Pollution diffusion, limited production factors, non-monotonic growth and the emergence of spatially heterogeneous steady states.

Carmen Camacho — *PSE and CNRS* *
Alexandre Cornet — *Université Paris 1*†
Weihua Ruan — *Purdue University Northwest*‡

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Abstract

We develop a spatial growth model for an agricultural economy where pollution diffuses in the soil. At each location, the only production factor is fertile soil, which is at the same time naturally bounded by the amount of available land, and eventually exposed to pollution diffused from neighboring locations. We develop a novel technique to obtain the policy maker's optimal solution, which is analytical in the case of an homogeneous economy and covers all cases, for impatience rates ranging from almost zero to extremely high. When agents are very patient, the policy maker starts by making fully fertile all land before allowing for positive consumption. For slightly more impatient agents, the policy maker will allow for some consumption from the beginning, in the cleaning-up stage. With time, abatement stops, consumption raises and land becomes fully polluted in the long-term. We provide with some general results for the general spatially heterogeneous economy and its long-run, completing our study with some numerical exercises. Worth noting, simulations reveal that also in the non-homogeneous economy optimal consumption may transit through four different stages in time, responding to changes in fertile land and not necessarily in a smooth manner.

Keywords: Economic growth, Diffusion, Soil Pollution, Optimal Control, Limited resources.
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*Paris School of Economics and CNRS, France. E-mail: carmen.camacho@psemail.eu.
†Université Paris 1. E-mail: alexandre.cornet1@univ-paris1.fr
‡Purdue University Northwest, USA. E-mail: Wruan@pnw.edu

1 Introduction

The Status of the World's Soil Resources Report identified soil pollution as one of the main threats to all the services provided by soils ecosystems (FAO and ITPS, 2015). Soil pollution can be defined as "the presence of a chemical or substance out of place and/or present at a higher than normal concentration that has adverse effects on any non-targeted organism" (ITPS, 2015),[1] and, generally speaking, anthropogenic activities are the main source of soil pollution: wars, mining, former factory sites, accidental oil leakage, over use of fertilizers, etc. We will focus here on diffuse soil pollution, which is mainly originated from intensive agricultural practices, and in particular from the use of nitrogenous fertilizers. Although we now have a precise understanding on how nitrogenous fertilizers pollute soils and even drinkable water, their use in agriculture keeps increasing every year. In 1980, the worldwide consumption of nitrogenous fertilizer attained 60 million tonnes, and almost doubled in less than 40 years, reaching 110 millions tonnes in 2014 (FAO, 2015). In this paper, we develop a spatial growth model with bounded production factors to account for the diffusion of pollution in agricultural soils.

The economics of soil conservation has a long tradition of interdisciplinary thinking. In the early days of soil economics, Ciriacy-Wantrup (1968) borrowed from ecology the concept of damage thresholds to study irreversible damages due to agricultural production. A second wave of theoretical research on the economics of soil was developed in the 1980's to analyze the complex interactions between agricultural practices and soil fertility. The seminal paper of Pope et al. (1983) analyzes the relationships among soil loss, topsoil depth, net farm income, and technological progress and provide with the optimal policies in soil conservation which maximizes net farm income (see also Saliba, 1985). Noteworthy, among the papers of this second wave, there are region specific case-studies like Segarra et al. (1987) and Barbier (1990). More recently in the 2000's, the question of soil conservation shifted to the context of developing countries where it is thought that better soil management practices could lead to the highest potential gains.[2]

Here we study the economics of soil conservation using a spatial growth model where the unique production factor, fertile land, is naturally bounded and possibly subject to pollution, which is either locally generated on production site, or coming from neighboring locations. Because production is linear in fertile soil, our model is structurally close to Boucekkine et al. (2013) and Boucekkine et al. (2019). Besides, since locally generated pollution can diffuse across locations, we are also close to Camacho and Perez-Barahona (2015), Brock and Xepapadeas (2008) and (2010), Smith et al. (2009), or La Torre et al. (2015). However,

[1] "Soil pollution" shall not be used as a synonym for "soil contamination". For the ITPS a soil is contaminated when "the concentration of a chemical or substance is higher than would occur naturally but is not necessarily causing harm" (ITPS, 2015).

[2] See for instance Antle et al. (2006), Hagos et al. (2006), de Graaf et al. (2008), Stephens et al. (2012), Barrett et al. (2015), Bevis et al. (2017), or Berazneva et al. (2018).

the works closer to ours by their theme and objective are Augeraud-Véron et al. (2019) and (2021). In that series of papers, they study the optimal solution to a growth model with groundwater pollution due to agricultural activities. Yet, there is a key difference between our paper and other existing spatial models which study the diffusion of wealth or air pollution: we take into account that the production factor, here fertile soil, is naturally bounded. As we will show, boundedness changes the economy dynamics in a non-trivial unprecedented way. Indeed, the standard practice would suggest to impose a constraint on fertile land in the policy maker optimization problem, and then, obtain the set of necessary conditions. The boundedness constraint would have appeared as a Kuhn-Tucker constraint and one could have concluded that the optimal solution for the economy would coincide with the unbounded well-known solution until the boundary is reached. Once the boundary reached, the economy would remain there. However, we prove here that this reasoning would be just wrong.

In order to deal with this novel challenge, we develop a new approach to obtain the optimal solution, which builds on optimal control techniques. In particular, this new technique will allow us to provide with the exact analytical solution in a spatially homogeneous economy. As we will later explain, in the general case of a heterogeneous economy our technique allows to study some characteristics of the optimal dynamics and to describe the optimal long-run behavior. Boucekkine et al. (2013) and Boucekkine et al. (2019) provide with the exact solution in an unbounded economy but only in the case of sufficiently impatient agents. Our new technique does provide with the optimal solution for all (positive) levels of impatience, and reveals the existence of four very different optimal strategies depending on the relative value of the time discount rate. Of these, only the case of highly impatient agents has been described in previous works, here case 3.

- If the decision maker is very patient, then she will start by devoting all resources to abatement to make the land fully fertile at all locations. That is, agents will not consume anything during an initial stage. Only once full fertility is recovered, will the decision maker allow for positive consumption, but only at the lowest possible level that ensures full fertility thereafter. Indeed, land will be fully fertile from then on.

- If the decision maker is a bit more impatient, she would start by a positive and low level of consumption, gradually increasing it with time. As a result, land pollution will first decrease, possibly even going down to zero. As consumption increases, pollution rises, and as a result, production decreases. Eventually, there will be a moment when consumption will equalize production. After this moment all production will be devoted to consumption, leaving no resources for pollution abatement. Ultimately, all land will be fully polluted as time tends to infinity.

- An even more impatient decision maker will start by setting consumption on a high and decreasing trajectory. Although there is some abatement, land becomes more and more polluted over time.

- Finally, a very impatient decision maker will simply let consume all production at all times, making no abatement of pollution. Land approaches full pollution steadily from beginning at a higher speed.

Our paper also studies heterogeneous economies, providing with some general properties of the optimal solution, and a detailed analysis of the long-term. We present a general method to obtain the optimal solution to the problem of a policy maker aiming at maximizing overall welfare in an economy with bounded production factors and subject to pollution diffusion. Besides, our optimization technique also offers a strategy for numerical purposes, which we employ in the numerical exercises that complete the paper. Among our results, we prove that land will never be completely polluted everywhere if no location was fully polluted from the beginning when the model parameters are spatially heterogeneous. As in the homogeneous economy, the discount rate plays an important role. If agents are patient, then either all land will asymptotically become fully polluted or fully fertile. If agents are impatient, then an heterogeneous steady state can emerge. Here we use heterogeneous in a large sense: not only it will be heterogeneous in space as in Boucekkine et al. (2019), driven by the heterogeneity in the model parameters, but also heterogeneous regions can emerge in the long term with different consumption policies. One region will be fully fertile while the second region will be partially polluted. And this despite diffusion. Hence, our results indicate that fertile locations stop absorbing pollution from its neighbors once they reach full fertility. Without this positive absorbing externality, polluted locations may not be able to reach full fertility by themselves.

We conclude the paper running some numerical exercises that shed light in the remaining open questions in the heterogeneous economy. In order to keep comparison possible with previous works, we study an economy with a technological pole and a population concentration, which are not centered and which are not located in the same area. Our exercises show that as in the homogeneous economy, optimal consumption may optimally transit through four different stages. Transition among these stages is not smooth, and adjustment is instantaneous.

The rest of the paper is organized as follows. The model is presented in Section 2. Section 3 provides with the optimal analytical solution in the homogeneous economy. Section 4 studies the general heterogeneous economy. In Section 5 we run numerical exercises, which complete our understanding of the optimal dynamic properties of the model. Finally, Section 6 concludes. All proofs are gathered in the Appendix.

2 Soil pollution diffusion in a linear growth model

We consider a closed economy, where both land and households are distributed over the unit circle on the plane, $\mathcal{S} = \{(\sin\theta, \cos\theta) \in \mathbb{R}^2 : \theta \in [0, 2\pi]\}$. Each location $\theta \in [0, 2\pi]$ is populated by $N(\theta)$ individuals and is endowed with an amount of land $L(t, \theta)$.

Assumption 1. *The spatial distribution of land, $L(\cdot)$, is a constant.*

Assumption 1 implies that locations cannot increase their total land allocation. We assume that land is composed of both fertile and polluted soil, L_F and L_P. That is, $L = L_P(t, \theta) + L_F(t, \theta)$. All locations produce a unique agriculture good from the labour of local fertile soil, according to the following linear production function $Y(t, \theta) = A(\theta)L_F(t, \theta)$, where $A(\theta)$ is the local production technology at location θ. Hence, a partially polluted location can still produce.

The dynamics of soil pollution at one location is explained by three factors. First, pollution diffuses according to Fick's law: pollution diffuses from more polluted locations to less polluted locations and its flux is proportional to the pollution gradient. Based on this law the diffusion of soil pollution is captured by $D\frac{\partial^2 L_P}{\partial \theta^2}(t, \theta)$, where D is the diffusion coefficient. For simplicity reasons, D is assumed both constant in time and homogeneous in space.[3] Second, fertile soil deteriorates locally. Indeed, local production generates some pollutant, which transforms fertile soil into polluted soil. The local effect is measured as $\nu(\theta)Y(t, \theta)$, where $\nu(\theta)$ is the local sensitivity of fertile soil to pollution. $\nu(\theta)$ can be related to more or less pollution producing technologies, to different levels of biodiversity, etc. And third, soil pollution is reversible. Indeed, polluted soil can be depolluted independently of the local pollution level. This last assumption is made for simplicity, being aware that it would be more accurate to consider that abatement is feasible only if pollution remains below a certain threshold.[4] Letting $C(t, \theta)$ denote total consumption at location θ at time t, the amount invested in abatement at location θ is $Y(t, \theta) - C(t, \theta) \geq 0$. Let $\phi(\theta)$ be the local pollution abatement efficiency. Then putting together the three factors behind local pollution, the spatial dynamics of polluted soil can be described as

$$\begin{aligned}\frac{\partial L_P}{\partial t} &= D\frac{\partial^2 L_P}{\partial \theta^2} + \nu A L_F - \phi[AL_F - C] \\ &= D\frac{\partial^2 L_p}{\partial \theta^2} + A\left[\phi - \nu\right]\left[L_P - L\right] + C\phi\end{aligned}$$

Under Assumption 1, one can write that $\frac{\partial L_F}{\partial t}(t, \theta) = -\frac{\partial L_P}{\partial t}(t, \theta)$, and writing total con-

[3] We do not consider any seasonal effect nor heterogeneity in soil porosity, which would lead to study time and space dependent diffusion coefficients. These more general specifications for the diffusion coefficient could be analyzed following Boucekkine et al. (2020), but it is out of the scope of this paper.

[4] The critical zone for pollution reversibility has been widely studied, see for instance Dupouey et al. (2002), Chartier et al. (2006), Gao et al. (2011) and Le Kama et al. (2014). Technically speaking, introducing irreversible pollution damages would lead us to impose that above a local pollution threshold concentration, the first partial derivative of fertile soils with respect to time should be negative, that is $\frac{\partial L_F(t,\theta)}{\partial t} \leq 0$.

sumption $C(t,\theta)$ as the product of per capita consumption, $c(t,\theta)$, and the location's time-independent population $N(\theta)$, the evolution of polluted soil becomes

$$\begin{cases} \frac{\partial L_P}{\partial t} = D\frac{\partial^2 L_P}{\partial \theta^2} - A\left[\phi - \nu\right]\left[L - L_P\right] + cN\phi & \text{for } t > 0, \quad \theta \in (0, 2\pi)\,, \\ \begin{array}{l} L_P\left(t,0\right) = L_P\left(t,2\pi\right), \\ \frac{\partial L_P}{\partial \theta}\left(t,0\right) = \frac{\partial L_P}{\partial \theta}\left(t,2\pi\right) \end{array} & \text{for } t > 0, \\ L_P(0,\theta) = L_P\left(0,\theta\right), & \text{for } \theta \in (0,2\pi) \end{cases} \tag{1}$$

for all $t > 0$ and $\theta \in [0, 2\pi]$. From now on we will only work with L_P. Obviously, it is straightforward to obtain the corresponding results for L_F using $L_F = L - L_P$.

In this economy there exists a policy maker whose aim is to maximize overall welfare. Welfare is measured as the present value of the spatial aggregate of individuals' utility. Here, utility depends solely on consumption per capita, c, and it is measured by a constant inter-temporal elasticity of substitution function of parameter $\sigma \in \mathbb{R}$. Knowing that the policy maker discounts time at a constant rate ρ, her problem writes as

$$\max_c \int_0^\infty e^{-\rho t}\left[\int_0^{2\pi} \frac{c(t,\theta)^{1-\sigma}}{1-\sigma} N(\theta) d\theta\right] dt, \tag{2}$$

subject to (1) and

$$\begin{array}{ll} 0 \le L_P(t,\theta) \ \le \ L, & \\ 0 \le c(t,\theta) \le \frac{A(\theta)[L - L_P(t,\theta)]}{N(\theta)} & \text{if } L_P\left(t,\theta\right) > 0, \\ \frac{A(\theta)[\phi(\theta)-\nu(\theta)]L}{\phi(\theta)N(\theta)} \le c\left(t,\theta\right) \le \frac{A(\theta)L}{N(\theta)} & \text{if } L_P\left(t,\theta\right) = 0, \end{array} \tag{3}$$

for any $\theta \in [0, 2\pi]$, $t \geq 0$. The constraint on c when $L_P\left(t,\theta\right) > 0$ is based on $0 \le C \le AL_F$, and the constraint when $L_P\left(t,\theta\right) = 0$ is based on $C \le AL_F$ and also on the fact that local consumption is bounded by local production at all times and locations

$$\nu AL - \phi[AL - C] \geq 0,$$

which ensures that pollution is nonnegative if land is pollution free.

Next, let us introduce the linear operator $\mathcal{L}$ defined for all functions $u \in H^2\left(\mathcal{S}\right)$ by

$$\mathcal{L}\left[u\right]\left(\theta\right) := Du''(\theta) + A(\theta)[\phi(\theta) - \nu(\theta)]u(\theta).$$

Using $\mathcal{L}$, the first equation in (1) can be rewritten as

$$\frac{\partial L_P}{\partial t} = \mathcal{L}\left[L_P\right] + cN\phi - A\left[\phi - \nu\right]L. \tag{4}$$

We say a function φ defined on $\mathcal{S}$, regular and non-identically zero, is an eigenfunction of $\mathcal{L}$, with associated eigenvalue $\lambda \in \mathbb{R}$ if $\mathcal{L}[\varphi] = \lambda\varphi$. Coddington and Levinson (1955) prove that there exists a countable set of eigenvalues $\{\lambda_n\}_{n\geq 0}$, which can be ordered as a decreasing sequence. It can be proven that the first eigenvalue of $\mathcal{L}$, λ_0, is an eigenvalue with multiplicity 1, all other eigenvalues have either multiplicity 1 or 2.[5] Besides, φ_0, the eigenfunction associated to λ_0, is strictly positive on the unit circle and one can set $\int_0^{2\pi} \varphi_0^2(\theta)d\theta = 1$. Moreover, the eigenfunctions of $\mathcal{L}$ form an orthogonal basis in $L^2(\mathcal{S})$ and the sequence of eigenvalues tends to $-\infty$, while the eigenfunction φ_n associated to λ_n has n zeros in $[0, 2\pi]$.[6]

Our model is structurally close to Boucekkine et al. (2013) and Boucekkine et al. (2018), since production is linear in fertile soil. As in previous works, all land is labored at all locations, independently of the distribution of population. Nevertheless, there is a key difference with all previous work in the literature: the production factor, here fertile soil, is naturally bounded. As we will show in the following sections, dealing with boundedness goes beyond imposing a condition on the production factor. First, it requires non standard changes in the method to obtain the set of optimal conditions. Second, we will prove how boundedness affects the definition, dynamics and long-term of the optimal solution, and this for all positive ρ.

3 Optimal solution in a homogeneous economy

Let us start our analysis studying an homogeneous economy, where A, ϕ, ν, and N are constants. We provide with the optimal trajectories for consumption and fertile land distinguising four different cases depending on the magnitude of the discount rate. Although previous works were limited to the case of moderately high discount rates, we are able here to cover all cases developing a new methodology.

The eigenvalues of $\mathcal{L}$ in the homogeneous economy are

$$\lambda_n = A(\phi - \nu) - Dn^2 \qquad \text{for } n = 0, 1, 2, \ldots$$

with the corresponding normalized eigenfunctions

$$\varphi_0(\theta) = \frac{1}{\sqrt{2\pi}}, \quad \varphi_n^1(\theta) = \frac{\cos n\theta}{\sqrt{\pi}}, \qquad \varphi_n^2(\theta) = \frac{\sin n\theta}{\sqrt{\pi}} \qquad \text{for } n \geq 1.$$

We divide this section in four parts depending on the relative size of the discount rate with respect to the first eigenvalue, $\lambda_0 = A(\phi - \nu)$.[7] Let us define constant g:

$$g = \frac{\lambda_0 - \rho}{\sigma}.$$

[5] The multiplicity of an eigenvalue is the number of times it appears in the sequence $\{\lambda_n\}_{n\geq 0}$.

[6] For further details see Coddinton and Levinson (1955) or Brown et al. (2013).

[7] We have relegated to the Appendix most technical material as well as all the article's proofs.

As it will be seen, g plays an important role when the discount rate is small because in all cases where there is growth or a monotonic trajectory, g appears as the growth rate of consumption.

Very small time discount. We say that the time discount rate is small if $\rho < \lambda_0$, and it is very small if $\rho < (1-\sigma)\lambda_0$ in the case where $0 < \sigma < 1$.

Proposition 1. *Let Assumption 1 hold. Suppose that $0 < \sigma < 1$ and that A, ϕ, ν, N and $L_P(0,\theta) \equiv L_P^0$ are positive constants. If*

$$0 < \rho < (1-\sigma)\lambda_0, \tag{5}$$

then the decision maker would optimally devote all production to abate land pollution until all land is pollution free. Once all land is fully fertile, optimal consumption will be positive, but will be kept at the minimum level that keeps all land pollution free. The optimal trajectories for consumption and polluted land are given by

$$c(t, L_P) = \frac{1}{N\phi}\begin{cases} 0 & \text{for } 0 \le t \le T_0, \\ \lambda_0 L & \text{for } t > T_0, \end{cases} \tag{6}$$

and

$$L_P(t) = \begin{cases} L - e^{\lambda_0 t}\left[L - L_P^0\right] & \text{for } 0 \le t \le T_0, \\ 0 & \text{for } t > T_0, \end{cases} \tag{7}$$

respectively, where

$$T_0 = \frac{1}{\lambda_0}\ln\frac{L}{L - L_P^0}.$$

Proposition 1 shows that an altruistic decision maker will always consume at the minimum level. Indeed, because the future is important to the policy maker, she optimally decides to start by setting consumption to zero during the period $[0, T_0]$. Polluted land decreases at a constant rate λ_0. Only once all land has become fully fertile (at time T_0), will the policy maker allow for consumption making sure that all land remains unpolluted forever. This is a rare result in the literature. As one may expect, Proposition 1 also proves that the higher the level of initial pollution, the longer it will take to clean all pollution from the soil and to start consuming.

Moderately small time discount. If the decision maker is less altruistic, even if slightly, then her optimal decision entails full degradation of all land in the long term:

Proposition 2. *Let Assumption 1 hold. Suppose that A, ϕ, ν, N and $L_P(0,\theta) \equiv L_P^0$ are positive constants and that*

$$(1-\sigma)\lambda_0 < \rho < \lambda_0 \le D. \tag{8}$$

Then, initially, optimal consumption follows $\hat{a}e^{gt}$ with some positive constant $\hat{a}$, which is below initial production, $A\phi\left(L-L_P^0\right)$, until the moment $\tilde{T}>0$ at which consumption equals production, i.e., when

$$\hat{a}e^{g\tilde{T}}=A\phi\left[L-L_P\left(\tilde{T}\right)\right],$$

After $\tilde{T}$, all production will be consumed. Specifically, there exist three moments in time T, $\hat{T}$ and $\tilde{T}$ that satisfy

$$\hat{a}\left[e^{\lambda_0 T}-e^{gT}\right]=\left(\lambda_0-g\right)\left[\left(L-L_P^0\right)e^{\lambda_0 T}-L\right]. \tag{9}$$

and

$$\hat{T}=\frac{1}{g}\ln\frac{\lambda_0 L}{\hat{a}}, \qquad \tilde{T}=\hat{T}+\frac{1}{\lambda_0-g}\ln\left[\frac{\lambda_0\left(g+A\nu\right)}{A\phi g}\right] \tag{10}$$

such that optimal consumption and polluted land are

$$c\left(t,L_P\right)=\frac{1}{N\phi}\begin{cases}\hat{a}e^{gt} & \text{for } 0\leq t<T,\\ \lambda_0 L & \text{for } T\leq t\leq\hat{T},\\ \hat{a}e^{gt} & \text{for } \hat{T}<t\leq\tilde{T},\\ A\phi\left[L-L_P(\tilde{T})\right]e^{-A\nu\left(t-\tilde{T}\right)} & \text{for } t>\tilde{T},\end{cases} \tag{11}$$

and

$$L_P\left(t\right)=\begin{cases}L+e^{\lambda_0 t}\left[L_P^0-L+\frac{\hat{a}}{\lambda_0-g}\right]-\frac{\hat{a}e^{gt}}{\lambda_0-g} & \text{for } 0<t<T,\\ 0 & \text{for } T\leq t<\hat{T},\\ L\left[1-e^{-\lambda_0\left(\hat{T}-t\right)}\right]-\frac{\hat{a}}{\lambda_0-g}\left[e^{gt}-e^{-\left(\lambda_0-g\right)\hat{T}+\lambda_0 t}\right] & \text{for } \hat{T}\leq t<\tilde{T},\\ L-\left[L-L_P\left(\tilde{T}\right)\right]e^{-A\nu\left(t-\tilde{T}\right)} & \text{for } t\geq\tilde{T}.\end{cases} \tag{12}$$

Proposition 2 describes the optimal dynamics of consumption as evolving through four different stages. Contrary to Proposition 1, consumption is always positive. Although consumption starts below production, it steadily increases with time. During this initial stage L_P first decreases until reaching zero at time T. Land remains fully fertile from T to $\hat{T}$. From $\hat{T}$ land starts getting polluted. As a result, production begins to fall. Eventually, at time $\tilde{T}$ consumption catches up with production, remaining equal to production thereafter. Thus there will be no abatement from then on. Consequently,

$$\lim_{t\to\infty}L_P\left(t\right)=L.$$

That is, all land will become fully polluted as time approaches infinity.

Propositions 1 and 2 together prove the existence of a threshold for the discount rate, $\left(1-\sigma\right)\lambda_0=A\left(\phi-\nu\right)\left(1-\sigma\right)$, beyond which all land will be optimally polluted in the long term. This level for the discount rate is actually very low and could not have been revealed using existing methods since the literature has focused on the case of relatively large discount rates. This result is of importance because without it, one could have concluded that all optimal solutions would imply full pollution at all locations.

Moderately large time discount. If the discount rate is higher than λ_0 but not too high, the following Proposition 3 proves that consumption decreases with time at a constant rate g from the initial date. In parallel, polluted land increases with time at the same rate until all land is fully polluted at all locations.

Proposition 3. *Let Assumption 1 hold. Suppose A, ϕ, ν, N and $L_P(0,\theta)$ are positive constants, and $D > \lambda_0$. Also suppose that*

$$\lambda_0 \leq \rho \leq \lambda_0 + \sigma A\nu \equiv A\left[\phi - (1-\sigma)\nu\right] \tag{13}$$

Then

$$\begin{aligned} c(t, L_P) &= \tfrac{\lambda_0 - g}{N\phi}\left[L - L_P^0\right]e^{gt} \\ L_P(t) &= L - \left[L - L_P^0\right]e^{gt} \end{aligned} \qquad \text{for all } t \geq 0. \tag{14}$$

Proposition 3 indicates that in the case where ρ satisfies (13), the decision maker strikes a balance between consumption and abatement, so that both consumption and land fertility diminish as $t \to \infty$ at the same exponential rate. This third case is the only of the four cases that has been studied in previous works.

Very large time discount. Finally, in the case where

$$\rho > \lambda_0 + \sigma A\nu, \tag{15}$$

we prove

Proposition 4. *Let Assumption 1 hold. Suppose A, ϕ, ν, N and $L_P(0,\theta)$ are positive constants, and $D > \lambda_0$. Also suppose that (15) holds. Then*

$$c(t, L_P) = \frac{A}{N}\left[L - L_P^0\right]e^{-A\nu t}, \quad L_P(t) = L - \left[L - L_P^0\right]e^{-A\nu t} \qquad \text{for } t > 0.$$

Clearly, $\lim_{t\to\infty} L_P(t) = L$.

Hence, consumption will optimally decrease with time from the initial date at a rate $-A\nu$ while polluted land increases at the very same rate. Worth noting, and putting together results obtained in the last two propositions, we have proven that the economy follows a Balanced Growth Path from $t = 0$ whenever $\rho > \lambda_0$.

We can now put our results in perspective to underline the importance of considering the boundedness of fertile land. In the case where the land is unlimited, both Boucekkine et al.

(2013) and Boucekkine et al. (2019) prove that when $\rho > \lambda_0(1-\sigma)$, the optimal trajectory for consumption grows at a constant and positive rate, g (which coincides with our g, previously defined). Here in our case, if the policy maker fails to acknowledge that land is limited, it would have implemented the optimal solution prescribed by previous works adopting an increasing consumption trajectory, which not only is not optimal and would provide lower welfare, but also would precipitate the economy towards its end even faster than necessary.

4 Optimal policy in the non homogeneous economy

This section provides with the optimal solution to the policy maker problem presented in Section 2 in the general, non homogeneous case. We first characterize the optimal policy of the decision maker in Subsection 4.1 describe our (novel) methodology, and then present dynamics and long-term of polluted soil in Subsection 4.2.

Subsection 4.1 presents all technical details starting with the presentation of the necessary optimal conditions. Although it gets technical, all these details are necessary in order to understand the paper's methodological innovation (that were already used to prove the results obtained in the simpler framework of the previous subsection).

4.1 Optimal policy

Using Ekeland's variational principle, one can show that there exists a $H^2(\mathcal{S})$ function $\psi(t,\theta)$ that satisfies the following necessary conditions at all locations

$$\begin{cases} \psi_t + \mathcal{L}[\psi] - \rho\psi = 0, & \text{for } t>0, \quad \theta \in (0,2\pi), \\ \psi(t,0) = \psi(t,2\pi), & \\ \psi_\theta(t,0) = \psi_\theta(t,2\pi), & \text{for } t > 0, \\ \lim_{t\to\infty} e^{-\rho t}\psi(t,\theta) = 0, & \text{for } \theta \in [0,2\pi]. \end{cases} \tag{16}$$

The way that the optimal control c is related to ψ is not as straightforward as in the case where the state variable, that is, the production factor, is unbounded. In previous works like Boucekkine et al. (2009) or Camacho and Pérez-Barahona (2016), there exists a direct relationship between them, namely $c = [\phi\psi]^{-1/\sigma}$. Then, in that case, using this equality one can recover a system of parabolic differential equations in c and the diffusive production factor. However, here the relationship between ϕ and c is described by $c = [\phi\psi]^{-1/\sigma}$, but only under certain circumstances that depend on the level of pollution in the soil. Again, neglecting the boundedness of the production function would have led to non-optimal outcomes. Let us see this in detail. When $L_P > 0$ then $c = [\phi\psi]^{-1/\sigma}$ if

$$0 < (\phi\psi)^{-1/\sigma} \le \frac{A[L-L_P]}{N}, \tag{17}$$

and when $L_P = 0$ if

$$\frac{A\left[\phi-\nu\right]L}{N\phi} \leq (\phi\psi)^{-1/\sigma} \leq \frac{AL}{N}. \tag{18}$$

Let us define the following function ξ that will help us present our results in a concise manner. Given two numbers $a < b$, we define the function

$$\xi\left(x;a,b\right) = \begin{cases} a & \text{if } x \leq a, \\ x & \text{if } a < x < b, \\ b & \text{if } x \geq b. \end{cases}$$

Therefore, if $L_P > 0$, optimal consumption is

$$c\left(t,\theta,L_P\right) = \begin{cases} \frac{A(\theta)}{N(\theta)}\left[L - L_P\left(t,\theta\right)\right] & \text{if } \psi\left(t,\theta\right) \leq 0, \\ \min\left\{\left[\phi\left(\theta\right)\psi\left(t,\theta\right)\right]^{-1/\sigma}, \frac{A(\theta)}{N(\theta)}\left[L - L_P\left(t,\theta\right)\right]\right\} & \text{if } \psi\left(t,\theta\right) > 0. \end{cases} \tag{19}$$

And if $L_P = 0$, we have that

$$c\left(t,\theta,0\right) = \begin{cases} \frac{A(\theta)}{N(\theta)}L & \text{if } \psi\left(t,\theta\right) \leq 0, \\ \xi\left(\left[\phi\left(\theta\right)\psi\left(t,\theta\right)\right]^{-1/\sigma}; \frac{A(\theta)\left[\phi(\theta)-\nu(\theta)\right]}{N(\theta)\phi(\theta)}L, \frac{A(\theta)}{N(\theta)}L\right) & \text{if } \psi\left(t,\theta\right) > 0. \end{cases} \tag{20}$$

Clearly, being functions defined on $\mathcal{S}$, c satisfies the periodic boundary conditions $c\left(t,0\right) = c\left(t,2\pi\right)$.

The co-state $\psi\left(t,\theta\right)$. Note that since the differential equation in (16) is linear, we can use eigenfunction expansion in the form

$$\psi\left(t,\theta\right) = \sum_{n=1}^{\infty} a_n\left(t\right)\varphi_n\left(\theta\right).$$

Substituting into the first equation of (16), we obtain

$$\sum_{n=1}^{\infty}\left[a_n'\left(t\right) + \left(\lambda_n - \rho\right)a_n\left(t\right)\right]\varphi_n\left(\theta\right) = 0.$$

Since the functions φ_n are not identically 0, it must necessarily be that the sequence of unknown functions a_n verifies

$$a_n\left(t\right) = a_n\left(0\right)e^{(\rho-\lambda_n)t}, \qquad \text{for all} \qquad n = 0, 1, 2, \ldots$$

As mentioned above, the eigenvalues λ_n of the operator $\mathcal{L}$ tend to $-\infty$ as $n \to \infty$. By the last equation in (16), only the positive eigenvalues can be included in the series. Let λ_k be the least positive eigenvalue of $\mathcal{L}$. Slightly abusing of the notation, we let a_n denote $a_n\left(0\right)$.

Then we have proven that the co-state variable ψ can be expressed as

$$\psi(t,\theta) = \sum_{\lambda_n>0} a_n e^{(\rho-\lambda_n)t} \varphi_n(\theta), \tag{21}$$

which is completely determined by the coefficients $\{a_n\}$. It follows that (16) is reduced to a system of these coefficients.

Let $\psi_n(\theta)$ be the sum of all the terms in (21) corresponding to the eigenvalue λ_n. We can rewrite (21) as

$$\psi(t,\theta) = \sum_{n=0}^{k} \psi_n(\theta) e^{(\rho-\lambda_n)t}. \tag{22}$$

Determining a_n. Note that each set of the coefficients $\{a_n\}$ determines $c(t,\theta,L_P)$ by (19) and (20) and also $L_P(t,\theta)$ by (1). As a result, once we substitute for c in the overall welfare function

$$J(c(\cdot)) = \int_0^{\infty} e^{-\rho t} \left[\int_0^{2\pi} \frac{c(t,\theta,L_P(t,\theta))^{1-\sigma}}{1-\sigma} N(\theta)\, d\theta \right] dt, \tag{23}$$

then J will also be determined by the sequence $\{a_n\}$. Following this line of thought, and since the aim of the policy maker is to maximize $J(c)$, we choose $\{a_n\}$ as the set that maximizes $J(c)$. Worth to note, this criterion has actually led to all analytic results in the homogeneous case, in particular those in Propositions 3 and 4, and the numerical examples in Section 5.

4.2 Dynamics of pollution. The general case.

We provide results on the optimal dynamics of polluted soil. Our first proposition shows that if there exists at least a single location which is not fully polluted at $t = 0$, not all land is initially polluted, then the economy will never be completely polluted (in finite time). That is, a single location could sustain the entire economy. Besides, we also prove that if diffusion is relatively low, meaning here that $D < \lambda_0$, then there will always be a share of polluted land and we can provide with a lower bound for polluted land location-wise.

Let λ_k be the least positive eigenvalue of $\mathcal{L}$ such that $\psi_k(\theta)$ is not identically zero. In this case the co-state variable $\psi(t,\theta)$ is dominated by $e^{(\rho-\lambda_k)t}\psi_k(\theta)$ for t sufficiently large, determining the long-run of the optimal solution. Also, let us divide space in three regions according to whether ψ_k is positive or negative

$$\begin{aligned} S_k^+ &= \{\theta \in [0,2\pi] : \psi_k(\theta) > 0\}, \\ S_k^- &= \{\theta \in [0,2\pi] : \psi_k(\theta) < 0\}, \\ S_k^0 &= \{\theta \in [0,2\pi] : \psi_k(\theta) = 0\}. \end{aligned}$$

Furthermore, let us introduce a new variable $\hat{L}_P(\theta)$ defined in S_k^- and satisfying

$$\begin{aligned} -D\hat{L}_P''(\theta) &= A(\theta)\nu(\theta)\left[L - \hat{L}_P(\theta)\right] && \text{for } \theta \in S_k^-, \\ \hat{L}_P(\theta) &= 0 && \text{if } \theta \in S_k^0. \end{aligned} \tag{24}$$

$\hat{L}_P$ is a time independent variable and it will turn out to be a lower bound for polluted land in S_k^-.

In the case where $k = 0$, then $S_k^0 = \varnothing$ meaning that ψ_k is either strictly positive or strictly negative for all θ. In this case of $k = 0$, we define $\hat{L}_P(\theta) = L$. Hence it can be seen that $\hat{L}_P(\theta)$ exists for any $k \geq 0$ and it satisfies $0 < \hat{L}_P(\theta) \leq L$ in S_k^-.

We first prove

Proposition 5. *Let Assumption 1 hold.*

1. *If $L_P(0,\theta) < L$ for at least one $\theta \in [0, 2\pi]$, then $L_P(t,\theta) < L$ for all $t > 0$, $\theta \in [0, 2\pi]$.*

2. *For any $\theta \in S_k^-$ there is $T_\theta > 0$ such that $L_P(t,\theta) > 0$ for any $t > T_\theta$. Furthermore,*

$$\liminf_{t\to\infty} L_P(t,\theta) \geq \hat{L}_P(\theta) \qquad \text{for } \theta \in S_k^-. \tag{25}$$

Proposition 5 proves that land will never be completely polluted unless all land is polluted from the beginning. If there exists a single location at which there is some fertile land, then there will be some fertile land at all locations at all times. Additionally, since S_k^- is nonempty if $k \geq 1$, and this inequality is equivalent to $D < \lambda_0$, Proposition 5 indicates that if the diffusion rate is low, then there will always be a region in which pollution will never be fully abated. In this case, $\hat{L}_P$ is a lower bound for L_P in then long run.

In order to study the long term of the optimal solution, we distinguish two cases again depending on whether the discount rate is relatively small or large as in Section 3. The threshold value here for ρ will be λ_k, $\mathcal{L}$'s smallest positive eigenvalue.

If the policy maker weights the future generations enough, here meaning if $\rho < \lambda_k$, then land either becomes pollution free in finite time, or it becomes completely polluted as time progresses.

Proposition 6 (Small discount rate $\rho < \lambda_k$)**.** *Suppose $\rho < \lambda_k$. Then either there is $T > 0$ such that $L_P(t,\theta) = 0$ for $t > T$ and $\theta \in [0, 2\pi]$, or $\lim_{t\to\infty} L_P(t,\theta) = L$.*

Proposition 6 proves that there are only two possible long-term outcomes for pollution, full or zero fertility, independently of the distributions of production and abatement technology, sensitivity, etc.

The following proposition shows that a spatially heterogeneous steady state can emerge when the time discount rate is large. Here heterogeneous means that regions can emerge, juxtaposed and alternated: fertile regions next to and surrounded by polluted regions. In fully fertile regions, consumption reaches its maximum at all locations. Obviously, within each of these regions, whether polluted or fully fertile, the optimal solution is spatially heterogeneous and it depends on production technology, abatement technology and population.

Proposition 7 (Large discount rate $\rho > \lambda_k$)**.** *Suppose $\rho > \lambda_k$ and $k \geq 1$. Then there exists a steady state*

$$\bar{L}_P(\theta) = \begin{cases} 0 & \text{if } \theta \in S_k^+, \\ \hat{L}_P(\theta) & \text{if } \theta \in S_k^-, \end{cases} \tag{26}$$

where $\hat{L}(\theta)$ is the solution of (24). That is, there are two nonempty distinct regions, $\mathcal{S} = \bar{\mathcal{S}} \cup \underline{\mathcal{S}}$ with

$$\bar{\mathcal{S}} = \left\{ (\cos\theta, \sin\theta) : \theta \in S_k^+ \right\}, \qquad \underline{\mathcal{S}} = \mathcal{S} \diagdown \bar{\mathcal{S}}.$$

Fertile soil equals total land in $\overline{\mathcal{S}}$, and in $\underline{\mathcal{S}}$ the soil is polluted to various extent. Furthermore, optimal consumption is given by

$$\bar{c}(\theta) = \frac{1}{\phi(\theta) N(\theta)} \begin{cases} A(\theta)\left[\phi(\theta) - \nu(\theta)\right] L & \text{in } \overline{\mathcal{S}}, \\ A(\theta)\phi(\theta)\left[L - \hat{L}(\theta)\right] & \text{in } \underline{\mathcal{S}}. \end{cases}.$$

Proposition 7 presents one of the main findings of the paper: a polluted region $\underline{\mathcal{S}}$ may fall in an environmental poverty trap and never catch up with the fertile region. This result may come as a surprise since it shows the existence of an heterogeneous steady state even in an homogeneous economy with diffusion. Let us describe one of the possibly many genesis of an heterogeneous steady state: when the most advanced locations become fully fertile, they stop absorbing others' pollution. From that moment onwards still polluted locations face an increased challenge since they will have to satisfy consumers' demand with their inferior land endowment but without the external help of the leader region. The polluted region may fall into an environmental poverty trap if its production or abatement technologies are not advanced enough to cope with their own pollution plus, probably, some additional pollution from their neighbors.

5 Numerical experiments

We develop next some numerical exercises to illustrate our results and shed light on some of the remaining open questions. In particular, we illustrate how taking into account that the economy operates with bounded production factors does change the optimal dynamics of the economy.

Overall, we find that our results crucially depend not only on the time discount rate as shown in Propositions **??**-4 in Section 3, but also on household's preferences. For this reason, we will show results for two values for σ, namely $\sigma = 5$ as in Boucekkine et al. (2019), and $\sigma = 0.5$ to contrast results. The calibration for the remaining of the parameters is shown in Table 1. From ethical grounds, the choice of the time discount rate has always been a delicate issue in the literature (see Stern, 2007, or Fleurbaey and Zuber, 2012, among others). We have additionally shown throughout the paper that the choice of the time discount rate not only determines quantitatively optimal consumption, but most importantly, it leads and dictates the economy dynamics. Here, ρ is set to 3% as in Boucekkine et al. (2019) and Lopez (2008).[8] The amount of total available land is 1 at all locations, and we assume in the benchmark scenario that everywhere in space there is a 10% of total land which is polluted. Finally, we set the diffusion parameter D to 0.1.

ϕ	Abatement efficiency	0.3
ν	Pollution sensitivity	0.2
D	Diffusion parameter	0.1
L	Maximum Fertile land	1
$L_P(0)$	Initial polluted land	0.1

Table 1: General calibration

This section serves various purposes. From a pure technical point of view, it allows us to reveal the qualitative properties of the optimal dynamics of our model in the general case, when the model parameters are not homogeneous in space. And second, it underlines the importance of acknowledging the boundedness of the production factor by running a similar numerical exercise to those in Boucekkine et al. (2019), allowing for a comparison of both the dynamics and the long-run distribution. Like them, we shall study the roles of technological poles and population concentration. Finally, we will close the section with a brief exercise that underlines the powerful role of the initial condition. As in standard models, the impact of the initial condition vanishes with time. However, the initial condition impacts transitional dynamics changing the behavior of the policy maker qualitatively. Note that our optimization strategy allows to compute the unique optimal solution and to conclude without any doubt about the transitional dynamics of our economy.

Let us assume there is a technological pole around location $\theta = 5\pi/4$ and a population concentration around $\theta = 3\pi/4$. In particular, we assume the following functional forms:

$$A(\theta) = \begin{cases} 2, \text{ if } \theta \in [0, \pi) \cup (3\pi/2, 2\pi], \\ 2.2 - 7.2/\pi^2(\theta - \pi)^2, \text{ elsewhere} \end{cases}$$

[8] We have run complementary exercises varying the value of the time discount factor within a feasible range of values. Results confirm the robustness of the exercises presented here.

and

$$N(\theta) = \begin{cases} 1, & \text{if } \theta \in [0, \pi/2) \cup (\pi, 2\pi], \\ 1.1 - 3.6/\pi^2(\theta - \pi)^2, & \text{elsewhere.} \end{cases}$$

Figure 1 shows our results for polluted land and consumption per capita when $\sigma = 5$ and 0.5 respectively. When $\sigma = 5$ (top line in Figure 1), polluted land increases monotonically at all locations, being the most polluted area the technologically advanced region. In the long-term pollution diffuses almost everywhere, and only the most distant locations will keep some fertile land. Note that all economy's consumption will be produced at those locations, which are also highly polluted. There is a remarkable change when $\sigma = 0.5$ (bottom line in Figure 1). Our results reveal that when the economy is not spatially homogeneous, the optimal trajectory for consumption can undergo non-monotone dynamics, which are actually similar to those described in Proposition 1. Indeed, the policy maker decides to start by abating intensively to make all land fertile at all locations. Once full fertility achieved, consumption increases and becomes constant at each location for some time. Across space however, we observe some heterogeneity between the locations. More populated locations will consume less while technologically advanced locations will consume more. Again, after some time-60 time units here-consumption will fall to a minimum again at all locations until $t = 100$. At $t = 100$, consumption jumps and reaches its maximum at all locations. From there onwards, it will decrease monotonically until reaching zero while land becomes fully polluted. In this last time interval, consumption becomes again spatially heterogeneous following technology and population as before.

As mentioned, let us conclude this section underlining the role of the initial condition. Here, we assume that $\sigma = 0.5$. Keeping the same spatial distribution for technology and population, let us assume that sixty percent of all available land is initially polluted in all locations. Note that the level of initial pollution does change optimal dynamics. Here, we do not observe consumption going through four different stages, but rather two, and dynamics do not seem directly linked to the previous exercises. The optimal solution starts at a low level for consumption, which increases with time while being able to improve fertility at all locations. Low fertility pushes the policy maker to start by abating enough pollution so as to ensure future consumption. However, after $t = 60$ polluted land reaches a minimum accompanied by a maximum in consumption. After that point, polluted land will increase until reaching a maximum at all locations. As in the first example, at the steady state for polluted land, only peripherical locations will still be able to produce. At the same time, consumption will steadily decrease. The steady state for consumption is also spatially heterogeneous and locations with more fertile land will consume more.

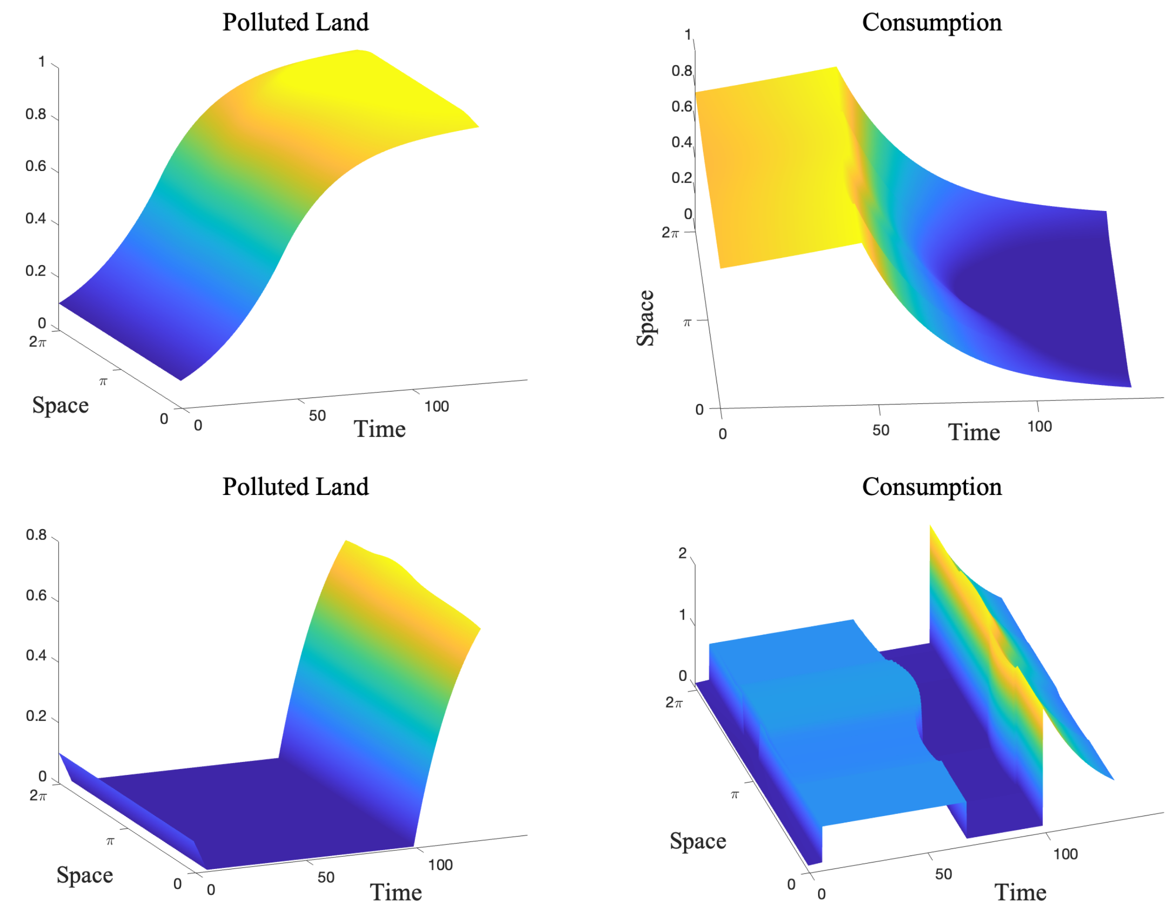

Figure 1: L_P and c under a central technological pole. Top: $\sigma = 5$. Bottom: $\sigma = 0.05$

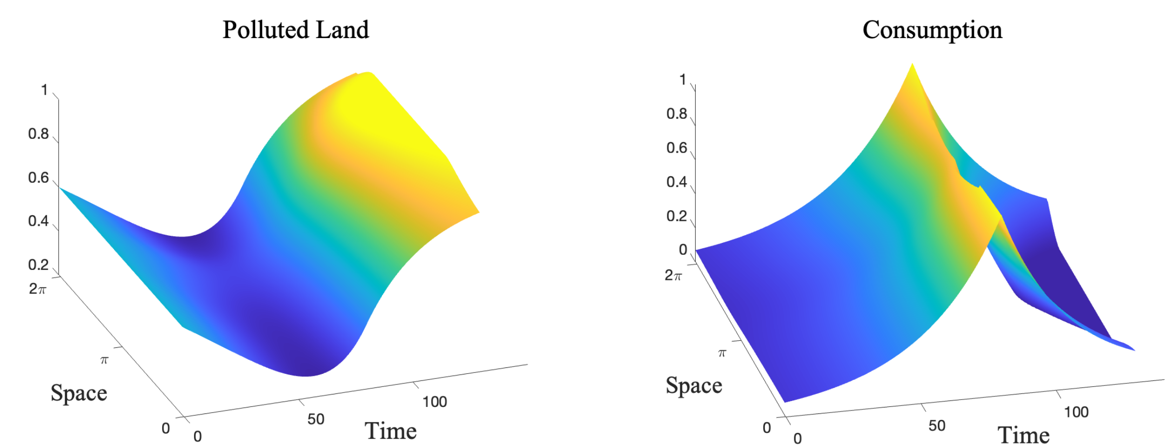

Figure 2: L_P and c when $L_P(0) = 0.6$.

6 Conclusion

This paper has developed a spatial growth model accounting for the diffusion of pollution in soils, and it has provided with the optimal trajectory for polluted soil and consumption. We have faced a major and new challenge: the production factor, fertile soil, is naturally bounded by the amount of available land at each location.

After obtaining the optimal solution for the co-state variable associated to the policy maker problem, we define and work on its eigenvalue expansion. In the case of a homogeneous economy, we provide with the unique optimal solution to the policy maker problem. When households are sufficiently altruistic, the optimal solution dictates to begin by cleaning all polluted land before start consuming at the highest rate. If on the contrary, households are strongly biased for the present, then they will optimally adopt high standards of consumption at the beginning, followed by a sustained decreasing trajectory. For intermediate values of altruism, consumption follows a non monotonic trajectory. Our technique also addresses the non homogeneous economy, and shows that the economy will never become fully polluted if there exists at time 0 a single location with some fertile land. We also show that heterogeneous steady states can emerge even from an initially homogeneous economy, characterizing the role of pollution diffusion. Fully fertile regions can co-exist forever next to partially polluted areas, which would remain in a pollution poverty trap.

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Appendices

A. Optimal consumption

Although our problem is very close to Boucekkine et al. (2013) and Boucekkine et al. (2019), the production factor is here bounded. Most importantly, the policy maker needs to take into account the eventuality of reaching one of the boundaries. We resort to an optimal control approach to sort this problem. Using Ekeland's variational principle, we first obtain the set of necessary conditions for our problem.

In order to apply Ekeland's variational principle, let us write the value function $V(\cdot)$ associated to problem (4)

$$V(c, L_P, \psi) = \int_0^\infty \int_0^{2\pi} e^{-\rho t} \frac{c(t,\theta)^{1-\sigma}}{1-\sigma} N(\theta) d\theta dt + \int_0^\infty \int_0^{2\pi} e^{-\rho t} \psi(t,\theta) \left[\frac{\partial L_P}{\partial t}(t,\theta) - \mathcal{L}\left[L_P\right](t,\theta) - c(t,\theta) N(\theta) \phi(\theta) + A(\theta) \left[\phi(\theta) - \nu(\theta)\right] L \right] d\theta dt$$

Then, assuming there exists an optimal solution and that any solution to our problem can be written as a deviation from the optimal solution we obtain

$$\begin{aligned} c(t,\theta) &= c^*(t,\theta)+\epsilon\mathcal{C}(t,\theta), \\ L_P(t,\theta) &= L_P^*(t,\theta)+\epsilon l_P(t,\theta) \end{aligned}$$

where $\mathcal{C}$ satisfies

$$\begin{array}{ll} \mathcal{C}(t,\theta)=0 & \text{if } L_P^*(t,\theta)=L, \\ \mathcal{C}(t,\theta)\le 0 & \text{if } c^*(t,\theta)=A(\theta)\left[L-L_P(t,\theta)\right]/N(\theta), \\ \mathcal{C}(t,\theta)\ge 0 & \text{if } L_P^*(t,\theta)=0 \text{ and } c^*(t,\theta)=A(\theta)\left[\phi(\theta)-\nu(\theta)\right]L/\left(\phi(\theta)N(\theta)\right), \end{array} \tag{27}$$

and l_P satisfies

$$l_P(t,\theta)\ge 0 \quad \text{if } L_P^*(t,\theta)=0, \qquad l_P(t,\theta)\le 0 \qquad \text{if } L_P^*(t,\theta)=L. \tag{28}$$

V becomes then a function of ϵ. At the optimal point (c^*, L_P^*), $\left.\frac{\partial V(\epsilon)}{\partial\epsilon}\right|_{\varepsilon=0}=0$. Assuming that ψ is a $H^1(\mathcal{S})$ function, then it satisfies

$$\psi(t,0)=\psi(t,2\pi), \qquad \psi_\theta(t,0)=\psi_\theta(t,2\pi).$$

Recall that L_P being a $H^2(\mathcal{S})$ function also satisfies these periodic conditions. After applying integration by parts, we can compute $\frac{\partial V(\epsilon)}{\partial\epsilon}$ as

$$\begin{aligned} \frac{\partial V(\epsilon)}{\partial\epsilon} &= \int_0^\infty\int_0^{2\pi} e^{-\rho t}\left[c(t,\theta)^{-\sigma}-\psi(t,\theta)\phi(\theta)\right]N(\theta)\mathcal{C}(t,\theta)d\theta dt \\ &\quad+\int_0^\infty\int_0^{2\pi} e^{-\rho t}\left[-\psi_t(t,\theta)-\mathcal{L}[\psi](t,\theta)+\rho\psi\right]l_P(t,\theta)\,d\theta dt. \end{aligned}$$

with the terminal condition

$$\lim_{t\to\infty} e^{-\rho t}\psi(t,\theta)=0 \qquad \text{for } \theta\in[0,2\pi].$$

Since $\mathcal{C}$ and l_p are arbitrary, subject only to (27) and (28) respectively, we obtain the necessary conditions for $\psi(t,\theta)$ given by (16) where c and ψ are related by

$$\psi(t,\theta)=\frac{c(t,\theta)^{-\sigma}}{\phi(\theta)}$$

if (17) holds when $L_P>0$ and if (18) holds when $L_P=0$. Note that $cN\le Y=A[L-L_P]$ in any circumstances. This leads to (19) and (20).

B. Proof of Propositions 1 and 2.

There are three cases depending on the size of $\hat{a}$ is: (1) $\hat{a}=0$, (2) $0<\hat{a}<A\phi\left[L-L_P^0\right]$, and (3) $\hat{a}\ge A\phi\left[L-L_P^0\right]$.

We first consider Cases (1) and (2). In these two cases

$$c(t, L_P) = \frac{\hat{a}}{N\phi} e^{gt}$$

and $L_P(t)$ satisfies

$$L_P' = -\lambda_0 [L - L_P] + \hat{a} e^{gt} \tag{29}$$

for $t < \min\left\{T, \tilde{T}\right\}$, where T is the time when $L_P(T) = 0$, and $\tilde{T}$ is the time when

$$\hat{a} e^{g\tilde{T}} = A\phi \left[L - L_P\left(\tilde{T}\right)\right] \tag{30}$$

The solution of (29) takes the form

$$L_P(t) = L + e^{\lambda_0 t}\left[L_P^0 - L + \frac{\hat{a}}{\lambda_0 - g}\right] - \frac{\hat{a}}{\lambda_0 - g} e^{gt} \qquad \text{for } 0 \le t < T \tag{31}$$

Thus, T satisfies (9).

In Case (1),

$$c(t, \theta) = 0, \qquad L_P(t) = L - \left[L - L_P^0\right] e^{\lambda_0 t} \qquad \text{for } 0 \le t \le T_0$$

where

$$T_0 = \frac{1}{\lambda_0} \ln \frac{L}{L - L_P^0}.$$

For $t > T_0$, by (20),

$$c(t, \theta) = \frac{\lambda_0 L}{N\phi}, \qquad L_P(t) = 0.$$

Hence, by (23), one can compute

$$J(c; T_0) = \frac{2\pi N}{(1-\sigma)\rho} \left(\frac{\lambda_0}{N\phi}\right)^{1-\sigma} \left(L - L_P^0\right)^{\rho/\lambda_0} L^{1-\sigma-\rho/\lambda_0}. \tag{32}$$

In Case (2), either $T < \tilde{T}$ or $T \ge \tilde{T}$. In the former case, $\hat{a}$ and T are related by (9). We can represent $\hat{a}$ as a function of T by

$$\hat{a}(T) = \frac{(\lambda_0 - g)\left[\left(L - L_P^0\right) e^{\lambda_0 T} - L\right]}{e^{\lambda_0 T} - e^{gT}}. \tag{33}$$

Thus,

$$c(t, \theta, L_P) = \frac{\hat{a}(T)}{N\phi} e^{gt} \qquad \text{for } 0 \le t < T.$$

It can be shown that $T > T_0$. Also, by differentiation,

$$\hat{a}'(T) = \frac{\lambda_0 - g}{e^{\lambda_0 T} - e^{gT}} \left\{\lambda_0 \left(L - L_P^0\right) e^{\lambda_0 T} - \frac{\lambda_0 e^{\lambda_0 T} - g e^{gT}}{e^{\lambda_0 T} - e^{gT}} \left[\left(L - L_P^0\right) e^{\lambda_0 T} - L\right]\right\}.$$

In particular,

$$\hat{a}'(T_0) = \frac{\lambda_0(\lambda_0 - g)}{e^{\lambda_0 T_0} - e^{gT_0}} \left(L - L_P^0\right) e^{\lambda_0 T_0} > 0$$

and $\hat{a}'(T) < 0$ for large T. Thus, there is a T_1 such that

$$\hat{a}(T_1) = \max_{T \geq T_0} \hat{a}(T).$$

It follows that $T_0 < T \leq T_1$.

Since $g > 0$ and $\hat{a}(T) > 0$, there is a $\hat{T}$ that satisfies

$$\hat{a}(T)e^{g\hat{T}} = \lambda_0 L. \tag{34}$$

It can be seen that

$$\hat{T} = \frac{1}{g} \ln \frac{\lambda_0 L}{\hat{a}(T)}, \qquad \text{and} \ \ c(t, \theta, L_P) = \frac{\lambda_0 L}{N\phi} \text{ for } T \leq t < \hat{T}. \tag{35}$$

It can be shown that $T < \hat{T}$. Note that

$$\hat{a}e^{gt} > \lambda_0 L \qquad \text{for } t > \hat{T},$$

it follows that $c(t, \theta, L_P) = \hat{a}(T)e^{gt}$ for $t > \hat{T}$ and is near $\hat{T}$. Therefore, $L_P(t) > 0$ for such t and it satisfies (29) together with the initial condition $L_P(\hat{T}) = 0$. We can rewrite (34) as

$$L_P(t) = L\left[1 - e^{-\lambda_0\left(\hat{T}-t\right)}\right] - \frac{\hat{a}(T)}{\lambda_0 - g}\left[e^{gt} - e^{-(\lambda_0-g)\hat{T}+\lambda_0 t}\right]. \tag{36}$$

This solution is valid until t reaches $\tilde{T}$, the moment at which

$$\hat{a}(T)e^{g\tilde{T}} = A\phi\left[L - L_P(\tilde{T})\right].$$

Using (36), we can compute $L_P(\tilde{T})$. Substituting it into the equality above

$$\hat{a}e^{g\tilde{T}} = A\phi\left\{Le^{\lambda_0\left(\tilde{T}-\hat{T}\right)} + \frac{\hat{a}}{\lambda_0 - g}\left[e^{g\tilde{T}} - e^{-(\lambda_0-g)\hat{T}+\lambda_0\tilde{T}}\right]\right\}.$$

Multiplying both sides by $e^{-g\left(\tilde{T}-\hat{T}\right)}$ and using (34), we obtain

$$\lambda_0 L = A\phi\left\{Le^{(\lambda_0-g)\left(\tilde{T}-\hat{T}\right)} + \frac{\lambda_0 L}{\lambda_0 - g}\left[1 - e^{(\lambda_0-g)\left(\tilde{T}-\hat{T}\right)}\right]\right\}.$$

This leads to

$$\frac{g}{\lambda_0 - g}e^{(\lambda_0-g)\left(\tilde{T}-\hat{T}\right)} = \frac{\lambda_0}{A\phi} - \frac{\lambda_0}{\lambda_0 - g}.$$

As a result, we obtain $\tilde{T}$ as a function of $\hat{T}$:

$$\tilde{T} = \hat{T} + \frac{1}{\lambda_0 - g} \ln\left[\frac{\lambda_0 (g + A\nu)}{A\phi g}\right] \tag{37}$$

and

$$c(t, \theta, L_P) = \frac{\hat{a}(T)}{N\phi}e^{gt} \qquad \text{for } \hat{T} < t \leq \tilde{T}.$$

For $t > \tilde{T}$, L_P satisfies

$$L_P'(t) = -\lambda_0 [L - L_P(t)] + A\phi [L - L_P(t)] = A\nu [L - L_P(t)]$$

for $t > \tilde{T}$ and the initial condition

$$L_P(\tilde{T}) = L - \frac{\hat{a}(T)}{A\phi} e^{g\tilde{T}}.$$

As a result,

$$c(t, \theta, L_P) = \frac{A}{N}\left[L - L_P\left(\tilde{T}\right)\right] = \frac{\hat{a}(T)}{N\phi} e^{g\tilde{T} - A\nu(t-\tilde{T})}.$$

Finally, using (23) we can compute overall welfare as

$$\begin{aligned} J(c;T) &= \frac{2\pi N}{(1-\sigma)(N\phi)^{1-\sigma}} \left\{ \int_0^T \hat{a}(T)^{1-\sigma} e^{-(\rho - g(1-\sigma))t} dt + \int_T^{\hat{T}} (\lambda_0 L)^{1-\sigma} e^{-\rho t} dt \right. \\ &\quad \left. + \int_{\hat{T}}^{\tilde{T}} \hat{a}(T)^{1-\sigma} e^{-(\rho - g(1-\sigma))t} dt + \int_{\tilde{T}}^{\infty} \hat{a}(T)^{1-\sigma} e^{-(1-\sigma)(A\nu - g)\tilde{T}} e^{-(\rho + A\nu(1-\sigma))t} dt \right\} \\ &= \frac{2\pi N}{(1-\sigma)(N\phi)^{1-\sigma}} \left\{ \frac{\hat{a}(T)^{1-\sigma}}{\rho - g(1-\sigma)} \left[1 - e^{-(\rho - g(1-\sigma))T} + e^{-(\rho - g(1-\sigma))\hat{T}} - e^{-(\rho - g(1-\sigma))\tilde{T}}\right] \right. \\ &\quad \left. + \frac{(\lambda_0 L)^{1-\sigma}}{\rho}\left[e^{-\rho T} - e^{-\rho \hat{T}}\right] + \frac{\hat{a}(T)^{1-\sigma} e^{-[\rho - (1-\sigma)g]\tilde{T}}}{\rho + A\nu(1-\sigma)} \right\}. \end{aligned}$$

We show that $J(c, T)$ is decreasing if $T_0 < T \leq T_1$. By differentiation,

$$\begin{aligned} \frac{dJ(c;T)}{dT} &= \frac{2\pi N}{(1-\sigma)(N\phi)^{1-\sigma}} \left\{ \hat{a}(T)^{1-\sigma} \left[e^{-(\rho + g(1-\sigma))T} - e^{-(\rho + g(1-\sigma))\hat{T}} \hat{T}'(T) \right.\right. \\ &\quad \left. + e^{-(\rho + g(1-\sigma))\tilde{T}} \tilde{T}'(T) \right] - (\lambda_0 L)^{1-\sigma} \left[e^{-\rho T} - e^{-\rho \hat{T}} \hat{T}'(T) \right] \\ &\quad - \frac{\hat{a}(T)^{1-\sigma}(\rho - (1-\sigma)g)}{\rho + A\nu(1-\sigma)} e^{-[\rho - (1-\sigma)g]\tilde{T}} \tilde{T}'(T) \\ &\quad + \frac{(1-\sigma)\hat{a}(T)^{-\sigma}\hat{a}'(T)}{\rho - g(1-\sigma)} \left[1 - e^{-(\rho - g(1-\sigma))T} + e^{-(\rho - g(1-\sigma))\hat{T}} \right. \\ &\quad \left.\left. + e^{-(\rho - g(1-\sigma))\tilde{T}} \right] + \frac{(1-\sigma)\hat{a}(T)^{-\sigma}\hat{a}'(T) e^{-[\rho - (1-\sigma)g]\tilde{T}}}{\rho + A\nu(1-\sigma)} \right\}. \end{aligned}$$

From (33) we find

$$\frac{\hat{a}'(T)}{\hat{a}(T)} = \frac{\lambda_0 \left(L - L_P^0\right) e^{\lambda_0 T}}{\left(L - L_P^0\right) e^{\lambda_0 T} - L} - \frac{\lambda_0 e^{\lambda_0 T} - g e^{gT}}{e^{\lambda_0 T} - e^{gT}}$$

and from (35) and (37),

$$\hat{T}'(T) = \tilde{T}'(T) = \frac{\hat{a}'(T)}{g\hat{a}(T)}.$$

It follows that,

$$\begin{aligned}
\frac{dJ(c;T)}{dT} &= \frac{2\pi N\hat{a}(T)^{-\sigma}}{(1-\sigma)(N\phi)^{1-\sigma}g}\Big\{\Big[g\hat{a}(T)e^{-(\rho-g(1-\sigma))T}+e^{-(\rho-g(1-\sigma))\hat{T}}\hat{a}'(T) \\
&\quad - e^{-(\rho-g(1-\sigma))\tilde{T}}\hat{a}'(T)\Big] - (\lambda_0 L)^{1-\sigma}e^{-\rho\hat{T}}\hat{a}'(T)\,\hat{a}(T)^{-1+\sigma} \\
&\quad + \frac{\rho-(1-\sigma)g}{\rho+A\nu(1-\sigma)}e^{-[\rho-(1-\sigma)g]\tilde{T}}\hat{a}'(T) \\
&\quad + \frac{(1-\sigma)g\hat{a}'(T)}{\rho-g(1-\sigma)}\Big[1-e^{-(\rho-g(1-\sigma))T}+e^{-(\rho-g(1-\sigma))\hat{T}} \\
&\quad + e^{-(\rho-g(1-\sigma))\tilde{T}}\Big] + \frac{(1-\sigma)ge^{-[\rho-(1-\sigma)g]\tilde{T}}\hat{a}'(T)}{\rho+A\nu(1-\sigma)}\Big\} \\
&\quad - \frac{2\pi N}{(1-\sigma)(N\phi)^{1-\sigma}}(\lambda_0 L)^{1-\sigma}e^{-\rho T}.
\end{aligned}$$

Note that $g>0$ and as $T\to T_0$, $\hat{a}(T)\to 0$, $\hat{T}\to\infty$, $\tilde{T}\to\infty$, and $\hat{a}'(T)\to\hat{a}(T_0)>0$. Thus, $dJ/dT\to+\infty$ or $-\infty$. The sign is the opposite of the sign of the quantity

$$\begin{aligned}
&-\frac{(1-\sigma)g}{\rho-g(1-\sigma)}\Big[1-e^{-(\rho-g(1-\sigma))T}\Big]-\frac{\rho}{\rho-(1-\sigma)g}e^{-(\rho-g(1-\sigma))\hat{T}} \\
&+\left[\frac{\rho}{\rho-g(1-\sigma)}-\frac{\rho}{\rho+A\nu(1-\sigma)}\right]e^{-(\rho-(1-\sigma)g)\tilde{T}}+\left(\frac{\lambda_0 L}{\hat{a}(T)}\right)^{1-\sigma}e^{-\rho\hat{T}}.
\end{aligned}$$

By (34), the last term is $e^{-(\rho-g(1-\sigma))\hat{T}}$. Also, by (37),

$$e^{-(\rho-(1-\sigma)g)\tilde{T}}=e^{-(\rho-(1-\sigma)g)\hat{T}}\frac{A\phi g}{\lambda_0(g+A\nu)}$$

Thus, the above quantity can be combined to

$$-\frac{(1-\sigma)g}{\rho-g(1-\sigma)}\Big[1-e^{-(\rho-g(1-\sigma))T}\Big]+\frac{\sigma g(1-\sigma)A\nu}{\lambda_0(1+A\nu(1-\sigma))}e^{-(\rho-(1-\sigma)g)\hat{T}}.$$

If (5) holds, then

$$\rho-(1-\sigma)g=\frac{1}{\sigma}[\rho-(1-\sigma)\lambda_0]<0.$$

Hence, the above quantity is positive if T is sufficiently close to T_0. As a result, $dJ/dT<0$. On the other hand, if (8) holds. Then $\rho-(1-\sigma)\lambda_0>0$. The above quantity is negative for T close to T_0. Hence $dJ/dT>0$. In the formal case, $J(c;T)$ is the maximum at T_0, but in the latter case $J(c;T)$ is the maximum for some $T>T_0$.

It remains to consider Case (3). In this case $\hat{a}e^{gt}\geq A\phi[L-L_P(t)]$ for all $t\geq 0$. Thus

$$c(t,L_P)=\frac{A}{N}[L-L_P(t)]$$

and L_P satisfies the equation

$$L_P'=A\nu[L-L_P].$$

The differential equation has the solution

$$L_P(t) = L - \left[L - L_P^0\right] e^{-A\nu t}.$$

Thus

$$c(t, L_P) = \frac{A}{N}\left[L - L_P^0\right] e^{-A\nu t}.$$

As a result,

$$J(c) = \frac{2\pi N}{(1-\sigma)(\rho + A\nu(1-\sigma))}\left[\frac{A}{N}\left(L - L_P^0\right)\right]^{1-\sigma}.$$

It can be shown that

$$\frac{1}{\rho + A\nu(1-\sigma)}\left[\frac{A}{N}\right]^{1-\sigma} \leq \frac{1}{\rho}\left[\frac{A(\phi-\nu)}{N\phi}\right]^{1-\sigma}$$

if $\rho < \lambda_0 = A(\phi - \nu)$. This leads to $J(c) \leq J(c; T_0)$ which is defined in (32). Thus no such $\hat{a}$ will be chosen.

This proves the both propositions.

C. Proof of Proposition 3

The assumption $D > \lambda_0$ implies that λ_0 is the only positive eigenvalue. Thus, ψ is independent of θ and

$$\psi(t) = a_0 e^{(\rho - \lambda_0)t}$$

for some constant a_0. Furthermore, equation (1) becomes

$$\frac{\partial L_P}{\partial t} - D\frac{\partial^2 L_P}{\partial \theta^2} = -\lambda_0\left[L - L_P\right] + c(t, L_P) N\phi \quad \text{for } t > 0, \tag{38}$$

with

$$c(t, L_P) = \frac{1}{N\phi}\begin{cases} \min\left\{\hat{b}e^{gt}, A\phi\left[L - L_P\right]\right\} & \text{if } L_P > 0 \\ \xi\left(\hat{b}e^{gt}; \lambda_0 L, A\phi L\right) & \text{if } L_P = 0 \end{cases} \tag{39}$$

where $\hat{b} = N\phi(a_0\phi)^{-1/\sigma}$. Since L_P^0 is constant and $c(t, L_P)$ is independent of θ, it follows that L_P is independent of θ.

By the definition of g and the Proposition assumptions,

$$g = \frac{\lambda_0 - \rho}{\sigma} \leq A\nu,$$

it follows that

$$\lambda_0 - g = -g + A\phi - A\nu \leq A\phi.$$

There are three cases:

1. $\hat{b} \geq A\phi\left(L - L_P^0\right)$;

2. $(\lambda_0 - g)\left(L - L_P^0\right) \leq \hat{b} < A\phi\left(L - L_P^0\right)$; and

3. $0 \leq \hat{b} < (\lambda_0 - g)\left(L - L_P^0\right)$.

In Case 1,

$$\frac{\hat{b}}{N\phi}e^{gt} > \frac{A}{N}\left[L - L_P^0\right]e^{-A\nu t} \qquad \text{for } t > 0.$$

Hence, (38) takes the form

$$L_P' = -\lambda_0\left[L - L_P\right] + A\phi\left[L - L_P\right] = A\nu\left[L - L_P\right].$$

As a result,

$$\begin{aligned} L_P(t) &= L - \left(L - L_P^0\right)e^{-A\nu t}, \\ c(t, L_P) &= \frac{A\left[L - L_P(t)\right]}{N} = \frac{A}{N}\left[L - L_P^0\right]e^{-A\nu t}. \end{aligned}$$

Overall welfare is

$$\begin{aligned} &\frac{2\pi N}{1-\sigma}\left(\frac{A}{N}\left[L - L_P^0\right]\right)^{1-\sigma}\int_0^\infty e^{-(\rho + A\nu(1-\sigma))t}dt \\ = &\frac{2\pi N}{(1-\sigma)\left[\rho + A\nu(1-\sigma)\right]}\left(\frac{A}{N}\left[L - L_P^0\right]\right)^{1-\sigma}. \end{aligned}$$

In Case 2,

$$\frac{\hat{b}}{N\phi}e^{gt} < \frac{A}{N}\left[L - L_P^0\right]e^{-A\nu t} \qquad \text{if } 0 \leq t < T$$

for some T. Thus, L_P satisfies (29) and the solution is (31) for $t < T$. If

$$\frac{\hat{b}}{\lambda_0 - g} = L - L_P^0$$

then

$$L_P(t) = L - \frac{\hat{b}}{\lambda_0 - g}e^{gt}.$$

In this case

$$\frac{A}{N}\left[L - L_P(t)\right] = \frac{A\hat{b}}{N(\lambda_0 - g)}e^{gt} > \frac{\hat{b}}{N\phi}e^{gt} \qquad \text{for all } t > 0.$$

Therefore,

$$c(t, L_P) = \frac{\hat{b}}{N\phi}e^{gt} = \frac{\lambda_0 - g}{N\phi}\left[L - L_P^0\right]e^{gt} \qquad \text{for } t > 0,$$

and overall welfare is

$$\begin{aligned} &\frac{2\pi N}{1-\sigma}\left(\frac{\lambda_0 - g}{N\phi}\left[L - L_P^0\right]\right)^{1-\sigma}\int_0^\infty e^{-(\rho - g(1-\sigma))t}dt \\ = &\frac{2\pi N}{(1-\sigma)\left[\rho - g(1-\sigma)\right]}\left(\frac{\lambda_0 - g}{N\phi}\left[L - L_P^0\right]\right)^{1-\sigma}. \end{aligned}$$

It can be shown that the above quantity is greater than the corresponding quantity in Case 1. Therefore at the optimal solution, it must be that $\hat{b} < A\phi\left[L - L_P^0\right]$.

If, as in Case 2,

$$\frac{\hat{b}}{\lambda_0 - g} > L - L_P^0, \tag{40}$$

then, by (31)

$$\frac{A}{N}\left[L - L_P(t)\right] = \frac{A\hat{b}}{N(\lambda_0 - g)} e^{gt} + \frac{A}{N} e^{\lambda_0 t}\left[L - L_P^0 - \frac{\hat{b}}{\lambda_0 - g}\right].$$

At $t = 0$, the right-hand side is

$$\frac{A}{N}\left[L - L_P^0\right] > \frac{\hat{b}}{N\phi}$$

and it is negative for t sufficiently large. Hence, there exists a value $T_1 > 0$ such that

$$\frac{A}{N}\left[L - L_P(T_1)\right] = \frac{\hat{b}}{N\phi} e^{gT_1}, \tag{41}$$

and according to (39)

$$c(t, L_P) = \frac{1}{N\phi}\begin{cases} \hat{b}e^{gt} & \text{if } t < T_1, \\ A\phi\left[L - L_P\right] & \text{if } t \geq T_1. \end{cases}$$

For any $t > T_1$, L_P satisfies

$$L_P' = -\lambda_0\left[L - L_P\right] + A\phi\left[L - L_P\right] = A\nu\left[L - L_P\right].$$

Solving this equation, we find

$$L_P(t) = L - e^{-A\nu(t - T_1)}\left[L - L_P(T_1)\right].$$

Therefore,

$$c(t, L_P) = \frac{A}{N}\left[L - L_P(T_1)\right] e^{-A\nu(t - T_1)} \qquad \text{for } t \geq T_1.$$

overall welfare in this case is

$$\begin{aligned} &\frac{2\pi N}{1-\sigma}\left\{\int_0^{T_1}\left(\frac{\hat{b}}{N\phi}\right)^{1-\sigma} e^{-(\rho - g(1-\sigma))t} dt + \int_{T_1}^{\infty}\left[\frac{A}{N}\left(L - L_P(T_1)\right)\right]^{1-\sigma} e^{-(\rho + A\nu(1-\sigma))t} dt\right\} \\ = \quad &\frac{2\pi N}{1-\sigma}\left\{\left(\frac{\hat{b}}{N\phi}\right)^{1-\sigma} \frac{1 - e^{-(\rho - g(1-\sigma))T_1}}{\rho - g(1-\sigma)} + \left[\frac{A}{N}\left(L - L_P(T_1)\right)\right]^{1-\sigma} \frac{e^{-\rho T_1}}{\rho + A\nu(1-\sigma)}\right\}. \end{aligned}$$

Note that $L_P(T_1)$ satisfies (31) for $t < T_1$. Let

$$I(T) = \frac{\hat{b}(T)^{1-\sigma}}{\rho - g(1-\sigma)}\left[1 - e^{-(\rho - g(1-\sigma))T}\right] + \frac{\left[A\phi\left(L - L_P(T)\right)\right]^{1-\sigma}}{\rho + A\nu(1-\sigma)} e^{-\rho T} \tag{42}$$

where here we underline the dependence of $\hat{b}$ on T, that is,

$$\hat{b}(T) = \frac{(\lambda_0 - g)\left[L - L_P^0\right]}{1 - \left(1 - \frac{\lambda_0 - g}{A\phi}\right) e^{-(\lambda_0 - g)T}} \tag{43}$$

and $L_P(T)$ is given by (31). Then the total benefit is equal to

$$\frac{2\pi N}{(1-\sigma)(N\phi)^{1-\sigma}} I(T_1).$$

We show that $I(T)$ is increasing in T. By (41),

$$A\phi\left[L - L_P(T)\right] = \hat{b}(T) e^{gT}.$$

Note that $\rho - (1-\sigma) g = \lambda_0 - g$. Thus

$$I(T) = \hat{b}(T)^{1-\sigma} \left\{ \frac{1 - e^{-(\lambda_0 - g)T}}{\lambda_0 - g} + \frac{e^{-(\lambda_0 - g)T}}{\rho + A\nu(1-\sigma)} \right\}$$

Taking derivatives with respect to T:

$$\begin{aligned} I'(T) &= (1-\sigma)\hat{b}(T)^{-\sigma}\hat{b}'(T)\left\{ \frac{1 - e^{-(\lambda_0 - g)T}}{\lambda_0 - g} + \frac{e^{-(\lambda_0 - g)T}}{\rho + A\nu(1-\sigma)} \right\} \\ &+ \hat{b}(T)^{1-\sigma} e^{-(\lambda_0 - g)T} \left\{ 1 - \frac{\lambda_0 - g}{\rho + (1-\sigma)A\nu} \right\}. \end{aligned}$$

The last term on the right-hand side is equal to

$$\hat{b}(T)^{1-\sigma} e^{-(\lambda_0 - g)T} \frac{(1-\sigma)(A\nu + g)}{\rho + (1-\sigma)A\nu}.$$

Also, from (43) it is true that

$$\frac{\hat{b}'(T)}{\hat{b}(T)} = -\frac{(\lambda_0 - g)\left(1 - \frac{\lambda_0 - g}{A\phi}\right) e^{-(\lambda_0 - g)T}}{1 - \left(1 - \frac{\lambda_0 - g}{A\phi}\right) e^{-(\lambda_0 - g)T}} = -\frac{(\lambda_0 - g)(A\nu + g) e^{-(\lambda_0 - g)T}}{A\phi - (A\nu + g) e^{-(\lambda_0 - g)T}}$$

since

$$A\phi + g - \lambda_0 = A\nu + g.$$

Thus,

$$\begin{aligned} I'(T) &= \frac{(1-\sigma)(A\nu + g)\hat{b}(T)^{1-\sigma} e^{-(\lambda_0 - g)T}}{A\phi - (A\nu + g) e^{-(\lambda_0 - g)T}} \\ &\cdot \left\{ -1 + e^{-(\lambda_0 - g)T} - \frac{(\lambda_0 - g) e^{-(\lambda_0 - g)T}}{\rho + A\nu(1-\sigma)} + \frac{A\phi - (A\nu + g) e^{-(\lambda_0 - g)T}}{\rho + (1-\sigma)A\nu} \right\} \\ &= \frac{(1-\sigma)(A\nu + g)\hat{b}(T)^{1-\sigma} e^{-(\lambda_0 - g)T}}{A\phi - (A\nu + g) e^{-(\lambda_0 - g)T}} \frac{\sigma(A\nu + g) + (\lambda_0 - g) e^{-(\lambda_0 - g)T}}{\rho + (1-\sigma)A\nu}. \end{aligned}$$

Since $A\nu + g > 0$, it follows that $I'(T) > 0$. This proves that $I(T)$ is increasing.

As a result,

$$\frac{2\pi N}{(1-\sigma)(N\phi)^{1-\sigma}} I(T_1) \leq \frac{2\pi N}{(1-\sigma)(N\phi)^{1-\sigma}} \lim_{T \to \infty} I(T).$$

Taking the limit in (42), we find that

$$\lim_{T\to\infty} I(T) = \frac{\left(\lim_{T\to\infty} \hat{b}(T)\right)^{1-\sigma}}{\lambda_0 - g} = \frac{\left[(\lambda_0 - g)\left(L - L_P^0\right)\right]^{1-\sigma}}{\lambda_0 - g}.$$

Hence, overall welfare is bounded by

$$\frac{2\pi N}{(1-\sigma)(N\phi)^{1-\sigma}} \frac{\left[(\lambda_0 - g)\left(L - L_P^0\right)\right]^{1-\sigma}}{\lambda_0 - g}$$

which is the same as overall welfare with $\hat{b} = (\lambda_0 - g)\left(L - L_P^0\right)$.

In Case 3,

$$\hat{b}e^{gt} \leq (\lambda_0 - g)\left[L - L_P(t)\right].$$

By the proposition assumptions,

$$\lambda_0 - g = \frac{\rho + (\sigma - 1) A [\phi - \nu]}{\sigma} \leq A\phi.$$

Thus

$$\hat{b}e^{gt} \leq A\phi \left[L - L_P(t)\right] \qquad \text{for all } t \geq 0.$$

Therefore, by (39),

$$c(t, L_P) = \frac{1}{N\phi} \begin{cases} \hat{b}e^{gt} & \text{if } L_P > 0, \\ \max\left\{\hat{b}e^{gt}, \lambda_0 L\right\} & \text{if } L_P = 0 \end{cases} \tag{44}$$

if $\hat{b}$ satisfies (40). Hence, optimal consumption solely depends on $\hat{b}$ and the sign of L_P.

We choose $\hat{b}$ to maximize overall welfare in (2). Since all quantities are independent of θ, it suffices to maximize

$$J(T) = \int_0^T \hat{b}^{1-\sigma} e^{-(\rho - (1-\sigma)g)t} + \int_T^\infty (\lambda_0 L)^{1-\sigma} e^{-\rho t} dt \tag{45}$$

where T is the minimum of t such that $L_P(t) = 0$ and accordingly

$$c(t, L_P) = \frac{\lambda_0 L}{N\phi}.$$

We can compute $J(T)$ as

$$J(T) = \frac{\hat{b}^{1-\sigma}}{\rho - (1-\sigma) g} \left[1 - e^{-(\rho - (1-\sigma) gT)}\right] + \frac{(\lambda_0 L)^{1-\sigma}}{\rho} e^{-\rho T}. \tag{46}$$

It follows from (31) and (44) that

$$\frac{\hat{b}}{\lambda_0 - g}\left[e^{\lambda_0 T} - e^{gT}\right] \leq \left[L - L_P^0\right] e^{\lambda_0 T} - L, \qquad \hat{b}e^{gT} \leq \lambda_0 L$$

and T is the minimum for both inequalities to hold. This implies that

$$\hat{b} = \min\left\{\frac{(\lambda_0 - g)\left[\left(L - L_P^0\right) e^{\lambda_0 T} - L\right]}{e^{\lambda_0 T} - e^{gT}}, \lambda_0 L e^{-gT}\right\}. \tag{47}$$

We show that
$$\frac{(\lambda_0 - g)\left[\left(L - L_P^0\right)e^{\lambda_0 T} - L\right]}{e^{\lambda_0 T} - e^{gT}} \leq \lambda_0 L e^{-gT} \tag{48}$$
for all T at which the left-hand side is nonnegative. The inequality is equivalent to
$$(\lambda_0 - g)\left(L - L_P^0\right) \leq L\left(\lambda_0 e^{-gT} + g e^{-\lambda_0 T}\right). \tag{49}$$
The derivative of the right-hand side is
$$L\lambda_0 g\left(e^{-gT} - e^{-\lambda_0 T}\right) < 0,$$
which is decreasing in T. The left-hand side of (48) nonnegative if
$$e^{\lambda_0 T} \geq \frac{L}{L - L_P^0}.$$
For such a T we find
$$\begin{aligned} L\left(\lambda_0 e^{-gT} - g e^{-\lambda_0 T}\right) &\geq L\lambda_0 e^{-gT} - g\left(L - L_P^0\right) \\ &\geq L\lambda_0 - g\left(L - L_P^0\right) \geq (\lambda_0 - g)\left(L - L_P^0\right). \end{aligned}$$
This proves (49) and thus also (48), implying that
$$\hat{b} = \frac{(\lambda_0 - g)\left[\left(L - L_P^0\right)e^{\lambda_0 T} - L\right]}{e^{\lambda_0 T} - e^{gT}}. \tag{50}$$
As a result of (48), $\hat{b}$ depends on T by (47). Thus T maximizes $J(T)$ in (45).

We show next that $J(T)$ is increasing in T. Differentiating $J(T)$ with respect to T, we find
$$\begin{aligned} J'(T) &= \frac{(1-\sigma)\hat{b}(T)^{-\sigma}}{\rho - (1-\sigma)g}\left[1 - e^{-(\rho-(1-\sigma)g)T}\right]\hat{b}'(T) \\ &\quad + \hat{b}(T)^{1-\sigma} e^{-(\rho-(1-\sigma)g)T} - (\lambda_0 L)^{1-\sigma} e^{-\rho T}. \end{aligned}$$
Log differentiating (50), we find
$$\begin{aligned} \frac{\hat{b}'(T)}{\hat{b}(T)} &= \frac{\lambda_0 L e^{-\lambda_0 T}}{L - L_P^0 - L e^{-\lambda_0 T}} - \frac{(\lambda_0 - g)e^{-(\lambda_0 - g)T}}{1 - e^{-(\lambda_0 - g)T}} \\ &= \frac{\lambda_0 L e^{-\lambda_0 T}\left[e^{(\lambda_0 - g)T} - 1\right] - (\lambda_0 - g)\left[L - L_P^0 - L e^{-\lambda_0 T}\right]}{\left(L - L_P^0 - L e^{-\lambda_0 T}\right)\left(e^{(\lambda_0 - g)T} - 1\right)}. \end{aligned} \tag{51}$$
Hence,
$$\begin{aligned} J'(T) &= \frac{(1-\sigma)\hat{b}(T)^{-\sigma}\hat{b}'(T)}{\rho - (1-\sigma)g}\left[1 - e^{-(\rho-(1-\sigma)g)T}\right] \\ &\quad + \hat{b}(T)^{1-\sigma} e^{-(\rho+(1-\sigma)g)T} - (\lambda_0 L)^{1-\sigma} e^{-\rho T}. \end{aligned} \tag{52}$$
Using (51), and that
$$\rho - (1-\sigma)g = \lambda_0 - g, \tag{53}$$

the first two terms on the right-hand side of (52) can be combined to

$$\hat{b}^{1-\sigma}e^{-(\lambda_0-g)T}\left\{\frac{(1-\sigma)\,\lambda_0 Le^{-gT}\left[1-e^{-(\lambda_0-g)T}\right]}{(\lambda_0-g)\,(L-L_P^0-Le^{-\lambda_0 T})}+\sigma\right\}.$$

In view of (50), the above quantity can be written as

$$\hat{b}^{-\sigma}e^{-(\lambda_0-g)T}\left\{(1-\sigma)\,\lambda_0 Le^{-gT}+\sigma\hat{b}\right\}.$$

Using the inequality

$$(1-\sigma)\,a_1+\sigma a_2\geq a_1^{1-\sigma}a_2^{\sigma}$$

for any positive numbers a_1 and a_2 and (53), we obtain

$$\begin{aligned}\hat{b}^{-\sigma}e^{-(\lambda_0-g)T}\left\{(1-\sigma)\,\lambda_0 Le^{-gT}+\sigma\hat{b}\right\} &\geq \hat{b}^{-\sigma}e^{-(\lambda_0-g)T}\,(\lambda_0 L)^{1-\sigma}\,\hat{b}^{\sigma}e^{-g(1-\sigma)T}\\ &= (\lambda_0 L)^{1-\sigma}\,e^{-\rho T}.\end{aligned}$$

This proves that $J'(T)\geq 0$.

It follows that $J(T)$ is increasing in T. However, by (31), this can happen only if

$$\frac{\hat{b}}{\lambda_0-g}=L-L_P^0.$$

Thus,

$$c\,(t,L_P)=\frac{\hat{b}e^{gt}}{N\phi}=\frac{(\lambda_0-g)\left(L-L_P^0\right)}{N\phi}e^{gt}\qquad\text{for } t>0,$$

and, by (31),

$$L_P\,(t)=L-\frac{\hat{b}}{\lambda_0-g}e^{gt}.$$

This leads to (14).

This completes the proof.

D. Proof of Proposition 4

As in the proof of Proposition 3, $c\,(t,L_P)$ is given by (39). Either $\hat{b}>A\phi\left[L-L_P^0\right]$ or $\hat{b}\leq A\phi\left[L-L_P^0\right]$.

We first consider the case where $\hat{b}>A\phi\left[L-L_P^0\right]$. By the continuity of $L_P\,(t)$, there is $\hat{T}>0$ such that

$$\hat{b}e^{gt}\geq A\phi\,[L-L_P\,(t)]\qquad 0\leq t\leq\hat{T}. \tag{54}$$

Thus

$$L_P'=-\lambda_0\,[L-L_P]+A\phi\,[L-L_P]=A\nu\,[L-L_P]\qquad\text{for } t<\hat{T}.$$

This implies that

$$L_P\,(t)=L-e^{-A\nu t}\left[L-L_P^0\right]\qquad\text{for } t<\hat{T}. \tag{55}$$

By (15), $0 > A\nu + g$. Hence

$$\hat{b}e^{gt} < A\phi e^{-A\nu t}\left[L - L_P^0\right] = A\phi\left[L - L_P(t)\right] \tag{56}$$

for large t. Therefore, $\hat{T}$ satisfies

$$\hat{b}e^{g\hat{T}} = A\phi\left[L - L_P(\hat{T})\right] \tag{57}$$

and (56) holds for $t > \hat{T}$. Thus L_P satisfies

$$L_P' = -\lambda_0\left[L - L_P\right] + \hat{b}e^{gt} \qquad \text{for } t > \hat{T} \quad \text{while } L_P(t) > 0.$$

It follows that

$$L_P(t) = L + e^{\lambda_0(t-\hat{T})}\left[L_P(\hat{T}) - L + \frac{\hat{b}}{\lambda_0 - g}e^{g\hat{T}}\right] - \frac{\hat{b}}{\lambda_0 - g}e^{gt} \tag{58}$$

if $t > \hat{T}$ and $L_P(t) \geq 0$. Since $0 > A\nu + g$, it follows that

$$\lambda_0 - g = A\phi - A\nu - g > A\phi.$$

Hence, (57),

$$\frac{\hat{b}}{\lambda_0 - g}e^{g\hat{T}} = \frac{A\phi}{\lambda_0 - g}\left[L - L_P(\hat{T})\right] < L - L_P(\hat{T}).$$

Thus, by (58), there exists $\tilde{T}$ such that $L_P(t) = 0$ for $t \geq \tilde{T}$. By (39),

$$c(t,0) = \frac{1}{N\phi}\max\left\{\hat{b}e^{gt}, \lambda_0 L\right\} \qquad \text{for } t \geq \tilde{T}.$$

Let

$$T = \max\left\{\tilde{T}, \frac{1}{g}\ln\left(\frac{\lambda_0 L}{\hat{b}}\right)\right\}. \tag{59}$$

Then

$$c(t, L_P) = \frac{1}{N\phi}\begin{cases} A\phi\left[L - L_P^0\right]e^{-A\nu t} & \text{for } t < \hat{T}, \\ \hat{b}e^{gt} & \text{for } \hat{T} \leq t < T, \\ \lambda_0 L & \text{for } t \geq T. \end{cases}$$

overall welfare is

$$\begin{aligned} & \frac{2\pi N}{(1-\sigma)(N\phi)^{1-\sigma}}\left\{\int_T^\infty (\lambda_0 L)^{1-\sigma} e^{-\rho t}dt + \int_{\hat{T}}^T \hat{b}^{1-\sigma}e^{-(\rho - g(1-\sigma))t}dt \right. \\ & \qquad \left. + \int_0^{\hat{T}} \left[A\phi\left(L - L_P^0\right)\right]^{1-\sigma} e^{-(\rho + A\nu(1-\sigma))t}dt\right\} \\ = \; & \frac{2\pi N}{(1-\sigma)(N\phi)^{1-\sigma}}\left\{\frac{\left[A\phi\left(L - L_P^0\right)\right]^{1-\sigma}}{\rho + A\nu(1-\sigma)}\left[1 - e^{-(\rho + A\nu(1-\sigma))\hat{T}}\right] + \frac{(\lambda_0 L)^{1-\sigma}}{\rho}e^{-\rho T}\right. \\ & \qquad \left. \frac{\hat{b}^{1-\sigma}}{\rho - g(1-\sigma)}\left[e^{-(\rho - g(1-\sigma))\hat{T}} - e^{-(\rho - g(1-\sigma))T}\right]\right\}. \end{aligned}$$

Let us rewrite the expression between the brackets as a function of $\hat{b}$

$$J(\hat{b}) = \frac{\left[A\phi\left(L-L_P^0\right)\right]^{1-\sigma}}{\rho+A\nu(1-\sigma)}\left[1-e^{-(\rho+A\nu(1-\sigma))\hat{T}\left(\hat{b}\right)}\right]+\frac{(\lambda_0 L)^{1-\sigma}}{\rho}e^{-\rho T(\hat{a})} \\ +\frac{\hat{b}^{1-\sigma}}{\rho-g(1-\sigma)}\left[e^{-(\rho-g(1-\sigma))\hat{T}(\hat{a})}-e^{-(\rho-g(1-\sigma))T\left(\hat{b}\right)}\right]. \tag{60}$$

By (15), in the case where $\hat{b} \leq A\phi\left[L-L_P^0\right]$,

$$\hat{b}e^{gt} \leq A\phi e^{-A\nu t}\left[L-L_P^0\right] \qquad \text{for all } t>0.$$

Hence,

$$L_P' = -\lambda_0\left[L-L_P\right]+\hat{b}e^{gt} \qquad \text{for } t>0$$

as long as $L_P(t)>0$. The solution is given by (31). Since

$$\frac{\hat{b}}{\lambda_0-g} \leq \frac{A\phi}{\lambda_0-g}\left[L-L_P^0\right] < L-L_P^0,$$

it follows that there exists $\tilde{T}>0$ such that $L_P(t)=0$ for $t \geq \tilde{T}$. Hence,

$$c(t,L_P) = \frac{1}{N\phi}\begin{cases}\hat{b}e^{gt} & \text{for } t<T,\\ \lambda_0 L & \text{for } t\geq T\end{cases}$$

where T is given by (59). Overall welfare is

$$\frac{2\pi N}{(1-\sigma)(N\phi)^{1-\sigma}}\left\{\int_0^T \hat{b}^{1-\sigma}e^{-(\rho+g(1-\sigma))t}dt+\int_T^\infty(\lambda_0 L)^{1-\sigma}e^{-\rho t}dt\right\}$$
$$= \frac{2\pi N}{(1-\sigma)(N\phi)^{1-\sigma}}\left\{\frac{\hat{b}^{1-\sigma}}{\rho+g(1-\sigma)}\left[1-e^{-(\rho+g(1-\sigma))T\left(\hat{b}\right)}\right]+\frac{(\lambda_0 L)^{1-\sigma}}{\rho}e^{-\rho T\left(\hat{b}\right)}\right\}.$$

Clearly, the right-hand side coincides with

$$\frac{2\pi N}{(1-\sigma)(N\phi)^{1-\sigma}}J\left(\hat{b}\right)$$

with $\hat{T}=0$. Thus let us define

$$\hat{T}\left(\hat{b}\right) = \max\left\{\frac{\ln\left[A\phi\left(L-L_P^0\right)\right]-\ln\hat{b}}{g+A\nu},0\right\}$$

and show that J is increasing in $\hat{b}$.

We first show that $J(\hat{b})$ is increasing in $\hat{b}$ for $\hat{b}_1 \leq A\phi\left(L-L_P^0\right)$. In this case $J(\hat{b})$ takes the form of the right-hand side of (46) in Proposition 3 with $\hat{b}$ and T satisfy (**??**) and (10) if

$$L-L_P^0-Le^{-\lambda_0 T}>0$$

which holds if

$$T > \frac{\ln\left(1-L_P^0/L\right)}{\lambda_0} \equiv T_0. \tag{61}$$

Differentiating the right-hand side of (61) with respect to T, we find $J'(T)$ is given by (52). Using the same argument as in the proof of Proposition 3, we find that $J'(T) \geq 0$. Hence,

$$J(\hat{b}) \leq J\left(A\phi\left[L - L_P^0\right]\right) \qquad \text{for } \hat{b} \leq A\phi\left[L - L_P^0\right].$$

We next consider the case where $\hat{b} > A\phi\left[L - L_P^0\right]$. In such a case $\hat{T} > 0$ and

$$\hat{b}e^{g\hat{T}} = A\phi\left[L - L_P(\hat{T})\right] = A\phi e^{-A\nu\hat{T}}\left[L - L_P^0\right]$$

and hence,

$$\hat{b} = e^{-g\hat{T}}A\phi\left[L - L_P(\hat{T})\right] = e^{-(g+A\nu)\hat{T}}A\phi\left[L - L_P^0\right] \tag{62}$$

which is increasing in $\hat{T}$. Recall that T satisfies

$$L_P(T) = 0, \qquad \hat{b}e^{gT} \leq \lambda_0 L.$$

The first relation implies that

$$\frac{\hat{b}}{\lambda_0 - g}\left[e^{\lambda_0\left(T-\hat{T}\right)+g\hat{T}} - e^{gT}\right] \leq \left[L - L_P(\hat{T})\right]e^{\lambda_0\left(T-\hat{T}\right)} - L$$

Thus

$$\hat{b} = \min\left\{\frac{(\lambda_0 - g)\left[\left(L - L_P(\hat{T})\right)e^{\lambda_0\left(T-\hat{T}\right)} - L\right]}{e^{\lambda_0\left(T-\hat{T}\right)+g\hat{T}} - e^{gT}}, \lambda_0 L e^{-gT}\right\}.$$

We show that

$$(\lambda_0 - g)\left[\left(L - L_P(\hat{T})\right)e^{\lambda_0\left(T-\hat{T}\right)} - L\right] \leq \lambda_0 L e^{-gT}\left[e^{\lambda_0\left(T-\hat{T}\right)+g\hat{T}} - e^{gT}\right] \tag{63}$$

for any $T \geq \hat{T} \geq 0$ as long as the left-hand side is nonnegative. The above inequality is equivalent to

$$(\lambda_0 - g)\left(L - L_P(\hat{T})\right) \leq L\left[\lambda_0 e^{-g\left(T-\hat{T}\right)} - g e^{-\lambda_0\left(T-\hat{T}\right)}\right].$$

It is easy to see that the right-hand side of the above inequality is increasing in $T - \hat{T}$. The left-hand side of (63) is nonnegative if and only if the difference $T - \hat{T}$ is such that

$$e^{\lambda_0\left(T-\hat{T}\right)} \geq \frac{L}{L - L_P\left(\hat{T}\right)}.$$

For such $T - \hat{T}$ we find

$$\begin{aligned} L\left[\lambda_0 e^{-g\left(T-\hat{T}\right)} - g e^{-\lambda_0\left(T-\hat{T}\right)}\right] &\geq L\lambda_0 e^{-g\left(T-\hat{T}\right)} - g\left(L - L_P(\hat{T})\right) \\ &\geq L\lambda_0 - g\left(L - L_P(\hat{T})\right) \geq (\lambda_0 - g)\left(L - L_P(\hat{T})\right). \end{aligned}$$

This proves (63).

As a result,

$$\hat{b} = e^{g\hat{T}} \frac{(\lambda_0 - g)\left[\left(L - L_P(\hat{T})\right) e^{\lambda_0\left(T-\hat{T}\right)} - L\right]}{e^{\lambda_0\left(T-\hat{T}\right)} - e^{g\left(T-\hat{T}\right)}}. \tag{64}$$

By (62), we find

$$e^{-g\hat{T}} A\phi\left[L - L_P(\hat{T})\right] = \frac{(\lambda_0 - g)\left[\left(L - L_P(\hat{T})\right) e^{\lambda_0\left(T-\hat{T}\right)} - L\right]}{e^{\lambda_0\left(T-\hat{T}\right)+g\hat{T}} - e^{gT}}$$

which leads to

$$\left[L - L_P^0\right] e^{-A\nu\hat{T}} = \frac{(\lambda_0 - g)\, L e^{-\lambda_0 u}}{A\phi e^{-(\lambda_0-g)u} - g - A\nu}$$

where we define u as $u = T - \hat{T}$. Differentiating the above equality with respect to u

$$\hat{T}'(u) = -\frac{\lambda_0 (g + A\nu) - A\phi g e^{-(\lambda_0-g)u}}{A\nu\left[-g - A\nu + A\phi e^{-(\lambda_0-g)u}\right]}. \tag{65}$$

It can be shown that there is $u_0 > 0$ such that

$$\frac{(\lambda_0 - g)\, L e^{-\lambda_0 u}}{-g - A\nu + A\phi e^{-(\lambda_0-g)u}} \leq L \qquad \text{for } u \geq u_0$$

and $\hat{T}'(u) > 0$ for $u \geq u_0$. Furthermore, from (57) and (62) we find $\hat{b}$ and T are functions of u given by

$$\begin{aligned} \hat{b}(u) &= e^{-g\hat{T}(u)} A\phi\left[L - L_P(\hat{T})\right] = e^{-(g+A\nu)\hat{T}(u)} A\phi\left[L - L_P^0\right], \\ T(u) &= \hat{T}(u) + u, \end{aligned}$$

respectively. Their derivatives are

$$\hat{b}'(u) = -\hat{b}(u)(g + A\nu)\hat{T}'(u), \qquad T'(u) = 1 + \hat{T}'(u) \tag{66}$$

for $u \geq u_0$. Clearly both $\hat{b}'(u)$ and $T'(u)$ are positive. As a result, function J defined in (60) is also a function of u. Its derivative is

$$\begin{aligned} J'(u) &= \left[A\phi\left(L - L_P^0\right)\right]^{1-\sigma} e^{-(\rho + A\nu(1-\sigma))\hat{T}}\hat{T}' - (\lambda_0 L)^{1-\sigma} e^{-\rho T} T' \\ &\quad + \frac{(1-\sigma)\hat{b}^{-\sigma}\hat{b}'}{\lambda_0 - g}\left[e^{-(\lambda_0-g)\hat{T}} - e^{-(\lambda_0-g)T}\right] - \hat{b}^{1-\sigma}\left[e^{-(\lambda_0-g)\hat{T}}\hat{T}' - e^{-(\lambda_0-g)T}T'\right]. \end{aligned}$$

Using (66), the sum of the last two terms on the right-hand side is

$$\begin{aligned} &\hat{b}^{1-\sigma}\left\{-\frac{(1-\sigma)(g + A\nu)\hat{T}'}{\lambda_0 - g}\left[e^{-(\lambda_0-g)\hat{T}} - e^{-(\lambda_0-g)T}\right] - e^{-(\lambda_0-g)\hat{T}}\hat{T}' + e^{-(\lambda_0-g)T}T'\right\} \\ = \; &\hat{b}^{1-\sigma}\hat{T}' e^{-(\lambda_0-g)\hat{T}}\frac{\sigma(g + A\nu) - A\phi}{\lambda_0 - g} + \frac{\hat{b}^{1-\sigma}e^{-(\lambda_0-g)T}}{\lambda_0 - g}\left\{\left[-\sigma(g + A\nu) + A\phi\right]\hat{T}' + \lambda_0 - g\right\}. \end{aligned}$$

Since

$$\begin{aligned}\left[A\phi\left(L-L_P^0\right)\right]^{1-\sigma}e^{-(\rho+A\nu(1-\sigma))\hat{T}} &= \left[A\phi\left(L-L_P^0\right)e^{-A\nu\hat{T}}\right]^{1-\sigma}e^{-\rho\hat{T}} \\ &= \left[\hat{b}e^{g\hat{T}}\right]^{1-\sigma}e^{-\rho\hat{T}}=\hat{b}^{1-\sigma}e^{-(\lambda_0-g)\hat{T}},\end{aligned}$$

it follows that

$$\begin{aligned}J'(u) &= \hat{b}^{1-\sigma}e^{-(\lambda_0-g)\hat{T}}\hat{T}'-(\lambda_0L)^{1-\sigma}e^{-\rho T}T'+\hat{b}^{1-\sigma}\hat{T}'e^{-(\lambda_0-g)\hat{T}}\frac{\sigma(g+A\nu)-A\phi}{\lambda_0-g} \\ &\quad+\frac{\hat{b}^{1-\sigma}e^{-(\lambda_0-g)T}}{\lambda_0-g}\left\{\left[-\sigma(g+A\nu)+A\phi\right]\hat{T}'+\lambda_0-g\right\} \\ &= -\frac{(1-\sigma)(g+A\nu)}{\lambda_0-g}\hat{a}^{1-\sigma}e^{-(\lambda_0-g)\hat{T}}\hat{T}'-(\lambda_0L)^{1-\sigma}e^{-\rho T}\left(\hat{T}'+1\right) \\ &\quad+\frac{\hat{b}^{1-\sigma}e^{-(\lambda_0-g)T}}{\lambda_0-g}\left\{\left[-\sigma(g+A\nu)+A\phi\right]\hat{T}'+\lambda_0-g\right\}.\end{aligned}$$

Then, from (57) and (64) it can be shown that

$$\hat{b}=\frac{A\phi L(\lambda_0-g)e^{-gT-(\lambda_0-g)u}}{\lambda_0-g-A\phi\left[1-e^{-(\lambda_0-g)u}\right]}.$$

Hence,

$$\hat{b}\left[(\lambda_0-g)e^{(\lambda_0-g)u}-A\phi e^{(\lambda_0-g)u}+A\phi\right]=A\phi L(\lambda_0-g)e^{-gT}$$

Using (65) it can be shown that

$$-\hat{b}\left[e^{(\lambda_0-g)u}(g+A\nu)\hat{T}'(u)+A\phi\hat{T}'(u)+\lambda_0-g\right]=(\lambda_0-g)\left(\hat{T}'(u)+1\right)\lambda_0Le^{-gT}.$$

As a result,

$$\begin{aligned}&\frac{\hat{b}}{(\lambda_0-g)\left(\hat{T}'(u)+1\right)}\left[-e^{(\lambda_0-g)u}(1-\sigma)(g+A\nu)\hat{T}'(u)+\left[\rho+(1-\sigma)A\nu\right]\hat{T}'(u)+\lambda_0-g\right] \\ =\;&(1-\sigma)\lambda_0Le^{-gT}+\sigma\hat{b}.\end{aligned}$$

Thus,

$$\begin{aligned}J'(u) &= \left\{\hat{b}^{-\sigma}e^{-(\lambda_0-g)T}\left[(1-\sigma)\lambda_0Le^{-gT}+\sigma\hat{a}\right]-(\lambda_0L)^{1-\sigma}e^{-\rho T}\right\}\left(\hat{T}'+1\right) \\ &\geq \left\{\hat{b}^{-\sigma}e^{-(\lambda_0-g)T}(\lambda_0L)^{1-\sigma}e^{-g(1-\sigma)T}\hat{b}^{\sigma}-(\lambda_0L)^{1-\sigma}e^{-\rho T}\right\}\left(\hat{T}'+1\right)=0.\end{aligned}$$

It follows that

$$\sup_{u\geq 0}J(u)=\lim_{u\to\infty}J(u).$$

This implies that J is maximized as $u\longrightarrow\infty$, i.e. $T-\hat{T}\longrightarrow\infty$. That is, $T\longrightarrow\infty$ and $\hat{T}\longrightarrow 0$. Since $\hat{T}'(u)>0$ for $u\geq 0$, we would need $\hat{T}(u)=0$, which by (64) implies that $\hat{b}\to\infty$.

This completes the proof.

E. Proof of Proposition 5.

We first show that the optimal solution is feasible, that is

$$0 \leq L_P(t,\theta) \leq L \qquad \text{for all } t \geq 0, \quad \theta \in [0, 2\pi]. \tag{67}$$

Let us define f as the right hand side of (1):

$$f(t,\theta,u) = A(\theta)[\phi(\theta) - \nu(\theta)][u - L] + c(t,\theta,u) N(\theta)\phi(\theta). \tag{68}$$

where $c(t,\theta,u)$ is given by (19) and (20). By (1), $L_P(t,\theta)$ satisfies the equation

$$\frac{\partial L_P}{\partial t} - D\frac{\partial^2 L_P}{\partial \theta^2} = f(t,\theta,L_P).$$

It is easy to see from (20) that

$$c(t,\theta,0) N(\theta)\phi(\theta) \geq A(\theta)[\phi(\theta) - \nu(\theta)] L.$$

Thus

$$f(t,\theta,0) \geq 0.$$

This implies that 0 is a lower solution of (1). Also, from (19) we see that

$$c(t,\theta,L) = 0.$$

Hence, by (68),

$$f(t,\theta,L) = c(t,\theta,L) N(\theta)\phi(\theta) = 0.$$

This implies that L is an upper solution of (1). By the comparision principle, (67) follows.

Part 1. Let $\tilde{L}_P(t,\theta)$ satisfy the following initial-boundary value problem

$$\begin{cases} \frac{\partial \tilde{L}_P}{\partial t} - D\frac{\partial^2 \tilde{L}_P}{\partial \theta^2} = a\left[L - \tilde{L}_P\right] & \text{for } t > 0, \quad \theta \in (0, 2\pi), \\ \tilde{L}_P(t,0) = \tilde{L}_P(t,2\pi), & \\ \frac{\partial \tilde{L}_P}{\partial \theta}(t,0) = \frac{\partial \tilde{L}_P}{\partial \theta}(t,2\pi) & \text{for } t > 0, \\ \tilde{L}_P(0,\theta) = L_P(0,\theta) & \text{for } \theta \in [0,2\pi], \end{cases} \tag{69}$$

where a is a constant such that

$$a \geq A(\theta)\nu(\theta) \qquad \text{for } \theta \in [0, 2\pi].$$

We show that $\tilde{L}_P$ is an upper solution to (1). Using the maximum principle it can be shown that

$$0 \leq \tilde{L}_P(t,\theta) \leq L \qquad \text{for } t \geq 0, \qquad \theta \in [0, 2\pi].$$

Also, by (19) and (20),

$$c\left(t,\theta,L_P\right) \le \frac{A\left(\theta\right)}{N\left(\theta\right)}\left[L - L_P\right],$$

it follows that

$$f\left(t,\theta,L_P\right) \le A\left(\theta\right)\nu\left(\theta\right)\left[L - L_P\right].$$

As a result,

$$\frac{\partial \tilde{L}_P}{\partial t} - D\frac{\partial^2 \tilde{L}_P}{\partial \theta^2} \ge A\left(\theta\right)\nu\left(\theta\right)\left[L - \tilde{L}_P\right] \ge f(t,\theta,\tilde{L}_P).$$

This proves that $\tilde{L}_P$ is an upper solution to (1).

Let

$$V\left(t,\theta\right) = e^{at}\left[L - \tilde{L}_P\left(t,\theta\right)\right].$$

It is easy to see that $V\left(t,\theta\right)$ is a solution to the initial-boundary value problem

$$\begin{aligned} &V_t - DV_{\theta\theta} = 0 && \text{for } t>0, \quad \theta \in \left(0,2\pi\right), \\ &V\left(t,0\right) = V\left(t,2\pi\right), && \\ &V\left(t,0\right) = V\left(t,2\pi\right) && \text{for } t>0, \\ &V\left(0,\theta\right) = L - L_P\left(0,\theta\right) && \text{for } \theta \in \left[0,2\pi\right]. \end{aligned}$$

By assumption, $V\left(0,\theta\right) \ge 0$ and is positive for at least one θ. Therefore, by the maximum principle, $V\left(t,\theta\right) > 0$ for all $t > 0$ and $\theta \in [0, 2\pi]$. This implies that $L > \tilde{L}_P\left(t,\theta\right)$ for all $t > 0$ and $\theta \in [0, 2\pi]$. As a result,

$$L_P\left(t,\theta\right) \le \tilde{L}_P\left(t,\theta\right) < L \qquad \text{for all } t>0, \quad \theta \in \left[0,2\pi\right],$$

proving that L_P remains below L for all t and θ.

Part 2. By (19) and (20),

$$c\left(t,\theta,L_P\right) = \frac{A\left(\theta\right)}{N\left(\theta\right)}\left[L - L_P\right] \qquad \text{if } \psi\left(t,\theta\right) \le 0. \tag{70}$$

Since $\psi_k\left(\theta\right)e^{\left(\rho-\lambda_k\right)}$ dominates $\psi\left(t,\theta\right)$, for any $\theta \in S_k^-$, there exists $T_\theta > 0$ such that $\psi\left(t,\theta\right) < 0$ for $t \ge T_\theta$. Hence, the above inequality holds for $\theta \in S_k^-$, $t \ge T_\theta$. As a result, $L_P\left(t,\theta\right)$ satisfies

$$\frac{\partial L_P}{\partial t} - D\frac{\partial^2 L_P}{\partial \theta^2} = A\left(\theta\right)\nu\left(\theta\right)\left[L - L_P\right] \qquad \text{if } \theta \in S_k^-, \quad t > T_\theta. \tag{71}$$

If $L_P\left(t^*,\theta^*\right) = 0$ for some $\theta^* \in S_k^-$ and $t^* \ge T_\theta$, then, since $L_P\left(t,\theta\right) \ge 0$ for all $\left(t,\theta\right)$, it follows that

$$\frac{\partial L_P}{\partial t}\left(t^*,\theta^*\right) \le 0, \qquad \frac{\partial^2 L_P\left(t^*,x^*\right)}{\partial \theta^2} \ge 0.$$

Hence, the left-hand side of (71) is nonpositive at $\left(t^*,\theta^*\right)$. However, the right-hand side is

$$A\left(\theta^*\right)\nu\left(\theta^*\right)L > 0.$$

This is a contradiction. Therefore, $L_P\left(t,\theta\right) > 0$ for all $\left(\theta,t\right)$ with $\theta \in S_k^-$, $t > T_\theta$.

Next, let us prove (25). Since $\psi_k(\theta)\, e^{(\rho-\lambda_k)t}$ dominates $\psi(t,\theta)$ in S_k^-, by (70), for any compact subset $S_k^\delta \Subset S_k^-$ there exists $T_\delta > 0$ such that $L_P(t,\theta)$ satisfies

$$\frac{\partial L_P}{\partial t} - D\frac{\partial^2 L_P}{\partial \theta^2} = A\nu\,[L - L_P] \qquad \text{for } t > T_\delta,\ \theta \in S_k^\delta.$$

Let $L_P^\delta(t,\theta)$ be the unique solution of the initial-boundary value problem

$$\begin{aligned}
\frac{\partial L_P^\delta}{\partial t} - D\frac{\partial^2 L_P^\delta}{\partial \theta^2} &= A\nu\left[L - L_P^\delta\right] \qquad \text{for } t > T_\delta, \quad \theta \in S_k^\delta,\\
L_P^\delta(t,\theta) &= 0 \qquad \text{for } t > T_\delta, \quad \theta \in \partial S_k^\delta,\\
L_P^\delta(T,\theta) &= 0 \qquad \text{for } \theta \in S_k^\delta.
\end{aligned}$$

It is easy to see that

$$0 \le L_P^\delta(t,\theta) \le L \qquad \text{for } t > T_\delta, \quad \theta \in S_k^\delta$$

and

$$\lim_{t\to\infty} L_P^\delta(t,\theta) = \hat{L}_P^\delta\ (\theta) \qquad \text{for } \theta \in S_k^\delta,$$

where $\hat{L}_P^\delta$ is the solution to the (stationary) boundary value problem

$$\begin{aligned}
-D\frac{d^2\hat{L}_P^\delta}{d\theta^2} &= A(\theta)\,\nu(\theta)\left[L - \hat{L}_P^\delta\right] \qquad \text{for } \theta \in S_k^\delta,\\
\hat{L}_P^\delta(\theta) &= 0 \qquad \text{for } \theta \in \partial S_k^\delta.
\end{aligned}$$

Furthermore, since L_P and L_P^δ satisfies the same parabolic PDE in S_k^δ and

$$\begin{aligned}
L_P(t,\theta) &\ge L_P^\delta(t,\theta) \qquad \text{for } t > T_\delta, \quad \theta \in \partial S_k^\delta,\\
L_P(T_\delta,\theta) &\ge L_P^\delta(T_\delta,\theta) \qquad \text{for } \theta \in S_k^\delta,
\end{aligned}$$

by the comparision principle, we have that $L_P(t,\theta) \ge L_P^\delta(t,\theta)$ for $t > T_\delta$, $\theta \in S_k^\delta$. It follows that

$$\liminf_{t\to\infty} L_P(t,\theta) \ge \hat{L}_P^\delta(\theta) \qquad \text{for } \theta \in S_k^\delta. \tag{72}$$

This relation holds true for any δ. Hence, taking the limit of (72) as $\delta \to 0$, we obtain (25).

This completes the proof.

F. Proof of Proposition 6

By (22), $\psi(t,\theta) \to 0$ as $t \to \infty$ for any $\theta \in [0, 2\pi]$. Hence, $\mathcal{S} = \mathcal{S}_0$ and by (19) and (20)

$$\bar{c}\left(\theta, \bar{L}_P\right) = \frac{A(\theta)}{N(\theta)}\left[L - \bar{L}_P(\theta)\right] \qquad \text{for } \theta \in [0, 2\pi]\,.$$

As a result, any steady state solution for L_P should verify

$$\begin{aligned}
&-D\bar{L}_P'' = A(\theta)\,\nu(\theta)\left[L - \bar{L}_P(\theta)\right] \qquad \text{for } \theta \in (0, 2\pi)\,,\\
&\bar{L}_P(0) = \bar{L}_P(2\pi)\,, \qquad \bar{L}_P'(0) = \bar{L}_P'(2\pi)\,.
\end{aligned} \tag{73}$$

It can be seen that the constant function L is a steady state solution. Furthermore, from (73) we see that any steady state, $\bar{L}_P$, is convex in $[0, 2\pi]$. Since L satisfies the periodic boundary conditions, the only possible steady state is $\bar{L}_P(\theta) = L$ for all θ.

Suppose there is no $T > 0$ such that $L_P(t, \theta) = 0$ for $t \geq T$, $\theta \in [0, 2\pi]$. Since $\psi(t, \theta) \to 0$ as $t \to \infty$, by (19) and (20),

$$c(t, \theta, L_P) = \frac{A(\theta)}{N(\theta)}[L - L_P] \tag{74}$$

if T is sufficiently large. Hence, substituting (74) in (1), $L_P(t, \theta)$ satisfies

$$\frac{\partial L_P(t, \theta)}{\partial t} - D\frac{\partial^2 L_P(t, \theta)}{\partial \theta^2} = A(\theta)\nu(\theta)[L - L_P(t, \theta)] \qquad \text{for } t > T. \tag{75}$$

Let b be a positive constant such that

$$b \leq A(\theta)\nu(\theta) \qquad \text{for } \theta \in [0, 2\pi].$$

We let $\hat{L}_P(t, \theta)$ be the solution to the initial-boundary value problem

$$\begin{array}{ll} \frac{\partial \hat{L}_P}{\partial t} - D\frac{\partial_P^2 \hat{L}}{\partial \theta^2} = b\left[L - \hat{L}_P\right] & \text{for } t > 0, \quad \theta \in (0, 2\pi), \\ \hat{L}_P(t, 0) = \hat{L}_P(t, 2\pi), & \\ & \text{for } t > 0, \\ \frac{\partial \hat{L}_P}{\partial \theta}(t, 0) = \frac{\partial \hat{L}_P}{\partial \theta}(t, 2\pi) & \\ \hat{L}_P(0, \theta) = L_P(0, \theta) & \text{for } \theta \in [0, 2\pi]. \end{array} \tag{76}$$

Then $\hat{L}_P$ is a lower solution of (1) and thus $\hat{L}_P(t, \theta) \leq L_P(t, \theta)$. Let $u(t)$ be defined by

$$u(t) = \frac{1}{2\pi}\int_0^{2\pi} \hat{L}_P(t, \theta)\, d\theta.$$

By integrating both sides of equation (76) over $[0, 2\pi]$, we find

$$u'(t) = b[L - u], \qquad u(0) = \frac{1}{2\pi}\int_0^{2\pi} L_P(0, \theta)\, d\theta \leq L.$$

It is clear that

$$\lim_{t\to\infty} u(t) = L.$$

Thus,

$$L \geq \lim_{t\to\infty} \frac{1}{2\pi}\int_0^{2\pi} L_P(t, \theta)\, d\theta \geq u(t) \geq L.$$

Since $L_P(t, \theta) \leq L$ for $t > 0$, $\theta \in [0, 2\pi]$, the above relation implies that $\lim_{t\to\infty} L_P(t, \theta) = L$. This proves Proposition 6.

G. Proof of Proposition 7

It suffices to show that $\bar{L}_P$ defined by (26) is a steady state solution. Since $k \geq 1$ and $\rho > \lambda_k$, both S_k^+ and S_k^- are nonempty and $\psi(t, \theta) \to +\infty$ in S_k^+ and $\psi(t, \theta) \to -\infty$ in S_k^-. Hence, by (19) and

(20)

$$\bar{c}\left(\theta, \bar{L}_P\right) = \begin{cases} 0 & \text{in } S_k^+, \\ \frac{A(\theta)}{N(\theta)}\left[L - \bar{L}_P\right] & \text{in } S_k^-, \end{cases}$$

if $\bar{L}_P > 0$ and

$$\bar{c}\left(\theta, 0\right) = \begin{cases} \frac{A(\theta)[\phi(\theta)-\nu(\theta)]}{N(\theta)\phi(\theta)}L & \text{in } S_k^+, \\ \frac{A(\theta)}{N(\theta)}L & \text{in } S_k^-. \end{cases}$$

Hence, $\bar{L}_P\left(\theta\right)$ satisfies

$$-D\bar{L}_P''\left(\theta\right) = \begin{cases} -A\left(\theta\right)\left[\phi\left(\theta\right) - \nu\left(\theta\right)\right]\left[L - \bar{L}_P\left(\theta\right)\right] & \text{if } \bar{L}_P\left(\theta\right) > 0, \\ 0 & \text{if } \bar{L}_P\left(\theta\right) = 0, \end{cases} \qquad \text{in } S_k^+$$

and

$$-D\bar{L}_P''\left(\theta\right) = A\left(\theta\right)\nu\left(\theta\right)\left[L - \bar{L}_P\left(\theta\right)\right] \qquad \text{in } S_k^+.$$

It is clear that $\bar{L}_P\left(\theta\right)$ defined by (26) satisfies the above two equations. This completes the proof.