Optimal Switching and the Abandonment of Promising Ventures

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Abstract

This paper uses the real options approach to analyze a decision maker's choice to switch from an ongoing project to another one. In both projects, there is uncertainty about the arrival and magnitude of the potential profit. Interestingly, increasing the probability of receiving a payoff in the initial project does not always make this project more attractive. Instead, it may be better to switch to the alternative project, even if the latter is less likely to succeed. This implies that the option value of waiting can be either increased or decreased by the uncertainty about the timing of a payoff.

Keywords: real options, optimal switching, uncertainty, research and development, innovation, upside potential
JEL Classification: D21, D81, O31

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1 Introduction

Economic activities are typically subject to various forms of uncertainty. For instance, firms conducting R&D usually do not know in advance when a project will result in a payoff (technological uncertainty). In addition, there is uncertainty about the magnitude of the profit that may be generated from a project (price uncertainty). Thus, even if a project appeared promising at the outset, the firm may at some point consider abandoning its current project and switching to an alternative one. This decision is particularly interesting if the new alternative was not yet available at the time the original activity was chosen.

This paper aims to answer the question of when it is optimal to switch from one project to another mutually exclusive project in the presence of technological and price uncertainty. While decision problems of this type may typically arise in the realm of R&D, one might also consider the general case of switching when technological and price uncertainty are present, i.e., when *someone does something where a stochastic profit can be obtained at an unknown time.*

The model used to answer this question is based on Dixit (1993) and Weeds (1999). In Dixit (1993), a decision maker (DM) chooses among several alternatives that all depend on the same stochastic variable. I extend Dixit's model by incorporating technological uncertainty, i.e. uncertainty about the arrival of potential rewards. Weeds (1999) analyzes a monopolist's decision to start or abandon a single R&D project under the two forms of uncertainty considered here. In contrast to her model, I assume that switching is irreversible. On the one hand, this simplifies the analysis; on the other hand, it could also reflect the idea that the DM may have forgotten the technical details of a project after it was abandoned. Therefore, the resumption of a project at a later date may be considered as the start of an entirely new project. Furthermore, the assumption that a DM (e.g. a large firm) is already carrying out a project may seem somewhat more realistic than the isolated decision about a single project from the perspective of an idle state.

The analysis shows that it is optimal to switch earlier if the probability to succeed in the alternative project increases. While this seems intuitive, the effect of increasing the probability of receiving a payoff in the original project is less obvious. It is shown that, in some cases, switching is accelerated when the initial project becomes more likely to be profitable. This is not straightforward, since one might expect that a higher probability to receive a payoff should be perceived as a good thing. The finding is due to price uncertainty: While the payoff that the DM can obtain is bounded from below (since it cannot be negative), there is no limit as to how much the potential profit can increase. Thus, there is a chance that the DM may obtain a substantial payoff if the project is successful at a later date. As a result of this upside potential, projects with a higher probability of success may be perceived as less attractive compared to those having a smaller probability to yield a reward, at least if the current value of the potential profit is relatively low.

This is remarkable, as it implies that the DM is willing to give up a relatively certain payoff in favor of more uncertainty about the timing of a reward. This result is significant since it indicates that not only the probability of receiving a payoff should be considered when deciding to switch, but also the profit's upside potential.

In addition to the model discussed so far, I also consider the case where the potential payoffs of both projects each follow a separate stochastic process. While previously both potential payoffs were perfectly correlated, they can now move in opposite directions, which is more realistic for some decision problems. This extension builds on Adkins and Paxson (2011b), who develop a model for analyzing the optimal time to replace an asset under several forms of uncertainty. I show that the same effect regarding the probability of success of the initial project as described above can also be observed in this framework.

The present paper is related to the literature on the optimal choice among several alternatives. Weitzman (1979) considers the sequential search among a number of different alternatives with uncertain outcome. In the context of real options, Roberts and Weitzman (1981), McDonald and Siegel (1986), and Majd and Pindyck (1987) consider firms that are able to integrate new information into their decision-making processes. Dixit (1989) considers entry and exit decisions under uncertainty in a product market model. The property of nondeterministic success for a project with uncertain costs is formulated by Pindyck (1993). For a general treatment of the application of the real options method to investment decisions under uncertainty, see Dixit and Pindyck (1994). The work of Dixit (1993) also serves as the foundation of Décamps et al. (2006), where a decision maker chooses among several projects of different scales. Similar to the approach in my paper, they also consider the case of optimal switching from one project to another as an extension, but they do not include technological uncertainty.

Optimal investment, abandonment, or switching decisions in the presence of one or multiple forms of uncertainty are also studied, among others, by Geltner et al. (1996), Alvarez and Stenbacka (2004), Jun (2005), Malchow-Møller and Thorsen (2005), Ly Vath and Pham (2007), Gryglewicz et al. (2008), Pham et al. (2009), Bobtcheff and Villeneuve (2010), Kort et al. (2010), Siddiqui and Fleten (2010), Guerra et al. (2018), Kauppinen et al. (2018), Zervos et al. (2018), Olsen and Hagspiel (2019), Compernolle et al. (2021), and Kravchenko et al. (2023). These models mostly consider uncertainty regarding the price or cost, whereas the present paper studies both technological and price uncertainty. The general approach of Adkins and Paxson (2011b) has been applied in further work by the same authors (Adkins and Paxson 2011a, 2013a,b, 2017) and has also been adapted by other authors (e.g., Heydari et al. 2012, Armada et al. 2013, Støre et al. 2018).

Morris et al. (1991) argue, that firms should prefer to pursue riskier R&D projects (in terms of the likelihood of making a discovery). While this is similar to the main result found in the present paper, the argument in Morris et al. (1991) is based on the distinction of an R&D phase and the subsequent commercialization phase, where the latter is only reached, and thus the cost only paid, if the research is successful. This differs from my approach, since I simplify the analysis by not differentiating between the two phases. McCardle et al. (2018) analyze a researcher's choice to either continue working on her current project or to search for a new idea. While in their paper the search for an alternative project is endogenized, I assume that the alternative already exists. The idea that some R&D projects may be abandoned is also addressed in Ganglmair et al. (2019), who study the selection between several ideas in a standards development framework.

The remainder of this paper is organized as follows. Section 2 considers the optimal decision to switch when there is one stochastic variable that affects both project values. Section 3 presents an extension of this model where each project value is influenced by a separate stochastic factor. Section 4 concludes.

2 Optimal Switching Between Two Projects

2.1 The Model

A risk-neutral decision maker (DM) is pursuing a project M. While working on this project, the DM has the option, but not the obligation, to switch to another project N

by incurring a sunk cost $S > 0$. Furthermore, switching is assumed to be irreversible, i.e., once the DM has switched to N, she cannot return to M. In addition, there is no possibility that further projects will become available at a later date. Project N must therefore be completed if a switch is made. Following Dixit (1993), denote by $X_M P$ the potential lump-sum payoff of the current project, while $X_N P$ is the potential payoff for the alternative activity. Thus, the possible payoff to each project consists of the variable $P > 0$ scaled by a constant factor $X_M > 0$ or $X_N > 0$, respectively. P is stochastic in continuous time and follows a geometric Brownian motion (GBM) with drift, that is,

$$dP = \alpha P dt + \sigma P dz. \tag{1}$$

Here, α is the drift rate, $\sigma > 0$ is the volatility parameter, and dz is the increment of a Wiener process, with $\mathbb{E}(dz) = 0$ and $\mathbb{E}[(dz)^2] = dt$. For simplicity, I restrict the analysis to the case where there are no recurring costs, i.e., the cost per period for each project is zero. Since the payoffs to both projects are determined by the same stochastic variable, they are perfectly correlated. Depending on which project the DM is pursuing at each point in time t, a payoff-generating event occurs, where the random time of the event follows an exponential distribution with parameter $\lambda_i > 0$. That is, the probability that an event occurs in project i in the interval $(t, t + dt)$, provided that no event has yet occurred by t, is $\lambda_i dt$. Consequently, the expected payoff for the DM in $(t, t + dt)$ is $\lambda_i X_i P dt$, $i \in \{M, N\}$. It is assumed that an event can only occur when a project is being actively pursued, i.e., while the DM is working on M, there can be no event in N, and vice versa.

In the context of an R&D project, an event can be understood as a discovery that can be patented and sold on the market at a price that depends on the current value of a stochastic variable, such as the demand for the new technology.

2.2 Analysis

Following standard practice in the real options literature (e.g., Dixit and Pindyck 1994, p. 185, Weeds 1999), the value of the initial project M is described by the following Bellman equation:

$$V_M(P) = \lambda_M X_M P dt + \mathrm{e}^{-(r+\lambda_M)dt} \left\{ V_M(P) + \mathbb{E}[dV_M(P)] \right\} \tag{2}$$

The first term on the right-hand side (RHS) is the expected immediate profit from project M, while the second term on the RHS is the continuation value. This is the value the DM receives if no event occurs in M in the next instant. Furthermore, $r > 0$ is a discount rate and $\mathbb{E}[dV_M(P)]$ is the expected change in value of the project associated with changes in the GBM described in equation (1). To avoid speculative bubbles, it is assumed that $r > \alpha$. Using Itô's Lemma, this dynamic programming equation can be expanded to

$$\frac{1}{2}\sigma^2 P^2 V_M''(P) + \alpha P V_M'(P) - (r + \lambda_M)V_M(P) + \lambda_M X_M P = 0, \tag{3}$$

where $V_M''(P) \equiv \partial^2 V_M(P)/\partial P^2$, and $V_M'(P) \equiv \partial V_M(P)/\partial P$ are derivatives of the value function. This differential equation is solved by

$$V_M(P) = BP^\beta + \frac{\lambda_M X_M P}{r + \lambda_M - \alpha}. \tag{4}$$

The first term on the RHS is the solution to the homogeneous part of equation (3) and captures the value of the option to switch from project M to project N, where $B > 0$ is a constant yet to be determined. Furthermore,

$$\beta = \frac{1}{2}\left[\left(1 - \frac{2\alpha}{\sigma^2}\right) + \sqrt{\left(1 - \frac{2\alpha}{\sigma^2}\right)^2 + \frac{8(r + \lambda_M)}{\sigma^2}}\right] > 1$$

is the positive root of the quadratic equation

$$\frac{1}{2}\sigma^2\beta(\beta - 1) + \alpha\beta - (r + \lambda_M) = 0.$$

The latter equation is obtained by substituting the equation (4) and its derivatives into (3). Here we only use the positive root, since it ensures that the value of the switching option increases with P and that the boundary condition $V_M(0) = 0$ is fulfilled. This last condition means that once P reaches the lower bound of zero, it does not increase anymore and both the switching option and the current project become worthless. Conversely, the value of the option grows infinitely large with $P \to \infty$, which is also guaranteed by $\beta > 1$. The second term on the RHS of equation (4) is the solution of the heterogeneous part of equation (3) and denotes the discounted expected payoff from project M.

In the same way as above, the value of project N can be determined. Since it is assumed that once the DM has switched to N, switching back to M is not possible, the value function only consists of the expected discounted profit from this project minus the cost of switching,

$$V_N(P) = \frac{\lambda_N X_N P}{r + \lambda_N - \alpha} - S. \tag{5}$$

The DM switches from M to N if the value of the alternative project is at least as large as the value of the current project, i.e. $V_N(P) \geq V_M(P)$. Thus, the switch occurs as soon as a critical value P^* is reached from below. Considering the typical value matching and smooth pasting conditions,

$$V_M(P)\bigg|_{P=P^*} = V_N(P)\bigg|_{P=P^*}, \quad \text{and} \quad \frac{\partial V_M(P)}{\partial P}\bigg|_{P=P^*} = \frac{\partial V_N(P)}{\partial P}\bigg|_{P=P^*},$$

the switching threshold is given by

$$P^* = \left(\frac{\beta}{\beta - 1}\right)\left[\frac{S}{h(\lambda_M, \lambda_N)}\right], \tag{6}$$

where

$$h(\lambda_M, \lambda_N) \equiv \frac{\lambda_N X_N}{r + \lambda_N - \alpha} - \frac{\lambda_M X_M}{r + \lambda_M - \alpha}.$$

To ensure that P^* is positive, it must hold that $h(\cdot) > 0$. Thus, the expected discounted payoff from project N must be greater than the one from the current project M. This is obvious, because otherwise there would be no reason for the DM to consider switching. However, note that this condition does not necessarily require that $\lambda_N > \lambda_M$ and $X_N > X_M$ are simultaneously satisfied, so there may be a trade-off between the certainty and the magnitude of the potential payoff.

The value matching and smooth pasting conditions also enable us to determine the constant B, which is given by

$$B = \left[\frac{h(\lambda_M, \lambda_N)}{\beta}\right]^{\beta}\left(\frac{\beta - 1}{S}\right)^{\beta - 1}. \tag{7}$$

Because of $B > 0$, the value of the switching option, BP^{β}, is indeed positive.

The switching threshold given by equation (6) allows us to derive some insights into the optimal decision to switch from M to N. All the proofs can be found in Appendix

A. The first result considers the effect of the parameters from the GBM described by equation (1).

Proposition 1. *The switching threshold P^* is increasing in σ and non-monotonic in α.*

The result with respect to the volatility parameter σ is consistent with findings from the real options literature: When the potential profit is more volatile, the option value increases and the DM demands a higher value of the alternative project before switching. This is due to the risk that the potential profit suddenly decreases, which is more pronounced with a large degree of price uncertainty.

The result regarding the drift rate α is less straightforward. Usually, one would expect a positive (negative) relationship between the drift rate in the current (alternative) project and the switching threshold. The non-monotonic effect observed here can be attributed to the assumption that both project values are driven by the same GBM, so the actual parameter values determine whether the positive or negative effect of α on P^* predominates. The intuition behind this result can be explained by considering the role of the likelihood of receiving a payoff in the current project, λ_M. If λ_M is relatively low, there is more time for the potential profit to increase due to the drift rate before an event occurs. This increases the switching threshold as the current project becomes more attractive. If λ_M is large, on the other hand, it is more likely that an event occurs in the near future, meaning that the potential payoff will probably only be influenced to a small extent by the drift rate. This lowers the switching threshold, as more time can pass before an event occurs in the other, (possibly) more uncertain project.

Now turn to the case where the DM can be certain that one of the projects leads to a payoff in the next instant.

Proposition 2. *The prospect that an event occurs instantly in only one of the projects ($\lambda_i \to \infty, \lambda_{j \neq i} < \infty$, $i, j \in \{M, N\}$) does not necessarily imply that it is optimal for the DM to pursue and thus complete this project i in $(t, t + dt)$.*

This result illustrates the trade-off between the respective probability to succeed and the size of the potential payoffs. It shows that the absence of technological uncertainty is not sufficient to choose one project over the other. Depending on the current value of the stochastic variable P, the DM may prefer to choose the project where she does not know when a payoff is obtained. This is because the value of the project that is certain to result in a payoff is fixed at the current level, while the value of the other project may increase in the future. Thus, if the value of the certain project is relatively low, it can be optimal to choose the uncertain project and hope for a larger profit at a later date.

Additionally, it is straightforward to consider the optimal behavior when both projects lead to a payoff in the next instant, i.e. $(\lambda_M, \lambda_N) \to (\infty, \infty)$. In this case, the threshold converges to $P^* = S/(X_N - X_M)$. Thus, intuitively, the switching costs and the difference between the scaling factors influence whether the DM should switch and get $X_N P - S$, or stay at the initial project and receive $X_M P$ with certainty.

The upside potential in the stochastic process also plays an important role when considering the impact of technological uncertainty on the switching threshold:

Proposition 3. *The value of the option to switch, Bp^β, is increasing in λ_N and non-monotonic in λ_M. Consequently, the switching threshold P^* is decreasing in λ_N and non-monotonic in λ_M.*

The alternative project always becomes more attractive when the probability of success increases, such that it is optimal to switch sooner. Similarly, the initial project may be perceived as more valuable when the probability of receiving a payoff increases. Interestingly, however, this is not always the case. Instead, it can sometimes be optimal to switch earlier if the initial project is more likely to succeed. This implies that the original project may be perceived as less attractive when it is increasingly likely to yield a profit. [1] To understand the intuition, note that the DM has to consider whether the upside potential in the price process has time to unfold before a payoff is realized. The larger the probability of success, the less room there is for P to increase. Subsequently, the DM has to trade-off the likelihood of the arrival of a payoff (technological uncertainty) and the size of the potential payoff (price uncertainty).

2.3 Numerical Examples

Consider the following parameter values: $X_M = 1, X_N = 2, \lambda_M = \lambda_N = 0.5, \alpha = 0, \sigma = 0.2, r = 0.05,$ and $S = 1$. This means that the potential payoff to project M is P while

[1] Weeds (1999) shows a resembling connection between the probability to receive a payoff and the decision to abandon a project. However, the results differ insofar as the decision to terminate the current project is made in light of different outside options. While Weeds allows the DM to switch back and forth between working and not working on the (single) project, abandoning the more promising project in my model in favor of the less certain project is a definitive decision.

λ_M	P^*	λ_M	P^*
0.1	1.2487	0.6	1.3311
0.2	1.3020	0.7	1.3304
0.3	1.3208	0.8	1.3291
0.4	1.3281	0.9	1.3275
0.5	1.3307	1.0	1.3258

λ_N	P^*	λ_N	P^*
0.1	2.8515	0.6	1.2910
0.2	1.7509	0.7	1.2633
0.3	1.5024	0.8	1.2430
0.4	1.3926	0.9	1.2274
0.5	1.3307	1.0	1.2150

Table 1: Exemplary switching thresholds with $X_M = 1, X_N = 2, \alpha = 0, \sigma = 0.2, r = 0.05$, and $S = 1$. In the left (right) panel, λ_N (λ_M) is held fixed at 0.5 and λ_M (λ_N) is varied.

the potential payoff to project N is $2P$. Table 1 shows some examples for the switching threshold P^* using these values, where only the parameter λ_i is varied while $\lambda_{j \neq i}$ is held constant, $i, j \in \{M, N\}$. First consider the right panel, which displays switching thresholds for different λ_N. Starting from $\lambda_N = 0.1$, which can be interpreted as a relatively large degree of technological uncertainty in project N, increasing λ_N results in a considerable decline of the switching boundary. While initially the DM would only switch for $P \geq 2.8515$, increasing λ_N to 0.2 reduces P^* to 1.7509. The more the technological uncertainty in N is reduced, the smaller is the impact of further changes on the switching threshold.

The left panel of table 1 varies technological uncertainty in the initial project. It can be seen that initially P^* increases until around $P^* = 1.3311$ but then decreases, illustrating the non-monotonic effect of λ_M on P^* described by Proposition 3.

Below, these effects are also shown graphically. Figure 1b shows the switching threshold P^* as a function of λ_N with different values of the parameter λ_M. For all λ_M chosen here, the threshold shows a similar L-shaped form. As long as technological uncertainty in the alternative project is very pronounced, irrespective of the degree of technological uncertainty in the current project, the switching threshold is very large. Once λ_N increases beyond a certain level, P^* drops abruptly.

Likewise, figure 1a shows switching thresholds as a function of λ_M with different values of the parameter λ_N. The switching thresholds depicted here appear to be increasing in λ_M. Taking a closer look at some of the plots shows, however, that they may decrease at some point. The lowest plot in this figure corresponds to the values in the left panel of table 1, where we have already seen a negative correspondence between λ_M and P^*.

As can be seen in figure 2a, in this numerical example, the negative effect of λ_M on

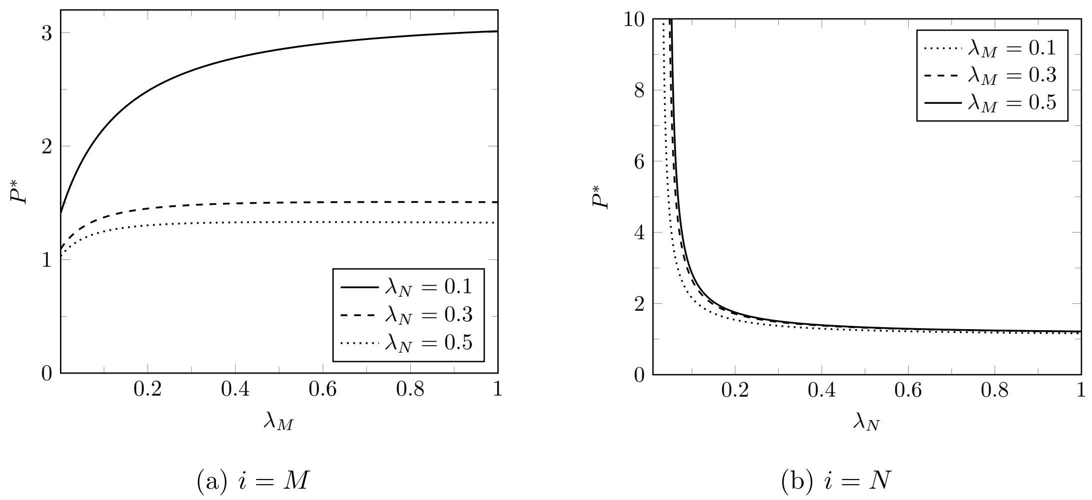
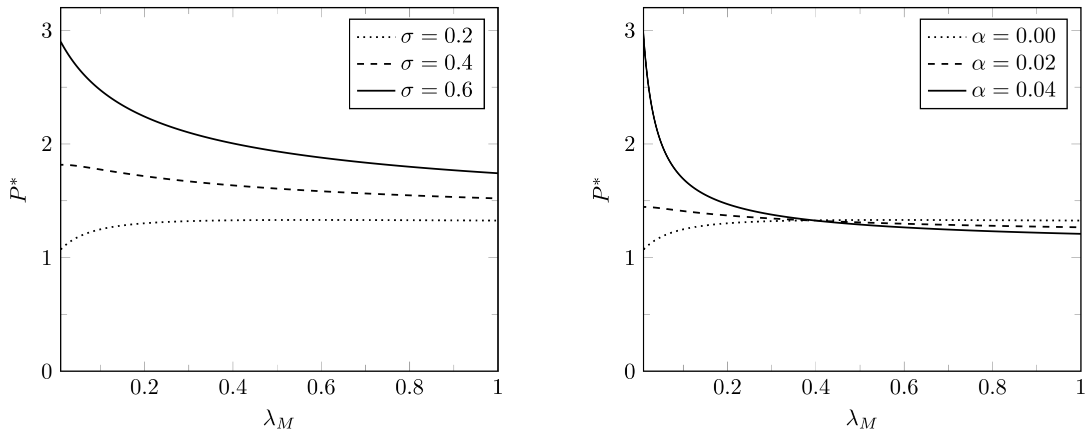


Figure 1: Switching threshold P^* displayed as a function of λ_i for different $\lambda_{j \neq i}$, with $X_M = 1, X_N = 2, \alpha = 0, \sigma = 0.2, S = 1, r = 0.05$; $i, j \in \{M, N\}$.

the switching threshold is more pronounced when the volatility of the potential profit (price uncertainty) is higher. While the negative effect for $\sigma = 0.2$ is barely visible and only present for relatively large λ_M, P^* appears to be strictly negative for $\sigma = 0.4$ and $\sigma = 0.6$.

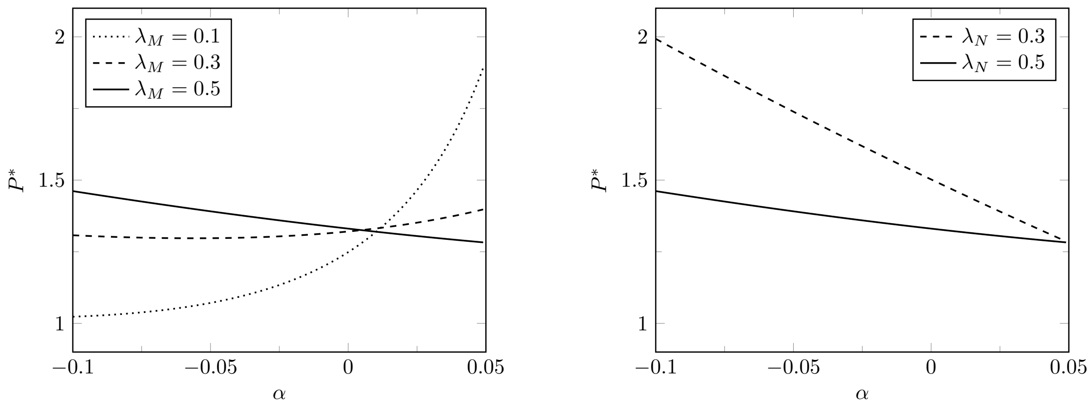
Furthermore, the non-monotonic effect of λ_M can also be observed in figures 3a and 4a, where P^* is displayed as a function of α and σ, respectively. In both cases, the switching threshold is increasing (decreasing) in λ_M for relatively small (large) parameter values. Figure 4a shows again what could already be observed in figure 2a, namely that the effect of technological uncertainty of the initial project on the switching threshold becomes more apparent when the volatility of the market price is relatively large. Furthermore, figure 3a shows the ambiguous effect of α on P^*.



(a) $\lambda_N = 0.5, \alpha = 0$

(b) $\lambda_N = 0.5, \sigma = 0.2$

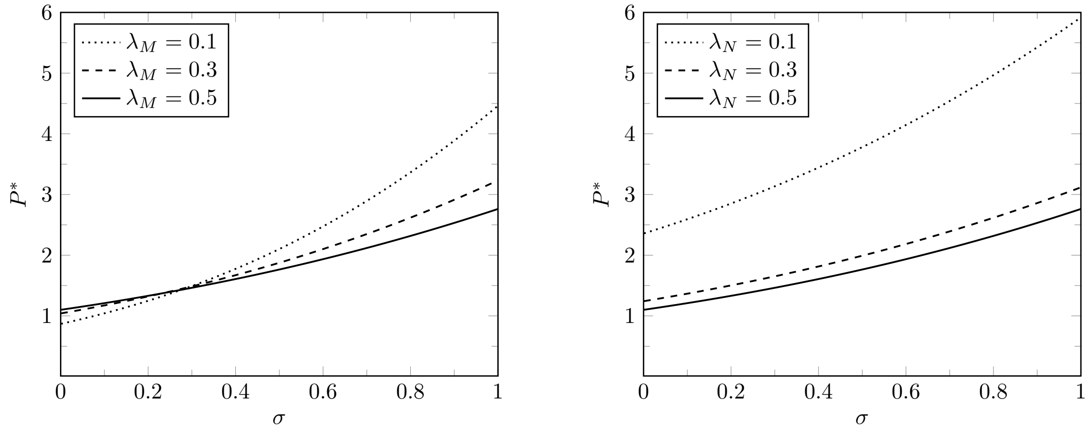
Figure 2: Ambiguous effect of λ_M on the switching threshold P^*. Parameter values as in figure 1. In the left figure σ is additionally varied, while in the right figure different levels of α are considered.



(a) $\lambda_N = 0.5$

(b) $\lambda_M = 0.5$

Figure 3: Switching threshold P^* displayed as a function of α with different levels of λ_M and λ_N, respectively. $X_M = 1, X_N = 2, \sigma = 0.2, S = 1, r = 0.05$. In the left figure, the ambiguous effect of α on P^* is visible.



(a) $\lambda_N = 0.5$

(b) $\lambda_M = 0.5$

Figure 4: Switching threshold P^* displayed as a function of σ with different levels of λ_M and λ_N, respectively. $X_M = 1, X_N = 2, \alpha = 0, S = 1, r = 0.05$.

3 Two Stochastic Processes

3.1 Model Extension

In this section, I extend the model from section 2 by allowing the potential payoffs from each of the two projects M and N to evolve according to a separate geometric Brownian motion. The extended model builds on the quasi-analytical approach of Adkins and Paxson (2011b) and is solved by the methodology applied in Støre et al. (2018).

To better distinguish between the two models discussed in this paper, let the potential payoffs now be denoted by π_M and π_N, respectively. In contrast to the model with one variable, the stochastic values here are not scaled by an additional factor X_i. Changes in π_M and π_N are described by

$$d\pi_i = \alpha_i \pi_i dt + \sigma_i \pi_i dz_i. \tag{8}$$

Note that in comparison to equation (1), here each process has its own parameter $\alpha_i < r$ and $\sigma_i > 0$, with dz_i being the increment of the respective Wiener process governing π_i, $i \in \{M, N\}$. Furthermore, $\mathbb{E}(dz_i) = 0$, $\mathbb{E}[(dz_i)^2] = dt$, and $\mathbb{E}(dz_M dz_N) = \rho_{MN} dt$, where $|\rho_{MN}| \leq 1$ describes the correlation between π_M and π_N. As before, let $\lambda_M > 0$ and $\lambda_N > 0$ denote the parameter describing the intensity of an event occurring in the respective project. The cost of switching from M to N is now given by $S \geq 0$. Other than in the model in section 2, there is no need for S to be strictly positive.

3.2 Analysis

In general, the DM's value functions for each project can be derived by the same approach as equations (4) and (5) in the previous section. The important difference is that the functions now depend on two stochastic variables instead of one. Thus, it does not suffice to determine a single threshold value similar to the one given by equation (6). Instead, we are looking for a *switching boundary* that represents an infinite number of combinations of values of the two stochastic variables that, when reached simultaneously, cause the DM to switch from M to N. Similar to Adkins and Paxson (2011b), these pairs of values are denoted by $\{\pi_M^*, \pi_N^*\}$. Thus, the switching boundary $\pi_N^*(\pi_M^*)$ maps a π_N^* on every π_M^*. In other words, holding the value of the potential payoff from the current project constant at some level π_M^* allows the DM to determine the value π_N^* which has to be reached to make switching optimal. The value function of the DM is given by

$$W(\pi_M, \pi_N) = \begin{cases} A\pi_M^\gamma \pi_N^\delta + \dfrac{\lambda_M \pi_M}{r + \lambda_M - \alpha_M}, & \pi_N < \pi_N^*(\pi_M^*), \\ \dfrac{\lambda_N \pi_N}{r + \lambda_N - \alpha_N} - S, & \pi_N \geq \pi_N^*(\pi_M^*). \end{cases} \tag{9}$$

A detailed derivation of (9) and of the parameters γ and δ can be found in Appendix B. The first line represents the expected discounted payoff before the switching boundary is reached, such that the DM remains at M, that is, $\pi_N < \pi_N^*(\pi_M^*)$. Again, the first term is the value of the option to switch to N. This option value now depends on two stochastic variables instead of one, which can be explained by the idea that an increasing π_M makes the current project more attractive, reducing the value of the option to switch, while increasing π_N makes the alternative project more attractive, which increases the value of the option. Thus, it has to hold that $\gamma < 0$ and $\delta > 0$. Using the value matching and the two smooth pasting conditions (one for each stochastic variable), we obtain the switching boundary,

$$\pi_N^*(\pi_M^*) = -\pi_M^* \left[\frac{\lambda_M \delta (r + \lambda_N - \alpha_N)}{\lambda_N \gamma (r + \lambda_M - \alpha_M)} \right]. \tag{10}$$

The switching costs are reflected in the boundary through the following expression:

$$\delta = 1 - \gamma \left[1 + S \left(\frac{r + \lambda_M - \alpha_M}{\pi_M^* \lambda_M} \right) \right] > 1.$$

Once again, a more detailed derivation can be found in Appendix B.

Even though π_M and π_N evolve according to separate GBMs, the drift and volatility parameters, as well as the likelihood that an event occurs, may be the same for both projects:

Proposition 4. *If the parameters of both projects are identical, that is,* $\alpha_i = \alpha, \sigma_i = \sigma$, *and* $\lambda_i = \lambda$, *it follows that, for* $\lambda < \infty$, $\pi_N^*(\pi_M^*) > \pi_M^*$.

In the case of identical parameter values, the DM should only switch to N if π_N is strictly larger than π_M. While this finding can, to some degree, be explained by the presence of switching costs, it can also be observed for $S = 0$. Another explanation is thus the value of the option to switch: As long as the DM pursues project M, she has the flexibility to abandon M in favor of N. After switching, however, a reversion of the decision is not possible. Thus, to compensate for the loss of flexibility that results from giving up the option value, the DM requires the potential payoff from the alternative project to be larger than the one of the current project, before she pays S and switches.

Additionally, it can be shown that, if both projects are certain to result in a payoff in $(t, t+dt)$, that is, $\lambda \to \infty$, the optimal switching boundary is equal to $\pi_N^*(\pi_M^*) = \pi_M^* + S$. This also holds if the parameters of the projects are not identical, i.e. for $\alpha_M \neq \alpha_N$ and (or) $\sigma_M \neq \sigma_N$. This simply means that, similar to the findings in section 2.2, when both projects immediately yield a profit, the DM chooses the larger net profit.

Finally, we can show that the result of Proposition 3 also carries over to the case of two stochastic variables. To make the proof of the following statement tractable, assume for simplicity that π_M and π_N are uncorrelated, i.e., $\rho_{MN} = 0$.

Proposition 5. *Assuming that* $\rho_{MN} = 0$, *the switching boundary* $\pi_M^*(\pi_N^*)$ *is*

(i.) monotonically decreasing in λ_N, *and*

(ii.) non-monotonic in λ_M.

Again, the first result states that increasing the probability that the DM will receive a payoff from project N in the near future lowers the switching boundary. Thus, switching is accelerated because π_N does not have to increase as much as before to trigger switching. Proposition 5(ii) corresponds to the insight from Proposition 3 that the DM might abandon the current project M in favor of the alternative project N if the probability of receiving the payoff in M increases.

4 Conclusion

This paper discusses the question of when to switch from one project to another when there are two forms of uncertainty. Technological uncertainty means that the decision maker (DM) does not know when the particular project will result in a profit, while price uncertainty captures the uncertainty about the actual amount of profit. An example, where this type of decision problem may arise, is a firm's R&D.

I show that it may be optimal to abandon an ongoing project sooner when the probability that the project results in a profit increases. While it is a well-established result in the real options literature that uncertainty about the size of a payoff increases the option value of waiting, uncertainty about the timing of a payoff can either accelerate or decelerate investment. This finding has significance for individual and firm decision making in the presence of uncertainty about the timing and size of a payoff. It implies that it may not always be optimal to choose a project solely on the basis of the probability of receiving a payoff.

A Proofs

A.1 Proof of Proposition 1

Differentiating P^* with respect to α gives

$$\frac{\partial P^*}{\partial \alpha} = -\frac{S}{(\beta-1)h(\cdot)}\left[\frac{\partial \beta}{\partial \alpha}\frac{1}{\beta-1} + \frac{\partial h(\cdot)}{\partial \alpha}\frac{\beta}{h(\cdot)}\right],$$

with

$$\frac{\partial \beta}{\partial \alpha} = -\frac{2\beta}{\sigma^2}\left[\left(1-\frac{2\alpha}{\sigma^2}\right)^2 + \frac{8(r+\lambda_M)}{\sigma^2}\right]^{-\frac{1}{2}} < 0.$$

The sign of $\partial h(\cdot)/\partial\alpha$ is ambiguous and varies with parameter values. Subsequently, the term in brackets can be positive or negative and thus P^* is non-monotonic in α.

Similarly, differentiating P^* with respect to σ gives

$$\frac{\partial P^*}{\partial \sigma} = -\frac{S}{h(\cdot)(\beta-1)^2}\frac{\partial \beta}{\partial \sigma}.$$

Taking the derivative of β with respect to σ yields

$$\frac{\partial \beta}{\partial \sigma} = \frac{4[\alpha\beta - (r+\lambda_M)]}{\sigma\sqrt{(\sigma^2-2\alpha)^2 + 8\sigma^2(r+\lambda_M)}}. \tag{A1}$$

Because of $\beta > 1$, the derivative is clearly negative for $\alpha \leq 0$. To see that it is also negative for $\alpha > 0$, first consider the limiting case $\alpha \to r$. Then, for $\partial\beta/\partial\sigma < 0$ to be true it has to hold that

$$\lim_{\alpha\to r}[\alpha\beta - (r+\lambda_M)] < 0 \qquad \Leftrightarrow \qquad \lim_{\alpha\to r}\beta < 1 + \frac{\lambda_M}{r}, \tag{A2}$$

where

$$\lim_{\alpha\to r}\beta = \frac{1}{2}\left[\left(1-\frac{2r}{\sigma^2}\right) + \sqrt{\left(1-\frac{2r}{\sigma^2}\right)^2 + \frac{8(r+\lambda_M)}{\sigma^2}}\right].$$

Using this expression, we see that (A2) is negative. Therefore, (A1) is negative for $\alpha \to r$, as well. Finally, we have to make sure that $\partial\beta/\partial\sigma$ can not be positive for any $\alpha \in (0, r)$. Because of $\partial\beta/\partial\sigma < 0$ for $\alpha \leq 0$ and for $\alpha \to r$, this is true if $[\alpha\beta-(r+\lambda_M)]$ is monotonic in α. Taking the derivative gives

$$\frac{\partial}{\partial \alpha}[\alpha\beta - (r + \lambda_M)] = \beta + \alpha\frac{\partial \beta}{\partial \alpha}.$$

By rearranging, we see that this derivative is strictly positive. Therefore, it can be concluded that $\partial P^*/\partial \sigma > 0$.

□

A.2 Proof of Proposition 2

Consider the limiting cases:

$$\lim_{\lambda_M \to \infty} P^* = \frac{S}{\frac{\lambda_N X_N}{r+\lambda_N-\alpha_N} - X_M}, \tag{A3}$$

and

$$\lim_{\lambda_N \to \infty} P^* = \left(\frac{\beta}{\beta - 1}\right)\left(\frac{S}{X_N - \frac{\lambda_M X_M}{r+\lambda_M-\alpha_M}}\right) \tag{A4}$$

Due to $h(\cdot) > 0$, it follows that the denominator is positive in both cases, ensuring that $P^* > 0$. From these relations we can conclude that it is not always rational for the DM to choose the certain project. If the DM always preferred to remain at M for $\lambda_M \to \infty$, it would have to be the case that $\lim_{\lambda_M \to \infty} P^* = \infty$. Then the DM would never change to N, no matter how large P is. However, this can only be the case if $S \to \infty$ or $h(\cdot) \to 0$. In all other cases, P^* converges, so it may be optimal to choose the project that does not immediately lead to a payoff.

A similar argument can be used with regard to the technological uncertainty of the alternative project. Here one might expect that $\lim_{\lambda_N \to \infty} P^* = S/X_N$, i.e. if the alternative activity immediately leads to a profit, the DM always switches when $P \geq S/X_N$ and realizes a profit of $X_N P - S \geq 0$. However, as can be easily verified, $\lim_{\lambda_N \to \infty} P^* > S/X_N$, so it might be better to stay with M even though $\lambda_N \to \infty$. (Because of $\beta > 1$ it suffices to show that the second fraction in (A4) is greater than S/X_N).

□

A.3 Proof of Proposition 3

The option value is given by

$$O(\lambda_M, \lambda_N) \equiv Bp^\beta = \left[\frac{h(\lambda_M, \lambda_N)}{\beta}\right]^\beta \left(\frac{\beta - 1}{S}\right)^{\beta-1} p^\beta.$$

Differentiating this expression with respect to λ_N yields

$$\frac{\partial O(\cdot)}{\partial \lambda_N} = \beta \frac{\partial h(\cdot)}{\partial \lambda_N} h(\cdot)^{\beta-1} \left(\frac{p}{\beta}\right)^\beta \left(\frac{\beta - 1}{S}\right)^{\beta-1}.$$

Because of $\partial h(\cdot)/\partial \lambda_N > 0$ for $\lambda_N < \infty$ (and $\lim_{\lambda_N \to \infty} \partial h(\cdot)/\partial \lambda_N = 0$), and since the remaining factors are positive, the value of the option to switch is (weakly) decreasing in λ_N.

Now consider the derivative of Bp^β with respect to λ_M. Since not only $h(\cdot)$ but also β depends on λ_M, first reformulate the option value as

$$O(\cdot) = \exp(\beta(\ln h(\cdot) + \ln p - \ln \beta) + (\beta - 1)(\ln(\beta - 1) - \ln S)).$$

The derivative with respect to λ_M is

$$\frac{\partial O(\cdot)}{\partial \lambda_M} = Bp^\beta \left[\frac{\partial}{\partial \lambda_M}\Big(\beta(\ln h(\cdot) + \ln p - \ln \beta) + (\beta - 1)(\ln(\beta - 1) - \ln S)\Big)\right],$$

which is equivalent to

$$\frac{\partial O(\cdot)}{\partial \lambda_M} = Bp^\beta \left[\frac{\partial \beta}{\partial \lambda_M} \ln \frac{(\beta - 1)h(\cdot)p}{\beta S} + \frac{\partial h(\cdot)}{\partial \lambda_M} \frac{\beta}{h(\cdot)}\right].$$

The first term in brackets is positive. However, due to $\partial h(\cdot)/\partial \lambda_M \leq 0$ the derivative might become negative if the absolute value of the second term is larger than that of the first one. Thus, it can be concluded that the value of the option to switch can either increase or decrease in λ_M.

Now turning to the impact of technological uncertainty on the switching threshold, first differentiating P^* with respect to λ_N gives

$$\frac{\partial P^*}{\partial \lambda_N} = -\frac{S}{h(\cdot)^2}\left(\frac{\beta}{\beta-1}\right)\frac{\partial h(\cdot)}{\partial \lambda_N}.$$

Every term behind the minus sign is non-negative, thus the derivative itself is (weakly) negative.

Similarly, the derivative of P^* with respect to λ_M is

$$\frac{\partial P^*}{\partial \lambda_M} = -\frac{S}{h(\cdot)(\beta-1)}\left[\frac{\partial \beta}{\partial \lambda_M}\frac{1}{\beta-1} + \frac{\partial h(\cdot)}{\partial \lambda_M}\frac{\beta}{h(\cdot)}\right].$$

We have $\partial\beta/\partial\lambda_M > 0$ for $\lambda_M < \infty$ and $\lim_{\lambda_M \to \infty} \partial\beta/\partial\lambda_M = 0$. Furthermore, $\partial h(\cdot)/\partial\lambda_M \leq 0$ (again, this derivative is strictly negative for $\lambda_M < \infty$). Subsequently, the term in brackets can be positive or negative. In fact, it can be shown that, for various parameter combinations it holds that $\partial P^*/\partial\lambda_M = 0$, that is,

$$\frac{\partial \beta}{\partial \lambda_M}\frac{1}{\beta-1} = -\frac{\partial h(\cdot)}{\partial \lambda_M}\frac{\beta}{h(\cdot)}.$$

Consequently, the sign of $\partial P^*/\partial\lambda_M$ can be ambiguous.

$\square$

A.4 Proof of Proposition 4

Inserting the parameter values as described in the proposition and rearranging gives

$$\pi_N^*(\pi_M^*) = \pi_M^*\left(1 - \frac{1}{\gamma}\right) + S\left(\frac{r+\lambda-\alpha}{\lambda}\right).$$

As long as $\lambda < \infty$ it follows that $\gamma > -\infty$ and thus $\pi_N^*(\pi_M^*) > \pi_M^*$.

$\square$

A.5 Proof of Proposition 5(i)

To prove Proposition 5(i), show that $\partial\pi_N^*(\pi_M^*)/\partial\lambda_N \leq 0$. Rearranging (10) gives

$$\pi_N^*(\pi_M^*) = \pi_M^* \left(\frac{\lambda_M}{r+\lambda_M-\alpha_M}\right)\left(H-\frac{1}{\gamma}\right)\left(\frac{r+\lambda_N-\alpha_N}{\lambda_N}\right), \tag{A5}$$

where

$$H \equiv 1 + S\left(\frac{r+\lambda_M-\alpha_M}{\pi_M^*\lambda_M}\right).$$

Only the last fraction on the RHS of (A5) depends on λ_N and the remainder is positive, therefore

$$\operatorname{sgn}\left[\frac{\partial\pi_N^*(\pi_M^*)}{\partial\lambda_N}\right] = \operatorname{sgn}\left[\frac{\partial}{\partial\lambda_N}\left(\frac{r+\lambda_N-\alpha_N}{\lambda_N}\right)\right].$$

Since

$$\frac{\partial}{\partial\lambda_N}\left(\frac{r+\lambda_N-\alpha_N}{\lambda_N}\right) = -\frac{r-\alpha_N}{\lambda_N^2} \leq 0,$$

the statement is true.

□

A.6 Proof of Proposition 5(ii)

Similar to the previous proof, show that the sign of $\partial\pi_N^*(\pi_M^*)/\partial\lambda_M$ is ambiguous. From (A5) it can be seen that

$$\operatorname{sgn}\left[\frac{\partial\pi_N^*(\pi_M^*)}{\partial\lambda_M}\right] = \operatorname{sgn}\left[\frac{\partial J(\lambda_M)}{\partial\lambda_M}\right], \tag{A6}$$

where

$$J(\lambda_M) \equiv \left(\frac{\lambda_M}{r+\lambda_M-\alpha_M}\right)\left(H(\lambda_M)-\frac{1}{\gamma(\lambda_M)}\right).$$

In the following, $H(\lambda_M)$ and $\gamma(\lambda_M)$ are abbreviated as H and γ, respectively. Differentiating $J(\lambda_M)$ results in

$$\frac{\partial J(\lambda_M)}{\partial\lambda_M} = \frac{r-\alpha_M}{(r+\lambda_M-\alpha_M)^2}\left(H-\frac{1}{\gamma}\right) + \left(\frac{\lambda_M}{r+\lambda_M-\alpha_M}\right)\left(H'+\frac{\gamma'}{\gamma^2}\right), \tag{A7}$$

with $H' \equiv \partial H / \partial \lambda_M \leq 0$, and

$$\gamma' \equiv \frac{\partial \gamma}{\partial \lambda_M} = \frac{1}{2g}\left(f'g - fg' - \frac{f(f'g - fg') - 4gg'(r + \lambda_M - \alpha_N)}{\sqrt{f^2 + 8g(r + \lambda_M - \alpha_N)}} \right) \tag{A8}$$

is the partial derivative of equation (B16) with respect to λ_M. From the definitions of f and g in Appendix B, it can be seen that, assuming $\rho_{MN} = 0$, $f' \equiv \partial f / \partial \lambda_M < 0, g > 0$, and $g' \equiv \partial g / \partial \lambda_M < 0$, while the sign of f is ambiguous. We can now use this information to show that $\gamma' < 0$. First, consider $f \leq 0$. In this case, it is straightforward to see that (A8) is negative. Now turn to the case where $f > 0$ and consider the three possible signs of the term $(f'g - fg')$, that is, $f'g \gtreqless fg'$. For $f'g = fg'$, we see directly that (A8) is negative. The same holds for $f'g < fg'$: Assuming that $\gamma' \geq 0$ and rearranging, we get

$$\sqrt{f^2 + 8g(r + \lambda_M - \alpha_N)} - f \leq -\frac{4gg'(r + \lambda_M - \alpha_N)}{f'g - fg'}.$$

Since the LHS of this equation is positive, while the RHS is negative, this is a contradiction. Finally, consider $f'g > fg'$. Again assuming that $\gamma' \geq 0$ and rearranging yields

$$f'g(f'g - fg') \geq 2g(g')^2(r + \lambda_M - \alpha_N). \tag{A9}$$

The LHS is negative while the RHS is positive, so this statement is a contradiction, as well. Thus, it can be concluded, that $\gamma' < 0$.

This result can be used to prove Proposition 5(ii). Substituting H and H' in (A7) and rearranging yields

$$\frac{\partial J(\lambda_M)}{\partial \lambda_M} = \frac{r - \alpha_M}{r + \lambda_M - \alpha_M}\left(\gamma^2 - \gamma\right) + \lambda_M \gamma'.$$

This term is positive (negative) for

$$\gamma' > (<) - \frac{r - \alpha_M}{\lambda_M(r + \lambda_M - \alpha_M)}\left(\gamma^2 - \gamma\right).$$

Since $\gamma' < 0$, both cases can be obtained by choosing appropriate parameter values. Because of (A6), it can be concluded that $\pi_N^*(\pi_M^*)$ is non-monotonic in λ_M.

□

B Derivation of the Value Function and Switching Boundary from Section 3

The derivation of equations, (9), (10), and the parameters γ and δ, as well as the constant A, is adapted from Adkins and Paxson (2011a) and Støre et al. (2018). Let

$$F(\pi_M, \pi_N) = \lambda_M \pi_M dt + \mathrm{e}^{-(r+\lambda_M)dt} \{F(\pi_M, \pi_N) + \mathbb{E}[dF(\pi_M, \pi_N)]\} \tag{B1}$$

be the value that the DM receives from the current project M. Expanding equation (B1) by applying Itô's Lemma in two dimensions results in the following partial differential equation (PDE):

$$\begin{gathered}\frac{1}{2}\sigma_M^2 \pi_M^2 F_{MM} + \frac{1}{2}\sigma_N^2 \pi_N^2 F_{NN} + \rho_{MN}\sigma_M\sigma_N\pi_M\pi_N F_{MN} + \alpha_M \pi_M F_M \\ +\alpha_N \pi_N F_N - (r+\lambda_M)F + \lambda_M \pi_M = 0.\end{gathered} \tag{B2}$$

Subscripts in this expression denote partial derivatives of $F(\cdot)$, that is, $F_M \equiv \partial F/\partial \pi_M$, $F_{MM} \equiv \partial^2 F/\partial \pi_M^2$, and so on. A possible solution to this PDE, as suggested by Adkins and Paxson (2011b), is

$$F(\pi_M, \pi_N) = A\pi_M^\gamma \pi_N^\delta + \frac{\lambda_M \pi_M}{r+\lambda_M-\alpha_M}. \tag{B3}$$

The parameters $\gamma < 0$ and $\delta > 0$ are determined below. By substituting $F(\cdot)$ and its derivatives into equation (B2), we obtain the following expression, similar to Adkins and Paxson (2011b):

$$\frac{1}{2}\gamma(\gamma-1)\sigma_M^2 + \frac{1}{2}\delta(\delta-1)\sigma_N^2 + \rho_{MN}\sigma_M\sigma_N\gamma\delta + \gamma\alpha_M + \delta\alpha_N - (r+\lambda_M) = 0. \tag{B4}$$

When determining the value $G(\pi_N)$ for project N, one has to consider that there is no option value after switching. Thus, the value of project N is simply the expected discounted profit minus switching cost, which is given by

$$G(\pi_N) = \mathbb{E}\left(\int_0^\infty \mathrm{e}^{-(r+\lambda_N)t}\lambda_N \pi_N dt\right) - S = \frac{\lambda_N \pi_N}{r+\lambda_N-\alpha_N} - S. \tag{B5}$$

Combining (B3) and (B5) yields $W(\pi_M, \pi_N)$ described by (9). The value matching condition is

$$A{\pi_M^*}^{\gamma}{\pi_N^*}^{\delta} + \frac{\lambda_M \pi_M^*}{r + \lambda_M - \alpha_M} = \frac{\lambda_N \pi_N^*}{r + \lambda_N - \alpha_N} - S. \tag{B6}$$

The two smooth pasting conditions are

$$A\gamma{\pi_M^*}^{\gamma-1}{\pi_N^*}^{\delta} + \frac{\lambda_M}{r + \lambda_M - \alpha_M} = 0, \tag{B7}$$

$$A\delta{\pi_M^*}^{\gamma}{\pi_N^*}^{\delta-1} = \frac{\lambda_N}{r + \lambda_N - \alpha_N}. \tag{B8}$$

Following the approach of Støre et al. (2018), combining (B7) and (B8) gives

$$\frac{\lambda_M \pi_M^*}{\gamma(r + \lambda_M - \alpha_M)} = -\frac{\lambda_N \pi_N^*}{\delta(r + \lambda_N - \alpha_N)}. \tag{B9}$$

Subsequently,

$$\pi_N^* = -\pi_M^* \frac{\lambda_M \delta(r + \lambda_N - \alpha_N)}{\lambda_N \gamma(r + \lambda_M - \alpha_M)}, \tag{B10}$$

and

$$A = -\frac{\lambda_M {\pi_M^*}^{1-\gamma}}{\gamma(r + \lambda_M - \alpha_M){\pi_N^*}^{\delta}}. \tag{B11}$$

Substitution of (B10) and (B11) into (B6) leads to the condensed equation

$$\pi_M^* \left(\frac{\lambda_M}{r + \lambda_M - \alpha_M}\right)\left(\frac{\gamma + \delta - 1}{\gamma}\right) + S = 0, \tag{B12}$$

which, by eliminating A, reduces the number of unknowns from four to three. This relation and the conditions described by (B4) and (B10) are sufficient to obtain π_N^*, γ and δ for any value $\pi_M^* > 0$. Continuing to follow Støre et al. (2018), the equations (B4), (B10), and (B12) can be solved analytically. First define

$$H(\pi_M^*) \equiv 1 + S\left(\frac{r + \lambda_M - \alpha_M}{\pi_M^* \lambda_M}\right). \tag{B13}$$

Rearranging (B13) and substituting into (B12) results in

$$\left(\frac{\delta - 1}{\gamma}\right) + H(\pi_M^*) = 0. \tag{B14}$$

Solving this expression for δ and substituting into (B4) yields

$$g(\pi_M^*)\gamma^2 - f(\pi_M^*)\gamma + 2(r + \lambda_M - \alpha_N) = 0, \tag{B15}$$

where

$$\begin{aligned} f(\pi_M^*) &= \sigma_M^2 - 2\alpha_M + H(\pi_M^*)(2\alpha_N + \sigma_N^2) - 2\rho_{MN}\sigma_M\sigma_N, \\ g(\pi_M^*) &= \sigma_M^2 + H(\pi_M^*)^2\sigma_N^2 - \rho_{MN}\sigma_M\sigma_N H(\pi_M^*). \end{aligned}$$

Solving the quadratic expression (B15) yields

$$\gamma(\pi_M^*) = \frac{f(\pi_M^*)}{2g(\pi_M^*)} - \sqrt{\left(\frac{f(\pi_M^*)}{2g(\pi_M^*)}\right)^2 + \frac{2(r + \lambda_M - \alpha_N)}{g(\pi_M^*)}}. \tag{B16}$$

This result is used to obtain the remaining unknowns in terms of π_M^*. Substituting (B16) in (B14) gives

$$\delta(\pi_M^*) = 1 - \gamma(\pi_M^*)H(\pi_M^*). \tag{B17}$$

From substituting (B16) in (B10) and (B11) it follows that

$$\pi_N^*(\pi_M^*) = -\pi_M^* \frac{\lambda_M \left[1 - \gamma(\pi_M^*)H(\pi_M^*)\right](r + \lambda_N - \alpha_N)}{\lambda_N \gamma(\pi_M^*)(r + \lambda_M - \alpha_M)}, \tag{B18}$$

$$A(\pi_M^*) = -\frac{\lambda_M {\pi_M^*}^{1-\gamma(\pi_M^*)}}{\gamma(\pi_M^*)(r + \lambda_M - \alpha_M){\pi_N^*}^{1-\gamma(\pi_M^*)H(\pi_M^*)}}. \tag{B19}$$

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