Unveiling Proportional Play Equilibrium: Understanding the Dispersion of Contest Success

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Abstract

We explore how the distribution of individual success probabilities in an asymmetric contest varies with the perceived relationship between efforts and success chances. Based on the empirically well-supported illusion of proportionality, we propose a Proportional Play Equilibrium (PPE) as an alternative behavioral theory to the traditional Nash Equilibrium (NE). Our PPE predicts a more extreme dispersion of success chances compared to NE. We test our theory in a laboratory experiment using a standard Tullock contest and find substantial support for PPE, where PPE aligns more closely with the empirical data and NE significantly deviates from it.

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1 Introduction

Many competitive situations in economics and politics can aptly be described as contests, wherein involved parties engage in competition over scarce resources. These contests manifest in various forms, encompassing scenarios ranging from agents competing for monopoly franchises and political candidates striving for electoral victory, to researchers vying for funding, athletes and teams battling in sports tournaments, and firms competing for market share. Central to the nature of these competitions is strategic decision-making, critically hinging on the anticipation of opponents' actions and a nuanced understanding of the probabilities of success in relation to the efforts and costs involved. The study of contests therefore is essential for decoding these complex interactions and predicting the related outcomes.

Traditionally, much of the focus in contest theory has been placed on studying the efforts exerted by participants. However, an equally critical aspect, often overlooked, is the analysis of success probabilities and market shares. In many real-world scenarios, from corporate strategies to political campaigns, the primary concern is not just the effort or resources expended but the actual chances of achieving a desired outcome. For instance, in political elections, while campaign efforts in terms of funding and rallies are important, what ultimately matters is the probability of winning – the share of votes a candidate is likely to secure. Similarly, in the business world, companies are deeply interested in their market shares as a measure of success, which often dictates future strategy and investment more than the mere calculation of resources spent. The human resource division of a company might care about designing an assessment center as a contest that maximizes the chance that the most able participant wins. In the realm of sports, athletes and teams strategize not merely based on the effort they put in but also on their perceived chances of victory. In reaction, designers of sports tournaments are often concerned with ensuring unpredictability and competitiveness, striving to create conditions that offer reasonably balanced chances of success among all contestants.

The prevailing theoretical analysis in contest theory has predominantly focused on modeling efforts. However, empirical evidence from various contest experiments robustly indicates that the effort levels predicted by traditional Nash Equilibrium (NE) are lower than what is exerted by real players. This discrepancy suggests that NE may not fully encapsulate the behavioral dynamics in contests.

In response, this paper proposes an alternative behavioral concept – Proportional Play Equilibrium (PPE) – to study the distribution of success probabilities in complete-information imperfectly discriminatory contests. PPE is rooted in the empirically well-supported idea that participants often rely on proportional reasoning when making decisions. Essentially, in PPE, players base their strategies on a presumed proportionate relationship between their efforts and success chances, albeit overlooking the true, highly non-linear nature of this relationship. This heuristic approach, intuitive yet simplistic, leads to predictions that markedly diverge from established theory: PPE inherently predicts larger efforts and a more pronounced dispersion of success chances compared to NE, thereby holding promise for better reconciling theoretical predictions with actual participant behavior.

By reconceptualizing the contest as a competition for market shares, our approach offers a direct method to analyze crucial yet often overlooked aspects in contest theory: success probabilities and market shares. This novel methodological framework, while maintaining equivalence with traditional equilibrium analysis, significantly streamlines the formal examination of success chance dispersion. It provides a clear, straightforward means to juxtapose the predictions of PPE with those of NE and facilitates a broader study of the comparative-statics of success probabilities. Furthermore, PPE not only introduces a novel behavioral perspective but also finds alignment with principles exogenously assumed for large aggregative games, particularly in how players perceive and react to aggregate actions. This congruence subtly bridges behavioral insights with existing economic theories, enriching our understanding of strategic decision-making in contests.

As we delve deeper into the comparative study of PPE and NE, our analysis reveals that PPE and NE are continuously related in the degree to which contestants account for the feedback effect of their choices; they are in fact homotopic to each other. This tight formal relation is not only of theoretical interest, but provides a valuable analytical path for systematically studying the relationship between the equilibrium success chances that these different equilibrium notions imply.

Our theoretical exploration reveals that proportional play not only leads to higher equilibrium efforts but also engenders in a more extreme dispersion of success chances compared to NE, a pattern that becomes increasingly pronounced in contests with a smaller number of players. Our theoretical results underscore the potential of PPE in offering a comprehensive, behaviorally grounded framework for studying the dynamics of contests, regardless of their scale.

These insights motivate our empirical testing of PPE in a controlled laboratory setting using a traditional Tullock contest with a small number of players. Our findings provide clear evidence that PPE aligns more closely with empirical data: While the distribution of success chances implied by NE significantly deviates from the empirical distribution, the predictions of PPE demonstrate an almost perfect alignment, thereby underscoring its effectiveness in capturing the nuanced dynamics of success probabilities. This observation suggests that the behavioral patterns often associated with large-scale aggregative games may be equally pertinent in smaller contests.

Regarding efforts, we observe that PPE tends to over-predict effort levels compared to the behavioral data. However, we propose that this deviation can be reconciled by integrating a success-based notion of risk aversion: When players exhibit risk aversion with respect to contest success, valuing the uncertainty of rewards differently from the certainty of costs, a modified equilibrium emerges. This equilibrium maintains the same dispersion of success chances as predicted by PPE but implies globally lower efforts, in line with the observed behavior. These insights highlight the intricate interplay between risk perceptions, effort allocation, and success probabilities in contests, offering a more complete picture of strategic decision-making and equilibrium outcome in such settings.

The remainder of this article is structured as follows. Section 2 reviews the relevant literature. We present the formal model, the notion of Proportional Play Equilibrium and the equilibrium analysis in Section 3. Section 4 explains our experimental design and Section 5 contains the corrsponding empirical analysis. Finally, Section 6 draws a conclusion.

2 Related Literature

Experimental studies on Tullock contests (Tullock et al., 1980) consistently reveal a compelling phenomenon: participants tend to expend more effort than predicted by Nash equilibrium (NE). This trend of overbidding, first documented by (Millner and Pratt, 1989), has been robustly observed across a diverse array of experimental designs and conditions. Comprehensive literature reviews by (Sheremeta, 2013, 2015; Dechenaux et al., 2015) detail these findings, underscoring the persistence of overbidding regardless of variations in experimental setups. Consistent with the previous contest experiments, we also find significant overbidding in all of our experimental conditions.

The consistent observation of overbidding invites closer examination of the underlying behavioral causes. Research in behavioral economics has proposed several explanations to rationalize overbidding, concentrating on the case of symmetric contests. These include Joy of Winning (Parco et al., 2005; Sheremeta, 2010; Mago et al., 2016; Boosey et al., 2017; Bruner et al., 2022), unsystematic evaluation errors as conceptualized by the Quantal Response Equilibrium (QRE) Sheremeta (2011), possibly combined with level-k depth of reasoning (Lim et al., 2014), asymmetric probability-weighting aligned with Prospect Theory (Parco et al., 2005; Baharad and Nitzan, 2008) risk preferences (Hillman and Katz, 1984; Millner and Pratt, 1991), loss and inequality aversion (Eisenkopf and Teyssier, 2013), competitive maximization of relative payoff (Mago et al., 2016), impulsiveness (Sheremeta, 2018; Bruner et al., 2022), and demographic and religious differences (Price and Sheremeta, 2015).[1]

While these behavioral studies mark significant strides in aligning theoretical models with psychological realities, they remain anchored in the analysis of symmetric equilibria due to the commonly presumed homogeneity of all contestants. This symmetry assumption, while simplifying the theoretical analysis, overlooks the heterogeneity that typifies real-world contests. Whether in economics, politics, sports or social settings, contest-like competitions commonly feature contestants

[1] The paper by Bruner et al. (2022) includes an incisive summary of what these different behavioral approaches imply in terms of payoff functions or decision behavior.

who differ in their abilities, resource endowments, risk appetites, information or strategic acumen. Such heterogeneity inevitably leads to a non-trivial equilibrium dispersion of success chances – a key aspect not captured by symmetric analysis.[2] Our approach aims to bridge this gap by adequately predicting the empirical dispersion of winning chances among heterogeneous contestants. This focus is particularly pertinent given that previous studies on heterogeneous contestants, compactly survey by March and Sahm (2017), have primarily examined the discouragement effect, where asymmetries reduce individual and aggregate efforts. While these studies offer valuable insights, they leave the dispersion of success probabilities unexplored, an aspect that March and Sahm (2018) began to address in relation to risk attitudes and effort costs in two-player Tullock contests. As articulated in the introduction, our emphasis on success probabilities is driven by the recognition that understanding their equilibrium dispersion is key to achieving a realistic comprehension of how contests unfold. Shifting the focus from quantifying efforts to a nuanced analysis of success probabilities is not merely a theoretical advancement but also carries significant practical implications for policymakers or contest designers, who often have reasons to prioritize the dispersion of success chances as the relevant contest outcome. Moreover, acknowledging heterogeneity in context of the equilibrium success chances not only captures a relevant feature in most real-world scenarios but also opens avenues for exploring more intricate strategies and outcomes, thereby enriching our understanding of the equilibrium forces at play and their impact on success chances or the dispersion of market shares.[3]

In contests, players choose their actions to influence their probability of success, which naturally makes the way how people process probabilistic information a key concern. Despite the prevalence of cognitive biases in decision-making, existing contest theories have not fully explored systematic biases in probability perception,

[2]Symmetric contests fall into a category of games that generally lack asymmetric equilibria (Hefti, 2017).

[3]With risk-neutral players, Tullock contests are isomorphic to models where different players (firms) can invest some form of resource to influence their market shares (Hefti and Teichgräber, 2022). The Nash predictions are analogous in both scenarios, the formalistic difference being that firms deterministically earn a portion of the total market revenue in the latter model, while this portion corresponds to a success probability in the conventional Tullock contest.

a gap that our concept of Proportional Play Equilibrium (PPE) seeks to fill. It is a well-documented fact that, when processing complex quantitative information, people pervasively apply proportional reasoning, even in highly non-linear contexts. The *Illusion of Proportionality*, a term going back to Freudenthal (1983), refers to a common misconception where individuals incorrectly assume direct proportionality in non-linear relationships.[4] A simple example from elementary geometry is that people tend to incorrectly generalize changes in linear dimensions to changes in area and volume. Early studies established a strong prevalence of the bias in case of students De Bock et al. (1998) that persists over time (Esteley et al., 2010) and was found in many different fields (Christodoulou, 2022), in particular also in non-scientific and routine-based tasks (Duma, 2021). Most crucial for our approach is that the illusion of proportionality extends to the realm of probabilistic reasoning (Van Dooren et al., 2003), demonstrating that inappropriate proportional thinking significantly influences the understanding of probabilities, which is a key element in strategic decision-making in contests.[5]

The evident inclination towards proportional reasoning suggests a pertinent and systematic bias specifically in the type of probabilistic decision that is characteristic for contests, and its integration in contest theory can offer a new perspective about how contestants form perceptions about success probabilities in relation to own actions. We conceptualize the paradigm of proportional thinking in our notion of PPE. PPE predicts that players estimate success chances to vary in proportions with their efforts. This systematic tendency to proportionally extrapolate probabilities stands in a stark contrast to the probabilistic distortions implied by models based on Prospect Theory Parco et al. (2005); Baharad and Nitzan (2008). While probability-weighting affects how the obtained probabilities are weighted in the expected utility framework utility, PPE directly affects the perceived likelihood of success based on effort. This difference also means that PPE can offer novel predictions that other models may not easily replicate. For example, PPE consistently predicts aggregate over-bidding relative to conventional Nash equilibrium due to

[4]The terms "Illusion of Proportionality" and "Illusion of Linearity" are used interchangeably.

[5]Van Dooren et al. (2003) show that the illicit application of proportionality is the heuristic that best explains the prevalent misconception of probability. Moreover, this bias remains a persistent trait even after study participants obtained specific training in probability theory.

its linear perception bias, while over- or under-bidding may occur with probability-weighting depending on details of the assumed weighting function and the contest itself (Baharad and Nitzan, 2008).[6]

Beyond its behavioral underpinning, PPE is notable for its exceptional tractability, particularly when analyzing complex equilibrium properties like the dispersion of success probabilities. This is most pertinent in contexts involving an arbitrary number of heterogeneous players and non-linear effort costs, where traditional approaches often favored symmetric models or numerical simulations to overcome the analytical challenges posed by such complexities. The methodology we employ in our paper not only masters these complexities but also yields direct and clear insights into how the equilibrium success probability dispersion under PPE qualitatively diverges from that predicted by conventional NE. More generally, the tractability of our approach not only yields a powerful concept for studying equilibrium properties in static contests as accomplished by this paper, but extends its applicability to dynamic contests with heterogeneous players (Hefti et al., 2020) or contests with agent entry (Hefti and Lareida, 2022).

In sum, while significant advancements have been made in aligning contest theory with psychological insights, a conspicuous gap remains in addressing contestants heterogeneity, the dispersion of their success chances and the intricacies of probabilistic information processing by human players. Our introduction of PPE, based on the concept of proportional reasoning, seeks to interconnect these aspects. PPE provides a comprehensive and tractable framework for analyzing the distribution of success with heterogeneous contestants. This approach delivers testable predictions and promises to illuminate new aspects of strategic behavior, particularly how contestants strategize in pursuit of specific success chances, enriching contest theory with theoretical depth and practical implications. To our knowledge, we also are the first paper to empirically explore how well equilibrium theories, like NE or PPE, are able to predict the empirical distribution of success probabilities in contests.

[6] PPE also differs from models of noisy evaluations like QRE. The latter introduces distortions as stochastic variations of the chosen actions due to unsystematic subject confusion, while PPE posits a consistent, predictable manner in which contestants misjudge probabilities based on their actions.

3 The Model

Contest theory has been instrumental to encapsulate the strategic interactions among contestants who can invest resources ("efforts") to affect their chances of achieving some objective of a given valuation $V > 0$. Such situations span a myriad of applications, from political competition and R&D races to legal disputes and competitive marketing strategies.[7] The payoff function represents the expected net gain of a contestant j and is commonly of the form:

$$\Pi_j = p_j V - C_j. \tag{1}$$

Given a finite set of contestants $\mathcal{J}_n = \{1, ..., n\}$, each p_j is a probability such that $\sum_{\mathcal{J}_n} p_j = 1$ and C_j denotes the overall costs incurred by j.[8] The traditional interpretation of (1) is that n risk-neutral contestants compete for a prize worth V that eventually is seized by a single agent, such that p_j is the individual probability of success. Political competition or patent races are standard examples for this. In the former, the payoff function can be understood as the political power or prestige gained by the winner (the prize V) minus campaigning costs. Patent races are common in high-technology industries where firms compete to innovate and patent new technologies. The "race" aspect arises because being first to patent can confer significant competitive advantages. In this case, the prize V quantifies the market advantage or expected financial gain from securing a patent first, p_j reflects a firm's likelihood of being the first to innovate and patent, while C_j summarizes the costs (R&D investments, human capital expenditures,...) associated with the efforts.

The model with payoff (1) is isomorphic to a setting where the probability p_j represents the market share of a global revenue V that contestant j manages to secure. The formalistic difference is that the revenue $\pi_j V$ is deterministically earned, while π_V amounted to the expected revenue in the former as the contest features the "winner-takes-it-all" property.[9] One example is that firms are engaged in some

[7] See Konrad (2009) for a textbook treatment.

[8] The model can be reformulated for a continuum of contestants $\mathcal{J}_n = [0, n]$, in which case $p : \mathcal{J}_n \to \mathbb{R}_+$, $j \mapsto p_j$, is a density with support $\mathcal{J}_n$ (Hefti and Teichgräber, 2022).

[9] Surprisingly many models of agent competition can be subsumed within such a generalized contest framework, including models of perfect or monopolistic price competition (Hefti and Teichgräber, 2022), showing that the contest structure may be far more ubiquitous than traditionally assumed.

form of combative advertising. The effectiveness of these advertising campaigns is often measured by the increase in market share or sales revenue they generate. In this context, V represents the total potential revenue in the market that firms are competing for and C_j correspond to the marketing expenses incurred by each firm. The market share p_j each firm achieves then is a direct result of how effective its advertising strategy is relative to its competitors.

In the digital economy, online platforms (such as social media networks, e-commerce marketplaces, or streaming services) compete for users or customers. The value of these platforms typically increases as more users join, a phenomenon known as network effects. Unlike traditional winner-takes-all contests, these markets often sustain multiple successful platforms, each with a significant share of the market. Success in this context is defined by market share, aligning with the p_j interpretation as a share of common gross revenue V. Each platform's goal is to maximize its portion of the total market value. In this scenario, V represents the total potential value generated by the market – for instance, the total advertising revenue in social media or total sales in e-commerce marketplaces, and C_j pertains to the costs associated with investments in technology, user experience, marketing, and content aimed at attracting and retaining users. The market share p_j for each platform is a key metric for its relative success in attracting users compared to its competitors. Another example again stems from political economics. In many democratic systems, political parties or candidates compete in elections to win seats in a legislative body, such as a parliament or congress. The total number of seats available (V) is fixed, and the objective of each competitor is to secure as many of these seats as possible. The costs C_j captures the resources expended by parties or candidates in their election campaigns, including financial spending, manpower, and strategic planning. The market share p_j for each party or candidate is then the proportion of seats they win in the legislature.

3.1 Determining Contest Success

We suppose that the different contestants can invest resources to influence their chance of success p_j. The resources invested by $j \in \mathcal{J}_n$ generate a certain impact

$e_j \in \mathbb{R}_+$. The profile of impacts $\mathbf{e} = (e_1, ..., e_n)$ determines the distribution of success chances $\mathbf{p} = (p_1, ..., p_n) \in \Delta^{n-1}$, where Δ^{n-1} is the $(n-1)$-dimensional simplex. In what follows we assume that each p_j is determined by the own impact e_j relative to total impact $\sum_i e_i$, i.e.,

$$p_j = P(e_j, \sum_i e_i). \tag{2}$$

The function $P : \mathbb{R}_+ \times \mathbb{R}_+ \to [0,1]$ in (2) is called a Contest Success Function (CSF). We discipline the CSF by assuming that it is zero homogeneous in (e, Σ), assuring scale-invariance of the impacts for shaping success probabilities. A highly convenient implication of zero homogeneity is that it uniquely pins down the functional form of the CSF: Given that P is differentiable and integrates to one for any non-zero impact profile $(e_1, ..., e_n) \in \mathbb{R}_{++}$ with $\Sigma = \sum_i e_i > 0$, the function P is determined as

$$p_j = P(e_j, \sum_i e_i) = \frac{e_j}{\sum_i e_i}. \tag{3}$$

if and only if P is zero homogeneous in (e_j, Σ).[10]

A key property of (3) is that it satisfies the principle of replicability. Two contestants i, j obtain exactly the same success chances, $p_i = p_j$, if and only if $e_i = e_j$. Likewise, $p_1 = ... = p_n = 1/n$ if and only if all contestants obtain the same impact $e \geq 0$.[11] In essence, this property states that a Buridanic outcome of competition, where all contestants split the pie in equal proportions, is always a theoretical possibility and occurs if and only if all impacts are identical.[12] This property can be illustrated, e.g., in case of a two-candidate election: If one candidate manages to fully replicate the impact of the other, this equates success chances as voters are completely undecided between them.

While we regard the property of replicability as a highly natural theoretical benchmark to impose on the nature of competition, this property does not imply that it is similarly feasible for both candidates to obtain the same impact. For

[10]See Hefti and Lareida (2022).

[11]For the case where $e_1 = ...e_n = 0$ we extend (3) by defining $P(0,0) = 1/n$.

[12]Buridan's competition refers to a philosophical paradox known as "Buridan's Donkey", attributed to the 14th-century philosopher Jean Buridan. The paradox illustrates a situation where a donkey, placed exactly midway between two identical bales of hay, dies of starvation because it is perfectly undecided between the two bales.

example, one candidate could have access to more funding, allowing for more efficient or voluminous campaigning, or voters may be differentially receptive towards the two candidates. We capture this key notion of possible agent heterogeneity by distinguishing between *impact* and *investment*, where impact e_j is a consequence of investment expenditures worth $f_j \geq 0$ and determined by a non-negative, (production) function

$$e_j = h_j(f_j). \tag{4}$$

Thus, the extent to which a contestant j may replicate or exceed a given impact of its competitors is governed by her h_j-function, and the functions h_j, $j \in \mathcal{J}_n$ generally accommodate the ex-ante heterogeneity of the contestants in producing a desired impact. To illustrate, if f_j amounts to campaigning expenditures, a political candidate j would need to expend comparably more to possibly replicate competitor i's impact if voters ex-ante are more inclined towards candidate i, as captured by $h_j(f) < h_i(f)$.[13]

Replacing impacts with investment, the expected contest payoff (1) is $\Pi_j = \frac{h_j(f_j)}{\sum_i h_i(f_i)} V - f_j$. It is analytically more convenient, however, to restate Π_j in terms of impacts rather than investments. This can always be accomplished if each h_j is strictly increasing, which we assume. Then, $h_j(f)$ has a well-defined inverse $f_j = h_j^{-1}(e_j) \equiv C_j(e_j)$, where we interpret $C_j(e_j)$ as the investment costs of j associated with achieving impact e_j. Restating the payoff in terms of impacts we obtain

$$\Pi_j = \frac{e_j}{\sum_i e_i} V - C_j(e_j), \tag{5}$$

where the cost functions $C_j(e_j)$ encode all information about the ex-ante agent heterogeneity.

Leading Example: Power Function Costs. The case $h_j(f) = \theta_j f^\mu$ has received most attention in the applied literature.[14] The corresponding impact costs

[13]For example, if $h_j(f_j) = \theta f_j$ and $h_i(f_i) = 2\theta f_i$, then j needs to incur twice the expenditure of i to replicate impact e_i. Whether replication is among the *feasible* options depends on the h_j-function. For instance, one could assume that $h_j(f_j)$ is bounded from above, such that a certain impact is not achievable even at infinite investment. Similarly, one could include a budget constraint $f_j \in \{0 \leq f_j \leq k_j\}$.

[14]The symmetric ratio form $f_j^\mu / \sum_i f_i^\mu$ was first employed by Tullock et al. (1980) in the context of rent seeking, and axiomatic and probabilistic foundations have been identified by the literature for this particular form; see Jia et al. (2013) for an overview.

then also adopt the power function property: $C_j(e_j) = c_j e_j^\eta$, with $\eta = \frac{1}{\mu}$ and $c_j = \theta_j^{-1}$. The elasticity parameter μ quantifies how sensitive the success probabilities are to relative investments f_j/f_i or, equivalently, how sensitively investment expenditures respond to changes in impacts.[15] It is fairly straightforward to show that the ratio $h_j(f_j)/\sum_i h_i(f_i)$ is zero homogeneous in the investment profile $(f_1, ..., f_n)$ if and only if h_j has the power function property. Thus, the power function form of h_j implies that the CSF is zero homogeneous also in investments, not only in impacts.

Beyond the foundation based on transformed investments, the case of power function costs bears further content. Suppose that $C_j(e)$ is a non-negative twice continuously differentiable function. The cost function $C(e)$ allows for a power function representation $C(e) = ce^\eta$ if and only if it is homogeneous of some degree $k \in \mathbb{R}$. Homogeneity encapsulates an intrinsic invariance to scale in relation to the sensitivity of (marginal) costs. In particular, it implies that the responses of (marginal) costs to proportional changes in e are proportionally consistent across all levels of $e > 0$. Thus, despite that costs can be subject to increasing returns to scale (which happens for $\eta > 1$) there are no scale effects in the sensitivity of (marginal) costs to proportional increases in e. Consequently, contestants, regardless of their impact level e, are equally susceptible to proportional changes in e. This property makes power function costs a relevant benchmark for analyzing contest equilibria, as it reflects a commonality in cost responsiveness irrespective of the scale of contestant involvement.

3.2 Proportional Play

Each contestant can influence her payoff Π_j (5) by choosing her impact $e_j \geq 0$. The set of players $\mathcal{J}_n$, the joint strategy space $\mathbb{R}_+^n$ and payoff functions $\{\Pi_i\}_{i \in \mathcal{J}_n}$ in (5) define a complete information game. A Nash Equilibrium (NE) then is a joint impact profile $\mathbf{e}^* = (e_1^*, ..., e_n^*)$, where each e_j^* maximizes Π_j given the opponent

[15] If $\mu \to 0$, then relative differences become irrelevant or, equivalently, impact costs become perfectly elastic. If $\mu \to \infty$, then the contest approximates a perfectly discriminatory all-pay auction or, equivalently, impact costs become perfectly inelastic. Generally, smaller values of μ (larger values of η) embody contests, where it becomes more difficult for contestants to influence their success probabilities.

impact profile $\mathbf{e}^*_{-j}$. NE imposes that each contestant j can correctly anticipate the equilibrium impact profile of her opponents $\mathbf{e}^*_{-j}$ and can also identify her best response e^*_j to these impacts.

Nash play requires that each contestant correctly accounts for how own impact e_j affects the probability of success p_j. The main behavioral hypothesis of this paper, based on the Illusion of Proportionality outlined in Section 2, is that actual human players evaluate their influence on their success probability in a proportional way. That is, they misconceptually perceive their success probability $\tilde{p}_j$ to scale with their impact according to $\tilde{p}_j(\lambda e_j) = \lambda \tilde{p}_j(e_j)$, $\lambda > 0$. This implies that their perceived success probability must be of the form $\tilde{p}_j(e_j) = e_j/\Sigma$, where $\Sigma > 0$ is a constant.[16]

3.2.1 Main Conjectures

In this paper we are primarily interested in the distribution $\mathbf{p} = (p_1, ..., p_n) \in \Delta^{n-1}$ of the equilibrium success chances. Drawing on the distinctive characteristics of proportional thinking, we put forward two main conjectures to guide our analysis.

Conjecture 1: Overinvestment with Proportional Play. Proportionally thinking contestants tend to overestimate their impact on contest success compared to Nash behavior. Specifically, such contestants overlook the diminishing returns of their marginal success chances, making them prone to exert more effort than what Nash behavior would dictate. Consequently, we conjecture that proportional play encourages contestants to exert more effort on average compared to Nash play.

Conjecture 2: Proportional Play unbalances the Success Distribution Our second conjecture addresses the uneven impact of the probabilistic misjudgment induced by proportional thinking across contestants with varying cost structures. We anticipate that this effect will be most significant for low-cost players, as these tend to exert most effort and achieve the highest success chances; high-cost

[16]Proportional thinking suggests that the estimated probability $\tilde{p}_j(e_j)$ might extrapolate beyond 1. This aligns with empirical findings from Support Theory, where human subjects' perceived probabilities can extend to more than one (Rottenstreich and Tversky, 1997). However, it is essential to remark that this is a pure extrapolation property in our case only; in equilibrium, chosen probabilities will invariably fall within the 0 to 1 range.

players, contributing minimally, are less affected. Given that only Nash players adjust their behavior to the diminishing marginal success chance, we conjecture that Nash behavior results in a more balanced success distribution compared to proportional play.

3.2.2 Proportional Play Equilibrium

In order to systematically assess the above conjectures about the influence of proportional play, we need to define an equilibrium notion with proportionally thinking players. Our equilibrium definition of a Proportional Play Equilibrium (PPE) seeks to reconcile the behavioral notion that individual contestants aim at maximizing their expected profits, while being subjected to the Illusion of Proportionality, with the consistency requirement that people hold consistent beliefs about the average choice of impact $\bar{e} = \sum_i e_i/n$.

Definition 1 (PPE) *A PPE is a an impact profile* $\mathbf{e} = (e_1^*, ..., e_n^*)$ *such that each* e_j^* *maximizes* $\Pi_j = \frac{e_j}{n\bar{e}^*}V - C_j(e_j)$ *for given average effort* $\bar{e}^* = \sum_i e_i^*/n$.

Note that our consistency assumption about beliefs assures that each contestant holds a correct anticipation of her success probability p_j^* in equilibrium. Beyond assuring that the essential bias, relative to NE, is due to the Illusion of Proportionality,[17] requiring that players merely hold correct estimates about average play may align more closely with how people process information. Human cognition is more attuned to recognizing average patterns, trends or "market sentiments", rather than fully processing complex matrices of individual strategies and payoffs. For example, psychological research has documented evidence in favor of a cognitive process, called "perceptual averaging" (Albrecht and Scholl, 2010), according to which complex information is mentally summarized by obtaining an average representation of a set of objects or features. For example, when looking at a flock of birds, instead of noting the size and position of each bird, the brain computes an average size and direction of motion.[18] In behavioral finance, it is

[17]The difference to the fully rational probability perception imputed by NE is that proportionally thinking contestants do not account for the feedback effect their choice of impact causes on p_j via $\sum_i e_i$.

[18]Perceptual averaging is part of what researchers have called "Statistical Summary Representations"; see Haberman and Whitney (2011) for an overview. This notion refers to the brain's

a well-acknowledged notion that individual traders often respond to a common market sentiment, rather than to individual actions or motivations of other market participants.[19]

Beyond being more reflective of the cognitive processes involved in real-world decision-making processes than NE, as evidenced by the human tendency to collapse complex information into typical or representative quantities, PPE also offers a simpler representation of important equilibrium objects. Due to the Illusion of Proportionality, the marginal success probability remains constant irrespective of the own impact e_j, and diminishes monotonically with an increase in average effort. This intuitive property captures that influencing one's success probability becomes increasingly challenging as the collective efforts of all contestants escalate. As individual players essentially best-respond to the average behavior, it follows that best replies are monotonically decreasing in average impact, a stark contrast to the complex and non-monotonic nature of best reply functions under Nash behavior. Furthermore, the simple, monotonic structure of best replies in PPE has positive implications for the strategic stability of the equilibrium.[20]

3.3 Equilibrium Analysis

In this study we examine the distribution of the success probabilities (or market shares) $\mathbf{p} \in \Delta^{n-1}$ across the n contestants in $\mathcal{J}_n$. Note that $\mathbf{p}$ can be represented by a probability density function $p : \mathcal{J}_n \to [0, 1]$, assigning each contestant j a success probability $p_j = p(j) = \pi_j(\mathbf{p})$, where $\pi_j(\mathbf{p})$ is the j-th component of $\mathbf{p}$. The main challenge in analyzing both PPE and NE arises from the model's lack of closed-form solutions, complicating the comparison of equilibrium success distributions without assuming specific numeric details about the number of players or their costs. To

general ability to abstract statistical information from groups of similar objects, often without explicit awareness, allowing for a quick understanding of the aggregate or average properties of the group.

[19]This sentiment represents the average mood or the typical market behavior prevailing in the market. Market sentiments generally align with the concept of the Representative Heuristic (Kahneman and Tversky, 1974). This heuristic suggests that in complex situations, people may use simple proxies to encapsulate the 'typical' or average scenario they face.

[20]As established in Hefti (2016), response behavior targeted at averages, as opposed to individual strategies, exhibits stronger stability properties with respect to standard dynamics based on best-reply functions or gradients associated with optimality conditions.

address most these analytical difficulties, we explore an equivalent representation of the model, where contestants are engaged in a competition for market shares. This approach reveals a single framework that includes PPE and NE as boundary cases, enabling a unified examination of their equilibrium success distributions.

For our equilibrium analysis we assume in what follows that $C_j(e) = c_j C(e)$ with $c_j > 0$ for each $j \in \mathcal{J}$, where $C(e)$ is C^2-function that satisfies the following standard assumptions. 1) Zero (marginal) cost at zero impact: $C(0) = C'(0) = 0$. This assumption embodies the standard economic notion that inactivity ($e = 0$) should not lead to any choice-relevant costs. 2) Positive, increasing marginal costs: $C'(e), C''(e) > 0$ for all $e > 0$. Conditions 1) and 2) assure that the first-order conditions are sufficient for characterizing optimal behavior. More generally, they are not only fundamental in ensuring mathematical tractability but also in guaranteeing existence of a unique interior equilibrium (NE and PPE), which is pivotal for comparative statics analysis.

3.3.1 Representation as Market Share Game

Consider the CSF $p_j = \frac{e_j}{\Sigma_j + \alpha e_j}$, where the quantity $\Sigma_j > 0$ is exogenous to contestant j. The corresponding payoff function then is

$$\Pi_j(e_j; \Sigma_j) = \frac{e_j}{\Sigma_j + \alpha e_j} V - c_j C(e_j), \tag{6}$$

We interpret the parameter $\alpha \in [0, 1]$ as a measure of the extent to which contestants recognize the feedback effect of their own efforts relative to a presumed joint impact Σ_j of the rivals. An equilibrium is defined as a situation, where each contestant chooses e_j to maximize (6), correctly anticipating the average equilibrium impact $\bar{e}^*$.

Definition 2 *An equilibrium is an impact profile* $(e_1^*, ..., e_n^*) \in \mathbb{R}_+^n$ *such that each* e_j^* *maximizes* (6) *for given* Σ_j^**, where* $\Sigma_j^* = n\bar{e}^* - \alpha e_j^*$ *and* $\bar{e}^* = \sum_i e_i^*/n$.

It is straightforward to verify that this equilibrium notion encapsulates PPE and NE as special cases at $\alpha = 0$ and $\alpha = 1$, respectively.[21]

[21]Formulation (6) illustrates a nuanced view on how contestants perceive their impact versus the combined impacts of their rivals. At the one extreme, they perfectly distinguish between total impact and others' aggregate contributions under NE, while this distinction is entirely obscured under PPE, indicating a stark contrast in strategic awareness between the two equilibrium notions.

For studying the equilibrium distribution of success chances $(p_1^*, ..., p_n^*)$ associated with an equilibrium impact profile, formulation (6) is far from practical. Therefore, we introduce an equivalent representation of the game, where contestants directly choose their success probabilities, rather than indirectly via their choices of impacts. The essence of this approach is to note that, for any given $\Sigma_j > 0$, the CSF $p_j = \frac{e_j}{\Sigma_j + \alpha e_j}$ implies a bijective relation between p_j and e_j, where $e_j = \frac{p_j}{1-\alpha p_j}\Sigma_j$. Writing the payoff as a function of p_j then yields

$$\Pi_j(p_j; \Sigma_j) = p_j V - c_j C\left(\frac{p_j}{1-\alpha p_j}\Sigma_j\right) \tag{7}$$

We now consider the game, referred to as a Market Share Game by Hefti and Teichgräber (2022), where each player $j \in \mathcal{J}_n$ chooses an aspired success chance $p_j \in [0, \frac{1}{\alpha})$ to maximize payoff (7), treating $\Sigma_j > 0$ as an exogenous parameter. An equilibrium of the Market Share Game is as a success distribution $(p_1, ..., p_n) \in \Delta^{n-1}$ and an average impact $\bar{e}$ such that each p_j maximizes (7) taking the quantity $\Sigma_j = (1 - \alpha p_j)n\bar{e}$ as given.

Definition 3 *An equilibrium in the Market Share Game is a density $\hat{p}_\alpha : \mathcal{J}_n \to [0, 1]$ and an average impact $\bar{e}_\alpha$ such that each $p_j \equiv \hat{p}_\alpha(j)$ maximizes (7) for given $\Sigma_j \in \mathbb{R}_+$, and the number Σ_j is determined as $\Sigma_j = (1 - \alpha p_j)n\bar{e}$.*

In Appendix A.1, we rigorously establish three pivotal results. First, we show that the equilibrium set of the Market Share Game aligns with that of the original contest model. A similar equivalence has been demonstrated by Hefti and Teichgräber (2022) in context of other equilibrium models, and essentially reflects that there is a one-to-one relation between market shares p_j and the "action" variable e_j. The central implication is that we can rely entirely on the equilibrium notion in Definition 3 to study the equilibrium success distribution in PPE and NE of the contest model. Second, we prove that the Market Share Game invariably leads to a unique equilibrium $(\hat{p}_\alpha, \bar{e}_\alpha)$ under the assumptions we imposed on the cost function. Moreover, this equilibrium satisfies $\bar{e}_\alpha > 0$ and $\hat{p}_\alpha(j) > 0$ for every $j \in \mathcal{J}_n$. These findings are crucial for our comparative analysis of the success distributions induced by NE and PPE.[22] Third, we show that that the equilibrium success dis-

[22]Equilibrium existence and uniqueness is fundamentally ensured by the strictly monotonic nature of players' optimal responses to the assumed quantity Σ_j.

tribution of NE and PPE can be continuously deformed into each other – they are homotopic form a mathematical viewpoint – despite the distinct behaviors underlying PPE and NE. In terms of Definition 3, the equilibrium success densities of PPE and NE, respectively, are the two boundary cases $\alpha \in \{0, 1\}$, where $p^{PPE} = \hat{p}_0$ and $p^{NE} = \hat{p}_1$. We extensively leverage this tight analytical relationship between PPE and NE in the next section to derive the majority of our formal results.

It is easily checked that $\Pi_j(p_j; \Sigma_j)$ in (7) is strictly concave in $p_j \in [0, \frac{1}{\alpha})$ for any given $\Sigma_j > 0$, such that the FOC

$$V(1 - \alpha p_j) = c_j C'(p_j \Sigma)\Sigma, \qquad \Sigma \equiv n\bar{e} \tag{8}$$

characterizes the optimal choice of the aspired market share $p_j > 0$ for every contestant j, given the requirement that $\Sigma_j = (1 - \alpha p_j)\Sigma$.

Let $\mathbf{p}(\alpha) \in \Delta^{n-1}$, where $\pi_j(\mathbf{p}(\alpha)) = \hat{p}_\alpha(j)$, denote the vector of equilibrium success probabilities. In the equilibrium $(\hat{p}_\alpha, \bar{e}_\alpha)$, the quantity $\Sigma_\alpha = n\bar{e}_\alpha > 0$ and the vector $\mathbf{p}(\alpha) > 0$ must jointly solve the system of n equations as specified by (8). The major advantage of working with the above Market Share representation is that powerful analytical tools, developed in Hefti and Teichgräber (2022), can be used to study the comparative statics of the distribution of success chances. This approach only relies on straightforward properties of the representative equilibrium equation (8), circumventing the need to analyze the entire system of n equations. Crucially, we can leverage this comparative static analysis across the line $\alpha \in [0, 1]$ to uncover the relation between p^{PPE} and p^{NE}.

3.3.2 Comparative Analysis

In the following we use the Market Share Game with general payoff (7) to assess our main two intuitive conjectures about how the PPE $(p^{PPE}, \bar{e}^{PPE})$ relates to the NE $(p^{NE}, \bar{e}^{NE})$.

Comparing Average Impacts Our first finding confirms that PPE results in a higher average impact compared to NE.

Proposition 1 *In any contest, it holds that $\bar{e}^{PPE} > \bar{e}^{NE}$.*

We prove this claim by showing that the aggregate equilibrium impact $\Sigma(\alpha)$ is strictly decreasing. The underlying intuition is that contestants are inclined to increase their investments the more they overlook the reduced sensitivity of the CSF to the own impact brought about by the feedback effect.

Comparing Success Distributions To address our second conjecture, we need to compare the entire success distributions $\mathbf{p}^{PPE}$ and $\mathbf{p}^{NE}$, which is a challenging task in view of the complexity of the equilibrium objects. As first step, we identify key similarities of PPE and NE. These commonalities highlight the core elements that define contest outcome, independent of the specific equilibrium concept. Furthermore, they also set the stage for discerning the differences between the success distributions of PPE and NE.

In our analysis, we assume, without loss of generality, that contestants are ex-ante sorted in ascending order of their cost functions, meaning $c_1 \leq c_2 \leq ... \leq c_n$. Our first result shows that PPE and NE have identical success distributions if and only if all contestants have identical cost functions, where $c_i = c_j = c \; \forall i, j \in \mathcal{J}_n$.

Proposition 2 *For any $\alpha \in [0,1]$, $\hat{p}_\alpha(i) = \hat{p}_\alpha(j)$ iff $c_i = c_j$. Thus $p^{PPE} = p^{NE}$ iff all contestants have identical cost functions. Conversely, $p^{PPE}(j)$ and $p^{NE}(j)$ are non-constant, decreasing densities iff at least two contestants have different cost functions.*

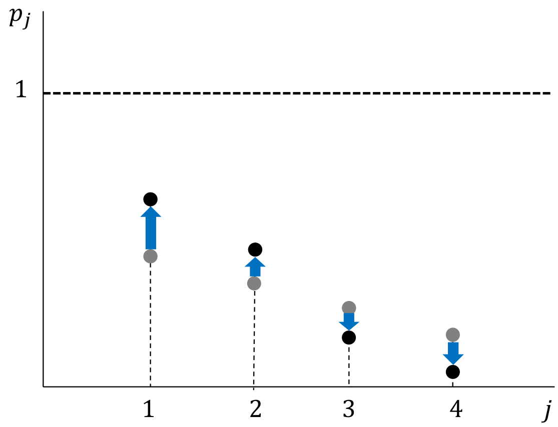
If all contestants have identical costs, a symmetric game results, implying that the unique equilibrium must be symmetric, i.e., given by the uniform density $p(j) = 1/n$, $j \in \mathcal{J}_n$.[23] By contrast, the equilibrium success distribution cannot be uniform whenever there is agent heterogeneity, indicacted by $c_1 < c_n$. The non-constant, decreasing nature of the equilibrium densities show that PPE and NE both satisfy a common "No-Leap-Frogging" property: Contestants with lower cost coefficients invariably secure higher equilibrium success probabilities. This intuitive result essentially reflects individual rationality. A low-cost contestant has the potential to match the success probability of a higher-cost rival at lower expenses, yet pursues a greater success chance as a consequence of optimal marginal reasoning.

[23] Symmetric games with a differentiable payoff function commonly possess symmetric equilibria (Hefti, 2017).

To deepen our understanding how the equilibrium success distributions of PPE and NE compare, we focus on scenarios where $C(e) = e^\eta$, $\eta > 1$, is given by an arbitrary power function. As elucidated earlier, this specification entails the key property of a proportionally sensitive cost function, and it encapsulates the "Tullock" contest framework, a benchmark in the field due to its extensive application across theoretical and empirical studies. We consider the case of an arbitrary cost function in the Appendix.

The formal challenges remain significant with the power function specification as the equilibrium equations (8) do not admit closed form solutions for $\alpha > 0$.Despite this challenge, our main result below shows that with heterogeneous contestants, p^{PPE} must be a *monotonic clockwise rotation* of p^{NE}. By a monotonic clockwise

Figure 1: Monotonic Rotations



Notes. The black distribution is a clockwise rotation of the gray distribution. The rotation is monotonic as indicated by the blue arrows, which increase in length towards the extremes.

rotation, we mean that

$$\frac{p^{PPE}(i)}{p^{PPE}(j)} > \frac{p^{NE}(i)}{p^{NE}(j)} \tag{9}$$

for any two contestants i, j with $c_i < c_j$.

Theorem 1 *Let $c_1 < c_n$. Then p^{PPE} is a monotonic clockwise rotation of p^{NE}. Therefore, PPE entails a greater disparity than NE according to any Lorenz-consistent inequality measure.*

The principle of a clockwise rotation is illustrated in Figure 1, where PPE's rotation of NE signifies that contestants towards either end of the advantage spectrum –

most advantaged or most disadvantaged – experience an amplification of their success probabilities in PPE relative to NE. A monotonic rotation is an even stronger property, in that the absolute gap between any two contestants i, j that obtain different success probabilities in a NE must widen in a PPE, indicated by the lengthening arrows in Figure 1. That is, even if the success chances of, say, the second strongest contestant increases in a PPE relative to NE, then the success chances of the strongest contestant must increase by even more.

Theorem 1 rosutly supports our second conjecture, demonstrating that the inherent bias of proportional thinking unbalances contestants' success probabilities relative to the Nash benchmark. More generally, the fact that PPE is a clockwise rotation of NE implies that the inequality of the success chances embedded in p^{PPE} must strictly exceed the inequality entailed in p^{NE} according to higher inequality measure, e.g., of the (S-)Gini coefficients, Theil indexes or the Coefficient of Variations. In the Appendix, we show that this key property is not specific to Power Functions, but necessarily holds for all cost functions $C(\cdot)$ that feature a non-increasing elasticity of marginal costs.

Unified Comparative Statics The intrinsic connection between PPE and NE exemplified by our previous results suggests a natural expectation for these equilibria to exhibit similar comparative statics behaviors. Specifically, with Power Function Costs $C(e) = e^{\eta}$, the contest model (7) has two key parameters: the value of the prize V and η, capturing how sensitive costs (or equivalently the CSF) respond to changes in individual behavior. The ensuing analysis confirms that the qualitative impacts of these central parameters on contest outcomes are not sensitive to the equilibrium concept.

Proposition 3 *For every $\alpha \in [0, 1]$, the equilibrium density $\hat{p}_\alpha$ is invariant to $V > 0$ and $\partial_V \Sigma(\alpha) > 0$. Further, an increase in contest sensitivity $d\eta < 0$ induces a clockwise rotation of $\hat{p}_\alpha$.*

The first result implies that contest designers cannot levearge the prize incentive as a means to re-balance the success distribution, independent of the equilibrium notion. The sole effect of increasing V is to escalate each contestant's impact (and hence total impact) while relative impacts e_i/e_j remain constant. The underlying rationale is that an increase in V affects all contestants by the same proportion, which is counterbalanced by the proportional sensitivity inherent in Power Function costs.[24]

Our second result highlights the CSF's critical role in shaping the competitive outcome. By "inducing a clockwise rotation", we mean the existence of a $\delta > 0$ such that $\hat{p}_\alpha(i;\eta')$ is a clockwise rotation of $\hat{p}_\alpha(i;\eta)$ for any $\eta' \in (\eta - \delta, \eta)$. Our result thus indicates that, regardless of the equilibrium concept, contestants with cost advantages are increasingly capable of converting these advantages into higher success probabilities as the CSF becomes more sensitive to their efforts.

4 Experimental Design and Procedures

We test our predictions for a standard Tullock contest (Tullock et al., 1980) as follows. Subjects compete for a prize in a contest by choosing their effort level x, which can be any number between 0 and 100. Within a contest, only one subject could win the prize. The winning probability of each subject is determined by the ratio of his/her effort relative to the total effort made by all subjects in that contest. Regardless of their winning outcome, each subject is presented with an effort cost associated with a quadratic function. Specifically, for a certain level of effort x, the associated cost is αx^2, where α is a randomly assigned cost coefficient.

We implement contests with group sizes N=2 and N=3. In each contest, subjects receive different cost coefficients with equal probabilities. For a contest with group size N=2, one subject receives a cost coefficient of 1, while the other receives a cost coefficient of 3. For contests with group size N=3, we adopt two different sets of cost coefficients, i.e., (1, 3, 6) and (1, 2, 3). The design allows us to not only test the theoretical predictions in different group sizes but also to compare them in different relative cost-competitive environments.

To ensure no negative payment for experimental participation, each subject is endowed with 60,000 experimental points at the beginning of a contest, and the prize size is set at 30,000 experimental points. The group compositions and

[24]We expand on this result in Appendix A.1.

theoretical predictions are shown in Table 1.

Table 1: Contest conditions and equilibrium predictions

	Gorup size		
	N=2	N=3	N=3
Cost coefficient	(1, 3)	(1, 3, 6)	(1 ,2, 3)
No. of groups /round	15	10	5
No. of round	20	20	18
	Equilibrium predictions		
Winning probability			
NE	(0.63, 0.37)	(0.55, 0.29, 0.17)	(0.47, 0.31, 0.23)
PPE	(0.75, 0.25)	(0.67, 0.22, 0.11)	(0.55, 0.27, 0.18)
Aggregate effort			
NE	93.1	111.6	130.6
PPE	141	150	165.8
Individual effort			
NE	(59, 34.06)	(60.97, 31.97, 18.66)	(61.11, 39.89, 29.61)
PPE	(106.06, 35.56)	(100, 33.3, 16.67)	(90.45, 45.23, 30.15)

Notes. Parameters of cost coefficient, equilibrium predictions of winning probability, and individual effort in a group are shown in tuples in parentheses.

In every session, subjects played the contest in 20 or 18 rounds. In each round, subjects were randomly matched into a new group, therefore, the composition of groups and the cost coefficients may change across rounds. Subjects were informed of the number of competitors, their own cost coefficient, the structure of the cost coefficients of their competitors, and how the winner is determined randomly according to their winning probabilities before selecting their effort level. At the end of each round, they were also informed of the total effort of that contest, their own winning probability, winning outcome, cost expenditure of their selected effort, and their earnings (in points) in that contest round.

The experiment was conducted at the Wuhan University Center for Behavior and Economic Research Laboratory. At the beginning of each session, the instructions were distributed to subjects (available upon request) and were read aloud by an experimenter. Subjects first completed a set of comprehension questions online and then started the actual experiment when all subjects correctly answered the questions. The experiment was programmed and conducted in oTree (Chen et al., 2016). A total of 75 subjects participated in three sessions. No subjects participated in more than one session. Each subject's final payoff was one randomly selected round-payoff in addition to 15 CNY show-up fee. The experimental earnings were converted to Chinese Yuan at the rate of 1,800 points to 1 CNY. On average, subjects earned 65 CNY each, and each session lasted about one hour.

5 Empirical Results

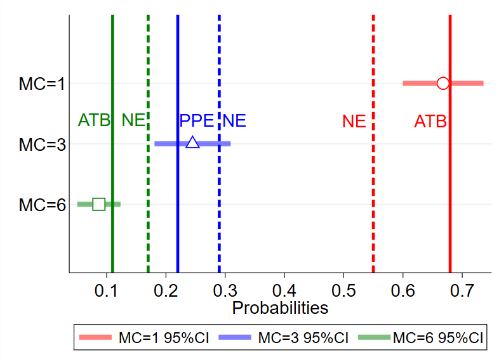
5.1 Main Findings

We begin by analyzing the winning probabilities under each contest situation and examining which theoretical prediction is more accurate in capturing the actual observation of winning probabilities. According to Table 1, both the NE and PPE predict that the low-cost subjects should have the highest winning probability, while the high-cost subjects have the lowest winning probability in a contest. Nevertheless, the PPE predicts a more extreme dispersion of individual winning probabilities, where the low-cost (high-cost) subjects across all the contest conditions are predicted to have consistently higher (lower) winning probabilities than the corresponding NE predictions.

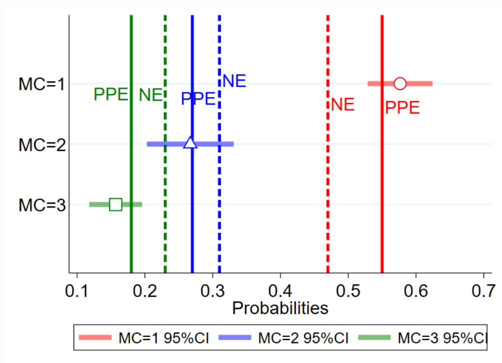
Figure 2 displays the average winning probability for each type of subject, where observations for the same type of cost coefficient in different rounds are pooled, and 95% confidence intervals are based on standard errors clustered at the individual level. For comparison, we also plot the NE and the PPE predictions for each subject type. While our theories are static, learning experience may influence subjects' behaviour. Therefore, we also explore the dynamic process of winning probability per round for each subject type in Figure 3.

Winning probabilities. Figure 2 shows point estimates of the average winning probability across subject types and contest situations. Comparing the empirical winning probabilities to the equilibrium predictions, we observe a striking difference in predictability between the NE and PPE. Among all the low-cost types (in red), the PPE predictions are always closer to the actual means and reside in the corresponding confidence intervals. In contrast, the NE predictions are never close to the actual means or their confidence intervals. The plots for all the high-cost types (in green with N=3, in blue with N=2) show similar differences between the

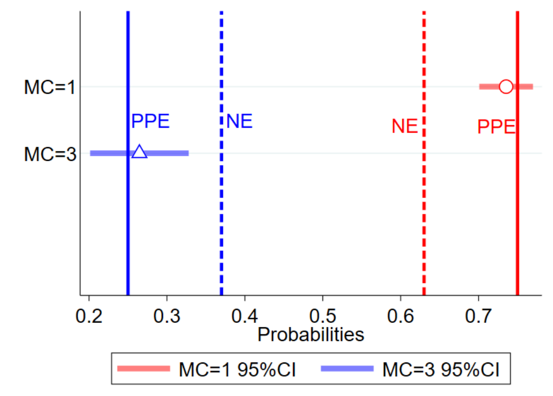
Figure 2: Average Winning Probabilities by Contest Conditions



(a) N=3, (1, 3, 6)



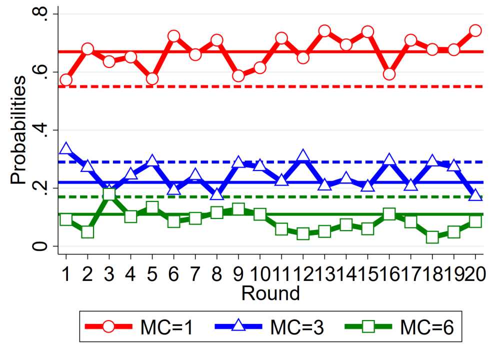
(b) N=3, (1, 2, 3)



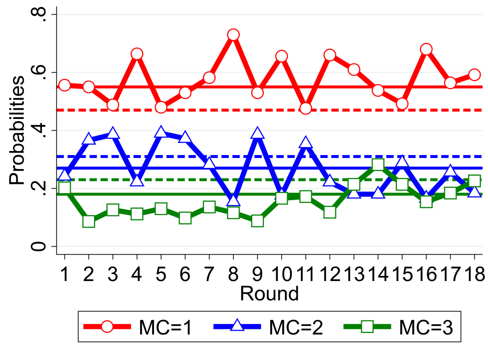
(c) N=2, (1, 3)

Notes. Each figure plots the means of winning probabilities for each type of cost coefficient over the whole session, while standard errors are clustered at the individual level. The vertical solid lines indicate the PPE predictions, and the vertical dashed lines indicate the NE predictions.

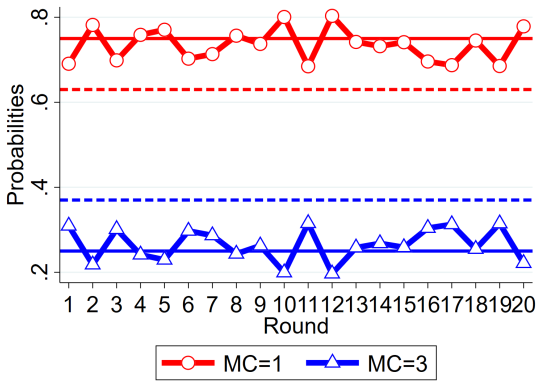
Figure 3: Average Winning Probabilities Over Rounds



(a) N=3, (1, 3, 6)



(b) N=3, (1, 2, 3)



(c) N=2, (1, 3)

Notes. Each figure plots the average winning probabilities per round for each type of cost coefficient. The horizontal solid lines indicate the PPE predictions, and the horizontal dashed lines indicate the NE predictions.

PPE and the NE predictions. For the middle-cost types (blue in Figure 2a, 2b), although the actual means seem to be closer to the PPE predictions, both the PPE and NE predictions are within the confidence intervals. The effectiveness of PPE prediction is also persistently shown in Figure 3. Although the winning probabilities of the middle-cost types in group size N=3 fluctuate above and below both the NE and PPE predictions (in blue in Figure 3a, 3b), the average winning probability of the high-cost and low-cost types remain closer to the PPE predictions in all rounds.

To measure the accuracy of the theoretical predictions, we first regress the individual winning probability on cost dummies while clustering standard errors at the individual level. We then test the equality of estimated coefficients with their corresponding NE and PPE predictions. In the upper panel of Table 2, the coefficient estimate of each cost variable reports the average winning probability of each cost type under each contest situation. The result is consistent with the prediction that low-cost subjects, for example, cost coefficient=1 in all contests, have the highest winning probability, and vice versa. The F-tests for the equality of estimated coefficients and the theoretical predictions reported at the lower panel of Table 2 show that, on average, the observed winning probability of both the high and low-cost types are significantly different from the NE predictions in all contest situations, while no significant difference is observed for the comparisons with the PPE predictions.

5.2 Average Efforts

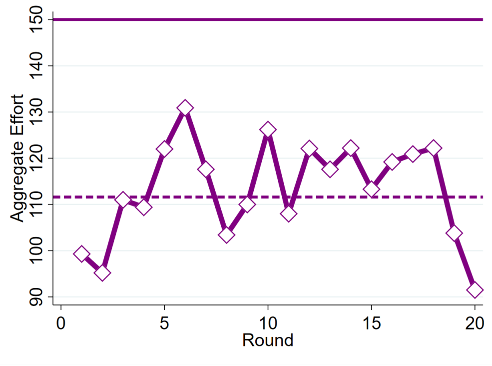
Figure 4 shows the average aggregate level of impact per round for the different settings we studied. The analysis of the figure indicates that contestant efforts frequently surpass the Nash equilibrium predictions, aligning with findings from previous research. However, it is also notable that these efforts consistently fall short of the levels predicted by PPE. In light of the preceding section, we deduce that while PPE, under our assumption of risk neutrality, accurately predicts the relative efforts among contestants, it tends to overestimate the absolute levels of effort.

Table 2: Winning probability of asymmetric contests

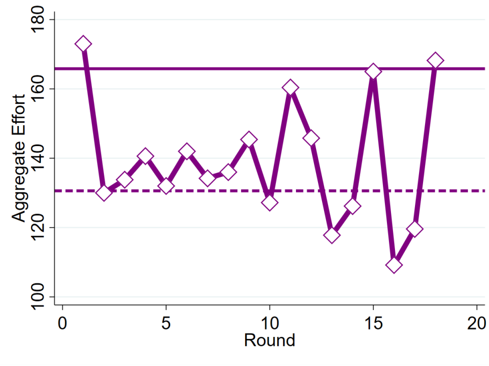
	N=3, (1, 3, 6) (1)	N=3, (1, 2, 3) (2)	N=2, (1, 3) (3)
Cost coeff=1	0.67	0.58	0.74
	(0.033)	(0.024)	(0.017)
Cost coeff=2		0.27	
		(0.031)	
Cost coeff=3	0.24	0.157	0.26
	(0.031)	(0.019)	(0.031)
Cost coeff=6	0.09		
	(0.018)		
R^2	0.798	0.866	0.826
Observ.	600	270	600
Winning probability = NE?			
Cost coeff=1	F=12.46***	F=20.52***	F=39.05***
Cost coeff=2		F=1.89	
Cost coeff=3	F=2.05	F=14.67***	F=11.58***
Cost coeff=6	F=21.63***		
Winning probability = PPE?			
Cost coeff=1	F=0.00	F=1.27	F=0.75
Cost coeff=2		F=0.01	
Cost coeff=3	F=0.63	F=1.47	F=0.22
Cost coeff=6	F=1.67		

Notes. OLS regressions of individual winning probability on all three (two) different cost coefficient variables for asymmetric contests with group size N=3 (N=2). Each cell reports the mean of the winning probability. Standard errors in parentheses are clustered at the individual level. The lower part of the table shows the F-tests of equality of coefficients and the corresponding NE and PPE predictions. We have repeated all the regression analyses by including either a round variable or round fixed effects and found similar results.

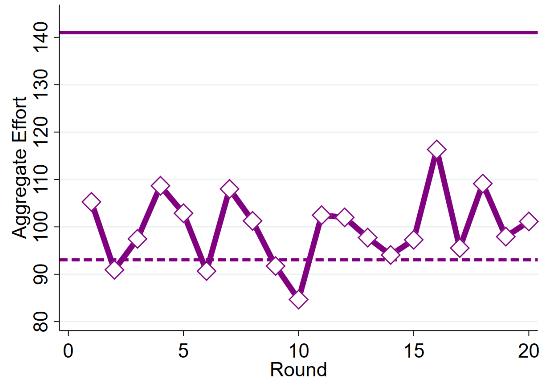
Figure 4: Average Aggregate Effort Over Rounds



(a) N=3, (1, 3, 6)



(b) N=3, (1, 2, 3)



(c) N=2, (1, 3)

Notes. Each figure plots the average of group's total efforts per round. The horizontal solid lines indicate the PPE predictions, and the horizontal dashed lines indicate the NE predictions. The F-test p-value rejects equality of the estimated average aggregate effort with the corresponding NE and PPE prediction across all contests with one exception. The F-test cannot reject the equality of NE and average aggregate effort in the contest of N=3, (1, 3, 6).

To reconcile this disparity, we propose an explanation grounded in a behavioral model of risk aversion. We consider the payoff function maximized by contestants, which comprises two distinct components: the 'revenue part,' characterized by uncertainty (as contestants either win V or nothing), and the 'cost part,' which remains certain due to the contest's 'All-Pay' structure. Our forthcoming result posits that if contestants exhibit risk aversion solely with respect to the uncertain revenue part of their payoff, while maintaining risk neutrality towards the certain cost part, the resultant PPE framework would yield an identical success distribution as previously derived, but with a notably lower aggregate level of effort.

Suppose that contestants evaluate the possible revenues ("V" or "0") according to a CRRA Bernoulli utility function $u(x) = x^{\gamma}$, $\gamma \in (0, 1]$, such that the maximize the payoff function

$$\Pi_j(p_j;\Sigma_j) = p_j V^\gamma - c_j C\left(\frac{p_j}{1-p_j}\Sigma_j\right) \tag{10}$$

Proposition 4 *The success distribution of the PPE and NE associated with* (10) *are invariant to the degree of risk aversion* $\gamma \in (0,1]$, *while the average equilibrium efforts increases strictly in* γ.

We can further use the insight in Proposition 4 to obtain the level of γ that aligns the PPE prediction of the aggregate effort with the observed averages by using a back-of-the-envelope calculation. This calculation allows us to deduce the corresponding relative risk aversion parameter, which is $1-\gamma$. Table 3 indicates that, across all the contest cases, the level of CRRA is close to zero, which implies a close to risk neutral behavior. This finding is broadly consistent with a literature that

Table 3: Back-of-the-envelope calculation of CRRA

Cost coefficients	K	Aggregate effort	V^γ	$1-\gamma$
(1,3)	1.33	99.76	14927.19	0.07
(1,3,6)	1.5	113.29	17112.83	0.05
(1,2,3)	1.83	139.24	21150.30	0.03

estimates the degree of risk aversion for varying stake sizes, which find evidence that subjects are close to risk neutral in low-stake choice experiments (Fehr-Duda et al., 2010; Rabin, 2013; Bombardini and Trebbi, 2012) to which our experiment clearly belongs. Moreover, we remark that NE could not be amended in this way, as aggregate efforts also were to decline while Nash efforts already are too low, and NE success chances are not matching well with the data.[25]

6 Conclusion

Proportional Play Equilibrium (PPE), grounded in the empirically supported illusion of proportionality, provides a novel behavioral perspective that enriches our

[25]Including standard risk aversion, i.e., applying the Bernoulli utility uniformly to the payoff function, may lead to lower predicted efforts as well, but would also change the dispersion of success chances, possibly breaking the strong empirical tie to PPE we find otherwise.

theoretical toolkit. While it clearly presents an advancement in terms of effort prediction, its potential in dissecting and explaining the distribution of success chances represents an exciting and empirically relevant frontier. Our study delves into both these aspects – not only theoretically establishing PPE as a robust model but also empirically testing its predictions against those of NE, especially in the context of success chance dispersion.

In conclusion, our investigation into PPE reveals significant insights into the dynamics of contest theory. PPE's alignment with empirical data, especially in terms of predicting success probabilities, underscores its potential as a more accurate and comprehensive framework compared to traditional Nash Equilibrium.

We observe that while PPE tends to over-predict effort levels, integrating a success-based notion of risk aversion provides a meaningful resolution, aligning theoretical predictions about efforts more closely with observed behavior. These findings highlight the complex interplay between risk perceptions, effort allocation, and success probabilities in contests, challenging and expanding our current understanding of strategic interactions.

Ultimately, this paper contributes to a deeper and more nuanced comprehension of contest theory, emphasizing the significance of considering both efforts and success probabilities for a holistic view of strategic behavior in economic and political arenas.

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APPENDIX

A Additional Results

A.1 Equilibrium: Existence, Uniqueness, Representation

In this section, we detail the formal results outlined with intuitive explanations in the main text. To maintain readability, all proofs are deferred to Section A.3.

We refer the game with equilibrium concept described in Definition 2 as to the Original Game. We first establish that, for any given $\alpha \in [0,1]$, the equilibrium sets of the Original Game and the Market Share Game coincide.

Theorem 2 *If* $(\mathbf{e}, \bar{e})$ *is an equilibrium in the Original Game, then the success distribution* $\hat{p}$ *determined by* $p_j = \frac{e_j}{\Sigma_j + \alpha e_j}$ *for each* $j \in \mathcal{J}_n$*, along with* $\bar{e}$*, constitutes an equilibrium in the Market Share Game. Conversely, if* $(\hat{p}, \bar{e})$ *is an equilibrium in the in the Market Share Game, then the effort profile* $\mathbf{e}$ *derived from* $e_j = \frac{p_j}{1-\alpha p_j}\Sigma_j$*,* $p_j = \hat{p}(j)$*, for each* $j \in \mathcal{J}_n$*, alongside* $\bar{e}$*, is an equilibrium in the Original Game.*

We next show that for every given $\alpha \in [0,1]$ the Market Share Game with payoff function (7) has a unique and strictly positive equilibrium $(\hat{p}_\alpha, \bar{e}_\alpha)$ under the assumptions imposed on the cost function $C(\cdot)$. By Theorem 2, the success distribution $\hat{p}_\alpha$ must correspond to the one implied by the unique equilibrium of the Original Game.

Theorem 3 *For any* $\alpha \in [0,1]$*, the Market Share Game has a unique equilibrium* $(\hat{p}_\alpha, \bar{e}_\alpha)$*, where* $\hat{p}_\alpha(j), \bar{e}_\alpha > 0$ *for each* $j \in \mathcal{J}_n$*.*

By Theorem 3, the Market Share Game with payoff (7) has a unique equilibrium $(\hat{p}_\alpha, \bar{e}_\alpha)$ for every $\alpha \in [0,1]$. Define the functions $\mathbf{p} : [0,1] \to \Delta^{n-1}$, $\pi_j(\mathbf{p}(\alpha)) \equiv \hat{p}_\alpha(j)$, $\bar{e} : [0,1] \to \mathbb{R}_+$, $\bar{e}(\alpha) \equiv \bar{e}_\alpha$. These mappings are well-defined by Theorem 3, and the following result establishes that these are continuously differentiable in α.

Corollary 1 $\mathbf{p}(\alpha)$ *and* $\bar{e}(\alpha)$ *are* C^1*-functions of* α*.*

The equilibrium density for each α is represented by the bivariate function $p : \mathcal{J}_n \times [0,1] \to [0,1]$, defined as $p(j,\alpha) \equiv \pi_j(\mathbf{p}(\alpha))$. Because $\mathbf{p}(\alpha)$ is continuous in α, $p(j,\alpha)$ is continuous.[26] As the endpoints satisfy $p(j,0) = p^{PPE}(j)$ and $p(j,1) = p^{NE}(j)$, the functions $p^{PPE}(j)$ and $p^{NE}(j)$ are homotopic, showing that the equilibrium distributions of PPE and NE can be continuously transformed into each other.

A.2 Comparative Statics

This analysis extends the comparative statics of p^{PPE} and p^{NE} beyond the scenario where $C(\cdot)$ is represented by a power function. We find that the impact of a marginal increase of V on these equilibrium success probabilities is similarly influenced by the characteristics of the cost function in both cases. This is most evident in the case where the elasticity of marginal costs, $\varepsilon(e) \equiv \frac{C''(e)e}{C'(e)}$, adopts a monotonic pattern.

Proposition 5 *Let $V > 0$. If $\varepsilon(e)$ is strictly decreasing, then $\exists\delta > 0$ such that $p_{V'}^{PPE}$ and $p_{V'}^{NE}$ are clockwise rotations of p_V^{PPE} and p_V^{NE}, respectively for any $V' \in (V, V+\delta)$. If $\varepsilon(e)$ is strictly increasing, then $\exists\gamma > 0$ such that $p_{V'}^{PPE}$ and $p_{V'}^{NE}$ are counter-clockwise rotations of p_V^{PPE} and p_V^{NE}, respectively for any $V' \in (V, V+\gamma)$.*

A.3 Proofs

Proof Theorem 2 Fix $\alpha \in [0,1]$. For given $\Sigma_j > 0$, define the function $h(e_j;\Sigma_j) = \frac{e_j}{\Sigma_j + \alpha e_j}$. Note that $h(\cdot;\Sigma_j) : \mathbb{R}_+ \to [0, 1/\alpha)$, $\frac{1}{\alpha} \geq 1$ is bijective and $p_j = h(e_j;\Sigma_j)$, with inverse $h^{-1}(p_j;\Sigma_j) = \frac{p_j}{1-\alpha p_j}\Sigma_j = e_j$. That is, to every given $p_j \in [0, \frac{1}{\alpha})$ we can assign a unique $e_j \in [0,\infty)$ and vice-versa. Define the function $\hat{\Pi}(p_j;\Sigma_j) \equiv \Pi(h^{-1}(p_j;\Sigma_j);\Sigma_j) = \Pi(e_j(p_j);\Sigma_j)$. Note that $\hat{\Pi}(p_j;\Sigma_j)$ amounts to the payoff function (7) in the Market Share Game.

Let $(\mathbf{e}^*, \bar{e}^*)$ be equilibrium in the Original Game. Let $\Sigma^* = n\bar{e}^*$ and note that $\Sigma_j^* = \Sigma^* - \alpha e_j^*$. The implied success probabilities are $p_j^* = \frac{e_j^*}{\Sigma_j^* + \alpha e_j^*}$. By contradiction, suppose that $(p_1^*, ..., p_n^*)$ formed in this way and $\bar{e}^*$ do not form an equilibrium of the Market Share Game. Then $\hat{\Pi}_j(p_j;\Sigma_j^*) > \Pi_j(p_j^*;\Sigma_j^*)$ for some $p_j \in [0, \frac{1}{\alpha})$. But

[26] Continuity of $p(j,\alpha)$ follows from continuity of $p(j,\cdot)$ because $\mathcal{J}_n$ is a discrete set. Here, $\mathcal{J}_n$ is a discrete metric space, $[0,1]$ is equipped with the natural metric and $X \times [0,1]$ is equipped with the metric defined by the sum of the two metrics.

as $e_j = h^{-1}(p_j; \Sigma_j^*)$ we have

$$\hat{\Pi}_j(p_j; \Sigma_j^*) = \Pi_j(e_j(p_j); \Sigma_j^*) > \hat{\Pi}(p_j^*; \Sigma_j^*) = \Pi_j(e_j^*; \Sigma_j^*)$$

as $e_j^* = e_j(p_j^*)$. That is, there is e_j such that $\Pi_j(e_j; \Sigma_j^*) > \Pi_j(e_j^*; \Sigma_j^*)$ which contradicts that $(\mathbf{e}^*, \bar{e}^*)$ is an equilibrium in the Original Game.

For the converse, let $(\hat{p}, \bar{e})$ be equilibrium in the Market Share Game, with $p_j^* = \hat{p}(j)$, $\Sigma^* = n\bar{e}^*$ and $\Sigma_j^* = (1 - \alpha p_j)\Sigma^*$. For every p_j^* there is a unique impact $e_j^* = e_j(p_j^*)$. Suppose that the corresponding impact profile $(e_1^*, ..., e_n^*)$ together with average effort $\bar{e}$ and Σ_j^* as determined above do not constitute an equilibrium in the Original Game. Then $\Pi(e_j; \Sigma_j^*) > \Pi(e_j^*; \Sigma_j^*)$ for at least one contestant j and some effort $e_j \neq e_j^*$. Hence also

$$\Pi(e_j; \Sigma_j^*) = \hat{\Pi}(p_j(e_j); \Sigma_j^*) > \Pi(e_j^*; \Sigma_j^*) = \hat{\Pi}(p_j^*; \Sigma_j^*)$$

as $p_j^* = e_j(p_j^*)$. Thus there is $p_j \neq p_j^*$ such that $= \hat{\Pi}(p_j; \Sigma_j^*) > \hat{\Pi}(p_j^*; \Sigma_j^*)$, contradicting that $(\hat{p}, \bar{e})$ is an equilibrium in the Market Share Game. ■

Proof Theorem 3 Fix an arbitrary $\alpha \in [0, 1]$. Note that for any given $\Sigma_j \geq 0$, $p_j = 0$ cannot maximize (7) because $V > 0$ and $C'(\cdot)$ is continuous with $C'(0) = 0$. Next, note that $\Pi(p_j; \Sigma_j)$ is strictly quasi-concave in $p_j \in (0, \frac{1}{\alpha})$ for any given $\Sigma_j \geq 0$. Thus, the FOC pertaining to maximizing (7) are sufficient, and given by

$$V = c_j C'\left(\frac{p_j}{1 - \alpha p_j}\Sigma_j\right) \frac{\Sigma_j}{(1 - \alpha p_j)^2}. \tag{11}$$

Thus, if p_j^* solves (11), then p_j^* is the global maximizer of $\Pi_j(p_j; \Sigma_j)$ on $[0, \frac{1}{\alpha})$. In equilibrium, the quantity Σ_j by definition is determined by $\Sigma_j = (1 - \alpha p_j)\Sigma$, where $\Sigma \equiv n\bar{e}$ is defined as the aggregate impact. Plugging this value in (11) yields the equilibrium condition (8). The remainder of the proof is organized in the following three steps. We first establish that this equation has a unique solution $p_j(\Sigma) > 0$ for each $j \in \mathcal{J}$ and any given $\Sigma \geq 0$. This solution is a C^1-function that verifies $p_j'(\Sigma) < 0$. Next, we will show that there is a unique value $\Sigma^* > 0$ such that $\sum_i p_i(\Sigma^*) = 1$. Finally, we will establish that $p_j(\Sigma^*) = p_j^*$, i.e., $p_j(\Sigma^*)$ indeed maximizes $\Pi(p_j; \Sigma_j)$ at $\Sigma_j = (1 - \alpha p_j^*)\Sigma^*$.

Let $\Sigma \geq 0$, and note that $p_j = 0$ cannot solve (8). Hence, any solution to (8) must verify $p_j > 0$. Next, note that, for any given $p_j > 0$, the RHS of (8) is continuous in Σ, equal to zero for $\Sigma = 0$, strictly increasing in Σ and grows arbitrarily large as $\Sigma \to \infty$. The LHS of (8) is continuous and non-increasing in p_j. If $\Sigma = 0$ there either is no solution of (8) (if $\alpha = 0$) or the solution is $p_j(0) = \frac{1}{\alpha} \geq 1$ (if $0 < \alpha \leq 1$). For any $\Sigma > 0$, the above arguments assure the existence of a unique solution $p_j(\Sigma) > 0$. Moreover, the Implicit Function Theorem assures that this solution is a C^1-function of Σ and satisfies $p_j'(\Sigma) < 0$. Finally, we have that $\lim_{\Sigma \to 0} p_j(\Sigma) \geq 1$ and $\lim_{\Sigma \to \infty} p_j(\Sigma) = 0$. For the solutions $p_1(\Sigma), ..., p_n(\Sigma)$ to be an equilibrium, they must integrate to one. Define $G(\Sigma) = \sum_{i=1}^{n} p_j(\Sigma)$ and note that $G'(\Sigma) < 0$ for any $\Sigma > 0$. The previous arguments about $p_j(\Sigma)$ imply that $\lim_{\Sigma \to 0} G(\Sigma) > 1$ and $\lim_{\Sigma \to \infty} G(\Sigma) = 0$. The existence of a unique $\Sigma^* > 0$ that solves $G(\Sigma) = 1$ follows from these facts. By construction, $p_1(\Sigma^*), ..., p_n(\Sigma^*)$ must maximize each $\Pi_j(p_j; \Sigma_j^*)$ if the value of Σ_j^* is given by $\Sigma_j^* = (1 - \alpha p_j(\Sigma^*))$. This concludes the proof. ■

Proof Corollary 1 For any $\alpha \in [0,1]$ define the aggregate impact by $\Sigma(\alpha) \equiv n\bar{e}(\alpha)$. Thus $\bar{e}(\alpha)$ is C^1 in α iff $\Sigma(\alpha)$ is C^1 in α. Lemma 1 in Appendix B establishes that the solution function $\Sigma(\alpha)$ must be C^1 with $\Sigma'(\alpha) < 0$. For each $j \in \mathcal{J}_n$, define the function

$$F_j(p_j; \alpha) \equiv V(1 - \alpha p_j) - c_j C'(p_j \Sigma(\alpha))\Sigma(\alpha).$$

Thus, $F_j(p_j \alpha) = 0$ amounts to equilibrium equation (8), and it is easy to check that $F_j(p_j; \alpha)$ is C^1 on $\mathbb{R}_{++} \times [0,1]$ with

$$\partial p_j F_j(p_j(\alpha); \alpha) = -\alpha V - c_j C''(p_j(\alpha)\Sigma(\alpha))\Sigma(\alpha)^2 < 0. \tag{12}$$

The solution $\mathbf{p}(\alpha) = (p_1(\alpha), ..., p_n(\alpha))$ solves the system of equations

$$F(\mathbf{p}; \alpha) \equiv \begin{pmatrix} F_1(p_1; \alpha) \\ \vdots \\ F_1(p_n; \alpha) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Because

$$D_pF\left(\mathbf{p}(\alpha);\alpha\right)=\begin{pmatrix}\partial p_1F_1\left(p_1(\alpha);\alpha\right) & 0 & 0 & 0\\ 0 & \partial p_2F_2\left(p_2(\alpha);\alpha\right) & \vdots & \vdots\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \partial p_nF_n\left(p_n(\alpha);\alpha\right)\end{pmatrix},$$

condition (12) implies $|D_pF\left(\mathbf{p}(\alpha);\alpha\right)|\neq 0$ for every $\alpha\in[0,1]$, and the Implicit Function Theorem therefore assures that the solution $\mathbf{p}(\alpha)$ must be continuously differentiable at each given $\alpha\in[0,1]$. ■

Proof Proposition 5 We prove the claim by showing its validity for arbitrary $\alpha\in[0,1]$ in the general Market Share Game. First, note that V is a level variable for any $\alpha\in[0,1]$, and $\Sigma'_\alpha(V)>$, which implies that the equilibrium behavior of $\hat{p}_\alpha$ is entirely governed by the properties of the marginal cost function $C'(\cdot)$. By Theorem 2 of Hefti and Teichgräber (2022), to establish the existence of the claimed (counter-)clockwise rotation of $\hat{p}_\alpha(j)$, we need to show that the direct-aggregative effect induced by $dV>0$ verifies $R_{ij}>(<)0$ for any two players i,j with $c_i<c_j$. Specifically we have

$$sign(R_{ij})=sign\left(\frac{C''(p_j\Sigma)p_j\Sigma}{C'(p_j\Sigma)}-\frac{C''(p_i\Sigma)p_i\Sigma}{C'(p_i\Sigma)}\right)$$

B Proofs

Proof Proposition 1 The proof relies on the following Lemma.

Lemma 1 *The equilibrium aggregate $\Sigma(\alpha)$ is a C^1-function of $\alpha\in[0,1]$ and $\Sigma'(\alpha)<0$.*

<u>Proof</u>: We follow the general procedure outlined in Hefti and Teichgräber (2022) for how to derive the comparative-statics of the aggregate quantity in a Market Share Game. For any $j\in\mathcal{J}$ the optimality condition is (8). Treating $\Sigma>0$ and $\alpha\in[0,1]$ as parameters, this equation has a unique solution $p_j(\Sigma,\alpha)>0$, and the Implicit Function Theorem assures that this solution is continuously differentiable

in α and Σ, where $\partial_\Sigma p_j(\Sigma,\alpha) < 0$ and $\partial_\alpha p_j(\Sigma,\alpha) < 0$ again by the Implicit Function Theorem. Define $G(\Sigma,\alpha) = \sum_i p_i(\Sigma,\alpha)$, and note that the equilibrium $\Sigma(\alpha)$ is determined by $G(\Sigma,\alpha) = 1$ for any given $\alpha \in [0,1]$. Applying the Implicit Function Theorem to this equation yields $\Sigma'(\alpha) < 0$. □

Proposition 1 follows from Lemma 1 as $\bar{e}(\alpha) = \Sigma(\alpha)/n$ and $\Sigma(0) = \Sigma^{PPE}$ and $\Sigma(1) = \Sigma^{NE}$. ■

Proof Proposition 2 The claim is a direct consequence of the broader analysis provided in Hefti and Teichgräber (2022), because the equilibrium equation (8) is consistent with the general equilibrium equation of their Proposition A.1, assuring that the equilibrium sorting of success chances (or market shares) must align with the ex-ante sorting of the agents. That result directly implies that $\hat{p}_\alpha(i) = \hat{p}_\alpha(j)$ iff $c_i = c_j$, which, by extension, implies that $\hat{p}_\alpha(i) = 1/n \ \forall i \in \mathcal{J}_n$ iff $c_1 = ... = c_n$. Conversely, if $c_i < c_j$ for two agents $i < j$, then Proposition A.1 in Hefti and Teichgräber (2022) further assures that $\hat{p}_\alpha(i) > \hat{p}_\alpha(j)$ across the entire range $\alpha \in [0,1]$. As contestants are ex-ante sorted by increasing costs, the previous result necessitates that the equilibrium density $\hat{p}_\alpha(j)$ is decreasing in j, and non-constant whenever $c_1 < c_n$. ■

Proof Theorem 1 Let i, j be such that $c_i < c_j$. Then, by Proposition 2, $\hat{p}_\alpha(i) > \hat{p}_\alpha(j)$ for any $\alpha \in [0,1]$. Note that to establish that p^{PPE} is a monotonic clockwise rotation of p^{NE}, it suffices to show that $\frac{\hat{p}_0(i)}{\hat{p}_0(j)} > \frac{\hat{p}_1(i)}{\hat{p}_1(j)}$.[27] Now, for $C(e) = e^\eta$, (8) implies that

$$\frac{\hat{p}_0(i)}{\hat{p}_1(i)} = \left(\frac{\Sigma(1)}{\Sigma(0)}\right)^{\frac{\eta}{\eta-1}} \left(\frac{1}{1-\hat{p}_1(i)}\right)^{\frac{1}{\eta-1}}. \tag{13}$$

Because $\hat{p}_1(i) > \hat{p}_1(j)$, (13) implies that $\frac{\hat{p}_0(i)}{\hat{p}_1(i)} > \frac{\hat{p}_0(j)}{\hat{p}_1(j)}$, which yields the requested condition. For the remaining assertion it is straightforward to verify that the Lorenz curve associated with p^{NE} must Lorenz-dominate the one implied by p^{PPE}, from which the claim follows. ■

Proof Proposition 3 The first claim directly follows from Proposition 5 in Hefti and Teichgräber (2022) as costs are neutral and V is a level variable for any given

[27] See Definition 4 of Hefti and Teichgräber (2022).

$\alpha \in [0,1]$. For the second claim, we need to derive the direct-aggregative effect R_{ij} associated with a changing value of η. We obtain that

$$sign(A(i) - A(j)) = sign\left(\frac{\partial_\eta \varphi(j)}{\varphi(j)} - \frac{\partial_\eta \varphi(i)}{\varphi(i)}\right) = sign\left(Ln(p(j;\alpha) - Ln(p(i;\alpha)\right)$$

As $p(j,\alpha) < p(i,\alpha)$ whenever $c_i < c_j$, we obtain $R_{ij} < 0$, and the claim follows from Theorem 2 in Hefti and Teichgräber (2022). ■

Proof Proposition 4 Follows directly from Proposition 3. ■.