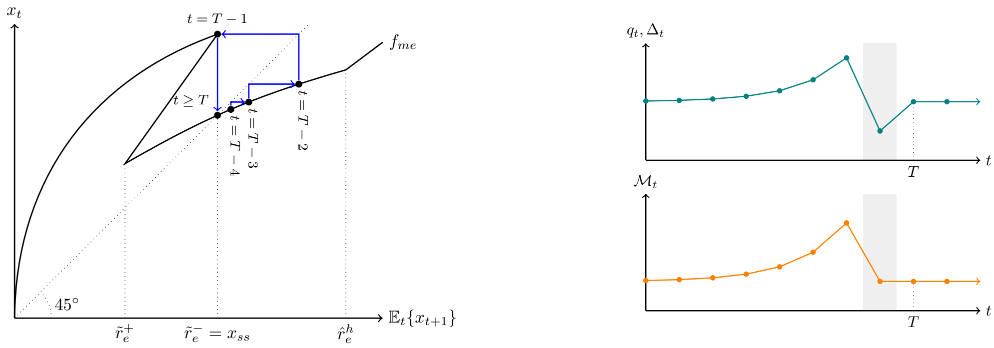


(b) Graph of g and g^{-1}.



(c) Two-cycle dynamics.

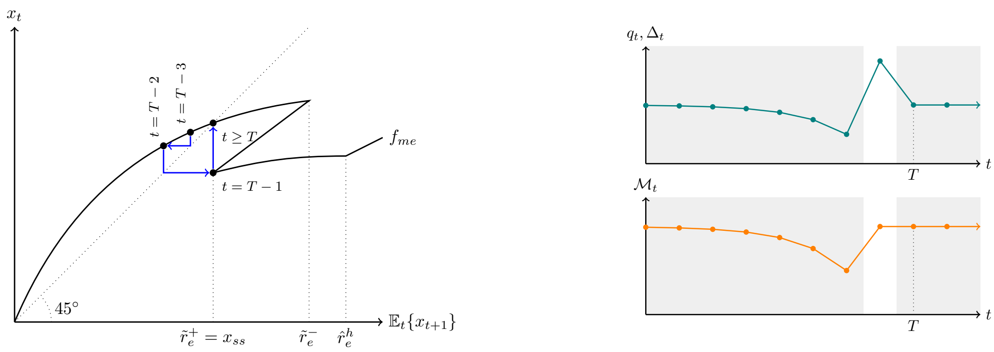
Figure 6: The case for a stochastic two cycle with occasional booms. Gray shaded areas in panel 5c are busts.



(a) Graph of f_{me}.

(b) Transition dynamics.

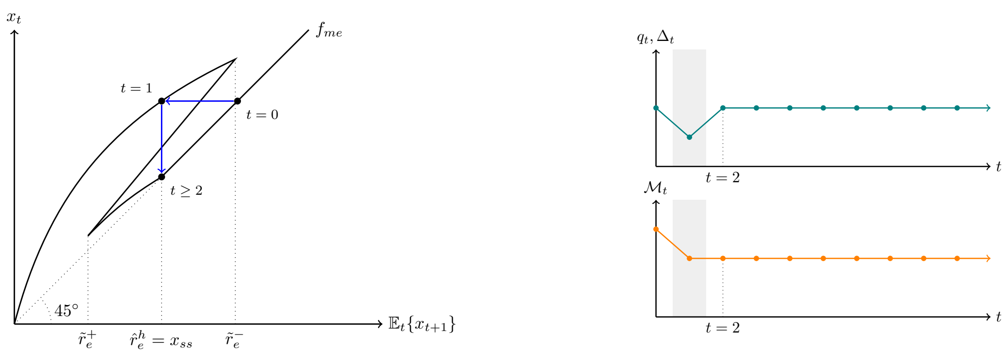
Figure 7: Transition dynamics to the steady state when $\mu = \beta(1 + \iota^h(\tilde{r}_e^-))$ and $\tilde{r}_e^- < \hat{r}_e^h$. Gray shaded areas in Panel 7b are busts.



(a) Graph of f_{me}.

(b) Transition dynamics.

Figure 8: Transition dynamics to the steady state when $\mu = \beta(1 + \iota^l(\tilde{r}_e^+))$ and $\tilde{r}_e^+ > 0$. Gray shaded areas in Panel 8b are busts.



(a) Graph of f_{me}.

(b) Transition dynamics.

Figure 9: Transition dynamics to the steady state at the Friedman rule. Gray shaded areas in Panel 9b are busts.

B Value functions and bargaining

This appendix details the derivation of the buyers' Bellman Equation (4), the firms' dividend (7), and the surplus of bilateral matches. I consider first CM_t and then DM_t, after which the Bellman equation can be derived.

Centralized market. An incumbent firm, born in CM_{t-1}, pays dividends and subsequently dies. A firm that holds an asset portfolio worth z_t^f CM goods as well as an inventory o_t of CM goods will therefore pay a dividend

$$\delta(z_t^f, o_t) = z_t^f + o_t. \tag{B.1}$$

Let (m_{t+1}, e_{t+1}) be the amount of currency and equity shares that the buyer carries into DM_{t+1}, and let $V_{t+1}(m_{t+1}, e_{t+1})$ be the associated utility value of entering DM_{t+1}, to be characterized later. The utility value of entering CM_t with currency and equity shares (m_t, e_t) is

$$\begin{aligned} W_t(m_t, e_t) = & \max_{x_t,(m_{t+1},e_{t+1})\forall\mathcal{H}_{t+1}} \{x_t + \beta\mathbb{E}_t\{V_{t+1}(m_{t+1}, e_{t+1})\}\} \\ \text{s.t.} \quad & x_t + \mathbb{E}_t\{\phi_{t+1}m_{t+1}\} + \mathbb{E}_t\{\psi_{t+1}e_{t+1}\} \leq \Phi_t m_t + \Delta_t e_t + \tau_t + \Psi_t, \\ & m_{t+1}, e_{t+1} \geq 0 \ \forall\mathcal{H}_{t+1}, \end{aligned} \tag{B.2}$$

where τ_t is the government transfer and Ψ_t the lump-sum transfer arising from the issuance of new equity shares. The Arrow-like structure of the market allows the buyer to choose (m_{t+1}, e_{t+1}) contingent on $\mathcal{H}_{t+1}$. The budget constraint in (B.2) binds for the optimal choices and since utility is linear in x_t, we can write W_t as

$$\begin{aligned} W_t(m_t, e_t) = & \Phi_t m_t + \Delta_t e_t + \tau_t + \Psi_t \\ & + \mathbb{E}_t\left\{\max_{m_{t+1},e_{t+1}\geq 0}\{\beta V_{t+1}(m_{t+1}, e_{t+1}) - \phi_{t+1}m_{t+1} - \psi_{t+1}e_{t+1}\}\right\}. \end{aligned} \tag{B.3}$$

The buyer's ability to choose (m_{t+1}, e_{t+1}) contingent on $\mathcal{H}_{t+1}$ allows to write the optimization w.r.t. (m_{t+1}, e_{t+1}) inside of the expectations operator.

Decentralized market. Buyers are randomly matched to the firms and the probability that a buyer ends up in a match with a firm equals the search devoted by the buyer.

Communication within bilateral matches is limited due to spatial separation; the buyer-firm pair cannot observe what happens in other matches.

Bargaining. The buyer-firm pair negotiates terms of trade (q, p), with q the DM_t goods received by the buyer and p the payment (in CM_t goods) received by the firm. This payment must be made with liquid assets, as detailed below. The utility surplus for the buyer is $u(q) - p$, as follows from the linearity of W_t in Equation (B.3). The trade increases the firm's divided payment in (B.1) by $p - c(q)$ since the firm uses $c(q)$ CM_t goods to produce q DM_t goods in exchange for liquid wealth worth p CM_t goods.

Firms are interested in maximizing the utility of their shareholders. The firm and the buyer disregard the effects of changes in the firm's dividend on other matches due to limited communication. Changes in the dividend of the firm also leave the buyer's (with which the firm negotiates) wealth unaffected because there is a continuum of firms and matching is random. The dividend change from the transaction thus directly represents the shareholders' utility gain since it is expressed in CM_t goods.

The total surplus from negotiated terms of trade (q, p) is $u(q) - c(q)$. With payment protocol v, mapping q into p, the buyer's surplus is $L(q) = u(q) - v(q)$ and the firm's surplus is $\Pi(q) = v(q) - c(q)$. A buyer chooses q to maximize $L(q)$ subject to $v(q) \leq z_t(m_t, e_t) \equiv \Phi_t m_t + \chi \Delta_t e_t$ and $c(q) \leq y$. It follows that the negotiated terms of trade are given by Equation (5) if the capacity constraint is slack.

Value functions and dividends. Expected dividends that an incumbent firm will pay in CM_t, contingent on the aggregate uncertainty being resolved, i.e., $\mathbb{E}\{\delta(z_t^f, o_t)|\mathcal{H}_t\}$, equal the dividend payment Δ_t of equity by the law of large numbers. If a firm is matched to a buyer with currency and equity holdings (m_t, e_t), its CM_t dividend payment will be

$$\delta_t = \Pi\left(\min\{v^{-1}(\Phi_t m_t + \chi \Delta_t e_t), \hat{q}\}\right) + y; \tag{B.4}$$

its endowment of CM_t goods plus the match surplus, where I use $q_t = \min\{v^{-1}(\Phi_t m_t + \chi \Delta_t e_t), \hat{q}\}$ as implied by Equation (5). Accounting for the distribution G_t of search and asset holdings across buyers, the firm's expected dividend payment $\Delta_t = \mathbb{E}\{\delta(z_t^f, o_t)|\mathcal{H}_t\}$ upon entering DM_t is then given by Equation (7).

If a buyer holds assets (m_t, e_t), its value when matched to a firm is

$$L\left(\min\{v^{-1}(\Phi_t m_t + \chi\Delta_t e_t), \hat{q}\}\right) + \Phi_t m_t + \Delta_t e_t + W_t(0,0), \tag{B.5}$$

as follows from the linearity of (B.3) and the specification of q_t in Equation (5). The buyer chooses search σ_t optimally and since σ_t equals the probability of being matched, the value of entering DM_t with assets (m_t, e_t) is

$$\begin{aligned} V_t(m_t, e_t) = \max_{\sigma_t \in \{l,h\}} &\left\{e_t\left(\min\{v^{-1}(\Phi_t m_t + \chi\Delta_t e_t), \hat{q}\}\right) - s(\sigma_t)\right\} \\ &+ \Phi_t m_t + \Delta_t e_t + W_t(0,0). \end{aligned} \tag{B.6}$$

Bellman equation. Using (B.3) to substitute out the term $W_t(0,0)$ in Equation (B.6) gives a recursive expression for $V_t(m_t, e_t)$:

$$\begin{aligned} V_t(m_t, e_t) = \max_{\sigma_t \in \{l,h\}} &\left\{\sigma_t L\left(\min\{v^{-1}(\Phi_t m_t + \chi\Delta_t e_t), \hat{q}\}\right) - s(\sigma_t)\right\} \\ &+ \Phi_t m_t + \Delta_t e_t + \tau_t + \Psi_t \\ &+ \mathbb{E}_t\left\{\max_{m_{t+1}, e_{t+1} \geq 0} \{\beta V_{t+1}(m_{t+1}, e_{t+1}) - \phi_{t+1} m_{t+1} - \psi_{t+1} e_{t+1}\}\right\}, \end{aligned} \tag{B.7}$$

Since $q_t = \min\{v^{-1}(\Phi_t m_t + \chi\Delta_t e_t), \hat{q}\}$ solves $\max_{q_t \geq 0} L(q_t)$ subject to the constraints $v(q_t) \leq z_t(m_t, e_t) \equiv \Phi_t m_t + \chi\Delta_t e_t$ and $c(q_t) \leq y$, we have

$$L(\min\{v^{-1}(\Phi_t m_t + \chi\Delta_t e_t), \hat{q}\}) = \max_{q_t \geq 0} \{L(q_t) | \text{ s.t. } v(q_t) \leq z_t(m_t, e_t) \text{ and } c(q_t) \leq y\}. \tag{B.8}$$

Using (B.8) in (B.7) gives the Bellman Equation (4).

C Two-sided search

This appendix shows that the results from the model can be generalized to a setup with two-sided search in the DM. Particularly, I introduce a unit mass of identical, infinitely-lived *workers* that value the net consumption $x_t^w \in \mathbb{R}$ of CM_t goods and that can devote search $\sigma_t^w \in \Sigma \subseteq [0,1]$ on behalf of the firms. Time-t flow utility for a worker is given by

$$\mathcal{U}(\sigma_t^w, x_t^w) = -s(\sigma_t^w) + x_t^w \tag{C.1}$$

and the time-discount rate is β. The CM is as in the baseline model and workers have no reason to hold assets since they do not consume DM goods.

Workers and firms form worker-firm pairs in DM_t which disband after DM_t has convened. Every worker is matched to a firm and *vice versa*. The workers devote search σ_t^w on behalf of the worker-firm pair. The mass of matches between buyers and workers in DM_t is given by a constant-returns-to-scale matching function $\mathcal{N}(\tilde{\sigma}_t^b, \tilde{\sigma}_t^w)$, where $\tilde{\sigma}_t^b$ and $\tilde{\sigma}_t^w$ is average search across the buyers respectively the workers.

A buyer devoting search σ_t^b in DM_t finds a match with a worker with probability $\sigma_t^b \mathcal{N}(1, 1/\kappa_t)$, where $\kappa_t = \tilde{\sigma}_t^b / \tilde{\sigma}_t^w$ is *market tightness*. A worker devoting search σ_t^w likewise finds a match with a buyer with probability $\sigma_t^w \mathcal{N}(\kappa_t, 1)$. Once matched with a buyer, the worker can connect the buyer to the firm.

Assumption C.1. *Search devoted by the worker is private information and the firm cannot incentive the worker to search. Moreover, the worker's decision to connect the buyer to the firm cannot be contracted ex ante.*

Assumption C.1 implies that firms negotiate with workers after the matching of buyers to workers has taken place. A worker matched to a buyer negotiates a real payment w_t from the firm in return for connecting the buyer with the firm. The buyer's liquid wealth is observable to both the worker and the firm during the negotiation process. The firm can settle the payment w_t instantaneously with ownership shares in its profits, and I assume that w_t follows from a protocol $\omega : \Pi \to w$, mapping the firm's surplus Π from being connected with the buyer into w. Hence, σ_t^w follows from

$$\max_{\sigma_t^w \in \Sigma} \left\{ \frac{\sigma_t^w \mathcal{N}(\kappa_t, 1)}{\tilde{\sigma}_t^b} \iiint \sigma \left[\omega \circ \Pi \left(\min\{ v^{-1}(\Phi_t m + \chi \Delta_t e), \hat{q} \} \right) \right] G_t(\mathrm{d}\sigma, \mathrm{d}m, \mathrm{d}e) - s(\sigma_t^w) \right\}, \tag{C.2}$$

and σ_t^b follows from

$$\max_{\sigma_t^b \in E} \left\{ \sigma_t^b \mathcal{N}(1, 1/\kappa_t) L \left(\min\{ v^{-1}(\Phi_t m_t + \chi \Delta_t e_t), \hat{q} \} \right) - s(\sigma_t^b) \right\}, \tag{C.3}$$

with (m_t, e_t) the buyer's asset holdings.

The dividend paid by equity becomes

$$\Delta_t = \mathcal{N}(1, 1/\kappa_t) \iiint \sigma \left[(1-\omega) \circ \Pi \left(\min\{v^{-1}(\Phi_t m + \chi \Delta_t e), \hat{q}\}\right)\right] G_t(\mathrm{d}\sigma, \mathrm{d}m, \mathrm{d}e) + y. \tag{C.4}$$

Assumption C.2. *Buyers and workers obtain the same share $\theta < 1/2$ of total match surplus $u(q) - c(q)$. That means, $v(q) = (1-\theta)u(q) + \theta c(q)$ and $\omega \circ \Pi(q) = \theta[u(q) - c(q)]$.*

Given Assumption C.2, we obtain $\kappa_t = 1$ in a symmetric equilibrium, i.e., when $\sigma_t^b = \tilde{\sigma}_t^b$ for all buyers and $\sigma_t^w = \tilde{\sigma}_t^w$ for all workers. To see this, note that all buyers then carry liquid wealth worth z_t^d into DM_t. Workers then anticipate $q_t = \min\{v^{-1}(z_t^d), \hat{q}\}$ and therefore choose σ_t^w to maximize $\sigma_t^w \mathcal{N}(\kappa_t, 1)\theta[u(q_t) - c(q_t)] - s(\sigma_t^w)$. Buyers choose σ_t^b to maximize $\sigma_t^b \mathcal{N}(1, 1/\kappa_t)\theta[u(q_t) - c(q_t)] - s(\sigma_t^b)$. We thus obtain unique σ_t^b and σ_t^w, except for knife-edge cases. When $\kappa_t = 1$, we have $\sigma_t^b = \sigma_t^w$ and this rationalizes $\kappa_t = 1$ as an equilibrium outcome. When $\kappa_t > 1$, we need $\sigma_t^b > \sigma_t^w$. But high κ_t is especially beneficial for the workers—they get matched to a buyer with a high probability so that the search incentives imply $\sigma_t^b < \sigma_t^w$. Likewise, when $\kappa_t < 1$, we need $\sigma_t^b < \sigma_t^w$ but a low κ_t is especially beneficial for the buyers. The search incentives would then imply $\sigma_t^b > \sigma_t^w$.

Taking stock, in symmetric equilibria a buyer is matched to a worker with probability $\sigma_t^b \mathcal{N}(1, 1)$. One can then normalize $\mathcal{N}(1, 1) = 1$ to obtain the same Bellman equation for the buyer as in the baseline model. The only difference arises when calculating the value of equity since firms now earn lower profits due to the payment w_t to workers, but this does not affect the main properties of the baseline model. Further, the main results about equilibrium multiplicity and endogenous dynamics do not rely on equilibria in which buyers use mixed strategies for their search. These results therefore hold true under the setup with two-sided search laid out above.

D Proofs

Proof of Proposition 1. I first characterize the set $\mathcal{S}_e$ and prove $\tilde{r}_e^+ < 0 < \tilde{r}_e^- \Leftrightarrow (k, y) \in \mathcal{S}_e$, after which I prove that this set has positive mass under both the parameter restriction $c(\hat{q}) \leq y$ and Assumption 3. Throughout, I restrict attention to $(y, k) \in \mathbb{R}_+^2$.

Define $q^\sigma(\iota) : \iota = \sigma L'(q)/v'(q)$. Also define $q^\sigma(y) \leq \hat{q}$ as the unique solution of $\sigma\Pi(q^\sigma) + y \geq v(q^\sigma)$ with $=$ if $q^\sigma < \hat{q}$. Note that $q^\sigma(\iota)$ is strictly decreasing in ι and that $q^\sigma(y)$ is strictly increasing in y if $y < v(\hat{q}) - \sigma\Pi(\hat{q})$ and constant in y for $y \geq v(\hat{q}) - \sigma\Pi(\hat{q})$. Further, $q^\sigma(y) \geq 0$ with $=$ if and only if $y = 0$. By the definition of $\tilde{r}_e^-, \tilde{r}_e^+$, we have

$$\tilde{r}_e^- = (v - l\Pi) \circ q^l(\tilde{\iota}) - y \quad \text{and} \quad \tilde{r}_e^+ = (v - h\Pi) \circ q^h(\tilde{\iota}) - y. \tag{D.1}$$

The properties of $q^\sigma(\iota)$ and $q^\sigma(y)$ then directly imply

$$\tilde{r}_e^- > 0 \;\Leftrightarrow\; \tilde{\iota} < \frac{l\Pi' \circ q^l(y)}{v' \circ q^l(y)} \quad \text{and} \quad \tilde{r}_e^+ < 0 \;\Leftrightarrow\; \tilde{\iota} > \frac{h\Pi' \circ q^h(y)}{v' \circ q^h(y)}. \tag{D.2}$$

Hence, $\tilde{r}_e^+ < 0 < \tilde{r}_e^- \;\Leftrightarrow\; \frac{h\Pi' \circ q^h(y)}{v' \circ q^h(y)} < \tilde{\iota} < \frac{l\Pi' \circ q^l(y)}{v' \circ q^l(y)}$. Define the set

$$\mathcal{Y} = \left\{ y : \quad \frac{h\Pi' \circ q^h(y)}{v' \circ q^h(y)} < \frac{l\Pi' \circ q^l(y)}{v' \circ q^l(y)} \right\}. \tag{D.3}$$

Recall that $I \equiv \lim_{q\to 0}[L'(q)/v'(q)]$ so that $0 \notin \mathcal{Y}$ since $h > l$, i.e., $\mathcal{Y} \subseteq \mathbb{R}_{++}$.

We can back out k from $\tilde{\iota}$ by using that (13) holds with equality at $\tilde{\iota}$:

$$k = \kappa(\tilde{\iota}) \equiv \max_{q\geq 0}\{hL(q) - \tilde{\iota}v(q)\} - \max_{q\geq 0}\{lL(q) - \tilde{\iota}v(q)\}, \tag{D.4}$$

where I use that $\max_{q\geq 0}\{\sigma L(q) - \iota v(q)\} = \max_{z\geq 0}\{\sigma L(\min\{v^{-1}(z), \hat{q}\}) - \iota z\}$. It follows that $\kappa(\tilde{\iota})$ is a strictly decreasing function of $\tilde{\iota}$ on the domain $(0, hI)$, and satisfies $\kappa(\tilde{\iota}) \geq 0$ with $>$ if and only if $\tilde{\iota} < hI$. Further note that $\kappa(\tilde{\iota}) > \max_z\{hL \circ v^{-1}(z) - lIz\}$ if and only if $\tilde{\iota} < lI$ and $\kappa(\tilde{\iota}) < (h-l)L(\hat{q})$ if and only if $\tilde{\iota} > 0$.

For $\tilde{\iota} \in (0, hI)$ we have $k < (>)\kappa(\iota) \;\Leftrightarrow\; \tilde{\iota} > (<)\iota$, so define the set

$$\mathcal{K}(y) = \left(\kappa\left(\frac{l\Pi' \circ q^l(y)}{v' \circ q^l(y)}\right), \kappa\left(\frac{h\Pi' \circ q^h(y)}{v' \circ q^h(y)}\right) \right). \tag{D.5}$$

The set $\mathcal{K}(y)$ has positive mass if and only if $y \in \mathcal{Y}$, and we have $\frac{h\Pi' \circ q^h(y)}{v' \circ q^h(y)} < \tilde{\iota} < \frac{l\Pi' \circ q^l(y)}{v' \circ q^l(y)}$ if and only if $k \in \mathcal{K}(y)$. It thus follows directly that

$$\mathcal{S}_e = \{(k, y) : \quad y \in \mathcal{Y} \quad \text{and} \quad k \in \mathcal{K}(y)\}. \tag{D.6}$$

I next show that $(k,y) \in \mathcal{S}_e$ implies that Assumption 2 is satisfied. Since $\mathcal{Y} \subseteq \mathbb{R}_{++}$, we have that $\frac{l\Pi' \circ q^l(y)}{v' \circ q^l(y)} < lI \ \forall y \in \mathcal{Y}$, where I use that $q^\sigma(y)$ is strictly increasing in y on the domain $(0, v(\hat{q}) - \sigma\Pi(\hat{q}))$ and satisfies $q^\sigma(0) = 0$. This implies that $(k,y) \in \mathcal{S}_e \Rightarrow k > \max_{z \geq 0}\{hL \circ v^{-1}(z) - lIz\}$. Further, we have that $\frac{\sigma\Pi' \circ q^l \sigma(y)}{v' \circ q^\sigma(y)} \geq 0$, with $>$ if and only if $y < v(\hat{q}) - \sigma\Pi(\hat{q})$, so we also have that $(k,y) \in \mathcal{S}_e \Rightarrow k < (h-l)L(\hat{q})$.

It remains to show that $\mathcal{S}_e$ has positive mass under both the parameter restriction $c(\hat{q}) \leq y$ and Assumption 3. For this, it suffices to show that the set $\mathcal{Y}' \equiv \mathcal{Y} \cap [c(\hat{q}), \infty)$ has positive mass under Assumption 3.

With this objective in mind, define $\mathcal{Y}'' \equiv (v(\hat{q}) - h\Pi(\hat{q}), v(\hat{q}) - l\Pi(\hat{q}))$. From the definition of $q^\sigma(y)$, it follows directly that $y \in \mathcal{Y}'' \Rightarrow q^l(y) < q^h(y) = \hat{q}$. Since $L'(q)/v'(q) = 0$ for $q = \hat{q}$ and $L'(q)/v'(q) > 0$ for $q < \hat{q}$, it follows directly that $\mathcal{Y}'' \subseteq \mathcal{Y}$. Moreover, $y \in \mathcal{Y}'' \Rightarrow y > v(\hat{q}) - h\Pi(\hat{q})$, and in turn $v(\hat{q}) - h\Pi(\hat{q}) = v(\hat{q}) - h[v(\hat{q}) - c(\hat{q})] = (1-h)v(\hat{q}) + hc(\hat{q}) \geq c(\hat{q})$, where the first equality uses $\Pi(q) = v(q) - c(q)$ and the inequality follows from the fact that for all $q \in (0, \hat{q}]$, we have $\Pi(q) > 0$. It follows that $\mathcal{Y}'' \subseteq [\hat{c}(q), \infty)$ and combining with the previous result, we have $\mathcal{Y}'' \subseteq \mathcal{Y}'$.

The set $\mathcal{Y}''$ has positive mass since $h > l$ and $\Pi(q) > 0$ on the relevant domain $(0, \hat{q}]$. This result holds true under Assumption 3; the result only requires that $\mathcal{L}^\sigma(z) \equiv \sigma L'(q)/v'(q)|_{q=\min\{v^1(z), \hat{q}\}} \geq 0$ (with equality if and only if $q = \hat{q}$), which does not rule out Assumption 3. Concluding, $\mathcal{Y}'$ must have positive mass under Assumption 3 since $\mathcal{Y}'' \subseteq \mathcal{Y}'$ and $\mathcal{Y}''$ has positive mass under Assumption 3. Q.E.D.

Proof of Proposition 2. The first part is to prove that

$$\max\{0, \tilde{r}_e^+\} < \tilde{r}_e^- \quad \Leftrightarrow \quad \exists n > 0 \text{ s.t. } \tilde{r}_e^+ < n < \tilde{r}_e^- . \tag{D.7}$$

First note that $\tilde{r}_e^+ < n < \tilde{r}_e^- \Leftrightarrow n \in (\tilde{r}_e^+, \tilde{r}_e^-)$. Then, note that $(\tilde{r}_e^+, \tilde{r}_e^-) \cap \mathbb{R}_+ = (\max\{0, \tilde{r}_e^+\}, \tilde{r}_e^-)$. Hence $(\max\{0, \tilde{r}_e^+\}, \tilde{r}_e^-) \neq \emptyset \Leftrightarrow \exists n > 0$ s.t. $\tilde{r}_e^+ < n < \tilde{r}_e^-$. Clearly $\max\{0, \tilde{r}_e^+\} < \tilde{r}_e^- \Leftrightarrow (\max\{0, \tilde{r}_e^+\}, \tilde{r}_e^+) \neq \emptyset$, thus proving (D.7).

The next part is to prove that

$$\exists n > 0 \text{ s.t. } \tilde{r}_e^+ < n < \tilde{r}_e^- \quad \Leftrightarrow \quad (k,y) \in \mathcal{S}_{me} \equiv \{(k,y) : \; \exists n > 0 \text{ s.t. } (k, y+n) \in \mathcal{S}_e\} \,. \tag{D.8}$$

From the proof of Proposition 1 it is immediate that $\tilde{r}_e^+ - n < 0 < \tilde{r}_e^- - n \Leftrightarrow (k, y+n) \in \mathcal{S}_e$, which in turn proves (D.8).

Combing the two parts above gives

$$\max\{0, \tilde{r}_e^+\} < \tilde{r}_e^- \;\Leftrightarrow\; \quad (k,y) \in \mathcal{S}_{me} \equiv \{(k,y) : \; \exists n > 0 \text{ s.t. } (k, y+n) \in \mathcal{S}_e\} \,. \tag{D.9}$$

Because $\mathcal{S}_e$ is an open set, it follows that $\mathcal{S}_e \subseteq \mathcal{S}_{me}$. Therefore $\mathcal{S}_{me}$ has positive mass under both the parameter restriction $c(\hat{q}) \leq y$ and Assumption 3 since $\mathcal{S}_e$ exhibits this property, too.

Finally, I show that $(k,y) \in \mathcal{S}_e$ implies that Assumption 2 is satisfied. We have that $(k,y) \in \mathcal{S}_{em} \;\Rightarrow\; \exists n > 0$ such that $(k, y+n) \in \mathcal{S}_e$. For that n, it must hold that $y + n > 0$ and $k \in \mathcal{K}(y+n)$, as otherwise $(k, y+n) \notin \mathcal{S}_e$. It then follows directly from the proof of Proposition 1 that indeed Assumption 2 is satisfied. Q.E.D.

Proof of Propositions 3, 4, and 5. First, if $(k,y) \notin \mathcal{S}_{me}$, then f_{me} is a function on the relevant domain $\mathbb{R}_{++}$. From the properties of f_{me}^e it follows that f_{me} is monotonically increasing given Assumption 3. Hence, the only bounded monetary equilibria are steady states and the monetary steady state is generically unique unless $\mu = \beta(1+\iota)$.

Next, if $(k,y) \in \mathcal{S}_{me}$, then f_{me} is a correspondence on the domain $[\max\{0, \tilde{r}_e^+\}, \tilde{r}_e^-]$, which has positive mass. Define $i = \mu/\beta - 1$. To elucidate how search behaves in equilibrium, note that a dynamic equilibrium is characterized by a bounded process $\{(x_t, \eta_t)\}_{t=0}^{\infty}$ that satisfies

$$x_t = \begin{cases} f_{me}^l(\mathbb{E}_t\{x_{t+1}\}) & \text{if } \eta_t = 0, \\ \frac{1+\tilde{\iota}}{1+i} & \text{if } \eta_t \in (0,1), \\ f_{me}^h(\mathbb{E}_t\{x_{t+1}\}) & \text{if } \eta_t = 1; \end{cases} \qquad \eta_t \in \begin{cases} \{0\} & \text{if } \mathbb{E}_t\{x_{t+1}\} < \tilde{r}_e^+, \\ \left\{0, \frac{\tilde{r}_e^- - \mathbb{E}_t\{x_{t+1}\}}{\tilde{r}_e^- - \tilde{r}_e^+}, 1\right\} & \text{if } \tilde{r}_e^+ \leq \mathbb{E}_t\{x_{t+1}\} \leq \tilde{r}_e^-, \\ \{1\} & \text{if } \mathbb{E}_t\{x_{t+1}\} > \tilde{r}_e^-. \end{cases} \tag{D.10}$$

This follows directly from Equations (43) and (44). As established in Section 5, f^l_{me} and f^h_{me} are monotonically increasing functions given Assumption 3, which I impose throughout the proof.

Next, note that $\iota^\sigma(r)$, as defined in Equation (45), is: continuous; strictly decreasing in r on the domain $(-y, \hat{r}^\sigma_e)$, where $\hat{r}^\sigma_e \equiv v(\hat{q}) - \sigma\Pi(\hat{q}) - y$, since $v'(q) - \sigma\Pi'(q) > 0$ and $\partial z^\sigma / \partial \iota^\sigma < 0$; and satisfies $\iota^\sigma(r) > (=) 0 \;\Leftrightarrow\; r < (\geq) \hat{r}^\sigma_e$ and $\iota^\sigma(r) = \sigma I \;\Leftrightarrow\; r = -y$. Further, we have $\iota^h(r) \leq (<) \tilde{\iota} \; \forall r \geq (>) \tilde{r}^+_e$ and $\tilde{\iota} \leq (<) \iota^l(r) \; \forall r \leq (<) \tilde{r}^-_e$ since $\iota^h(\tilde{r}^+_e) = \iota^l(\tilde{r}^-_e) = \tilde{\iota}$ and $\tilde{\iota} \in (0, lI)$.

Then, note that the set $\mathcal{I}$, as defined in Equation (46), has positive mass since $(k, y) \in \mathcal{S}_{me}$ implies $\max\{0, \tilde{r}^+_e\} < \tilde{r}^-_e$; the fact that $\iota^l(\cdot)$ and $\iota^h(\cdot)$ are strictly decreasing on the domain $(-y, \hat{r}^l_e)$ and $(-y, \hat{r}^h_e)$, respectively, and $\tilde{r}^-_e < \hat{r}^l_e$ and $\tilde{r}^+_e < \hat{r}^h_e$ (since $\tilde{\iota} > 0$), therefore imply that $\iota^h(\tilde{r}^-_e) < \iota^h(\tilde{r}^+_e) = \tilde{\iota} = \iota^l(\tilde{r}^-_e) < \iota^l(\max\{0, \tilde{r}^+_e\})$. Note that $\mathrm{int}(\mathcal{I}) = (\iota^h(\tilde{r}^-_e), \iota^l(\max\{0, \tilde{r}^+_e\}))$ then has positive mass, too, and contains $\tilde{\iota}$.

It is now useful to study separately the cases: (a) $0 = i \leq \iota^h(\tilde{r}^-_e)$; (b) $0 < i \leq \iota^h(\tilde{r}^-_e)$; (c) $i \in (\iota^h(\tilde{r}^-_e), \iota^l(\max\{0, \tilde{r}^+_e\}))$; and (d) $i \geq \iota^l(\max\{0, \tilde{r}^+_e\})$.

Case a: $0 = i \leq \iota^h(\tilde{r}^-_e)$. We have on the relevant domain $\mathbb{R}_{++}$ that $f^h_{me}(x) > x \; \forall x < \hat{r}^h_e$ and $f^h_{me}(x) = x \; \forall x \geq \hat{r}^h_e$. Likewise, $f^l_{me}(x) > x \; \forall x < \hat{r}^l_e$ and $f^l_{me}(x) = x \; \forall x \geq \hat{r}^l_e$. Hence, since $\hat{r}^h_e < \hat{r}^l_e$, $\mathbb{E}_t\{x_{t+1}\} < \hat{r}^h_e \;\Rightarrow\; \mathbb{E}_t\{x_{t+1}\} < x_t$ by Equation (D.10).

If $\iota^h(\tilde{r}^-_e) > 0$, we must also have $\eta_t = 1$ if $\mathbb{E}_t\{x_{t+1}\} \geq \hat{r}^h_e$ since $\iota^h(\tilde{r}^-_e) > 0$ entails $\tilde{r}^-_e < \hat{r}^h_e$. Hence, $x_t < \hat{r}^h_e \;\Rightarrow\; \mathbb{E}_t\{x_{t+1}\} < \hat{r}^h_e$. So $x_t < \hat{r}^h_e \;\Rightarrow\; \mathbb{E}_t\{x_{t+1}\} < \hat{r}^h_e \;\Rightarrow \mathbb{E}_t\{x_{t+1}\} < x_t$; we cannot have $x_t < \hat{r}^h_e$ in a bounded monetary equilibrium since it would imply that $\{x_\tau\}^\infty_{\tau=0}$ would go to zero with positive probability. If $x_t \geq \hat{r}^h_e$, we can have $\eta_t < 1$ only if $\mathbb{E}_t\{x_{t+1}\} < \hat{r}^h_e$ since $\tilde{r}^-_e < \hat{r}^h_e$, so $x_t \geq \hat{r}^h_e$ and $\eta_t < 1$ would likewise imply that $\{x_\tau\}^\infty_{\tau=0}$ would go to zero with positive probability. Thus, in a bounded monetary equilibrium we must have $x_t \geq \hat{r}^h_e$ and $\eta_t = 1$. From Equation (D.10) it follows that $\mathbb{E}_t\{x_{t+1}\} = x_t$; in a bounded equilibrium we may have x_t developing stochastically over time but $\eta_t = 1 \; \forall t$ and $x_t \geq \hat{r}^h_e \; \forall t$, which implies that the real allocation is pinned down uniquely. I.e., all buyers search intensely $(e = h)$ and they consume $\hat{q}$ DM goods if matched to a firm.

If $\iota^h(\tilde{r}^-_e) = 0$, we have $\tilde{r}^-_e \geq \hat{r}^h_e$. Consider the sequence

$$\{(x_t, \eta_t)\}^\infty_{t=0} = \{(f^l_{me}(\hat{r}^h_e), 1), (f^l_{me}(\hat{r}^h_e), 0), (\hat{r}^h_e, 1), (\hat{r}^h_e, 1), \ldots\}. \tag{D.11}$$

Clearly, $\{(x_t,\eta_t)\}_{t=2}^{\infty}$ satisfies Equation (D.10) since it is a steady state. Further, we have that $\eta_1 = 0$ is feasible since $x_2 = \hat{r}_e^h \leq \tilde{r}_e^-$. From Equation (D.10) this then indeed implies $x_1 = f_{me}^l(\hat{r}_e^h)$; $\{(x_t,\eta_t)\}_{t=1}^{\infty}$ satisfies Equation (D.10), too. Then, note that $f_{me}^l(\hat{r}_e^h) > \hat{r}_e^h$ because $\hat{r}_e^h < \hat{r}_e^l$. Therefore we can have $\eta_0 = 1$, which through Equation (D.10) then implies $x_0 = f_{me}^h \circ f_{me}^l(\hat{r}_e^h) = f_{me}^l(\hat{r}_e^h)$, were the last equality uses $f_{me}^h(x) = x\ \forall x \geq \hat{r}_e^h$; the proposed sequence in Equation (D.11) is indeed an equilibrium. Note the equilibrium features a one-time boom-bust cycle; the economy starts in a boom, then experiences a bust, and subsequently remains in a boom (the steady state).

Case b: $0 < i \leq \iota^h(\tilde{r}_e^-)$. It follows that $i < \tilde{\imath}$ in this case because $\tilde{\imath} \in \text{int}(\mathcal{I})$. We have a unique monetary steady state at $x_{ss} \equiv r_e(i) \in (\max\{0, \tilde{r}_e^+\}, \hat{r}_e^h)$, entailing high search. Moreover, $x_{ss} \geq \tilde{r}_e^-$, with $=$ if and only if $i = \iota^h(\tilde{r}_e^-)$ since $\iota^h(r)$ is strictly decreasing on $(-y, \hat{r}_e^h)$ and $0 < \iota^h(x_{ss}) = i \leq \iota^h(\tilde{r}_e^-)$. The unique monetary steady state at x_{ss} implies we have $f_{me}^h(x) > (<)x \Leftrightarrow x < (>)x_{ss}$ on the relevant domain $\mathbb{R}_{++}$, as well as $f_{me}^l(x) > x$ on the relevant domain $(0, \tilde{r}_e^-]$.

Equation (D.10) implies $\eta_t = 1\ \forall \mathbb{E}_t\{x_{t+1}\} > x_{ss}$ since $x_{ss} \geq \tilde{r}_e^-$. This property implies $x_{ss} < x_t < \mathbb{E}_{t+1}\{x_{t+1}\}\ \forall \mathbb{E}_t\{x_{t+1}\} > x_{ss}$ due to monotonicity of f_{me}^h. At the same time, since $f_{me}^l(x) > x\ \forall x \in (0, \tilde{r}_e^-]$ and $f_{me}^h(x) > x\ \forall x \in (0, x_{ss})$, $x_t < x_{ss} \Rightarrow \mathbb{E}_t\{x_{t+1}\} < x_{ss} \Rightarrow \mathbb{E}_t\{x_{t+1}\} < x_t$; it must be that $\{x_\tau\}_{\tau=t}^{\infty}$ grows goes to zero with positive probability if $x_t < x_{ss}$. We must thus have $x_t \geq x_{ss}$ in a bounded monetary equilibrium.

On the other hand, if $x_t > x_{ss}$, then if $\eta_t = 1$ (feasible since $x_{ss} > \tilde{r}_e^+$) we have for sure that $\mathbb{E}_{t+1}\{x_{t+1}\} > x_t$ by the monotonicity of f_{me}^h. Since $x_{ss} \geq \tilde{r}_e^-$, other $\mathbb{E}_t\{x_{t+1}\}$ that satisfy Equation (D.10) for $x_t > x_{ss}$ must induce $\eta_t < 1$ and thus $\mathbb{E}_{t+1}\{x_{t+1}\} \leq \tilde{r}_e^-$, which in turn satisfies $\tilde{r}_e^- \leq x_{ss}$. If $\tilde{r}_e^- < x_{ss}$ it therefore follows directly that $x_t > x_{ss}$ implies that $\{x_\tau\}_{\tau=t}^{\infty}$ grows either unbounded or to zero with positive probability; we must have $x_t \leq x_{ss}$ in a bounded monetary equilibrium. For the knife edge case $\tilde{r}_e^- = x_{ss}$, we have that $\mathbb{E}_t\{x_{t+1}\} = x_{ss}$ only for $(x_t,\eta_t) = (x_{ss}, 1)$ and $(x_t,\eta_t) = (f_{me}^l(\tilde{r}_e^-), 0)$.

Taking stock, if $\tilde{r}_e^- < x_{ss}$, we must have $(x_t,\eta_t) = (x_{ss}, 1)\ \forall t$ in a bounded monetary equilibrium. For the special case $\tilde{r}_e^- = x_{ss}$ we can also have a deterministic sequence

$$\{(x_t,\eta_t)\}_{t=0}^{\infty} = \left\{ \begin{array}{l} (f_{me}^{h,T-1}(x_{T-1}), 1), (f_{me}^{h,T-2}(x_{T-1}), 1), \\ \quad ..., (f_{me}^h(x_{T-1}), 1), (x_{T-1}, 0), (x_{ss}, 1), (x_{ss}, 1), ... \end{array} \right\}, \tag{D.12}$$

where $x_{T-1} = f^l_{me}(x_{ss})$ and $T \in \mathbb{N}$. The sequence $\{(x_t, \eta_t)\}_{t=T}^{\infty}$ satisfies Equation (D.10) since it is the monetary steady state. Further, $x_T = x_{ss}$ implies we can have $\eta_{T-1} = 0$ since $\mathbb{E}_{T-1}\{x_T\} = x_{ss} = \tilde{r}_e^-$. In turn, to satisfy Equation (D.10), this requires $x_{T-1} = f^l_{me}(x_{ss})$; the sequence $\{(x_t, \eta_t)\}_{t=T-1}^{\infty}$ also satisfies Equation (D.10). Then note that $f^l_{me}(x_{ss}) > x_{ss}$. In turn, this implies we can have $\eta_{T-2} = 1$ since $\mathbb{E}_{T-2}\{x_{T-1}\} = f^l_{me}(x_{ss}) > x_{ss} > \max\{0, \tilde{r}_+\}$. To satisfy Equation (D.10), this requires $x_{T-2} = f^h_{me}(x_{T-1})$; the sequence $\{(x_t, \eta_t)\}_{t=T-2}^{\infty}$ also satisfies Equation (D.10). Since $x_{T-1} > x_{ss} \Rightarrow x_{ss} < x_{T-2} < x_{T-1}$, as established before, we can have $\eta_{T-3} = 1$, too. We can then backward iterate further to conclude that the entire sequence $\{(x_t, \eta_t)\}_{t=0}^{\infty}$ characterized in Equation (D.12) satisfies Equation (D.10) $\forall T \in \mathbb{N}$.

Case c: $i \in (\iota^h(\tilde{r}_e^-), \iota^l(\max\{0, \tilde{r}_e^+\}))$. We have that the set

$$\mathcal{X} = \left\{x \in (\max\{0, \tilde{r}_e^+\}, \tilde{r}_e^-) : \quad f^h_{me}(x) < x < f^l_{me}(x)\right\} \tag{D.13}$$

is non-empty. To see this, note that

$$f^{\sigma}_{me}(x) = \frac{1 + \iota^{\sigma}(x)}{1 + i} x, \quad \sigma \in \{l, h\}. \tag{D.14}$$

For some arbitrary $x \in (\max\{0, \tilde{r}_e^+\}, \tilde{r}_e^-)$ we therefore have $f^h_{me}(x) < x < f^l_{me}(x)$ if and only if $i \in (\iota^h(x), \iota^l(x))$, where it has to be noted that $x < \tilde{r}_e^- \;\Rightarrow\; \iota^l(x) > \tilde{\iota}$ and $x > \max\{0, \tilde{r}_e^+\} \;\Rightarrow\; \iota^h(x) < \tilde{\iota}$; the set $(\iota^h(x), \iota^l(x))$ has positive mass for all $x \in (\max\{0, \tilde{r}_e^+\}, \tilde{r}_e^-)$. It follows that for an arbitrary $i \in (\iota^h(\tilde{r}_e^-), \iota^l(\max\{0, \tilde{r}_e^+\}))$, there exists an $x \in (\max\{0, \tilde{r}_e^+\}, \tilde{r}_e^-)$ such that $f^h_{me}(x) < x < f^l_{me}(x)$ since $\iota^l(\cdot), \iota^h(\cdot)$ are continuous and decreasing in x; $\mathcal{X}$ has positive mass and is, in fact, a convex set.

Pick an arbitrary $\hat{x} \in \mathcal{X}$. Suppose first that $f^l_{me} \circ f^h_{me}(\hat{x}) < \hat{x}$. It follows that $\exists x' \in (f^h_{me}(\hat{x}), \hat{x})$ such that $f^l_{me}(x') = x'$ by the intermediate value theorem since $f^l_{me}(\hat{x}) > \hat{x}$. Consider therefore the following process for (x_t, η_t):

$$(x_t, \eta_t) = \begin{cases} \begin{cases} (f^l_{me}(\hat{x}), 0) & \text{with prob. } \rho, \\ (f^h_{me}(\hat{x}), 1) & \text{with prob. } 1 - \rho, \end{cases} & \text{if } t \text{ odd} \\ (\hat{x}, 0) & \text{if } t \text{ even;} \end{cases} \qquad \rho \equiv \frac{x' - f^h_{me}(\hat{x})}{f^l_{me}(\hat{x}) - f^h_{me}(\hat{x})}. \tag{D.15}$$

Note that $\rho \in (0,1)$ since $f^h_{me}(\hat{x}) < x' < \hat{x} < f^l_{me}(\hat{x})$. Given process (D.15), Equation (D.10) is satisfied for odd t by construction since we then have $\mathbb{E}_t\{x_{t+1}\} = \hat{x} \in \mathcal{X} \subseteq (\max\{0, \tilde{r}^+_e\}, \tilde{r}^-_e)$, thus allowing for both $(x_t, \eta_t) = (f^l_{me}(\hat{x}), 0)$ and $(x_t, \eta_t) = (f^h_{me}(\hat{x}), 1)$. Further, for even t, we have $\mathbb{E}_t\{x_{t+1}\} = x'$, as follows from the definition of ρ. Equation (D.10) is then satisfied for even t, too, since $x' < \hat{x} < \tilde{r}^-_e$, thus allowing for $(x_t, \eta_t) = (f^l_{me}(\mathbb{E}_t\{x_{t+1}\}), 0) = (\hat{x}, 0)$; we have found a stochastic two cycle with $\mathcal{M}_t = \mathbb{E}_t\{x_{t+1}\} = x' < \hat{x}$ and $\eta_t = 0$ for even t; and $\mathcal{M}_t = \mathbb{E}_t\{x_{t+1}\} = \hat{x}$ with $\eta_t = 0$ with prob. ρ and $\eta_t = 1$ with prob. $1 - \rho$ for odd t. Real currency balances are thus pro cyclical and inflation is counter cyclical.

Suppose next that $f^l_{me} \circ f^h_{me}(\hat{x}) \geq \hat{x}$ and $f^{h,-1}_{me}(\hat{x}) \geq f^l_{me}(\hat{x})$. It follows that $\exists x' \in [f^h_{me}(\hat{x}), \hat{x}]$ such that $f^{h,-1}_{me}(x') = f^l_{me}(x')$ by the intermediate value theorem since $f^l_{me} \circ f^h_{me}(\hat{x}) \geq \hat{x} \;\Rightarrow\; f^{h,-1}_{me} \circ f^h_{me}(\hat{x}) \leq f^l_{me} \circ f^h_{me}(\hat{x})$. By construction, $x' = f^h_{me} \circ f^l_{me}(x')$, so consider the process

$$(x_t, \eta_t) = \begin{cases} (x', 1) & \text{if } t \text{ odd,} \\ (x'', 0) & \text{if } t \text{ even;} \end{cases} \qquad \text{where} \quad x'' \equiv f^l_{me}(x'). \tag{D.16}$$

For even t, we have $\mathbb{E}_t\{x_{t+1}\} = x'$. It follows that $\eta_t = 0$ for even t is in line with (D.10) because $x' \leq \hat{x} < \tilde{r}^-_e$. Given $\eta_t = 0$ and $\mathbb{E}_t\{x_{t+1}\} = x'$ for even t, it follows that (D.10) implies $x_t = x''$ for even t since $x'' \equiv f^l_{me}(x')$. For odd t, we have $\mathbb{E}_t\{x_{t+1}\} = x''$. It follows that $\eta_t = 1$ for odd t is in line with (D.10) because $x'' \equiv f^l_{me}(x') \geq f^l_{me} \circ f^h_{me}(\hat{x}) \geq \hat{x} > \max\{0, \tilde{r}^+_e\}$, where the first inequality follows from the fact that f^l_{me} is monotone increasing and $x' \geq f^h_{me}(\hat{x})$; the second is satisfied by supposition; and the third follows from the fact that $\hat{x} \in \mathcal{X} \subseteq (\max\{0, \tilde{r}^+_e\}, \tilde{r}^-_e)$. Given $\eta_t = 1$ and $\mathbb{E}_t\{x_{t+1}\} = x''$ for odd t, it follows that (D.10) implies $x_t = f^h_{me}(x'') = f^h_{me} \circ f^l_{me}(x') = x'$ for odd t; we have found a deterministic two cycle with $\mathcal{M}_t = \mathbb{E}_t\{x_{t+1}\} = x'$ and $\eta_t = 0$ for even t; and $\mathcal{M}_t = \mathbb{E}_t\{x_{t+1}\} = x''$ and $\eta_t = 1$ for odd t. Further, $x' < x''$; if $x' = x''$ we must have $x' = x'' = \hat{x}$ since the previous steps implied $x' \leq \hat{x} \leq x''$, but $x' = \hat{x}$ implies $x'' = f^l_{me}(\hat{x}) > \hat{x}$ since $\hat{x} \in \mathcal{X}$. Real currency balances are thus pro cyclical and inflation is counter cyclical.

Suppose finally that $f^l_{me} \circ f^h_{me}(\hat{x}) \geq \hat{x}$ and $f^{h,-1}_{me}(\hat{x}) < f^l_{me}(\hat{x})$. It follows that $f^h_{me}(\hat{x}) < \hat{x} < f^{h,-1}_{me}(\hat{x}) < f^l_{me}(\hat{x})$ since $\hat{x} \in \mathcal{X} \;\Rightarrow\; f^h_{me}(\hat{x}) < \hat{x} < f^l_{me}(\hat{x})$. Consider therefore the following process for (x_t, η_t):

$$(x_t,\eta_t) = \begin{cases} \begin{cases} (f^l_{me}(\hat{x}), 0) & \text{with prob. } \rho, \\ (f^h_{me}(\hat{x}), 1) & \text{with prob. } 1-\rho, \end{cases} & \text{if } t \text{ odd} \\ (\hat{x}, 1) & \text{if } t \text{ even;} \end{cases} \qquad \rho \equiv \frac{f^{h,-1}_{me}(\hat{x}) - f^h_{me}(\hat{x})}{f^l_{me}(\hat{x}) - f^h_{me}(\hat{x})}. \tag{D.17}$$

Note that $\rho \in (0,1)$ since $f^h_{me}(\hat{x}) < \hat{x} < f^{h,-1}_{me}(\hat{x}) < f^l_{me}(\hat{x})$. Given process (D.17), Equation (D.10) is satisfied for odd t by construction since we then have $\mathbb{E}_t\{x_{t+1}\} = \hat{x} \in \mathcal{X} \subseteq (\max\{0, \tilde{r}^+_e\}, \tilde{r}^-_e)$, thus allowing for both $(x_t, \eta_t) = (f^l_{me}(\hat{x}), 0)$ and $(x_t, \eta_t) = (f^h_{me}(\hat{x}), 1)$. Further, for even t, we have $\mathbb{E}_t\{x_{t+1}\} = f^{h,-1}_{me}(\hat{x})$, as follows from the definition of ρ. Equation (D.10) is then satisfied for even t, too, since $f^{h,-1}_{me}(\hat{x}) > \hat{x} > \max\{0, \tilde{r}^+_e\}$, thus allowing for $(x_t, \eta_t) = (f^h_{me}(\mathbb{E}_t\{x_{t+1}\}), 1) = (f^h_{me} \circ f^{h,-1}_{me}(\hat{x}), 1) = (\hat{x}, 1)$; we have found a stochastic two cycle with $\mathcal{M}_t = \mathbb{E}_t\{x_{t+1}\} = f^{h,-1}_{me}(\hat{x}) > \hat{x}$ and $\eta_t = 1$ for even t; and $\mathcal{M}_t = \mathbb{E}_t\{x_{t+1}\} = \hat{x}$ with $\eta_t = 0$ with prob. ρ and $\eta_t = 1$ with prob. $1-\rho$ for odd t. Real currency balances are thus pro cyclical and inflation is counter cyclical.

Case e: $i \geq \iota^l(\max\{0, \tilde{r}^+_e\})$. We have $i > \tilde{\imath}$ because $\tilde{\imath} \in \text{int}(\mathcal{I})$. There exists no monetary steady state if either $\tilde{r}^+_e \leq 0$, or $\tilde{r}^+_e > 0$ but $i \geq \iota^l(0)$ since we then have $i \geq \iota^l(0)$, entailing $f^l_{me}(x) < x \; \forall x > 0$ and $f^h_{me}(x) < x \; \forall x \geq \max\{\varepsilon, \tilde{r}^+_e\}$, where $\varepsilon > 0$ but infinitesimal. The monotonicity of f^l_{me} and f^h_{me} then imply that $x_t > 0 \Rightarrow x_t < \mathbb{E}_t\{x_{t+1}\}$ to satisfy (D.10); it must be that $\{x_\tau\}^\infty_{\tau=t}$ grows unbounded with positive probability if $x_t > 0$, entailing there is no bounded monetary equilibrium.

If $\tilde{r}^+_e > 0$ and $i > \iota^l(0)$, we have a unique monetary steady state at $x_{ss} \equiv r_e(i) \in (0, \tilde{r}^-_e)$, entailing low search. The procedure now develops analogous to case b. We have $x_{ss} \leq \tilde{r}^+_e$, with $=$ if and only if $i = \iota^l(\tilde{r}^+_e)$ since $\iota^l(r)$ is strictly decreasing on $(-y, \hat{r}^l_e)$ and $\iota^l(\tilde{r}^+_e) \leq i = \iota^l(x_{ss}) < \iota^l(0)$, where $\tilde{r}^+_e < \hat{r}^h_e < \hat{r}^l_e$. The unique monetary steady state at x_{ss} implies we have $f^l_{me}(x) > (<) x \Leftrightarrow x < (>) x_{ss}$ on the relevant domain $\mathbb{R}_{++}$, as well as $f^h_{me}(x) < x \; \forall x \geq \tilde{r}^+_e$.

Equation (D.10) implies $\eta_t = 0 \; \forall \mathbb{E}_t\{x_{t+1}\} < x_{ss}$ since $x_{ss} \leq \tilde{r}^+_e$. This property implies $\mathbb{E}_{t+1}\{x_{t+1}\} < x_t < x_{ss} \; \forall \mathbb{E}_t\{x_{t+1}\} < x_{ss}$ due to monotonicity of f^l_{me}. At the same time, since $f^h_{me}(x) < x \; \forall x \geq \tilde{r}^+_e$ and $f^l_{me}(x) < x \; \forall x > x_{ss}$, $x_t > x_{ss} \Rightarrow \mathbb{E}_t\{x_{t+1}\} > x_{ss} \Rightarrow \mathbb{E}_t\{x_{t+1}\} > x_t$; it must be that $\{x_\tau\}^\infty_{\tau=t}$ grows unbounded with positive probability if

$x_t > x_{ss}$. We must thus have $x_t \leq x_{ss}$ in a bounded monetary equilibrium.

On the other hand, if $x_t < x_{ss}$, then if $\eta_t = 0$ (feasible since $x_{ss} < \tilde{r}_e^-$) we have for sure that $\mathbb{E}_{t+1}\{x_{t+1}\} < x_t$ by the monotonicity of f^l_{me}. Since $x_{ss} \leq \tilde{r}_e^+$, other $\mathbb{E}_t\{x_{t+1}\}$ that satisfy Equation (D.10) for $x_t < x_{ss}$ must induce $\eta_t > 0$ and thus $\mathbb{E}_{t+1}\{x_{t+1}\} \geq \tilde{r}_e^+$, which in turn satisfies $\tilde{r}_e^+ \geq x_{ss}$. If $\tilde{r}_e^+ > x_{ss}$ it therefore follows directly that $x_t < x_{ss}$ implies that $\{x_\tau\}_{\tau=t}^\infty$ grows either unbounded or to zero with positive probability; we must have $x_t \geq x_{ss}$ in a bounded monetary equilibrium. For the knife edge case $\tilde{r}_e^+ = x_{ss}$, we have that $\mathbb{E}_t\{x_{t+1}\} = x_{ss}$ only for $(x_t, \eta_t) = (x_{ss}, 0)$ and $(x_t, \eta_t) = (f^h_{me}(\tilde{r}_e^+), 1)$.

Taking stock, if $\tilde{r}_e^+ > x_{ss}$, we must have $(x_t, \eta_t) = (x_{ss}, 0)$ $\forall t$ in a bounded monetary equilibrium. For the special case $\tilde{r}_e^+ = x_{ss}$ we can also have a deterministic sequence

$$\{(x_t, \eta_t)\}_{t=0}^\infty = \left\{ \begin{array}{l} (f^{l,T-1}_{me}(x_{T-1}), 0), (f^{l,T-2}_{me}(x_{T-1}), 0), \\ \quad \ldots, (f^l_{me}(x_{T-1}), 0), (x_{T-1}, 1), (x_{ss}, 0), (x_{ss}, 0), \ldots \end{array} \right\}, \tag{D.18}$$

where $x_{T-1} = f^h_{me}(x_{ss})$ and $T \in \mathbb{N}$. The sequence $\{(x_t, \eta_t)\}_{t=T}^\infty$ satisfies Equation (D.10) since it is the monetary steady state. Further, $x_T = x_{ss}$ implies we can have $\eta_{T-1} = 1$ since $\mathbb{E}_{T-1}\{x_T\} = x_{ss} = \tilde{r}_e^+$. In turn, to satisfy Equation (D.10), this requires $x_{T-1} = f^h_{me}(x_{ss})$; the sequence $\{(x_t, \eta_t)\}_{t=T-1}^\infty$ also satisfies Equation (D.10). Then note that $f^h_{me}(x_{ss}) < x_{ss}$. In turn, this implies we can have $\eta_{T-2} = 0$ since $\mathbb{E}_{T-2}\{x_{T-1}\} = f^h_{me}(x_{ss}) < x_{ss} < \tilde{r}_e^-$. To satisfy Equation (D.10), this requires $x_{T-2} = f^l_{me}(x_{T-1})$; the sequence $\{(x_t, \eta_t)\}_{t=T-2}^\infty$ also satisfies Equation (D.10). Since $x_{T-1} < x_{ss} \;\Rightarrow\; x_{T-1} < x_{T-2} < x_{ss}$, as established before, we can have $\eta_{T-3} = 0$, too. We can then backward iterate further to conclude that the entire sequence $\{(x_t, \eta_t)\}_{t=0}^\infty$ characterized in Equation (D.18) satisfies Equation (D.10) $\forall T \in \mathbb{N}$.

Combining insights from the cases *a*-*d*, we find that: (i) two cycles with boom-bust dynamics and counter-cyclical inflation exist if $i \in \text{int}(\mathcal{I})$, proving Proposition 3; (ii) equilibria that converge to the monetary steady state with a boom-bust cycle on the transition path exist if $i \in \mathcal{I}/\,\text{int}(\mathcal{I})$, proving Proposition 4; and (iii) bounded monetary equilibria other than steady states do not exist if $i \notin \mathcal{I}$, proving Proposition 5. Q.E.D.

Proof of Proposition 6. First observe that $\mathcal{M}_t$ is perfectly predicable at time $t-1$ since $\mathcal{M}_t = \Phi_t M_{t-1}$ and $\Phi_t = \Phi_{t-1}/\pi$. Thus, $\mathcal{M}_t = \mathbb{E}_{t-1}\{\mathcal{M}_t\}$. Define $\mathbb{M}_{t-1} = \mathbb{E}_{t-1}\{\mathcal{M}_t\}$

to capture this. Clearance of the market for liquid wealth implies that only ι_t which solve $r_e(\iota_t) = \mathbb{M}_{t-1}$ can occur on the equilibrium path due to the perfect predictability of $\mathcal{M}_t$.

If $(k, y) \in \mathcal{S}_{me}$ we know from Corollary 1 that there are three ι_t for which $r_e(\iota_t) = \mathbb{M}_{t-1}$ if and only if $\mathbb{M}_{t-1} \in [\max\{0, \tilde{r}_e^+\}, \tilde{r}_e^-]$, namely $\iota^l(\mathbb{M}_{t-1})$, $\iota^h(\mathbb{M}_{t-1})$, and $\tilde{\iota}$, where $\iota^l(\cdot), \iota^h(\cdot)$ are as defined in Equation (45).

Let $\mathbb{P}^l_{t-1} = \mathbb{P}_{t-1}\{\iota_t = \iota^l(\mathbb{M}_{t-1})\}$, $\mathbb{P}^h_{t-1} = \mathbb{P}_{t-1}\{\iota_t = \iota^h(\mathbb{M}_{t-1})\}$ and $\tilde{\mathbb{P}}_{t-1} = \mathbb{P}_{t-1}\{\iota_t = \tilde{\iota}\}$. Given that i is fixed in the inflation-targeting regime, it follows from Equation (48) that $\mathbb{M}_{t-1} > 0$ and $(\mathbb{P}^l_{t-1}, \tilde{\mathbb{P}}_{t-1}, \tilde{\mathbb{P}}^h_{t-1}) \in \Delta^2$, where Δ^2 is the 2-dimensional simplex, are determined endogenously by

$$i = \mathbb{E}_{t-1}\{\iota_t\} \equiv \mathbb{P}^l_{t-1}\iota^l(\mathbb{M}_{t-1}) + \tilde{\mathbb{P}}_{t-1}\tilde{\iota} + \mathbb{P}^h_{t-1}\iota^h(\mathbb{M}_{t-1}). \tag{D.19}$$

Because $\iota^\sigma(\mathbb{M}_{t-1})$ is decreasing in $\mathbb{M}_{t-1}$ and $\iota^h(\mathbb{M}_{t-1}) \leq \tilde{\iota} \leq \iota^l(\mathbb{M}_{t-1})$, it follows that there exists a non-degenerate probability distribution for ι_t, i.e., there exists an $\mathbb{M}_{t-1} \in [\max\{0, \tilde{r}_e^+\}, \tilde{r}_e^-]$ and a vector $(\mathbb{P}^l_{t-1}, \tilde{\mathbb{P}}_{t-1}, \tilde{\mathbb{P}}^h_{t-1}) \in \Delta^2/\{(1,0,0),(0,1,0),(0,0,1)\}$ that jointly solve Equation (D.19) if and only if

$$i \in (\iota^h(\tilde{r}_e^-), \iota^l(\max\{0, \tilde{r}_e^+\})) \equiv \text{int}(\mathcal{I}). \tag{D.20}$$

If $(k, y) \notin \mathcal{S}_{me}$, then $\nexists \mathcal{M} > 0$ s.t. $\tilde{r}_e^+ < \mathcal{M} < \tilde{r}_e^-$. It follows that there is a unique ι that solves $r_e(\iota) = \mathcal{M}\ \forall \mathcal{M} > 0$, as implied by Corollary 1. With $\mathcal{M}_t = \mathbb{M}_{t-1}$, i.e., $\mathcal{M}_t$ being perfectly predicable at time $t-1$, it follows that ι_t must be perfectly predictable at time $t-1$, too. Equation (48) therefore implies that $\iota_t = i_t$, entailing a degenerate distribution for ι_t. Q.E.D.

Proof of Proposition 7. First observe that there is only a scope for stochastic equilibrium multiplicity if $(k, y) \in \mathcal{S}_{me}$ and $i \in \text{int}(\mathcal{I})$ (see Proposition 6). Further, we focus on the case $i < \tilde{\iota}$, meaning that the objective is to implement the boom equilibrium with probability one. For future purposes, it is useful to define

$$\tilde{\Delta}^+ \equiv h\Pi \circ v^{-1}(\underline{z}^h) + y \quad \text{and} \quad \tilde{\Delta}^- \equiv l\Pi \circ v^{-1}(\overline{z}^l) + y. \tag{D.21}$$

Also, let $\mathbb{M}_{t-1}^{det}$ solve $i = \iota^h(\mathbb{M}_{t-1}^{det})$ ($\iota^\sigma(\cdot)$, $\sigma \in \{l, h\}$ is defined in Equation (45)), i.e., $\mathbb{M}_{t-1}^{det}$ is the value of currency balances in deterministic equilibrium where ι_t is degenerate at i. Note that $\mathbb{M}_{t-1}^{det} > \tilde{r}_e^+$ because $\mathbb{M}_{t-1}^{det}$ solves

$$\mathbb{M}_{t-1}^{det} = z^h(i) - h\Pi(\min\{v^{-1} \circ z^h(i), \hat{q}\}) - y \tag{D.22}$$

where the RHS is decreasing in i so that $i < \tilde{\iota}$ implies $\mathbb{M}_{t-1}^{det} \geq z^h(\tilde{\iota}) - h\Pi \circ v^{-1} \circ z^h(\tilde{\iota}) - y \equiv \tilde{r}_e^+$. We also have $\mathbb{M}_{t-1} < \tilde{r}_e^-$ because $\iota^h(\tilde{r}_e^-) < i = \iota^h(\mathbb{M}_{t-1})$ and $\iota^h(\mathbb{M}_{t-1})$ is decreasing in $\mathbb{M}_{t-1}$.

The first step is to prove that $\mathbb{M}_{t-1} > \mathbb{M}_{t-1}^{det}$ if ι_t is non-degenerate and $\underline{\Delta} \leq \tilde{\Delta}^+$. Because Δ_t is decreasing in ι_t (see Equation (19)), it follows that $z^s(\iota_t < \tilde{\iota}) = \mathbb{M}_{t-1} + \Delta_t$; TARP is not deployed in case $\iota_t < \tilde{\iota}$. We therefore have that $\iota_t = \iota^h(\mathbb{M}_{t-1})$ if $\iota_t < \tilde{\iota}$. We also have that $\mathbb{P}_{t-1}\{\iota_t < i\} > 0$ if ι_t is non-degenerate since $i = \mathbb{E}_{t-1}\{\iota_t\}$ by Equation (48). It follows that $\mathbb{M}_{t-1} > \mathbb{M}_{t-1}^{det}$ if ι_t is non-degenerate because for any $\iota_t < \tilde{\iota}$ we have $\iota_t = \iota^h(\mathbb{M}_{t-1})$ and $\iota^h(\mathbb{M}_{t-1})$ is decreasing in $\mathbb{M}_{t-1}$.

The next step is to prove that ι_t cannot be non-degenerate if

$$\underline{\Delta} = \Delta' \equiv \eta' h\Pi \circ v^{-1}(\underline{z}^h) + (1 - \eta') l\Pi \circ v^{-1}(\overline{z}^l) + y, \quad \text{where} \quad \eta' \equiv \frac{\tilde{r}_e^- - r_e(i)}{\tilde{r}_e^- - \tilde{r}_e^+}. \tag{D.23}$$

I do this by means of a contradiction. First, note that $\Delta' = \eta' \tilde{\Delta}^+ + (1 - \eta')\tilde{\Delta}^-$. Second, note that $\eta' \in (0, 1)$ because $\mathbb{M}_{t-1}^{det} = r_e(i)$ and $\mathbb{M}_{t-1}^{det} \in (\tilde{r}_e^-, \tilde{r}_e^+)$. Third, note that $\tilde{\Delta}^- < \tilde{\Delta}^+$ since $l < h$ and $\overline{z}^l < \underline{z}^h$. We thus have $\Delta' \in (\tilde{\Delta}^-, \tilde{\Delta}^+)$.

Because Δ_t is decreasing in ι_t it follows that TARP is not deployed in case $\iota_t < \tilde{\iota}$ since $\lim_{\iota_t \nearrow \tilde{\iota}} \Delta_t = \tilde{\Delta}^+$, whilst it is deployed in case $\iota_t > \tilde{\iota}$ since $\lim_{\iota_t \searrow \tilde{\iota}} \Delta_t = \tilde{\Delta}^-$. On the one hand we thus have $z^s(\iota_t < \tilde{\iota}) = \mathbb{M}_{t-1} + \Delta_t$, so that $\iota_t = \iota^h(\mathbb{M}_{t-1})$ if $\iota_t < \tilde{\iota}$. On the other hand, $z^s(\iota_t > \tilde{\iota}) = \mathbb{M}_{t-1} + \Delta'$ and also, $z^d(\iota_t > \tilde{\iota}) \leq \overline{z}^l$ since z^d is decreasing in ι_t and

$\lim_{\iota_t \searrow \tilde{\iota}} z^d(\iota_t) = \overline{z}^l$. Since $\overline{z}^l = \tilde{r}_e^- + \tilde{\Delta}^-$, we find that if $\iota_t > \tilde{\iota}$:

$$\begin{aligned} z^d(\iota_t) - z^s(\iota_t) &\leq \tilde{r}_e^- + \tilde{\Delta}^- - \Delta' - \mathbb{M}_{t-1} \\ &< \tilde{r}_e^- + \tilde{\Delta}^- - \Delta' - \mathbb{M}_{t-1}^{det} \\ &= \tilde{r}_e^- + \tilde{\Delta}^- - \eta'\tilde{\Delta}^+ - (1-\eta')\tilde{\Delta}^- - r_e(i) \\ &= (\tilde{r}_e^- - r_e(i)) \left(1 - \frac{\tilde{\Delta}^+ - \tilde{\Delta}^-}{\tilde{r}_e^- - \tilde{r}_e^+}\right) \\ &< 0, \end{aligned} \tag{D.24}$$

where the last line uses that $\tilde{r}_e^- + \tilde{\Delta}^- = \overline{z}^l < \underline{z}^h = \tilde{r}_e^+ + \tilde{\Delta}^+$. Thus, for $\underline{\Delta} = \Delta'$, having $\iota_t > \tilde{\iota}$ is inconsistent with clearance of the market for liquid wealth.

If $\iota_t = \tilde{\iota}$, we have that η_t adjusts to clear the market for liquid wealth. Particularly, η_t solves

$$\eta_t z^h(\tilde{\iota}) + (1-\eta_t) z^l(\tilde{\iota}) = \Delta_t + \mathbb{M}_{t-1} + \max\{\Delta' - \Delta_t, 0\}, \tag{D.25}$$

where $z^h(\tilde{\iota}) = \underline{z}^h$, $z^l(\tilde{\iota}) = \overline{z}^l$, and $\Delta_t = \eta_t\tilde{\Delta}^+ + (1-\eta_t)\tilde{\Delta}^-$. It follows that η_t satisfies

$$\begin{aligned} \eta_t &= \frac{\tilde{r}_e^- - \mathbb{M}_{t-1} - \max\{\eta' - \eta_t, 0\}(\tilde{\Delta}^+ - \tilde{\Delta}^-)}{\tilde{r}_e^- - \tilde{r}_e^+} \\ &< \frac{\tilde{r}_e^- - \mathbb{M}_{t-1}^{det}}{\tilde{r}_e^- - \tilde{r}_e^+} \\ &= \frac{\tilde{r}_e^- - r_e(i)}{\tilde{r}_e^- - \tilde{r}_e^+} \equiv \eta'. \end{aligned} \tag{D.26}$$

We therefore have that TARP is deployed if $\iota_t = \tilde{\iota}$ since $\eta_t < \eta' \Leftrightarrow \Delta_t < \Delta'$. It follows that η_t solves

$$\eta_t = \frac{\tilde{r}_e^- - \mathbb{M}_{t-1} - (\eta' - \eta_t)(\tilde{\Delta}^+ - \tilde{\Delta}^-)}{\tilde{r}_e^- - \tilde{r}_e^+}. \tag{D.27}$$

Hence, we obtain

$$\begin{aligned}\eta_t &= \frac{\mathbb{M}_{t-1} + \Delta' - \tilde{r}_e^- - \tilde{\Delta}^-}{\tilde{r}_e^+ + \tilde{\Delta}^+ - \tilde{r}_e^- - \tilde{\Delta}^-} \\ &= \frac{\mathbb{M}_{t-1} + \eta'(\tilde{\Delta}^+ - \tilde{\Delta}^-) - \tilde{r}_e^-}{\tilde{r}_e^+ + \tilde{\Delta}^+ - \tilde{r}_e^- - \tilde{\Delta}^-} \\ &> \frac{\mathbb{M}_{t-1}^{det} + \eta'(\tilde{\Delta}^+ - \tilde{\Delta}^-) - \tilde{r}_e^-}{\tilde{r}_e^+ + \tilde{\Delta}^+ - \tilde{r}_e^- - \tilde{\Delta}^-} \\ &= \frac{\tilde{r}_e^- - r_e(i)}{\tilde{r}_e^- - \tilde{r}_e^+} \equiv \eta';\end{aligned} \tag{D.28}$$

a contradiction.

Taking stock, on the one hand we can neither have $\iota_t > \tilde{\iota}$ nor $\iota_t = \tilde{\iota}$ if $\underline{\Delta} = \Delta'$ and ι_t is non-degenerate. On the other, we have $\iota_t = \iota^h(\mathbb{M}_{t-1})$ if $\iota < \tilde{\iota}$ and $\underline{\Delta} = \Delta'$. Hence, it must be that $\iota_t = \iota^h(\mathbb{M}_{t-1})$ if $\underline{\Delta} = \Delta'$, contradicting that ι_t is non-degenerate since it is perfectly predictable from $\mathbb{M}_{t-1}$.

The last step is to prove that ι_t can be non-degenerate if $\underline{\Delta} < \Delta'$. I do this by showing that

$$\iota_t = \begin{cases} \underline{\iota} & \text{with prob } 1-\rho \\ \tilde{\iota} & \text{with prob. } \rho; \end{cases} \quad \text{where} \quad \rho \equiv \frac{\tilde{\iota} - i}{\tilde{\iota} - \underline{\iota}}, \tag{D.29}$$

is an equilibrium distribution for ι_t if

$$\underline{\iota} \in \mathcal{I} \equiv \left(\iota^h \left(\tilde{r}_e^- - \max\left\{ \frac{(\underline{\Delta} - \tilde{\Delta}^-)(\tilde{r}_e^- - \tilde{r}_e^+)}{\tilde{\Delta}^+ - \tilde{\Delta}^-}, 0 \right\} \right), i \right). \tag{D.30}$$

Note that the set $\mathcal{I}$ has positive mass since $i > \iota^h(\tilde{r}_e^-)$—if not, i is such that there is no scope for a stochastic equilibrium in the first place—and

$$\underline{\Delta} < \Delta' \quad \Rightarrow \quad \tilde{r}_e^- - \max\left\{ \frac{(\underline{\Delta} - \tilde{\Delta}^-)(\tilde{r}_e^- - \tilde{r}_e^+)}{\tilde{\Delta}^+ - \tilde{\Delta}^-}, 0 \right\} \in \left[\tilde{r}_e^-, r_e(i)\right). \tag{D.31}$$

First, we have that $\mathbb{M}_{t-1}$, i.e., the perfectly predictable equilibrium value for currency balances, solves $\iota^h(\mathbb{M}_{t-1}) = \underline{\iota}$ since $\underline{\iota} < i < \tilde{\iota}$; with $\iota^h(\mathbb{M}_{t-1}) = \underline{\iota}$ the market for liquid wealth clears for $\iota_t = \underline{\iota}$. With $\iota^h(\mathbb{M}_{t-1})$ decreasing in $\mathbb{M}_{t-1}$ and $\underline{\iota} \in \mathcal{I}$ it follows that we also have

$$\mathbb{M}_{t-1} \in \mathcal{R} \equiv \left(r_e(i), r_- - \max\left\{ \frac{(\underline{\Delta} - \tilde{\Delta}^-)(\tilde{r}_e^- - \tilde{r}_e^+)}{\tilde{\Delta}^+ - \tilde{\Delta}^-}, 0 \right\} \right), \tag{D.32}$$

which is a set with positive mass since $\underline{\Delta} < \Delta'$.

Second, with $\iota_t = \tilde{\iota}$, the market for liquid wealth clears if and only if there is an $\eta_t \in [0,1]$ which solves

$$\eta_t z^h(\tilde{\iota}) + (1-\eta_t) z^l(\tilde{\iota}) = \Delta_t + \mathbb{M}_{t-1} + \max\{\underline{\Delta} - \Delta_t, 0\}, \tag{D.33}$$

where $\Delta_t = \eta_t \tilde{\Delta}^+ + (1-\eta_t)\tilde{\Delta}^-$. Suppose that η_t is such that $\underline{\Delta} \leq \Delta_t$. Then η_t

$$\eta_t = \frac{\tilde{r}_e^- - \mathbb{M}_{t-1}}{\tilde{r}_e^- - \tilde{r}_e^+} \tag{D.34}$$

and with $\mathbb{M}_{t-1} \in \mathcal{R}$ it follows that

$$\eta_t \in \mathcal{N} \equiv \left(\max\left\{ 0, \frac{(\underline{\Delta} - \tilde{\Delta}^-)(\tilde{r}_e^- - \tilde{r}_e^+)}{\tilde{\Delta}^+ - \tilde{\Delta}^-} \right\}, \eta' \right); \tag{D.35}$$

as set with positive mass since

$$\underline{\Delta} < \Delta' \quad \Rightarrow \quad \max\left\{ 0, \frac{(\underline{\Delta} - \tilde{\Delta}^-)(\tilde{r}_e^- - \tilde{r}_e^+)}{\tilde{\Delta}^+ - \tilde{\Delta}^-} \right\} < \eta'. \tag{D.36}$$

The last step is to verify that $\eta_t \in \mathcal{N} \Rightarrow \Delta_t > \overline{\Delta}$. Here,

$$\Delta_t > \overline{\Delta} \quad \Leftrightarrow \quad \eta_t > \frac{(\underline{\Delta} - \tilde{\Delta}^-)(\tilde{r}_e^- - \tilde{r}_e^+)}{\tilde{\Delta}^+ - \tilde{\Delta}^-} \tag{D.37}$$

if $\iota_t = \tilde{\iota}$. This is indeed satisfied since $\eta_t \in \mathcal{N}$; there exists an $\eta_t \in [0,1]$ that clears the market for liquid wealth if $\iota_t = \tilde{\iota}$.

Taking stock, we have that the market for liquid wealth clears if $\iota_t = \tilde{\iota}$ and if $\iota_t = \underline{\iota}$; the ι_t on the support of the distribution in Equation (D.29) can occur in equilibrium. From the definition of ρ it also follows that $\mathbb{E}_t\{\iota_t\} = i$ if we indeed take the probability distribution from Equation (D.29); we indeed found an equilibrium with ι_t non-degenerate. Q.E.D.

Proof of Proposition 8. We focus on the relevant case in which there is indeed a stochastic, i.e., a non-degenerate distribution for ι_t, and $i < \tilde{\iota}$, i.e., the deterministic equilibrium is a boom. From the proof of Proposition 7 we therefore observe that $\mathbb{M}_{t-1}^{det} \in (\tilde{r}_e^+, \tilde{r}_e^-)$, where $\tilde{r}_e^- < \tilde{r}_e^+$ and $\mathbb{M}_{t-1}^{det} = r_e(i)$.

It first has to be noted that Δ'' is determined uniquely and satisfies $\Delta'' < \tilde{\Delta}^-$, with $\tilde{\Delta}^-$ as defined in the proof of Proposition 7. Uniqueness follows from the fact that

$$0 = \Delta - l\Pi \circ v^{-1}(r_e(i) + \Delta) - y \tag{D.38}$$

is increasing in Δ since $l\Pi'(q) < v'(q)$. To prove that $\Delta'' < \tilde{\Delta}^-$ it therefore suffices to show that

$$0 < \tilde{\Delta}^- - l\Pi \circ v^{-1}(r_e(i) + \tilde{\Delta}^-) - y. \tag{D.39}$$

Since $\tilde{\Delta}^- \equiv l\Pi \circ v^{-1}(\overline{z}^l) + y$ and $\tilde{r}_e^- \equiv \overline{z}^l - l\Pi \circ v^{-1}(\overline{z}^l) - y$, it follows directly that

$$0 < \tilde{\Delta}^- - l\Pi \circ v^{-1}(r_e(i) + \tilde{\Delta}^-) - y \quad \Leftrightarrow \quad r_e(i) < \tilde{r}_e^-, \tag{D.40}$$

where the latter is satisfied since $r_e(i) = \mathbb{M}_{t-1}^{det} \in (\tilde{r}_e^+, \tilde{r}_e^-)$. It now follows from the proof of Proposition 7 that $\mathbb{M}_{t-1} > \mathbb{M}_{t-1}^{det}$, with $\mathbb{M}_{t-1}$ as defined before, since $\Delta'' < \tilde{\Delta}^- < \tilde{\Delta}^+$.

Next, consider the case $\underline{\Delta} \leq \Delta''$. It follows directly that TARP is never deployed when $\iota_t \leq \tilde{\iota}$ since Δ_t is decreasing in ι and satisfies $\Delta_t \geq \tilde{\Delta}^-$ if $\iota_t \leq \tilde{\iota}$. It remains to consider $\iota_t > \tilde{\iota}$, for which I prove that TARP is not deployed by means of a contradiction. I.e., suppose that TARP is deployed, which, in turn, requires that $\Delta_t \leq \underline{\Delta}$. With TARP deployed, supply of liquid wealth equals $\mathbb{M}_{t-1} + \underline{\Delta}$, entailing

$$\Delta_t = l\Pi \circ v^{-1}(\mathbb{M}_{t-1} + \underline{\Delta}) + y. \tag{D.41}$$

We therefore need

$$\begin{aligned} \underline{\Delta} &\geq l\Pi \circ v^{-1}(\mathbb{M}_{t-1} + \underline{\Delta}) + y \\ &> \Pi \circ v^{-1}(r_e(i) + \underline{\Delta}) + y, \end{aligned} \tag{D.42}$$

where the last line uses that $\mathbb{M}_{t-1} > \mathbb{M}_{t-1}^{det} = r_e(i)$. From Equation D.38 it follows directly that

$$\underline{\Delta} > \Pi \circ v^{-1}(r_e(i) + \underline{\Delta}) + y \quad \Leftrightarrow \quad \underline{\Delta} > \Delta''; \tag{D.43}$$

a contradiction. With $\underline{\Delta} \leq \Delta''$ it follows that TARP is never deployed in stochastic equilibrium, entailing the exact same result as in Proposition 6; i.e., the economy is not stabilized as a non-degenerate distribution for ι_t is feasible.

Then, consider the case $\underline{\Delta} \in (\Delta'', \tilde{\Delta}^- + \tilde{r}_e^- - r_e(i))$, for which I prove that TARP can be deployed with positive probability by supposing that $\mathbb{M}_{t-1} = \mathbb{M}_{t-1}^{det} + \varepsilon$, where $\varepsilon > 0$ but infinitesimal. I construct a two-point distribution for ι_t, with $\iota^h < \tilde{\iota}$ and $\iota^l > \tilde{\iota}$. For $\iota_t = \iota^l$, I first show that TARP is deployed, for which it suffices to show that $\Delta_t < \underline{\Delta}$. We have that

$$\Delta_t = l\Pi \circ v^{-1}(r_e(i) + \varepsilon + \underline{\Delta}) + y \tag{D.44}$$

if TARP is indeed deployed, where I use $\mathbb{M}_{t-1} = r_e(i) + \varepsilon$. It follows that $\Delta_t < \underline{\Delta}$ since $\underline{\Delta} > \Delta''$ and ε is infinitesimal; TARP is indeed deployed if $\iota_t > \tilde{\iota}$. On the other hand, as follows from the proof of Proposition 7, TARP is not deployed if $\iota_t < \tilde{\iota}$ since we have $\underline{\Delta} \leq \tilde{\Delta}^- + \tilde{r}_e^- - r_e(i) < \Delta' < \tilde{\Delta}^+$, where $\tilde{\Delta}^- + \tilde{r}_e^- - r_e(i) < \Delta'$ follows from the definition of Δ' in Proposition 7.

I next argue that the market for liquid wealth indeed clears for some $\iota_t > \tilde{\iota}$, which, since TARP is being deployed in this case, requires existence of an $\iota_t > \iota$ that solves

$$r_e(i) + \varepsilon + \underline{\Delta} \leq z^l(\iota_t). \tag{D.45}$$

Such an ι_t exists if $r_e(i) + \varepsilon + \underline{\Delta} < \tilde{r}_e^- + \tilde{\Delta}^-$ since $z^l(\iota_t)$ is decreasing in ι_t and $z^l(\tilde{\iota}) = \overline{z}^l = \tilde{r}_e^- + \tilde{\Delta}^-$ by the definition of $\tilde{r}_e^-$ and $\tilde{\Delta}^-$. In turn, $r_e(i) + \varepsilon + \underline{\Delta} < \tilde{r}_e^- + \tilde{\Delta}^-$ is satisfied because $\underline{\Delta} < \tilde{\Delta}^- + \tilde{r}_e^- - r_e(i)$ and ε is infinitesimal.

The market for liquid wealth also clears for some $\iota_t < \tilde{\iota}$, where ι_t solves

$$r_e(i) + \varepsilon + h\Pi(\max\{z^h(\iota_t), \hat{q}\}) = z^h(\iota_t), \quad \text{with} = \text{if } \iota_t > 0. \tag{D.46}$$

This follows directly from the fact that such an ι_t is decreasing ε and exists for sure if $\varepsilon = 0$; otherwise $i < \tilde{\iota}$ cannot hold. We have that $\iota^h = i - \delta$, where $\delta > 0$ but infinitesimal exactly because $\varepsilon > 0$ but infinitesimal and $i > 0$; otherwise we cannot have a non-degenerate distribution for ι_t in the first place.

It remains to construct a non-degenerate probability distribution over ι^l, ι^h such that $i = \mathbb{E}_{t-1}\{\iota_t\}$ (see Equation (48)) holds. This requires setting

$$\mathbb{P}_{t-1}\{\iota_t = \iota^l\} = \frac{i - \iota^h}{\iota^l - \iota^h} \quad \text{and} \quad \mathbb{P}_{t-1}\{\iota_t = \iota^h\} = 1 - \mathbb{P}_{t-1}\{\iota_t = \iota^l\}. \tag{D.47}$$

It follows that $\mathbb{P}_{t-1}\{\iota_t = \iota^l\} > 0$ but infinitesimal since $\frac{i-\iota^h}{\iota^l-\iota^h} = \frac{\delta}{\iota^l-i+\delta}$, where $\delta > 0$ but infinitesimal whilst $\iota^l - i > 0$ since $\iota^l > \tilde{\iota} > i$; we have that $\delta \to 0$ by letting $\varepsilon \to 0$, whilst $\iota^l - i$ remains fixed at some positive value. This proves existence of a stochastic equilibrium in which TARP is deployed with positive probability, in which case $\Delta_t < \underline{\Delta}$, entailing losses for the taxpayer.

Finally, consider the case $\underline{\Delta} \in [\tilde{\Delta}^- + \tilde{r}_e^- - r_e(i), \Delta'')$, for which I again prove that TARP can be deployed with positive probability by supposing that $\mathbb{M}_{t-1} = \mathbb{M}_{t-1}^{det} + \varepsilon$, where $\varepsilon > 0$ but infinitesimal. I now construct a two-point distribution for ι_t, with $\iota^h < \tilde{\iota}$ and $\tilde{\iota}$.

For $\iota_t = \tilde{\iota}$, I first show that TARP is deployed, for which it suffices to show that

$$\eta_t < \underline{\eta} \equiv \frac{\underline{\Delta} - \tilde{\Delta}^-}{\tilde{\Delta}^+ - \tilde{\Delta}^-}. \tag{D.48}$$

We have that, if TARP is deployed, η_t solves

$$r_e(i) + \varepsilon + \underline{\Delta} = \eta_t z^h(\tilde{\iota}) + (1 - \eta_t) z^l(\tilde{\iota}) \tag{D.49}$$

where I use $\mathbb{M}_{t-1} = r_e(i) + \varepsilon$. It follows that

$$\eta_t = \frac{\underline{\Delta} - \tilde{\Delta}^- - (\tilde{r}_e^- - r_e(i)) + \varepsilon}{\tilde{\Delta}^+ - \tilde{\Delta}^- - (\tilde{r}_e^- - \tilde{r}_e^+)} \tag{D.50}$$

where I use $z^h(\tilde{\iota}) = \tilde{r}_e^+ + \tilde{\Delta}^+$ and $z^l(\tilde{\iota}) = \tilde{r}_e^- + \tilde{\Delta}^-$. Note that the denominator in Equation (D.50) is positive since $z^l(\tilde{\iota}) = \overline{z}^l < \underline{z}^h = z^h(\tilde{\iota})$. Using η' as defined in Proposition 7, it follows that

$$\eta_t < \underline{\eta} \quad \Leftrightarrow \quad \underline{\eta} < \eta' - \frac{\varepsilon}{(\tilde{r}_e^- - \tilde{r}_e^+)(\tilde{\Delta}^+ - \tilde{\Delta}^-)} \tag{D.51}$$

Further, we have $\overline{\eta} < \eta' \;\Leftrightarrow\; \underline{\Delta} < \Delta'$ by the definition of Δ' in Proposition 7; it follows that indeed, $\eta_t < \underline{\eta}$ if ε is infinitesimal.

I next show that for $\iota_t = \tilde{\iota}$ we indeed have clearance of the market for liquid wealth. For this, it suffices to show that η_t, as given by Equation (D.50), is in the interval $[0, 1]$. For $\eta_t \geq 0$ we need

$$0 < \underline{\Delta} - \tilde{\Delta}^- - (\tilde{r}_e^- - r_e(i)) + \varepsilon. \tag{D.52}$$

Condition (D.52) is satisfied since $\varepsilon > 0$ and $\underline{\Delta} \geq \tilde{\Delta}^- - (\tilde{r}_e^- - r_e(i))$ by assumption. On the other hand, note that

$$\begin{aligned}\eta_t &= \frac{\underline{\Delta} - \tilde{\Delta}^- - (\tilde{r}_e^- - r_e(i)) + \varepsilon}{\tilde{\Delta}^+ - \tilde{\Delta}^- - (\tilde{r}_e^- - \tilde{r}_e^+)} \\ &= \frac{\underline{\Delta} - \tilde{\Delta}^- - \eta'(\tilde{r}_e^- \tilde{r}_e^+) + \varepsilon}{\tilde{\Delta}^+ - \tilde{\Delta}^- - (\tilde{r}_e^- - \tilde{r}_e^+)} \\ &< \eta' \frac{\varepsilon}{\tilde{\Delta}^+ - \tilde{\Delta}^- - (\tilde{r}_e^- - \tilde{r}_e^+)},\end{aligned} \tag{D.53}$$

where the first line uses the definition of η' in Proposition 7, and the second line uses $\underline{\Delta} < \Delta'$ and the definition of Δ' in Proposition 7. With ε infinitesimal and $\eta' \in (0, 1)$ since $r_e(i) \in (\tilde{r}_e^+, \tilde{r}_e^-)$, it follows that $\eta_t < 1$.

The market for liquid wealth also clears for $\iota_t = i - \delta$ for the exact same reason as explained for the case $\underline{\Delta} \in (\Delta'', \tilde{\Delta}^- + \tilde{r}_e^- - r_e(i))$. A non-degenerate probability distribution over $\tilde{\iota}, \iota^h$ such that $i = \mathbb{E}_{t-1}\{\iota_t\}$ (see Equation) is

$$\mathbb{P}_{t-1}\{\iota_t = \tilde{\iota}\} = \frac{i - \iota^h}{\tilde{\iota} - \iota^h} \quad \text{and} \quad \mathbb{P}_{t-1}\{\iota_t = \iota^h\} = 1 - \mathbb{P}_{t-1}\{\iota_t = \iota^l\}. \tag{D.54}$$

It follows that $\mathbb{P}_{t-1}\{\iota_t = \tilde{\iota}\} > 0$ but infinitesimal for the exact same reason as before, again proving existence of a stochastic equilibrium in which TARP is deployed with positive probability, entailing losses for the taxpayer. Q.E.D.

References

Altermatt, L. (2022). Inside money, investment, and unconventional monetary policy. *International Economic Review*, **63** (4), 1527–1560.

—, Iwasaki, K. and Wright, R. (2021). Asset pricing in monetary economies. *Journal of International Money and Finance*, **115**, 1023–1052.

Andolfatto, D., Berentsen, A. and Waller, C. J. (2016). Monetary policy with asset-backed money. *Journal of Economic Theory*, **164**, 166–186.

Angeletos, G.-M. and La'O, J. (2013). Sentiments. *Econometrica*, **81** (2), 739–779.

Azariadis, C. (1993). *Intertemporal macroeconomics*. Oxford, England: Blackwell.

Berentsen, A., Menzio, G. and Wright, R. (2011). Inflation and unemployment in the long run. *American Economic Review*, **101** (1), 371–398.

Bernanke, B. S. and Gertler, M. (2001). Should central banks respond to movements in asset prices? *American Economic Review*, **91** (2), 253–257.

Bordo, M. and Jeanne, O. (2002). Monetary policy and asset prices. *International Finance*, **5**, 139–164.

Branch, W. A. and Silva, M. (2022). Liquidity, unemployment, and the stock market. Mimeo.

Diamond, P. A. (1982). Aggregate demand management in search equilibrium. *Journal of Political Economy*, **90** (5), 881–894.

Fisher, I. (1936). 100% money and the public debt. *Economic Forum*, (Spring Numer, April-June 1936), 406–420.

Geromichalos, A. and Herrenbrueck, L. (2016). Monetary policy, asset prices, and liquidity in over-the-counter markets. *Journal of Money, Credit and Banking*, **48** (1), 35–79.

— and — (2017). A tractable model of indirect asset liquidity. *Journal of Economic Theory*, **168**, 252–260.

—, Jung, K. M., Lee, S. and Carlos, D. (2021). A model of endogenous direct and indirect asset liquidity. *European Economic Review*, **132**, 103627.

—, Licari, J. M. and Suárez-Lledó, J. (2007). Monetary policy and asset prices. *Review of Economic Dynamics*, **10** (4), 761–779.

Gorton, G. and Ordoñez, G. (2014). Collateral crises. *American Economic Review*, **104** (2), 343–378.

Greenspan, A. (2007). *The Age of Turbulence: Adventures in a New World*. London, England: Penguin Books.

Gu, C., Mattesini, F., Monnet, C. and Wright, R. (2013). Banking: A new monetarist approach. *The Review of Economic Studies*, **80** (2), 636–662.

—, Monnet, C., Nosal, E. and Wright, R. (2020). *On the instability of banking and other financial intermediation*. BIS Working Papers 862, Bank for International Settlements.

— and Wright, R. (2016). Monetary mechanisms. *Journal of Economic Theory*, **163**, 644–657.

Guerrieri, V. and Lorenzoni, G. (2009). Liquidity and trading dynamics. *Econometrica*, **77** (6), 1751–1790.

Howitt, P. and McAfee, R. P. (1987). Costly search and recruiting. *International Economic Review*, **28** (1), 89–107.

— and — (1992). Animal spirits. *American Economic Review*, **82** (3), 493–507.

Kalai, E. (1977). Proportional solutions to bargaining situations: Interpersonal utility comparisons. *Econometrica*, **45** (7), 1623–1630.

Kaplan, G. and Menzio, G. (2016). Shopping externalities and self-fulfilling unemployment fluctuations. *Journal of Political Economy*, **124** (3), 771–825.

Lagos, R. (2010). Asset prices, liquidity, and monetary policy in the search theory of money. *Federal Reserve Bank of Minneapolis Quarterly Review*, **33** (1), 14–20.

— and Rocheteau, G. (2008). Money and capital as competing media of exchange. *Journal of Economic Theory*, **142** (1), 247–258.

—, — and Wright, R. (2017). Liquidity: A new monetarist perspective. *Journal of Economic Literature*, **55** (2), 371–440.

— and Wright, R. (2005). A unified framework for monetary theory and policy analysis. *Journal of Political Economy*, **113** (3), 463–484.

Lettau, M. and Madhavan, A. (2018). Exchange-traded funds 101 for economists. *Journal of Economic Perspectives*, **32** (1), 135–154.

Lucas, R. E. (1978). Asset prices in an exchange economy. *Econometrica*, **46** (6), 1429–1445.

McCallum, B. T. (1985). On consequences and criticisms of monetary targeting. *Journal of Money, Credit and Banking*, **17** (4), 570–597.

Mortensen, D. T. and Pissarides, C. A. (1994). Job creation and job destruction in the theory of unemployment. *The Review of Economic Studies*, **61** (3), 397–415.

Nash, J. F. (1950). The bargaining problem. *Econometrica*, **18** (2), 155–162.

Nosal, E. and Rocheteau, G. (2011). *Money, Payments, and Liquidity.* The MIT Press.

Pagano, M., Sánchez Serrano, A. and Zechner, J. (2019). *Can ETFs contribute to systemic risk?* Report of the Advisory Scientific Committee 9, European Systemic Risk Board.

Peck, J. and Shell, K. (2003). Equilibrium bank runs. *Journal of Political Economy*, **111** (1), 103–123.

Pissarides, C. A. (1984). Search intensity, job advertising, and efficiency. *Journal of Labor Economics*, **2** (1), 128–143.

Reis, R. (2015). *Different Types of Central Bank Insolvency and the Central Role of Seignorage.* NBER Working Papers 21226, National Bureau of Economic Research, Inc.

Rocheteau, G., Hu, T.-W., Lebeau, L. and In, Y. (2021). Gradual bargaining in decentralized asset markets. *Review of Economic Dynamics*, **42**, 72–109.

— and Wright, R. (2005). Money in search equilibrium, in competitive equilibrium, and in competitive search equilibrium. *Econometrica*, **73** (1), 175–202.

— and — (2013). Liquidity and asset-market dynamics. *Journal of Monetary Economics*, **60** (2), 275–294.

Roubini, N. (2006). Why central banks should burst bubbles. *International Finance*, **9** (1), 87–107.

Rubinstein, A. and Wolinsky, A. (1987). Middlemen. *The Quarterly Journal of Economics*, **102** (3), 581–593.

Sargent, T. J. and Wallace, N. (1981). Some unpleasant monetarist arithmetic. *Quarterly Review*, **5** (3), 1–17.

Schwartz, A. (2003). Asset price inflation and monetary policy. *Atlantic Economic Journal*, **31** (1), 1–14.

Smets, F. (1997). *Financial asset prices and monetary policy: theory and evidence.* BIS Working Papers 47, Bank for International Settlements.

Tanaka, A. (2021). Central bank capital and credibility: A literature survey. *Compar-*

ative Economic Studies, **63** (2), 249–262.

White, W. R. and Borio, C. E. V. (2004). *Whither monetary and financial stability? the implications of evolving policy regimes*. BIS Working Papers 147, Bank for International Settlements.

Woodford, M. (2012). *Inflation Targeting and Financial Stability*. NBER Working Papers 17967, National Bureau of Economic Research, Inc.