Endogenous Fiscal and Monetary Interactions: The Curious Case of Brazil *

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Abstract

We investigate the unconventional interaction of fiscal and monetary policy in a two-period general equilibrium model with a representative lender and the government. Under logarithmic preferences, a sequential game with perfect information between the fiscal and the monetary authority emerges endogenously in equilibrium. The optimal responses depend on the level of the government consumption, which works as a threshold and defines either a positive or negative relationship between the tax rate and the interest rate over the business cycle. We confront the theory with data for Brazil. The effects of the base interest rate on the income-tax effective rate depends on whether government consumption is below or above the referred threshold. This pattern captures the unconventional interaction between fiscal and monetary policy in an emerging economy.

Keywords: General equilibrium; Fiscal policy; Monetary policy; Government consumption.

JEL Codes: D52; C81, H26.

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1 Introduction

The interaction between fiscal and monetary policy has attracted renewed interest since the Global Financial Crisis, as the accommodative fiscal and monetary policies become the standard response around the world to the downturn. Governments coupled decreases in the base interest rate on the monetary side with increases in public spending and selective tax cuts on the fiscal front.

In Brazil, public banks have also been used to expand the supply of credit by reducing lending rates and providing funds for riskier individuals and firms. Despite successfully stimulating economic activity in the short run, this unconventional policy combination has raised concerns about the sharp increase in the public debt to GDP ratio and the rise in inflation under the inflation targeting regime.[1]

Since then, fiscal policy has faced difficulties achieving a sound budget balance, while monetary policy has struggled to bring inflation back to target. The scenario worsened with the outbreak of the Covid-19 pandemic. To face the unprecedented downturn, the prescription was once again a combination of expansionist fiscal and monetary policies. This last act, however, worsened even more the fiscal imbalance and the monetary misalignment. Since the end of the pandemic, the Central Bank of Brazil has continuously increased the base interest rate to stabilize the inflation target regime and keep tracking of the inflationary pressures. Government spending has increased sharply, with the debt to GDP ratio rising from 52% in the aftermath of the 2008 crisis in January of 2011 to 88% in February of 2021. In parallel, the annual inflation rate measured by percent change in the economy-wide consumer price index (IPCA) rose from 6.5% in 2011 to 10.1% in 2021, well above the upper bound of 5.25% around the tolerance interval.[2]

Figure 1 provides a good picture of the scenario by reporting cyclical components estimated by the HP filter for government consumption and gross national income (GNI), in addition to the policy interest rate. The shaded regions refer to crises episodes as dated by the Brazilian Business Cycle Dating Committee (CODACE) of the Brazilian Institute of Economics from the Getulio Vargas Foundation (IBRE/FGV).

The cyclical components of government consumption and GNI do not always follow the

[1] See Elenev et al. (2021) for a broad discussion with a focus on the US economy.

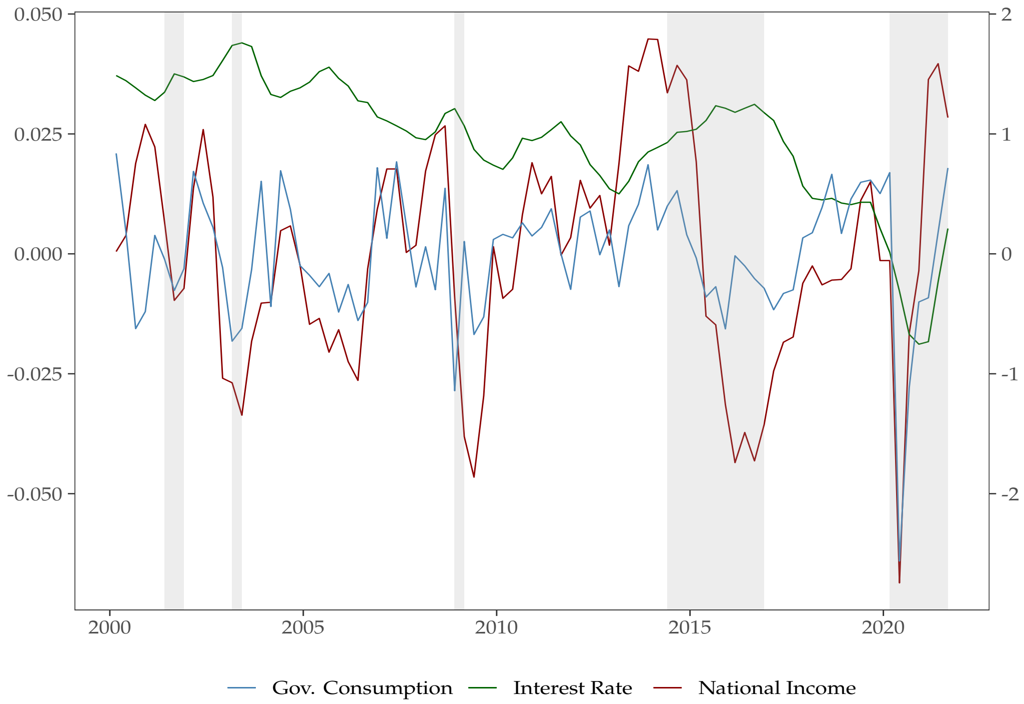
[2] For the year of 2021, the inflation target was 3.75% with a tolerance interval of ± 1.5 pp. See Ayres et al. (2019) for a good overview of the Brazilian economy since 1960.

same direction, as their correlation depends on the business cycle phase. In bust periods, they both seem to decrease as the policy interest rate increases, while in boom the opposite happens. However, this is not always the case, as we can observe decreases in all cyclical components and in the policy rate during some episodes across the sample period. This preliminary evidence suggests a potential incongruity between the Brazilian fiscal and monetary policy that departures from the conventional wisdom. We build a theoretical model to show that this apparently antagonist behavior depends on the relative level of the government consumption and the business cycle phase. We apply the model results to the Brazilian data and find a close matching with an endogenous threshold estimation.

Apart from the lack of theoretical background, the episodes highlighted in the figure suggest a practical coordination between fiscal and monetary policy during the downturn of the business cycle. However, the same practice is not adopted during boom episodes, whereupon these policies seem to behave in an independent manner. It is unclear the role that each policy is playing in the different phases of the business cycle nor how they might eventually strength coordination to improve the country's macroeconomic performance.

This paper addresses these issues by theoretically and empirically investigating the interaction between fiscal and monetary policy in a general equilibrium (GE) environment. We propose a simple and tractable two-period GE model with a representative lender and government. Under logarithmic preferences, we derive the equilibrium conditions, from which endogenously emerges a sequential game with perfect information played by the fiscal authority (the leader) and the monetary authority (the Follower), and vice-versa.

Figure 1: Cyclical behavior of selected variables



Notes: This figure shows the filtered values of Government Consumption and Gross National Income, and the logarithm of the Policy Interest Rate. Interest rate values are measured on the right-hand side axis. The gray vertical bars represents crisis periods as dated by CODACE/IBRE/FGV.
Source: Central Bank of Brazil and National Treasury.

Technically, the best response of the fiscal authority depends on the level of government consumption and is given by a hyperbole with vertical asymptote, such that the public spending works as a threshold in the relationship between the fiscal and monetary policy instruments. Thus, depending on the level of government consumption relative to the threshold, there might be a positive or negative relationship between the tax rate (fiscal policy instrument) and the base interest rate (monetary policy instrument) over the business cycle. In this context, we adopt two criteria to define the relative size of government consumption. According to the first, consumption is low when it lies below the private income gain. The second considers how close government consumption is to the private income in the first period.

We confront the theoretical results with the data for the Brazilian economy in the post-2000 period and find a narrow correspondence. The effects of changes in the interest rate on the effective income tax rate might be either positive or negative depending on whether public consumption is below or above the referred threshold, respectively. Specifically, if government consumption is too high relative to the private income, there is a negative relationship between interest rate and effective income tax rate. Otherwise, if public consumption is low, this relationship is positive. Taking together, these findings suggest a counter-cyclical coordination between the fiscal and the monetary policies over the business cycle that departures from the conventional wisdom.

This is because, under low government consumption, any increase in the interest rate aimed at fighting inflation, for instance, leads to a reduction in the private consumption due to the substitution effect. The effective income tax rate rises because the tax collection increases under higher private-financial gains. Given that government consumption is low relative to the threshold, so is public debt and the cost of debt rollover after the interest rate increase. The opposite occurs when public consumption is high, and so is public debt and the cost of debt rollover. Any increase in the interest rate would reduce the effective income tax rate because private financial gains are lower in this scenario.

Brazil is an emerging economy that went into the global financial crisis with an already high debt burden, and so issues of policy changes spillover and policy credibility are paramount. The country has experienced a wide range of stabilization policies since the edition of the Fiscal Responsibility Law in 2000 and its fiscal-monetary coordination framework has not followed any pattern usually observed in developed economies.[3] For instance, there is evidence of a pro-cyclical fiscal policy coupled with at a times an unreliable inflation-targeting monetary policy. Political interference usually threatens the Central Bank when the fight against inflation calls for a tighter monetary policy, as stressed by Divino and Haraguchi (2022). On the fiscal instance, the primary-surplus target officially guided fiscal policy from 2000 to 2015, but with lower enforcement after 2007 and a formal change in the fiscal regime to the expenditure ceiling in 2016. However, the new regime has not worked as a binding constraint due to repeated spending limit violations with the legal seal of the National Congress.

Understanding the interplay between these policies in an emerging economy like Brazil can help policymakers formulate effective strategies to manage inflation, balance the public budget, and boost economic growth simultaneously. Fiscal and monetary policies have been inadvertently used to achieve various economic objectives, including stabilizing inflation, promoting growth, and reducing social inequality. However, they have failed as the country still faces a range of economic challenges, including inflation above the target level, high public debt as a ratio of GDP, and weak long-run economic growth. This poor macroeconomic performance has been adversely affected by the fragile fiscal-monetary institutional nexus.

We contribute to the vast literature on the interaction of fiscal and monetary policy, which includes Sargent and Wallace (1981), Leeper (1991), Sims (1994), Woodford (1994, 1995, 2001, 2011), Cochrane (1998, 2001), Schmitt-Grohé and Uribe (2000), Bassetto (2002), Engwerda, van Aarle and Plasmans (2002), Reis (2016), Bianchi and Melosi (2019) among others. More recently, in an innovative study, Elenev et al. (2021) investigated the interaction of fiscal and monetary policy in the context of unconventional policies and offered conditions under which monetary policy can create fiscal capacity. The major difference from our framework to this literature is that a perfect sub-game equilibrium endogenously emerges from a sequential game with perfect information played by the fiscal authority and the Central Bank. In addition, by using data for the Brazilian economy, we find an empirical relationship between the base interest rate and the effective tax rate that closely resembles the theoretical

[3] The Fiscal Responsibility Law forced the public administration, in all levels, to comply with budgetary plans and respect limits on expenses and debts, according to its own revenue collection capacity. It also defined mandatory expenses and the specific purposes of certain public funds, not allowing the use of money set aside for one expense with other types of spending.

results on the counter cyclical interaction of the fiscal and monetary policy.

Organization Section 2 reports the model, describes the equilibrium, and provides an illustrative example to discuss policy interaction. Section 3.6 derives and explains the relationship between the interest rate and the effective tax rate under a sequential game with perfect information. Section 4 provides empirical evidence for the Brazilian economy, while Section 5 describes some caveats. Finally, Section 6 concludes. Additional material is in the Appendices.

2 The model

We construct a two-period general equilibrium model with incomplete financial markets in which there are H agents who are taxed or subsidized by a government. In the first period J nominal assets and one physical good are traded. In the second period, there are S states of nature which may affect both asset payoffs and the H individuals' preferences over each one of the $s-$ contingent physical goods.

The government is characterized by $C > 0$ standing for its exogenous expenditure which can be interpreted in quite ample way. It could be from a public good, represented by durable goods, to bureaucratic services or completely depreciated goods. Besides, C is assumed to be depreciated according to a linear rate represented by a state-contingent linear transformation[4] $Y_s \in R_+^{L\times L}$ for each state of nature.

The public expenditure incurred in $C > 0$ is financed by the issuance of nominal assets in the first period. In the second period, the government pays its debt by taxing both the non-financial and financial gains of all individuals, called taxpayers from now on.

The taxpayers also have tastes related to C as it is a public goods otherwise not. Thus, preferences are assumed to be dependent on C. Thus, the utility function of $U^h : R_+^{S+1} \to R$ is written as $U^h(x, C)$ where C is present in both periods in the case in which C is a durable. Taxpayers' endowments are modeled by a bundle $\omega^h \in R_+^{S+1}$, for $h \in H$.

Our economy is then defined by the generalised vector

$$\mathcal{E} = [H, L = 1, J, (u^h, \omega^h)_{h\in H}; V, (C, Y)] \tag{1}$$

[4]Technically, if C is completely depreciated Y is a singular linear transformation including C into its kernel.

where H is the number of taxpayers who are characterised by their utility functions and endowments, $L = 1$ is the number of commodities (which might be durable or not) and are depreciated according to a exogenously given rate, $Y \in R_{+}^{S}$, J is the number of securities whose nominal returns become given by matrix $V \in R_{+}^{S \times J}$, and the government is characterized by exogenous spending.

2.1 Individual Budget Set

We write down the budget constraints of taxpayer h, given the asset-price vector $\pi \in R^{J}$, and the tax rate system $\tau \in [0,1]^{S}$.

Thus the first-period budget constraint is

$$x_o + \pi\theta \leq \omega_o^h \tag{2}$$

The left hand side represents the expenditure in consumption and investment (asset purchasing) which is financed by their wealth.

The second-period budget constraints, for each state $s \in S$ of the nature, come given by

$$x_s + \tau_s R_s^h \leq \omega_s^h + V_s\theta \tag{3}$$

where $R_s^h = [w_s^h - w_o^h] + [r_s\theta - \pi\theta]$ represents both the gains or losses coming from the non-financial and financial wealth respectively. These gains are subject to a differentiated taxation. That is to say, $\tau_s = (\tau_{s1}, \tau_{s2}) \in [-1,1]^2$ implying that τ_{s1} or τ_{s2} could to a tax or or subsidy depending on whether or not there is a gain or loss.

Thus, the taxpayer $h's$ budget set is

$$B^h(\pi,\tau) = \{(x,\theta) \in R_{+}^{L(S+1)} \times R^{J} : (2) \text{ and } (3) \text{ are satisfied } \} \tag{4}$$

2.2 Government

The government spends in public goods, $C > 0$ which is financed by the issuance of securities, $\Theta \in R_{+}^{J}$. More precisely, one has the following first-period budget constraint of the government:

$$C = \pi\Theta \tag{5}$$

In the second period, in each state of nature $s \in S$, the budget constraint is

$$V_s \Theta = \tau_s \sum_{h \in H} R_s^h + Y_s C, \tag{6}$$

and says that the payment of the debt incurred in the first period is financed by the collection of taxes plus the market value of the deteriorated public good. Here, $\tau \in [-1, 1]^2$ is the tax rate system charged by the government.

2.3 Equilibrium

Before defining equilibrium, we must be aware that the variables in the this model are six: two macro variables, (p, π); two choice variables (consumption and investment) of each taxpayer $h \in H$, (x, θ); and two choice variables (sale of asset and taxes[5]) of the government , (Θ, τ).

Definition 1 *An equilibrium is a vector $[\pi; (x^h, \theta^h)_{h \in H}; (\Theta, \tau)]$ such that*

1. *For each consumer $h \in H$, (x^h, θ^h) maximizes $U^h(x, c)$ subject to (2) and (3).*
2. *The government balances its budgets constraints (5) and (6)*
3. *The asset markets clear:*

$$\sum_{h \in H} \theta^h = \Theta \tag{7}$$

Remark 1 *Definition 1 only says that taxpayers' choices are optimal, government's choices only balance its budget constraints and all assets markets clear. In addition, under strictly increasing utility functions all good markets clear as well. That is to say:*

$$\begin{aligned} \sum_{h \in H} x_o^h + C &= \sum_{h \in H} \omega_o^h \\ \sum_{h \in H} x_s^h &= \sum_{h \in H} \omega_s^h + Y_s C, s \in S \end{aligned} \tag{8}$$

2.4 Existence of equilibrium

In this section we list our main assumptions which will guarantee our main results.

[5]represented by the income-tax aliquot system

Assumptions

A1 Utility functions of taxpayers are continuous, strictly quasi-concave and strictly increasing.

A2 Initial endowments of each agents is strictly positive.

A3 For all $j \in J$, there exists $s \in S$ such that $r_s^j > 0$.

Theorem 1 *Under (A1) - (A3) the economy $\mathcal{E}$ always has an equilibrium.*

Proof The proof of Theorem (1) will be done in Appendix.

3 Fiscal and monetary policies

We specialize our earlier model, by considering $L = 1, S = 1$ and $H = 1$, to analyse the interaction between monetary and fiscal policies. Thus, the economy comprises a representative taxpayer and a government. The government is collectively represented by the monetary and fiscal authorities that are assumed to be institutionally independent. The former is represented by the *Central Bank*, which chooses the monetary-policy interest rate, while the latter is represented by *Treasury* and decides the tax rate on income gains (financial and non-financial) accruing to taxpayers.

The Central Bank sets the interest rate charged on the public debt according to the market equilibrium, meaning that the price of the debt equals the inverse of the policy rate.[6] Notice that although the policy rate is endogenous in the model and determined in equilibrium, the cost of the public debt is fixed by the government. However, to keep things simple, the government sets the interest rate on debt equal to the market equilibrium rate. Actually, there are several ways by which the government can determine the cost of the public debt, as discussed by Divino and Orrillo (2022) for instance.

For simplicity, we assume that there is only one asset available for trade, namely government debt, which is default-risk free and whose price, $q = (1 + r)^{-1}$, is determined in equilibrium with $r > -1$ being the risk-free interest rate (where $r = (1 - q)/q$). The consumption good is the numeraire, so that all values are expressed in terms of the unique economic good.

[6] See LeRoy and Werner (2014) for a definition.

The economy lasts for two periods. In the first one, the bond market is open for trading between the representative taxpayer and the government. Private consumption takes place in both periods while the exogenous government consumption occurs just in the first period.

The taxpayer is represented by a utility function $U : R^2_+ \to R$ and a strictly positive vector of initial endowments $\omega = (\omega_1, \omega_2) >> 0$, one for each period, while the government consumption is given by $G > 0$. Taxes are only levied in the second period. Therefore, the economy is summarized by:

$$\mathcal{E} = \Big(U, (\omega_1, \omega_2);\ G\Big) \tag{9}$$

3.1 Taxpayer decision

Given the income-tax (or subsidy) rate $\tau \in (-1, 1)$ and the bond price q, the representative taxpayer chooses a consumption-investment plan (c_1, c_2, θ) to maximize utility subject to the following budget constraints:[7]

– In the first period

$$c_1 = \omega_1 - q\theta, \tag{10}$$

– In the second period

$$c_2 + \tau R = \omega_2 + \theta, \tag{11}$$

where

$$R = (\omega_2 - \omega_1) + (1 - q)\theta \tag{12}$$

captures the total income gain (financial and non-financial) of the representative taxpayer.

In the case of an income loss, (11) implies that the government subsidizes rather than taxes; this decision is determined in equilibrium.

The first-period budget constraint (10) states that the private consumption is funded by initial endowment net on investment in bonds. The constraint (11) claims that, in the second-period, private consumption and tax paid on income gains equals initial endowment plus the face value of the risk free bond. After some simplification, (11) can be written as:

$$c_2 = \Big(\omega_2 - \tau(\omega_2 - \omega_1)\Big) + \theta\Big(1 - \tau(1 - q)\Big) \tag{13}$$

[7] It is an income-subsidy whenever there is an income loss: i.e., $\omega_2 - \omega_1 < 0$. Later we call such a second period outcome a bust, and its reverse inequality a boom.

Equation (13) states that the second-period consumption is funded by the after tax income, including both non-financial and financial.

3.2 Government decision

Given the price q of the risk-free security, the government sells Θ units of the security to the representative taxpayer in the first period and levies an income-tax rate τ on any possible income gain in the second period. That is, the government chooses (Θ, τ) in order to balance its budget constraints in both periods:

$$G = q\Theta, \tag{14}$$

and

$$\Theta = \tau R, \tag{15}$$

where G is government consumption and R is given by (12).

Equilibrium can be defined in the usual way. Beforehand, it is worth pointing out that the taxpayer rationally anticipates the tax rate to be levied. Accordingly, there is no need to consider two tax rates (e.g., unexpected and realized).

Definition 2 *An equilibrium for the economy $\mathcal{E}$ consists of an allocation of consumption-investment plan $(\bar{c}_1, \bar{c}_2, \bar{\theta})$ and a fiscal policy for debt and income-tax rate, $(\overline{\Theta}, \overline{\tau})$, such that:*

1. *The choices of the representative taxpayer are optimal. That is, the consumption-investment plan $(\bar{c}_1, \bar{c}_2, \bar{\theta})$ maximizes $U(c_1, c_2)$ subject to the budget constraints (10) and (11).*
2. *Fiscal policy, $(\overline{\Theta}, \overline{\tau})$, satisfies the budget constraints (14) and (15).*
3. *Markets clear, meaning that $\overline{\Theta} = \bar{\theta}$ and $\bar{c}_1 + G = \omega_1$ in the first period and $c_2 = \omega_2$ in the second period.*

3.3 Equilibrium with logarithmic utility

To obtain an analytical solution, in this subsection, we take the previous results and assume that the taxpayer's preferences are represented by a logarithmic utility function. The objective is to evaluate the interaction of the fiscal and monetary policies by using a standard parametrization of the preferences. That is,

$$U(c_1, c_2) = \ln c_1 + \beta \ln c_2 \tag{16}$$

where $\beta \in (0, 1)$ is the subjective discount factor.

To characterize the equilibrium, we begin by deriving the demand for bonds. Given q and τ, the taxpayer chooses $\theta \geq 0$ to maximize:

$$\ln(w_1 - q\theta) + \beta \ln \Big(\omega_2 + \theta\big(1 - \tau(1 - q)\big) - \tau(\omega_2 - \omega_1) \Big). \tag{17}$$

The first order condition implies that:

$$\frac{q}{\omega_1 - q\theta} = \frac{\beta(1 - \tau(1 - q))}{\omega_2 + \theta\big(1 - \tau(1 - q)\big) - \tau(\omega_2 - \omega_1)}. \tag{18}$$

Given q, the government chooses (Θ, τ) to balance its budget constraints:

$$G = q\Theta, \tag{19}$$

$$\Theta = \tau\big(\omega_2 - \omega_1 + (1 - q)\theta\big). \tag{20}$$

The market clearing condition requires that:

$$\Theta = \theta. \tag{21}$$

Equations (18)-(21) characterize the equilibrium in this economy.

3.4 Computation of Equilibrium

Solving for θ from (18) and Θ from (19), the demand and supply for bonds are, respectively:

$$\begin{aligned} \theta &= \frac{\beta\omega_1}{(1+\beta)q} - \frac{\omega_2 - \tau(\omega_2 - \omega_1)}{(1 - \tau(1 - q))(1 + \beta)}, \\ \Theta &= \frac{G}{q}. \end{aligned} \tag{22}$$

By using the market clearing condition (21), one obtains the equilibrium price (or equivalently the gross interest rate) as a function of τ. That is,

$$1/q = 1 + r = \frac{1}{1-\tau}\left(\frac{\omega_2 - \tau(\omega_2 - \omega_1)}{\beta\omega_1 - G(1+\beta)} - \tau\right). \tag{23}$$

Clearly, if we substitute (23) into (22), we derive $\theta = \frac{G}{q}$. Substituting this into (20), the income-tax rate levied by the government is:

$$\tau = \frac{G/q}{(\omega_2 - \omega_1) + \frac{G}{q} - G}. \tag{24}$$

Substituting the first relation of (23) in (24) and under the perfect foresight assumption, the equilibrium income-tax rate levied by the government is given by:

$$\overline{\tau} = \frac{G\omega_2}{\beta(\omega_1 - G)(\omega_2 - \omega_1)}. \tag{25}$$

Remark 2 *The equality (25) states that fiscal policy is counter cyclical. That is, the government taxes in a boom (when $\omega_2 > \omega_1$) and subsidizes in a bust (when $\omega_2 < \omega_1$).*

Inserting (25) into (23), we find that the equilibrium bond price (or equilibrium gross interest rate) is alternatively represented by:

$$1/\overline{q} = 1 + \overline{r} = \frac{\omega_2(\omega_2 - \omega_1) - G\omega_2}{\beta(\omega_1 - G)(\omega_2 - \omega_1) - G\omega_2}. \tag{26}$$

Thus, the amount of bonds in equilibrium is given by:

$$\overline{\theta} = \overline{\Theta} = (1 + \overline{r})G = \left(\frac{\omega_2(\omega_2 - \omega_1) - G\omega_2}{\beta(\omega_1 - G)(\omega_2 - \omega_1) - G\omega_2}\right)G \tag{27}$$

Finally, the levels of private consumption in equilibrium are equal to:

$$\begin{aligned} \overline{c}_1 &= \omega_1 - \overline{q}\overline{\theta} = \omega_1 - G \\ \overline{c}_2 &= \omega_2 \end{aligned} \tag{28}$$

3.5 Fiscal impact: Changing G

Having computed the equilibrium of the economy, a natural question to ask is how the equilibrium changes when some fundamentals change. We address this question by parameterizing the equilibrium monetary and fiscal policies in scenarios of a boom and a bust.

The choice of G as a comparative static of interest is natural because it has a direct impact on consumption, as shown in (28). Then, we analyze the behaviour of both policies in equilibrium. We begin by setting the following curve $\alpha : R \to R^2$ as follows:

$$\alpha(G) = \left(\overline{R}(G), \overline{\tau}(G) \right) \tag{29}$$

where $\overline{R} = 1 + \overline{r}$ and $\overline{\tau}$ are both defined as in (26) and (25), respectively.

Figure 2 geometrically illustrates the relationship described in (29) by plotting the equilibrium values of interest rate and tax rate that are consistent with different levels of the government consumption in either a boom or a bust period. Beforehand, we determine the feasible range of parameters (or fundamentals) for the equilibrium to be well defined and consider two phases of the business cycle:

1. **A bust:** $\omega_2 - \omega_1 < 0$.

   From (25), we clearly see that $\overline{\tau}(0) = 0$. In addition, notice that $\overline{\tau}$ as a function of G is strictly decreasing, so that the government subsidizes instead of taxing income gains. This implies that the upper bound of public expenditure in a bust, denoted by G^u_{bust} can be easily determined by solving $\overline{\tau}(G) = -1$. More precisely, we have that

$$0 < G^u_{bust} = \frac{\beta\omega_1(\omega_1 - \omega_2)}{\omega_2 + \beta(\omega_1 - \omega_2)} < \omega_1 \tag{30}$$

2. **A boom:** $\omega_2 - \omega_1 > 0$.

   By using a similar argument from the previous case, we have that $\overline{\tau}$ as a function of G is strictly increasing. Thus, solving for $\overline{\tau}(G) = 1$, we derive

$$0 < G^u_{boom} = \frac{\beta\omega_1(\omega_2 - \omega_1)}{\omega_2 + \beta(\omega_2 - \omega_1)} < \omega_1 \tag{31}$$

   Notice that, in a boom, the government taxes instead subsidizing income gains.

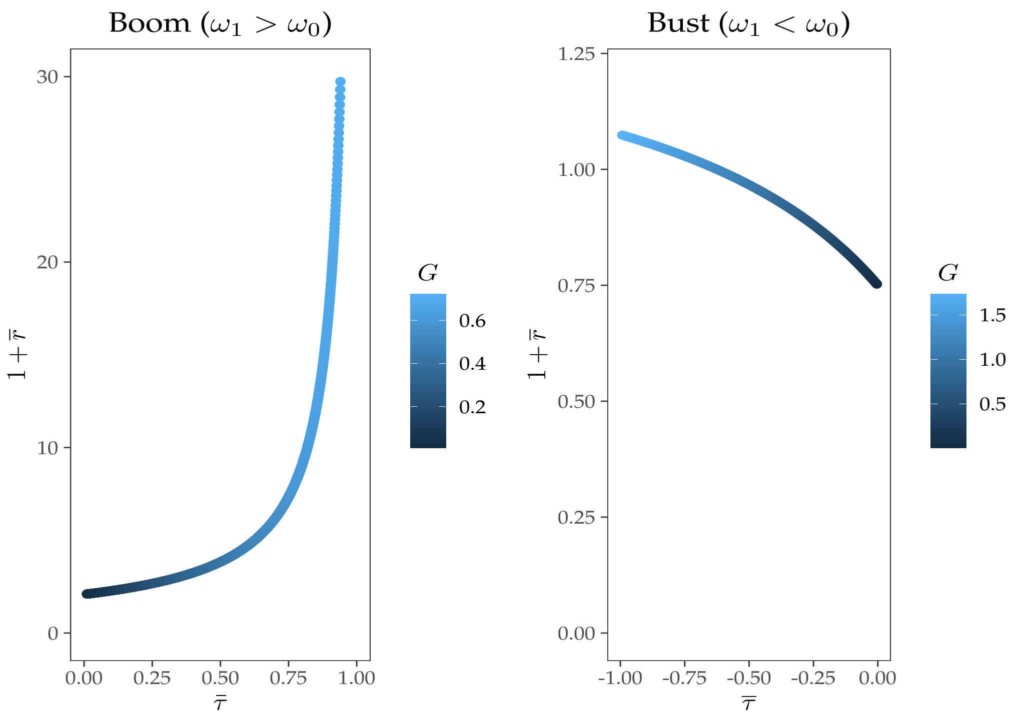
It is worth noticing that, in a bust, the government can increase public expenditure more than in a boom phase. Considering the numbers in Figure 2, the upper limit for government spending in a boom is 24% of the initial endowment, ω_1. On the other hand, in a bust period, this limit increases to 35% of ω_1.

We have seen how both policies respond to exogenous changes in the public expenditure G. However, how do these policies interact with each other in equilibrium? We address this issue by investigating how the equilibrium tax rate responds to changes in the equilibrium net interest rate which is obtained by solving from (26). That is to say,

$$\overline{r} = \frac{\omega_2(\omega_2 - \omega_1) - \beta(\omega_1 - G)(\omega_2 - \omega_1)}{\beta(\omega_1 - G)(\omega_2 - \omega_1) - G\omega_2}. \tag{32}$$

Appendix C reports both the functional forms between $\overline{\tau}$ and $\overline{r}$ given by (25) and (32) respectively. In it, we can also find the same equilibrium relationship as the policies were parameterized by other parameters of the economy.

Figure 2: Equilibrium path of monetary and fiscal policies parameterized by G



Notes: We plot the equilibrium path of the economy parameterized by the public expenditure, G. The left plot refers to a boom while the right one to a bust. The scale in blue represents an arbitrary increase in G, where the darker blue refers to a lower level of G. For the sake of the simulation, assuming $\beta = 0.8$ throughout, for a boom we choose

$$\omega_2 = 5 > \omega_1 = 3$$

while for a bust

$$\omega_2 = 3 < \omega_1 = 5$$

With these values, it is straightforward to see that the maximum G in a boom (given by (31)) is below 0.73 while the maximum G in a bust (given by (30)) is lower than 1.74.

Source: Authors' numerical simulations.

So far everything is straightforward because the economy is in equilibrium. However, it is worth asking what happens with the tax rate if the interest rate is not in equilibrium and vice-versa. In other words, what happens with the tax rate if the monetary authority does not follow the bond market and be set at the cost of public debt. We address this question in the next section by using intermediate aspects when seeking for the economic equilibrium. We use these aspects to build a sequential game with complete information played by the monetary and the fiscal authority.

3.6 Interaction of fiscal and monetary policy

Assume that monetary policy instrument is the base interest rate, while the fiscal instrument is the tax rate levied on private (financial and non-financial) income gains that are realized in the second period. To obtain a genuine strategic interaction between the authorities, we would need to construct a game between them. To do so, we start from the end of the story: namely, from the Nash equilibrium outcome.

Note that if we join (32) and (24) or (25) and (23); then whatever these couple of relations are, they resemble a Stackelberg-equilibrium outcome. Could it be that we can find payoffs for both the monetary and fiscal authorities in such a way that they playing a sequential game generate those equilibria? The answer is yes and we do it in in Section 5.

In the next section we set the Stackelberg-equilibrium outcomes by using the couple of above relations. The first one is the equilibrium when the monetary authority acts like leader while in the second one when it acts like a follower. But beforehand, we clarify some crucial elements in the resulting strategic interaction. The most important variable is the level of government consumption relative to private income, which rests on the assumption of what is considered a low or high public consumption. We adopt two criteria in this regard. The first establishes that there is either a low or a high public consumption if $G < \omega_2 - \omega_1$ or $G > \omega_2 - \omega_1$, respectively, since $\omega_2 - \omega_1 > 0$. The second criterion states that the size of government consumption depends on how close is G to ω_1. In the next section, we analyze these criteria sequentially.

3.7 Monetary authority as the Leader

We consider (24) to be the best response of fiscal policy to monetary policy setting, which sets the interest rate $r = (1-q)/q$ according to the market equilibrium. Then, we can define a sequential game with perfect information played between the monetary authority, the Leader, and the fiscal authority, the Follower. The perfect sub-game equilibrium is given by:

$$\begin{cases} \overline{r} & = \dfrac{(\omega_2 - \omega_1)\big(\omega_2 - \beta(\omega_1 - G)\big)}{\beta(\omega_2 - \omega_1)(\omega_1 - G) - G\omega_2} \\ \\ \tau(r) & = \dfrac{G(1+r)}{\omega_2 - \omega_1 + rG} \end{cases} \tag{33}$$

That the fiscal authority is the Follower is evinced by the reaction variable $\tau(r)$.

By substituting $\overline{r}$ in $\tau(r)$, we find that the backward-induction equilibrium $(\overline{r}, \overline{\tau})$ coincides with (25) and (26), which are part of the equilibrium.[8]

Proposition 1 *Assume that $0 < G < \omega_1$. Then, the following holds:*

1. *If $G < \omega_2 - \omega_1$, then $\dfrac{d\tau}{dr} > 0$.*
2. *If $G > \omega_2 - \omega_1 > 0$, then $\dfrac{d\tau}{dr} < 0$.*
3. *If $\omega_2 - \omega_1 < 0$, then $\dfrac{d\tau}{dr} < 0$.*

Proof By differentiating τ in (33) with respect to r, we obtain:

$$\frac{d\tau}{dr} = G\frac{\Big((\omega_2 - \omega_1) - G\Big)}{\Big((\omega_2 - \omega_1) + rG\Big)^2}$$

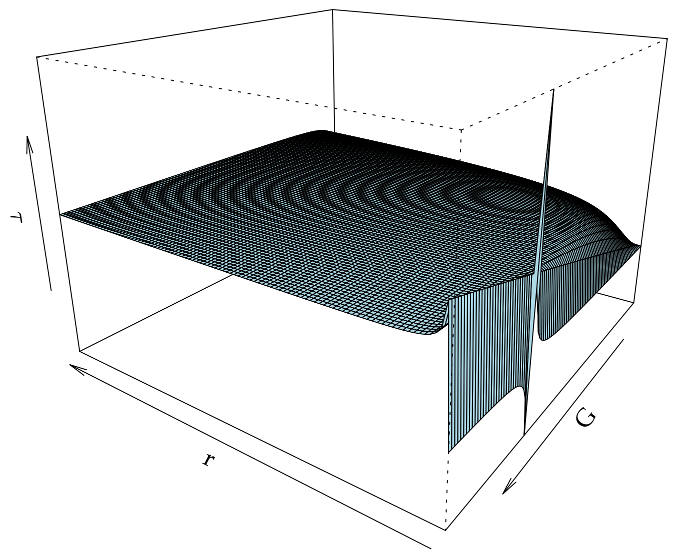
Then, Proposition 1 follows. ■

If the resource available to society increases over time, so that $\omega_2 > \omega_1$, we say that the (second-period) economy is in a boom, otherwise it is in a bust. Proposition 1 claims that if public consumption is low according to the first criterion, the impact that the interest rate

[8] An equilibrium for this economy consists of a price and tax rate $(\overline{\tau}, \overline{q})$ and an allocation of bonds $(\overline{\theta}, \overline{\Theta})$.

has on the income-tax rate is positive. Otherwise, the impact is negative. However, if the economy is in a bust, the effect is always negative. Figure 3 illustrates these relationships by plotting a surface of the tax rate, τ, interest rate, r, and public expenditure, G, in the different phases of the business cycle.

Figure 3: Graphical representation of equation (33), from Proposition 1



Notes: We plot the tax rate τ as function of the interest r and public expenditure G. The resulting figure is a surface showing the relationship between τ and r for all levels of G that belong to $[0, \omega_1]$. In accordance with Proposition 1, τ decreases with r in a bust. However, when $(\omega_2 - \omega_1) > 0$ the relationship between τ and r depends on the relative position of the government expenditure, G. The surface illustrates these relationships.

Source: Authors' numerical simulations.

This means that, in a boom episode with low public consumption, the optimal interaction of fiscal and monetary policy is an increase in the income-tax rate after any rise in the interest rate. This is because the public debt used to fund government consumption is also low, resulting in a small cost of debt rollover after the interest rate increase. However, if either government consumption is high or the economy is in a bust, the best strategy for the policy maker is to reduce the income-tax rate after any rise in the policy interest rate in an attempt to compensate for the negative effects of the restrictive monetary policy.

3.7.1 Optimal Monetary Policy

Although money does not explicitly enter into the model, we can nonetheless talk about a kind of unconventional monetary policy. More precisely, (1) embodies an optimal monetary rule in its essence. The choice of the monetary authority in the perfect sub-game equilibrium given by (33) represents an optimal interest rate rule in the following sense: in this sequential game, it seems that the fiscal authority has an advantage because its choice comes after the Leader. However, since this is a complete information game, the Leader can solve the follower's game in such a way that the optimal interest rate is essentially an optimal response to the choice of the fiscal authority. Nonetheless, if the authorities' roles were reversed, a similar argument could be applied to deliver an optimal fiscal rule. We address this issue below.

3.8 Fiscal authority as the Leader

Now, consider (23) as the reaction function of the monetary to the policy. More precisely, (23) defines r as being a function of τ, not necessarily the equilibrium one that is given by (25). In this case, we have that:

$$\begin{cases} \overline{\tau} &= \dfrac{G\omega_2}{\beta(\omega_1 - G)(\omega_2 - \omega_1)} \\[2ex] r(\tau) &= \dfrac{1}{1-\tau}\left(\dfrac{\omega_2 - \tau(\omega_2 - \omega_1)}{\beta\omega_1 - G(1+\beta)} - \tau\right) - 1 \end{cases} \tag{34}$$

The labelling $r(\tau)$ indicates that the monetary authority is the Follower, to the fiscal authority as Leader.

The system of equations in (34) can be thought of as being a perfect sub-game equilibrium of a sequential game with perfect information played by the fiscal authority (the leader) and the monetary authority (the Follower).[9] When $\overline{\tau}$ is replaced in $q(\tau)$, we obtain a backward-induction equilibrium $(\overline{\tau}, \overline{q})$ that coincides with the equilibrium price of this finance-fiscal economy.[10]

From a technical point of view, if we allow for the best response $q(\tau)$ to depend also on government consumption, G, we have a hyperbole with vertical asymptote at $\overline{G} = \frac{\beta}{1+\beta}\omega_1$. Thus, depending on the level of G, the best response of the monetary authority might be summarized by the following proposition.

Proposition 2 *Let $0 < G < \omega_1$. Then the following results hold:*

1. *if $G < \frac{\beta}{1+\beta}\omega_1$, then $\frac{dr}{d\tau} > 0$.*

2. *if $G > \frac{\beta}{1+\beta}\omega_1$, then $\frac{dr}{d\tau} < 0$.*

Proof By differentiating r in (34) with respect to τ, we obtain

$$\frac{dr}{d\tau} = \frac{(1-\beta)\omega_1 + G(1+\beta)}{(1-\tau)^2\big(\beta\omega_1 - G(1+\beta)\big)}$$

From this, Proposition 2 follows immediately. ■

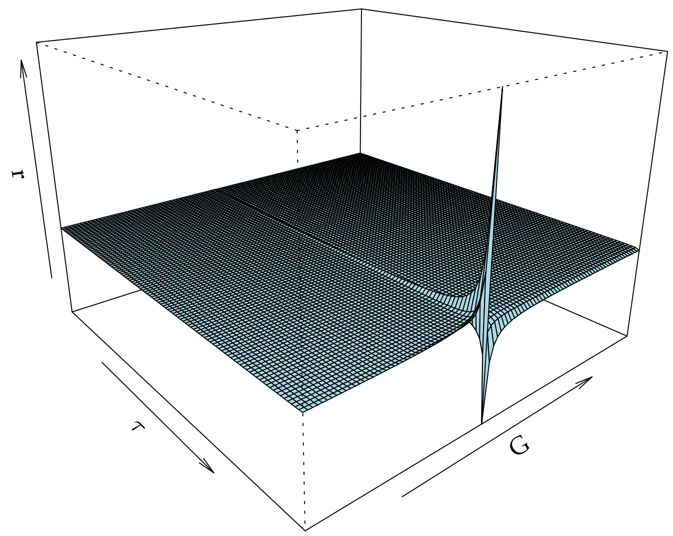
When the interest rate is given by the market equilibrium, Proposition 2 claims that there is a positive relationship between this rate and the income-tax rate when public consumption is low. However, the relationship is negative when government consumption is high relative to the initial endowment. Despite the fact the interest rate given by the second relation in (34), is discontinuous at $\tau = 1$, we can solve for τ of $r(\tau)$ by setting a theoretical fiscal rule.[11] Figure 4 illustrates these relationships by plotting a surface of the tax rate, τ, interest rate, r, and public expenditure, G, in the different phases of the business cycle.

[9] Appendix B describes this game in detail.

[10] Once again, an equilibrium for this economy consists of a price and tax rate $(\overline{\tau}, \overline{q})$ and an allocation of bonds $(\overline{\theta}, \overline{\Theta})$.

[11] See Section 5 for a discussion.

Figure 4: Graphical representation of equation (34), from Proposition 2



Notes: We plot the interest rate, r, as a function of the tax rate, τ, and public expenditure G. The resulting figure is a surface showing the relationship between r and τ for all levels of G that belong to $[0, \omega_1]$. In accordance with Proposition 2, r increases with τ for sufficiently small G. The surface illustrates this positive relationship if we fix a level of G close to 0. However, when G is slightly below the vertical asymptote $\frac{\beta}{1+\beta}\omega_1$, r increases and suddenly becomes too high. The other case, when $G > \frac{\beta}{1+\beta}\omega_1$, reveals the negative relation between r and τ.

Source: Authors' numerical simulation.

Intuitively, with low public consumption, the optimal interaction of fiscal and monetary policy shall still be a countercyclical response as the interest rate increases after any rise in the income-tax rate. This is because the public debt and government consumption are both low, leaving room for increases in income-tax rate and interest rate as best responses for both policies. However, if government consumption is high, the best response of the monetary authority is to decrease the interest rate after any increase in the income tax rate in an attempt to compensate for the negative effects of the restrictive fiscal policy.

4 Empirical evidence

We shall now take our theoretical framework to the Brazilian data. The goal is to investigate the empirical relationship between income-tax rate and interest rate in a threshold model that depends on the level of government consumption, G, as described in Proposition 2.

To empirically evaluate that relationships, we apply a threshold regression framework that captures the dependence of τ on the level of G:

$$\tau_t = \sum_{j=1}^{m+1} \boldsymbol{\alpha}_j \mathbf{x}_t \mathbb{I}_{(G_t, \gamma_j)} + \varepsilon_t \tag{35}$$

where $\mathbf{x_t} = (\mathbf{1}, \omega_t, r_t)$, and $\mathbb{I}_{(G_t,\gamma_j)} = \mathbb{I}(\gamma_{j-1} \leq G_t < \gamma_j)$ is an indicator function for the j^{th} threshold (or break), m is the number of ordered threshold with $\gamma_1 = -\infty$ and $\gamma_{m+1} = +\infty$, and $(\boldsymbol{\alpha}_j, \gamma_j)\ \forall\ i, j$ are parameters to be estimated. We use quarterly data from 2000q1 to 2021q3 to represent the variables in equation (35). The sources of the data were the Central Bank of Brazil and the Internal Revenue Service. The variables are described as follows:

τ_t : logarithm of the total income-tax revenue as a ratio of gross national income (seasonally adjusted). This give us the effective income-tax rate, i.e., the fraction of income that was effectively paid as income-tax by the private sector:

ω_t : logarithm of gross national income at constant prices and seasonally adjusted;

r_t : logarithm of the overnight Selic interest rate, which is the monetary policy instrument in the inflation target regime (the Brazilian equivalent of the Federal Funds Rate);

G_t : logarithm of total government consumption index in real terms.

We tested the time series for unit root by applying the new generation tests of MAD-FGLS and MPPGLS and found them non-stationary. However, according to the Johansen cointegration test, there is a unique cointegrating relation among them, a result that is robust to all test specifications. Thus, the residuals of the estimated threshold regressions are stationary and the asymptotic theory is still valid.

Table 1 reports the estimates. We consider $m = 5$ as the maximum number of breaks. However, taking into account the Bai-Perron F statistics, the optimal threshold number was found to be $m = 2$. Based on this result, we report estimates for $m = 1$ and $m = 2$.

For both regressions, the results of Proposition 2 are empirically validated. In the upper panel, with $m = 1$, for a government consumption level below the threshold $\gamma = 4.9$, the effect of the interest rate r on the income tax rate τ is positive, i.e. $d\tau/dr > 0$. On the contrary, for a government consumption level above this threshold, the effect is negative, that is, $d\tau/dr < 0$. The same result holds in the lower panel, with $m = 2$. For government consumption levels below $\gamma_2 = 4.99$, $d\tau/dr > 0$ and above this threshold, $d\tau/dr < 0$. The difference here is that there is an intermediary regime, between $\gamma_2 = \gamma$ and γ_2, where the $d\tau/dr > 0$ but the estimated response coefficient is lower than the estimate for government consumption threshold below $\gamma = 4.93$.

The rationale for this finding is that, for lower levels of government consumption, it is more likely that the private sector income gains are higher. Conversely, for higher levels of government consumption, it is more likely that the private sector income gains are lower. Thus, we can empirically confirm the results from Proposition 2 for the Brazilian economy by using a data set for the inflation target regime in the post-2000 period.

5 Discussion about equlibria

For the sake of comparison, we describe how the finance-fiscal economy with government works. In the first period, both the taxpayers and government decide how much to buy and sell in the bond market. In the second, taxpayers receive their payoffs, and the government pays its debt by taxing both financial and non-financial income gains.[12]

[12] We do not consider the taxpayers' consumption decisions since they are determined by the financial decisions.

Table 1: Threshold regression estimates for equation (35)

Variables	$m = 1$		
	$G_t < \gamma = 4.9351$	$G_t \geq \gamma = 4.9351$	
ω_t	0.6568*** (0.1234)	-0.0083 (0.2640)	
r_t	0.1828*** (0.0415)	-0.0628*** (0.0193)	
Constant	-6.8830*** (1.6256)	1.8970 (3.4831)	
% obs.	47%	53%	
$R^2 = 0.45$	AIC = -2.7245	BIC = -2.5545	HQ = -2.6561
	$m = 2$		
	$G_t < \gamma_2 = 4.9351$	$\gamma_2 \leq G_t < \gamma_2$	$G_t \geq \gamma_2 = 4.9858$
ω_t	0.6568*** (0.1258)	1.1190*** (0.1574)	-0.7414*** (0.2684)
r_t	0.1828*** (0.0423)	0.0328** (0.0137)	-0.0662** (0.0323)
Constant	-6.8830*** (1.6565)	-12.9659*** (2.0762)	11.5746*** (3.5395)
% obs.	47%	15%	38%
Threshold test	0 vs. 1	1 vs. 2	2 vs. 3
F-stat$^{(a)}$	12.5624***	26.2378***	4.542603
$R^2 = 0.57$	AIC = -2.9017	BIC = -2.6466	HQ = -2.7990

Notes: For details on threshold modeling and estimation procedures, see Hansen (1999, 2000). (a) See Bai and Perron (2003). Newey-West heteroskedasticity and autocorrelation-consistent standard errors in parentheses. *, ** and ***, denote statistical significance at the 1, 5 and 10% level, respectively.

The equilibrium for this finance-economy when the taxpayer's utility is arbitrary is guaranteed by the generalized game methodology due to Debreu (1952). This "generalized game" is played by 3 players: the taxpayers, who maximize their utility function subject to their budget set, the government that sets the policy given by (Θ, τ) to balance its budget set, and an auctioneer who chooses the bond price to clear the aforementioned bond market.[13]

Since taxpayers' preferences are known, the demand and supply of bonds can be easily determined.[14] By using them, we can build the two players' payoffs who will play a sequential game with perfect information.[15] The players are the fiscal and the monetary authority, whose respective payoffs are:

$$\mathcal{F}(q,\tau) = -\left(\Theta(q) - \tau\left((\omega_2 - \omega_1) + (1-q)\theta(q,\tau)\right)\right)^2 \tag{36}$$

$$\mathcal{M}(q,\tau) = -\left(\theta(q,\tau) - \Theta(q)\right)^2 \tag{37}$$

Now we restrict ourselves to the game where the fiscal authority is the leader, and the monetary authority is the Follower. Thus, the timing of game is as follows:

1. The fiscal authority, the leader, chooses a tax rate, τ, which is known by the monetary authority.
2. The monetary authority, the Follower, chooses r under full knowledge of τ.
3. The payoffs are realized, and the game ends.

The solution to this game is through backward induction. The follower solves its problem given the leader's action represented by τ. That is to say, given τ, the monetary authority, the Follower, chooses q in order to maximize

$$\mathcal{M}(q,\tau) = -\left(\theta(q,\tau) - \Theta(q)\right)^2. \tag{38}$$

Thus, the Follower's solution is $q(\tau)$. Given the perfect information assumption, the leader could play instead of the Follower by incorporating $q(\tau)$ and deciding the optimal tax rate

[13] The analysis is easily generalized to accommodate heterogeneous taxpayers.

[14] They are actually the best responses to the (τ, q).

[15] This game has nothing to do with the generalized game used to prove the existence of equilibrium. Actually, this sequential game emerges from conditions (23) or (24) that characterize the equilibrium of the economy.

$\overline{\tau}$. This would be the result of maximizing:

$$\mathcal{F}(q(\tau),\tau) = -\left(\Theta(q(\tau)) - \tau\left((\omega_2-\omega_1)+(1-q(\tau))\theta(q(\tau),\tau)\right)\right)^2. \tag{39}$$

This approach leads us to the unique sub-game perfect equilibrium solution given by (34).

Remark 3 *The aforementioned game is only restricted to the monetary authority's best-responses function, $q(\tau)$, given by the second relation in (34), leaving aside the non-credible strategies that do not come to the case in this study. Ultimately, this is the relationship that informed our threshold regressions using Brazilian data.*

6 Conclusions

The contribution of this paper is to investigate theoretically and empirically the interaction of fiscal and monetary policy in a two-period general equilibrium (GE) model with the government and a representative taxpayer. We compute the equilibrium with logarithmic preferences and endogenously derive a sequential game with perfect information played by the fiscal (the leader) and the monetary authority (the Follower). The best response of the fiscal authority depends on government consumption, which defines a threshold for the relationship between the effective income tax rate and the policy interest rate. We consider that government consumption is low when either it lies below private income gains or it is close enough to the first-period private income.

If the economy is in a boom, such that the second period income exceeds first period income, and public consumption is low according the first criterion, there is a positive relationship between the interest rate on the effective income tax rate rate. Otherwise, this relationship is negative. However, if the economy is in a bust, the relationship is always negative.

If the economy is in a boom with low public consumption, there is a counter-cyclical interaction between the fiscal and monetary policies as the income tax rate increases after any rise in the interest rate. In this scenario, the cost of debt rollover after the interest rate increase is small. However, under a high government consumption or an economic bust, the income-tax rate is reduced after any rise in the interest rate in an attempt to compensate for the negative effects of the restrictive monetary policy.

The threshold regression estimated for the Brazilian data after the inflation target regime closely resembled the theoretical findings. The policy interest rate had positive or negative effects on the effective income tax rate depending on whether public consumption is below or above the estimated threshold, respectively. When government consumption is too high relative to private income, the relationship between interest rate and effective income tax rate is negative. However, if public consumption is low, the relationship is positive. These theoretical and empirical findings suggest a counter-cyclical interaction between the fiscal and the monetary policies that departures from the conventional wisdom.

Intuitively, under low government consumption, interest rate increases lead to reductions in private consumption due to the increase in the effective tax rate under optimal monetary policy, regardless of which authority is the leader or follower. The effective income tax rate rises because the private financial gains increase. Low government consumption is coupled with low public debt and reduced cost of debt rollover after the interest rate increase. If public consumption is high, however, so are the public debt and the cost of debt rollover. Any increase in the policy interest rate reduces the effective income tax rate because financial gains are lower in this scenario.

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A Proof of Theorem 1

Before proving Theorem 1, we first establish the following lemma which allows us to bound the allocations satisfying the feasible conditions (market clear conditions) of the equilibrium definition. More precisely, we state and prove the following lemma.

Lemma 1 *Under hypotheses of Theorem 1, consumption allocations* $(x^h)_{h\in H}$ *in* $\mathcal{E}$ *that satisfy the feasibility conditions of the equilibrium definition are bounded.*

Proof From the first equation of (7) it follows that $x_o^h \leq \sum_{h\in H} w_o^h$ as both x_o^h and C are positive. Setting W_0 to be $\max_{h\in H} \omega_o^h$. Then, $x_o^h \in [0, M_0]$. Similarly, one has $x_s^h \in [0, W]$ where $W = \max_{s\in S} \left(\max_{h\in H} \omega_s^h + Y_s C\right)$. Therefore $x^h = (x_0^h, x_1^h, \ldots, x_S^h) \in [0, M]^{1+S}$ with $M = \max\{W_0, W\}$. ■

Lemma 1 says nothing about the boundedness of the asset allocations $(\theta^h)_{h\in H}$ satisfying the second equation of (7) as we have not imposed, a priory, any bound on government's short sale. In principle, the government could issue assets without limit. It makes us prevent to apply directly the generalized game approach due to Debreu (1952). To overcame this problem, we construct a sequence of economies $\mathcal{E}^n$ for which the asset prices as well as both sale and purchase of assets belong to a boxes depending on n. Then, we construct an equilibrium for $\mathcal{E}$ by taking limits as $n \to \infty$.

A.1 Truncated Economies

Consider a sequence of truncated economies $\{\mathcal{E}^n\}_{n\geq 1}$ in which the budget set of the taxpayer is,

$$B_n(\pi, \tau) = \{(x, \theta) \in [0, M]^{1+S} \times [0, n]^J : \text{ constraints (2) and (3) are satisfied}\} \tag{A.1}$$

where $\pi \in [\frac{1}{n}, 1]^S$ and $\tau \in [-1, 1]^S$. In addition, the government chooses $(\Theta, \tau) \in [0, n]^J \times [-1, 1]^S$ such that equations (5), (6) are satisfied.

A.2 Generalized Games

Define z to be

$$[(x^h, \theta^h)_{h\in H}; (\Theta, \tau); \pi] \in (\mathbb{R}_+^{1+S} \times \mathbb{R}_+^J) \times (\mathbb{R}_+^J \times \mathbb{R}_+^{2S}) \times \mathbb{R}_+^J \tag{A.2}$$

and $z_{-\kappa}$ the vector z in the κ-coordinate has been dropped. The sub-index, κ, is any coordinate of the vector z just defined.

For each $n \geq 1$ we define the following generalized game played by 4 players. We denote this game by $\mathcal{G}^n$ which is described as follows:

1. The first player is the taxpayer whose payoff is their utility function $u^h : \mathbb{R}_+^{1+S} \to \mathbb{R}$ and its strategy set is $B_n(\pi, \tau)$.
2. The second player, representing the government, chooses $\Theta \in [0, n]^J$ in order to maximize its payoff, denoted by $F_1(\cdot)$

$$F_1\big(\Theta; z_{-\Theta}\big) = -(\pi\Theta - C)^2$$

3. The third player is a tax collector who chooses $\tau \in [-1, 1]^{2S}$ in order to maximize its payoff, denoted $F_2(\cdot)$

$$F_2\big((\Theta, \tau); z_{-(\Theta,\tau)}\big) = -\sum_{s=1}^{S}\big(V_s\Theta - \sum_h \tau_{s1}(\omega_s^h - \omega_0^h) - \tau_{s2}(V_s\theta^h - \pi\theta^h) - Y_sC\big)^2$$

4. The fourth player is an auctioneer who chooses $\pi \in [\frac{1}{n}, 1]$ in order to maximize its payoff, denoted by $\ell(\cdot)$

$$\boldsymbol{\ell}(\pi, z_{-\pi}) = \pi(\sum_{h\in H}\theta^h - \Theta) + (\sum_{h\in H}(x_0^h - w_0^h) - C)$$

The objective functions of the taxpayers are continuous and quasi-concave in their strategies. In addition, the objective functions of the players, representing the government, are continuous and strictly concave. Lastly, the the auctioneer's objective function is continuous and linear in their own strategies, and therefore quasi-concave. The correspondence of admissible strategies, for the all players, has a compact domain and compact, convex, and nonempty values. Such correspondences[1] are upper semi-continuous, because they have compact values and a closed graph. We can then apply Kakutani's fixed point theorem to the correspondence of optimal strategies in order to find a pure strategy equilibrium for $\mathcal{G}^n : [\pi_n; (x_n^h, \theta_n^h)_{h\in H}; (\Theta_n, \tau_n)]$.

The following lemma guarantees an equilibrium for each truncated economy $\mathcal{E}^n$.

Lemma 2 *For each n, a pure strategy equilibrium for $\mathcal{G}^n$ is also an equilibrium for the truncated economy $\mathcal{E}^n$.*

Proof To avoid cluttering notation, all indexes n of equilibrium variables of $\mathcal{G}^n$ will momentarily be suppressed. Let $[\pi; (x^h, \theta^h)_{h\in H}; (\Theta, \tau)] \in [\frac{1}{n}, 1]^J \times \left([0, M] \times [0, n]^J\right)^H \times [0, n]^J \times [-1, 1]^{2S}$ be a pure strategy equilibrium for $\mathcal{G}^n$. Thus, (x^h, θ^h) is an optimal choice for each taxpayer so that it belongs to $B_n^h(\pi, \tau)$ and therefore

$$\begin{aligned} x_o^h + \pi\theta^h &= \omega_0^h \\ x_s^h &= \omega_s^h + V_s\theta - \tau_{s1}(\omega_s^h - \omega_0^h) - \tau_{s2}(V_s\theta^h - \pi\theta^h) \end{aligned} \tag{A.3}$$

Here we have used the fact that the utility functions are strictly increasing so that the budget constraints hold with equality.

Optimal conditions of the players representing the government imply that

$$\begin{aligned} \pi\Theta &= C \\ V_s\Theta &= \sum_{h\in H} \tau_{s1}(\omega_s^h - \omega_0^h) + \tau_{s2}(V_s\theta^h - \pi\theta^h) \end{aligned} \tag{A.4}$$

Adding the first equality in (A.3) and the first one in (A.4) one has

$$\pi(\sum_{h\in H} \theta^h - \Theta) + \sum_{h\in H} (x_0^h - \omega_0^h) - C = 0 \tag{A.5}$$

Thus, the optimality conditions of the auctioneer's problem imply that

$$\sum_{h\in H} \theta^h = \Theta \tag{A.6}$$

This ends the proof of Lemma 2

■

[1]The lower semi-continuity of the taxpayers' interior correspondences follows from Hildenbrand (1974, p. 26, fact 4) because the closure of a lower semi-continuity correspondence is also lower semi-continuous.

A.3 Asymptotics of Truncated Equilibria and Proof of Theorem 1

To prove Theorem 1 we analyze the asymptotic properties of the sequence of equilibria

$$\{e_n = [\pi_n; (x_n^h, \theta_n^h)_{h\in H}; (\Theta_n, \tau_n)]\}_{n\geq 1}$$

which exist from Lemma 2. Actually, we will demonstrate that the sequence of equilibria above is uniformly bounded, and therefore it will have a sub sequence that converges, say $\bar{e}$. Theorem 1 will then be shown if we prove that $\bar{e}$ corresponds to an equilibrium of our original economy $\mathcal{E}$.

First, $\pi_n \in [0,1]^J$ for every n so that the sequence $\{\pi_n\}_{n\geq 1}$ is bounded uniformly. Similarly, $\tau_{sn} = (\tau_{s1n}, \tau_{s2n}) \in [-1,1]^{2S}$ so that it is also bounded uniformly. Second, the fact that (x_n^h, θ_n^h) belongs to the budget set $B_n(\pi_n, \tau_n)$ implies that for each $h \in H$, $x^h \in [0, M]$ with $M = \max\{W_0, W\}$ guaranteed by Lemma 1.

Third, due to the fact that the government balances its first-period budget constraint, we have

$$\pi_n \Theta_n = C \tag{A.7}$$

Equality (A.7) and Lemma 1 together with the fact that π_n belongs to $[\frac{1}{n}, 1]^J$ imply the following

$$\frac{\Theta_{nj}}{n} \leq C \leq \sum_{j\in J} \Theta_{nj}$$

and

$$\theta_{nj}^h \leq nW_0$$

We claim that $\Theta_{nj} \leq C$ for if we had the contrary, say $\Theta_{nj} = C + \xi_j$ for some $\xi_j > 0$, we would contradict (A.6) for a n large enough. In fact, if we choose $\theta_{nj}^h = nW_0$ and $\Theta_{nj} = C + \xi_j$, we would have,

$$\sum_{h\in H} \theta_{nj} - \Theta_{nj} = nW_0 H - C - \xi_j$$

which for a n sufficiently large $\sum_{h\in H} \theta_{nj}^h - \Theta_{nj} > 0$, thus contradicting (A.6). Therefore, the sequences $\{\Theta_n, \theta_n^h\}_{n\geq 1}$ are uniformly bounded. That is, $\theta_n^h = \varphi_n \in [0, C]^J$.

From the earlier analysis, we have that the sequence $\{e^n\}_{n\geq 1}$ is uniformly bounded, so that it converges along a sub sequence, say to $\overline{e} = [\overline{\pi}; (\overline{x}^h, \overline{\theta}^h)_{h \in H}; (\overline{\Theta}, \overline{\tau})]$. Next, we shall prove that $\overline{e}$ is an equilibrium for $\mathcal{E}$.

We state that the array $\overline{e}$ satisfies Conditions 2 and 3 of Definition 1. This follows after taking limits in equations[2] (A.4) and (A.6). It remains to prove condition 1 of Definition 1. For that, we require the following claim.

Claim The budget set correspondence $B^h : [0,1]^J \times [-1,1]^{2S} \rightrightarrows [0,M]^{1+S} \times [0,C]^J$ defined by $B^h(\pi, \tau)$ is lower hemicontinuous (lhc) at the point $(\overline{\pi}, \overline{\tau})$.

Proof of Claim

To start with, we state that the correspondence defined by

$$Int\ B^h(q, \pi) = \{(x, \theta) \in [0,M]^{1+S} \times [0,C]^J : (2) \text{ and } (3) \text{ hold with strict inequality } \}$$

is lhc at the point $(\overline{\pi}, \overline{\tau})$. In fact, $Int\ B(\pi, \tau)$ is a non empty set, since $x = 0$ and $\theta = 0$ satisfy equations (2) and (3) with strict inequality as $\omega^h >> 0, \forall h$.

Let $\{(\pi_n, \tau_n)\}$ be a sequence converging to $(\overline{\pi}, \overline{\tau})$ and be $(x', \theta') \in Int\ B^h(p, \pi)$. Thus, for every sequence $\{(x'_n, \theta'_n)\}_{n \geq 1}$ converging to (x', θ') belonging to $Int\ B^h(\overline{q}, \overline{\pi})$ we have that for n large enough (x'_n, θ'_n) belonging to $Int\ B^h(q_n, \pi_n)$ which implies that $Int\ B(\cdot)$ is lhc.

Next, from Hildenbrand (1974, p.26, Fact 4), it follows that the correspondence $B^h(\cdot)$ defined in the earlier claim, which is the closure of $Int\ B^h(\cdot)$, is also lhc. Thus, the Claim follows. ■

To prove Item 1 of Definition 1, let us suppose the contrary. That is, $(\overline{x}^h, \overline{\theta}^h)$ does not maximize u^h subject to $B^h(\overline{\pi}, \overline{\tau})$ being $(\overline{\pi}, \overline{\tau})$ a the cluster point of $\{(\pi_n, \tau_n)\}_{n \geq 1}$. Hence, there exist $(x, \theta) \in B^h(\overline{\pi}, \overline{\tau})$ satisfying $u^h(x) > u(\overline{x}^h)$. The early claim implies that there exists a sequence $\{(x^h_n, \theta^h_n)\}_{n \geq 1} \subset B^h(\pi_n, \tau_n)$ such that $(x^h_n, \theta^h_n) \to (\overline{x}^h, \overline{\theta}^h)$. Notice that the arguments of $B^h(\cdot)$ are terms of the sequence that form part of the sequence of equilibria $\{e^n\}$ of the truncated economy. That is, $(\pi_n, \tau_n) \to (\overline{\pi}, \overline{\tau})$.

Since u^h is continuous one has that for n large enough, we obtain $u(x'_n) > u(x^h_n)$ contradicting the optimality of x^h_n in the truncated economy $\mathcal{E}^n$. This ends the proof of existence.

[2]Notice that we have have suppressed the index n in this equations.

To finish the proof of Theorem **??**, we shall show that there is trading.

From (A.7) (in the limit) and i) of Assumption **??**, we have the implication that $\overline{q\varphi} = -E_0 > 0$. This fact implies that both $\overline{q} > 0$ and $\overline{\varphi} > 0$. and thus that there is trading in equilibrium. That is, $\theta = \varphi > 0$. ■

B Proofs

Lemma 3 *Given τ, the function $\mathcal{M}(q,\tau) = -\Big(\theta(q,\tau) - \Theta(q)\Big)^2$ has a maximum at q satisfying* (23).

Proof In what follows, the sub-indexes stand for partial derivatives and the "prime" is for the ordinary derivatives. The first and second differentiation of $\mathcal{M}(q,\tau)$ with respect to q yields,

$$\mathcal{M}_2(q,\tau) = -2\Big(\theta(q,\tau) - \Theta(q)\Big)\Big(\theta_2(q,\tau) - \Theta'(q)\Big) \tag{A.8}$$

$$\mathcal{M}_{11}(q,\tau) = -2\Big(\theta(q,\tau) - \Theta(q)\Big)\Big(\theta_{11}(q,\tau) - \Theta''(q)\Big) - 2\Big(\theta_2(q,\tau) - \Theta'(q)\Big)^2 \tag{A.9}$$

We claim that the second factor in (A.8) is nonzero. In fact,

$$\Big(\theta_2(q,\tau) - \Theta'(q)\Big) = -\frac{\beta\omega_1}{1+\beta}\frac{1}{q^2} + \frac{\omega_2 - \tau(\omega_2-\omega_1)}{1+\beta}\frac{\tau}{\big(1-\tau(1-q)\big)^2} + \frac{G}{q^2} \tag{A.10}$$

By manipulating the second term of the previous condition, we obtain

$$\Big(\theta_2(q,\tau) - \Theta'(q)\Big) = \left(G - \frac{\beta\omega_1}{1+\beta}\right)\frac{1}{q^2} + \frac{\omega_2 - \tau(\omega_2-\omega_1)}{1+\beta}\frac{\tau}{\big(1-\tau(1-q)\big)^2}. \tag{A.11}$$

Condition (A.11) shows that $\mathcal{M}_2(q,\tau)$ is zero at q satisfying (23). In addition, the second derivative $\mathcal{M}_{11}(q,\tau)$ evaluated at q is strictly negative since the first factor of the first term in (A.9) is zero. This follows from the fact that the q which zeroes $\mathcal{M}_2(q,\tau)$ is solution of the market clearing condition (21). This implies that q, as a function of τ, is a maximum for $\mathcal{M}(q,\tau)$. Thus, Lemma 3 follows. ■

Lemma 4 *$\overline{\tau}$ satisfying* (25) *is a maximizer of $\mathcal{F}(q(\tau),\tau)$ provided that $q(\tau)$ satisfies* (23).

Proof First, we compute $\mathcal{F}(q(\tau),\tau)$ with $q(\tau)$ satisfying (23). In fact,

$$\mathcal{F}(q(\tau),\tau) = -\left(\frac{G}{q(\tau)} - \tau\left(\omega_2 - \omega_1 + \frac{G}{q(\tau)} - G\right)\right)^2. \tag{A.12}$$

Differentiating (A.12) with respect to τ yields

$$\frac{d\mathcal{F}(q(\tau),\tau)}{d\tau} = -2\left(\frac{G}{q(\tau)} - \tau\left(\omega_2 - \omega_1 + \frac{G}{q(\tau)} - G\right)\right)\left(-\frac{G}{q} - G(1-\tau)\frac{q'(\tau)}{q^2} - \left((\omega_2-\omega_1)-G\right)\right). \tag{A.13}$$

Differentiating the first equality in (23) with respect to τ and manipulating the result produces

$$-\frac{G}{q} - G(1-\tau)\frac{q'(\tau)}{q^2} = -\frac{G(\omega_2-\omega_1)}{\beta\omega_1 - G(1+\beta)} - G. \tag{A.14}$$

Substituting (A.14) in the second term of (A.13) yields

$$-\frac{(\omega_2-\omega_1)\beta(\omega_1 - G)}{\beta(\omega_1-G)-G},$$

which is different from zero provided that $\omega_2 \neq \omega_1$ and $G \neq \frac{\beta}{1+\beta}\omega_1$. This implies that the τ satisfying (25) makes $\mathcal{F}(q(\tau),\tau)$ equal zero. That is, $\overline{\tau}$.

To prove that $\overline{\tau}$ is a maximizer, it is sufficient to verify that $\frac{d^2\mathcal{F}(q(\tau),\tau)}{d\tau^2}$ evaluated at $\overline{\tau}$ is strictly negative. Replacing the function within parenthesis in (A.12) by $h(\tau)$, we have

$$\frac{d^2\mathcal{F}(q(\tau),\tau)}{d\tau^2} = -2h(\tau)h''(\tau) - 2\left(h'(\tau)\right)^2 \tag{A.15}$$

after differentiating (A.13) with respect to τ. Lemma 4 then follows, after remembering that $h(\overline{\tau}) = 0$. ■

C The Geometry of the fiscal and monetary policies

First, if (25) and (32) are parameterized by G, then by manipulating them we derive $\overline{\tau}$ as a function of $\overline{r}$. Namely,

$$\frac{1}{\overline{\tau}} = \left(\frac{\beta(\omega_2 - \omega_1) + \overline{r}(\beta(\omega_2 - \omega_1) + \omega_2)}{\beta\omega_1(1+\overline{r}) - \omega_2} - \frac{\omega_2 - \omega_1}{\omega_1} \right) \frac{\beta\omega_1}{\omega_2} \tag{B.1}$$

Differentiating $\overline{\tau}$ in (B.1) with respect to $\overline{r}$ yields

$$\frac{d\overline{\tau}}{d\overline{r}} = \left(\frac{\overline{\tau}}{\beta\omega_1(1+\overline{r}) - \omega_2} \right)^2 2\beta\omega_2(\omega_2 - \omega_1) \tag{B.2}$$

which is strictly positive in a boom ($\omega_2 - \omega_1 > 0$) and strictly negative in a bust ($\omega_2 - \omega_1 < 0$). Second, if (25) and (32) are parameterized by ω_2, then we derive

$$\overline{\tau} = \frac{G(1+\overline{r})}{\omega_2 - \omega_1 + \overline{r}G} \tag{B.3}$$

Differentiating (B.3) with respect to $\overline{r}$ produces

$$\frac{d\overline{\tau}}{d\overline{r}} = \frac{G(\omega_2 - \omega_1 - G)}{(\omega_2 - \omega_1 + \overline{r}G)^2} \tag{B.4}$$

which is strictly positive in a boom provided that $G < \omega_2 - \omega_1$ and strictly negative in a bust ($\omega_2 - \omega_1 < 0$).

Finally, if they are parameterized by $\omega_2 - \omega_1$, then we have

$$\overline{\tau} = \frac{(1+\overline{r})\beta(\omega_1 - G) - \omega_2}{\overline{r}\beta(\omega_1 - G)} \tag{B.5}$$

Differentiating (B.5) one has that

$$\frac{d\overline{\tau}}{d\overline{r}} = \frac{1}{\overline{r}^2}\left(\frac{\omega_2}{\beta(\omega_1 - G)} - 1 \right) \tag{B.6}$$

which is strictly positive in a boom – while in a bust it depends on the taxpayer's discount factor.