Bilateral Trade with Costly Information Acquisition*

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Abstract. We study a bilateral trade problem with flexible but costly information acquisition. There is a buyer and a seller who can trade a single unit of a good through an intermediary who designs a mechanism to facilitate their trade. In the beginning, the buyer, the seller and the intermediary share a common prior over a finite set of states of the world. The intermediary proposes a mechanism to the players, who can then acquire information about the true state by privately designing a signal device. Assuming that the information acquisition cost is proportional to the expected reduction in entropy, we characterize the set of implementable allocations. Using the implementability conditions, given by a finite-dimensional system of equations and inequalities, we maximize the intermediary's revenue over all implementable allocationally efficient mechanisms. Under certain symmetry conditions, our revenue maximization problem can be solved in closed form.

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1 Introduction

Traders dealing with complex objects often do not have enough relevant information to correctly estimate the object's value at the outset, and therefore may take potentially costly actions to acquire more information. Consider for example a landowner (a seller) who owns a plot of land which is known to likely have a commercially viable amount of oil under its surface. Suppose this landowner is not in the oil business and is thus considering selling the mineral rights to an oil company (a buyer). At the beginning neither party has a precise estimate of the amount of oil under the surface, but each party could order exploratory drilling to obtain better estimates. The outcomes of the two exploratory drilling studies could be more or less correlated depending on how much coordination between studies the landowner and the oil company achieve. It is possible that the parties decide to order a single study together, in which case the outcomes will be perfectly correlated, or two independent studies in different locations, in which case the outcomes can conceivably be independent conditional on the amount of oil under the surface.

We are interested in the problem of a third party who intermediates trade between the seller and the buyer, and is interested in maximizing her own revenue. The parties communicate with each other via the intermediary who determines the communication protocol and the resulting allocation and payments. The possibility of information acquisition by the parties presents a considerable challenge for the intermediary: in our example the landowner and the oil company may hide some aspects of their exploratory studies from the intermediary (and each other), and thus have to be incentivized to disclose what studies have been performed.

In order to understand the problem of such an intermediary, we build a model with two players: a buyer and a seller who can trade an indivisible object, and a mechanism designer who intermediates trade between them. The object's quality (*payoff-relevant state*) determines its value for the players. We assume that, conditional on knowing the true quality, the players would always like to trade. Our designer is interested in money and values the object at zero irrespective of the quality.

At the outset, neither the players nor the mechanism designer have any information about the object's quality beyond a commonly known prior. In the beginning, the mechanism designer commits to a *mechanism* which consists of messages to be sent by the players later and the allocation and payment functions defined on the messages. Once the mechanism designer has selected a mechanism, the players simultaneously generate *signals* to acquire more information about the quality of the object. To model the information acquisition process, we assume that there is a probability space of fundamental states of nature and that every random variable in the model is a measurable mapping form the sample space of fundamental states to another measurable space (e.g. the object's quality is a random variable that maps the fundamental states to the space of possible qualities). Before the game starts, nature draws a fundamental state but nobody observes it. A player's *signal* is a pair consisting of a finite space of possible *signal realizations* and a random variable that maps the fundamental states to the signal realizations. The signals generated by the players are *costly*. The cost of a signal is proportional to the expected reduction of entropy achieved by the player generating the signal (i.e. as in rational inattention, see Sims (2003) and Matejka and McKay (2015)). Information acquisition is thus costly but *flexible*, allowing for arbitrary correlations across signals and the object's quality. It is also *hidden* as neither player observes the signal chosen by the other player and the intermediary does not observe the signal chosen by either player.

After the players have chosen their signals, they *privately* observe the *signal realization* corresponding to the fundamental state chosen by nature. Having observed their signal realization, they select a message to report to the designer who then announces the allocations and payments. The quality of the object corresponding to the fundamental state is then revealed and the players' payoffs are determined. The players are interested in maximizing their payoffs net of information acquisition costs.

We consider *Nash equilibria* in pure strategies[1] of the resulting mechanisms and, in the case of multiplicity, select an equilibrium that maximizes the mechanism designer's revenue. One might wonder whether choosing an equilibrium concept that takes into account the dynamic nature of our environment (e.g. perfect Bayesian equilibrium) would change our results, but, fortunately, it is not the case. Intuitively, since players only observe their own signal realizations, they obtain no information on

[1]Whether the restriction to pure strategies is without loss of generality or not is an open problem that appears to be non-trivial. Appendix D discusses this issue in more detail.

the signal chosen by the other player. Hence, an off-equilibrium information set can only be achieved following a player's own deviation, which makes sure that every Nash equilibrium has an outcome-equivalent perfect Bayesian equilibrium (Proposition 2 proven in Appendix C formalizes this argument).

We establish a *revelation principle* (Proposition 1), which allows us to restrict attention to truthful-revelation equilibria of *direct mechanisms*. Direct mechanisms ask the players to report one of the signal realizations from the support of their equilibrium signal. Signals chosen by the players induce a joint distribution over object's qualities and signal realizations (*an information structure*) whose marginal on the set of qualities is equal to the prior. Moreover, we notice that any such information structure can be induced by a pair of signals (see Lemmas 1 and 2, which directly follow form Theorem 1 in Yang (2020)). This equivalence of signals and information structures allows us to state our implementability conditions in terms of information structures, which considerably simplifies the problem.

Simplifying the problem further, we show that we can consider a restricted class of deviations for each player without loss of generality (Lemma 4). Intuitively, if along the equilibrium path of a truthful-revelation Nash equilibrium a player has n signal realizations, then he only has $n+1$ available reporting deviations: he can choose to report a different signal realization or abstain from participation altogether. With only $n+1$ available actions the player would never want to deviate to an alternative signal with more than $n+1$ signal realizations since additional information would be learned but essentially wasted otherwise. Thus, when we want to check if a particular information structure with a given number of signal realizations can be induced in a truthful-revelation Nash equilibrium of a direct mechanism, it is enough to consider information structures which have one additional signal realization for each player.

Considering this restricted class of deviations, we derive implementability conditions. In order to show that a given information structure paired with truthful reporting is implementable, we divide the restricted class of deviations into two subclasses. In the first subclass, we consider deviation-induced information structures that preserve the set of signal realizations for each player. To show that the deviations in the first subclass are unprofitable, we solve a finite-dimensional payoff-maximization problem for each player, where the maximum is taken over information structures which preserve the set of signal realizations for that player. In the second subclass, we consider deviation-induced information structures augmented with an additional signal realization. We explicitly solve for the best deviation in this class (Lemma 6) and derive an unprofitability condition. Combining the unprofitability conditions from the two subclasses, we obtain our implementability conditions (Proposition 3).

We then turn to our application and consider the problem of an intermediary who seeks to implement allocationally efficient trade while simulataneously maximizing her own revenue (or, equivalently, minimizing the subsidy required to implement allocationally efficient trade). To simplify our analysis, we assume that the set of qualities is binary. One might think that, to maximize revenue, the intermediary must incentivize the players to learn uncorrelated information, as it would require a smaller subsidy for a larger amount of acquired information. The intermediary is however restricted by the players' incentive compatibilty constraints as the players might prefer to strategically acquire more correlated information. Surprisingly, Lemma 7 shows that incentive compatibility forces the intermediary to pay for information acquisition *twice*: allocationally efficient trade is implementable only if the players acquire *perfectly correlated* information. To maximize revenue among mechanisms with perfectly correlated information acquisition, we adapt the concavification approach commonly used to solve Bayesian persuasion problems (see Kamenica and Gentzkow (2011)) to our setting. Proposition 4 uses a concavification argument to derive optimality conditions for the distribution of posteriors that is induced by the revenue-maximizing mechanism. As we show in Subsection 4.3, this distribution of posteriors (and thus a full characterization of the revenue-maximizing mechanism) can be obtained in closed form under certain symmetry conditions.

1.1 Related literature

The literature on information acquisition in mechanism design goes back to Bergemann and Välimäki (2002) who study efficient implementation in transferable environments with exogenously restricted information acquisition. They show that the VCG mechanism achieves both *ex ante* and *ex post* efficiency if agents have private values, but not necessarily when they have common values. Bikhchandani (2010) points out that the full surplus extraction mechanism of Crémer and McLean (1988) may not be robust to information acquisition because agents presented with a Crémer-McLean lottery may have incentives to acquire additional information about their competitors' valuations. Bikhchandani and Obara (2017) study a mechanism design problem, in which (similarly to our paper) agents can acquire costly signals about a payoff-relevant state of nature. The space of signals available to each agent is, however, exogenously restricted. Bikhchandani and Obara (2017) provide conditions under which full surplus extraction is possible in their setting.

More recently, some consideration has been given to flexible information acquisition. In a paper closely related to ours, Mensch (2022) solves for a revenue-maximizing auction among buyers who, like the players in our paper, can acquire costly and hidden information about the value of an object sold in an auction. The cost of information acquisition belongs to the posterior-separable class, which contains, among others, the entropy cost we use in our paper. Unlike in our paper, however, the agents in Mensch (2022) have private values and are exogenously restricted to acquire information about their own values. Terstiege and Wasser (2022) solve for a revenue-maximizing auction with private values and flexible information acquisition but assume that information acquistion is *costless* and *public*. In their environment, the bidders' choice of signals is publicly revealed before the bidders privately observe a signal realization. Having observed the signals chosen by the bidders, the seller proposes a mechanism to maximize her revenue. Gleyze and Pernoud (2023) study a mechanism design problem with costly flexible information acquisition, transferable utility, and private values but allow the agents to acquire information both on their own preferences and the preferences of the other agents. Gleyze and Pernoud (2023) are interested in *informationally simple* mechanisms, i.e. those in which the agents have no incentive to acquire information about anyone's preferences but their own.

In our paper, the players choose an information structure to maximize their own payoffs, hence their choice may not necessarily be desirable from the designer's perspective. This feature of our model makes our paper somewhat close to the literature on "adverse" choice of information structures. Yamashita (2018) studies a private-value auction, in which for any mechanism proposed by the seller, nature chooses an information structure that minimizes the seller's revenue. Bergemann et al. (2017)

and Brooks and Du (2021) study analogous models of common-value auctions. Roesler and Szentes (2017) study a bilateral trade model, in which the buyer can acquire costless information about the good's value and the seller sets revenue-maximizing take-it-or-leave-it price in response. Ravid et al. (2022) consider the same setting as Roesler and Szentes (2017) but make the buyer's information acquisition costly.

We model information acquisition by giving the players access to a large space of signals which partition an underlying set of fundamental states. This way of modeling signals is introduced by Green and Stokey (1978) and is also used by Gentzkow and Kamenica (2017) in the context of Bayesian persuasion with multiple senders.

2 Model

2.1 Setup

An indivisible good, whose quality v is drawn from a finite set of payoff-relevant states of the world V, can be traded between two players: a seller and a buyer. The buyer's valuation for the good of quality $v \in V$ is given by $u^b(v)$, the seller's valuation for the good of quality $v \in V$ is given by $u^s(v)$. We assume that gains from trade always exist, i.e. $u^b(v) > u^s(v)$ for any v. To model information acquisition, we assume that there is a set of fundamental states of the world $x \in X = [0, 1]$ with an associated Borel σ-algebra $\mathcal{F}$ and the uniform measure $\mathbb{P}$, and a random variable $\mathbf{V} : X \to V$. At the beginning of the game, this structure is commonly known. $\mathbf{V}$ induces a common prior μ_0 on the set of qualities such that the probability of quality being equal to v is $\mu_0(v) \equiv \int_0^1 \mathbf{1}_{\{\mathbf{V}(x)=v\}} dx$. We assume that μ_0 has full support on V.

The players can acquire costly information about the good's quality by generating *signals*. We assume that each player p has access to a countably infinite set of possible *signal realizations*. Since the labels of signal realizations do not have any particular meaning in our setup, we assume that the set of signal realizations is the set of all natural numbers $\mathbb{N} \equiv \{1, 2, 3, \dots\}$. We use $\mathscr{P}(\mathbb{N})$ to denote the collection of all *finite non-empty* subsets of $\mathbb{N}$. A *signal* is a pair $\sigma^p = (S^p, \mathbf{S}^p)$, where $S^p \in \mathscr{P}(\mathbb{N})$ and $\mathbf{S}^p$ is a random variable that maps fundamental states to signal realizations in S^p, i.e. $\mathbf{S}^p : X \to S^p$. If the fundamental state is x, then player p observes the signal realization $s^p = \mathbf{S}^p(x)$. We use Σ^p to denote the set of all signals for player p. Signals are costly, the cost of a signal σ^p, denoted by $C(\sigma^p)$, is proportional to the reduction in entropy achieved by that signal. We introduce the cost function formally below. The players maximize their utilities net of information acquisition costs.

There is a mechanism designer who intermediates trade between the seller and the buyer. The designer commits to a *mechanism* at the *ex ante* stage. A mechanism is a tuple $\big(M, q, t\big)$, where $M = M^b \times M^s$ with M^p being a finite set of messages sent by player p to the mechanism designer. q is a tuple of allocation functions $\big(q^b, q^s\big)$, where $q^p : M \to [0, 1]$ determines the allocation for player p. t is a tuple of payment functions $\big(t^b, t^s\big)$, where $t^b : M \to \mathbb{R}$ is a payment made by the buyer to the mechanism designer, and $t^s : M \to \mathbb{R}$ is a payment made by the mechanism designer to the seller. The mechanism designer is interested in maximizing her revenue.

To summarize, the timing of the interaction is as follows:

1. Nature draws $x \in X$ uniformly, but nobody observes it.
2. The mechanism designer commits to a mechanism $\big(M, q, t\big)$.
3. Each player p privately chooses $\sigma^p = \big(S^p, \mathbf{S}^p\big)$.
4. Each player p privately observes $s^p = \mathbf{S}^p(x)$.
5. Each player privately sends $m^p \in M^p$ to the mechanism designer.
6. Allocations and transfers are determined according to (q, t).

The buyer gets $q^b(m)u^b(v) - t^b(m) - C(\sigma^b)$, the seller gets $t^s(m) - q^s(m)u^s(v) - C(\sigma^s)$, and the designer gets $t^b(m) - t^s(m)$, where $v = \mathbf{V}(x)$ and $m = (m^b, m^s)$.

We choose Nash equilibrium in pure strategies as our solution concept but this choice has little effect on our analysis as we show in Proposition 2 that any Nash equilibrium has an outcome-equivalent perfect Bayesian equilibrium (whether the restriction to pure strategies leads to loss of generality or not is an open problem; we elaborate on this issue in Appendix D). If there are multiple Nash equilibria, we select one that maximizes the mechanism designer's revenue. Naturally, we assume that the mechanism designer is restricted to choose mechanisms that satisfy physical feasibility, i.e. $0 \leq q^b(m) \leq q^s(m) \leq 1$ for any $m \in M$, and allow for voluntary participation both *ex ante* and at the *interim* stage, i.e. we assume that there exists a message $m_\emptyset \in M^p$ for any player p, such that $q^p(m_\emptyset, m^{-p}) = t^p(m_\emptyset, m^{-p}) = 0$ for any message $m^{-p} \in M^{-p}$ sent by by the other player.

2.2 Information structures

A signal σ^p chosen by player p induces a joint distribution on $S^p \times V$. We use α^p to denote this joint distribution and write $\alpha^p(s^p; v)$ for the probability of player p observing the signal realization s^p and the state of the world being v. When we want to emphasize the dependence of α^p on σ^p, we write $\alpha^p[\sigma^p]$. A pair of signals $\left(\sigma^b, \sigma^s\right)$ induces a joint distribution on $S^b \times S^s \times V$. We use α to denote this joint distribution, and use $\alpha(s^b, s^s; v)$ to denote the joint probability of the buyer observing the signal realization s^b, the seller observing the signal realization s^s, and state of the world being v. When we want to emphasize the dependence of α on $\left(\sigma^b, \sigma^s\right)$, we write $\alpha[\sigma^b, \sigma^s]$. Clearly, we have $\text{marg}_{S^p \times V}\alpha = \alpha^p$ for any player p. In what follows, we refer to α as *information structure*. Following Kamenica and Gentzkow (2011), we introduce the following definition:

Definition 1 (Bayes-plausibility). *α is Bayes-plausible if* $\text{marg}_V \alpha = \mu_0$.

Any information structure induced by a pair of signals must be Bayes-plausible. The following lemma shows that the converse is also true:

Lemma 1. *For any Bayes-plausible α there exists a profile of signals that induces it.*

Now suppose that player p deviates to an alternative signal $\tilde{\sigma}^p$. Which alternative information structures $\tilde{\alpha}$ can this deviation induce? Clearly, we must have $\text{marg}_{S^{-p} \times V}\ \tilde{\alpha} = \text{marg}_{S^{-p} \times V}\ \alpha$, i.e. a deviation by player p cannot change the joint distribution of player $-p$'s signal realizations and states of the world. The following lemma shows that the converse is also true:

Lemma 2. *Fix a signal profile $\left(\sigma^p, \sigma^{-p}\right)$ and the associated information structure α. Consider any joint distribution $\tilde{\alpha}$ on $\tilde{S}^p \times S^{-p} \times V$ such that* $\text{marg}_{\tilde{S}^{-p} \times V}\ \tilde{\alpha} = \text{marg}_{S^{-p} \times V}\ \alpha$*. There exists $\tilde{\sigma}^p \in \Sigma^p$ such that $\left(\tilde{\sigma}^p, \sigma^{-p}\right)$ induces $\tilde{\alpha}$.*

Lemmas 1 and 2 both follow immediately from Theorem 1 in Yang (2020), hence their proofs are omitted. Lemmas 1 and 2 will allow us to rewrite the mechanism designer's problem in terms of information structures, and thus avoid having to explicitly model players' signal choices. We return to this issue below in Subsection 2.4 after we discuss the cost of information acquisition.

2.3 Cost of information acquisition

Consider a signal σ^p chosen by player p and the distribution $\alpha^p[\sigma^p]$ induced by that signal. Having chosen σ^p, player p observes signal realization s^p with probability $\tau[\sigma^p](s^p) \equiv \sum_{v\in V} \alpha[\sigma^p](s^p; v)$. If $\tau[\sigma^p](s^p) > 0$, then the signal realization s^p induces a posterior distribution over states of the world $\mu^p[\sigma^p](s^p)$. The posterior probability of state v is $\mu^p[\sigma^p](s^p; v) \equiv \frac{\alpha^p[\sigma^p](s^p;v)}{\tau[\sigma^p](s^p)}$. The cost of signal σ^p is proportional to the expected reduction in entropy achieved by $\alpha^p[\sigma^p]$:

$$C(\sigma^p) = c(\alpha^p[\sigma^p]) \equiv \chi\Bigg(H(\mu_0) - \sum_{s^p|\tau[\sigma^p](s^p)>0} \tau[\sigma^p](s^p)H\big(\mu^p[\sigma^p](s^p)\big)\Bigg),$$

where $H(\mu) = -\sum_{v\in V} \mu(v)\log\big(\mu(v)\big)$ with the standard convention $0\log 0 = 0$.

In what follows, we normalize χ to 1. Whenever we consider a pair of signals $\big(\sigma^p, \sigma^{-p}\big)$ inducing a joint distribution α, we find it convenient to work with a cost function defined directly on information structures as follows: $c^p(\alpha) \equiv c(\text{marg}_{S^p\times V}\alpha)$. Clearly, we have $C(\sigma^p) = c^p(\alpha[\sigma^p, \sigma^{-p}])$ for each player p. The following lemma, whose proof is relegated to Appendix A, will be helpful later in our analysis:

Lemma 3. *$c^p(\alpha)$ is convex in α for any player $p \in \{b, s\}$.*

2.4 Strategies, equilibria, and direct mechanisms

Consider a mechanism $\big(M, q, t\big)$. A strategy is a tuple $\big(\sigma^p, \{\mathbf{m}^p[\hat\sigma^p]\}_{\hat\sigma^p\in\Sigma^p}\big)$, where σ^p is a signal chosen by player p on path, and $\{\mathbf{m}^p[\hat\sigma^p]\}_{\hat\sigma^p\in\Sigma^p}$ is a family of reporting functions, one for each $\hat\sigma^p \in \Sigma^p$, mapping signal realizations from $\hat\sigma^p$ to the mechanism's messages, i.e. $\mathbf{m}^p[\hat\sigma^p] : \hat S^p \to M^p$ for each $\hat\sigma^p \in \Sigma^p$. As player p chooses σ^p at the information acquisition stage, his on-path reports are given by $\mathbf{m}^p[\sigma^p]$, i.e. when player p observes a signal realization $s^p \in S^p$, then he sends the message $\mathbf{m}^p[\sigma^p](s^p)$. In what follows, we omit the dependence of the on-path reports on σ^p and simply write $\mathbf{m}^p(s^p)$ for the message sent by player p who has observed s^p.

We focus our attention on *direct mechanisms*. In a direct mechanism, each player p chooses a signal $\sigma^p = (\mathbf{S}^p, S^p)$, and the mechanism designer asks the players to report their signal realizations or send the abstention message, thus $M^p = S^p \cup \{m_\emptyset\}$ for each player p. Let α be the information structure induced by a pair of signals $\big(\sigma^b, \sigma^s\big)$ and let $\mathbf{m}^p_T$ be the truthful reporting function for player p, i.e. for any

$s^p \in S^p$ we have $\mathbf{m}^p_T(s^p) = s^p$, and consider truthful-revelation Nash equilibria in direct mechanisms. Using Lemmas 1 and 2, we can write the truthful-revelation Nash equilibrium conditions in a direct mechanism in terms of α and $\mathbf{m}^p_T$. α and $\mathbf{m}^p_T$ can arise in a truthful-revelation Nash equilibrium of a direct mechanism if and only if α Bayes-plausible (BP) and

- They are *ex ante* incentive compatible for the buyer[2]:

$$(\mathrm{IC}^b_{\mathrm{A}}) \quad (S^b, \alpha, \mathbf{m}^b_T) \in \underset{\tilde{\alpha}, \tilde{S}^b, \tilde{\mathbf{m}}^b}{\operatorname{argmax}} \sum_{s^b \in \tilde{S}^b} \sum_{s^s \in S^s} \sum_{v \in V} \tilde{\alpha}(s^b, s^s; v)\big(q^b(\tilde{\mathbf{m}}^b(s^b), s^s)u^b(v) - t^b(\tilde{\mathbf{m}}^b(s^b), s^s)\big) - c^b(\tilde{\alpha}),$$

$$\text{s.t.} \quad (1)\ \tilde{S}^b \in \mathscr{P}(\mathbb{N}), \ \ \tilde{\alpha} \in \Delta\big(\tilde{S}^b \times S^s \times V\big), \ \ \tilde{\mathbf{m}}^b : \tilde{S}^b \to S^b \cup \{m_\emptyset\};$$

$$(2)\ \operatorname{marg}_{S^s \times V} \tilde{\alpha} = \operatorname{marg}_{S^s \times V} \alpha.$$

- They are *ex ante* incentive compatible for the seller[2]:

$$(\mathrm{IC}^s_{\mathrm{A}}) \quad (S^s, \alpha, \mathbf{m}^s_T) \in \underset{\tilde{\alpha}, \tilde{S}^s, \tilde{\mathbf{m}}^s}{\operatorname{argmax}} \sum_{s^b \in S^b} \sum_{s^s \in \tilde{S}^s} \sum_{v \in V} \tilde{\alpha}(s^b, s^s; v)\big(t^s(s^b, \tilde{\mathbf{m}}^s(s^s)) - q^s(s^b, \tilde{\mathbf{m}}^s(s^s))u^b(v)\big) - c^s(\tilde{\alpha}),$$

$$\text{s.t.} \quad (1)\ \tilde{S}^s \in \mathscr{P}(\mathbb{N}), \ \ \tilde{\alpha} \in \Delta\big(S^b \times \tilde{S}^s \times V\big), \ \ \tilde{\mathbf{m}}^s : \tilde{S}^s \to S^s \cup \{m_\emptyset\};$$

$$(2)\ \operatorname{marg}_{S^b \times V} \tilde{\alpha} = \operatorname{marg}_{S^b \times V} \alpha.$$

We now show that our focus on direct mechanisms is without loss of generality:

Proposition 1 (**Revelation principle**)**.** *For any Nash equilibrium of an indirect mechanism there exists an outcome-equivalent truthful-revelation Nash equilibrium of a direct mechanism.*

The proof of the Revelation principle is relegated to Appendix B.

One could argue that we should have chosen perfect Bayesian equilibrium as our solution concept since our environment has dynamic structure. The following proposition shows that the two equilibrium concepts are outcome-equivalent in our setting.

Proposition 2. *Every truthful-revelation Nash equilibrium of a direct mechanism has an outcome-equivalent perfect Bayesian equilibrium in this direct mechanism.*

[2]Observe that this formulation of *ex ante* incentive compatibility takes care of *ex ante* individual rationality as well. Consider e.g. a deviation for the buyer $(\tilde{S}^b, \tilde{\alpha}, \tilde{\mathbf{m}}^b)$, where $\tilde{S}^b = \{1\}$, $\tilde{\alpha}(1, s^s; v) = \sum_{s^b \in S^b} \alpha(s^b, s^s; v)$, and $\tilde{\mathbf{m}}^b(1) = m_\emptyset$. Observe that $\operatorname{marg}_{S^s \times V} \tilde{\alpha} = \operatorname{marg}_{S^s \times V} \alpha$ The payoff from this deviation is $\sum_{s^s \in S^s} \sum_{v \in V} \tilde{\alpha}(1, s^s; v)0 - c^b(\tilde{\alpha}) = -c^b(\tilde{\alpha})$. Bayes-plausibility implies $\alpha^b(1, v) = \sum_{s^s \in S^s} \tilde{\alpha}(1, s^s; v) = \mu_0(v)$, hence $c^b(\tilde{\alpha}) = H(\mu_0) - H(\mu_0) = 0$. By *ex ante* incentive compatibility we have: $\sum_{s^b \in S^b} \sum_{s^s \in S^s} \sum_{v \in V} \alpha(s^b, s^s; v)\big(q^b(s^b, s^s)u^b(v) - t^b(s^b, s^s)\big) - c^b(\alpha) \geq 0$.

The proof of Proposition 2 (presented in Appendix C) is somewhat tedious but the intuition is straightforward. Since the players' signals are chosen simultaneously, player p has no information about the signal chosen by player $-p$. Moreover, the information that player p gets at the signal realization stage does not reveal any information about the signal chosen by player $-p$ either, hence a player can achieve an off-equilbrium information set only by deviating to a different signal himself. Then, if truthful reporting along the equilibrium path is not sequentially rational, it will not be optimal from the *ex ante* perspective either. In other words, if a player suddenly finds it profitable to misreport after observing a signal realization s^p, then he must have contingently planned to misreport following s^p from the start, but then of course truthful revelation could not have been a Nash equilibrium in the first place.

3 Implementability

The above *ex ante* incentive compatibility constraints are rather complicated. They prevent players from deviating to a possibly different information structure and misreporting their signal realizations at the same time. The class of such deviations is extremely large. In this section, we show that it is without loss of generality to consider a much smaller class of *ex ante* deviations.

3.1 Restricted *ex ante* deviations

We first show that it is without loss of generality to restrict attention to those *ex ante* deviations, in which a player augments his information structure with an additional signal realization $s^p_\emptyset$ and chooses a new joint distribution on the augmented signal realization space. The player abstains from participation after observing $s^p_\emptyset$ and reports truthfully otherwise. This idea is captured by *restricted ex ante incentive compatibility constraints.*

- The restricted *ex ante* incentive compatibility constraint for the buyer is:

$$(\text{R-IC}^b_{\text{A}}) \quad (S^b,\alpha) \in \operatorname*{argmax}_{\tilde{\alpha},\tilde{S}^b} \sum_{s^b\in S^b}\sum_{s^s\in S^s}\sum_{v\in V} \tilde{\alpha}(s^b,s^s;v)\big(q^b(s^b,s^s)u^b(v)-t^b(s^b,s^s)\big)-c^b(\tilde{\alpha}),$$

$$\text{s.t.} \quad (1)\ \tilde{S}^b = S^b\cup\{s^b_\emptyset\},\ \ \tilde{\alpha}\in\Delta\big(\tilde{S}^b\times S^s\times V\big);$$

$$(2)\ \operatorname{marg}_{S^s\times V}\tilde{\alpha} = \operatorname{marg}_{S^s\times V}\alpha.$$

- The restricted *ex ante* incentive compatibility constraint for the seller is:

$$(\text{R-IC}^s_\text{A}) \quad (S^s, \alpha) \in \operatorname*{argmax}_{\tilde{\alpha}, \tilde{S}^s} \sum_{s^b \in S^b} \sum_{s^s \in S^s} \sum_{v \in V} \tilde{\alpha}(s^b, s^s; v)\big(t^s(s^b, s^s) - q^s(s^b, s^s)u^s(v)\big) - c^s(\tilde{\alpha}),$$

$$\text{s.t.} \quad (1)\ \tilde{S}^s = S^s \cup \{s^s_\emptyset\},\ \ \tilde{\alpha} \in \Delta\big(S^b \times \tilde{S}^s \times V\big);$$

$$(2)\ \operatorname{marg}_{S^b \times V} \tilde{\alpha} = \operatorname{marg}_{S^b \times V} \alpha.$$

The following lemma, whose proof is relegated to Appendix E, establishes that restricted *ex ante* incentive compatibility implies *ex ante* incentive compatibility.

Lemma 4. $\text{R-IC}^p_\text{A} \Rightarrow \text{IC}^p_\text{A}$ *for both players* $p \in \{b, s\}$.

The argument at the core of Lemma 4's proof is straightforward. In any direct mechanism, a player, whose on-path signal has n possible signal realizations, can choose between $n+1$ possible actions: this player can report one of the signal realizations (possibly misreporting) or abstain from participation altogether. Suppose this player has a profitable unrestricted *ex ante* deviation, i.e. there is a pair consisting of an information structure and a reporting function that gives this player a strictly larger expected payoff. If the information structure in this unrestricted deviation has more than $n + 1$ signal realizations, than at least two signal realizations will lead to the same action. If we scramble all signal realizations leading to the same action, we will obtain an information structure with a one-to-one mapping between signal realizations and actions. Since the labels of signal realizations do not have any specific meaning, we can always relabel them to ensure that those that do not lead to an abstention are reported truthfully. In that way, we can construct a *restricted ex ante* deviation whose information structure is less informative in the Blackwell sense and is therefore less costly. Since the rest of the payoff is exactly the same, the restricted *ex ante* deviation is even more profitable than the unrestricted one.

Suppose now that the mechanism designer hopes that a particular information structure $\alpha \in \Delta\big(S^b \times S^s \times V\big)$ will be induced in the truthful-revelation Nash equilibrium of her direct mechanism $\big((S^b \cup \{m_\emptyset\}) \times (S^s \cup \{m_\emptyset\}), q, t\big)$. If $|S^b| = I$ and $|S^s| = J$, then α is essentially a collection of $I \times J$ matrices, one for each state (we adopt the convention that the buyer is the *row player* and the seller is the *column player*). From now on, let us use $\alpha_{ij}(v)$ to denote the joint probability of the buyer observing the signal realization s_i^b and the seller observing the signal realization s_j^s in state v (we will also use q_{ij}^p and t_{ij}^p to denote the respective allocation and transfer of player p). We also denote $\mu_i^b(v)$ the posterior probability of state v as evaluated by the buyer who receives the signal realization s_i^b, and $\mu_j^s(v)$ the posterior probability of state v as evaluated by the seller who receives the signal realization s_j^s.

Lemma 4 shows that to make sure that her desired α will indeed be induced, the mechanism designer should only check whether (R-IC$_\text{A}^s$) and (R-IC$_\text{A}^s$) constraints are satisfied, i.e. should only check deviations that augment α by not more than one signal realization. To analyze these deviations, we find it useful to split them into two classes. The first class of deviations consists of possibly different joint distributions over the same signal realizations while the deviations in the second class augment the set of the signal realizations by exactly one realization. The usefulness of this approach will become clear by the end of this section. Let us deal with the first class of the restricted *ex ante* deviations first.

3.1.1 Class 1 of restricted *ex ante* deviations

- Class 1-deviations are unprofitable for the buyer as long as α satisfies:

$$\text{(R-IC}_\text{A}^b\text{-1)} \quad \alpha \in \underset{\tilde{\alpha}}{\operatorname{argmax}} \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{v \in V} \tilde{\alpha}_{ij}(v)\big(q_{ij}^b u^b(v) - t_{ij}^b\big) - c^b(\tilde{\alpha}), \quad \text{s.t.}$$

$$(1)\ \ \tilde{\alpha} \in \Delta\big(S^b \times S^s \times V\big);$$

$$(2)\ \ \operatorname{marg}_{S^s \times V}\tilde{\alpha} = \operatorname{marg}_{S^s \times V}\alpha.$$

- Class 1-deviations are unprofitable for the seller as long as α satisfies:

$$\text{(R-IC}_\text{A}^s\text{-1)} \quad \alpha \in \underset{\tilde{\alpha}}{\operatorname{argmax}} \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{v \in V} \tilde{\alpha}_{ij}(v)\big(t_{ij}^s - q_{ij}^s u^s(v)\big) - c^s(\tilde{\alpha}), \quad \text{s.t.}$$

$$(1)\ \ \tilde{\alpha} \in \Delta\big(S^b \times S^s \times V\big);$$

$$(2)\ \ \operatorname{marg}_{S^b \times V}\tilde{\alpha} = \operatorname{marg}_{S^b \times V}\alpha.$$

Notice that (R-IC$_\text{A}^b$-1) and (R-IC$_\text{A}^s$-1) are finite dimensional maximization problems with concave objectives and affine constraints. Moreover, observe that a Bayes-plausible information structure that allows a player to put a posterior probability of zero on any of the states of the world can never be a solution to the maximization problems in (R-IC$_{\text{A}}^{b}$-1) and (R-IC$_{\text{A}}^{s}$-1). This is due to the properties of the entropy cost function, which makes sure that marginal costs of information acquisition go to infinity as soon as any of the posteriors approaches zero. The following lemma, whose proof is relegated to Appendix F, establishes the claim formally:

Lemma 5 (Strictly positive posteriors). *Suppose α is Bayes-plausible, and satisfies* (R-IC$_{\text{A}}^{p}$-1) *for both players $p \in \{b, s\}$, then for any $v \in V$ and for any $s_i^b \in S^b$ we have $\mu_i^b(v) > 0$, likewise for any $v \in V$ and for any $s_j^s \in S^s$ we have $\mu_j^s(v) > 0$.*

Lemma 5 makes sure that the objective functions in (R-IC$_{\text{A}}^{b}$-1) and (R-IC$_{\text{A}}^{s}$-1) are differentiable at any optimum, hence all deviations in the first class are unprofitable iff α satisfies the Karush-Kuhn-Tucker optimality conditions in both problems.

3.1.2 Class 2 of restricted *ex ante* deviations

- Class 2-deviations are unprofitable for the buyer as long as α satisfies:

$$\text{(R-IC}_{\text{A}}^{b}\text{-2)} \quad \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{v\in V}\alpha_{ij}(v)\big(q_{ij}^{b}u^{b}(v)-t_{ij}^{b}\big)-c^{b}(\alpha) \geq \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{v\in V}\tilde{\alpha}_{ij}(v)\big(q_{ij}^{b}u^{b}(v)-t_{ij}^{b}\big)-c^{b}(\tilde{\alpha}),$$

for all $\tilde{\alpha}$ such that:

(1) $\tilde{\alpha} \in \Delta\big(\tilde{S}^b \times S^s \times V\big)$, where $\tilde{S}^b = S^b \cup \{s_\emptyset^b\}$ and $\exists s_j^s \in S^s, v \in V$ s.t. $\alpha_{\emptyset,j}(v) > 0$;

(2) $\text{marg}_{S^s\times V}\tilde{\alpha} = \text{marg}_{S^s\times V}\alpha$.

- Class 2-deviations are unprofitable for the seller as long as α satisfies:

$$\text{(R-IC}_{\text{A}}^{s}\text{-2)} \quad \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{v\in V}\alpha_{ij}(v)\big(t_{ij}^{s}-q_{ij}^{s}u^{s}(v)\big)-c^{s}(\alpha) \geq \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{v\in V}\tilde{\alpha}_{ij}(v)\big(t_{ij}^{s}-q_{ij}^{s}u^{s}(v)\big)-c^{s}(\tilde{\alpha})$$

for all $\tilde{\alpha}$ such that:

(1) $\tilde{\alpha} \in \Delta\big(S^b \times \tilde{S}^s \times V\big)$, where $\tilde{S}^s = S^s \cup \{s_\emptyset^s\}$ and $\exists s_i^b \in S^b, v \in V$ s.t. $\alpha_{i,\emptyset}(v) > 0$;

(2) $\text{marg}_{S^b\times V}\tilde{\alpha} = \text{marg}_{S^b\times V}\alpha$.

The usefulness of splitting the deviations into these two classes is illustrated by the next lemma, which shows that if α satisfies (R-IC$_{\text{A}}^{b}$-1) and (R-IC$_{\text{A}}^{s}$-1), then (R-IC$_{\text{A}}^{b}$-2) and (R-IC$_{\text{A}}^{s}$-2) can be considerably simplified.

Lemma 6. *Suppose α satisfies* R-IC$_{\text{A}}^{p}$-1 *for both players $p \in \{b, s\}$, then*

- *α satisfies* R-IC$_{\text{A}}^{b}$-2 *if and only if $\sum_{v\in V}\exp\big(-y^b(v)\big) \leq 1$, where*

$$y^b(v) \equiv \min_{(i,j)|\alpha_{ij}(v)>0}\Big\{q_{ij}^{b}u^{b}(v)-t_{ij}^{b}-\log\big(\mu_i^b(v)\big)\Big\};$$

- α *satisfies* R-IC^s_A-2 *if and only if* $\sum_{v \in V} \exp\big(-y^s(v)\big) \leq 1$, *where*

$$y^s(v) \equiv \min_{(i,j)|\alpha_{ij}(v)>0} \left\{ t^s_{ij} - q^s_{ij} u^s(v) - \log\big(\mu^s_j(v)\big) \right\}.$$

The full proof of Lemma 6 is relegated to Appendix G, but we illustrate the main ideas of the proof below using a simple example with two states and two signal realizations for each player. Suppose there are indeed two payoff-relevant states of the world, i.e. $V = \{\underline{v}, \overline{v}\}$, and we would like to find out whether the following information structure (denoted α) satisfies (R-IC^b_A-2) for a given mechanism assuming that it satisfies (R-IC^b_A-1) for the same mechanism.

State $\underline{v}$	s^s_1	s^s_2
s^b_1	$\underline{\alpha}_{11}$	$\underline{\alpha}_{12}$
s^b_2	$\underline{\alpha}_{21}$	$\underline{\alpha}_{22}$

State $\underline{v}$	s^s_1	s^s_2
s^b_1	$\overline{\alpha}_{11}$	$\overline{\alpha}_{12}$
s^b_2	$\overline{\alpha}_{21}$	$\overline{\alpha}_{22}$

Let us suppose that it does not actually satisfy the constraint (R-IC^b_A-2), then we must be able to find a profitable deviation, which induces a different information structure, which transfers some probability mass from the existing signal realizations to $s^b_\emptyset$, after which the buyer abstains. For some $\epsilon > 0$ we can write down the information structure induced by this deviation as follows:

State $\underline{v}$	s^s_1	s^s_2
s^b_1	$\underline{\alpha}_{11} - \epsilon\underline{\beta}_{11}$	$\underline{\alpha}_{12} - \epsilon\underline{\beta}_{12}$
s^b_2	$\underline{\alpha}_{21} - \epsilon\underline{\beta}_{21}$	$\underline{\alpha}_{22} - \epsilon\underline{\beta}_{22}$
$s^b_\emptyset$	$\epsilon\sum_{i=1}^2 \underline{\beta}_{i1}$	$\epsilon\sum_{i=1}^2 \underline{\beta}_{i2}$

State $\overline{v}$	s^s_1	s^s_2
s^b_1	$\overline{\alpha}_{11} - \epsilon\overline{\beta}_{11}$	$\overline{\alpha}_{12} - \epsilon\overline{\beta}_{12}$
s^b_2	$\overline{\alpha}_{21} - \epsilon\overline{\beta}_{21}$	$\overline{\alpha}_{22} - \epsilon\overline{\beta}_{22}$
$s^b_\emptyset$	$\epsilon\sum_{i=1}^2 \overline{\beta}_{i1}$	$\epsilon\sum_{i=1}^2 \overline{\beta}_{i2}$

We denote the gain from this deviation $G_\alpha(\epsilon\beta)$. By assumption, $G_\alpha(\epsilon\beta) > 0$ for some $\epsilon > 0$. First of all, we notice that the payoff function of the buyer is concave and hence for any global profitable deviation there is a local deviation with a marginal gain $MG_\alpha(\beta) \equiv \lim_{\epsilon\to 0} \frac{1}{\epsilon} G_\alpha(\epsilon\beta)$ that is also strictly positive. Moreover, once we consider local deviations, we can without loss of generality take all β_{ij}'s to be weakly positive because, locally, any direction of improvement of α can be represented as a linear combination of a direction that is feasible in (R-IC^b_A-1) and another direction, in which all β_{ij}'s are weakly positive. Since α solves (R-IC^b_A-1) by assumption, any improvement must come from the second component of this linear combination.

Calculating the marginal gain $MG_\alpha(\beta)$, we obtain:

$$MG_\alpha(\beta) = -\sum_{i=1}^{2}\sum_{j=1}^{2}\Big(\underline{\beta}_{ij}\big[q^b_{ij}u^b(\underline{v}) - t^b_{ij} - \log\big(\mu^b_i(\underline{v})\big)\big] + \overline{\beta}_{ij}\big[q^b_{ij}u^b(\overline{v}) - t^b_{ij} - \log\big(\mu^b_i(\overline{v})\big)\big]\Big)$$
$$- \left[\underline{B}\log\left(\frac{\underline{B}}{\underline{B}+\underline{B}}\right) + \overline{B}\log\left(\frac{\overline{B}}{\underline{B}+\underline{B}}\right)\right],$$

where $\underline{B} \equiv \sum_{i=1}^{2}\sum_{j=1}^{2}\underline{\beta}_{ij}$ and $\overline{B} \equiv \sum_{i=1}^{2}\sum_{j=1}^{2}\overline{\beta}_{ij}$.

Since all β_{ij}'s are weakly positive, we can obtain a better direction of payoff improvement, whose marginal gain will be equal to:

$$-\underline{B}y^b(\underline{v}) - \overline{B}y^b(\overline{v}) - \left[\underline{B}\log\left(\frac{\underline{B}}{\underline{B}+\underline{B}}\right) + \overline{B}\log\left(\frac{\overline{B}}{\underline{B}+\underline{B}}\right)\right] > 0.$$

Defining $P \equiv \frac{\underline{B}}{\underline{B}+\overline{B}}$, we can write

$$-Py^b(\underline{v}) - (1-P)y^b(\overline{v}) - P\log\big(P\big) - (1-P)\log\big(1-P\big) > 0.$$

Maximizing over P, we can identify an even better direction of improvement:

$$\max_P\left\{-Py(\underline{v}) - (1-P)y(\overline{v}) - P\log\big(P\big) - (1-P)\log\big(1-P\big)\middle| P\in[0,1]\right\} > 0$$
$$\Leftrightarrow\ \exp\big(-y^b(\underline{v})\big) + \exp\big(-y^b(\overline{v})\big) > 1,$$

which establishes the "if" direction of our special case of Lemma 6 by contrapositive. To see why the "only if" direction also holds, observe that if $\exp\big(-y^b(\underline{v})\big) + \exp\big(-y^b(\overline{v})\big) > 1$, then we can construct a profitable local Class 2-deviation by taking away some probability mass from those (i,j) in each state v, for which $q^b_{ij}u^b(v) - t^b_{ij} - \log\big(\mu^b_i(v)\big)$ is minimal, and putting this probability mass on $s^b_\emptyset$.

3.2 Implementability conditions

Combining the Karush-Kuhn-Tucker optimality conditions from the maximization problems in (R-IC$^b_{\text{A}}$-1) and (R-IC$^s_{\text{A}}$-1) with the optimality conditions from Lemma 6, we obtain our main implementability result:

Proposition 3 (Implementability conditions). *The tuple (α, S^b, S^s) satisfies restricted ex ante incentive compatibility* R-IC$^p_{\text{A}}$ *for both players $p \in \{b, s\}$ if and only if there are multipliers $\lambda^b_i(v), \lambda^s_j(v)$ for all i and j respectively, and $\phi^b_{ij}(v), \phi^s_{ij}(v)$ for* *all pairs* (i, j) *such that the following conditions are satisfied:*

$$
\begin{aligned}
&(\text{ST}^b) \quad q^b_{ij}u^b(v) - t^b_{ij} - \log\big(\mu^b_i(v)\big) - \lambda^b_j(v) + \phi^b_{ij}(v) = 0 \ \forall (i,j),\ v \in V;\\
&(\text{ST}^s) \quad t^s_{ij} - q^s_{ij}u^s(v) - \log\big(\mu^s_j(v)\big) - \lambda^s_i(v) + \phi^s_{ij}(v) = 0 \ \forall (i,j),\ v \in V;\\
&(\text{DF}) \quad \phi^b_{ij}(v) \geq 0,\ \phi^s_{ij}(v) \geq 0 \ \forall (i,j),\ v \in V;\\
&(\text{CS}) \quad \alpha_{ij}(v)\phi^b_{ij}(v) = 0,\ \alpha_{ij}(v)\phi^s_{ij}(v) = 0 \ \forall (i,j),\ v \in V;\\
&(\text{NA}) \quad \sum_{v \in V} \exp\big(-\min_j\{\lambda^b_j(v)\}\big) \leq 1,\ \sum_{v \in V} \exp\big(-\min_i\{\lambda^s_i(v)\}\big) \leq 1.
\end{aligned}
$$

Proof. (ST^b) and (ST^s) are stationarity conditions in (R-IC$^b_{\text{A}}$-1) and (R-IC$^s_{\text{A}}$-1) respectively. (DF) are dual feasibility conditions, which make sure that the multipliers on non-negativity constraints on joint probabilities are themselves non-negative. (CS) are complementary slackness conditions. To obtain (NA^b), recall from Lemma 6 that if (R-IC$^b_{\text{A}}$-1) is satisfied, (R-IC$^b_{\text{A}}$-2) is equivalent to:

$$
\sum_{v \in V} \exp\left(-\min_{(i,j)|\alpha_{ij}(v)>0}\left\{q_{ij}(v)u^b(v) - t^b_{ij} - \log\big(\mu^b_i(v)\big)\right\}\right) \leq 1,
$$

which, combined with (ST^b), is equivalent to $\sum_{v \in V} \exp\big(-\min_{(i,j)|\alpha_{ij}(v)>0}\{\lambda^b_j(v) + \phi^b_{ij}(v)\}\big) \leq 1$. (DF^b) and (CS^b) together imply that $\phi^b_{ij}(v) = 0$ whenever $\alpha_{ij}(v) > 0$, which means that $\sum_{v \in V} \exp\big(-\min_{(i,j)|\alpha_{ij}(v)>0}\{\lambda^b_j(v)\}\big) \leq 1$. Lemma 5 says that all the seller's posteriors must be strictly positive, which means that in every column j there is at least one strictly positive $\alpha_{ij}(v)$ in every state v. We can therefore simply minimize over columns instead, hence $\sum_{v \in V} \exp\big(-\min_j\{\lambda^b_j(v)\}\big) \leq 1$. (NA^s) can be obtained using a similar argument. □

4 Application: revenue maximization among efficient mechanisms

We now turn to our application and consider the problem of maximizing revenue for a mechanism designer who is interested in allocational efficiency. We assume that the state of the world is binary, i.e. $V = \{\underline{v}, \overline{v}\}$. We will refer to $\underline{v}$ as the low state, and to $\overline{v}$ as the high state. To ease notation, we will use underlined letters to denote the respective state-$\underline{v}$ variables and parameters, and overlined letters to denote the respective state-$\overline{v}$ variables and parameters. We make the following assumption:

Assumption 1 (Parameter values). $\underline{u}^p < \overline{u}^p$ *for both* $p \in \{b, s\}$.

Assumption 1 imposes a common interpretation of the states on the players. The opposite case could be solved just as easily using our method, but is less natural, and hence omitted. Since we also assume that there are gains from trade in both states, the designer must always allocate the good to the buyer to achieve allocational efficiency. We allow the designer to implement information structures of arbitrary dimensionality. More precisely, the designer could choose any α on $S^b \times S^s \times V$, where $S^b = \{1, \dots, I\}$ and $S^s = \{1, \dots, J\}$ for any I, J. The implementability conditions for all pairs (i, j) can then be written as:

$$\begin{aligned}
&(\text{ST}^b_{ij}) \quad \underline{u}^b - t^b_{ij} - \log\left(\underline{\mu}^b_i\right) - \underline{\lambda}^b_j + \underline{\phi}^b_{ij} = 0, \qquad &&(\text{ST}^s_{ij}) \quad t^s_{ij} - \underline{u}^s - \log\left(\underline{\mu}^s_j\right) - \underline{\lambda}^s_i + \underline{\phi}^s_{ij} = 0,\\
&\qquad\qquad \overline{u}^b - t^b_{ij} - \log\left(\overline{\mu}^b_i\right) - \overline{\lambda}^b_j + \overline{\phi}^b_{ij} = 0; &&\qquad\qquad t^s_{ij} - \overline{u}^s - \log\left(\overline{\mu}^s_j\right) - \overline{\lambda}^s_i + \overline{\phi}^s_{ij} = 0;\\
&(\text{DF}^b_{ij}) \quad \underline{\phi}^b_{ij} \geq 0,\ \overline{\phi}^b_{ij} \geq 0; &&(\text{DF}^s_{ij}) \quad \underline{\phi}^s_{ij} \geq 0,\ \overline{\phi}^s_{ij} \geq 0;\\
&(\text{CS}^b_{ij}) \quad \underline{\alpha}_{ij}\underline{\phi}^b_{ij} = \overline{\alpha}_{ij}\overline{\phi}^b_{ij} = 0; &&(\text{CS}^s_{ij}) \quad \underline{\alpha}_{ij}\underline{\phi}^s_{ij} = \overline{\alpha}_{ij}\overline{\phi}^s_{ij} = 0;\\
&(\text{NA}^b) \quad \exp\left(-\underline{\Lambda}^b\right) + \exp\left(-\overline{\Lambda}^b\right) \leq 1, &&(\text{NA}^s) \quad \exp\left(-\underline{\Lambda}^s\right) + \exp\left(-\overline{\Lambda}^s\right) \leq 1,\\
&\qquad\qquad \underline{\Lambda}^b \equiv \min_j\{\underline{\lambda}^b_j\},\ \overline{\Lambda}^b \equiv \min_j\{\overline{\lambda}^b_j\}. &&\qquad\qquad \underline{\Lambda}^s \equiv \min_i\{\underline{\lambda}^s_i\},\ \overline{\Lambda}^s \equiv \min_i\{\overline{\lambda}^s_i\}.
\end{aligned}$$

The mechanism designer solves:

$$\sup_{\alpha, I, J; t; \phi, \lambda} \sum_{i=1}^{I} \sum_{j=1}^{J} \left(\underline{\alpha}_{ij} + \overline{\alpha}_{ij}\right)\left(t^b_{ij} - t^s_{ij}\right), \text{ s.t.}$$

$$(\alpha\text{-F}) \quad S^b = \{1, \dots, I\},\ S^s = \{1, \dots, J\},\ \alpha \in \Delta\left(S^b \times S^s \times V\right),\ \text{BP},\ \overline{\mu}^b_i, \overline{\mu}^s_i \in (0, 1);$$

$$(\text{Imp}) \quad \text{ST}^b,\ \text{ST}^s, \text{DF}^b,\ \text{DF}^s,\ \text{CS}^b,\ \text{CS}^s,\ \text{NA}^b,\ \text{NA}^s.$$

4.1 Perfect correlation of signal realizations

We start by establishing that it is without loss of generality to focus our attention on information structures that feature perfect correlation of signal realizations:

Lemma 7 (Perfect correlation of signal realizations). *If $(\alpha, I, J; t; \phi, \lambda)$ satisfies the implementability conditions and α is Bayes-plausible, there is $(\alpha', I', J'; t'; \phi', \lambda')$ where α' is Bayes-plausible, $I' = J'$, and $\underline{\alpha'}_{ij} = \overline{\alpha'}_{ij} = 0$ for all $i \neq j$, which also satisfies the implementability conditions and achieves the same revenue as $(\alpha, I, J; t; \phi, \lambda)$.*

Proof sketch. The complete proof is in Appendix H. Observe that the signal realizations can be ordered according to their posteriors without loss of generality, i.e. $\overline{\mu}^b_1 \geq \cdots \geq \overline{\mu}^b_I$ and $\overline{\mu}^s_1 \geq \cdots \geq \overline{\mu}^s_J$. We establish the following claims:

Claim 1. *For all* $j \in S^s$ *there exists* $\mathcal{I}^*(j) \subseteq S^b$ *such that*

1. $\underline{\alpha}_{ij} = 0$ *for all* $i > \max \mathcal{I}^*(j) \equiv \overline{i}^*(j)$,
2. $\overline{\alpha}_{ij} = 0$ *for all* $i < \min \mathcal{I}^*(j) \equiv \underline{i}^*(j)$.

Moreover, for any $i, i' \in \mathcal{I}^*(j)$ *we get* $\overline{\mu}_i^b = \overline{\mu}_{i'}^b$.

Claim 2. *For all* $i \in S^b$ *there exists* $\mathcal{J}^*(i) \subseteq S^s$ *such that*

1. $\underline{\alpha}_{ij} = 0$ *for all* $j > \max \mathcal{J}^*(i) \equiv \overline{j}^*(i)$,
2. $\overline{\alpha}_{ij} = 0$ *for all* $j < \min \mathcal{J}^*(i) \equiv \underline{j}^*(i)$.

Moreover, for any $j, j' \in \mathcal{J}^*(i)$ *we get* $\overline{\mu}_j^s = \overline{\mu}_{j'}^s$.

Claims 1 and 2 have analogous proofs, thus we only sketch the proof of Claim 1.

Proof sketch of Claim 1. Fix $j \in S^s$. Set $\overline{i}^*(j) \equiv \max\{i | \underline{\alpha}_{ij} > 0\}$, and define $\mathcal{I}^*(j) \equiv \{i | \overline{\mu}_i^b = \overline{\mu}_{\overline{i}^*(j)}^b\}$ and $\underline{i}^*(j) \equiv \min \mathcal{I}^*(j)$. Suppose for a contradiction that $\exists i < \underline{i}^*(j)$ such that $\overline{\alpha}_{ij} > 0$. Note that $\overline{\mu}_i^b > \overline{\mu}_{\overline{i}^*(j)}^b$ since $\overline{\mu}_i^b \geq \overline{\mu}_{\overline{i}^*(j)}^b$ by the ordering assumption and $i \notin \mathcal{I}^*(j)$. The combination of ($\mathrm{ST}_{ij}^b$) and ($\mathrm{ST}_{\overline{i}^*(j),j}^b$) then implies $\overline{\mu}_{\overline{i}^*(j)}^b \geq \overline{\mu}_i^b$, which is the desired contradiction. □

We proceed further in the inductive manner. We introduce the following sets: $\hat{\mathcal{I}}_1 \equiv \{i | \overline{\mu}_i^b = \overline{\mu}_1^b\}$, $\hat{\mathcal{J}}_1 \equiv \{j | \overline{\mu}_j^s = \overline{\mu}_1^s\}$, and $\tilde{\mathcal{I}}_1 \equiv \{i | \mathcal{J}^*(i) = \hat{\mathcal{J}}_1\}$, $\tilde{\mathcal{J}}_1 \equiv \{j | \mathcal{I}^*(i) = \hat{\mathcal{I}}_1\}$; and show that $\tilde{\mathcal{I}}_1$ and $\tilde{\mathcal{J}}_1$ are non-empty. We then establish the base case:

Claim 3 (**Base case of the induction**)**.**

1. *For all* $i \in \hat{\mathcal{I}}_1$ *and* $j \notin \hat{\mathcal{J}}_1$ *we have* $\underline{\alpha}_{ij} = \overline{\alpha}_{ij} = 0$.
2. *For all* $i \notin \hat{\mathcal{I}}_1$ *and* $j \in \hat{\mathcal{J}}_1$ *we have* $\underline{\alpha}_{ij} = \overline{\alpha}_{ij} = 0$.

Proof sketch of Claim 3. Suppose $i' \notin \tilde{\mathcal{I}}_1 = \{i | \mathcal{J}^*(i) = \hat{\mathcal{J}}_1\}$, then $\mathcal{J}^*(i') \neq \hat{\mathcal{J}}_1$. We must have $\min \mathcal{J}^*(i') > \max \hat{\mathcal{J}}_1$. Claim 2 then implies that $\overline{\alpha}_{i'j} = 0$ for all $j \in \hat{\mathcal{J}}_1$. Recall that $1 \in \hat{\mathcal{J}}_1$ and the posteriors of all signal realizations in $\hat{\mathcal{J}}_1$ are the same, hence $\overline{\mu}_1^s$ is equal to the weighted average of the posteriors of signal realizations from $\hat{\mathcal{J}}_1$. It can be shown that this weighted average does not exceed $\frac{\sum_{j \in \hat{\mathcal{J}}_1} \sum_{i \in \tilde{\mathcal{I}}_1} \overline{\alpha}_{ij}}{\sum_{j \in \hat{\mathcal{J}}_1} \sum_{i \in \tilde{\mathcal{I}}_1} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}$. Suppose now $i' \in \tilde{\mathcal{I}}_1$, i.e. $\mathcal{J}^*(i') = \hat{\mathcal{J}}_1$. Claim 2 implies that $\underline{\alpha}_{i'j} = 0$ for all $j \notin \hat{\mathcal{J}}_1 = \mathcal{J}^*(i')$. By the ordering assumption, $\overline{\mu}_1^b$ is higher than the weighted average of the posteriors of signal realizations from $\tilde{\mathcal{I}}_1$. This weighted average can then be shown to exceed $\frac{\sum_{i \in \tilde{\mathcal{I}}_1} \sum_{j \in \hat{\mathcal{J}}_1} \overline{\alpha}_{ij}}{\sum_{i \in \tilde{\mathcal{I}}_1} \sum_{j \in \hat{\mathcal{J}}_1} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}$, implying $\overline{\mu}_1^s \leq \overline{\mu}_1^b$. A similar argument, but for $\tilde{\mathcal{J}}_1$ and $\hat{\mathcal{I}}_1$, leads to

$\overline{\mu}^b_1 \leq \frac{\sum_{j\in\tilde{\mathcal{J}}_1}\sum_{i\in\hat{\mathcal{I}}_1}\overline{\alpha}_{ij}}{\sum_{j\in\tilde{\mathcal{J}}_1}\sum_{i\in\hat{\mathcal{I}}_1}(\overline{\alpha}_{ij}+\underline{\alpha}_{ij})} \leq \overline{\mu}^s_1$, implying $\overline{\mu}^b_1 \leq \overline{\mu}^s_1$, hence $\overline{\mu}^s_1 = \overline{\mu}^b_1$, which implies $\hat{\mathcal{I}}_1 = \tilde{\mathcal{I}}_1$ and $\hat{\mathcal{J}}_1 = \tilde{\mathcal{J}}_1$. Claim 1 then implies that $\underline{\alpha}_{ij} = 0$ for all $i \notin \hat{\mathcal{I}}_1$ and $j \in \hat{\mathcal{J}}_1$. Likewise, Claim 2 then implies that $\underline{\alpha}_{ij} = 0$ for all $i \in \hat{\mathcal{I}}_1$ and $j \notin \hat{\mathcal{J}}_1$. To establish the claim for $\overline{\alpha}_{ij}$, let $j' \notin \hat{\mathcal{J}}_1$. Since $\hat{\mathcal{J}}_1 = \tilde{\mathcal{J}}_1$, we have $j' \notin \tilde{\mathcal{J}}_1 = \{j|\mathcal{I}^*(j) = \hat{\mathcal{I}}_1\}$, i.e. $\mathcal{I}^*(j') \neq \hat{\mathcal{I}}_1$. We then have $\min \mathcal{I}^*(j') > \max \hat{\mathcal{I}}_1$ and Claim 1 implies $\overline{\alpha}_{ij'} = 0$ for all $i \in \hat{\mathcal{I}}_1$. A similar argument works for any $i' \notin \hat{\mathcal{I}}_1$ and $j \in \hat{\mathcal{J}}_1$. □

Since the posteriors of all signal realizations in $\hat{\mathcal{I}}_1$ and $\hat{\mathcal{J}}_1$ are the same, we can merge all signal realizations in $\hat{\mathcal{I}}_1$ and $\hat{\mathcal{J}}_1$ into a single signal realization for the buyer and a single signal realization for the seller respectively. We then show that the resulting mechanism is implementable and leads to the same revenue. Having shown that, we formulate our induction hypothesis:

Induction hypothesis 1. *There exists $k < \min\{I, J\}$ such that $\underline{\alpha}_{ij} = \overline{\alpha}_{ij} = 0$ if $i \neq j$ and ($i \leq k-1$ or $j \leq k-1$). Moreover, $\overline{\mu}^b_1 > \cdots > \overline{\mu}^b_k \geq \cdots \geq \overline{\mu}^b_I$ and $\overline{\mu}^s_1 > \cdots > \overline{\mu}^s_k \geq \cdots \geq \overline{\mu}^s_J$.*

We then introduce the following sets: $\hat{\mathcal{I}}_k \equiv \{i|\overline{\mu}^b_i = \overline{\mu}^b_k\}$, $\hat{\mathcal{J}}_k \equiv \{j|\overline{\mu}^s_j = \overline{\mu}^s_k\}$, and $\tilde{\mathcal{I}}_k \equiv \{i|\mathcal{J}^*(i) = \hat{\mathcal{J}}_k\}$, $\tilde{\mathcal{J}}_k \equiv \{j|\mathcal{I}^*(i) = \hat{\mathcal{I}}_k\}$; and show that $\hat{\mathcal{I}}_k = \tilde{\mathcal{I}}_k$ and $\hat{\mathcal{J}}_k = \tilde{\mathcal{J}}_k$ by essentially repeating the argument used in the proof of Claim 3 with appropriate modifications. $\hat{\mathcal{I}}_k = \tilde{\mathcal{I}}_k$ and $\hat{\mathcal{J}}_k = \tilde{\mathcal{J}}_k$ combined with Induction hypothesis 1 then implies $\underline{\alpha}_{ij} = \overline{\alpha}_{ij} = 0$ for all $i \in \hat{\mathcal{I}}_k$ and $j \notin \hat{\mathcal{J}}_k$, and $\underline{\alpha}_{ij} = \overline{\alpha}_{ij} = 0$ for all $i \notin \hat{\mathcal{I}}_k$ and $j \in \hat{\mathcal{J}}_k$. We then merge all signal realizations in $\hat{\mathcal{I}}_k$ and $\hat{\mathcal{J}}_k$ into a single signal realization for the buyer and a single signal realization for the seller respectively, and show that the resulting mechanism is implementable and leads to the same revenue. □

4.2 Revenue maximization under perfect correlation of signal realizations

We have shown above that it is without loss of generality to restrict attention to perfectly correlated information structures. From now on, let us maintain that $S^b = S^s = \{1, \ldots, I\}$, and let $\tilde{\Delta}(S^b \times S^s \times V)$ be the collection of such information structures. If $\alpha \in \tilde{\Delta}(S^b \times S^s \times V)$, then $\underline{\alpha}_{ij} = \overline{\alpha}_{ij} = 0$ unless $i = j$. We therefore drop the double subscripts and write $\cdot_i$ to mean $\cdot_{ii}$ for all variables with double subscripts. Since $\overline{\mu}^b_i = \overline{\mu}^s_i$ under perfect correlatation, we drop the player superscripts from the posteriors too. The implementability conditions are then as follows:

$$(\mathrm{ST}^b_i)\quad \underline{u}^b - t^b_i - \log\big(\underline{\mu}_i\big) = \underline{\lambda}^b_i, \qquad (\mathrm{ST}^s_i)\quad t^s_i - \underline{u}^s - \log\big(\underline{\mu}_i\big) = \underline{\lambda}^s_i,$$

$$\overline{u}^b - t^b_i - \log\big(\overline{\mu}_i\big) = \overline{\lambda}^b_i; \qquad t^s_i - \overline{u}^s - \log\big(\overline{\mu}_i\big) = \overline{\lambda}^s_i;$$

$$(\mathrm{NA}^b)\quad \exp\big(-\underline{\Lambda}^b\big) + \exp\big(-\overline{\Lambda}^b\big) \le 1, \qquad (\mathrm{NA}^s)\quad \exp\big(-\underline{\Lambda}^s\big) + \exp\big(-\overline{\Lambda}^s\big) \le 1,$$

$$\underline{\Lambda}^b \equiv \min_i\{\underline{\lambda}^b_i\},\ \overline{\Lambda}^b \equiv \min_i\{\overline{\lambda}^b_i\}. \qquad \underline{\Lambda}^s \equiv \min_i\{\underline{\lambda}^s_i\},\ \overline{\Lambda}^s \equiv \min_i\{\overline{\lambda}^s_i\}.$$

We take a relaxed program approach, and relax implementabilty as follows:

$$(\text{Rel-ST}^b_i)\quad \underline{u}^b - t^b_i - \log\big(\underline{\mu}_i\big) \ge \underline{\Lambda}^b, \qquad (\text{Rel-ST}^s_i)\quad t^s_i - \underline{u}^s - \log\big(\underline{\mu}_i\big) \ge \underline{\Lambda}^s,$$

$$\overline{u}^b - t^b_i - \log\big(\overline{\mu}_i\big) \ge \overline{\Lambda}^b; \qquad t^s_i - \overline{u}^s - \log\big(\overline{\mu}_i\big) \ge \overline{\Lambda}^s;$$

$$(\mathrm{NA}^b)\quad \exp\big(-\underline{\Lambda}^b\big) + \exp\big(-\overline{\Lambda}^b\big) \le 1. \qquad (\mathrm{NA}^s)\quad \exp\big(-\underline{\Lambda}^s\big) + \exp\big(-\overline{\Lambda}^s\big) \le 1.$$

The mechanism designer's relaxed problem can then be written as:

$$\sup_{\alpha,I,t;\Lambda} \sum_{i=1}^{I} \big(\underline{\alpha}_i + \overline{\alpha}_i\big)\big(t^b_i - t^s_i\big),\ \text{s.t.}$$

$$(\alpha\text{-F})\quad S^b = S^s = \{1,\dots,I\},\ \alpha \in \tilde{\Delta}\big(S^b \times S^s \times V\big),\ \mathrm{BP},\ \overline{\mu}_i \in (0,1);$$

$$(\mathrm{Imp})\quad \text{Rel-ST}^b,\ \text{Rel-ST}^s,\ \mathrm{NA}^b,\ \mathrm{NA}^s.$$

We solve this problem using a concavification approach, similar to the one used in the Bayesian persuasion literature. Let $\tau_i \equiv \underline{\alpha}_i + \overline{\alpha}_i$, and notice that (Rel-ST^b_i) is equivalent to $t^b_i \le \min\big\{\underline{u}^b - \log(\underline{\mu}_i) - \underline{\Lambda}^b,\ \overline{u}^b - \log(\overline{\mu}_i) - \overline{\Lambda}^b\big\}$, and (Rel-ST$^s_i$) is equivalent to $-t^s_i \le \min\big\{-\underline{u}^s - \log(\underline{\mu}_i) - \underline{\Lambda}^s,\ -\overline{u}^s - \log(\overline{\mu}_i) - \overline{\Lambda}^s\big\}$. It is immediately clear that, written this way, both (Rel-ST^b_i) and (Rel-ST^s_i) are binding at the optimum for all i. Define the following function:

$$T\big(\underline{x},\overline{x}\big) \equiv \min\big\{\underline{u}^b - \log(\underline{x}) - \underline{\Lambda}^b,\ \overline{u}^b - \log(\overline{x}) - \overline{\Lambda}^b\big\} + \min\big\{-\underline{u}^s - \log(\underline{x}) - \underline{\Lambda}^s,\ -\overline{u}^s - \log(\overline{x}) - \overline{\Lambda}^s\big\}.$$

The mechanism designer's relaxed revenue maximization problem can then be written as:

$$(\mathcal{RM})\quad \sup_{\mu,\tau;I;\Lambda} \left\{ \sum_{i=1}^{I} \tau_i T\big(\underline{\mu}_i, \overline{\mu}_i\big) \;\middle|\; \sum_{i=1}^{I} \tau_i \underline{\mu}_i = \underline{\mu}_0,\ \sum_{i=1}^{I} \tau_i \overline{\mu}_i = \overline{\mu}_0;\ \underline{\mu}_i, \overline{\mu}_i \in (0,1);\ \mathrm{NA}^b,\ \mathrm{NA}^s \right\}.$$

Note that $\mathcal{RM}$ could in principle achieve its supremum at a point, in which one of the posteriors is extreme, i.e. $\underline{\mu}_{i'} = 0$ or $\overline{\mu}_{i'} = 0$ for some i', which would not be implementable by Lemma 5. We show in Appendix J that, at least in the symmetric case (to be formally defined below), most such cases can be ruled out. We have not yet ruled out one solution candidate with extreme posteriors, but we conjecture, based on numerical computations, that it can be ruled out too (see Appendix J for details). However, even if it cannot be ruled out, one can still virtually implement a solution with extreme posteriors by allowing the appropriate off-path punishments to grow without bound[3]. In the remainder of this section, we focus on non-extreme posteriors and assume that $\mathcal{RM}$ achieves a maximum on its feasible set. We establish the following optimality conditions:

Proposition 4 (Optimality conditions). *Suppose $\mathcal{RM}$ achieves a maximum, then we can set $I = 2$ w.l.o.g., and moreover the optimal posteriors satisfy*

$$(\text{Opt}^b) \quad \underline{u}^b - \log(\underline{\mu}_1) - \underline{\Lambda}^b = \overline{u}^b - \log(\overline{\mu}_1) - \overline{\Lambda}^b,$$
$$(\text{Opt}^s) \quad \underline{u}^s + \log(\underline{\mu}_2) + \underline{\Lambda}^s = \overline{u}^s + \log(\overline{\mu}_2) + \overline{\Lambda}^s.$$

Proof. Suppose $\underline{\Lambda}^p$ and $\overline{\Lambda}^p$ are optimally chosen for both $p \in \{b, s\}$ and let us maximize $\sum_{i=1}^{I} \tau_i T(\underline{\mu}_i, \overline{\mu}_i)$ subject to BP. Let $(\underline{\mu}_1, \overline{\mu}_1)$ and $(\underline{\mu}_2, \overline{\mu}_2)$ be as defined in (Opt^b) and (Opt^s) respectively. It is easy to see that they are uniquely defined. Define $f(x) \equiv T(x, 1-x)$. If $\underline{\mu}_1 \le \overline{\mu}_2$, then

$$f(x) = \begin{cases} \overline{u}^b - \overline{u}^s - 2\log(1-x) - \overline{\Lambda}^b - \overline{\Lambda}^s & \text{if } x < \underline{\mu}_1, \\ \underline{u}^b - \overline{u}^s - \log(x) - \log(1-x) - \underline{\Lambda}^b - \overline{\Lambda}^s & \text{if } \underline{\mu}_1 \le x \le \underline{\mu}_2, \\ \underline{u}^b - \underline{u}^s - 2\log(x) - \underline{\Lambda}^b - \underline{\Lambda}^s & \text{if } x > \underline{\mu}_2. \end{cases}$$

Likewise, if $\underline{\mu}_2 \le \overline{\mu}_1$, then

$$f(x) = \begin{cases} \overline{u}^b - \overline{u}^s - 2\log(1-x) - \overline{\Lambda}^b - \overline{\Lambda}^s & \text{if } x < \underline{\mu}_2, \\ \overline{u}^b - \underline{u}^s - \log(x) - \log(1-x) - \overline{\Lambda}^b - \underline{\Lambda}^s & \text{if } \underline{\mu}_2 \le x \le \underline{\mu}_1, \\ \underline{u}^b - \underline{u}^s - 2\log(x) - \underline{\Lambda}^b - \underline{\Lambda}^s & \text{if } x > \underline{\mu}_1. \end{cases}$$

[3]Recall that the main idea behind the proof of Lemma 5 is that the marginal cost of an extreme posterior is infinite. A player, therefore, would always want to acquire slighgtly less information and potentially end up with an off-path report profile since the punishment for submitting such an off-path profile is assumed to be bounded. However, if we allow the respective off-path punishment to go to infinity (at an appropriately chosen speed of convergence) as the players' posterior approaches zero, then an extreme posterior can be "implemented in the limit", i.e. virtually implemented.

Consider $coc[f](x)$, the concave closure of $f(x)$. By the standard argument from Bayesian persuasion, the desired maximum is equal to $coc[f](\underline{\mu}_0)$. Observe that $-\log(1-x)$ is strictly increasing and strictly convex in x, $-\log(x)-\log(1-x)$ is strictly convex in x, and $-\log(x)$ is strictly decreasing and strictly convex in x. It implies that in either of the two cases the convex hull of the graph of $f(x)$ can have at most four extreme points: $\big(0, f(0)\big)$, $\big(\underline{\mu}_1, f(\underline{\mu}_1)\big)$, $\big(\underline{\mu}_2, f(\underline{\mu}_2)\big)$, and $\big(1, f(1)\big)$. Moreover $f(x) < coc[f](x)$ for any $x \notin \{0, \underline{\mu}_1, \underline{\mu}_2, 1\}$, which implies that other posteriors cannot be optimal. Since both 0 and 1 are infeasible, we are done. □

The following corollary is immediate:

Corollary 1. *If $\mathcal{RM}$ achieves a maximum, then its solution is implementable.*

Proof. Set $\underline{\lambda}_1^b \equiv \underline{\Lambda}^b$, $\overline{\lambda}_1^b \equiv \overline{\Lambda}^b$ and $\underline{\lambda}_2^s \equiv \underline{\Lambda}^s$, $\overline{\lambda}_2^s \equiv \overline{\Lambda}^s$. Set $\underline{\lambda}_2^b \equiv \underline{u}^b - t_2^b - \log(\underline{\mu}_2) \geq \underline{\Lambda}^b$, $\overline{\lambda}_2^b \equiv \overline{u}^b - t_2^b - \log(\overline{\mu}_2) \geq \overline{\Lambda}^b$ and $\underline{\lambda}_1^s \equiv t_1^s - \underline{u}^s - \log(\underline{\mu}_1) \geq \underline{\Lambda}^s$, $\overline{\lambda}_1^s \equiv t_1^s - \overline{u}^s - \log(\overline{\mu}_1) \geq \overline{\Lambda}^s$. Clearly, ($\text{ST}_i^b$) and ($\text{ST}_i^s$) are then satisfied for both $i \in \{1,2\} = S^b = S^s$. □

The objective in $\mathcal{RM}$ weakly decreases in $\underline{\Lambda}^p$ and $\overline{\Lambda}^p$ for both $p \in \{b, s\}$, hence both (NA^p) can be assumed to bind. Coupled with Proposition 4, it gives us the following optimality conditions:

$$\begin{cases} \underline{u}^b - \log(\underline{\mu}_1) - \underline{\Lambda}^b = \overline{u}^b - \log(\overline{\mu}_1) - \overline{\Lambda}^b, \\ \exp\big(-\underline{\Lambda}^b\big) + \exp\big(-\overline{\Lambda}^b\big) = 1. \end{cases} \qquad \begin{cases} \underline{u}^s + \log(\underline{\mu}_2) + \underline{\Lambda}^s = \overline{u}^s + \log(\overline{\mu}_2) + \overline{\Lambda}^s, \\ \exp\big(-\underline{\Lambda}^s\big) + \exp\big(-\overline{\Lambda}^s\big) = 1. \end{cases}$$

We solve for $\big(\underline{\Lambda}^p, \overline{\Lambda}^p\big)$ for both p and plug the resulting Λ's into the objective. Define $\Delta^b \equiv \overline{u}^b - \underline{u}^b$ and $\Delta^s \equiv \overline{u}^b - \underline{u}^s$. BP implies that there are three possible cases.

Case 0: $\underline{\mu}_1 = \underline{\mu}_0 = \underline{\mu}_2$, In this case, $I = 1$, and the designer's revenue is:

$$\begin{aligned} \mathcal{R}_{\text{Case 0}} &= -\log\left[\underline{\mu}_0 e^{-\underline{u}^b} + \overline{\mu}_0 e^{-\overline{u}^b}\right] - \log\left[\underline{\mu}_0 e^{\underline{u}^s} + \overline{\mu}_0 e^{\overline{u}^s}\right] \\ &= -\log\left[\underline{\mu}_0 e^{\Delta^b} + \overline{\mu}_0\right] - \log\left[\underline{\mu}_0 + \overline{\mu}_0 e^{\Delta^s}\right] + \overline{u}^b - \underline{u}^s. \end{aligned}$$

Case 1: $\underline{\mu}_2 < \underline{\mu}_0 < \underline{\mu}_1$. In this case, $I = 2$, and the designer's revenue is:

$$\begin{aligned} \mathcal{R}_{\text{Case 1}} &= \tau_1\left[\overline{u}^b - \log(\overline{\mu}_1) - \overline{\Lambda}^b - \underline{u}^s - \log(\underline{\mu}_1) - \underline{\Lambda}^s\right] + \tau_2\left[\overline{u}^b - \log(\overline{\mu}_2) - \overline{\Lambda}^b - \underline{u}^s - \log(\underline{\mu}_2) - \underline{\Lambda}^s\right] \\ &= \log\left[\overline{\mu}_1\underline{\mu}_2\right] - \tau_1 \log\left[\underline{\mu}_1\overline{\mu}_1\right] - \tau_2 \log\left[\underline{\mu}_2\overline{\mu}_2\right] - \log\left[\underline{\mu}_1 e^{-\underline{u}^b} + \overline{\mu}_1 e^{-\overline{u}^b}\right] - \log\left[\underline{\mu}_2 e^{\underline{u}^s} + \overline{\mu}_2 e^{\overline{u}^s}\right] \\ &= \log\left[\overline{\mu}_1\underline{\mu}_2\right] - \tau_1 \log\left[\underline{\mu}_1\overline{\mu}_1\right] - \tau_2 \log\left[\underline{\mu}_2\overline{\mu}_2\right] - \log\left[\underline{\mu}_1 e^{\Delta^b} + \overline{\mu}_1\right] - \log\left[\underline{\mu}_2 + \overline{\mu}_2 e^{\Delta^s}\right] + \overline{u}^b - \underline{u}^s. \end{aligned}$$

Case 2: $\underline{\mu}_1 < \underline{\mu}_0 < \underline{\mu}_2$. In this case, $I = 2$, and the designer's revenue is:

$$\begin{aligned}\mathcal{R}_{\text{Case 2}} &= \tau_1\left[\underline{u}^b - \log(\underline{\mu}_1) - \underline{\Lambda}^b - \overline{u}^s - \log(\overline{\mu}_1) - \overline{\Lambda}^s\right] + \tau_2\left[\underline{u}^b - \log(\underline{\mu}_2) - \underline{\Lambda}^b - \overline{u}^s - \log(\overline{\mu}_2) - \overline{\Lambda}^s\right]. \\ &= \log\left[\underline{\mu}_1\overline{\mu}_2\right] - \tau_1\log\left[\underline{\mu}_1\overline{\mu}_1\right] - \tau_2\log\left[\underline{\mu}_2\overline{\mu}_2\right] - \log\left[\underline{\mu}_1 e^{-\underline{u}^b} + \overline{\mu}_1 e^{-\overline{u}^b}\right] - \log\left[\underline{\mu}_2 e^{\underline{u}^s} + \overline{\mu}_2 e^{\overline{u}^s}\right] \\ &= \log\left[\underline{\mu}_1\overline{\mu}_2\right] - \tau_1\log\left[\underline{\mu}_1\overline{\mu}_1\right] - \tau_2\log\left[\underline{\mu}_2\overline{\mu}_2\right] - \log\left[\underline{\mu}_1 e^{\Delta^b} + \overline{\mu}_1\right] - \log\left[\underline{\mu}_2 + \overline{\mu}_2 e^{\Delta^s}\right] + \overline{u}^b - \underline{u}^s.\end{aligned}$$

We now show that Case 1 can never be optimal:

Proposition 5. *For any* $(\mu_1, \mu_2) \in$ Case 1*, there exists* $(\mu_1', \mu_2') \in$ Case 2 *such that the revenue from* (μ_1', μ_2') *exceeds the revenue from* (μ_1, μ_2).

Proof. Suppose (μ_1, μ_2) belongs to Case 1, i.e. $\underline{\mu}_2 < \underline{\mu}_0 < \underline{\mu}_1$, and let $\mathcal{R}_{\text{Case 1}}$ be the corresponding revenue. Let (μ_1', μ_2') be such that $\underline{\mu}_1' = \underline{\mu}_2$ and $\underline{\mu}_2' = \underline{\mu}_1$. Obviously, (μ_1', μ_2') belongs to Case 2. Bayes-plausibility then implies $\tau_1' = \tau_2$ and $\tau_2' = \tau_1$. The revenue at (μ_1', μ_2') is then given by:

$$\begin{aligned}\mathcal{R}'_{\text{Case 2}} &= \log\left[\underline{\mu}_2\overline{\mu}_1\right] - \tau_1\log\left[\underline{\mu}_1\overline{\mu}_1\right] - \tau_2\log\left[\underline{\mu}_2\overline{\mu}_2\right] - \log\left[\underline{\mu}_2 e^{-\underline{u}^b} + \overline{\mu}_2 e^{-\overline{u}^b}\right] - \log\left[\underline{\mu}_1 e^{\underline{u}^s} + \overline{\mu}_1 e^{\overline{u}^s}\right] \\ &= \mathcal{R}_{\text{Case 1}} + \underbrace{\log\left[\frac{\underline{\mu}_1 e^{-\underline{u}^b} + \overline{\mu}_1 e^{-\overline{u}^b}}{\underline{\mu}_2 e^{-\underline{u}^b} + \overline{\mu}_2 e^{-\overline{u}^b}}\right]}_{>0 \text{ as } \underline{\mu}_1 > \underline{\mu}_2 \ \& \ \overline{u}^b > \underline{u}^b} + \underbrace{\log\left[\frac{\underline{\mu}_2 e^{\underline{u}^s} + \overline{\mu}_2 e^{\overline{u}^s}}{\underline{\mu}_1 e^{\underline{u}^s} + \overline{\mu}_1 e^{\overline{u}^s}}\right]}_{>0 \text{ as } \underline{\mu}_1 > \underline{\mu}_2 \ \& \ \overline{u}^s > \underline{u}^s} > \mathcal{R}_{\text{Case 1}}.\end{aligned}$$

□

4.3 Symmetric revenue maximization problem

We now turn our attention to the symmetric problem, defined as follows:

Definition 2 (Symmetry). $\mathcal{RM}$ *is symmetric if* $\underline{\mu}_0 = \overline{\mu}_0 = 0.5$ *and* $\Delta^b = \Delta^s \equiv \Delta$.

Consider first Case 2. Under Definition 2, the Case 2-revenue can be written as

$$\mathcal{R}_{\text{Case 2}} = \log\left(\underline{\mu}_1\overline{\mu}_2\right) - \tau_1\log\left(\underline{\mu}_1\overline{\mu}_1\right) - \tau_2\log\left(\underline{\mu}_2\overline{\mu}_2\right) - \log\left[\underline{\mu}_1 e^{\Delta} + \overline{\mu}_1\right] - \log\left[\underline{\mu}_2 + \overline{\mu}_2 e^{\Delta}\right] + \overline{u}^b - \underline{u}^s.$$

Let us rewrite the Case 2-revenue as a function of $(\underline{\mu}_1, \overline{\mu}_2)$ only

$$\begin{aligned}\mathcal{R}_{\text{Case 2}}(\underline{\mu}_1, \overline{\mu}_2) = \log\left(\underline{\mu}_1\overline{\mu}_2\right) &- \frac{0.5 - \overline{\mu}_2}{1 - \underline{\mu}_1 - \overline{\mu}_2}\log\left(\underline{\mu}_1(1 - \underline{\mu}_1)\right) - \frac{0.5 - \underline{\mu}_1}{1 - \underline{\mu}_1 - \overline{\mu}_2}\log\left(\overline{\mu}_2(1 - \overline{\mu}_2)\right) \\ &- \log\left[1 - \underline{\mu}_1 + \underline{\mu}_1 e^{\Delta}\right] - \log\left[1 - \overline{\mu}_2 + \overline{\mu}_2 e^{\Delta}\right] + \overline{u}^b - \underline{u}^s.\end{aligned}$$

We show that any interior solution to our symmetric problem must be symmetric:

Lemma 8. *Let* $(\underline{\mu}_1^*, \overline{\mu}_2^*) \in \operatorname{argmax}_{(\underline{\mu}_1, \overline{\mu}_2) \in (0,0.5)^2} \mathcal{R}_{\text{Case 2}}(\underline{\mu}_1, \overline{\mu}_2)$, *then* $\underline{\mu}_1^* = \overline{\mu}_2^*$.

Proof. Follows from strict supermodularity of $\mathcal{R}_{\text{Case 2}}(\underline{\mu}_1, \overline{\mu}_2)$, see Appendix I. □

Lemma 8 allows us to write down the revenue of a symmetric problem as a function of μ_1 only:

$$\mathcal{R}_{\text{Sym}} = \begin{cases} -2\log\left[0.5e^{\Delta} + 0.5\right] + \overline{u}^b - \underline{u}^s, & \text{Case 0 } (\underline{\mu}_1 = \underline{\mu}_0 = 0.5) \\ \log\left[\underline{\mu}_1\right] - \log\left[\overline{\mu}_1\right] - 2\log\left[\underline{\mu}_1 e^{\Delta} + \overline{\mu}_1)\right] + \overline{u}^b - \underline{u}^s, & \text{Case 2 } (\underline{\mu}_1 < \underline{\mu}_0 = 0.5) \end{cases}$$

Maximizing $\mathcal{R}_{\text{Sym}}$ with respect to μ_1 is a basic calculus problem, whose solution can be obtained in closed form:

$$\overline{\mu}_1^* = \begin{cases} \overline{\mu}_0 = 0.5 & \text{if } 0 < \Delta \leq \Delta^*, \\ \frac{3}{4} + \frac{1}{4}\sqrt{9 + 8\frac{\exp(\Delta)}{1-\exp(\Delta)}} & \text{if } \Delta > \Delta^*, \end{cases}$$

where $\Delta^* \approx 2.366203279542585$, and is obtained numerically. The mechanism designer achieves:

$$\mathcal{R}^*_{\text{Sym}} = \begin{cases} -2\log\left[e^{\Delta} + 1\right] + 2\log 2 & \text{if } 0 < \Delta \leq \Delta^*, \\ \log\left[\frac{\sqrt{e^{\Delta}-1} - \sqrt{e^{\Delta}-9}}{3\sqrt{e^{\Delta}-1} + \sqrt{e^{\Delta}-9}}\right] - 2\log\left[3 + e^{\Delta} - \sqrt{(e^{\Delta}-1)(e^{\Delta}-9)}\right] + 4\log 2 & \text{if } \Delta > \Delta^*. \end{cases}$$

5 Concluding remarks

We have considered a mechanism design problem with information acquisition in a bilateral trade environment. At the beginning, the buyer, the seller, and the mechanism designer have no information about the good's quality beyond a common prior. The buyer and the seller can generate signals from a large signal space to acquire more information about the good's quality. The mechanism designer commits to a mechanism taking information acquisition by the players into account.

We characterize the set of implementable mechanisms. To check whether a particular tuple of allocations, transfers, and signals is implementable, one has to check whether these allocations, transfers, and the information structure induced by the signals satisfy a finite-dimensional system of equations and inequalities.

Using our characterization of implementability, we address the problem of maximizing revenue for an intermediary interested in implementing allocationally efficient trade. We show that implementability implies that the players will acquire perfectly correlated signals in any mechanism. We then use concavification to maximize the intermediary's revenue over mechanisms with perfectly correlated signals, and show that symmetric revenue maximization problems can be solved in closed form.

A Proof of Lemma 3

Proof. We prove the statement for the buyer only as the proof for the seller is analogous. Suppose the set of payoff-relevant states of the world is given by $V = \{\underline{v}, \ldots, \overline{v}\}$ and suppose that the proposed information structure has I signal realizations for the buyer and J signal realizations for the seller. If $|S^b| = I$ and $|S^s| = J$, then the information structure is a collection of $I \times J$ matrices, one for each state (we adopt the convention that the buyer is a *row player* and the seller is a *column player*):

State v	s_1^s	s_2^s	$\ldots$	s_J^s
s_1^b	$\alpha_{11}(v)$	$\alpha_{12}(v)$	$\ldots$	$\alpha_{1J}(v)$
s_2^b	$\alpha_{21}(v)$	$\alpha_{22}(v)$	$\ldots$	$\alpha_{2J}(v)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
s_I^b	$\alpha_{I1}(v)$	$\alpha_{I2}(v)$	$\ldots$	$\alpha_{IJ}(v)$

The cost of this information structure for the buyer is given by:

$$c^b(\alpha) = H(\mu_0) + \sum_{i=1}^{I} \sum_{v \in V} \left[\left(\sum_{j=1}^{J} \alpha_{ij}(v) \right) \log \left(\frac{\sum_{j=1}^{J} \alpha_{ij}(v)}{\sum_{j=1}^{J} \sum_{\hat{v} \in V} \alpha_{ij}(\hat{v})} \right) \right].$$

Define $f_i(\alpha) \equiv \sum_{v \in V} \left[\left(\sum_{j=1}^{J} \alpha_{ij}(v) \right) \log \left(\frac{\sum_{j=1}^{J} \alpha_{ij}(v)}{\sum_{j=1}^{J} \sum_{\hat{v} \in V} \alpha_{ij}(\hat{v})} \right) \right]$, the expected entropy component of signal realization s_i^b. The cost function can then be written as $c^b(\alpha) = H(\mu_0) + \sum_{i=1}^{I} f_i(\alpha)$. We first show the following:

Lemma 9. *$f_i(\alpha)$ is convex for every i.*

Proof. We first find $\nabla f_i(\alpha)$. To do that, note that the partial derivative of $f_i(\alpha)$ with respect to any $\alpha_{il}(v)$ is the same across all l and is given by:

$$\frac{\partial f_i(\alpha)}{\partial \alpha_{il}(v)} = \log \left(\frac{\sum_{j=1}^{J} \alpha_{ij}(v)}{\sum_{j=1}^{J} \sum_{\hat{v} \in V} \alpha_{ij}(\hat{v})} \right) + \left(\sum_{j=1}^{J} \alpha_{ij}(v) \right) \frac{\sum_{j=1}^{J} \sum_{\hat{v} \in V} \alpha_{ij}(\hat{v})}{\sum_{j=1}^{J} \alpha_{ij}(v)} \frac{\partial}{\partial \alpha_{il}(v)} \left(\frac{\sum_{j=1}^{J} \alpha_{ij}(v)}{\sum_{j=1}^{J} \sum_{\hat{v} \in V} \alpha_{ij}(\hat{v})} \right)$$
$$+ \sum_{\hat{v} \neq v} \left(\sum_{j=1}^{J} \alpha_{ij}(\hat{v}) \right) \frac{\sum_{j=1}^{J} \sum_{\hat{v} \in V} \alpha_{ij}(\hat{v})}{\sum_{j=1}^{J} \alpha_{ij}(\hat{v})} \frac{\partial}{\partial \alpha_{il}(v)} \left(\frac{\sum_{j=1}^{J} \alpha_{ij}(\hat{v})}{\sum_{j=1}^{J} \sum_{\hat{v} \in V} \alpha_{ij}(\hat{v})} \right),$$

which can be rewritten as:

$$\begin{aligned}
\frac{\partial f_i(\alpha)}{\partial \alpha_{il}(v)} &= \log\left(\frac{\sum_{j=1}^{J}\alpha_{ij}(v)}{\sum_{j=1}^{J}\sum_{\hat{v}\in V}\alpha_{ij}(\hat{v})}\right) + \left(\sum_{j=1}^{J}\sum_{\hat{v}\in V}\alpha_{ij}(\hat{v})\right)\frac{\partial}{\partial \alpha_{il}(v)}\left(\sum_{\hat{v}\in V}\frac{\sum_{j=1}^{J}\alpha_{ij}(\hat{v})}{\sum_{j=1}^{J}\sum_{\hat{v}\in V}\alpha_{ij}(\hat{v})}\right) \\
&= \log\left(\frac{\sum_{j=1}^{J}\alpha_{ij}(v)}{\sum_{j=1}^{J}\sum_{\hat{v}\in V}\alpha_{ij}(\hat{v})}\right) + \left(\sum_{j=1}^{J}\sum_{\hat{v}\in V}\alpha_{ij}(\hat{v})\right)\frac{\partial}{\partial \alpha_{il}(v)}1 \\
&= \log\left(\frac{\sum_{j=1}^{J}\alpha_{ij}(v)}{\sum_{j=1}^{J}\sum_{\hat{v}\in V}\alpha_{ij}(\hat{v})}\right).
\end{aligned}$$

To determine the Hessian of $f_i(\alpha)$ we have to take second-order derivatives. Note that for any l and r and for any state v the following is true:

$$\begin{aligned}
\frac{\partial^2 f_i(\alpha)}{\partial \alpha_{il}(v)\partial \alpha_{ir}(v)} &= \frac{\sum_{j=1}^{J}\sum_{\hat{v}\in V}\alpha_{ij}(\hat{v})}{\sum_{j=1}^{J}\alpha_{ij}(v)}\frac{1\sum_{j=1}^{J}\sum_{\hat{v}\in V}\alpha_{ij}(\hat{v}) - 1\sum_{j=1}^{J}\alpha_{ij}(v)}{\left[\sum_{j=1}^{J}\sum_{\hat{v}\in V}\alpha_{ij}(\hat{v})\right]^2} \\
&= \frac{1}{\sum_{j=1}^{J}\alpha_{ij}(v)}\frac{\sum_{j=1}^{J}\sum_{\hat{v}\in V}\alpha_{ij}(\hat{v}) - \sum_{j=1}^{J}\alpha_{ij}(v)}{\sum_{j=1}^{J}\sum_{\hat{v}\in V}\alpha_{ij}(\hat{v})}
\end{aligned}$$

Defining $A_i(v) \equiv \sum_{j=1}^{J}\alpha_{ij}(v)$, we can write:

$$\frac{\partial^2 f_i(\alpha)}{\partial \alpha_{il}(v)\alpha_{ir}(v)} = \frac{1}{A_i(v)}\frac{\sum_{\hat{v}\in V}A_i(\hat{v}) - A_i(v)}{\sum_{\hat{v}\in V}A_i(\hat{v})}$$

For every l and r and for any pair of states $v \neq \tilde{v}$ the following is true:

$$\begin{aligned}
\frac{\partial^2 f_i(\alpha)}{\partial \alpha_{il}(\tilde{v})\partial \alpha_{ir}(v)} &= \frac{\sum_{j=1}^{J}\sum_{\hat{v}\in V}\alpha_{ij}(\hat{v})}{\sum_{j=1}^{J}\alpha_{ij}(v)}\frac{0\sum_{j=1}^{J}\sum_{\hat{v}\in V}\alpha_{ij}(\hat{v}) - 1\sum_{j=1}^{J}\alpha_{ij}(v)}{\left[\sum_{j=1}^{J}\sum_{\hat{v}\in V}\alpha_{ij}(\hat{v})\right]^2} \\
&= \frac{-1}{\sum_{j=1}^{J}\sum_{\hat{v}\in V}\alpha_{ij}(\hat{v})} = \frac{-1}{\sum_{\hat{v}\in V}A_i(\hat{v})}
\end{aligned}$$

The Hessian of $f_i(\alpha)$ can then be written as $\nabla^2 f_i(\alpha) = \frac{1}{\sum_{\hat{v}\in V}A_i(\hat{v})}\mathcal{H}_i(\alpha)$, where $\mathcal{H}_i(\alpha)$ is the following matrix:

	$\alpha_{i1}(\underline{v})$	$\cdots$	$\alpha_{iJ}(\underline{v})$	$\cdots$	$\cdots$	$\alpha_{i1}(\overline{v})$	$\cdots$	$\alpha_{iJ}(\overline{v})$
$\alpha_{i1}(\underline{v})$	$\frac{\sum_{\hat{v}\in V}A_i(\hat{v})-A_i(\underline{v})}{A_i(\underline{v})}$	$\cdots$	$\frac{\sum_{\hat{v}\in V}A_i(\hat{v})-A_i(\underline{v})}{A_i(\underline{v})}$	$\cdots$	$\cdots$	-1	$\cdots$	-1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\alpha_{iJ}(\underline{v})$	$\frac{\sum_{\hat{v}\in V}A_i(\hat{v})-A_i(\underline{v})}{A_i(\underline{v})}$	$\cdots$	$\frac{\sum_{\hat{v}\in V}A_i(\hat{v})-A_i(\underline{v})}{A_i(\underline{v})}$	$\cdots$	$\cdots$	-1	$\cdots$	-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$\alpha_{i1}(\overline{v})$	-1	$\cdots$	-1	$\cdots$	$\cdots$	$\frac{\sum_{\hat{v}\in V}A_i(\hat{v})-A_i(\overline{v})}{A_i(\overline{v})}$	$\cdots$	$\frac{\sum_{\hat{v}\in V}A_i(\hat{v})-A_i(\overline{v})}{A_i(\overline{v})}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\alpha_{iJ}(\overline{v})$	-1	$\cdots$	-1	$\cdots$	$\cdots$	$\frac{\sum_{\hat{v}\in V}A_i(\hat{v})-A_i(\overline{v})}{A_i(\overline{v})}$	$\cdots$	$\frac{\sum_{\hat{v}\in V}A_i(\hat{v})-A_i(\overline{v})}{A_i(\overline{v})}$

We now show that $\nabla^2 f_i(\alpha)$ is positive semi-definite. To do that, consider an arbitrary vector $x \in \mathbb{R}^{J|V|}$ and evaluate $x^T\nabla^2 f_i(\alpha)x$. Let $x(v) \in \mathbb{R}^J$ for states $v \in V$

be such that x can be obtained by concatenating vectors $x(v)$ across all $v \in V$. Let $\mathbf{e}$ denote the vector consisting of J ones, i.e. $\mathbf{e}^T = [1, \ldots, 1] \in \mathbb{R}^J$. We then have:

$$\begin{aligned}
x^T \nabla^2 f_i(\alpha) x &= \frac{1}{\sum_{\hat{v}\in V} A_i(\hat{v})} \sum_{v\in V} \left(\frac{\sum_{\hat{v}\in V} A_i(\hat{v}) - A_i(v)}{A_i(v)} \big(\mathbf{e}^T x(v)\big)^2 - \mathbf{e}^T x(v) \sum_{\hat{v}\neq v} \mathbf{e}^T x(\hat{v}) \right) \\
&= \frac{1}{\sum_{\hat{v}\in V} A_i(\hat{v})} \sum_{v\in V} \left(\frac{\sum_{\hat{v}\in V} A_i(\hat{v})}{A_i(v)} \big(\mathbf{e}^T x(v)\big)^2 - \big(\mathbf{e}^T x(v)\big)^2 - \mathbf{e}^T x(v) \sum_{\hat{v}\neq v} \mathbf{e}^T x(\hat{v}) \right) \\
&= \sum_{v\in V} \frac{1}{A_i(v)} \big(\mathbf{e}^T x(v)\big)^2 - \frac{1}{\sum_{\hat{v}\in V} A_i(\hat{v})} \sum_{v\in V} \left(\big(\mathbf{e}^T x(v)\big)^2 + \mathbf{e}^T x(v) \sum_{\hat{v}\neq v} \mathbf{e}^T x(\hat{v}) \right) \\
&= \sum_{v\in V} \frac{1}{A_i(v)} \big(\mathbf{e}^T x(v)\big)^2 - \frac{1}{\sum_{v\in V} A_i(v)} \left(\sum_{v\in V} \mathbf{e}^T x(v) \right)^2.
\end{aligned}$$

Defining $X(v) \equiv \mathbf{e}^T x(v)$ for every $v \in V$, we can write:

$$x^T \nabla^2 f_i(\alpha) x = \sum_{v\in V} \frac{1}{A_i(v)} X^2(v) - \frac{1}{\sum_{v\in V} A_i(v)} \left(\sum_{v\in V} X(v) \right)^2$$

To show that $\nabla^2 f_i(\alpha)$ is positive semi-definite, we have to show that the above expression is weakly positive for all $\{X(v)\}_{v\in V}$. In order to do that, we show that

$$\min_{\{X(v)\}_{v\in V}} \left\{ \sum_{v\in V} \frac{1}{A_i(v)} X^2(v) - \frac{1}{\sum_{v\in V} A_i(v)} \left(\sum_{v\in V} X(v) \right)^2 \right\} \geq 0.$$

To that end, consider the restricted problem for some $\breve{X} \in \mathbb{R}$ given by:

$$\min_{\{X(v)\}_{v\in V}} \left\{ \sum_{v\in V} \frac{1}{A_i(v)} X^2(v) - \frac{1}{\sum_{v\in V} A_i(v)} \breve{X}^2 \text{ s.t. } \sum_{v\in V} X(v) = \breve{X} \right\}.$$

The restricted problem is clearly convex in $\{X(v)\}_{v\in V}$, hence the first order conditions are necessary and sufficient for minimization. The Lagrangian is given by:

$$\mathcal{L}(X;\eta) = \sum_{v\in V} \frac{1}{A_i(v)} X^2(v) - \frac{1}{\sum_{v\in V} A_i(v)} \breve{X}^2 - 2\eta \left(\sum_{v\in V} X(v) - \breve{X} \right).$$

The optimality conditions are given by:

$$\begin{cases} \frac{1}{A_i(v)} 2X^*(v) - 2\eta^* = 0 \quad \forall v \in V, \\ \sum_{v\in V} X^*(v) = \breve{X}. \end{cases}$$

The minimum is achieved at $X^*(v) = \frac{A_i(v)\breve{X}}{\sum_{\hat{v}\in V} A_i(\hat{v})}$, and the value of the objective achieved at the minimum is given by:

$$\begin{aligned}
\sum_{v\in V} \frac{1}{A_i(v)} \frac{A_i^2(v)\breve{X}^2}{\big(\sum_{\hat{v}\in V} A_i(\hat{v})\big)^2} - \frac{1}{\sum_{v\in V} A_i(v)} \breve{X}^2 &= \sum_{v\in V} \frac{A_i(v)\breve{X}^2}{\big(\sum_{\hat{v}\in V} A_i(\hat{v})\big)^2} - \frac{1}{\sum_{v\in V} A_i(v)} \breve{X}^2 \\
&= \breve{X}^2 \left[\frac{\sum_{v\in V} A_i(v)}{\big(\sum_{v\in V} A_i(v)\big)^2} - \frac{1}{\sum_{v\in V} A_i(v)} \right] = 0,
\end{aligned}$$

implying that the minimal value achieved in the restricted problem is zero for every $\breve{X} \in \mathbb{R}$, implying in turn that the minimal value achieved by the unrestricted problem is also zero, hence $x^T \nabla^2 f_i(\alpha) x \geq 0$ for every $x \in \mathbb{R}^{J|V|}$ and that $\nabla^2 f_i(\alpha)$ is positive semi-definite, which means that $f_i(\alpha)$ is convex. □

Recall that $c(\alpha) = H(\mu_0) + \sum_{i=1}^{I} f_i(\alpha)$ and hence is a sum of convex functions, implying that $c(\alpha)$ is convex. □

B Proof of Proposition 1 (Revelation principle)

Proof. Let $\big(M_{\text{IN}}, q_{\text{IN}}, t_{\text{IN}}\big)$ be a (possibly indirect) mechanism and $\Big[\big(\sigma^b, \big\{\mathbf{m}^b_{\text{IN}}[\hat{\sigma}^b]\big\}_{\hat{\sigma}^b \in \Sigma^b}\big), \big(\sigma^s, \big\{\mathbf{m}^s_{\text{IN}}[\hat{\sigma}^s]\big\}_{\hat{\sigma}^s \in \Sigma^s}\big)\Big]$, where $\sigma^b = \big(S^b, \mathbf{S}^b\big)$ and $\sigma^s = \big(S^s, \mathbf{S}^s\big)$, be its Nash equilibrium. Let α be the information structure induced by the signals $\big(\sigma^b, \sigma^s\big)$. Using Lemmas 1 and 2, we can write the equilibrium conditions as follows.

- For the buyer: $(S^b, \alpha, \mathbf{m}^b_{\text{IN}}[\sigma^b])$ solves the following problem:

$$\max_{\tilde{\alpha}, \tilde{S}^b, \tilde{\mathbf{m}}^b_{\text{IN}}} \sum_{s^b \in \tilde{S}^b} \sum_{s^s \in S^s} \sum_{v \in V} \tilde{\alpha}(s^b, s^s; v)\big(q^b_{\text{IN}}\big(\tilde{\mathbf{m}}^b_{\text{IN}}(s^b), \mathbf{m}^s_{\text{IN}}[\sigma^s](s^s)\big)u^b(v) - t^b_{\text{IN}}\big(\tilde{\mathbf{m}}^b_{\text{IN}}(s^b), \mathbf{m}^s_{\text{IN}}[\sigma^s](s^s)\big) - c^b(\tilde{\alpha}),$$

s.t. (1) $\tilde{S}^b \in \mathscr{P}(\mathbb{N}), \;\; \tilde{\alpha} \in \Delta(\tilde{S}^b \times S^s \times V), \;\; \tilde{\mathbf{m}}^b_{\text{IN}} : \tilde{S}^b \to M^b_{\text{IN}};$

(2) $\text{marg}_{S^s \times V} \tilde{\alpha} = \text{marg}_{S^s \times V} \alpha.$

- For the seller: $(S^s, \alpha, \mathbf{m}^s_{\text{IN}}[\sigma^s])$ solves the following problem:

$$\max_{\tilde{\alpha}, \tilde{S}^s, \tilde{\mathbf{m}}^s_{\text{IN}}} \sum_{s^b \in S^b} \sum_{s^s \in \tilde{S}^s} \sum_{v \in V} \tilde{\alpha}(s^b, s^s; v)\big(t^s_{\text{IN}}\big(\mathbf{m}^b_{\text{IN}}[\sigma^b](s^b), \tilde{\mathbf{m}}^s_{\text{IN}}(s^s)\big) - q^s_{\text{IN}}\big(\mathbf{m}^b_{\text{IN}}[\sigma^b](s^b), \tilde{\mathbf{m}}^s_{\text{IN}}(s^s)\big)u^s(v)\big) - c^s(\tilde{\alpha}),$$

s.t. (1) $\tilde{S}^s \in \mathscr{P}(\mathbb{N}), \;\; \tilde{\alpha} \in \Delta(S^b \times \tilde{S}^s \times V), \;\; \tilde{\mathbf{m}}^s_{\text{IN}} : \tilde{S}^s \to M^s_{\text{IN}};$

(2) $\text{marg}_{S^b \times V} \tilde{\alpha} = \text{marg}_{S^b \times V} \alpha.$

Consider the following direct mechanism $\big(M_{\text{D}}, q_{\text{D}}, t_{\text{D}}\big)$, where the message space is given by $M_{\text{D}} \equiv \big(S^b \cup \{m_\emptyset\}\big) \times \big(S^s \cup \{m_\emptyset\}\big)$; the allocation function is defined as $q^p_{\text{D}}(s^b, s^s) \equiv q^p_{\text{IN}}\big(\mathbf{m}^b_{\text{IN}}[\sigma^b](s^b), \mathbf{m}^s_{\text{IN}}[\sigma^s](s^s)\big)$, and the transfer function is given by $t^p_{\text{D}}(s^b, s^s) \equiv t^p_{\text{IN}}\big(\mathbf{m}^b_{\text{IN}}[\sigma^b](s^b), \mathbf{m}^s_{\text{IN}}[\sigma^s](s^s)\big)$ for both players $p \in \{b, s\}$. We claim that $\Big[\big(\sigma^b, \big\{\mathbf{m}^b_{\text{D}}[\hat{\sigma}^b]\big\}_{\hat{\sigma}^b \in \Sigma^b}\big), \; \big(\sigma^s, \big\{\mathbf{m}^s_{\text{D}}[\hat{\sigma}^s]\big\}_{\hat{\sigma}^s \in \Sigma^s}\big)\Big]$, where $\mathbf{m}^p_{\text{D}}[\hat{\sigma}^p] = \mathbf{m}^p_T$ for all $\hat{\sigma}^p \in \Sigma^p$ is a Nash equilibrium in the direct mechanism.

Suppose for a contradiction that this is not the case, then one of the players has a profitable deviation to untruthful reporting, a different signal, or both. Let us suppose that it is the buyer who can profitably deviate (the argument for the seller is identical), then the tuple $(S^b, \alpha, \mathbf{m}^b_T)$ violates the constraint IC^b_{A} for the direct mechanism $\big(M_{\text{D}}, q_{\text{D}}, t_{\text{D}}\big)$, i.e. there exists a signal $\tilde{\sigma}^b = \big(\tilde{S}^b, \tilde{\mathbf{S}}^b\big)$ inducing a new joint distribution $\tilde{\alpha} \in \Delta\big(\tilde{S}^b \times S^s \times V\big)$ and $\text{marg}_{S^s \times V}\tilde{\alpha} = \text{marg}_{S^s \times V}\alpha$, and a (not necessarily truthful) reporting function $\tilde{\mathbf{m}}^b_{\text{D}} : \tilde{S}^b \to S^b \cup \{m_\emptyset\}$ such that for signal realization s^b:

$$\sum_{s^b \in \tilde{S}^b} \sum_{s^s \in S^s} \sum_{v \in V} \tilde{\alpha}(s^b, s^s; v)\big(q^b_{\text{D}}(\tilde{\mathbf{m}}^b_{\text{D}}(s^b), s^s)u^b(v) - t^b_{\text{D}}(\tilde{\mathbf{m}}^b_{\text{D}}(s^b), s^s)\big) - c^b(\tilde{\alpha})$$
$$> \sum_{s^b \in S^b} \sum_{s^s \in S^s} \sum_{v \in V} \alpha(s^b, s^s; v)\big(q^b_{\text{D}}(s^b, s^s)u^b(v) - t^b_{\text{D}}(s^b, s^s)\big) - c^b(\alpha),$$

implying (by definition of allocation and transfer functions in the direct mechanism):

$$\sum_{s^b \in \tilde{S}^b} \sum_{s^s \in S^s} \sum_{v \in V} \tilde{\alpha}(s^b, s^s; v)\big(q^b_{\text{IN}}(\mathbf{m}^b_{\text{IN}}[\sigma^b](\tilde{\mathbf{m}}^b_{\text{D}}(s^b)), \mathbf{m}^s_{\text{IN}}[\sigma^s](s^s))u^b(v) - t^b_{\text{IN}}(\mathbf{m}^b_{\text{IN}}[\sigma^b](\tilde{\mathbf{m}}^b_{\text{D}}(s^b)), \mathbf{m}^s_{\text{IN}}[\sigma^s](s^s))\big) - c^b(\tilde{\alpha})$$
$$> \sum_{s^b \in S^b} \sum_{s^s \in S^s} \sum_{v \in V} \alpha(s^b, s^s; v)\big(q^b_{\text{IN}}(\mathbf{m}^b_{\text{IN}}[\sigma^b](s^b), \mathbf{m}^s_{\text{IN}}[\sigma^s](s^s))u^b(v) - t^b_{\text{IN}}(\mathbf{m}^b_{\text{IN}}[\sigma^b](s^b), \mathbf{m}^s_{\text{IN}}[\sigma^s](s^s))\big) - c^b(\alpha),$$

which in turn means that the tuple $(S^b, \alpha, \mathbf{m}^b_{\text{IN}}[\sigma^b])$ violates the buyer's equilibrium conditions, hence a contradiction. □

C Proof of Proposition 2

Let $\big[\big(\sigma^b, \{\mathbf{m}^b[\hat{\sigma}^b]\}_{\hat{\sigma}^b \in \Sigma^b}\big),\ \big(\sigma^s, \{\mathbf{m}^s[\hat{\sigma}^s]\}_{\hat{\sigma}^s \in \Sigma^s}\big)\big]$ be a truthful-revelation Nash equilibrium of a direct mechanism. It implies in particular $\mathbf{m}^b[\sigma^b] = \mathbf{m}^b_T$ and $\mathbf{m}^s[\sigma^s] = \mathbf{m}^s_T$. Use $\alpha \in \Delta(S^b \times S^s \times V)$ to denote the joint distribution of signal realizations and states of the world induced by the on-path profile of signals $\big(\sigma^b, \sigma^s\big)$. By assumption $\big(\alpha, S^p, \mathbf{m}^p_T\big)$ satisfies IC^p_{A} for player p and α is Bayes-plausible.

Consider a profile $\big[\big(\sigma^b, \{\mathbf{m}^b_{\text{PBE}}[\hat{\sigma}^b]\}_{\hat{\sigma}^b \in \Sigma^b}\big),\ \big(\sigma^s, \{\mathbf{m}^s_{\text{PBE}}[\hat{\sigma}^s]\}_{\hat{\sigma}^s \in \Sigma^s}\big)\big]$ where $\mathbf{m}^p_{\text{PBE}}[\sigma^p] \equiv \mathbf{m}^p_T$ and $\mathbf{m}^p_{\text{PBE}}[\hat{\sigma}^p]$ for $\hat{\sigma} \neq \sigma^p$ are to be defined below. By construction, this strategy profile is outcome-equivalent to the original Nash equilibrium strategy profile $\big[\big(\sigma^b, \{\mathbf{m}^b[\hat{\sigma}^b]\}_{\hat{\sigma}^b \in \Sigma^b}\big),\ \big(\sigma^s, \{\mathbf{m}^s[\hat{\sigma}^s]\}_{\hat{\sigma}^s \in \Sigma^s}\big)\big]$. We are now going to show that it can be a perfect Bayesian equilibrium profile of the same direct mechanism. To do that, let us first specify the players' beliefs. Let $\mathcal{I}^p\big(\hat{\sigma}^p, s^p\big)$ denote the information set achieved by player p who has played $\hat{\sigma}^p \in \Sigma^p$ and observed a signal realization $s^p \in \mathbb{N}$. Let $\gamma^p\big(\hat{\sigma}^{-p}, s^{-p}, v|\mathcal{I}^p\big(\hat{\sigma}^p, s^p\big)\big)$ denote the belief of player p that player $-p$ has played $\hat{\sigma}^{-p} \in \Sigma^{-p}$, has observed the signal realization $s^{-p} \in \mathbb{N}$; and the state of the world is $v \in V$. We specify the players' beliefs as follows:

1. The beliefs at information sets $\mathcal{I}^p\big(\hat{\sigma}^p, s^p\big)$ such that $\hat{\sigma}^p \neq \sigma^p$ are derived using Bayes rule for the buyer from $\alpha[\hat{\sigma}^b, \sigma^s]$ and for the seller from $\alpha[\sigma^b, \hat{\sigma}^s]$. These

beliefs are given by:

$$\gamma^b\big(\hat{\sigma}^s, s^s; v|\mathcal{I}^b\big(\sigma^b, s^b\big)\big) = \begin{cases} \frac{\alpha[\hat{\sigma}^b,\sigma^s](s^b,s^s;v)}{\sum_{i=1}^{+\infty}\alpha[\hat{\sigma}^b,\sigma^s](i,s^s;v)} & \text{for } \hat{\sigma}^s = \sigma^s. \\ 0 & \text{otherwise.} \end{cases}$$

$$\gamma^s\big(\hat{\sigma}^b, s^b; v|\mathcal{I}^s\big(\sigma^s, s^s\big)\big) = \begin{cases} \frac{\alpha[\sigma^b,\hat{\sigma}^s](s^b,s^s;v)}{\sum_{j=1}^{+\infty}\alpha[\sigma^b,\hat{\sigma}^s](s^b,j;v)} & \text{for } \hat{\sigma}^b = \sigma^b, \\ 0 & \text{otherwise.} \end{cases}$$

2. The beliefs at $\mathcal{I}^p\big(\sigma^p, s^p\big)$ are derived using Bayes rule from α. These beliefs are:

$$\gamma^b\big(\hat{\sigma}^s, s^s; v|\mathcal{I}^b\big(\sigma^b, s^b\big)\big) = \begin{cases} \frac{\alpha(s^b,s^s;v)}{\sum_{i=1}^{+\infty}\alpha(i,s^s;v)} & \text{for } \hat{\sigma}^s = \sigma^s. \\ 0 & \text{otherwise.} \end{cases}$$

$$\gamma^s\big(\hat{\sigma}^b, s^b; v|\mathcal{I}^s\big(\sigma^s, s^s\big)\big) = \begin{cases} \frac{\alpha(s^b,s^s;v)}{\sum_{j=1}^{+\infty}\alpha(s^b,j;v)} & \text{for } \hat{\sigma}^b = \sigma^b, \\ 0 & \text{otherwise.} \end{cases}$$

Let us now show that $\big[\big(\sigma^b, \big\{\mathbf{m}^b_{\text{PBE}}[\hat{\sigma}^b]\big\}_{\hat{\sigma}^b\in\Sigma^b}\big),\ \big(\sigma^s, \big\{\mathbf{m}^s_{\text{PBE}}[\hat{\sigma}^s]\big\}_{\hat{\sigma}^s\in\Sigma^s}\big)\big]$ is sequentially rational given the beliefs specified above.

C.1 Reporting after off-path information acquisition

Let us start with the off-path signals. Suppose the buyer has arrived at the information set $\mathcal{I}^b\big(\hat{\sigma}^b, s^b\big)$ with $\hat{\sigma}^b \neq \sigma^b$, obtain the report following $\big(\hat{\sigma}^b, s^b\big)$ by solving (in case there are many solutions, pick any):

$$\mathbf{m}^b_{\text{PBE}}[\hat{\sigma}^b](s^b) \equiv \operatorname*{argmax}_{m\in S^b\cup\{m_\emptyset\}} \sum_{s^s\in S^s}\sum_{v\in V} \frac{\alpha[\hat{\sigma}^b,\sigma^s](s^b,s^s;v)}{\sum_{i=1}^{+\infty}\alpha[\hat{\sigma}^b,\sigma^s](i,s^s;v)}\big(q^b(m,s^s)u^b(v) - t^b(m,s^s)\big)$$

The resulting reporting function $\mathbf{m}^b_{\text{PBE}}[\hat{\sigma}^b]$ is sequentially rational.

Likewise, suppose the seller has arrived at the information set $\mathcal{I}^s\big(\hat{\sigma}^s, s^s\big)$ with $\hat{\sigma}^s \neq \sigma^s$, obtain the report following $\big(\hat{\sigma}^s, s^s\big)$ by solving:

$$\mathbf{m}^s_{\text{PBE}}[\hat{\sigma}^s](s^s) \equiv \operatorname*{argmax}_{m\in S^s\cup\{m_\emptyset\}} \sum_{s^b\in S^b}\sum_{v\in V} \frac{\alpha[\sigma^b,\hat{\sigma}^s](s^b,s^s;v)}{\sum_{j=1}^{+\infty}\alpha[\sigma^b,\hat{\sigma}^s](s^b,j;v)}\big(t^s(s^b,m) - q^s(s^b,m)u^s(v)\big)$$

The resulting reporting function $\mathbf{m}^s_{\text{PBE}}[\hat{\sigma}^s]$ is sequentially rational.

It remains to show sequential rationality of truthful reporting after choosing the on-path signal.

C.2 Reporting after on-path information acquisition

Let us now move on to the on-path signals. Suppose player p has arrived at the information set $\mathcal{I}^p(\sigma^p, s^p)$. At this information set player p believes that player $-p$ has taken his on-path action as well with probability 1, and player p's beliefs about signal realizations are derived from α using Bayes' rule. The proposed perfect Bayesian equilibrium strategy prescribes truthful reporting after playing the on-path information acquisition action. There are two ways, in which player p could deviate from truthful reporting: he could misreport a particular signal realization, or he could abstain following a particular signal realization. In what follows, we show that these deviations are not profitable.

C.2.1 Misreporting a signal realization

If a signal realization s^b occurs with positive probability given α, then the buyer is willing to report it truthfully as long as the following *interim* incentive compatibility condition is satisfied:

$$(\text{IC}_{\text{I}}^b) \quad \sum_{s^s \in S^s} \sum_{v \in V} \alpha(s^b, s^s; v)\big(q^b(s^b, s^s)u^b(v) - t^b(s^b, s^s)\big) \geq \sum_{s^s \in S^s} \sum_{v \in V} \alpha(s^b, s^s; v)\big(q^b(\tilde{s}^b, s^s)u^b(v) - t^b(\tilde{s}^b, s^s)\big)$$

for all $\tilde{s}^b \in S^b$.

Likewise if a signal realization s^s occurs with positive probability given α, then the buyer is willing to report it truthfully as long as the following *interim* incentive compatibility condition is satisfied:

$$(\text{IC}_{\text{I}}^s) \quad \sum_{s^b \in S^b} \sum_{v \in V} \alpha(s^b, s^s; v)\big(t^s(s^b, s^s) - q^s(s^b, s^s)u^b(v)\big) \geq \sum_{s^b \in S^b} \sum_{v \in V} \alpha(s^b, s^s; v)\big(t^s(s^b, \tilde{s}^s) - q^s(s^b, \tilde{s}^s)u^s(v)\big)$$

for all $\tilde{s}^s \in S^s$.

The following lemma shows that the *interim* incentive compatibility conditions are implied by *ex ante* incentive compatibility conditions:

Lemma 10. $\text{IC}_{\text{A}}^p \Rightarrow \text{IC}_{\text{I}}^p$ *for both players* $p \in \{b, s\}$

Proof. We show that $\neg IC_I^b \Rightarrow \neg IC_A^b$. The argument for the seller is again identical. Suppose that the mechanism is not *interim* incentive compatible for the buyer, i.e. there exists a signal realization $x^b \in S^b$, which occurs with positive probability, and a non-truthful report $\tilde{x}^b \in S^b$ such that:

$$\sum_{s^s \in S^s} \sum_{v \in V} \alpha(x^b, s^s; v)\big(q^b(x^b, s^s)u^b(v) - t^b(x^b, s^s)\big) < \sum_{s^s \in S^s} \sum_{v \in V} \alpha(x^b, s^s; v)\big(q^b(\tilde{x}^b, s^s)u^b(v) - t^b(\tilde{x}^b, s^s)\big)$$

Consider an *ex ante* deviation to $(S^b, \alpha, \tilde{\mathbf{m}}^b)$, where:

$$\tilde{\mathbf{m}}^b(s^b) = \begin{cases} s^b, & \text{if } s^b \neq x^b; \\ \tilde{x}^b, & \text{if } s^b = x^b. \end{cases}$$

The payoff from this deviation is given by:

$$\sum_{s^b \in S^b} \sum_{s^s \in S^s} \sum_{v \in V} \alpha(s^b, s^s; v)\big(q^b(\tilde{\mathbf{m}}^b(s^b), s^s)u^b(v) - t^b(\tilde{\mathbf{m}}^b(s^b), s^s)\big) - c^b(\alpha) > \sum_{s^b \in S^b} \sum_{s^s \in S^s} \sum_{v \in V} \alpha(s^b, s^s; v)\big(q^b(s^b, s^s)u^b(v) - t^b(s^b, s^s)\big) - c^b(\alpha),$$

implying that the mechanism is not *ex ante* incentive compatible for the buyer. The argument for the seller is identical. □

C.2.2 Abstaining instead of reporting a signal realization

If a signal realization s^b occurs with positive probability given α, the buyer is willing to report it instead of abstaining if the following *interim* individual rationality condition is satisfied:

$$(\text{IR}_\text{I}^b) \qquad \sum_{s^s \in S^s} \sum_{v \in V} \alpha(s^b, s^s; v)\big(q^b(s^b, s^s)u^b(v) - t^b(s^b, s^s)\big) \geq 0.$$

Likewise if a signal realization s^s occurs with positive probability given α, the seller is willing to report it instead of abstaining if the following *interim* individual rationality condition is satisfied:

$$(\text{IR}_\text{I}^s) \qquad \sum_{s^b \in S^b} \sum_{v \in V} \alpha(s^b, s^s; v)\big(t^s(s^b, s^s) - q^s(s^b, s^s)u^b(v)\big) \geq 0.$$

The following lemma shows that the *interim* individual rationality conditions are implied by *ex ante* incentive compatibility conditions:

Lemma 11. $\mathrm{IC}^p_{\mathrm{A}} \Rightarrow \mathrm{IR}^p_{\mathrm{I}}$ *for both players* $p \in \{b, s\}$

Proof. We show that $\neg IR^b_I \Rightarrow \neg IC^b_A$. The argument for the seller is again identical. Suppose that the mechanism is not *interim* individually rational for the buyer, i.e. there exists a signal realization $x^b \in S^b$, which occurs with positive probability, such that

$$\sum_{s^s \in S^s} \sum_{v \in V} \alpha(x^b, s^s; v)\big(q^b(x^b, s^s)u^b(v) - t^b(x^b, s^s)\big) < 0$$

Consider an *ex ante* deviation to $(S^b, \alpha, \tilde{\mathbf{m}}^b)$, where:

$$\tilde{\mathbf{m}}^b(s^b) = \begin{cases} s^b, & \text{if } s^b \neq x^b; \\ m_\emptyset, & \text{if } s^b = x^b. \end{cases}$$

The payoff from this deviation is given by:

$$\sum_{s^b \in S^b} \sum_{s^s \in S^s} \sum_{v \in V} \alpha(s^b, s^s; v)\big(q^b(\tilde{\mathbf{m}}^b(s^b), s^s)u^b(v) - t^b(\tilde{\mathbf{m}}^b(s^b), s^s)\big) - c^b(\alpha)$$
$$> \sum_{s^b \in S^b} \sum_{s^s \in S^s} \sum_{v \in V} \alpha(s^b, s^s; v)\big(q^b(s^b, s^s)u^b(v) - t^b(s^b, s^s)\big) - c^b(\alpha),$$

implying that the mechanism is not *ex ante* incentive compatible for the buyer. The argument for the seller is identical. □

D Mixed and correlated strategies

In this appendix, we provide an argument suggesting that the treatment of mixed and correlated strategies in our environment might require altogether different methods. In particular, we explore a natural approach one could take to prove that mixed and correlated strategies are outcome-equivalent to pure strategies, and show, by providing a counterexample, that this approach does not yield the desired result.

Suppose that the players randomize over the sets of signals $R^b = \{\sigma^b_1, \sigma^b_2, \ldots, \sigma^b_K\}$ and $R^s = \{\sigma^s_1, \sigma^s_2, \ldots, \sigma^s_N\}$. Their strategy profile gives rise to the following joint distribution over signals

	σ^s_1	σ^s_2	$\ldots$	σ^s_N
σ^b_1	$\mathcal{P}[\sigma^b_1, \sigma^s_1]$	$\mathcal{P}[\sigma^b_1, \sigma^s_2]$	$\ldots$	$\mathcal{P}[\sigma^b_1, \sigma^s_N]$
σ^b_2	$\mathcal{P}[\sigma^b_2, \sigma^s_1]$	$\mathcal{P}[\sigma^b_2, \sigma^s_2]$	$\ldots$	$\mathcal{P}[\sigma^b_2, \sigma^s_N]$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
σ^b_K	$\mathcal{P}[\sigma^b_K, \sigma^s_1]$	$\mathcal{P}[\sigma^b_K, \sigma^s_2]$	$\ldots$	$\mathcal{P}[\sigma^b_K, \sigma^s_N]$

Note that these randomizations could in principle be correlated if we enriched our setup with an additional communication stage at the beginning of the game, in which the mechanism designer would issue correlated recommendations to the players. We show below, however, that even independent randomizations cause difficulties.

If one wanted to prove that our restriction to pure strategies is without loss of generality, one could define a new information structure by finding the average over the information structures given above as follows

$$\hat{\alpha}(s_i^b, s_j^s; v) \equiv \sum_{\sigma^b \in R^b} \sum_{\sigma^s \in R^s} \mathcal{P}[\sigma^b, \sigma^s] \alpha[\sigma^b, \sigma^s](s_i^b, s_j^s; v),$$

and notice that, due to Bayes-plausibility of the new information structure, the new information structure can be induced by a pure strategy profile $(\hat{\sigma}^b, \hat{\sigma}^s)$. One could then hope that if the original distribution of the information structures arises in some equilibrium, then the new information structure could also arise in an outcome-equivalent equilbrium of a possibly different mechanism. The next counterexample shows that this strategy will not work: it is possible to construct a deviation from the resulting pure strategy profile $(\hat{\sigma}^b, \hat{\sigma}^s)$ that induces an information structure that cannot be induced by a deviation from the original mixed/correlated strategy profile (see Gentzkow and Kamenica (2017) and Li and Norman (2018) who point out a similar issue in the context of multisender Bayesian persuasion).

D.1 Counterexample

Consider the following strategy profile:

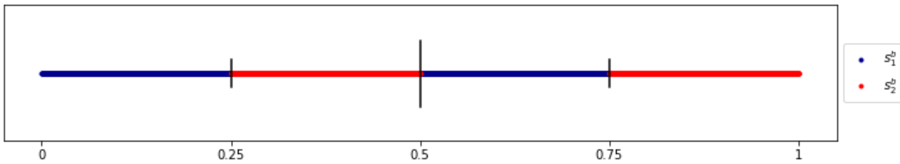
	σ_1^s	σ_2^s
σ^b	$\frac{1}{2}$	$\frac{1}{2}$

In words, the seller mixes between σ_1^s and σ_2^s with equal probabilities. The buyer plays σ^b with probability 1. The strategies are defined as follows:

- $\sigma^b = (S^b, \mathbf{S}^b)$, where $S^b = \{s_1^b, s_2^b\}$ and $\mathbf{S}^b$ is given by:

$$\mathbf{S}^b(x) = \begin{cases} s_1^b & \text{if } x \in [0, 0.25] \cup (0.5, 0.75], \\ s_2^b & \text{if } x \in (0.25, 0.5] \cup (0.75, 1]. \end{cases}$$

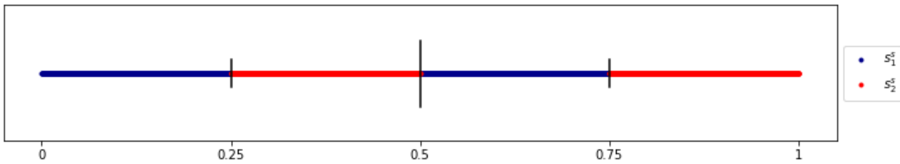
The corresponding partition of $X = [0, 1]$ is illustrated by:



- $\sigma_1^s = \left(S_1^s, \mathbf{S}_1^s\right)$, where $S_1^s = \{s_1^s, s_2^s\}$ and $\mathbf{S}_1^s$ is given by:

$$\mathbf{S}_1^s(x) = \begin{cases} s_1^s & \text{if } x \in [0, 0.25] \cup (0.5, 0.75], \\ s_2^s & \text{if } x \in (0.25, 0.5] \cup (0.75, 1]. \end{cases}$$

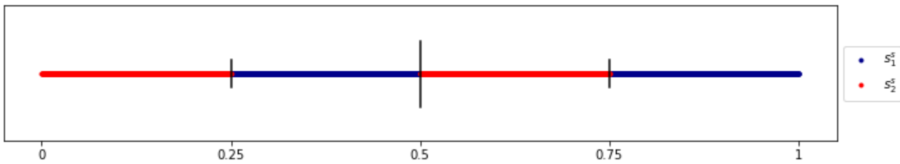
The corresponding partition of $X = [0, 1]$ is illustrated by:



- $\sigma_2^s = \left(S_2^s, \mathbf{S}_2^s\right)$, where $S_2^s = \{s_1^s, s_2^s\}$ and $\mathbf{S}_2^s$ is given by:

$$\mathbf{S}_2^s(x) = \begin{cases} s_1^s & \text{if } x \in (0.25, 0.5] \cup (0.75, 1], \\ s_2^s & \text{if } x \in [0, 0.25] \cup (0.5, 0.75]. \end{cases}$$

The corresponding partition of $X = [0, 1]$ is illustrated by:



Observe that if the players play the signal profile $\left(\sigma^b, \sigma_1^s\right)$, they induce the information structure $\alpha[\sigma^b, \sigma_1^s]$ given by:

State $\underline{v}$	s_1^s	s_2^s
s_1^b	$\frac{1}{4}$	0
s_2^b	0	$\frac{1}{4}$

State $\overline{v}$	s_1^s	s_2^s
s_1^b	$\frac{1}{4}$	0
s_2^b	0	$\frac{1}{4}$

Likewise, if the players play the signal profile $\left(\sigma^b, \sigma_2^s\right)$, they induce the information structure $\alpha[\sigma^b, \sigma_2^s]$ given by

State $\underline{v}$	s_1^s	s_2^s
s_1^b	0	$\frac{1}{4}$
s_2^b	$\frac{1}{4}$	0

State $\overline{v}$	s_1^s	s_2^s
s_1^b	0	$\frac{1}{4}$
s_2^b	$\frac{1}{4}$	0

The average over the two information structures $\frac{1}{2}\alpha[\sigma^b, \sigma_1^s] + \frac{1}{2}\alpha[\sigma^b, \sigma_2^s]$ is:

State $\underline{v}$	s_1^s	s_2^s
s_1^b	$\frac{1}{8}$	$\frac{1}{8}$
s_2^b	$\frac{1}{8}$	$\frac{1}{8}$

State $\overline{v}$	s_1^s	s_2^s
s_1^b	$\frac{1}{8}$	$\frac{1}{8}$
s_2^b	$\frac{1}{8}$	$\frac{1}{8}$

Lemma 1 in the main text ensures that $\frac{1}{2}\alpha[\sigma^b, \sigma_1^s] + \frac{1}{2}\alpha[\sigma^b, \sigma_2^s]$ can be induced by a profile of pure signals. Let $(\hat{\sigma}^b, \hat{\sigma}^s)$ be such a profile of pure signals. Lemma 2 in the main text shows that, by deviating to some $\tilde{\sigma}^b$ (i.e. to the pure signal profile $(\tilde{\sigma}^b, \hat{\sigma}^s)$), the buyer can induce any information structure that has the same seller-marginals as $\frac{1}{2}\alpha[\sigma^b, \sigma_1^s] + \frac{1}{2}\alpha[\sigma^b, \sigma_2^s]$. In particular, there exists $\tilde{\sigma}^b$ such that $(\tilde{\sigma}^b, \hat{\sigma}^s)$ induces $\alpha[\sigma^b, \sigma_1^s]$ since $\alpha[\sigma^b, \sigma_1^s]$ has the same seller-marginals as $\frac{1}{2}\alpha[\sigma^b, \sigma_1^s] + \frac{1}{2}\alpha[\sigma^b, \sigma_2^s]$. The next proposition, however, shows that it's impossible to obtain $\alpha[\sigma^b, \sigma_1^s]$ by taking averages over the information structures induced by any deviation from σ^b when the seller plays his original mixed strategy $\frac{1}{2}\sigma_1^s + \frac{1}{2}\sigma_s^2$:

Proposition 6. *There is no $\tilde{\sigma}^b$ such that $\alpha[\sigma^b, \sigma_1^s] = \frac{1}{2}\alpha[\tilde{\sigma}^b, \sigma_1^s] + \frac{1}{2}\alpha[\tilde{\sigma}^b, \sigma_2^s]$.*

Proof. Suppose for a contradiction that such $\tilde{\sigma}^b$ exists, and recall that $\alpha[\sigma^b, \sigma_1^s]$ is:

State $\underline{v}$	s_1^s	s_2^s
s_1^b	$\frac{1}{4}$	0
s_2^b	0	$\frac{1}{4}$

State $\overline{v}$	s_1^s	s_2^s
s_1^b	$\frac{1}{4}$	0
s_2^b	0	$\frac{1}{4}$

Since only signal realizations s_1^b and s_2^b occur with positive probability under $\tilde{\sigma}^b$, it is without loss of generality to restrict attention to $\tilde{\sigma}^b = (\tilde{S}^b, \tilde{\mathbf{S}}^b)$ such that $\tilde{S}^b = \{s_1^b, s_2^b\}$ and $\tilde{\mathbf{S}}^b : X \to \tilde{S}^b$. To obtain a contradiction, note the following:

- $\alpha[\sigma^b, \sigma_1^s](s_1^b, s_2^s; \underline{v}) = 0$, hence it is true that $\alpha[\tilde{\sigma}^b, \sigma_1^s](s_1^b, s_2^s; \underline{v}) = \alpha[\tilde{\sigma}^b, \sigma_2^s](s_1^b, s_2^s; \underline{v}) = 0$. Given the above definitions of σ_1^s and σ_2^s these imply that $[\tilde{\mathbf{S}}^b]^{-1}(s_1^b) \cap (0.25, 0.5] = \emptyset$ and $[\tilde{\mathbf{S}}^b]^{-1}(s_1^b) \cap [0, 0.25] = \emptyset$ respectively, which in turn means that $[\tilde{\mathbf{S}}^b]^{-1}(s_1^b) \cap [0, 0.5] = \emptyset$.
- $\alpha[\sigma^b, \sigma_1^s](s_2^b, s_1^s; \underline{v}) = 0$, hence it is true that $\alpha[\tilde{\sigma}^b, \sigma_1^s](s_2^b, s_1^s; \underline{v}) = \alpha[\tilde{\sigma}^b, \sigma_2^s](s_2^b, s_1^s; \underline{v}) = 0$. Given the above definitions of σ_1^s and σ_2^s these imply that $[\tilde{\mathbf{S}}^b]^{-1}(s_2^b) \cap [0, 0.25] = \emptyset$ and $[\tilde{\mathbf{S}}^b]^{-1}(s_2^b) \cap (0.25, 0.5] = \emptyset$ respectively, which in turn means that $[\tilde{\mathbf{S}}^b]^{-1}(s_2^b) \cap [0, 0.5] = \emptyset$.

Hence $\left([\tilde{\mathbf{S}}^b]^{-1}(s_1^b) \cup [\tilde{\mathbf{S}}^b]^{-1}(s_2^b) \right) \cap [0, 0.5] = \emptyset$ implying that $[\tilde{\mathbf{S}}^b]^{-1}(s_1^b) \cup [\tilde{\mathbf{S}}^b]^{-1}(s_2^b) \neq X$, implying in turn that $\tilde{\mathbf{S}}^b$ cannot be a function from X to $\tilde{S}^b$. □

E Proof of Lemma 4

Proof. We show that $\neg \text{IC}^b_{\text{A}} \Rightarrow \neg \text{R-IC}^b_{\text{A}}$. The argument for the seller is identical. Suppose that the mechanism violates *ex ante* incentive compatibility for the buyer, i.e. there exists an *ex ante* deviation $(\tilde{S}^b, \tilde{\alpha}, \tilde{\mathbf{m}}^b)$ where $\tilde{S}^b \in \mathscr{P}(\mathbb{N})$, $\tilde{\alpha} \in \Delta(\tilde{S}^b \times S^s \times V)$, $\tilde{\mathbf{m}}^b : \tilde{S}^b \to S^b \cup \{m_\emptyset\}$, and $\text{marg}_{S^s \times V} \tilde{\alpha} = \text{marg}_{S^s \times V} \alpha$, such that

$$\sum_{s^b \in \tilde{S}^b} \sum_{s^s \in S^s} \sum_{v \in V} \tilde{\alpha}(s^b, s^s; v)\big(q^b(\tilde{\mathbf{m}}^b(s^b), s^s) u^b(v) - t^b(\tilde{\mathbf{m}}^b(s^b), s^s)\big) - c^b(\tilde{\alpha})$$
$$> \sum_{s^b \in S^b} \sum_{s^s \in S^s} \sum_{v \in V} \alpha(s^b, s^s; v)\big(q^b(s^b, s^s) u^b(v) - t^b(s^b, s^s)\big) - c^b(\alpha).$$

We are going to show that there is a restricted deviation $(\tilde{S}^b_{\text{R}}, \tilde{\alpha}_{\text{R}})$ where $\tilde{S}^b_{\text{R}} = S^b \cup \{s^b_\emptyset\}$, $\tilde{\alpha}_{\text{R}} \in \Delta(\tilde{S}^b_{\text{R}} \times S^s \times V)$, and $\text{marg}_{S^s \times V} \tilde{\alpha}_{\text{R}} = \text{marg}_{S^s \times V} \alpha$, such that

$$\sum_{s^b \in S^b} \sum_{s^s \in S^s} \sum_{v \in V} \tilde{\alpha}_{\text{R}}(s^b, s^s; v)\big(q^b(s^b, s^s) u^b(v) - t^b(s^b, s^s)\big) - c^b(\tilde{\alpha}_{\text{R}})$$
$$> \sum_{s^b \in S^b} \sum_{s^s \in S^s} \sum_{v \in V} \alpha(s^b, s^s; v)\big(q^b(s^b, s^s) u^b(v) - t^b(s^b, s^s)\big) - c^b(\alpha).$$

Define $\tilde{\mathcal{X}}^b(s^b) \equiv \{x^b \in \tilde{S}^b | \tilde{\mathbf{m}}^b(x^b) = s^b\}$ and $\tilde{\mathcal{X}}^b(m_\emptyset) \equiv \{x^b \in \tilde{S}^b | \tilde{\mathbf{m}}^b(x^b) = m_\emptyset\}$, i.e. the set of all signal realizations $x^b \in \tilde{S}^b$ such that the reports of s^b and $m_\emptyset$ are submitted respectively under $\tilde{\mathbf{m}}^b$. Define the restricted information structure as:

$$\tilde{\alpha}_R(s^b, s^s; v) \equiv \sum_{x^b \in \tilde{\mathcal{X}}^b(s^b)} \tilde{\alpha}(x^b, s^s; v) \qquad \forall\ s^b \in S^b,$$
$$\tilde{\alpha}_R(s^b_\emptyset, s^s; v) \equiv \sum_{x^b \in \tilde{\mathcal{X}}^b(m_\emptyset)} \tilde{\alpha}(x^b, s^s; v).$$

The restricted information structure respects the marginals of the seller by construction, and thus also can be a part of a feasible deviation. Indeed,

$$\sum_{s^b \in S^b} \tilde{\alpha}_R(s^b, s^s; v) + \tilde{\alpha}_R(s^b_\emptyset, s^s; v) = \sum_{s^b \in S^b} \Bigg[\sum_{x^b \in \tilde{\mathcal{X}}^b(s^b)} \tilde{\alpha}(x^b, s^s; v) + \sum_{x^b \in \tilde{\mathcal{X}}^b(m_\emptyset)} \tilde{\alpha}(x^b, s^s; v) \Bigg]$$
$$= \sum_{x^b \in \tilde{S}^b} \tilde{\alpha}(x^b, s^s; v)$$

for every $s^s \in S^s$

Clearly by construction we also obtain

$$\sum_{s^b \in S^b} \sum_{s^s \in S^s} \sum_{v \in V} \tilde{\alpha}_{\text{R}}(s^b, s^s; v)\big(q^b(s^b, s^s) u^b(v) - t^b(s^b, s^s)\big)$$
$$= \sum_{s^b \in \tilde{S}^b} \sum_{s^s \in S^s} \sum_{v \in V} \tilde{\alpha}(s^b, s^s; v)\big(q^b(\tilde{\mathbf{m}}^b(s^b), s^s) u^b(v) - t^b(\tilde{\mathbf{m}}^b(s^b), s^s)\big).$$

By construction $\tilde{\alpha}_R$ is Blackwell-less-informative than $\tilde{\alpha}$ for the buyer, which means that the expected entropy of $\tilde{\alpha}$ is lower than that of $\tilde{\alpha}_R$, implying $c^b(\tilde{\alpha}_{\mathrm{R}}) \leq c^b(\tilde{\alpha})$, which in turn implies

$$\sum_{s^b \in S^b} \sum_{s^s \in S^s} \sum_{v \in V} \tilde{\alpha}_{\mathrm{R}}(s^b, s^s; v)\big(q^b(s^b, s^s)u^b(v) - t^b(s^b, s^s)\big) - c^b(\tilde{\alpha}_{\mathrm{R}})$$
$$\geq \sum_{s^b \in \tilde{S}^b} \sum_{s^s \in S^s} \sum_{v \in V} \tilde{\alpha}(s^b, s^s; v)\big(q^b(\tilde{\mathbf{m}}^b(s^b), s^s)u^b(v) - t^b(\tilde{\mathbf{m}}^b(s^b), s^s)\big) - c^b(\tilde{\alpha}),$$

establishing the claim. □

F Proof of Lemma 5 (Strictly positive posteriors)

Proof. We prove the statement of the lemma for the buyer only. The proof for the seller is analogous. We have to distinguish two cases. **Case 1.** $I = 1$: by Bayes-plausibility we have $\mu_1^b(v) = \mu_0(v) > 0$ for any $v \in V$, hence the statement of the lemma holds trivially. **Case 2.** $I > 1$: suppose for a contradiction that there exists a state $v' \in V$ such that after receiving signal realization s_1^b the buyer puts probability zero on state v', i.e. $\mu_1^b(v') = 0$. Note that since the labels of signal realizations do not have any particular meaning in our analysis, choosing s_1^b is without loss of generality. Since s_1^b leads to a zero posterior on v', the information structure at v' is written as:

State v'	s_1^s	s_2^s	$\ldots$	s_l^s	$\ldots$	s_J^s
s_1^b	0	0	$\ldots$	0	$\ldots$	0
s_2^b	$\alpha_{21}(v')$	$\alpha_{22}(v')$	$\ldots$	$\alpha_{2l}(v')$	$\ldots$	$\alpha_{2J}(v')$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
s_r^b	$\alpha_{r1}(v')$	$\alpha_{r2}(v')$	$\ldots$	$\alpha_{rl}(v')$	$\ldots$	$\alpha_{rJ}(v')$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
s_I^b	$\alpha_{I1}(v')$	$\alpha_{I2}(v')$	$\ldots$	$\alpha_{Il}(v')$	$\ldots$	$\alpha_{IJ}(v')$

The payoff from this information structure is given by:

$$\sum_{j=1}^{J} \sum_{v \in V \setminus \{v'\}} \alpha_{1j}(v)\big(q_{1j}^b u^b(v) - t_{1j}^b\big) + \sum_{i=2}^{I} \sum_{j=1}^{J} \sum_{v \in V} \alpha_{ij}(v)\big(q_{ij}^b u^b(v) - t_{ij}^b\big) - H(\mu_0)$$
$$- \sum_{v \in V \setminus \{v'\}} \left(\sum_{j=1}^{J} \alpha_{1j}(v)\right) \log\left[\frac{\sum_{j=1}^{J} \alpha_{1j}(v)}{\sum_{j=1}^{J} \sum_{\hat{v} \in V \setminus \{v'\}} \alpha_{1j}(\hat{v})}\right] - \underbrace{0 \log 0}_{=0}$$
$$- \sum_{i=2}^{I} \sum_{v \in V} \left(\sum_{j=1}^{J} \alpha_{ij}(v)\right) \log\left[\frac{\sum_{j=1}^{J} \alpha_{ij}(v)}{\sum_{j=1}^{J} \sum_{\hat{v} \in V} \alpha_{ij}(\hat{v})}\right].$$

Observe that at least one of the $\alpha_{ij}(v')$ for some $i \neq 1$ must be strictly positive by Bayes-plausibility. Otherwise Bayes -plausibility would imply $\mu_0(v') = 0$ contradicting the full support assumption. Assume without loss of generality that $\alpha_{rl}(v') > 0$ and consider now an alternative information structure $\tilde{\alpha}$, in which every $\tilde{\alpha}_{ij}(v) = \alpha_{ij}(v)$ for all pairs (i, j) in all states $v \neq v'$. In state v', we transfer a small probability mass from (s^b_r, s^s_l) to (s^b_1, s^s_l), the alternative information structure in state v' is written as:

State v'	s^s_1	s^s_2	$\dots$	s^s_l	$\dots$	s^s_J
s^b_1	0	0	$\dots$	ϵ	$\dots$	0
s^b_2	$\alpha_{21}(v')$	$\alpha_{22}(v')$	$\dots$	$\alpha_{2l}(v')$	$\dots$	$\alpha_{2J}(v')$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
s^b_r	$\alpha_{r1}(v')$	$\alpha_{r2}(v')$	$\dots$	$\alpha_{rl}(v') - \epsilon$	$\dots$	$\alpha_{rJ}(v')$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
s^b_I	$\alpha_{I1}(v')$	$\alpha_{I2}(v')$	$\dots$	$\alpha_{Il}(v')$	$\dots$	$\alpha_{IJ}(v')$

for some small $\epsilon > 0$. Observe that $\text{marg}_{S^s \times V}\tilde{\alpha} = \text{marg}_{S^s \times V}\alpha$, and hence $\tilde{\alpha}$ can be a feasible deviation for the buyer in (R-IC$^b_{\text{A}}$-1). The payoff from this deviation is:

$$
\begin{aligned}
&\epsilon\big(q^b_{1l}u^b(v') - t^b_{1l}\big) + \sum_{j=1}^{J} \sum_{v \in V \setminus \{v'\}} \alpha_{1j}(v)\big(q^b_{1j}u^b(v) - t^b_{1j}\big) \\
&+ \sum_{i=2}^{I} \sum_{j=1}^{J} \sum_{v \in V} \alpha_{ij}(v)\big(q^b_{ij}u^b(v) - t^b_{ij}\big) - \epsilon\big(q^b_{rl}u^b(v') - t^b_{rl}\big) - H(\mu_0) \\
&- \sum_{v \in V \setminus \{v'\}} \left(\sum_{j=1}^{J} \alpha_{1j}(v)\right) \log\left[\frac{\sum_{j=1}^{J} \alpha_{1j}(v)}{\sum_{j=1}^{J} \sum_{\hat{v} \in V \setminus \{v'\}} \alpha_{1j}(\hat{v}) + \epsilon}\right] \\
&- \epsilon \log\left[\frac{\epsilon}{\sum_{j=1}^{J} \sum_{\hat{v} \in V \setminus \{v'\}} \alpha_{1j}(\hat{v}) + \epsilon}\right] \\
&- \sum_{v \in V \setminus \{v'\}} \left(\sum_{j=1}^{J} \alpha_{rj}(v)\right) \log\left[\frac{\sum_{j=1}^{J} \alpha_{rj}(v)}{\sum_{j=1}^{J} \sum_{\hat{v} \in V} \alpha_{rj}(\hat{v}) - \epsilon}\right] \\
&- \left(\sum_{j=1}^{J} \alpha_{rj}(v') - \epsilon\right) \log\left[\frac{\sum_{j=1}^{J} \alpha_{rj}(v')}{\sum_{j=1}^{J} \sum_{\hat{v} \in V} \alpha_{rj}(\hat{v}) - \epsilon}\right] \\
&- \sum_{i \neq 1,r} \sum_{v \in V} \left(\sum_{j=1}^{J} \alpha_{ij}(v)\right) \log\left[\frac{\sum_{j=1}^{J} \alpha_{ij}(v)}{\sum_{j=1}^{J} \sum_{\hat{v} \in V} \alpha_{ij}(\hat{v})}\right].
\end{aligned}
$$

The gain from this deviation as a function of ϵ is given by:

$$
\begin{aligned}
G(\epsilon) \equiv\ & \epsilon\big(q_{1l}^b u^b(v') - t_{1l}^b\big) - \epsilon\big(q_{rl}^b u^b(v') - t_{rl}^b\big) \\
& - \sum_{v\in V\setminus\{v'\}} \left(\sum_{j=1}^J \alpha_{1j}(v)\right) \log\left[\frac{\sum_{j=1}^J \alpha_{1j}(v)}{\sum_{j=1}^J \sum_{\hat{v}\in V\setminus\{v'\}} \alpha_{1j}(\hat{v}) + \epsilon}\right] \\
& - \epsilon \log\left[\frac{\epsilon}{\sum_{j=1}^J \sum_{\hat{v}\in V\setminus\{v'\}} \alpha_{1j}(\hat{v}) + \epsilon}\right] \\
& + \sum_{v\in V\setminus\{v'\}} \left(\sum_{j=1}^J \alpha_{1j}(v)\right) \log\left[\frac{\sum_{j=1}^J \alpha_{1j}(v)}{\sum_{j=1}^J \sum_{\hat{v}\in V\setminus\{v'\}} \alpha_{1j}(\hat{v})}\right] \\
& - \sum_{v\in V\setminus\{v'\}} \left(\sum_{j=1}^J \alpha_{rj}(v)\right) \log\left[\frac{\sum_{j=1}^J \alpha_{rj}(v)}{\sum_{j=1}^J \sum_{\hat{v}\in V} \alpha_{rj}(\hat{v}) - \epsilon}\right] \\
& - \left(\sum_{j=1}^J \alpha_{rj}(v') - \epsilon\right) \log\left[\frac{\sum_{j=1}^J \alpha_{rj}(v') - \epsilon}{\sum_{j=1}^J \sum_{\hat{v}\in V} \alpha_{rj}(\hat{v}) - \epsilon}\right] \\
& + \sum_{v\in V\setminus\{v'\}} \left(\sum_{j=1}^J \alpha_{rj}(v)\right) \log\left[\frac{\sum_{j=1}^J \alpha_{rj}(v)}{\sum_{j=1}^J \sum_{\hat{v}\in V} \alpha_{rj}(\hat{v})}\right] \\
& + \left(\sum_{j=1}^J \alpha_{rj}(v')\right) \log\left[\frac{\sum_{j=1}^J \alpha_{rj}(v')}{\sum_{j=1}^J \sum_{\hat{v}\in V} \alpha_{rj}(\hat{v})}\right].
\end{aligned}
$$

Define the function $\psi(\epsilon) \equiv \frac{1}{\epsilon}G(\epsilon)$. It can be written as

$$
\begin{aligned}
\psi(\epsilon) \equiv\ & \big(q_{1l}^b u^b(v') - t_{1l}^b\big) - \big(q_{rl}^b u^b(v') - t_{rl}^b\big) - \log\left[\frac{\epsilon}{\sum_{j=1}^J \sum_{\hat{v}\in V\setminus\{v'\}} \alpha_{1j}(\hat{v}) + \epsilon}\right] \\
& - \sum_{v\in V\setminus\{v'\}} \rho_1(\epsilon; v) - \sum_{v\in V\setminus\{v'\}} \rho_r(\epsilon; v) - \xi(\epsilon),
\end{aligned}
$$

where

$$
\begin{aligned}
\rho_1(\epsilon; v) &\equiv \frac{1}{\epsilon}\left(\sum_{j=1}^J \alpha_{1j}(v)\right)\left(\log\left[\frac{\sum_{j=1}^J \alpha_{1j}(v)}{\sum_{j=1}^J \sum_{\hat{v}\in V\setminus\{v'\}} \alpha_{1j}(\hat{v}) + \epsilon}\right] - \log\left[\frac{\sum_{j=1}^J \alpha_{1j}(v)}{\sum_{j=1}^J \sum_{\hat{v}\in V\setminus\{v'\}} \alpha_{1j}(\hat{v})}\right]\right), \\
\rho_r(\epsilon; v) &\equiv \frac{1}{\epsilon}\left(\sum_{j=1}^J \alpha_{rj}(v)\right)\left(\log\left[\frac{\sum_{j=1}^J \alpha_{rj}(v)}{\sum_{j=1}^J \sum_{\hat{v}\in V} \alpha_{rj}(\hat{v}) - \epsilon}\right] - \log\left[\frac{\sum_{j=1}^J \alpha_{rj}(v)}{\sum_{j=1}^J \sum_{\hat{v}\in V} \alpha_{rj}(\hat{v})}\right]\right), \\
\xi(\epsilon) &\equiv \frac{1}{\epsilon}\left(\sum_{j=1}^J \alpha_{rj}(v') - \epsilon\right) \log\left[\frac{\sum_{j=1}^J \alpha_{rj}(v') - \epsilon}{\sum_{j=1}^J \sum_{\hat{v}\in V} \alpha_{rj}(\hat{v}) - \epsilon}\right] - \frac{1}{\epsilon}\left(\sum_{j=1}^J \alpha_{rj}(v')\right) \log\left[\frac{\sum_{j=1}^J \alpha_{rj}(v')}{\sum_{j=1}^J \sum_{\hat{v}\in V} \alpha_{rj}(\hat{v})}\right].
\end{aligned}
$$

We are now going to determine the right-limit of $\psi(\epsilon)$ as ϵ approaches zero.

Lemma 12. $\lim_{\epsilon\to 0^+} \psi(\epsilon) = +\infty$.

Proof. Observe first that since $\sum_{j=1}^J \sum_{\hat{v}\in V\setminus\{v'\}} \alpha_{1j}(\hat{v}) > 0$, we have:

$$
\lim_{\epsilon\to 0^+} \left(\big(q_{1l}^b u^b(v') - t_{1l}^b\big) - \big(q_{rl}^b u^b(v') - t_{rl}^b\big) - \log\left[\frac{\epsilon}{\sum_{j=1}^J \sum_{\hat{v}\in V\setminus\{v'\}} \alpha_{1j}(\hat{v}) + \epsilon}\right]\right) = +\infty.
$$

It thus remains to show that the remaining terms converge to a finite value.

- Consider $\rho_1(\epsilon; v)$ for some $v \in V \setminus \{v'\}$. There are two possibilities:

(i) If $\sum_{j=1}^{J} \alpha_{1j}(v) = 0$, then $\rho_1(\epsilon) = \frac{1}{\epsilon}\big[0 \log 0 - 0 \log 0\big] = 0$, hence $\lim_{\epsilon \to 0^+} \rho_1(\epsilon) = 0$.

(ii) If $\sum_{j=1}^{J} \alpha_{1j}(v) > 0$, then $\lim_{\epsilon \to 0^+} \rho_1(\epsilon) = -\frac{\sum_{j=1}^{J} \alpha_{1j}(v)}{\sum_{j=1}^{J} \sum_{\hat{v} \in V \setminus \{v'\}} \alpha_{1j}(\hat{v})}$, which is finite.

- Consider $\rho_r(\epsilon; v)$ for some $v \in V \setminus \{v'\}$. There are again two possibilities:

(i) If $\sum_{j=1}^{J} \alpha_{rj}(v) = 0$, then $\rho_1(\epsilon) = \frac{1}{\epsilon}\big[0 \log 0 - 0 \log 0\big] = 0$, hence $\lim_{\epsilon \to 0^+} \rho_1(\epsilon) = 0$.

(ii) If $\sum_{j=1}^{J} \alpha_{rj}(v) > 0$, then $\lim_{\epsilon \to 0^+} \rho_r(\epsilon) = \frac{\sum_{j=1}^{J} \alpha_{rj}(v)}{\sum_{j=1}^{J} \sum_{\hat{v} \in V} \alpha_{rj}(\hat{v})}$, which is finite.

- Consider $\xi(\epsilon)$. Recall that by assumption $\sum_{j=1}^{J} \alpha_{rj}(v') > 0$, hence we have

$$\lim_{\epsilon \to 0^+} \xi(\epsilon) = \frac{\sum_{j=1}^{J} \alpha_{rj}(v')}{\sum_{j=1}^{J} \sum_{\hat{v} \in V} \alpha_{rj}(\hat{v})} - \log\left[\frac{\sum_{j=1}^{J} \alpha_{rj}(v')}{\sum_{j=1}^{J} \sum_{\hat{v} \in V} \alpha_{rj}(\hat{v})}\right] - 1,$$

which is finite. □

Since $\lim_{\epsilon \to 0^+} \psi(\epsilon) = +\infty$, we conclude that for all $n > 0$ there exists $\epsilon > 0$ small enough such that $\psi(\epsilon) = \frac{1}{\epsilon} G(\epsilon) > n$, implying $G(\epsilon) > \epsilon n > 0$, implying that the constructed deviation $\tilde{\alpha}$ is profitable for all ϵ small enough, and thus contradicting the optimality of α. □

G Proof of Lemma 6

Proof. We prove the statement of the lemma for the buyer only (the proof for the seller is almost identical). **"If".** To establish the "if" direction of the claim we prove the contrapositive statement. Consider an $I \times J$ information structure α given by:

State v	s_1^s	s_2^s	$\dots$	s_J^s
s_1^b	$\alpha_{11}(v)$	$\alpha_{12}(v)$	$\dots$	$\alpha_{1J}(v)$
s_2^b	$\alpha_{21}(v)$	$\alpha_{22}(v)$	$\dots$	$\alpha_{2J}(v)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
s_I^b	$\alpha_{I1}(v)$	$\alpha_{I2}(v)$	$\dots$	$\alpha_{IJ}(v)$

The buyer's payoff from this information structure is given by:

$$\sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{v \in V} \alpha_{ij}(v)\big(q_{ij}^b u^b(v) - t_{ij}^b\big) - c^b(\alpha)$$

$$= \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{v \in V} \alpha_{ij}(v)\big(q_{ij}^b u^b(v) - t_{ij}^b\big) - H(\mu_0) - \sum_{i=1}^{I} \sum_{v \in V} \left(\sum_{j=1}^{J} \alpha_{ij}(v)\right) \log\left[\frac{\sum_{j=1}^{J} \alpha_{ij}(v)}{\sum_{j=1}^{J} \sum_{\hat{v} \in V} \alpha_{ij}(\hat{v})}\right].$$

Suppose that α satisfies R-IC^b_A-1 but does not satisfy R-IC^b_A-2, then there exists a profitable deviation for the buyer which involves augmenting α with a an $I+1$-st signal realization $s^b_\emptyset$. This deviation has the following form:

State v	s^s_1	s^s_2	$\dots$	s^s_J
s^b_1	$\alpha_{11}(v)-\beta_{11}(v)$	$\alpha_{12}(v)-\beta_{12}(v)$	$\dots$	$\alpha_{1J}(v)-\beta_{1J}(v)$
s^b_2	$\alpha_{21}(v)-\beta_{21}(v)$	$\alpha_{22}(v)-\beta_{22}(v)$	$\dots$	$\alpha_{2J}(v)-\beta_{2J}(v)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
s^b_I	$\alpha_{I1}(v)-\beta_{I1}(v)$	$\alpha_{I2}(v)-\beta_{I2}(v)$	$\dots$	$\alpha_{IJ}(v)-\beta_{IJ}(v)$
$s^b_\emptyset$	$\sum_{i=1}^I \beta_{i1}(v)$	$\sum_{i=1}^I \beta_{i2}(v)$	$\dots$	$\sum_{i=1}^I \beta_{iJ}(v)$

where $\sum_{i=1}^I\sum_{j=1}^J\sum_{v\in V}\beta_{ij}(v)>0$. The payoff from this deviation is given by:

$$\sum_{i=1}^I\sum_{j=1}^J\sum_{v\in V}\big(\alpha_{ij}(v)-\beta_{ij}(v)\big)\big(q^b_{ij}u^b(v)-t^b_{ij}\big)-H(\mu_0)$$
$$-\sum_{i=1}^I\sum_{v\in V}\left(\sum_{j=1}^J\big(\alpha_{ij}(v)-\beta_{ij}(v)\big)\right)\log\left[\frac{\sum_{j=1}^J\big(\alpha_{ij}(v)-\beta_{ij}(v)\big)}{\sum_{j=1}^J\sum_{\hat v\in V}\big(\alpha_{ij}(\hat v)-\beta_{ij}(\hat v)\big)}\right]$$
$$-\sum_{v\in V}\left(\sum_{i=1}^I\sum_{j=1}^J\beta_{ij}(v)\right)\log\left[\frac{\sum_{i=1}^I\sum_{j=1}^J\beta_{ij}(v)}{\sum_{i=1}^I\sum_{j=1}^J\sum_{\hat v\in V}\beta_{ij}(\hat v)}\right],$$

which can be rewritten as:

$$\sum_{i=1}^I\sum_{j=1}^J\sum_{v\in V}\big(\alpha_{ij}(v)-\beta_{ij}(v)\big)\big(q^b_{ij}u^b(v)-t^b_{ij}\big)-c^b(\alpha-\beta)$$
$$-\sum_{v\in V}\left(\sum_{i=1}^I\sum_{j=1}^J\beta_{ij}(v)\right)\log\left[\frac{\sum_{i=1}^I\sum_{j=1}^J\beta_{ij}(v)}{\sum_{i=1}^I\sum_{j=1}^J\sum_{\hat v\in V}\beta_{ij}(\hat v)}\right].$$

We now define the gain-from-deviation function as the difference between the payoff from the deviation and the payoff from α:

$$G_\alpha(\beta)\equiv-\sum_{i=1}^I\sum_{j=1}^J\sum_{v\in V}\beta_{ij}(v)\big(q^b_{ij}u^b(v)-t^b_{ij}\big)-c^b(\alpha-\beta)+c^b(\alpha)$$
$$-\sum_{v\in V}\left(\sum_{i=1}^I\sum_{j=1}^J\beta_{ij}(v)\right)\log\left[\frac{\sum_{i=1}^I\sum_{j=1}^J\beta_{ij}(v)}{\sum_{i=1}^I\sum_{j=1}^J\sum_{\hat v\in V}\beta_{ij}(\hat v)}\right].$$

Since the deviation under consideration is profitable, we have $G_\alpha(\beta)>0$, We now define $\psi(\epsilon)\equiv\frac{1}{\epsilon}G_\alpha(\epsilon\beta)$ for $\epsilon>0$. Clearly $\psi(1)=G_\alpha(\beta)>0$. $\psi(\epsilon)$ is written as:

$$\psi(\epsilon)=-\sum_{i=1}^I\sum_{j=1}^J\sum_{v\in V}\beta_{ij}(v)\big(q^b_{ij}u^b(v)-t^b_{ij}\big)+\frac{c^b(\alpha-\epsilon\beta)-c^b(\alpha)}{-\epsilon}$$
$$-\sum_{v\in V}\left(\sum_{i=1}^I\sum_{j=1}^J\beta_{ij}(v)\right)\log\left[\frac{\sum_{i=1}^I\sum_{j=1}^J\beta_{ij}(v)}{\sum_{i=1}^I\sum_{j=1}^J\sum_{\hat v\in V}\beta_{ij}(\hat v)}\right].$$

We establish the following lemma:

Lemma 13. *$\psi(\epsilon)$ is weakly decreasing.*

Proof. It is enough to show that $\frac{c^b(\alpha-\epsilon\beta)-c^b(\alpha)}{-\epsilon}$ is weakly decreasing. To that end, take $0<\epsilon_1<\epsilon_2<1$ and observe that $\alpha-\epsilon_1\beta=\left(1-\frac{\epsilon_1}{\epsilon_2}\right)\alpha+\frac{\epsilon_1}{\epsilon_2}(\alpha-\epsilon_2\beta)$. Recall that $c^b(\alpha)$ is convex by Lemma 3, hence $c^b(\alpha-\epsilon_1\beta)\leq\left(1-\frac{\epsilon_1}{\epsilon_2}\right)c^b(\alpha)+\frac{\epsilon_1}{\epsilon_2}c^b(\alpha-\epsilon_2\beta)$, or, equivalently, $\frac{c^b(\alpha-\epsilon_1\beta)-c^b(\alpha)}{-\epsilon_1}\geq\frac{c^b(\alpha-\epsilon_2\beta)-c^b(\alpha)}{-\epsilon_2}$. □

We now define the marginal gain-from-deviation $MG_\alpha(\beta)\equiv\lim_{\epsilon\to0}\frac{1}{\epsilon}G_\alpha(\epsilon\beta)$. Recall that α satisfies R-IC^b_A-1 by assumption, hence Lemma 5 ensures that all the posteriors induced by α are strictly positive, which in turn makes sure that the limit $\lim_{\epsilon\to0}\frac{1}{\epsilon}G_\alpha(\epsilon\beta)$ is well-defined and given by:

$$\begin{aligned}MG_\alpha(\beta)=&-\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{v\in V}\beta_{ij}(v)\big(q^b_{ij}u^b(v)-t^b_{ij}\big)\\&+\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{v\in V}\beta_{ij}(v)\log\left[\frac{\sum_{j=1}^{J}\alpha_{ij}(v)}{\sum_{j=1}^{J}\sum_{\hat{v}\in V}\alpha_{ij}(\hat{v})}\right]\\&-\sum_{v\in V}\left(\sum_{i=1}^{I}\sum_{j=1}^{J}\beta_{ij}(v)\right)\log\left[\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\beta_{ij}(v)}{\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{\hat{v}\in V}\beta_{ij}(\hat{v})}\right],\end{aligned}$$

which can be rewritten as:

$$\begin{aligned}MG_\alpha(\beta)=&-\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{v\in V}\beta_{ij}(v)\big(q^b_{ij}u^b(v)-t^b_{ij}-\log\big(\mu^b_i(v)\big)\big)\\&-\sum_{v\in V}\left(\sum_{i=1}^{I}\sum_{j=1}^{J}\beta_{ij}(v)\right)\log\left[\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\beta_{ij}(v)}{\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{\hat{v}\in V}\beta_{ij}(\hat{v})}\right].\end{aligned}$$

Defining $y^b_{ij}(v)\equiv q^b_{ij}u^b(v)-t^b_{ij}-\log\big(\mu^b_i(v)\big)$, we can rewrite $MG_\alpha(\beta)$ as follows:

$$MG_\alpha(\beta)=-\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{v\in V}\beta_{ij}(v)y^b_{ij}(v)-\sum_{v\in V}\left(\sum_{i=1}^{I}\sum_{j=1}^{J}\beta_{ij}(v)\right)\log\left[\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}\beta_{ij}(v)}{\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{\hat{v}\in V}\beta_{ij}(\hat{v})}\right].$$

The following lemma holds:

Lemma 14. *$MG_\alpha(\beta)>0$.*

Proof. Recall that $MG_\alpha(\beta)=\lim_{\epsilon\to0}\frac{1}{\epsilon}G_\alpha(\epsilon\beta)=\lim_{\epsilon\to0}\frac{1}{\epsilon}\psi(\epsilon)$. By Lemma 13, $\psi(\epsilon)\geq\psi(1)$ for every $0<\epsilon<1$, hence $\lim_{\epsilon\to0}\psi(\epsilon)\geq\psi(1)>0$. □

Decompose the marginal deviation under consideration into two parts, β' and β'':

$$\beta'_{ij}(v) \equiv \begin{cases} \beta_{ij}(v) - \frac{1}{z_j(v)} \sum_{i=1}^{I} \beta_{ij}(v) & \text{if } \alpha_{ij}(v) > 0 \\ \beta_{ij}(v), & \text{otherwise} \end{cases};$$

$$\beta''_{ij}(v) \equiv \begin{cases} \frac{1}{z_j(v)} \sum_{i=1}^{I} \beta_{ij}(v), & \text{if } \alpha_{ij}(v) > 0 \\ 0, & \text{otherwise} \end{cases};$$

where $z_j(v)$ is the number of zero elements in the vector $[\alpha_{1j}(v), \ldots, \alpha_{Ij}(v)]$. Observe that by construction we have for every j:

$$\sum_{i=1}^{I} \beta'_{ij}(v) = \sum_{i=1}^{I} \beta_{ij}(v) - z_j(v) \frac{1}{z_j(v)} \sum_{i=1}^{I} \beta_{ij}(v) = 0$$
$$\sum_{i=1}^{I} \beta''_{ij}(v) = z_j(v) \frac{1}{z_j(v)} \sum_{i=1}^{I} \beta_{ij}(v) = \sum_{i=1}^{I} \beta_{ij}(v).$$

We can now rewrite the marginal gain in terms of β' and β'' as follows:

$$\begin{aligned} MG_\alpha(\beta) = &- \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{v \in V} \beta'_{ij}(v) y^b_{ij}(v) \\ &- \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{v \in V} \beta''_{ij}(v) y^b_{ij}(v) - \sum_{v \in V} \left(\sum_{i=1}^{I} \sum_{j=1}^{J} \beta''_{ij}(v) \right) \log \left[\frac{\sum_{i=1}^{I} \sum_{j=1}^{J} \beta''_{ij}(v)}{\sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{\hat{v} \in V} \beta''_{ij}(\hat{v})} \right], \end{aligned}$$

which in turn implies $MG_\alpha(\beta) = -\sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{v \in V} \beta'_{ij}(v) y^b_{ij}(v) + MG_\alpha(\beta'')$. We establish the following lemma:

Lemma 15. $MG_\alpha(\beta'') > 0$.

Proof. Observe that $-\sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{v \in V} \beta'_{ij}(v) y^b_{ij}(v)$ is the directional derivative of the objective function in R-IC^b_A-1 at α in the direction $-\beta'$. Since α satisfies the constraint R-IC^b_A-1 by assumption and $-\beta'$ is a feasible direction in R-IC^b_A-1, we must have $-\sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{v \in V} \beta'_{ij}(v) y^b_{ij}(v) \leq 0$. Since $MG_\alpha(\beta) > 0$, we have $MG_\alpha(\beta'') > 0$. □

Defining $B''(v) = \sum_{i=1}^{I} \sum_{j=1}^{J} \beta''(v)$, we can rewrite $MG_\alpha(\beta'')$ as follows:

$$MG_\alpha(\beta'') = -\sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{v \in V} \beta''_{ij}(v) y^b_{ij}(v) - \sum_{v \in V} B''(v) \log \left[\frac{B''(v)}{\sum_{\hat{v} \in V} B''(\hat{v})} \right].$$

Recall that $y^b(v) = \min_{(i,j)|\alpha_{ij}>0} \{y^b_{ij}(v)\}$. The following lemma holds:

Lemma 16. $-\sum_{v \in V} B''(v) y^b(v) - \sum_{v \in V} B''(v) \log \left[\frac{B''(v)}{\sum_{\hat{v} \in V} B''(\hat{v})} \right] > 0$.

Proof. Recall that all $\beta''_{ij}(v)$ are weakly positive by construction, and are equal to zero whenever $\alpha_{ij}(v) = 0$. We then have $\sum_{v \in V} B''(v) y^b(v) \leq \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{v \in V} \beta''_{ij}(v) y^b_{ij}(v)$, i.e. $-\sum_{v \in V} B''(v) y^b(v) - \sum_{v \in V} B''(v) \log\left[\frac{B''(v)}{\sum_{\hat{v} \in V} B''(\hat{v})}\right] \geq MG_\alpha(\beta'')$, which together with the previous lemma establishes the claim. □

Dividing the expression in the lemma above by $\sum_{\hat{v} \in V} B''(\hat{v})$, we get

$$-\sum_{v \in V} \frac{B''(v)}{\sum_{\hat{v} \in V} B''(\hat{v})} y^b(v) - \sum_{v \in V} \frac{B''(v)}{\sum_{\hat{v} \in V} B''(\hat{v})} \log\left[\frac{B''(v)}{\sum_{\hat{v} \in V} B''(\hat{v})}\right] > 0.$$

Defining $P(v) \equiv \frac{B''(v)}{\sum_{\hat{v} \in V} B''(\hat{v})}$, we can rewrite the above inequality as:

$$-\sum_{v \in V} P(v) y^b(v) - \sum_{v \in V} P(v) \log\big(P(v)\big) > 0,$$

which clearly implies:

$$0 < \max_{\{P(v)\}_{v \in V}} \Big\{ -\sum_{v \in V} P(v) y^b(v) - \sum_{v \in V} P(v) \log\big(P(v)\big) \text{ s.t. } \sum_{v \in V} P(v) = 1,\ P(v) \geq 0\ \forall v \Big\}.$$

To evaluate the right-hand side, relax the non-negativity constraints and write down the Lagrangian of the relaxed problem:

$$\mathcal{L}(P; \nu) = -\sum_{v \in V} P(v) y^b(v) - \sum_{v \in V} P(v) \log\big(P(v)\big) - \nu\Big(\sum_{v \in V} P(v) - 1\Big).$$

Observe that the objective function in the relaxed problem is strictly concave and the feasible set is convex, implying that the first order conditions are necessary and sufficient for optimality. The optimality conditions are therefore given by:

$$\begin{cases} -y^b(v) - \log\big(P^*(v)\big) - 1 - \nu^* = 0 \quad \forall v \in V, \\ \sum_{v \in V} P^*(v) = 1. \end{cases}$$

The optimum is achieved at $P^*(v) = \frac{\exp(-y^b(v))}{\sum_{\hat{v} \in V} \exp(-y^b(\hat{v}))}$. We then have:

$$\begin{aligned}
& -\sum_{v \in V} P^*(v) y^b(v) - \sum_{v \in V} P^*(v) \log\big(P^*(v)\big) > 0 \\
\Leftrightarrow & -\sum_{v \in V} \frac{\exp\big(-y^b(v)\big)}{\sum_{\hat{v} \in V} \exp\big(-y^b(\hat{v})\big)} y^b(v) - \sum_{v \in V} \frac{\exp\big(-y^b(v)\big)}{\sum_{\hat{v} \in V} \exp\big(-y^b(\hat{v})\big)} \log\left(\frac{\exp\big(-y^b(v)\big)}{\sum_{\hat{v} \in V} \exp\big(-y^b(\hat{v})\big)}\right) > 0 \\
\Leftrightarrow & -\sum_{v \in V} \exp\big(-y^b(v)\big) y^b(v) - \sum_{v \in V} \exp\big(-y^b(v)\big) \log\left(\frac{\exp\big(-y^b(v)\big)}{\sum_{\hat{v} \in V} \exp\big(-y^b(\hat{v})\big)}\right) > 0 \\
\Leftrightarrow & \sum_{v \in V} \exp\big(-y^b(v)\big) \log\left(\sum_{\hat{v} \in V} \exp\big(-y^b(\hat{v})\big)\right) > 0 \\
\Leftrightarrow & \log\left(\sum_{\hat{v} \in V} \exp\big(-y^b(\hat{v})\big)\right) > 0 \\
\Leftrightarrow & \sum_{\hat{v} \in V} \exp\big(-y^b(\hat{v})\big) > 1,
\end{aligned}$$

which establishes the contrapositive claim.

"Only if". To establish the "only if" direction, we again prove the contrapositive statement. Suppose that α satisfies R-IC$^b_{\text{A}}$-1 and $\sum_{\hat{v}\in V}\exp\big(-y^b(\hat{v})\big) > 1$. The above calculations show that it is possible to construct profitable local deviation from α that involves augmenting α with a $I+1$-st signal realization. This deviation involves transferring probability mass to the $I+1$-st signal realization from those $\alpha_{rl}(v)$ that satisfy $(r,l) = \operatorname{argmax}_{(i,j)|\alpha_{ij}(v)>0}\{y^b_{ij}(v)\}$ for each state $v \in V$. The profitability of this deviation implies that α violates R-IC$^b_{\text{A}}$-2. $\square$

H Proof of Lemma 7 (Perfect correlation of signal realizations)

Proof. Observe first that the signal realizations can be ordered according to the ordering of their posteriors without loss of generality, hence in what follows we will assume that $\overline{\mu}^b_1 \geq \cdots \geq \overline{\mu}^b_I$ and $\overline{\mu}^s_1 \geq \cdots \geq \overline{\mu}^s_J$. We will present the proof in a series of auxiliary lemmas starting with the following pair:

Lemma 17. *For all $j \in S^s$ there exists $\mathcal{I}^*(j) \subseteq S^b$ such that*

1. $\underline{\alpha}_{ij} = 0$ *for all* $i > \max \mathcal{I}^*(j) \equiv \overline{i}^*(j)$,
2. $\overline{\alpha}_{ij} = 0$ *for all* $i < \min \mathcal{I}^*(j) \equiv \underline{i}^*(j)$.

Moreover, for any $i, i' \in \mathcal{I}^*(j)$ we get $\overline{\mu}^b_i = \overline{\mu}^b_{i'}$.

Lemma 18. *For all $i \in S^b$ there exists $\mathcal{J}^*(i) \subseteq S^s$ such that*

1. $\underline{\alpha}_{ij} = 0$ *for all* $j > \max \mathcal{J}^*(i) \equiv \overline{j}^*(i)$,
2. $\overline{\alpha}_{ij} = 0$ *for all* $j < \min \mathcal{J}^*(i) \equiv \underline{j}^*(i)$.

Moreover, for any $j, j' \in \mathcal{J}^*(i)$ we get $\overline{\mu}^s_j = \overline{\mu}^s_{j'}$.

Lemmas 17 and 18 have analogous proofs, thus we only prove Lemma 17 here.

Proof of Lemma 17. Fix $j \in S^s$. Set $\overline{i}^*(j) \equiv \max\{i|\underline{\alpha}_{ij} > 0\}$, and define $\mathcal{I}^*(j) \equiv \{i|\overline{\mu}^b_i = \overline{\mu}^b_{\overline{i}^*(j)}\}$ and $\underline{i}^*(j) \equiv \min \mathcal{I}^*(j)$. Suppose for a contradiction that $\exists i < \underline{i}^*(j)$ such that $\overline{\alpha}_{ij} > 0$. Note that $\overline{\mu}^b_i > \overline{\mu}^b_{\overline{i}^*(j)}$ since $\overline{\mu}^b_i \geq \overline{\mu}^b_{\overline{i}^*(j)}$ by the ordering assumption and $i \notin \mathcal{I}^*(j)$. Consider the following stationarity conditions of the buyer:

$$(\text{ST}^b_{ij})\quad \underline{u}^b - t^b_{ij} - \log\big(\underline{\mu}^b_i\big) - \underline{\lambda}^b_j = -\underline{\phi}^b_{ij}, \quad (\text{ST}^b_{\overline{i}^*(j),j})\quad \underline{u}^b - t^b_{\overline{i}^*(j),j} - \log\big(\underline{\mu}^b_{\overline{i}^*(j)}\big) - \underline{\lambda}^b_j = -\underline{\phi}^b_{\overline{i}^*(j),j},$$
$$\overline{u}^b - t^b_{ij} - \log\big(\overline{\mu}^b_i\big) - \overline{\lambda}^b_j = -\overline{\phi}^b_{ij}. \qquad \overline{u}^b - t^b_{\overline{i}^*(j),j} - \log\big(\overline{\mu}^b_{\overline{i}^*(j)}\big) - \overline{\lambda}^b_j = -\overline{\phi}^b_{\overline{i}^*(j),j}.$$

By complementary slackness we have $\overline{\phi}^b_{ij} = 0$ since $\overline{\alpha}_{ij} > 0$, likewise $\underline{\phi}^b_{\overline{i}^*(j),j} = 0$ since $\underline{\alpha}_{\overline{i}^*(j),j} > 0$ by definition of $\overline{i}^*(j)$. Dual feasibility implies $\underline{\phi}^b_{ij} \geq 0$, hence (ST^b_{ij}) imply

$$\overline{u}^b - \log\left(\overline{\mu}^b_i\right) - \overline{\lambda}^b_j \geq \underline{u}^b - \log\left(\underline{\mu}^b_i\right) - \underline{\lambda}^b_j. \tag{1}$$

Similarly, dual feasibility implies $\overline{\phi}^b_{\overline{i}^*(j),j} \geq 0$, hence $(\mathrm{ST}^b_{\overline{i}^*(j),j})$ imply

$$\underline{u}^b - \log\left(\underline{\mu}^b_{\overline{i}^*(j)}\right) - \underline{\lambda}^b_j \geq \overline{u}^b - \log\left(\overline{\mu}^b_{\overline{i}^*(j)}\right) - \overline{\lambda}^b_j. \tag{2}$$

Adding (1) and (2), we get

$$\overline{u}^b - \log\left(\overline{\mu}^b_i\right) - \overline{\lambda}^b_j + \underline{u}^b - \log\left(\underline{\mu}^b_{\overline{i}^*(j)}\right) - \underline{\lambda}^b_j \geq \underline{u}^b - \log\left(\underline{\mu}^b_i\right) - \underline{\lambda}^b_j + \overline{u}^b - \log\left(\overline{\mu}^b_{\overline{i}^*(j)}\right) - \overline{\lambda}^b_j,$$

which implies $\frac{\overline{\mu}^b_{\overline{i}^*(j)}}{\underline{\mu}^b_{\overline{i}^*(j)}} \geq \frac{\overline{\mu}^b_i}{\underline{\mu}^b_i}$, implying $\overline{\mu}^b_{\overline{i}^*(j)} \geq \overline{\mu}^b_i$, which is the desired contradiction. $\square$

We proceed further in the inductive manner. Let us introduce the following sets: $\hat{\mathcal{I}}_1 \equiv \{i | \overline{\mu}^b_i = \overline{\mu}^b_1\}$, $\hat{\mathcal{J}}_1 \equiv \{j | \overline{\mu}^s_j = \overline{\mu}^s_1\}$; and $\tilde{\mathcal{I}}_1 \equiv \{i | \mathcal{J}^*(i) = \hat{\mathcal{J}}_1\}$, $\tilde{\mathcal{J}}_1 \equiv \{j | \mathcal{I}^*(i) = \hat{\mathcal{I}}_1\}$. Define $\overline{\hat{i}}_1 \equiv \max \hat{\mathcal{I}}_1$ and $\overline{\hat{j}}_1 \equiv \max \hat{\mathcal{J}}_1$. We prove the following pair of auxiliary lemmas:

Lemma 19. $\tilde{\mathcal{I}}_1 = \{i | \mathcal{J}^*(i) = \hat{\mathcal{J}}_1\} \neq \emptyset$.

Lemma 20. $\tilde{\mathcal{J}}_1 = \{j | \mathcal{I}^*(i) = \hat{\mathcal{I}}_1\} \neq \emptyset$.

Lemmas 19 and 20 have analogous proofs, hence we only prove Lemma 19 here.

Proof of Lemma 19. Suppose for a contradiction that $\tilde{\mathcal{I}}_1 = \emptyset$, i.e. for all i we have $\mathcal{J}^*(i) \neq \hat{\mathcal{J}}_1$. Since both $\mathcal{J}^*(i)$ and $\hat{\mathcal{J}}_1$ are sets of signal realizations with equal posteriors and the posteriors are ordered, we have $\min \mathcal{J}^*(i) > \max \hat{\mathcal{J}}_1$ for all i. Lemma 18 then implies that $\overline{\alpha}_{i1} = 0$ for all i, which implies $\overline{\mu}^s_1 = 0$ contradicting Lemma 5. $\square$

In the following lemma we establish the base case of our induction argument:

Lemma 21 (**Base case of the induction**)**.**

1. *For all* $i \in \hat{\mathcal{I}}_1$ *and* $j \notin \hat{\mathcal{J}}_1$ *we have* $\underline{\alpha}_{ij} = \overline{\alpha}_{ij} = 0$.
2. *For all* $i \notin \hat{\mathcal{I}}_1$ *and* $j \in \hat{\mathcal{J}}_1$ *we have* $\underline{\alpha}_{ij} = \overline{\alpha}_{ij} = 0$.

Proof of Lemma 21. Suppose $i' \notin \tilde{\mathcal{I}}_1 = \{i | \mathcal{J}^*(i) = \hat{\mathcal{J}}_1\}$, then $\mathcal{J}^*(i') \neq \hat{\mathcal{J}}_1$. Since both $\mathcal{J}^*(i')$ and $\hat{\mathcal{J}}_1$ are sets of signal realizations with equal posteriors and the posteriors are ordered, we must have $\min \mathcal{J}^*(i') > \max \hat{\mathcal{J}}_1$. Lemma 18 then implies that $\overline{\alpha}_{i'j} = 0$ for all $j \in \hat{\mathcal{J}}_1$. Recall that $1 \in \hat{\mathcal{J}}_1$ and the posteriors of all signal realizations in $\hat{\mathcal{J}}_1$

are the same, hence $\overline{\mu}_1^s$ is equal to the weighted average of the posteriors of signal realizations from $\hat{\mathcal{J}}_1$:

$$\overline{\mu}_1^s = \frac{\sum_{j\in\hat{\mathcal{J}}_1} \overline{\mu}_j^s \sum_{i=1}^I (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}{\sum_{j\in\hat{\mathcal{J}}_1} \sum_{i=1}^I (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} = \frac{\sum_{j\in\hat{\mathcal{J}}_1} \sum_{i=1}^I \overline{\alpha}_{ij}}{\sum_{j\in\hat{\mathcal{J}}_1} \sum_{i=1}^I (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}$$

$$= \frac{\sum_{j\in\hat{\mathcal{J}}_1} \left[\sum_{i\in\tilde{\mathcal{I}}_1} \overline{\alpha}_{ij} + \sum_{i\notin\tilde{\mathcal{I}}_1} \overbrace{\cancel{\overline{\alpha}_{ij}}}^{=0 \text{ by L18}} \right]}{\sum_{j\in\hat{\mathcal{J}}_1} \left[\sum_{i\in\tilde{\mathcal{I}}_1} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij}) + \sum_{i\notin\tilde{\mathcal{I}}_1} (\underbrace{\cancel{\overline{\alpha}_{ij}}}_{=0 \text{ by L18}} + \underline{\alpha}_{ij}) \right]}$$

$$= \frac{\sum_{j\in\hat{\mathcal{J}}_1} \sum_{i\in\tilde{\mathcal{I}}_1} \overline{\alpha}_{ij}}{\sum_{j\in\hat{\mathcal{J}}_1} \sum_{i\in\tilde{\mathcal{I}}_1} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij}) + \sum_{j\in\hat{\mathcal{J}}_1} \sum_{i\notin\tilde{\mathcal{I}}_1} \underline{\alpha}_{ij}} \le \frac{\sum_{j\in\hat{\mathcal{J}}_1} \sum_{i\in\tilde{\mathcal{I}}_1} \overline{\alpha}_{ij}}{\sum_{j\in\hat{\mathcal{J}}_1} \sum_{i\in\tilde{\mathcal{I}}_1} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}.$$

Suppose now $i' \in \tilde{\mathcal{I}}_1$, i.e. $\mathcal{J}^*(i') = \hat{\mathcal{J}}_1$. Lemma 19 guarantees that such i' exist. Lemma 18 implies that $\underline{\alpha}_{i'j} = 0$ for all $j \notin \hat{\mathcal{J}}_1 = \mathcal{J}^*(i')$. By the ordering assumption, $\overline{\mu}_1^b$ is higher than the weighted average of the posteriors of signal realizations from $\tilde{\mathcal{I}}_1$:

$$\overline{\mu}_1^b \ge \frac{\sum_{i\in\tilde{\mathcal{I}}_1} \overline{\mu}_i^b \sum_{j=1}^J (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}{\sum_{i\in\tilde{\mathcal{I}}_1} \sum_{j=1}^J (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} = \frac{\sum_{i\in\tilde{\mathcal{I}}_1} \sum_{j=1}^J \overline{\alpha}_{ij}}{\sum_{i\in\tilde{\mathcal{I}}_1} \sum_{j=1}^J (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}$$

$$= \frac{\sum_{i\in\tilde{\mathcal{I}}_1} \left[\sum_{j\in\hat{\mathcal{J}}_1} \overline{\alpha}_{ij} + \sum_{j\notin\hat{\mathcal{J}}_1} \overline{\alpha}_{ij} \right]}{\sum_{i\in\tilde{\mathcal{I}}_1} \left[\sum_{j\in\hat{\mathcal{J}}_1} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij}) + \sum_{j\notin\hat{\mathcal{J}}_1} (\overline{\alpha}_{ij} + \underbrace{\cancel{\underline{\alpha}_{ij}}}_{= \text{ by L18}}) \right]}$$

$$= \frac{\sum_{i\in\tilde{\mathcal{I}}_1} \sum_{j\in\hat{\mathcal{J}}_1} \overline{\alpha}_{ij} + \sum_{i\in\tilde{\mathcal{I}}_1} \sum_{j\notin\hat{\mathcal{J}}_1} \overline{\alpha}_{ij}}{\sum_{i\in\tilde{\mathcal{I}}_1} \sum_{j\in\hat{\mathcal{J}}_1} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij}) + \sum_{i\in\tilde{\mathcal{I}}_1} \sum_{j\notin\hat{\mathcal{J}}_1} \overline{\alpha}_{ij}}$$

$$\ge \frac{\sum_{i\in\tilde{\mathcal{I}}_1} \sum_{j\in\hat{\mathcal{J}}_1} \overline{\alpha}_{ij}}{\sum_{i\in\tilde{\mathcal{I}}_1} \sum_{j\in\hat{\mathcal{J}}_1} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}.$$

Combining the above, we get

$$\overline{\mu}_1^s \le \frac{\sum_{i\in\tilde{\mathcal{I}}_1} \sum_{j\in\hat{\mathcal{J}}_1} \overline{\alpha}_{ij}}{\sum_{i\in\tilde{\mathcal{I}}_1} \sum_{j\in\hat{\mathcal{J}}_1} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} \le \overline{\mu}_1^b,$$

implying that $\overline{\mu}_1^s \le \overline{\mu}_1^b$.

Suppose now $j' \notin \tilde{\mathcal{J}}_1 = \{j | \mathcal{I}^*(j) = \hat{\mathcal{I}}_1\}$, i.e. $\mathcal{I}^*(j') \neq \hat{\mathcal{I}}_1$. Since both $\mathcal{I}^*(j')$ and $\hat{\mathcal{I}}_1$ are sets of signal realizations with equal posteriors, and the posteriors are ordered, we have $\min \mathcal{I}^*(j') > \max \hat{\mathcal{I}}_1$. Lemma 17 then implies that $\overline{\alpha}_{ij'} = 0$ for all $i \in \hat{\mathcal{I}}_1$. Recall that $1 \in \hat{\mathcal{I}}_1$ and the posteriors of all signal realizations in $\hat{\mathcal{I}}_1$ are the same, hence $\overline{\mu}_1^b$

is equal to the weighted average of the posteriors of signal realizations in $\hat{\mathcal{I}}_1$:

$$\overline{\mu}_1^b = \frac{\sum_{i\in\hat{\mathcal{I}}_1} \overline{\mu}_i^b \sum_{j=1}^J (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}{\sum_{i\in\hat{\mathcal{I}}_1} \sum_{i=1}^I (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} = \frac{\sum_{i\in\hat{\mathcal{I}}_1} \sum_{j=1}^J \overline{\alpha}_{ij}}{\sum_{i\in\hat{\mathcal{I}}_1} \sum_{j=1}^J (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}$$

$$= \frac{\sum_{i\in\hat{\mathcal{I}}_1} \left[\sum_{j\in\tilde{\mathcal{J}}_1} \overline{\alpha}_{ij} + \sum_{j\notin\tilde{\mathcal{J}}_1} \overbrace{\cancel{\overline{\alpha}_{ij}}}^{=0 \text{ by L17}} \right]}{\sum_{i\in\hat{\mathcal{I}}_1} \left[\sum_{j\in\tilde{\mathcal{J}}_1} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij}) + \sum_{j\notin\tilde{\mathcal{J}}_1} (\underbrace{\cancel{\overline{\alpha}_{ij}}}_{=0 \text{ by L17}} + \underline{\alpha}_{ij}) \right]}$$

$$= \frac{\sum_{i\in\hat{\mathcal{I}}_1} \sum_{j\in\tilde{\mathcal{J}}_1} \overline{\alpha}_{ij}}{\sum_{i\in\hat{\mathcal{I}}_1} \sum_{j\in\tilde{\mathcal{J}}_1} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij}) + \sum_{i\in\hat{\mathcal{I}}_1} \sum_{j\notin\tilde{\mathcal{J}}_1} \underline{\alpha}_{ij}} \leq \frac{\sum_{i\in\hat{\mathcal{I}}_1} \sum_{j\in\tilde{\mathcal{J}}_1} \overline{\alpha}_{ij}}{\sum_{i\in\hat{\mathcal{I}}_1} \sum_{j\in\tilde{\mathcal{J}}_1} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}.$$

Suppose now $j' \in \tilde{\mathcal{J}}_1$, i.e. $\mathcal{I}^*(j') = \hat{\mathcal{I}}_1$. Lemma 20 guarantees that such j' exist. Lemma 17 implies that $\underline{\alpha}_{ij'} = 0$ for all $i \notin \hat{\mathcal{I}}_1 = \mathcal{I}^*(j')$. By the ordering assumption, $\overline{\mu}_1^s$ is higher than the weighted average of the posteriors of signal realizations from $\tilde{\mathcal{J}}_1$:

$$\overline{\mu}_1^s \geq \frac{\sum_{j\in\tilde{\mathcal{J}}_1} \overline{\mu}_j^s \sum_{i=1}^I (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}{\sum_{j\in\tilde{\mathcal{J}}_1} \sum_{i=1}^I (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} = \frac{\sum_{j\in\tilde{\mathcal{J}}_1} \sum_{i=1}^I \overline{\alpha}_{ij}}{\sum_{j\in\tilde{\mathcal{J}}_1} \sum_{i=1}^I (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}$$

$$= \frac{\sum_{j\in\tilde{\mathcal{J}}_1} \left[\sum_{i\in\hat{\mathcal{I}}_1} \overline{\alpha}_{ij} + \sum_{i\notin\hat{\mathcal{I}}_1} \overline{\alpha}_{ij} \right]}{\sum_{j\in\tilde{\mathcal{J}}_1} \left[\sum_{i\in\hat{\mathcal{I}}_1} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij}) + \sum_{i\notin\hat{\mathcal{I}}_1} (\overline{\alpha}_{ij} + \underbrace{\cancel{\underline{\alpha}_{ij}}}_{=0 \text{ by L17}}) \right]}$$

$$= \frac{\sum_{j\in\tilde{\mathcal{J}}_1} \sum_{i\in\hat{\mathcal{I}}_1} \overline{\alpha}_{ij} + \sum_{j\in\tilde{\mathcal{J}}_1} \sum_{i\notin\hat{\mathcal{I}}_1} \overline{\alpha}_{ij}}{\sum_{j\in\tilde{\mathcal{J}}_1} \sum_{i\in\hat{\mathcal{I}}_1} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij}) + \sum_{j\in\tilde{\mathcal{J}}_1} \sum_{i\notin\hat{\mathcal{I}}_1} \overline{\alpha}_{ij}}$$

$$\geq \frac{\sum_{j\in\tilde{\mathcal{J}}_1} \sum_{i\in\hat{\mathcal{I}}_1} \overline{\alpha}_{ij}}{\sum_{j\in\tilde{\mathcal{J}}_1} \sum_{i\in\hat{\mathcal{I}}_1} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}.$$

Combining the above, we get

$$\overline{\mu}_1^b \leq \frac{\sum_{j\in\tilde{\mathcal{J}}_1} \sum_{i\in\hat{\mathcal{I}}_1} \overline{\alpha}_{ij}}{\sum_{j\in\tilde{\mathcal{J}}_1} \sum_{i\in\hat{\mathcal{I}}_1} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} \leq \overline{\mu}_1^s,$$

implying $\overline{\mu}_1^b \leq \overline{\mu}_1^s$. But we have also shown above that $\overline{\mu}_1^s \leq \overline{\mu}_1^b$, hence $\overline{\mu}_1^s = \overline{\mu}_1^b$. The next claim is almost immediate:

Claim 4. *$\hat{\mathcal{I}}_1 = \tilde{\mathcal{I}}_1$ and $\hat{\mathcal{J}}_1 = \tilde{\mathcal{J}}_1$.*

Proof. To see that $\hat{\mathcal{I}}_1 = \tilde{\mathcal{I}}_1$, recall that we have shown above that

$$\frac{\sum_{i\in\tilde{\mathcal{I}}_1} \overline{\mu}_i^b \sum_{j=1}^J (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}{\sum_{i\in\tilde{\mathcal{I}}_1} \sum_{j=1}^J (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} \geq \frac{\sum_{i\in\tilde{\mathcal{I}}_1} \sum_{j\in\hat{\mathcal{J}}_1} \overline{\alpha}_{ij}}{\sum_{i\in\tilde{\mathcal{I}}_1} \sum_{j\in\hat{\mathcal{J}}_1} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} \geq \overline{\mu}_1^s = \overline{\mu}_1^b,$$

i.e. that the weighted average of the posteriors in $\tilde{\mathcal{I}}_1$ exceeds $\overline{\mu}_1^b$. But from the ordering assumption we know that $\overline{\mu}_1^b \geq \overline{\mu}_{i'}^b$ for any $i' \in \tilde{\mathcal{I}}_1$, which then immediately implies

$\overline{\mu}_1^b = \overline{\mu}_{i'}^b$ for any $i' \in \tilde{\mathcal{I}}_1$ and therefore $\hat{\mathcal{I}}_1 = \tilde{\mathcal{I}}_1$. The proof of $\hat{\mathcal{J}}_1 = \tilde{\mathcal{J}}_1$ follows exactly the same lines and therefore omitted. □

Claim 4 combined with Lemma 17 then immediately implies that $\underline{\alpha}_{ij} = 0$ for all $i \notin \hat{\mathcal{I}}_1$ and $j \in \hat{\mathcal{J}}_1$. Likewise, Claim 4 combined with Lemma 18 then immediately implies that $\underline{\alpha}_{ij} = 0$ for all $i \in \hat{\mathcal{I}}_1$ and $j \notin \hat{\mathcal{J}}_1$.

We conclude the proof of the base case by establishing the following two claims:

Claim 5. $\overline{\alpha}_{ij} = 0$ *for all* $i \in \hat{\mathcal{I}}_1$ *and* $j \notin \hat{\mathcal{J}}_1$.

Claim 6. $\overline{\alpha}_{ij} = 0$ *for all* $i \notin \hat{\mathcal{I}}_1$ *and* $j \in \hat{\mathcal{J}}_1$.

The proofs of Claims 5 and 6 are analogous, hence we only prove Claim 5.

Proof of Claim 5. Let $j' \notin \hat{\mathcal{J}}_1$. Since $\hat{\mathcal{J}}_1 = \tilde{\mathcal{J}}_1$ by Claim 4 we have $j' \notin \tilde{\mathcal{J}}_1 = \{j | \mathcal{I}^*(j) = \hat{\mathcal{I}}_1\}$, i.e. $\mathcal{I}^*(j') \neq \hat{\mathcal{I}}_1$. Since both $\mathcal{I}^*(j')$ and $\hat{\mathcal{I}}_1$ are sets of signal realizations with equal posteriors and the posteriors are ordered, we have $\min \mathcal{I}^*(j') > \max \hat{\mathcal{I}}_1$. Lemma 17 then immediately implies $\overline{\alpha}_{ij'} = 0$. □

This concludes the proof of Lemma 21. □

Lemma 21 has the following corollary:

Corollary 2. *Let* $i \in \hat{\mathcal{I}}_1$, $j \in \hat{\mathcal{J}}_1$. *If* $\underline{\alpha}_{ij} + \overline{\alpha}_{ij} > 0$, *then* $t_{ij}^b = \underline{u}^b - \log(\underline{\mu}_1^b) - \underline{\lambda}_j^b = \overline{u}^b - \log(\overline{\mu}_1^b) - \overline{\lambda}_j^b$ *and* $t_{ij}^s = \underline{u}^s + \log(\underline{\mu}_1^s) + \underline{\lambda}_j^s = \overline{u}^s + \log(\overline{\mu}_1^s) + \overline{\lambda}_j^s$.

Proof. We only show the claim for the buyer, the proof for the seller is analogous. Lemma 21 implies that $\overline{\mu}_i^b = \overline{\mu}_1^b$ for all $i \in \hat{\mathcal{I}}_1$, thus $\underline{\alpha}_{ij} + \overline{\alpha}_{ij} > 0$, combined with stationarity and complementary slackness, implies either $t_{ij}^b = \underline{u}^b - \log(\underline{\mu}_1^b) - \underline{\lambda}_j^b$ or $t_{ij}^b = \overline{u}^b - \log(\overline{\mu}_1^b) - \overline{\lambda}_j^b$, or both. We now show that $\underline{u}^b - \log(\underline{\mu}_1^b) - \underline{\lambda}_j^b = \overline{u}^b - \log(\overline{\mu}_1^b) - \overline{\lambda}_j^b$.

For any $j \in \hat{\mathcal{J}}_1$ there is $i' \in \hat{\mathcal{I}}_1$ such that $\underline{\alpha}_{i'j} > 0$ (if not, Lemma 21 would imply $\underline{\alpha}_{ij} > 0$ for all i, implying $\underline{\mu}_j^s = 0$ and contradicting Lemma 5). Also, for any $j \in \hat{\mathcal{J}}_1$ there is $i'' \in \hat{\mathcal{I}}_1$ such that $\overline{\alpha}_{i''j} > 0$ (if not, Lemma 21 would imply $\overline{\alpha}_{ij} > 0$ for all i, implying $\overline{\mu}_j^s = 0$ and contradicting Lemma 5). Consider $(\text{ST}_{i'j}^b)$ and $(\text{ST}_{i''j}^b)$:

$$(\text{ST}_{i'j}^b) \quad \begin{aligned} &\underline{u}^b - t_{i'j}^b - \log\left(\underline{\mu}_1^b\right) - \underline{\lambda}_j^b = 0, \\ &\overline{u}^b - t_{i'j}^b - \log\left(\overline{\mu}_1^b\right) - \overline{\lambda}_j^b = -\overline{\phi}_{i'j}^b \leq 0. \end{aligned} \qquad (\text{ST}_{i''j}^b) \quad \begin{aligned} &\underline{u}^b - t_{i''j}^b - \log\left(\underline{\mu}_1^b\right) - \underline{\lambda}_j^b = -\underline{\phi}_{i''j}^b \leq 0, \\ &\overline{u}^b - t_{i''j}^b - \log\left(\overline{\mu}_1^b\right) - \overline{\lambda}_j^b = 0. \end{aligned}$$

$(\text{ST}_{i'j}^b)$ implies that $\overline{u}^b - \log(\overline{\mu}_1^b) - \overline{\lambda}_j^b \leq \underline{u}^b - \log(\underline{\mu}_1^b) - \underline{\lambda}_j^b$ and $(\text{ST}_{i''j}^b)$ implies that $\underline{u}^b - \log(\underline{\mu}_1^b) - \underline{\lambda}_j^b \leq \overline{u}^b - \log(\overline{\mu}_1^b) - \overline{\lambda}_j^b$, hence $\underline{u}^b - \log(\underline{\mu}_1^b) - \underline{\lambda}_j^b = \overline{u}^b - \log(\overline{\mu}_1^b) - \overline{\lambda}_j^b$. □

We can now merge the buyer's and the seller's signal realizations in $\hat{\mathcal{I}}_1$ and $\hat{\mathcal{J}}_1$ respectively, and still achieve the same revenue. More formally, define a new information structure α' on $\{\hat{1}, \bar{\hat{i}}_1+1, \ldots, I\} \times \{\hat{1}, \bar{\hat{j}}_1+1, \ldots, J\} \times V$, such that $\underline{\alpha}'_{\hat{1}\hat{1}} \equiv \sum_{i=1}^{\bar{\hat{i}}_1} \sum_{j=1}^{\bar{\hat{j}}_1} \underline{\alpha}_{ij}$ and $\overline{\alpha}'_{\hat{1}\hat{1}} \equiv \sum_{i=1}^{\bar{\hat{i}}_1} \sum_{j=1}^{\bar{\hat{j}}_1} \overline{\alpha}_{ij}$, while $\underline{\alpha}'_{\hat{1}j} = \overline{\alpha}'_{\hat{1}j} \equiv 0$ for all $j \neq \hat{1}$ and $\underline{\alpha}'_{i\hat{1}} = \overline{\alpha}'_{i\hat{1}} \equiv 0$ for all $i \neq \hat{1}$; and the remaining probabilities are the same: $\underline{\alpha}'_{ij} \equiv \underline{\alpha}_{ij}$ and $\overline{\alpha}'_{ij} \equiv \overline{\alpha}_{ij}$ for $(i,j) \in \{\bar{\hat{i}}_1 + 1, \ldots, I\} \times \{\bar{\hat{j}}_1 + 1, \ldots, J\}$. α' can be illustrated as follows:

$\underline{v}$	$s^s_{\hat{1}}$	$s^s_{\bar{\hat{j}}_1+1}$	$\ldots$	s^s_J	$\overline{v}$	$s^s_{\hat{1}}$	$s^s_{\bar{\hat{j}}_1+1}$	$\ldots$	s^s_J
$s^b_{\hat{1}}$	$\sum_{i=1}^{\bar{\hat{i}}_1} \sum_{j=1}^{\bar{\hat{j}}_1} \underline{\alpha}_{ij}$	0	$\ldots$	0	$s^b_{\hat{1}}$	$\sum_{i=1}^{\bar{\hat{i}}_1} \sum_{j=1}^{\bar{\hat{j}}_1} \overline{\alpha}_{ij}$	0	$\ldots$	0
$s^b_{\bar{\hat{i}}_1+1}$	0	$\underline{\alpha}_{\bar{\hat{i}}_1+1,\bar{\hat{j}}_1+1}$	$\ldots$	$\underline{\alpha}_{\bar{\hat{i}}_1+1,J}$	$s^b_{\bar{\hat{i}}_1+1}$	0	$\overline{\alpha}_{\bar{\hat{i}}_1+1,\bar{\hat{j}}_1+1}$	$\ldots$	$\overline{\alpha}_{\bar{\hat{i}}_1+1,J}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
s^b_I	0	$\underline{\alpha}_{I,\bar{\hat{j}}_1+1}$	$\ldots$	$\underline{\alpha}_{IJ}$	s^b_I	0	$\overline{\alpha}_{I,\bar{\hat{j}}_1+1}$	$\ldots$	$\overline{\alpha}_{IJ}$

Define further $t'^p_{ij} \equiv t^p_{ij}$ for all $(i,j) \in \{\bar{\hat{i}}_1 + 1, \ldots, I\} \times \{\bar{\hat{j}}_1 + 1, \ldots, J\}$ and both $p \in \{b, s\}$, and set $t'^b_{\hat{1}j} = t'^b_{i\hat{1}} \equiv T$ and $t'^s_{\hat{1}j} = t'^s_{i\hat{1}} \equiv -T$ for all $i \neq \hat{1}$ and $j \neq \hat{1}$, where T is a large number, and define for both $p \in \{b, s\}$:

$$t'^p_{\hat{1}\hat{1}} \equiv \frac{\sum_{i \in \hat{\mathcal{I}}_1, j \in \hat{\mathcal{J}}_1 | \overline{\alpha}_{ij} + \underline{\alpha}_{ij} > 0} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij}) t^p_{ij}}{\sum_{i \in \hat{\mathcal{I}}_1, j \in \hat{\mathcal{J}}_1 | \overline{\alpha}_{ij} + \underline{\alpha}_{ij} > 0} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}.$$

Clearly, with transfers so defined, (α', t') achieves the same revenue as (α, t). If we define $\underline{\lambda}'^b_{\hat{1}} \equiv \underline{u}^b - t'^b_{\hat{1}\hat{1}} - \log\left(\underline{\mu}'^b_{\hat{1}}\right)$ and $\overline{\lambda}'^b_{\hat{1}} \equiv \overline{u}^b - t'^b_{\hat{1}\hat{1}} - \log\left(\overline{\mu}'^b_{\hat{1}}\right)$, we will make sure that the $(\mathrm{ST}^b_{\hat{1}\hat{1}})$-conditions are satisfied. Likewise, if we define $\underline{\lambda}'^s_{\hat{1}} \equiv t'^s_{\hat{1}\hat{1}} - \underline{u}^s - \log\left(\underline{\mu}'^s_{\hat{1}}\right)$ and $\overline{\lambda}'^s_{\hat{1}} \equiv t'^s_{\hat{1}\hat{1}} - \overline{u}^s - \log\left(\overline{\mu}'^s_{\hat{1}}\right)$, we will make sure that all the $(\mathrm{ST}^s_{\hat{1}\hat{1}})$-conditions are satisfied. By complementary slackness we then have $\underline{\phi}'^b_{\hat{1}\hat{1}} = \overline{\phi}'^b_{\hat{1}\hat{1}} = \underline{\phi}'^s_{\hat{1}\hat{1}} = \overline{\phi}'^s_{\hat{1}\hat{1}} = 0$.

Choosing T to be large enough makes sure that $\underline{\phi}'^b_{ij} \equiv -\underline{u}^b + T + \log\left(\underline{\mu}'^b_i\right) + \underline{\lambda}'^b_j \geq 0$ and $\overline{\phi}'^b_{ij} \equiv -\overline{u}^b + T + \log\left(\overline{\mu}'^b_i\right) + \overline{\lambda}'^b_j \geq 0$ for all $(i,j) = (\hat{1}, j)$ and $(i,j) = (i, \hat{1})$ thus satisfying the (ST^b_{ij}), (CS^b_{ij}), and (DF^b_{ij})-conditions for all such (i,j). Likewise, choosing T to be large enough makes sure that $\underline{\phi}'^s_{ij} \equiv \underline{u}^s + T + \log\left(\underline{\mu}'^s_j\right) + \underline{\lambda}'^s_j \geq 0$ and $\overline{\phi}'^s_{ij} \equiv \overline{u}^b + T + \log\left(\overline{\mu}'^s_j\right) + \overline{\lambda}'^s_j \geq 0$ for all $(i,j) = (\hat{1}, j)$ and $(i,j) = (i, \hat{1})$ thus satisfying the (ST^s_{ij}), (CS^s_{ij}), and (DF^s_{ij})-conditions for all such (i,j).

The remaining values of λ' and ϕ' are equal to their respective values in λ and ϕ, and thus, by construction, the remaining (ST), (CS), and (DF)-conditions are left unchanged under the new structure.

To show that $(\alpha', I', J'; t'; \phi', \lambda')$ satisfy the implementability conditions, it remains to show that the $(\mathrm{NA})^b$ and $(\mathrm{NA})^s$- conditions are satisfied:

Lemma 22. *The* (NA^b)*-condition is satisfied by the new structure, i.e.*

$$\exp\big(-\min_{j\in(S^s\setminus\hat{\mathcal{J}}_1)\cup\{\hat{1}\}}\{\underline{\lambda}'^b_j\}\big)+\exp\big(-\min_{j\in(S^s\setminus\hat{\mathcal{J}}_1)\cup\{\hat{1}\}}\{\overline{\lambda}'^b_j\}\big)\le 1.$$

Lemma 23. *The* (NA^s)*-condition is satisfied by the new structure, i.e.*

$$\exp\big(-\min_{i\in(S^b\setminus\hat{\mathcal{I}}_1)\cup\{\hat{1}\}}\{\underline{\lambda}'^s_i\}\big)+\exp\big(-\min_{i\in(S^b\setminus\hat{\mathcal{I}}_1)\cup\{\hat{1}\}}\{\overline{\lambda}'^s_i\}\big)\le 1.$$

Lemmas 22 and 23 have analogous proofs, hence we only prove Lemma 22 here.

Proof of Lemma 22. Consider first $\min_{j\in(S^s\setminus\hat{\mathcal{J}}_1)\cup\{\hat{1}\}}\{\underline{\lambda}'^b_j\}$. Either $\min_{j\in(S^s\setminus\hat{\mathcal{J}}_1)\cup\{\hat{1}\}}\{\underline{\lambda}'^b_j\} = \min_{j\in S^s\setminus\hat{\mathcal{J}}_1}\{\underline{\lambda}'^b_j\} = \min_{j\in S^s\setminus\hat{\mathcal{J}}_1}\{\underline{\lambda}^b_j\}\ge\min_{j\in S^s}\{\underline{\lambda}^b_j\}$; or $\min_{j\in(S^s\setminus\hat{\mathcal{J}}_1)\cup\{\hat{1}\}}\{\underline{\lambda}'^b_j\}=\hat{1}$. We will show that $\underline{\lambda}'^b_{\hat{1}}\ge\min_{j\in S^s}\{\underline{\lambda}^b_j\}$ too:

$$\begin{aligned}
\underline{\lambda}'^b_{\hat{1}} &= \underline{u}^b - t'^b_{\hat{1}\hat{1}} - \log\big(\overbrace{\underline{\mu}'^b_{\hat{1}}}^{=\underline{\mu}^b_1}\big)\\
&= \underline{u}^b - \frac{\sum_{i\in\hat{\mathcal{I}}_1,j\in\hat{\mathcal{J}}_1|\overline{\alpha}_{ij}+\underline{\alpha}_{ij}>0}(\overline{\alpha}_{ij}+\underline{\alpha}_{ij})t^b_{ij}}{\sum_{i\in\hat{\mathcal{I}}_1,j\in\hat{\mathcal{J}}_1|\overline{\alpha}_{ij}+\underline{\alpha}_{ij}>0}(\overline{\alpha}_{ij}+\underline{\alpha}_{ij})} - \log\big(\underline{\mu}^b_1\big)\\
&= \underline{u}^b - \frac{\sum_{i\in\hat{\mathcal{I}}_1,j\in\hat{\mathcal{J}}_1|\overline{\alpha}_{ij}+\underline{\alpha}_{ij}>0}(\overline{\alpha}_{ij}+\underline{\alpha}_{ij})\overbrace{\big(\underline{u}^b-\log(\underline{\mu}^b_1)-\underline{\lambda}^b_j\big)}^{\text{by Corollary 2}}}{\sum_{i\in\hat{\mathcal{I}}_1,j\in\hat{\mathcal{J}}_1|\overline{\alpha}_{ij}+\underline{\alpha}_{ij}>0}(\overline{\alpha}_{ij}+\underline{\alpha}_{ij})} - \log\big(\underline{\mu}^b_1\big)\\
&= \frac{\sum_{i\in\hat{\mathcal{I}}_1,j\in\hat{\mathcal{J}}_1|\overline{\alpha}_{ij}+\underline{\alpha}_{ij}>0}(\overline{\alpha}_{ij}+\underline{\alpha}_{ij})\underline{\lambda}^b_j}{\sum_{i\in\hat{\mathcal{I}}_1,j\in\hat{\mathcal{J}}_1|\overline{\alpha}_{ij}+\underline{\alpha}_{ij}>0}(\overline{\alpha}_{ij}+\underline{\alpha}_{ij})} \ge \frac{\sum_{i\in\hat{\mathcal{I}}_1,j\in\hat{\mathcal{J}}_1|\overline{\alpha}_{ij}+\underline{\alpha}_{ij}>0}(\overline{\alpha}_{ij}+\underline{\alpha}_{ij})\min_{j\in S^s}\{\underline{\lambda}^b_j\}}{\sum_{i\in\hat{\mathcal{I}}_1,j\in\hat{\mathcal{J}}_1|\overline{\alpha}_{ij}+\underline{\alpha}_{ij}>0}(\overline{\alpha}_{ij}+\underline{\alpha}_{ij})} = \min_{j\in S^s}\{\underline{\lambda}^b_j\}.
\end{aligned}$$

Using an analogous argument, we can establish $\min_{j\in(S^s\setminus\hat{\mathcal{J}}_1)\cup\{\hat{1}\}}\{\overline{\lambda}'^b_j\}\ge\min_{j\in S^s}\{\overline{\lambda}^b_j\}$. Hence we have $\exp\big(-\min_{j\in(S^s\setminus\hat{\mathcal{J}}_1)\cup\{\hat{1}\}}\{\underline{\lambda}'^b_j\}\big)+\exp\big(-\min_{j\in(S^s\setminus\hat{\mathcal{J}}_1)\cup\{\hat{1}\}}\{\overline{\lambda}'^b_j\}\big)\le\exp\big(-\min_{j\in S^s}\{\underline{\lambda}^b_j\}\big)+\exp\big(-\min_{j\in S^s}\{\overline{\lambda}^b_j\}\big)\le 1$. □

Let us now formulate our induction hypothesis

Induction hypothesis 2. *There exists* $k<\min\{I,J\}$ *such that* $\underline{\alpha}_{ij}=\overline{\alpha}_{ij}=0$ *if* $i\ne j$ *and* ($i\le k-1$ *or* $j\le k-1$)*:*

$\underline{v}$	s^s_1	$\dots$	s^s_{k-1}	s^s_k	$\dots$	s^s_J
s^b_1	$\underline{\alpha}_{11}$	$\dots$	0	0	$\dots$	0
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
s^b_{k-1}	0	$\dots$	$\underline{\alpha}_{k-1,k-1}$	0	$\dots$	0
s^b_k	0	$\dots$	0	$\underline{\alpha}_{kk}$	$\dots$	$\underline{\alpha}_{kJ}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
s^b_I	0	$\dots$	0	$\underline{\alpha}_{Ik}$	$\dots$	$\underline{\alpha}_{IJ}$

$\overline{v}$	s^s_1	$\dots$	s^s_{k-1}	s^s_k	$\dots$	s^s_J
s^b_1	$\overline{\alpha}_{11}$	$\dots$	0	0	$\dots$	0
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
s^b_{k-1}	0	$\dots$	$\overline{\alpha}_{k-1,k-1}$	0	$\dots$	0
s^b_k	0	$\dots$	0	$\overline{\alpha}_{kk}$	$\dots$	$\overline{\alpha}_{kJ}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
s^b_I	0	$\dots$	0	$\overline{\alpha}_{Ik}$	$\dots$	$\overline{\alpha}_{IJ}$

Moreover, $\overline{\mu}^b_1>\dots>\overline{\mu}^b_k\ge\dots\ge\overline{\mu}^b_I$ *and* $\overline{\mu}^s_1>\dots>\overline{\mu}^s_k\ge\dots\ge\overline{\mu}^s_J$.

We now show the following two lemmas:

Lemma 24. *For every $i' \leq k-1$ we have $\mathcal{J}^*(i') = \{i'\}$.*

Lemma 25. *For every $j' \leq k-1$ we have $\mathcal{I}^*(j') = \{j'\}$.*

Lemmas 24 and 25 have analogous proofs thus we only prove Lemma 24 here.

Proof of Lemma 24. Suppose $\mathcal{J}^*(i') \neq \{i'\}$. The following cases are possible:

Case 1: $\mathcal{J}^*(i') = \{j^*\}$ for some $j^* < i'$. If $i' = 1$, then this case does not apply, hence suppose $i' \neq 1$. Induction hypothesis 2 then implies that $\underline{\alpha}_{i'j} = 0$ for all $j \neq i'$. Since $j^* < i'$, Lemma 18 implies $\underline{\alpha}_{i'i'} = 0$, hence $\underline{\mu}^b_{i'} = 0$, contradicting Lemma 5.

Case 2: $\min \mathcal{J}^*(i') > i'$. Induction hypothesis 2 implies that $\overline{\alpha}_{i'j} = 0$ for all $j \neq i'$. Lemma 18 implies $\overline{\alpha}_{i'i'} = 0$, hence $\overline{\mu}^b_{i'} = 0$, contradicting Lemma 5. □

We also establish the following:

Lemma 26. *If $i \geq k$, then $\mathcal{J}^*(i) \neq \{j^*\}$ for any $j^* \leq k-1$.*

Lemma 27. *If $j \geq k$, then $\mathcal{I}^*(j) \neq \{i^*\}$ for any $i^* \leq k-1$.*

Lemmas 26 and 27 have analogous proofs, hence we only prove Lemma 26 here.

Proof of Lemma 26. Suppose for a contradiction that there is $i \geq k$ and $\mathcal{J}^*(i) = \{j^*\}$ for some $j^* \leq k-1$. Induction hypothesis 2 implies $\underline{\alpha}_{ij} = 0$ for all $1 \leq j \leq k-1$, and Lemma 18 implies $\underline{\alpha}_{ij} = 0$ for all $j^*+1 \leq j \leq J$, hence $\underline{\mu}^b_i = 0$, which means that $\underline{\alpha}_{ij} = 0$ for all j, contradicting Lemma 5. □

Let us introduce the following sets: $\hat{\mathcal{I}}_k \equiv \{i | \overline{\mu}^b_i = \overline{\mu}^b_k\}$, $\hat{\mathcal{J}}_k \equiv \{j | \overline{\mu}^s_j = \overline{\mu}^s_k\}$; and $\tilde{\mathcal{I}}_k \equiv \{i | \mathcal{J}^*(i) = \hat{\mathcal{J}}_k\}$, $\tilde{\mathcal{J}}_k \equiv \{j | \mathcal{I}^*(i) = \hat{\mathcal{I}}_k\}$. Define $\bar{\hat{i}}_k \equiv \max \hat{\mathcal{I}}_k$ and $\bar{\hat{j}}_k \equiv \max \hat{\mathcal{J}}_k$. We prove the following pair of auxiliary lemmas:

Lemma 28. $\tilde{\mathcal{I}}_k = \{i | \mathcal{J}^*(i) = \hat{\mathcal{J}}_k\} \neq \emptyset$.

Lemma 29. $\tilde{\mathcal{J}}_k = \{j | \mathcal{I}^*(i) = \hat{\mathcal{I}}_k\} \neq \emptyset$.

Lemmas 28 and 29 have analogous proofs, hence we only prove Lemma 28 here.

Proof of Lemma 28. Suppose for a contradiction that $\hat{\mathcal{I}}_k = \emptyset$, i.e. for any i we have $\mathcal{J}^*(i) \neq \hat{\mathcal{J}}_k$. From Lemma 24 we know that $\mathcal{J}^*(i) = \{i\} \neq \hat{\mathcal{J}}_k$ for $i \leq k-1$, hence let us consider cases $i \geq k$. If $i \geq k$, then $\mathcal{J}^*(i) \neq \{j^*\}$ for any $j^* \leq k-1$ by Lemma 26, hence the only possibility is $\min \mathcal{J}^*(i) > \max \hat{\mathcal{J}}_k$, which implies $\overline{\alpha}_{ik} = 0$ by Lemma 18. Since $i \geq k$ was arbitrarily chosen, we get $\overline{\alpha}_{ik} = 0$ for all $i \geq k$. Induction hypothesis 2 implies $\overline{\alpha}_{ik} = 0$ for all $i \leq k-1$, implying $\overline{\mu}^s_k = 0$, contradicting Lemma 5. □

We can now establish the step case of our induction:

Lemma 30 (**Step case of the induction**).

1. *For all* $i \in \hat{\mathcal{I}}_k$ *and* $j \notin \hat{\mathcal{J}}_k$ *we have* $\underline{\alpha}_{ij} = \overline{\alpha}_{ij} = 0$.
2. *For all* $i \notin \hat{\mathcal{I}}_k$ *and* $j \in \hat{\mathcal{J}}_k$ *we have* $\underline{\alpha}_{ij} = \overline{\alpha}_{ij} = 0$.

Proof. Suppose $i' \geq k$ and $i' \notin \tilde{\mathcal{I}}_k = \{i | \mathcal{J}^*(i) = \hat{\mathcal{J}}_k\}$, i.e. $\mathcal{J}^*(i') \neq \hat{\mathcal{J}}_k$. Lemma 26 implies that $\mathcal{J}^*(i') \neq \{j^*\}$ for any $j^* \leq k-1$, hence the only remaining possibility is $\min \mathcal{J}^*(i') > \max \hat{\mathcal{J}}_k$. Lemma 18 then implies $\overline{\alpha}_{i'j} = 0$ for any $j \in \hat{\mathcal{J}}_k$. Moreover, $\overline{\alpha}_{i'j} = 0$ for all $j \leq k-1$ by Induction hypothesis 2. Recall that $k \in \hat{\mathcal{J}}_k$ and the posteriors of all signal realizations in $\hat{\mathcal{J}}_k$ are the same, hence $\overline{\mu}^s_k$ is equal to the weighted average of the posteriors of signal realizations from $\hat{\mathcal{J}}_k$:

$$\overline{\mu}^s_k = \frac{\sum_{j \in \hat{\mathcal{J}}_k} \overline{\mu}^s_j \sum_{i=1}^I (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}{\sum_{j \in \hat{\mathcal{J}}_k} \sum_{i=1}^I (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} = \frac{\sum_{j \in \hat{\mathcal{J}}_k} \sum_{i=1}^I \overline{\alpha}_{ij}}{\sum_{j \in \hat{\mathcal{J}}_k} \sum_{i=1}^I (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}$$

$$= \frac{\sum_{j \in \hat{\mathcal{J}}_k} \left[\sum_{i \leq k-1} \overbrace{\cancel{\overline{\alpha}_{ij}}}^{=0 \text{ by IH2}} + \sum_{i \in \tilde{\mathcal{I}}_k} \overline{\alpha}_{ij} + \sum_{i \geq k, i \notin \tilde{\mathcal{I}}_k} \overbrace{\cancel{\overline{\alpha}_{ij}}}^{=0 \text{ by L18}} \right]}{\sum_{j \in \hat{\mathcal{J}}_k} \left[\sum_{i \leq k-1} \underbrace{\cancel{(\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}}_{=0 \text{ by IH2}} + \sum_{i \in \tilde{\mathcal{I}}_k} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij}) + \sum_{i \geq k, i \notin \tilde{\mathcal{I}}_k} (\underbrace{\cancel{\overline{\alpha}_{ij}}}_{=0 \text{ by L18}} + \underline{\alpha}_{ij}) \right]}$$

$$= \frac{\sum_{j \in \hat{\mathcal{J}}_k} \sum_{i \in \tilde{\mathcal{I}}_k} \overline{\alpha}_{ij}}{\sum_{j \in \hat{\mathcal{J}}_k} \sum_{i \in \tilde{\mathcal{I}}_k} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij}) + \sum_{j \in \hat{\mathcal{J}}_k} \sum_{i \geq k, i \notin \tilde{\mathcal{I}}_k} \underline{\alpha}_{ij}} \leq \frac{\sum_{j \in \hat{\mathcal{J}}_k} \sum_{i \in \tilde{\mathcal{I}}_k} \overline{\alpha}_{ij}}{\sum_{j \in \hat{\mathcal{J}}_k} \sum_{i \in \tilde{\mathcal{I}}_k} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}.$$

Suppose now that $i' \in \tilde{\mathcal{I}}_k$, i.e. $\mathcal{J}^*(i') = \hat{\mathcal{J}}_k$. Lemma 28 guarantees that such i' exist, and Lemma 24 implies that $i' \geq k$ (otherwise we would have $\mathcal{J}^*(i') = \{i'\} \neq \hat{\mathcal{J}}_k$). Lemma 18 and Induction hypothesis 2 imply that $\underline{\alpha}_{i'j} = 0$ for all $j \notin \hat{\mathcal{J}}_k = \mathcal{J}^*(i')$. By the ordering assumption, $\overline{\mu}^b_k$ is then higher than the weighted average of the posteriors of signal realizations from $\tilde{\mathcal{I}}_k$:

$$\begin{aligned}\overline{\mu}^b_k \geq \frac{\sum_{i\in\tilde{\mathcal{I}}_k} \overline{\mu}^b_i \sum_{j=1}^J(\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}{\sum_{i\in\tilde{\mathcal{I}}_k}\sum_{j=1}^J(\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} &= \frac{\sum_{i\in\tilde{\mathcal{I}}_k}\sum_{j=1}^J \overline{\alpha}_{ij}}{\sum_{i\in\tilde{\mathcal{I}}_k}\sum_{j=1}^J(\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} \\ &= \frac{\sum_{i\in\tilde{\mathcal{I}}_k}\left[\sum_{j\in\hat{\mathcal{J}}_k}\overline{\alpha}_{ij} + \sum_{j\notin\hat{\mathcal{J}}_k}\overline{\alpha}_{ij}\right]}{\sum_{i\in\tilde{\mathcal{I}}_k}\left[\sum_{j\in\hat{\mathcal{J}}_k}(\overline{\alpha}_{ij} + \underline{\alpha}_{ij}) + \sum_{j\notin\hat{\mathcal{J}}_k}(\overline{\alpha}_{ij} + \underbrace{\cancel{\underline{\alpha}_{ij}}}_{=0 \text{ by L18 \& IH2}})\right]} \\ &= \frac{\sum_{i\in\tilde{\mathcal{I}}_k}\sum_{j\in\hat{\mathcal{J}}_k}\overline{\alpha}_{ij} + \sum_{i\in\tilde{\mathcal{I}}_k}\sum_{j\notin\hat{\mathcal{J}}_k}\overline{\alpha}_{ij}}{\sum_{i\in\tilde{\mathcal{I}}_k}\sum_{j\in\hat{\mathcal{J}}_k}(\overline{\alpha}_{ij} + \underline{\alpha}_{ij}) + \sum_{i\in\tilde{\mathcal{I}}_k}\sum_{j\notin\hat{\mathcal{J}}_k}\overline{\alpha}_{ij}} \\ &\geq \frac{\sum_{i\in\tilde{\mathcal{I}}_k}\sum_{j\in\hat{\mathcal{J}}_k}\overline{\alpha}_{ij}}{\sum_{i\in\tilde{\mathcal{I}}_k}\sum_{j\in\hat{\mathcal{J}}_k}(\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}.\end{aligned}$$

Combining the above, we get

$$\overline{\mu}^s_k \leq \frac{\sum_{i\in\tilde{\mathcal{I}}_k}\sum_{j\in\hat{\mathcal{J}}_k}\overline{\alpha}_{ij}}{\sum_{i\in\tilde{\mathcal{I}}_k}\sum_{j\in\hat{\mathcal{J}}_k}(\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} \leq \overline{\mu}^b_k,$$

implying that $\overline{\mu}^s_k \leq \overline{\mu}^b_k$.

Suppose now $j' \geq k$ and $j' \notin \tilde{\mathcal{J}}_k = \{j | \mathcal{I}^*(j) = \hat{\mathcal{I}}_k\}$, i.e. $\mathcal{I}^*(j') \neq \hat{\mathcal{I}}_k$. Lemma 27 implies that $\mathcal{I}^*(j') \neq \{i^*\}$ for any $i^* \leq k-1$, hence the only remaining possibility is $\min \mathcal{I}^*(j') > \max \hat{\mathcal{I}}_k$. Lemma 17 then implies $\overline{\alpha}_{ij'} = 0$ for any $i \in \hat{\mathcal{I}}_k$. Moreover, $\overline{\alpha}_{ij'} = 0$ for all $i \leq k-1$ by Induction hypothesis 2. Recall that $k \in \hat{\mathcal{I}}_k$ and the posteriors of all signal realizations in $\hat{\mathcal{I}}_k$ are the same, hence $\overline{\mu}^b_k$ is equal to the weighted average of the posteriors of signal realizations from $\hat{\mathcal{I}}_k$:

$$\begin{aligned}\overline{\mu}^b_k = \frac{\sum_{i\in\hat{\mathcal{I}}_k}\overline{\mu}^b_i\sum_{j=1}^J(\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}{\sum_{i\in\hat{\mathcal{I}}_k}\sum_{j=1}^J(\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} &= \frac{\sum_{i\in\hat{\mathcal{I}}_k}\sum_{j=1}^J\overline{\alpha}_{ij}}{\sum_{i\in\hat{\mathcal{I}}_k}\sum_{j=1}^J(\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} \\ &= \frac{\sum_{i\in\hat{\mathcal{I}}_k}\left[\sum_{j\leq k-1}\overbrace{\cancel{\overline{\alpha}_{ij}}}^{=0 \text{ by IH2}} + \sum_{j\in\tilde{\mathcal{J}}_k}\overline{\alpha}_{ij} + \sum_{j\geq k, j\notin\tilde{\mathcal{J}}_k}\overbrace{\cancel{\overline{\alpha}_{ij}}}^{=0 \text{ by L17}}\right]}{\sum_{i\in\hat{\mathcal{I}}_k}\left[\sum_{j\leq k-1}\underbrace{\cancel{(\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}}_{=0 \text{ by IH2}} + \sum_{j\in\tilde{\mathcal{J}}_k}(\overline{\alpha}_{ij} + \underline{\alpha}_{ij}) + \sum_{j\geq k, j\notin\tilde{\mathcal{J}}_k}(\underbrace{\cancel{\overline{\alpha}_{ij}}}_{=0 \text{ by L17}} + \underline{\alpha}_{ij})\right]} \\ &= \frac{\sum_{i\in\hat{\mathcal{I}}_k}\sum_{j\in\tilde{\mathcal{J}}_k}\overline{\alpha}_{ij}}{\sum_{i\in\hat{\mathcal{I}}_k}\sum_{j\in\tilde{\mathcal{J}}_k}(\overline{\alpha}_{ij} + \underline{\alpha}_{ij}) + \sum_{i\in\hat{\mathcal{I}}_k}\sum_{j\geq k, j\notin\tilde{\mathcal{J}}_k}\underline{\alpha}_{ij}} \leq \frac{\sum_{i\in\hat{\mathcal{I}}_k}\sum_{j\in\tilde{\mathcal{J}}_k}\overline{\alpha}_{ij}}{\sum_{i\in\hat{\mathcal{I}}_k}\sum_{j\in\tilde{\mathcal{J}}_k}(\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}.\end{aligned}$$

Suppose now that $j' \in \tilde{\mathcal{J}}_k$, i.e. $\mathcal{I}^*(j') = \hat{\mathcal{I}}_k$. Lemma 29 guarantees that such j' exist, and Lemma 25 implies that $j' \geq k$ (otherwise we would have $\mathcal{I}^*(j') = \{j'\} \neq \hat{\mathcal{I}}_k$). Lemma 17 and Induction hypothesis 2 imply that $\underline{\alpha}_{ij'} = 0$ for all $i \notin \hat{\mathcal{I}}_k = \mathcal{I}^*(j')$. By the ordering assumption, $\overline{\mu}^s_k$ is then higher than the weighted average of the posteriors of signal realizations from $\tilde{\mathcal{J}}_k$:

$$\overline{\mu}^s_k \geq \frac{\sum_{j\in\tilde{\mathcal{J}}_k} \overline{\mu}^s_j \sum_{i=1}^I (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}{\sum_{j\in\tilde{\mathcal{J}}_k} \sum_{i=1}^I (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} = \frac{\sum_{j\in\tilde{\mathcal{J}}_k} \sum_{i=1}^I \overline{\alpha}_{ij}}{\sum_{i\in\tilde{\mathcal{J}}_k} \sum_{i=1}^I (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}$$

$$= \frac{\sum_{j\in\tilde{\mathcal{J}}_k} \left[\sum_{i\in\hat{\mathcal{I}}_k} \overline{\alpha}_{ij} + \sum_{i\notin\hat{\mathcal{I}}_k} \overline{\alpha}_{ij}\right]}{\sum_{j\in\tilde{\mathcal{J}}_k} \left[\sum_{i\in\hat{\mathcal{I}}_k} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij}) + \sum_{i\notin\hat{\mathcal{I}}_k} (\overline{\alpha}_{ij} + \underbrace{\cancel{\underline{\alpha}_{ij}}}_{=0 \text{ by L17 \& IH2}})\right]}$$

$$= \frac{\sum_{j\in\tilde{\mathcal{J}}_k} \sum_{i\in\hat{\mathcal{I}}_k} \overline{\alpha}_{ij} + \sum_{j\in\tilde{\mathcal{J}}_k} \sum_{i\notin\hat{\mathcal{I}}_k} \overline{\alpha}_{ij}}{\sum_{j\in\tilde{\mathcal{J}}_k} \sum_{i\in\hat{\mathcal{I}}_k} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij}) + \sum_{j\in\tilde{\mathcal{J}}_k} \sum_{i\notin\hat{\mathcal{I}}_k} \overline{\alpha}_{ij}}$$

$$\geq \frac{\sum_{j\in\tilde{\mathcal{J}}_k} \sum_{i\in\hat{\mathcal{I}}_k} \overline{\alpha}_{ij}}{\sum_{j\in\tilde{\mathcal{J}}_k} \sum_{i\in\hat{\mathcal{I}}_k} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}.$$

Combining the above, we get

$$\overline{\mu}^b_k \leq \frac{\sum_{j\in\tilde{\mathcal{J}}_k} \sum_{i\in\hat{\mathcal{I}}_k} \overline{\alpha}_{ij}}{\sum_{j\in\tilde{\mathcal{J}}_k} \sum_{i\in\hat{\mathcal{I}}_k} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} \leq \overline{\mu}^s_k,$$

implying that $\overline{\mu}^b_k \leq \overline{\mu}^s_k$. Above, we have established $\overline{\mu}^s_k \leq \overline{\mu}^b_k$, hence $\overline{\mu}^s_k = \overline{\mu}^b_k$. The next claim is analogous to Claim 4.

Claim 7. $\hat{\mathcal{I}}_k = \tilde{\mathcal{I}}_k$ *and* $\hat{\mathcal{J}}_k = \tilde{\mathcal{J}}_k$.

Proof. To see that $\hat{\mathcal{I}}_k = \tilde{\mathcal{I}}_k$ recall that we have shown above that

$$\frac{\sum_{i\in\tilde{\mathcal{I}}_k} \overline{\mu}^b_i \sum_{j=1}^J (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})}{\sum_{i\in\tilde{\mathcal{I}}_k} \sum_{j=1}^J (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} \geq \frac{\sum_{i\in\tilde{\mathcal{I}}_k} \sum_{j\in\hat{\mathcal{J}}_k} \overline{\alpha}_{ij}}{\sum_{i\in\tilde{\mathcal{I}}_k} \sum_{j\in\hat{\mathcal{J}}_k} (\overline{\alpha}_{ij} + \underline{\alpha}_{ij})} \geq \overline{\mu}^s_k = \overline{\mu}^b_k,$$

i.e. that the weighted average of the posteriors in $\tilde{\mathcal{I}}_k$ exceeds $\overline{\mu}^b_k$. If $i' \in \tilde{\mathcal{I}}_k$, then Lemma 24 implies that $i' \geq k$ (otherwise we would have $\mathcal{J}^*(i') = \{i'\} \neq \hat{\mathcal{J}}_k$), but then we know from the ordering assumption that $\overline{\mu}^b_k \geq \overline{\mu}^b_{i'}$ for any $i' \in \tilde{\mathcal{I}}_k$, which then immediately implies $\overline{\mu}^b_k = \overline{\mu}^b_{i'}$ for any $i' \in \tilde{\mathcal{I}}_k$, and therefore $\hat{\mathcal{I}}_k = \tilde{\mathcal{I}}_k$. The proof of $\hat{\mathcal{J}}_k = \tilde{\mathcal{J}}_k$ follows exactly the same lines. □

Claim 7 combined with Lemma 17 and Induction hypothesis 2 then immediately implies that $\underline{\alpha}_{ij} = 0$ for all $i \notin \hat{\mathcal{I}}_k$ and $j \in \hat{\mathcal{J}}_k$. Claim 7 combined with Lemma 18 and Induction hypothesis 2 implies that $\underline{\alpha}_{ij} = 0$ for all $i \in \hat{\mathcal{I}}_k$ and $j \notin \hat{\mathcal{J}}_k$.

We conclude the proof of the base step by establishing the following two claims:

Claim 8. $\overline{\alpha}_{ij} = 0$ *for all* $i \in \hat{\mathcal{I}}_k$ *and* $j \notin \hat{\mathcal{J}}_k$.

Claim 9. $\overline{\alpha}_{ij} = 0$ *for all* $i \notin \hat{\mathcal{I}}_k$ *and* $j \in \hat{\mathcal{J}}_k$.

The proofs of Claims 8 and 9 are analogous, hence we only prove Claim 8.

Proof of Claim 8. Let $j' \notin \hat{\mathcal{J}}_k$. If $j' \leq k-1$, then $\overline{\alpha}_{ij'} = 0$ for all $i \in \hat{\mathcal{I}}_k$ by Induction hypothesis 2. Let $j' > \max \hat{\mathcal{J}}_k$. Since $\hat{\mathcal{J}}_k = \tilde{\mathcal{J}}_k$ by Claim 7, we have $j' \notin \tilde{\mathcal{J}}_k = \{j | \mathcal{I}^*(j) = \hat{\mathcal{I}}_k\}$, i.e. $\mathcal{I}^*(j') \neq \hat{\mathcal{I}}_k$. Recall that Lemma 27 implies $\mathcal{I}^*(j) \neq \{i^*\}$ for any $i^* \leq k-1$, and thus the only possible case is $\min \mathcal{I}^*(j') > \max \hat{\mathcal{I}}_k$. Lemma 17 then implies $\overline{\alpha}_{ij'} = 0$ for all $i \in \hat{\mathcal{I}}_k$. □

This concludes the proof of Lemma 30. □

Lemma 30 has the following corollary, analogous to Corollary 2.

Corollary 3. *Let $i \in \hat{\mathcal{I}}_k$, $j \in \hat{\mathcal{J}}_k$. If $\underline{\alpha}_{ij} + \overline{\alpha}_{ij} > 0$, then $t^b_{ij} = \underline{u}^b - \log(\underline{\mu}^b_k) - \underline{\lambda}^b_j = \overline{u}^b - \log(\overline{\mu}^b_k) - \overline{\lambda}^b_j$ and $t^s_{ij} = \underline{u}^s + \log(\underline{\mu}^s_k) + \underline{\lambda}^s_j = \overline{u}^s + \log(\overline{\mu}^s_k) + \overline{\lambda}^s_j$.*

Proof. We only prove the claim for the buyer, the proof for the seller is analogous. Lemma 30 implies that $\overline{\mu}^b_i = \overline{\mu}^b_k$ for all $i \in \hat{\mathcal{I}}_k$, thus $\underline{\alpha}_{ij} + \overline{\alpha}_{ij} > 0$, combined with stationarity and complementary slackness, implies either $t^b_{ij} = \underline{u}^b - \log(\underline{\mu}^b_k) - \underline{\lambda}^b_j$ or $t^b_{ij} = \overline{u}^b - \log(\overline{\mu}^b_k) - \overline{\lambda}^b_j$, or both. We now show that $\underline{u}^b - \log(\underline{\mu}^b_k) - \underline{\lambda}^b_j = \overline{u}^b - \log(\overline{\mu}^b_k) - \overline{\lambda}^b_j$.

For any $j \in \hat{\mathcal{J}}_k$ there is $i' \in \hat{\mathcal{I}}_k$ such that $\underline{\alpha}_{i'j} > 0$ (if not, Lemma 30 would imply $\underline{\alpha}_{ij} > 0$ for all i, implying $\underline{\mu}^s_j = 0$ and contradicting Lemma 5). Also, for any $j \in \hat{\mathcal{J}}_k$ there is $i'' \in \hat{\mathcal{I}}_k$ such that $\overline{\alpha}_{i''j} > 0$ (if not, Lemma 30 would imply $\overline{\alpha}_{ij} > 0$ for all i, implying $\overline{\mu}^s_j = 0$ and contradicting Lemma 5). Consider $(\mathrm{ST}^b_{i'j})$ and $(\mathrm{ST}^b_{i''j})$:

$$(\mathrm{ST}^b_{i'j}) \quad \begin{aligned} &\underline{u}^b - t^b_{i'j} - \log\left(\underline{\mu}^b_k\right) - \underline{\lambda}^b_j = 0, \\ &\overline{u}^b - t^b_{i'j} - \log\left(\overline{\mu}^b_k\right) - \overline{\lambda}^b_j = -\overline{\phi}^b_{i'j} \leq 0. \end{aligned} \qquad (\mathrm{ST}^b_{i''j}) \quad \begin{aligned} &\underline{u}^b - t^b_{i''j} - \log\left(\underline{\mu}^b_k\right) - \underline{\lambda}^b_j = -\underline{\phi}^b_{i''j} \leq 0, \\ &\overline{u}^b - t^b_{i''j} - \log\left(\overline{\mu}^b_k\right) - \overline{\lambda}^b_j = 0. \end{aligned}$$

$(\mathrm{ST}^b_{i'j})$ implies that $\overline{u}^b - \log(\overline{\mu}^b_k) - \overline{\lambda}^b_j \leq \underline{u}^b - \log(\underline{\mu}^b_k) - \underline{\lambda}^b_j$ and $(\mathrm{ST}^b_{i''j})$ implies that $\underline{u}^b - \log(\underline{\mu}^b_k) - \underline{\lambda}^b_j \leq \overline{u}^b - \log(\overline{\mu}^b_k) - \overline{\lambda}^b_j$, hence $\underline{u}^b - \log(\underline{\mu}^b_k) - \underline{\lambda}^b_j = \overline{u}^b - \log(\overline{\mu}^b_k) - \overline{\lambda}^b_j$. □

Corollary 3 allows us to merge the buyer's and the seller's signal realizations in $\hat{\mathcal{I}}_k$ and $\hat{\mathcal{J}}_k$ respectively, and still achieve the same revenue. The construction of the merger is analogous to that of the base case and thus omitted. □

I Proof of Lemma 8

Proof. To simplify the notation let $(x, y) \equiv (\underline{\mu}_1, \overline{\mu}_2)$ and define the following function:

$$f(x,y) \equiv \log(xy) - \frac{0.5-y}{1-x-y}\log\big(x(1-x)\big) - \frac{0.5-x}{1-x-y}\log\big(y(1-y)\big)$$
$$- \log\left[1-x-ye^{\Delta}\right] - \log\left[1-y+ye^{\Delta}\right] + \overline{u}^b - \underline{u}^s.$$

Our problem is then equivalent to $\max_{(x,y)\in(0,0.5)^2} f(x,y)$, which can only have symmetric solutions because $f(x,y)$ is symmetric and strict-supermodular[4]. Symmetry is immediate, to check strict-supermodularity, let us compute $\frac{\partial^2 f}{\partial x \partial y}$ and show that it is strictly positive for all $(x,y) \in (0.0.5)^2$. Start with $\frac{\partial f}{\partial x}$:

$$\frac{\partial f}{\partial x} = \frac{1}{x} + \frac{0.5-y}{(1-x-y)^2}\log\left[\frac{y(1-y)}{x(1-x)}\right] - \frac{0.5-y}{1-x-y}\frac{1-2x}{x(1-x)} - \frac{e^{\Delta}-1}{1-x+xe^{\Delta}}.$$

Now compute $\frac{\partial^2 f}{\partial x \partial y}$:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{-(1-x-y)^2 + 2(1-x-y)(0.5-y)}{(1-x-y)^4}\log\left[\frac{y(1-y)}{x(1-x)}\right]$$
$$+ \frac{0.5-y}{(1-x-y)^2}\frac{1-2y}{y(1-y)} - \frac{-(1-x-y)+0.5-y}{(1-x-y)^2}\frac{1-2x}{x(1-x)},$$

which simplifies to:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{x-y}{(1-x-y)^3}\log\left[\frac{y(1-y)}{x(1-x)}\right] + \frac{1}{2}\frac{(1-2y)^2}{y(1-y)(1-x-y)^2} + \frac{1}{2}\frac{(1-2x)^2}{x(1-x)(1-x-y)^2}.$$

[4]Strict-supermodular functions cannot have asymmetric maxima. Indeed suppose for a contradiction that $(x^*, y^*) \in \max_{(x,y)\in(0,0.5)^2} f(x,y)$ for some symmetric and strict-supermodular $f(x,y)$, and $x^* > y^*$ wlog. By symmetry, we have $f(x^*, y^*) = f(y^*, x^*)$. Optimality then implies

$$\begin{cases} f(y^*, x^*) \geq f(x^*, x^*), \\ f(y^*, y^*) \leq f(x^*, y^*), \end{cases}$$

which implies $f(x^*, x^*) - f(y^*, x^*) \leq 0 \leq f(x^*, y^*) - f(y^*, y^*)$. To obtain a contradiction, observe

$$f(x^*, x^*) - f(y^*, x^*) - \left[f(x^*, y^*) - f(y^*, y^*)\right] = \int_{y^*}^{x^*} \frac{\partial f}{\partial x}(z, x^*)dz - \int_{y^*}^{x^*} \frac{\partial f}{\partial x}(z, y^*)dz$$
$$= \int_{x^*}^{y^*} \left[\frac{\partial f}{\partial x}(z, x^*) - \frac{\partial f}{\partial x}(z, y^*)\right]dz = \int_{x^*}^{y^*}\int_{x^*}^{y^*} \frac{\partial^2 f}{\partial x \partial y}(z, \omega)d\omega dz > 0.$$

Direct calculation shows that for $x = y$ we get $\frac{\partial^2 f}{\partial x \partial y} = \frac{1}{x(1-x)} > 0$. Let $x < y$ wlog:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &\propto (x-y)\log\left[\frac{y(1-y)}{x(1-x)}\right] + \frac{1-x-y}{2}\left[\frac{(1-2y)^2}{y(1-y)} + \frac{(1-2x)^2}{x(1-x)}\right] \\
&\geq (x-y)\left[\frac{y(1-y)}{x(1-x)} - 1\right] + \frac{1-x-y}{2}\left[\frac{(1-2y)^2}{y(1-y)} + \frac{(1-2x)^2}{x(1-x)}\right] \\
&= (x-y)\left[\frac{y(1-y)}{x(1-x)} - 1\right] + \frac{1-x-y}{2}\frac{x(1-x)(1-2y)^2 + y(1-y)(1-2x)^2}{x(1-x)y(1-y)} \\
&\propto y(1-y)(x-y)\big[y(1-y) - x(1-x)\big] + \frac{1-x-y}{2}\big[x(1-x)(1-2y)^2 + y(1-y)(1-2x)^2\big] \\
&= -y(1-y)(y-x)(y-x)(1-x-y) + \frac{1-x-y}{2}\big[x(1-x)(1-2y)^2 + y(1-y)(1-2x)^2\big] \\
&\propto -2y(1-y)(y-x)^2 + (1-2y)^2 x(1-x) + (1-2x)^2 y(1-y) \\
&\geq -2y(1-y)(y-x)^2 + (1-2x)^2 y(1-y) \\
&= y(1-y)\big[(1-2x)^2 - 2(y-x)^2\big] \\
&\propto (1-2x)^2 - 2(y-x)^2 > (1-2x)^2 - 2(0.5-x)^2 = 2(0.5-x)^2 > 0.
\end{aligned}$$

□

J Sufficient optimality conditions for strictly positive posteriors

Rewrite the designer's revenue maximization problem as follows:

$$\max_{n,\alpha,t,\lambda} \sum_{i=1}^{n} \underbrace{\left(\underline{\alpha}_i + \overline{\alpha}_i\right)}_{\tau_i}\left(t_i^b - t_i^s\right) \quad \text{s.t.}$$

$$\begin{aligned}
&(\text{ST}^b) \quad && \underline{u}^b - t_i^b - \log\left(\underline{\mu}_i\right) = \underline{\lambda}_i^b \geq \underline{\Lambda}^b \quad \forall i, \\
& && \overline{u}^b - t_i^b - \log\left(\overline{\mu}_i\right) = \overline{\lambda}_i^b \geq \overline{\Lambda}^b \quad \forall i, \\
&(\text{ST}^s) \quad && t_i^s - \underline{u}^s - \log\left(\underline{\mu}_i\right) = \underline{\lambda}_i^s \geq \underline{\Lambda}^s \quad \forall i, \\
& && t_i^s - \overline{u}^s - \log\left(\overline{\mu}_i\right) = \overline{\lambda}_i^s \geq \overline{\Lambda}^s \quad \forall i, \\
&(\text{NA}^b) \quad && \exp\Big(-\underbrace{\min_i\{\underline{\lambda}_i^b\}}_{\equiv \underline{\Lambda}^b}\Big) + \exp\Big(-\underbrace{\min_i\{\overline{\lambda}_i^b\}}_{\equiv \overline{\Lambda}^b}\Big) \leq 1, \\
&(\text{NA}^s) \quad && \exp\Big(-\underbrace{\min_i\{\underline{\lambda}_i^s\}}_{\equiv \underline{\Lambda}^s}\Big) + \exp\Big(-\underbrace{\min_i\{\overline{\lambda}_i^s\}}_{\equiv \overline{\Lambda}^s}\Big) \leq 1, \\
&(\alpha\text{-F}) \quad && S^b = S^s = \{1, \ldots, n\}, \alpha \in \tilde{\Delta}\big(S^b \times S^s \times V\big), \text{ BP}, \\
& && 0 < \underline{\mu}_i < 1, \quad 0 < \overline{\mu}_i < 1.
\end{aligned}$$

The problem can then be rewritten as follows:

$$\max_{n,\tau,\mu,t,\Lambda} \sum_{i=1}^{n} \tau_i \left(t_i^b - t_i^s\right) \quad \text{s.t.}$$

$$\begin{aligned}
(\text{ST}^b) \quad & \underline{u}^b - \underline{\Lambda}^b \geq t_i^b + \log\left(\underline{\mu}_i\right) \ \ \forall i, \\
& \overline{u}^b - \overline{\Lambda}^b \geq t_i^b + \log\left(\overline{\mu}_i\right) \ \ \forall i, \\
(\text{ST}^s) \quad & -\underline{u}^s - \underline{\Lambda}^s \geq -t_i^s + \log\left(\underline{\mu}_i\right) \ \ \forall i, \\
& -\overline{u}^s - \overline{\Lambda}^s \geq -t_i^s + \log\left(\overline{\mu}_i\right) \ \ \forall i, \\
(\text{NA}^b) \quad & \exp\left(-\underline{\Lambda}^b\right) + \exp\left(-\overline{\Lambda}^b\right) \leq 1, \\
(\text{NA}^s) \quad & \exp\left(-\underline{\Lambda}^s\right) + \exp\left(-\overline{\Lambda}^s\right) \leq 1, \\
(\alpha\text{-F}) \quad & \text{BP}, \ 0 < \underline{\mu}_i < 1, \quad 0 < \overline{\mu}_i < 1.
\end{aligned}$$

Take exponentials on both sides of the ST^b and ST^s constraints:

$$\max_{n,\tau,\mu,t,\Lambda} \sum_{i=1}^{n} \tau_i \left(t_i^b - t_i^s\right) \quad \text{s.t.}$$

$$\begin{aligned}
(\text{ST}^b) \quad & \exp\left(\underline{u}^b\right)\underbrace{\exp\left(-\underline{\Lambda}^b\right)}_{\equiv \underline{\zeta}^b} \geq \underline{\mu}_i \exp\left(t_i^b\right) \ \ \forall i, \\
& \exp\left(\overline{u}^b\right)\underbrace{\exp\left(-\overline{\Lambda}^b\right)}_{\equiv \overline{\zeta}^b} \geq \overline{\mu}_i \exp\left(t_i^b\right) \ \ \forall i, \\
(\text{ST}^s) \quad & \exp\left(-\underline{u}^s\right)\underbrace{\exp\left(-\underline{\Lambda}^s\right)}_{\equiv \underline{\zeta}^s} \geq \underline{\mu}_i \exp\left(-t_i^s\right) \ \ \forall i, \\
& \exp\left(-\overline{u}^s\right)\underbrace{\exp\left(-\overline{\Lambda}^s\right)}_{\equiv \overline{\zeta}^s} \geq \overline{\mu}_i \exp\left(-t_i^s\right) \ \ \forall i, \\
(\text{NA}^b) \quad & \underbrace{\exp\left(-\underline{\Lambda}^b\right)}_{\equiv \underline{\zeta}^b} + \underbrace{\exp\left(-\overline{\Lambda}^b\right)}_{\equiv \overline{\zeta}^b} \leq 1, \ \ \underline{\zeta}^b > 0, \ \overline{\zeta}^b > 0, \\
(\text{NA}^s) \quad & \underbrace{\exp\left(-\underline{\Lambda}^s\right)}_{\equiv \underline{\zeta}^s} + \underbrace{\exp\left(-\overline{\Lambda}^s\right)}_{\equiv \overline{\zeta}^s} \leq 1, \ \ \underline{\zeta}^s > 0, \ \overline{\zeta}^s > 0, \\
(\alpha\text{-F}) \quad & \text{BP}, \ 0 < \underline{\mu}_i < 1, \quad 0 < \overline{\mu}_i < 1.
\end{aligned}$$

Finally, relax the strict inequalities:

$$\max_{n,\tau,\mu,t,\zeta} \sum_{i=1}^{n} \tau_i \left(t_i^b - t_i^s\right) \quad \text{s.t.} \tag{3}$$

$$\begin{aligned}
(\text{ST}^b) \quad & \underline{\zeta}^b \exp\left(\underline{u}^b\right) \geq \underline{\mu}_i \exp\left(t_i^b\right) \ \ \forall i, \\
& \overline{\zeta}^b \exp\left(\overline{u}^b\right) \geq \overline{\mu}_i \exp\left(t_i^b\right) \ \ \forall i, \\
(\text{ST}^s) \quad & \underline{\zeta}^s \exp\left(-\underline{u}^s\right) \geq \underline{\mu}_i \exp\left(-t_i^s\right) \ \ \forall i, \\
& \overline{\zeta}^s \exp\left(-\overline{u}^s\right) \geq \overline{\mu}_i \exp\left(-t_i^s\right) \ \ \forall i, \\
(\text{NA}^b) \quad & \underline{\zeta}^b + \overline{\zeta}^b \leq 1, \ \ \underline{\zeta}^b \geq 0, \ \overline{\zeta}^b \geq 0, \\
(\text{NA}^s) \quad & \underline{\zeta}^s + \overline{\zeta}^s \leq 1, \ \ \underline{\zeta}^s \geq 0, \ \overline{\zeta}^s \geq 0, \\
(\alpha\text{-F}) \quad & \text{BP}, \ \ 0 \leq \underline{\mu}_i \leq 1, \quad 0 \leq \overline{\mu}_i \leq 1.
\end{aligned}$$

Let us now solve the relaxed Revenue maximization problem (3). The interior cases were considered above. We will now consider the cases, in which one or both posteriors are extreme.

J.1 Case E0: both posteriors are extreme

In this case we have $\underline{\mu}_1 = \overline{\mu}_2 = 0$ and $\underline{\mu}_2 = \overline{\mu}_1 = 1$. The Bayes-plausibilty conditions can be written as:

$$\begin{cases} \tau_1 \underline{\mu}_1 + \tau_2 \underline{\mu}_2 = \tau_1 0 + \tau_2 1 = \tau_2 = \underline{\mu}_0, \\ \tau_1 \overline{\mu}_1 + \tau_2 \overline{\mu}_2 = \tau_1 1 + \tau_2 0 = \tau_1 = \overline{\mu}_0. \end{cases}$$

The relaxed stationarity conditions can then be written as:

$$\begin{aligned}
(\text{ST}^b) \quad & \underline{\zeta}^b \underbrace{\exp\left(\underline{u}^b\right) \geq 0 \exp\left(t_1^b\right)}_{\text{always holds}}, & (\text{ST}^s) \quad & \underbrace{\underline{\zeta}^s \exp\left(-\underline{u}^s\right) \geq 0 \exp\left(-t_1^s\right)}_{\text{always holds}}, \\
& \overline{\zeta}^b \exp\left(\overline{u}^b\right) \geq 1 \exp\left(t_1^b\right), & & \overline{\zeta}^s \exp\left(-\overline{u}^s\right) \geq 1 \exp\left(-t_1^s\right), \\
& \underline{\zeta}^b \exp\left(\underline{u}^b\right) \geq 1 \exp\left(t_2^b\right), & & \underline{\zeta}^s \exp\left(-\underline{u}^s\right) \geq 1 \exp\left(-t_2^s\right), \\
& \underbrace{\overline{\zeta}^b \exp\left(\overline{u}^b\right) \geq 0 \exp\left(t_2^b\right)}_{\text{always holds}}. & & \underbrace{\overline{\zeta}^s \exp\left(-\overline{u}^s\right) \geq 0 \exp\left(-t_2^s\right)}_{\text{always holds}}.
\end{aligned}$$

The relaxed maximization problem can therefore be rewritten as:

$$\max_{t,\zeta} \ \overline{\mu}_0\big(t_1^b - t_1^s\big) + \underline{\mu}_0\big(t_2^b - t_2^s\big) \quad \text{s.t.} \tag{4}$$

$$\begin{aligned}
&(\text{ST}^b) \quad \overline{\zeta}^b \exp\big(\overline{u}^b\big) \geq \exp\big(t_1^b\big), \quad \underline{\zeta}^b \exp\big(\underline{u}^b\big) \geq \exp\big(t_2^b\big),\\
&(\text{ST}^s) \quad \overline{\zeta}^s \exp\big(-\overline{u}^s\big) \geq \exp\big(-t_1^s\big), \quad \underline{\zeta}^s \exp\big(-\underline{u}^s\big) \geq \exp\big(-t_2^s\big),\\
&(\text{NA}^b) \quad \underline{\zeta}^b + \overline{\zeta}^b \leq 1, \quad \underline{\zeta}^b \geq 0, \ \overline{\zeta}^b \geq 0,\\
&(\text{NA}^s) \quad \underline{\zeta}^s + \overline{\zeta}^s \leq 1, \quad \underline{\zeta}^s \geq 0, \ \overline{\zeta}^s \geq 0.
\end{aligned}$$

The following claim applies:

Claim 10. *At the optimum of the revenue maximization problem (4) we have* $\underline{\zeta}^p > 0$ *and* $\overline{\zeta}^p > 0$ *for both* $p \in \{b, s\}$. *Suppose not, then the respective payment has to be equal to* $-\infty$ *to satisfy the relaxed stationarity constraints. But* (NA^p) *implies that* $\underline{\zeta}^p \leq 1$ *and* $\overline{\zeta}^p \leq 1$, *which implies that the remaining payments are bounded from above, implying in turn that the revneue has to be equal to* $-\infty$ *as well, which obviously cannot be optimal.*

The revenue maximization problem (4) can therefore be rewritten in terms of the original variables:

$$\max_{t,\Lambda} \ \overline{\mu}_0\big(t_1^b - t_1^s\big) + \underline{\mu}_0\big(t_2^b - t_2^s\big) \quad \text{s.t.}$$

$$\begin{aligned}
&(\text{ST}^b) \quad t_1^b \leq \overline{u}^b - \overline{\Lambda}^b, \quad t_2^b \leq \underline{u}^b - \underline{\Lambda}^b\\
&(\text{ST}^s) \quad -t_1^s \leq -\overline{u}^s - \overline{\Lambda}^s, \quad -t_2^s \leq -\underline{u}^s - \underline{\Lambda}^s,\\
&(\text{NA}^b) \quad \exp\big(-\underline{\Lambda}^b\big) + \exp\big(-\overline{\Lambda}^b\big) \leq 1,\\
&(\text{NA}^s) \quad \exp\big(-\underline{\Lambda}^s\big) + \exp\big(-\overline{\Lambda}^s\big) \leq 1.
\end{aligned}$$

The constraints $(\text{ST})^b$ and $(\text{ST})^s$ are obviously binding at the optimum, hence the revenue can be written as:

$$\begin{aligned}
\mathcal{R}_{\text{Case E0}} &= \overline{\mu}_0\big(\overline{u}^b - \overline{\Lambda}^b - \overline{u}^s - \overline{\Lambda}^s\big) + \underline{\mu}_0\big(\underline{u}^b - \underline{\Lambda}^b - \underline{u}^s - \underline{\Lambda}^s\big)\\
&= \overline{\mu}_0\big(\overline{u}^b - \overline{u}^s\big) + \underline{\mu}_0\big(\underline{u}^b - \underline{u}^s\big) - \overline{\mu}_0\overline{\Lambda}^b - \underline{\mu}_0\underline{\Lambda}^b - \overline{\mu}_0\overline{\Lambda}^s - \underline{\mu}_0\underline{\Lambda}^s.
\end{aligned}$$

Additive separability implies that it is enough to solve for both $p \in \{b, s\}$:

$$\max_{\underline{\Lambda}^p,\overline{\Lambda}^p} \ -\overline{\mu}_0\overline{\Lambda}^p - \underline{\mu}_0\underline{\Lambda}^p \ \text{s.t.} \ \exp\big(-\underline{\Lambda}^p\big) + \exp\big(-\overline{\Lambda}^p\big) \leq 1.$$

The optimal solution is given by $\underline{\Lambda}^p = -\log\left(\underline{\mu}^0\right)$ and $\overline{\Lambda}^p = -\log\left(\overline{\mu}^0\right)$. The optimal revenue can then be written as:

$$\begin{aligned}
\mathcal{R}^*_{\text{Case E0}} &= \overline{\mu}_0\left(\overline{u}^b - \overline{u}^s\right) + \underline{\mu}_0\left(\underline{u}^b - \underline{u}^s\right) + 2\Big(\overline{\mu}_0 \log\left(\overline{\mu}^0\right) + \underline{\mu}_0 \log\left(\underline{\mu}^0\right)\Big) \\
&= \overline{u}^b - \underline{u}^s + \Big[\overline{\mu}_0\left(\overline{u}^b - \overline{u}^s\right) + \underline{\mu}_0\left(\underline{u}^b - \underline{u}^s\right) - \overline{u}^b + \underline{u}^s\Big] + 2\Big(\overline{\mu}_0 \log\left(\overline{\mu}^0\right) + \underline{\mu}_0 \log\left(\underline{\mu}^0\right)\Big) \\
&= \overline{u}^b - \underline{u}^s - \underline{\mu}_0\left(\overline{u}^b - \underline{u}^b\right) - \overline{\mu}_0\left(\overline{u}^s - \underline{u}^s\right) + 2\Big(\overline{\mu}_0 \log\left(\overline{\mu}^0\right) + \underline{\mu}_0 \log\left(\underline{\mu}^0\right)\Big) \\
&= -\underline{\mu}_0\Delta^b - \overline{\mu}_0\Delta^s + 2\Big(\overline{\mu}_0 \log\left(\overline{\mu}^0\right) + \underline{\mu}_0 \log\left(\underline{\mu}^0\right)\Big) + \overline{u}^b - \underline{u}^s.
\end{aligned}$$

In the symmetric case we get:

$$\begin{aligned}
\mathcal{R}^*_{\text{Case E0}} &= -\frac{1}{2}\Delta - \frac{1}{2}\Delta + 2\left(\frac{1}{2}\log\left(\frac{1}{2}\right) + \frac{1}{2}\log\left(\frac{1}{2}\right)\right) + \overline{u}^b - \underline{u}^s \\
&= -\Delta - 2\log 2 + \overline{u}^b - \underline{u}^s.
\end{aligned}$$

J.2 Case E1: μ_1 is extreme, μ_2 is interior

Suppose $\underline{\mu}_1 = 0, \overline{\mu}_1 = 1$ and $0 < \underline{\mu}_2, \overline{\mu}_2 < 1$. From Bayes-plausibility we then have:

$$\begin{cases} \tau_1\underline{\mu}_1 + \tau_2\underline{\mu}_2 = \tau_1 0 + \tau_2\underline{\mu}_2 = \underline{\mu}_0, \\ \tau_1\overline{\mu}_1 + \tau_2\overline{\mu}_2 = \tau_1 1 + \tau_2\overline{\mu}_2 = \overline{\mu}_0. \end{cases} \Rightarrow \begin{cases} \tau_1 = 1 - \frac{\underline{\mu}_0}{\underline{\mu}_2}, \\ \tau_2 = \frac{\underline{\mu}_0}{\underline{\mu}_2}. \end{cases}$$

The relaxed stationarity conditions are then given by:

$$(\text{ST}^b) \quad \underbrace{\underline{\zeta}^b \exp\left(\underline{u}^b\right) \geq 0 \exp\left(t_1^b\right)}_{\text{always holds}}, \qquad (\text{ST}^s) \quad \underbrace{\underline{\zeta}^s \exp\left(-\underline{u}^s\right) \geq 0 \exp\left(-t_1^s\right)}_{\text{always holds}},$$

$$\overline{\zeta}^b \exp\left(\overline{u}^b\right) \geq 1 \exp\left(t_1^b\right), \qquad \overline{\zeta}^s \exp\left(-\overline{u}^s\right) \geq 1 \exp\left(-t_1^s\right),$$

$$\underline{\zeta}^b \exp\left(\underline{u}^b\right) \geq \underline{\mu}_2 \exp\left(t_2^b\right), \qquad \underline{\zeta}^s \exp\left(-\underline{u}^s\right) \geq \underline{\mu}_2 \exp\left(-t_2^s\right),$$

$$\overline{\zeta}^b \exp\left(\overline{u}^b\right) \geq \overline{\mu}_2 \exp\left(t_2^b\right). \qquad \overline{\zeta}^s \exp\left(-\overline{u}^s\right) \geq \overline{\mu}_2 \exp\left(-t_2^s\right).$$

We first establish the following claim:

Claim 11. *t_i^b are bounded from above, t_i^s are bounded from below for all i.*

Proof. (NA^b) implies $\underline{\zeta}^b \leq 1$ and $\overline{\zeta}^b \leq 1$, and (NA^s) implies $\underline{\zeta}^s \leq 1$ and $\overline{\zeta}^s \leq 1$. It then follows from (ST^b) that $t_1^b \leq \overline{u}^b$ and $t_2^b \leq \log\min\left\{\frac{1}{\underline{\mu}_2}\exp(\underline{u}^b), \frac{1}{\overline{\mu}_2}\exp(\overline{u}^b)\right\}$, and it follows from (ST^s) that $t_1^s \geq \overline{u}^s$ and $t_2^s \geq -\log\min\left\{\frac{1}{\underline{\mu}_2}\exp(-\underline{u}^s), \frac{1}{\overline{\mu}_2}\exp(-\overline{u}^s)\right\}$. □

We then establish the following:

Claim 12. *At the optimum* $\underline{\zeta}^b > 0$, $\overline{\zeta}^b > 0$ *and* $\underline{\zeta}^s > 0$, $\overline{\zeta}^s > 0$.

Proof. If $\underline{\zeta}^b = 0$ or $\overline{\zeta}^b = 0$, then at least one of the buyer's payments would be equal to $-\infty$. Since the remaining payments of the buyer are bounded from above and the remaining payments to the seller are bounded from above, it would lead to revenue equal to $-\infty$, which cannot be optimal. If $\underline{\zeta}^s = 0$ or $\overline{\zeta}^s = 0$, then at least one of the seller's payments would be equal to ∞. Since the remaining payments of the buyer are bounded from above and the remaining payments to the seller are bounded from above, it would lead to revenue equal to $-\infty$, which cannot be optimal. □

Observe that at the optimum we must have $\overline{\zeta}^b \exp\left(\overline{u}^b\right) = \exp\left(t_1^b\right)$. Moreover, we must have either $\underline{\zeta}^b \exp\left(\underline{u}^b\right) = \underline{\mu}_2 \exp\left(t_2^b\right)$ or $\overline{\zeta}^b \exp\left(\overline{u}^b\right) = \overline{\mu}_2 \exp\left(t_2^b\right)$, or both. We establish the following lemma:

Lemma 31. $\underline{\zeta}^b \exp\left(\underline{u}^b\right) = \underline{\mu}_2 \exp\left(t_2^b\right)$ *at the optimum.*

Proof. Suppose for a contradiction that this is not the case, then the other constraint in the pair must be binding, i.e. $\overline{\zeta}^b \exp\left(\overline{u}^b\right) = \overline{\mu}_2 \exp\left(t_2^b\right)$. Suppose now that $t_1^s, t_2^s, \underline{\zeta}^s, \overline{\zeta}^s, \underline{\mu}_2, \overline{\mu}_2$ are optimally chosen. To make sure that t_1^b, t_2^b and $\underline{\zeta}^b, \overline{\zeta}^b$ are optimally chosen, we solve:

$$
\begin{aligned}
& \max_{t^b, \zeta^b} \; \tau_1\left(t_1^b - t_1^s\right) + \tau_2\left(t_2^b - t_2^s\right) \quad \text{s.t.} \\
(\mathrm{ST}^b) \quad & \exp\left(t_1^b\right) = \overline{\zeta}^b \exp\left(\overline{u}^b\right), \\
& \exp\left(t_2^b\right) = \frac{1}{\overline{\mu}_2} \overline{\zeta}^b \exp\left(\overline{u}^b\right), \\
(\mathrm{NA}^b) \quad & \underline{\zeta}^b + \overline{\zeta}^b \leq 1, \quad \underline{\zeta}^b \geq 0, \; \overline{\zeta}^b \geq 0.
\end{aligned}
$$

The optimal solution to the above problem is to set $\underline{\zeta}^b = 0$ and $\overline{\zeta}^b = 1$, which contradicts Claim 12. □

Likewise, we must have at the optimum $\overline{\zeta}^s \exp\left(-\overline{u}^s\right) = \exp\left(-t_2^s\right)$. Moreover, we must have either $\underline{\zeta}^s \exp\left(-\underline{u}^s\right) = \underline{\mu}_1 \exp\left(-t_1^s\right)$ or $\overline{\zeta}^s \exp\left(-\overline{u}^s\right) = \overline{\mu}_1 \exp\left(-t_1^s\right)$, or both. We establish the following lemma:

Lemma 32. $\underline{\zeta}^s \exp\left(-\underline{u}^s\right) = \underline{\mu}_2 \exp\left(-t_1^s\right)$ *at the optimum.*

Proof. Analogous to Lemma 31 for the buyer. □

The revenue maximization problem can then be rewtitten in terms of the original variables as follows:

$$\max_{\tau,\mu,t,\Lambda} \; \tau_1\big(t_1^b - t_1^s\big) + \tau_2\big(t_2^b - t_2^s\big) \quad \text{s.t.}$$

$$
\begin{aligned}
(\mathrm{ST}^b) \quad & t_1^b = \overline{u}^b - \overline{u}^b, \\
& t_2^b = \underline{u}^b - \log\big(\underline{\mu}_2\big) - \underline{\Lambda}^b, \\
& t_2^b \le \overline{u}^b - \log\big(\overline{\mu}_2\big) - \overline{\Lambda}^b; \\
(\mathrm{ST}^s) \quad & -t_1^s = -\overline{u}^s - \overline{\Lambda}^s, \\
& -t_2^s = -\underline{u}^s - \log\big(\underline{\mu}_2\big) - \underline{\Lambda}^s, \\
& -t_2^s \le -\overline{u}^s - \log\big(\overline{\mu}_2\big) - \overline{\Lambda}^s; \\
(\mathrm{NA}^b) \quad & \exp\big(-\underline{\Lambda}^b\big) + \exp\big(-\overline{\Lambda}^b\big) \le 1, \\
(\mathrm{NA}^s) \quad & \exp\big(-\underline{\Lambda}^s\big) + \exp\big(-\overline{\Lambda}^s\big) \le 1, \\
(\alpha\text{-F}) \quad & \mathrm{BP}, \; 0 < \underline{\mu}_2 < 1, \quad 0 < \overline{\mu}_2 < 1.
\end{aligned}
$$

The revenue in Case E1 can then be written as follows:

$$\mathcal{R}_{\text{Case E1}} = \tau_1\Big[\overline{u}^b - \overline{\Lambda}^b - \overline{u}^s - \overline{\Lambda}^s\Big] + \tau_2\Big[\underline{u}^b - \underline{u}^s - 2\log\big(\underline{\mu}_2\big) - \underline{\Lambda}^b - \underline{\Lambda}^s\Big].$$

Since $\mathcal{R}_{\text{Case E1}}$ is strictly decreasing in Λ, both (NA^b) and (NA^s) are binding.

From our consideration of the concave closures above, it follows that we must consider the following three cases:

Case E1.1 $-t_2^s = -\underline{u}^s - \log\big(\underline{\mu}_2\big) - \underline{\Lambda}^s = -\overline{u}^s - \log\big(\overline{\mu}_2\big) - \overline{\Lambda}^s$.

Since (NA^s) is binding, we can solve for Λ^s:

$$\begin{cases} -\underline{\Lambda}^s = \underline{u}^s + \log\big(\underline{\mu}_2\big) - \log\big[\underline{\mu}_2\exp(\underline{u}^s) + \overline{\mu}_2\exp(\overline{u}^s)\big], \\ -\overline{\Lambda}^s = \overline{u}^s + \log\big(\overline{\mu}_2\big) - \log\big[\underline{\mu}_2\exp(\underline{u}^s) + \overline{\mu}_2\exp(\overline{u}^s)\big]. \end{cases}$$

The revenue can then be written as follows:

$$\mathcal{R}_{\text{Case E1.1}} = \tau_1\big[\overline{u}^b + \log\big(\overline{\mu}_2\big) - \overline{\Lambda}^b\big] + \tau_2\big[\underline{u}^b - \log\big(\underline{\mu}_2\big) - \underline{\Lambda}^b\big] - \log\big[\underline{\mu}_2\exp(\underline{u}^s) + \overline{\mu}_2\exp(\overline{u}^s)\big].$$

Maximizing $\mathcal{R}_{\text{Case E1.1}}$ with respect to $\underline{\Lambda}^b$ and $\overline{\Lambda}^b$ subject to (NA^b), we get $-\underline{\Lambda}^b = \log\tau_2$ and $-\overline{\Lambda}^b = \log\tau_1$. The revenue can then be written as:

$$
\begin{aligned}
\mathcal{R}_{\text{Case E1.1}} &= \tau_1\big[\overline{u}^b + \log\big(\overline{\mu}_2\big) + \log\tau_1\big] + \tau_2\big[\underline{u}^b - \log\big(\underline{\mu}_2\big) + \log\tau_2\big] - \log\big[\underline{\mu}_2\exp(\underline{u}^s) + \overline{\mu}_2\exp(\overline{u}^s)\big] \\
&= \log\big(\overline{\mu}_2\big) - \tau_2\log\big(\underline{\mu}_2\overline{\mu}_2\big) - \tau_2\Delta^b + \tau_1\log\tau_1 + \tau_2\log\tau_2 - \log\big[\underline{\mu}_2 + \overline{\mu}_2\exp(\Delta^s)\big] + \overline{u}^b - \underline{u}^s.
\end{aligned}
$$

Recall that Case E1.1 applies only if the (ST^b)-constraints are satisfied, i.e.

$$\begin{aligned}
&\underline{u}^b - \log\left(\underline{\mu}_2\right) - \underline{\Lambda}^b \le \overline{u}^b - \log\left(\overline{\mu}_2\right) - \overline{\Lambda}^b \\
\Leftrightarrow\ &\underline{u}^b - \log\left(\underline{\mu}_2\right) + \log\tau_2 \le \overline{u}^b - \log\left(\overline{\mu}_2\right) + \log\tau_1 \\
\Leftrightarrow\ &\log\left[\frac{\overline{\mu}_2}{\underline{\mu}_2}\right] + \log\left[\frac{\tau_2}{\tau_1}\right] \le \Delta^b.
\end{aligned}$$

Case E1.2 $t_2^b = \underline{u}^b - \log\left(\underline{\mu}_2\right) - \underline{\Lambda}^b = \overline{u}^b - \log\left(\overline{\mu}_2\right) - \overline{\Lambda}^b$.

Since (NA^b) is binding, we can solve for Λ^b:

$$\begin{cases}
-\underline{\Lambda}^b = -\underline{u}^b + \log\left(\underline{\mu}_2\right) - \log\left[\underline{\mu}_2\exp(-\underline{u}^b) + \overline{\mu}_2\exp(-\overline{u}^b)\right], \\
-\overline{\Lambda}^b = -\overline{u}^b + \log\left(\overline{\mu}_2\right) - \log\left[\underline{\mu}_2\exp(-\underline{u}^b) + \overline{\mu}_2\exp(-\overline{u}^b)\right].
\end{cases}$$

The revenue can then be written as follows:

$$\mathcal{R}_{\text{Case E1.2}} = \tau_1\left[-\overline{u}^s + \log\left(\overline{\mu}_2\right) - \overline{\Lambda}^s\right] + \tau_2\left[-\underline{u}^s - \log\left(\underline{\mu}_2\right) - \underline{\Lambda}^s\right] - \log\left[\underline{\mu}_2\exp(-\underline{u}^b) + \overline{\mu}_2\exp(-\overline{u}^b)\right].$$

Maximizing $\mathcal{R}_{\text{Case E1.2}}$ with respect to $\underline{\Lambda}^s$ and $\overline{\Lambda}^b$ subject to (NA^b), we get $-\underline{\Lambda}^s = \log\tau_2$ and $-\overline{\Lambda}^s = \log\tau_1$. The revenue can then be written as:

$$\begin{aligned}
\mathcal{R}_{\text{Case E1.2}} &= \tau_1\left[-\overline{u}^s + \log\left(\overline{\mu}_2\right) + \log\tau_1\right] + \tau_2\left[-\underline{u}^s - \log\left(\underline{\mu}_2\right) + \log\tau_2\right] - \log\left[\underline{\mu}_2\exp(-\underline{u}^b) + \overline{\mu}_2\exp(-\overline{u}^b)\right] \\
&= \log\left(\overline{\mu}_2\right) - \tau_1\log\left(\underline{\mu}_2\overline{\mu}_2\right) - \tau_1\Delta^s + \tau_1\log\tau_1 + \tau_2\log\tau_2 - \log\left[\underline{\mu}_2\exp(\Delta^b) + \overline{\mu}_2\right] + \overline{u}^b - \underline{u}^s.
\end{aligned}$$

Recall that Case E1.2 applies only if the (ST^b)-constraints are satisfied, i.e.

$$\begin{aligned}
&-\underline{u}^s - \log\left(\underline{\mu}_2\right) - \underline{\Lambda}^s \le -\overline{u}^s - \log\left(\overline{\mu}_2\right) - \overline{\Lambda}^s \\
\Leftrightarrow\ &-\underline{u}^s - \log\left(\underline{\mu}_2\right) + \log\tau_2 \le -\overline{u}^s - \log\left(\overline{\mu}_2\right) + \log\tau_1 \\
\Leftrightarrow\ &\log\left[\frac{\overline{\mu}_2}{\underline{\mu}_2}\right] + \log\left[\frac{\tau_2}{\tau_1}\right] \le -\Delta^s.
\end{aligned}$$

Case E1.3: all constraints are binding

In this case we have:

$$\begin{cases}
-\underline{\Lambda}^b = -\underline{u}^b + \log\left(\underline{\mu}_2\right) - \log\left[\underline{\mu}_2\exp(-\underline{u}^b) + \overline{\mu}_2\exp(-\overline{u}^b)\right], \\
-\overline{\Lambda}^b = -\overline{u}^b + \log\left(\overline{\mu}_2\right) - \log\left[\underline{\mu}_2\exp(-\underline{u}^b) + \overline{\mu}_2\exp(-\overline{u}^b)\right], \\
-\underline{\Lambda}^s = \underline{u}^s + \log\left(\underline{\mu}_2\right) - \log\left[\underline{\mu}_2\exp(\underline{u}^s) + \overline{\mu}_2\exp(\overline{u}^s)\right], \\
-\overline{\Lambda}^s = \overline{u}^s + \log\left(\overline{\mu}_2\right) - \log\left[\underline{\mu}_2\exp(\underline{u}^s) + \overline{\mu}_2\exp(\overline{u}^s)\right].
\end{cases}$$

The revenue then becomes:

$$\begin{aligned}
\mathcal{R}_{\text{Case E1.3}} &= 2\tau_1\log\left(\overline{\mu}_2\right) - \log\left[\underline{\mu}_2\exp(-\underline{u}^b) + \overline{\mu}_2\exp(-\overline{u}^b)\right] - \log\left[\underline{\mu}_2\exp(\underline{u}^s) + \overline{\mu}_2\exp(\overline{u}^s)\right] \\
&= 2\tau_1\log\left(\overline{\mu}_2\right) - \log\left[\underline{\mu}_2\exp(\Delta^b) + \overline{\mu}_2\right] - \log\left[\underline{\mu}_2 + \overline{\mu}_2\exp(\Delta^s)\right] + \overline{u}^b - \underline{u}^s.
\end{aligned}$$

J.3 Case E2: μ_1 is interior, μ_2 is extreme

Suppose $0 \leq \underline{\mu}_1, \overline{\mu}_1 < 1$ and $\underline{\mu}_2 = 1,\ \overline{\mu}_2 = 0$. From Bayes-plausibility, we have:

$$\begin{cases} \tau_1 \underline{\mu}_1 + \tau_2 \underline{\mu}_2 = \tau_1 \underline{\mu}_1 + \tau_2 1 = \underline{\mu}_0, \\ \tau_1 \overline{\mu}_1 + \tau_2 \overline{\mu}_2 = \tau_1 \overline{\mu}_1 + \tau_2 0 = \overline{\mu}_0. \end{cases} \Rightarrow \begin{cases} \tau_1 = \frac{\overline{\mu}_0}{\overline{\mu}_1}, \\ \tau_2 = 1 - \frac{\overline{\mu}_0}{\overline{\mu}_1}. \end{cases}$$

The relaxed stationarity conditions are then given by:

$$(\mathrm{ST}^b) \quad \underbrace{\begin{aligned} &\underline{\zeta}^b \exp\left(\underline{u}^b\right) \geq \underline{\mu}_1 \exp\left(t_1^b\right), \\ &\overline{\zeta}^b \exp\left(\overline{u}^b\right) \geq \overline{\mu}_1 \exp\left(t_1^b\right), \\ &\underline{\zeta}^b \exp\left(\underline{u}^b\right) \geq 1 \exp\left(t_2^b\right), \\ &\overline{\zeta}^b \exp\left(\overline{u}^b\right) \geq 0 \exp\left(t_2^b\right) \end{aligned}}_{\text{always holds}}. \qquad (\mathrm{ST}^s) \quad \underbrace{\begin{aligned} &\underline{\zeta}^s \exp\left(-\underline{u}^s\right) \geq \underline{\mu}_1 \exp\left(-t_1^s\right), \\ &\overline{\zeta}^s \exp\left(-\overline{u}^s\right) \geq \overline{\mu}_1 \exp\left(-t_1^s\right), \\ &\underline{\zeta}^s \exp\left(-\underline{u}^s\right) \geq 1 \exp\left(-t_2^s\right), \\ &\overline{\zeta}^s \exp\left(-\overline{u}^s\right) \geq 0 \exp\left(-t_2^s\right) \end{aligned}}_{\text{always holds}}.$$

We first establish the following claim:

Claim 13. *t_i^b are bounded from above, t_i^s are bounded from below for all i.*

Proof. (NA^b) implies $\underline{\zeta}^b \leq 1$ and $\overline{\zeta}^b \leq 1$, and ($\mathrm{NA}^s$) implies that $\underline{\zeta}^s \leq 1$ and $\overline{\zeta}^s$. It then follows from (ST^b) that $t_1^b \leq \log\min\left\{\frac{1}{\underline{\mu}_1}\exp(\underline{u}^b), \frac{1}{\overline{\mu}_1}\exp(\overline{u}^b)\right\}$ and $t_2^b \leq \underline{u}^b$ and it follows from (ST^s) that $t_1^s \geq -\log\min\left\{\frac{1}{\underline{\mu}_1}\exp(-\underline{u}^s), \frac{1}{\overline{\mu}_1}\exp(-\overline{u}^s)\right\}$ and $t_2^s \geq \underline{u}^s$. □

We then establish the following:

Claim 14. *At the optimum $\underline{\zeta}^b > 0$, $\overline{\zeta}^b > 0$ and $\underline{\zeta}^s > 0$, $\overline{\zeta}^s > 0$.*

Proof. If $\underline{\zeta}^b = 0$ or $\overline{\zeta}^b = 0$, then at least one of the buyer's payments would be equal to $-\infty$. Since the remaining payments of the buyer are bounded from above and the remaining payments to the seller are bounded from above, it would lead to revenue equal to $-\infty$, which cannot be optimal. If $\underline{\zeta}^s = 0$ or $\overline{\zeta}^s = 0$, then at least one of the seller's payments would be equal to ∞. Since the remaining payments of the buyer are bounded from above and the remaining payments to the seller are bounded from above, it would lead to revenue equal to $-\infty$, which cannot be optimal. □

Observe that at the optimum we must have $\underline{\zeta}^b \exp\left(\underline{u}^b\right) = \exp\left(t_2^b\right)$. Moreover, we must have either $\underline{\zeta}^b \exp\left(\underline{u}^b\right) = \underline{\mu}_1 \exp\left(t_1^b\right)$ or $\overline{\zeta}^b \exp\left(\overline{u}^b\right) = \overline{\mu}_1 \exp\left(t_1^b\right)$, or both. We establish the following lemma:

Lemma 33. $\overline{\zeta}^b \exp\left(\overline{u}^b\right) = \overline{\mu}_1 \exp\left(t_1^b\right)$ *at the optimum.*

Proof. Suppose for a contradiction that this is not the case, then the other constraint in the pair must be binding, i.e. $\underline{\zeta}^b \exp\left(\overline{u}^b\right) = \underline{\mu}_1 \exp\left(t_1^b\right)$. Suppose now that t_1^s, t_2^s, $\underline{\zeta}^s$, $\overline{\zeta}^s$ and $\underline{\mu}_1$, $\overline{\mu}_1$ are optimally chosen. To make sure that t_1^b, t_2^b and $\underline{\zeta}^b$, $\overline{\zeta}^b$ are optimally chosen, we solve:

$$
\begin{aligned}
& \max_{t^b, \zeta^b} \ \tau_1\left(t_1^b - t_1^s\right) + \tau_2\left(t_2^b - t_2^s\right) \quad \text{s.t.} \\
(\mathrm{ST}^b) \qquad & \exp\left(t_1^b\right) = \frac{1}{\underline{\mu}_1} \underline{\zeta}^b \exp\left(\underline{u}^b\right), \\
& \exp\left(t_2^b\right) = \underline{\zeta}^b \exp\left(\underline{u}^b\right), \\
(\mathrm{NA}^b) \qquad & \underline{\zeta}^b + \overline{\zeta}^b \leq 1, \quad \underline{\zeta}^b \geq 0, \ \overline{\zeta}^b \geq 0.
\end{aligned}
$$

The optimal solution to the above problem is to set $\underline{\zeta}^b = 1$ and $\overline{\zeta}^b = 0$, which contradicts Claim 14. □

Likewise, we must have at the optimum $\underline{\zeta}^s \exp\left(-\underline{u}^s\right) = \exp\left(-t_2^s\right)$. Moreover, we must have either $\underline{\zeta}^s \exp\left(-\underline{u}^s\right) = \underline{\mu}_1 \exp\left(-t_1^s\right)$ or $\overline{\zeta}^s \exp\left(-\overline{u}^s\right) = \overline{\mu}_1 \exp\left(-t_1^s\right)$, or both. We establish the following lemma:

Lemma 34. $\overline{\zeta}^s \exp\left(-\overline{u}^s\right) = \overline{\mu}_1 \exp\left(-t_1^s\right)$ *at the optimum.*

Proof. Analogous to Lemma 33 for the buyer. □

The revenue maximization problem can then be rewtitten in terms of the original variables as follows:

$$
\begin{aligned}
& \max_{\tau,\mu,t,\Lambda} \ \tau_1\big(t_1^b - t_1^s\big) + \tau_2\big(t_2^b - t_2^s\big) \ \text{ s.t.} \\
(\text{ST}^b) \quad & t_1^b \le \underline{u}^b - \log\big(\underline{\mu}_1\big) - \underline{\Lambda}^b, \\
& t_1^b = \overline{u}^b - \log\big(\overline{\mu}_1\big) - \overline{\Lambda}^b, \\
& t_2^b = \underline{u}^b - \underline{\Lambda}^b; \\
(\text{ST}^s) \quad & -t_1^s \le -\underline{u}^s - \log\big(\underline{\mu}_1\big) - \underline{\Lambda}^s, \\
& -t_1^s = -\overline{u}^s - \log\big(\overline{\mu}_1\big) - \overline{\Lambda}^s, \\
& -t_2^s = -\underline{u}^s - \underline{\Lambda}^s; \\
(\text{NA}^b) \quad & \exp\big(-\underline{\Lambda}^b\big) + \exp\big(-\overline{\Lambda}^b\big) \le 1, \\
(\text{NA}^s) \quad & \exp\big(-\underline{\Lambda}^s\big) + \exp\big(-\overline{\Lambda}^s\big) \le 1, \\
(\alpha\text{-F}) \quad & \text{BP}, \ 0 < \underline{\mu}_1 < 1, \quad 0 < \overline{\mu}_1 < 1.
\end{aligned}
$$

The revenue in Case E2 can then be written as follows:

$$
\mathcal{R}_{\text{Case E2}} = \tau_1\Big[\overline{u}^b - \overline{u}^s - 2\log\big(\overline{\mu}_1\big) - \overline{\Lambda}^b - \overline{\Lambda}^s\Big] + \tau_2\Big[\underline{u}^b - \underline{u}^s - \underline{\Lambda}^b - \underline{\Lambda}^s\Big].
$$

Since $\mathcal{R}_{\text{Case 2E}}$ is strictly decreasing in Λ, both (NA^b) and (NA^s) are binding.

From our consideration of the concave closures above, it follows that we must consider the following three cases:

Case E2.1: $t_1^b = \underline{u}^b - \log\big(\underline{\mu}_1\big) - \underline{\Lambda}^b = \overline{u}^b - \log\big(\overline{\mu}_1\big) - \overline{\Lambda}^b$

Since (NA^b) is binding, we can solve for Λ^b:

$$
\begin{cases}
-\underline{\Lambda}^b = -\underline{u}^b + \log\big(\underline{\mu}_1\big) - \log\big[\underline{\mu}_1 \exp(-\underline{u}^b) + \overline{\mu}_1 \exp(-\overline{u}^b)\big], \\
-\overline{\Lambda}^b = -\overline{u}^b + \log\big(\overline{\mu}_1\big) - \log\big[\underline{\mu}_1 \exp(-\underline{u}^b) + \overline{\mu}_1 \exp(-\overline{u}^b)\big].
\end{cases}
$$

The revenue can then be written as follows:

$$
\mathcal{R}_{\text{Case E2.1}} = \tau_1\big[-\overline{u}^s - \log\big(\overline{\mu}_1\big) - \overline{\Lambda}^s\big] + \tau_2\big[-\underline{u}^s + \log\big(\underline{\mu}_1\big) - \underline{\Lambda}^s\big] - \log\big[\underline{\mu}_1 \exp(-\underline{u}^b) + \overline{\mu}_1 \exp(-\overline{u}^b)\big]
$$

Maximizing $\mathcal{R}_{\text{Case E2.1}}$ with respect to $\underline{\Lambda}^s$ and $\overline{\Lambda}^s$ subject to (NA^s), we get $-\underline{\Lambda}^s = \log\tau_2$ and $-\overline{\Lambda}^s = \log\tau_1$. The revenue can then be written as:

$$
\begin{aligned}
\mathcal{R}_{\text{Case E2.1}} &= \tau_1\big[-\overline{u}^s - \log\big(\overline{\mu}_1\big) + \log\tau_1\big] + \tau_2\big[-\underline{u}^s + \log\big(\underline{\mu}_1\big) + \log\tau_2\big] - \log\big[\underline{\mu}_1 \exp(-\underline{u}^b) + \overline{\mu}_1 \exp(-\overline{u}^b)\big] \\
&= \log\big(\underline{\mu}_1\big) - \tau_1 \log\big(\underline{\mu}_1\overline{\mu}_1\big) - \tau_1\Delta^s + \tau_1\log\tau_1 + \tau_2\log\tau_2 - \log\big[\underline{\mu}_1 \exp(\Delta^b) + \overline{\mu}_1\big] + \overline{u}^b - \underline{u}^s
\end{aligned}
$$

Recall that Case E2.1 applies only if the (ST^s)-constraints are satisfied, i.e.

$$\begin{aligned}
&-\overline{u}^s-\log\left(\overline{\mu}_1\right)-\overline{\Lambda}^s\leq-\underline{u}^s-\log\left(\underline{\mu}_1\right)-\underline{\Lambda}^s\\
\Leftrightarrow&-\overline{u}^s-\log\left(\overline{\mu}_1\right)+\log\tau_1\leq-\underline{u}^s-\log\left(\underline{\mu}_1\right)+\log\tau_2\\
\Leftrightarrow&-\Delta^s\leq\log\left[\frac{\overline{\mu}_1}{\underline{\mu}_1}\right]+\log\left[\frac{\tau_2}{\tau_1}\right]
\end{aligned}$$

Case E2.2: $t_1^s=-\underline{u}^s-\log\left(\underline{\mu}_1\right)-\underline{\Lambda}^s=-\overline{u}^s-\log\left(\overline{\mu}_1\right)-\overline{\Lambda}^s$

Since (NA^s) is binding, we can solve for Λ^b:

$$\begin{cases}
-\underline{\Lambda}^s=\underline{u}^s+\log\left(\underline{\mu}_1\right)-\log\left[\underline{\mu}_1\exp(\underline{u}^s)+\overline{\mu}_1\exp(\overline{u}^s)\right],\\
-\overline{\Lambda}^s=\overline{u}^s+\log\left(\overline{\mu}_1\right)-\log\left[\underline{\mu}_1\exp(\underline{u}^s)+\overline{\mu}_1\exp(\overline{u}^s)\right].
\end{cases}$$

The revenue can then be written as follows:

$$\mathcal{R}_{\text{Case E2.2}}=\tau_1\left[\overline{u}^b-\log\left(\overline{\mu}_1\right)-\overline{\Lambda}^b\right]+\tau_2\left[\underline{u}^b+\log\left(\underline{\mu}_1\right)-\underline{\Lambda}^b\right]-\log\left[\underline{\mu}_1\exp(\underline{u}^s)+\overline{\mu}_1\exp(\overline{u}^s)\right].$$

Maximizing $\mathcal{R}_{\text{Case E2.2}}$ with respect to $\underline{\Lambda}^b$ and $\overline{\Lambda}^b$subject to (NA^b), we get $-\underline{\Lambda}^b=\log\tau_2$ and $-\overline{\Lambda}^b=\log\tau_1$. The revenue can then be written as:

$$\begin{aligned}
\mathcal{R}_{\text{Case E2.2}}&=\tau_1\left[\overline{u}^b-\log\left(\overline{\mu}_1\right)+\log\tau_1\right]+\tau_2\left[\underline{u}^b+\log\left(\underline{\mu}_1\right)+\log\tau_2\right]-\log\left[\underline{\mu}_1\exp(\underline{u}^s)+\overline{\mu}_1\exp(\overline{u}^s)\right]\\
&=\log\left(\underline{\mu}_1\right)-\tau_1\log\left(\underline{\mu}_1\overline{\mu}_1\right)-\tau_2\Delta^b+\tau_1\log\tau_1+\tau_2\log\tau_2-\log\left[\underline{\mu}_1+\overline{\mu}_1\exp(\Delta^s)\right]+\overline{u}^b-\underline{u}^s
\end{aligned}$$

Recall that Case E2.2 applies only if the (ST^s)-constraints are satisfied, i.e.

$$\begin{aligned}
&\overline{u}^b-\log\left(\overline{\mu}_1\right)-\overline{\Lambda}^b\leq\underline{u}^b-\log\left(\underline{\mu}_1\right)-\underline{\Lambda}^b\\
\Leftrightarrow&\overline{u}^b-\log\left(\overline{\mu}_1\right)+\log\tau_1\leq\underline{u}^b-\log\left(\underline{\mu}_1\right)+\log\tau_2\\
\Leftrightarrow&\Delta^b\leq\log\left[\frac{\overline{\mu}_1}{\underline{\mu}_1}\right]+\log\left[\frac{\tau_2}{\tau_1}\right]
\end{aligned}$$

Case E2.3: all constraints are binding

In this case we have:

$$\begin{cases}
-\underline{\Lambda}^b=-\underline{u}^b+\log\left(\underline{\mu}_1\right)-\log\left[\underline{\mu}_1\exp(-\underline{u}^b)+\overline{\mu}_1\exp(-\overline{u}^b)\right],\\
-\overline{\Lambda}^b=-\overline{u}^b+\log\left(\overline{\mu}_1\right)-\log\left[\underline{\mu}_1\exp(-\underline{u}^b)+\overline{\mu}_1\exp(-\overline{u}^b)\right],\\
-\underline{\Lambda}^s=\underline{u}^s+\log\left(\underline{\mu}_1\right)-\log\left[\underline{\mu}_1\exp(\underline{u}^s)+\overline{\mu}_1\exp(\overline{u}^s)\right],\\
-\overline{\Lambda}^s=\overline{u}^s+\log\left(\overline{\mu}_1\right)-\log\left[\underline{\mu}_1\exp(\underline{u}^s)+\overline{\mu}_1\exp(\overline{u}^s)\right].
\end{cases}$$

The revenue then becomes:

$$\begin{aligned}
\mathcal{R}_{\text{Case E2.3}}&=2\tau_2\log\left(\underline{\mu}_1\right)-\log\left[\underline{\mu}_1\exp(-\underline{u}^b)+\overline{\mu}_1\exp(-\overline{u}^b)\right]-\log\left[\underline{\mu}_1\exp(\underline{u}^s)+\overline{\mu}_1\exp(\overline{u}^s)\right]\\
&=2\tau_2\log\left(\underline{\mu}_1\right)-\log\left[\underline{\mu}_1\exp(\Delta^b)+\overline{\mu}_1\right]-\log\left[\underline{\mu}_1+\overline{\mu}_1\exp(\Delta^s)\right]+\overline{u}^b-\underline{u}^s.
\end{aligned}$$

J.4 Ruled-out cases

Lemma 35. $\mathcal{R}_{\text{Case E2.1}} \geq \mathcal{R}_{\text{Case E2.2}}$ *for all* $\Delta \leq \log\left(\frac{\overline{\mu}_1}{\underline{\mu}_1}\right) + \log\left(\frac{\tau_2}{\tau_1}\right)$.

Proof. $\mathcal{R}_{\text{Case E2.1}} \geq \mathcal{R}_{\text{Case E2.2}}$ can equivalently be written as

$$\tau_1\Delta + \log\left[\underline{\mu}_1 \exp(\Delta) + \overline{\mu}_1\right] \leq \tau_2\Delta + \log\left[\underline{\mu}_1 + \overline{\mu}_1 \exp(\Delta)\right],$$

which in turn can be rewritten as

$$\frac{\underline{\mu}_1}{\overline{\mu}_1}\Delta + \log\left[\underline{\mu}_1 \exp(\Delta) + \overline{\mu}_1\right] - \log\left[\underline{\mu}_1 + \overline{\mu}_1 \exp(\Delta)\right] \leq 0$$

The condition $\Delta \leq \log\left(\frac{\overline{\mu}_1}{\underline{\mu}_1}\right) + \log\left(\frac{\tau_2}{\tau_1}\right)$ can be rewritten as:

$$\exp(\Delta) \leq \frac{\overline{\mu}_1}{\underline{\mu}_1}\frac{\tau_2}{\tau_1} = \frac{\overline{\mu}_1}{\underline{\mu}_1}\left(\frac{\overline{\mu}_1}{\overline{\mu}_0} - 1\right) = \frac{\overline{\mu}_1}{\underline{\mu}_1}(2\overline{\mu}_1 - 1) = \frac{\overline{\mu}_1}{\underline{\mu}_1}(\overline{\mu}_1 - \underline{\mu}_1).$$

Define the function $f(\Delta) \equiv \frac{\underline{\mu}_1}{\overline{\mu}_1}\Delta + \log\left[\underline{\mu}_1 \exp(\Delta) + \overline{\mu}_1\right] - \log\left[\underline{\mu}_1 + \overline{\mu}_1 \exp(\Delta)\right]$. Our goal is to show $f(\Delta) \leq 0$ for all $\Delta \geq 0$ such that $\exp(\Delta) \leq \frac{\overline{\mu}_1}{\underline{\mu}_1}(\overline{\mu}_1 - \underline{\mu}_1)$. Observe first that $f(0) = \frac{\underline{\mu}_1}{\overline{\mu}_1}0 + \log\left[\underline{\mu}_1 + \overline{\mu}_1\right] - \log\left[\underline{\mu}_1 + \overline{\mu}_1\right] = 0$. We will now show that $f(\Delta)$ is decreasing for all $\Delta \geq 0$ such that $\exp(\Delta) \leq \frac{\overline{\mu}_1}{\underline{\mu}_1}(\overline{\mu}_1 - \underline{\mu}_1)$. First, let us compute $f'(\Delta)$:

$$\begin{aligned}
f'(\Delta) &= \frac{\underline{\mu}_1}{\overline{\mu}_1} + \frac{\underline{\mu}_1 \exp(\Delta)}{\underline{\mu}_1 \exp(\Delta) + \overline{\mu}_1} - \frac{\overline{\mu}_1 \exp(\Delta)}{\underline{\mu}_1 + \overline{\mu}_1 \exp(\Delta)} \\
&= \frac{\underline{\mu}_1}{\overline{\mu}_1} + \exp(\Delta)\left[\frac{\underline{\mu}_1^2 + \underline{\mu}_1\overline{\mu}_1 \exp(\Delta) - \underline{\mu}_1\overline{\mu}_1 \exp(\Delta) - \overline{\mu}_1^2}{\left(\underline{\mu}_1 \exp(\Delta) + \overline{\mu}_1\right)\left(\underline{\mu}_1 + \overline{\mu}_1 \exp(\Delta)\right)}\right] \\
&= \frac{\underline{\mu}_1}{\overline{\mu}_1} + \exp(\Delta)\left[\frac{(\underline{\mu}_1 - \overline{\mu}_1)(\underline{\mu}_1 + \overline{\mu}_1)}{\left(\underline{\mu}_1 \exp(\Delta) + \overline{\mu}_1\right)\left(\underline{\mu}_1 + \overline{\mu}_1 \exp(\Delta)\right)}\right] \\
&= \frac{\underline{\mu}_1}{\overline{\mu}_1} - \exp(\Delta)\left[\frac{\overline{\mu}_1 - \underline{\mu}_1}{\left(\underline{\mu}_1 \exp(\Delta) + \overline{\mu}_1\right)\left(\underline{\mu}_1 + \overline{\mu}_1 \exp(\Delta)\right)}\right].
\end{aligned}$$

We now show that $f'(\Delta) \leq 0$ for all $\Delta \geq 0$ such that $\exp(\Delta) \leq \frac{\overline{\mu}_1}{\underline{\mu}_1}(\overline{\mu}_1 - \underline{\mu}_1)$. The condition $f'(\Delta) \leq 0$ can be rewritten as:

$$\begin{aligned}
&\frac{\underline{\mu}_1}{\overline{\mu}_1} - \exp(\Delta)\left[\frac{\overline{\mu}_1 - \underline{\mu}_1}{\left(\underline{\mu}_1 \exp(\Delta) + \overline{\mu}_1\right)\left(\underline{\mu}_1 + \overline{\mu}_1 \exp(\Delta)\right)}\right] \leq 0 \\
\Leftrightarrow\ & 1 - \exp(\Delta)\frac{\frac{\overline{\mu}_1}{\underline{\mu}_1}\left(\overline{\mu}_1 - \underline{\mu}_1\right)}{\left(\underline{\mu}_1 \exp(\Delta) + \overline{\mu}_1\right)\left(\underline{\mu}_1 + \overline{\mu}_1 \exp(\Delta)\right)} \leq 0 \\
\Leftrightarrow\ & \exp(-\Delta)\left(\underline{\mu}_1 \exp(\Delta) + \overline{\mu}_1\right)\left(\underline{\mu}_1 + \overline{\mu}_1 \exp(\Delta)\right) \leq \frac{\overline{\mu}_1}{\underline{\mu}_1}(\overline{\mu}_1 - \underline{\mu}_1). \qquad (5)
\end{aligned}$$

Since $\exp(\Delta) \leq \frac{\overline{\mu}_1}{\underline{\mu}_1}(\overline{\mu}_1 - \underline{\mu}_1)$, to show that (5) is true, it is enough to show that

$$\begin{aligned} &\exp(-\Delta)\big(\underline{\mu}_1 \exp(\Delta) + \overline{\mu}_1\big)\big(\underline{\mu}_1 + \overline{\mu}_1 \exp(\Delta)\big) \leq \exp(\Delta) \\ \Leftrightarrow\ &\big(\underline{\mu}_1 \exp(\Delta) + \overline{\mu}_1\big)\big(\underline{\mu}_1 + \overline{\mu}_1 \exp(\Delta)\big) - \exp(2\Delta) \leq 0. \end{aligned}$$

To that end, define the function $g(x) \equiv \big(\underline{\mu}_1 x + \overline{\mu}_1\big)\big(\underline{\mu}_1 + \overline{\mu}_1 x\big) - x^2$. We show that $g(x) \leq 0$ for all $x \geq 1$. First, observe that $g(1) = \big(\underline{\mu}_1 + \overline{\mu}_1\big)\big(\underline{\mu}_1 + \overline{\mu}_1\big) - 1^2 = 0$. We now show that $g(x)$ is strictly decreasing for all $x \geq 1$. Let us compute $g'(x)$:

$$g'(x) = \underline{\mu}_1(\underline{\mu}_1 + \overline{\mu}_1) + \overline{\mu}_1(\underline{\mu}_1 x + \overline{\mu}_1) - 2x = \underline{\mu}_1^2 + \overline{\mu}_1^2 - 2(1 - \underline{\mu}_1\overline{\mu}_1)x.$$

To show that $g'(x) < 0$ for all $x \geq 1$, observe that $g'(1) = \underline{\mu}_1^2 + \overline{\mu}_1^2 - 2 + 2\underline{\mu}_1\overline{\mu}_1 = (\underline{\mu}_1 + \overline{\mu}_1)^2 - 2 = 1^2 - 2 = -1 < 0$, which, combined with $g''(x) = -2(1 - \underline{\mu}_1\overline{\mu}_1) < 0$, gives the desired result. ☐

Lemma 36. $\mathcal{R}_{\text{Case E2.1}} > \mathcal{R}_{\text{Case E2.3}}$ *for* $-\Delta \leq \log\left(\frac{\overline{\mu}_1}{\underline{\mu}_1}\right) + \log\left(\frac{\tau_2}{\tau_1}\right)$.

Proof. $\mathcal{R}_{\text{Case E2.1}} > \mathcal{R}_{\text{Case E2.3}}$ can be equivalently written as

$$-\tau_2 \log(\underline{\mu}_1) - \tau_1 \log(\overline{\mu}_1) - \tau_1\Delta + \tau_1 \log(\tau_1) + \tau_2 \log \tau_2 + \log\left[\underline{\mu}_1 + \overline{\mu}_1 \exp(\Delta)\right] \geq 0.$$

The condition $-\Delta \leq \log\left(\frac{\overline{\mu}_1}{\underline{\mu}_1}\right) + \log\left(\frac{\tau_2}{\tau_1}\right)$ can be rewritten as:

$$\exp(-\Delta) \leq \frac{\overline{\mu}_1}{\underline{\mu}_1}\frac{\tau_2}{\tau_1} = \frac{\overline{\mu}_1}{\underline{\mu}_1}\left(\frac{\overline{\mu}_1}{\overline{\mu}_0} - 1\right) = \frac{\overline{\mu}_1}{\underline{\mu}_1}(2\overline{\mu}_1 - 1) = \frac{\overline{\mu}_1}{\underline{\mu}_1}(\overline{\mu}_1 - \underline{\mu}_1).$$

Define $f(\Delta) \equiv -\tau_2 \log(\underline{\mu}_1) - \tau_1 \log(\overline{\mu}_1) - \tau_1\Delta + \tau_1 \log(\tau_1) + \tau_2 \log \tau_2 + \log\big[\underline{\mu}_1 + \overline{\mu}_1 \exp(\Delta)\big]$. Our goal is to show that $f(\Delta) \geq 0$ for all $\Delta \geq -\log\left(\frac{\overline{\mu}_1}{\underline{\mu}_1}\right) - \log\left(\frac{\tau_2}{\tau_1}\right)$. Let us first evaluate f at $\hat{\Delta} \equiv -\log\left(\frac{\overline{\mu}_1}{\underline{\mu}_1}\right) - \log\left(\frac{\tau_2}{\tau_1}\right)$ (note that $\exp(\hat{\Delta}) = \frac{\underline{\mu}_1}{\overline{\mu}_1}\frac{1}{\overline{\mu}_1 - \underline{\mu}_1}$):

$$\begin{aligned} f(\hat{\Delta}) &= -\tau_2 \log(\underline{\mu}_1) - \tau_1 \log(\overline{\mu}_1) + \tau_1\left[\log\left(\frac{\overline{\mu}_1}{\underline{\mu}_1}\right) + \log\left(\frac{\tau_2}{\tau_1}\right)\right] + \tau_1 \log(\tau_1) + \tau_2 \log \tau_2 + \log\left[\underline{\mu}_1 + \overline{\mu}_1 e^{\hat{\Delta}}\right] \\ &= -(\tau_2 + \tau_1)\log(\underline{\mu}_1) + (\tau_1 + \tau_2)\log \tau_2 + \log\left[\underline{\mu}_1 + \overline{\mu}_1 \exp(\hat{\Delta})\right] \\ &= -\log(\underline{\mu}_1) + \log \tau_2 + \log\left[\underline{\mu}_1 + \overline{\mu}_1 \frac{\underline{\mu}_1}{\overline{\mu}_1}\frac{1}{\overline{\mu}_1 - \underline{\mu}_1}\right] \\ &= -\log(\underline{\mu}_1) + \log \tau_2 + \log(\underline{\mu}_1) + \log\left[1 + \frac{1}{\overline{\mu}_1 - \underline{\mu}_1}\right] \\ &= \log \tau_2 + \log\left[1 + \frac{1}{\overline{\mu}_1 - \underline{\mu}_1}\right] = \log\left[\frac{\overline{\mu}_1 - \underline{\mu}_1}{2\overline{\mu}_1}\right] + \log\left[\frac{2\overline{\mu}_1}{\overline{\mu}_1 - \underline{\mu}_1}\right] = 0. \end{aligned}$$

Since $f(\hat{\Delta}) = 0$, it remains to show that $f(\Delta)$ is increasing for all $\Delta \geq \hat{\Delta}$, i.e. whenever $\exp(-\Delta) \leq \frac{\overline{\mu}_1}{\underline{\mu}_1}(\overline{\mu}_1 - \underline{\mu}_1)$. To that end, we compute the derivative of f:

$$f'(\Delta) = -\tau_1 + \frac{\overline{\mu}_1 \exp(\Delta)}{\underline{\mu}_1 + \overline{\mu}_1 \exp(\Delta)} = -\frac{1}{2\overline{\mu}_1} + \frac{\overline{\mu}_1}{\underline{\mu}_1 \exp(-\Delta) + \overline{\mu}_1}.$$

Direct calculation shows that $f'(\Delta) \geq 0$ is equivalent to $\exp(-\Delta) \leq \frac{\overline{\mu}_1}{\underline{\mu}_1}(\overline{\mu}_1 - \underline{\mu}_1)$, which establishes the claim. □

Lemma 37. $\mathcal{R}_{\text{No-learning}} \geq \mathcal{R}_{\text{Case E2.3}}$.

Proof. The condition $\mathcal{R}_{\text{No-learning}} \geq \mathcal{R}_{\text{Case E2.3}}$ can equivalently be written as:

$$-2\log\big[\exp(\Delta)+1\big]+2\log 2 \geq 2\tau_2 \log(\underline{\mu}_1) - \log\big[\underline{\mu}_1 \exp(\Delta)+\overline{\mu}_1\big] - \log\big[\underline{\mu}_1 + \overline{\mu}_1 \exp(\Delta)\big].$$

Define the function f as follows:

$$f(x) \equiv \log\big[\underline{\mu}_1 x + \overline{\mu}_1\big] + \log\big[\underline{\mu}_1 + \overline{\mu}_1 x\big] - 2\log(x+1) + 2\log 2 - 2\tau_2 \log(\underline{\mu}_1).$$

The statement of the lemma is equivalent to showing that $f(x) \geq 0$ for all $x \geq 1$. We first show that f is strictly decreasing for all $x > 1$ by computing its derivative:

$$\begin{aligned}
f'(x) &= \frac{\underline{\mu}_1}{\underline{\mu}_1 x + \overline{\mu}_1 x} + \frac{\overline{\mu}_1}{\underline{\mu}_1 + \overline{\mu}_1 x} - \frac{2}{x+1} \\
&= \frac{\underline{\mu}_1}{\underline{\mu}_1 x + \overline{\mu}_1} - \frac{1}{x+1} + \frac{\overline{\mu}_1}{\underline{\mu}_1 + \overline{\mu}_1 x} - \frac{1}{x+1} \\
&= \frac{\underline{\mu}_1 - \overline{\mu}_1}{(\underline{\mu}_1 x + \overline{\mu}_1)(x+1)} + \frac{\overline{\mu}_1 - \underline{\mu}_1}{(\underline{\mu}_1 + \overline{\mu}_1 x)(x+1)} \\
&= \frac{\overline{\mu}_1 - \underline{\mu}_1}{x+1}\left[\frac{1}{\underline{\mu}_1 + \overline{\mu}_1 x} - \frac{1}{\underline{\mu}_1 x + \overline{\mu}_1}\right] = \frac{\left(\overline{\mu}_1 - \underline{\mu}_1\right)^2 (1-x)}{(x+1)(\underline{\mu}_1 + \overline{\mu}_1 x)(\underline{\mu}_1 x + \overline{\mu}_1)} < 0.
\end{aligned}$$

We now show that $\lim_{x\to+\infty} f(x) \geq 0$:

$$\begin{aligned}
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} \log\left[\frac{(\underline{\mu}_1 x + \overline{\mu}_1)(\underline{\mu}_1 + \overline{\mu}_1 x)}{(x+1)^2}\right] + 2\log 2 - 2\tau_2 \log(\underline{\mu}_1) \\
&= \lim_{x\to+\infty} \log\left[\frac{\underline{\mu}_1^2 + \underline{\mu}_1\overline{\mu}_1 + \underline{\mu}_1\overline{\mu}_1 x^2 + \overline{\mu}_1^2}{x^2+2x+1}\right] + 2\log 2 - 2\tau_2 \log(\underline{\mu}_1) \\
&= \log(\underline{\mu}_1\overline{\mu}_1) + 2\log 2 - 2\tau_2 \log(\underline{\mu}_1) \\
&= (1-2\tau_2)\log(\underline{\mu}_1) + \log(\overline{\mu}_1) + 2\log 2 \\
&= \frac{\underline{\mu}_1}{\overline{\mu}_1}\log(\underline{\mu}_1) + \log(\overline{\mu}_1) + 2\log 2.
\end{aligned}$$

Define the function $g(x) \equiv \frac{x}{1-x}\log(x) + \log(1-x) + 2\log 2$. To conclude the proof, we need to establish the following lemma:

Lemma 38. *$g(x) \geq 0$ for all $0 < x \leq \frac{1}{2}$.*

Proof. First observe that $g(\frac{1}{2}) = \log(\frac{1}{2}) + \log(\frac{1}{2}) + 2\log 2 = 0$. To conclude the proof, we show that g is strictly decreasing on $0 < x \geq \frac{1}{2}$:

$$g'(x) = \frac{1 - x + x}{(1 - x)^2} \log(x) + \frac{x}{1 - x} \frac{1}{x} - \frac{1}{1 - x} = \frac{\log(x)}{(1 - x)^2} < 0.$$

□

□

J.5 Conjectures

Recall that at the (interior) optimum we have:

$$\begin{aligned}\mathcal{R}^*_{\text{interior}} = &\log\left[\sqrt{e^\Delta - 1} - \sqrt{e^\Delta - 9}\right] - \log\left[3\sqrt{e^\Delta - 1} + \sqrt{e^\Delta - 9}\right] \\ &- 2\log\left[3 + e^\Delta - \sqrt{(e^\Delta - 1)(e^\Delta - 9)}\right] + 4\log 2.\end{aligned}$$

Numerical computations suggest that the following conjecture is true:

Conjecture 1. *For all $\Delta \geq \log 9$ and $-\Delta \leq \log\left[\frac{\overline{\mu}_1}{\underline{\mu}_1}\right] + \log\left[\frac{\tau_2}{\tau_1}\right]$, we have*

$$\mathcal{R}^*_{\text{interior}} \geq \mathcal{R}_{\text{Case E2.1}} = \log(\underline{\mu}_1) - \tau_1 \log(\underline{\mu}_1 \overline{\mu}_1) - \tau_1 \Delta + \tau_1 \log \tau_1 + \tau_2 \log \tau_2 - \log\left[\underline{\mu}_1 e^\Delta + \overline{\mu}_1\right],$$

where $\underline{\mu}_1 \in (0, 0.5)$ and the τ's are given by:

$$\begin{cases} \tau_1 \underline{\mu}_1 + \tau_2 \underline{\mu}_2 = \tau_1 \underline{\mu}_1 + \tau_2 1 = \underline{\mu}_0, \\ \tau_1 \overline{\mu}_1 + \tau_2 \overline{\mu}_2 = \tau_1 \overline{\mu}_1 + \tau_2 0 = \overline{\mu}_0. \end{cases} \Rightarrow \begin{cases} \tau_1 = \frac{\overline{\mu}_0}{\overline{\mu}_1} = \frac{1}{2\overline{\mu}_1}, \\ \tau_2 = 1 - \frac{\overline{\mu}_0}{\overline{\mu}_1} = 1 - \frac{1}{2\overline{\mu}_1}. \end{cases}$$

Recall that the no-learning revenue is given by:

$$\mathcal{R}_{\text{No-learning}} = -2\log\left[\exp\left(\Delta\right) + 1\right] + 2\log 2.$$

Numerical computations suggest that the following conjecture is true:

Conjecture 2. *For all $\Delta \leq \log 9$ and $-\Delta \leq \log\left[\frac{\overline{\mu}_1}{\underline{\mu}_1}\right] + \log\left[\frac{\tau_2}{\tau_1}\right]$, we have*

$$\mathcal{R}_{\text{No-learning}} \geq \mathcal{R}_{\text{Case E2.1}} = \log(\underline{\mu}_1) - \tau_1 \log(\underline{\mu}_1 \overline{\mu}_1) - \tau_1 \Delta + \tau_1 \log \tau_1 + \tau_2 \log \tau_2 - \log\left[\underline{\mu}_1 e^\Delta + \overline{\mu}_1\right],$$

where $\underline{\mu}_1 \in (0, 0.5)$ and the τ's are given by:

$$\begin{cases} \tau_1 \underline{\mu}_1 + \tau_2 \underline{\mu}_2 = \tau_1 \underline{\mu}_1 + \tau_2 1 = \underline{\mu}_0, \\ \tau_1 \overline{\mu}_1 + \tau_2 \overline{\mu}_2 = \tau_1 \overline{\mu}_1 + \tau_2 0 = \overline{\mu}_0. \end{cases} \Rightarrow \begin{cases} \tau_1 = \frac{\overline{\mu}_0}{\overline{\mu}_1} = \frac{1}{2\overline{\mu}_1}, \\ \tau_2 = 1 - \frac{\overline{\mu}_0}{\overline{\mu}_1} = 1 - \frac{1}{2\overline{\mu}_1}. \end{cases}$$

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