Leading by Example Among Equals*

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Abstract

I examine the factors that determine whether a grassroots social movement reaches the necessary size to achieve its goal. I study a collective action problem where identical individuals who value the common goal sequentially decide whether to join the movement. The model has two key ingredients: (i) The movement is facing a free-riding problem (i.e., while individuals want the movement to succeed, they would rather have others bear the cost of participation) and (ii) The necessary number of members to achieve success is ex-ante unknown but it can be revealed as the movement grows in size. The central insight is that an increase in cost of participation, such as harsher and more likely punishment for members of the movement, can lead to a drastic surge in membership.

Keywords: Social movements, repression, free-riding, threshold uncertainty, dynamic games

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1 Introduction

It is difficult to predict whether a grassroots social movement will succeed. While the supporters of a cause fail to mobilize in one case, they take active responsibility in mass under similar circumstances in another. When we look at movements that rapidly grow and reach the critical level of participation to achieve their common goal, we see that they often follow lengthy periods where supporters of the cause were unable to coordinate in collective action. A significant example of such an abrupt spark in participation is the fall of the Berlin Wall in 1989. In some cases, rapid mass mobilization is paradoxically triggered by changes that are meant to deter it. An example of this is repressive action against a movement's members reaching a point where it backfires. Such instances range from the Amritsar massacre of 1919 prompting the mass mobilization of those who oppose the British rule in India, to more recent cases such as those who oppose the ongoing deforestation of Istanbul taking to the streets after the use of excessive police force in Gezi Park in 2013.

In this paper, I study when and how the supporters of a cause overcome the problem of collective action. I argue that while the variation in a movement's ability to mobilize may appear arbitrary at first glance, it can be explained in a unified manner through a threshold phenomenon. A small change in the environment can convince a large number of "ordinary" individuals who value the cause to pay the personal cost of taking action instead of free-riding off others. Surprisingly, this small change can be in the form of a higher personal cost.

To demonstrate this phenomenon, I model the formation of a movement as an ongoing process. As the movement forms, different individuals encounter the decision of whether to participate at different points in time (e.g., as they come in contact with the local chapters of an organization). The movement succeeds if the necessary number of participants is reached. Participation comes with a cost independent of the outcome, such as the time dedicated to the movement or the risk of being arrested in the process. These individuals are identical except for the time at which they encounter the decision: They value the cause equally, they have to pay the same cost to participate, and all information is public. The key ingredient of the setting is that initially, the necessary number of participants for success is unknown. As the movement continues to grow, however, their progress (or lack thereof) informs those who value the goal about how many participants are needed to achieve it. In particular, when the goal itself is reached, the necessary number is revealed. The lack of success despite a growing movement, on the other hand, makes individuals believe that more people are needed than they initially thought.

This game yields a unique equilibrium. In this equilibrium, the chance of success is determined by when the initial participation takes place, which depends on the cost of participation and the (common) prior belief about how many participants are necessary. Once an individual decides to participate, those who follow her do so until either the goal is reached or the base of supporters is exhausted (i.e., all individuals who support the cause have made their decisions). Thus, the earlier participation starts, the higher is the chance of success for the movement. Individuals who decide before the initial participant choose to stay out, knowing that those who will be in their position later will bear the cost. The structure of the equilibrium suggests that due to free-riding, it can take time for a movement to get off the ground.

I capture the shifts in the environment surrounding the movement through changes in cost of participation. The central finding is that for a large set of prior beliefs about the necessary number of participants, there are critical intervals of cost where the equilibrium chance of success makes an upward jump. When there is an increase in cost such that it crosses the lower bound and enters one of these intervals, it becomes optimal for a large number of early decision makers to participate instead of free-riding off later movers. This is because in these intervals, they are not only responsible for their own participation in the movement: the participation of future decision makers depends on theirs as well. By participating, they can lead the future movers by example, even though they have no private information to signal and their preferences are the same as everyone else.

The dependence on the actions of earlier movers occurs because the participation decision of an individual is affected by the group size (i.e., number of people already participating) at the time of her move. In particular, observing a larger group leading up to her move has two potential effects on her incentive to participate that counteract each other. First, joining a larger group can mean that the difference that her participation makes (i.e., her marginal contribution to success) is lower. We can consider this a deterrent "crowding out" effect. Second, if success has not yet been achieved despite a large group, she updates her beliefs about the number of participants it takes to achieve the common goal.[1] This makes her think that her participation is more likely to be necessary. We can consider this an encouraging "information effect". If the latter effect dominates the former, then a bigger group incentivizes this individual to participate. Then, for some levels of cost, she participates if and only if sufficiently many have done so before her without success. That is, her decision is contingent on a large enough group size for a "critical interval" of cost.

A cost increase leads to higher participation because if an individual is contingent, earlier movers anticipate it: If the cost lies below a critical interval, early movers know that the participation of later movers is independent of their own, so they free-ride. If, however, the cost is raised to an interval where one or more future decision makers are contingent, these earlier movers start a chain of participation such that each participant puts the next one in a position where future individuals depend on her. Throughout the paper, I refer to this phenomenon as a "participation cascade".

I provide a sufficient condition on prior beliefs about the necessary group size for success such that these cascade intervals exist. I show that when there is a large population playing the game, this condition holds for a large set of commonly used distributions, such as the log-normal. Then I provide two examples to demonstrate the extent of the change in success probability caused by participation cascades, including an extreme case where a small increase in cost leads to a jump from a single participant to all individuals who value the goal participating. In other words, a slightly higher cost can allow an essentially non-existent movement to mobilize all of its supporters.

[1] Specifically, she concludes that the necessary group size is greater than the level she observes. Therefore, she truncates her beliefs about necessary group size from below.

Relation to the Literature

My framework has three main contributions to the study of mass mobilization in social movements. First, I provide an explanation as to how a successful movement can be triggered by adverse developments, such as intensifying persecution of its members.[2] This phenomenon is referred to as the "paradox of repression" (Kurz & Smithey (2018)).

Second, I show that observable participation by "ordinary individuals" who possess no private information can convince others to participate, triggering a surge in movement size. Existing research involving actions that spark mass mobilization focuses on heterogeneity among the supporters. These actions have been attributed to "extremists" (i.e., those who value the common goal the most) mobilizing the moderates (Kuran (1991), Lohman (1994 a,b), Kricheli et al. (2011)). Another proposed source is agents who possess better information regarding the value of the common goal (Ginkel & Smith (1999), Loeper et al. (2014)). Finally, it is suggested that observable actions can serve as an aggregator of private information dispersed across the population (Lohmann (1994 a,b), Chwe (2000), Bueno de Mesquita (2010), Battaglini (2017), Barbera & Jackson (2020)). I focus on an environment where all information is public and all supporters are identical to show that the actions which mobilize the masses need not come from extremists, fringe groups, or insiders. They can take place purely by virtue of their observability.[3]

Third, I provide conditions on prior beliefs such that the participation of others encour-

[2] Kricheli et al. (2011) predict that repression makes protests less likely to occur, but more likely to lead to a revolution once they do. However, it unambiguously decreases the ex-ante probability of a revolution. Winter (2009) describes a related result for a different setting: Optimal reward schemes in a team project. He finds that if efforts are complementary in the production technology, we can design two (heterogeneous across members) reward schedules such that one induces more effort although it rewards all team members a lower amount for a successful project.

[3] This insight also extends to the general study of public good provision. Contributions by early movers aimed at manipulating the decisions of others are referred to as "leading by example" (see Drouvelis (2021) for a survey of experimental evidence). As in the social movement literature, this is mainly attributed to heterogeneity (e.g., asymmetric information in Hermalin (1998), heterogeneous preferences in Winter (2009)).

ages or deters a supporter of the movement from participating herself. Much of existing work on social movements abstracts away from a potential deterrence by imposing strategic complementarity. Examples include Kuran (1991), Kricheli et al. (2011), Edmond (2013), Hsieh et al. (2023), and the literature using a global game approach (e.g., Angeletos et al. (2007)). I make no a priori restriction on the complementarity of individual participations. Instead, I examine movement formation as a collective action problem, which allows for free-riding incentives (via strategic substitutability) in addition to the usual coordination concerns (Olson (1971)).

Through this approach, I show that the willingness of a supporter to participate can increase with the number of participants they observe, while decreasing with how many others they expect to participate in the future. This discrepancy between the effects of knowledge and expectation regarding the participation of others is in line with recent empirical evidence on protest behavior: While improved information transmission is found to increase protest attendance (Enikopolov et al. (2020), Manacorda & Tesei (2020)), when only the beliefs about the number of participants are varied, evidence of strategic substitutes is found (Cantoni et al. (2019)).

Regarding the payoff structure, this paper is related to Basak et al. (2023) and Matta (2024) who model the success of a protest as a public good. Basak et al. (2023) take a simultaneous setting under uncertainty, and examine the effects of information similarity between groups of potential participants. Matta (2024) looks at a dynamic setting with a known success threshold, and argues the existence of equilibria where a positive number of people participate even when there are arbitrarily many potential participants. I study a public good problem that combines a sequential structure and (common) threshold uncertainty. The consequent collective learning is at the heart of the backfiring repression result, as well as the opposing incentive effects of observing and expecting others' participation.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 provides a sufficient condition for participation cascades and discusses when it applies for large populations. Section 4 analyzes the model for two example threshold distributions.

Section 5 concludes.

2 Model

Consider the following game where a finite number N of identical individuals are to form a group in order to produce a public good. The game consists of N periods of time. In each period, one individual is randomly selected to move without replacement (i.e., each individual is drawn in exactly one period). Let Player i denote the individual who is drawn to move in period $i \in \{1, ..., N\}$. Player i chooses an action $a_i \in \{0, 1\}$, where actions 0 and 1 stand for *pass* and *participate* respectively. Prior to her move, Player i observes the action of all players $\{1, ..., i-1\}$ who have moved before her. After the move of Player N, the game ends. [4]

If the number of players who choose to participate reaches threshold $t \in \mathbb{N}_+$, a public good is produced and all players obtain utility 1 regardless of their action. If a player participates, she pays cost $c \in (0, 1)$ with no refunds. We can summarize the payoff of Player i as follows.

$$u_i = \mathbb{1}\left\{\sum_{j=1}^{N} a_j \geq t\right\} - \mathbb{1}\{a_i = 1\}c$$

Throughout the analysis, it is assumed that players choose to participate in case of indifference.[5]

Ex-ante, threshold t is unknown. It is common knowledge that this threshold will be revealed to all as soon as the number of players i who play $a_i = 1$ reaches t (before

[4]All results presented in the paper carry through if we allow players to only observe the *number of players who have chosen to participate*, instead of the complete action profile. The optimal action at a given period depends on the history of play only through the number of participants. Therefore, the game yields the same equilibrium path under this alternative specification. This is shown in Appendix A.1.

[5]This assumption is required only for finitely many values of cost c in the interval $(0, 1)$. For generic cost, all results discussed in the analysis continue to hold without this assumption.

the remaining players make their move). The game effectively ends as soon as t players choose to participate, because it is revealed that the public good has already been produced and there is no more gains from participating for those who move later.

It is common knowledge that ex-ante, t follows prior probability distribution F with $supp(F) \subseteq \mathbb{N}_+$. Denote by p and F the probability mass function and cumulative distribution function of t respectively. That is, $p(n)$ dentoes the prior probability that $t = n$ for $n \in \mathbb{N}_+$. Given that t has not yet been reached, the belief of Player i is the Bayesian posterior as a function of the number of players who chose to participate up to her turn. In particular, if she observes $k \in \{1, ..., i-1\}$ participants prior to her turn without success (i.e., without the threshold having been reached), she conditions her belief on $t > k$. The solution concept is subgame perfect equilibrium.

3 General Case

In this section, I first discuss the structure of the unique equilibrium path in the above model. Then I argue that raising the cost c above certain levels can lead to an upward jump in the equilibrium group size (that is, the number of players who choose to participate on the equilibrium path), and thus the probability of producing the public good. I provide a sufficient condition on distribution F such that these upward jumps occur and discuss the underlying mechanism. Finally, I describe some cases where this condition holds when the game is played by a large number of players.

3.1 Equilibrium Path

As described in the previous section, once threshold t has been reached the public good has already been produced and there is no gain from participating. Thus, any strategy that chooses to participate after the threshold is reached is strictly dominated. Here I discuss the action of a player given that the threshold has not yet been reached by the time of her move. Let “potential participant” describe a player who participates if the threshold is not reached by the time of her move (i.e., Player i is a potential participant if she participates given that the threshold is not reached by period i). If a player is not a potential participant, then she passes regardless of whether the threshold has been reached.

The ex-ante probability of success (i.e., producing the public good) is equal to the probability that the number of potential participants is greater than or equal to threshold t. That is, if the number of potential participants is $s \in \{1, ..., N\}$, the success probability is given by $F(s)$. The marginal return (in terms of added ex-ante success probability) to one further participant when there are s potential participants is given by $F(s+1) - F(s) = p(s+1)$: This additional participant will be pivotal to success if and only if the threshold is $s+1$.

While the equilibrium number of potential participants and the resulting success probability depends on the exact specification of distribution F, we can make the following observations on the structure of the unique equilibrium path. Proofs of all results can be found in the Appendix.

Proposition 1. *The game yields a unique equilibrium. The following hold on the equilibrium path.*

(a) *If there are any potential participants, then there exists an initial participant* $n^* \in \{1, ..., N\}$ *such that Player* i *is a potential participant if and only if* $i \in \{n^*, ..., N\}$

(b) *If* $c \leq p(N-i+1)$*, then Player* i *is a potential participant.*

The first observation states that on the equilibrium path, all players who move before some period n^* pass. Player n^* is the first one to participate and after her, the remaining players participate until either the threshold is reached or all N players have made their moves without reaching the threshold. In other words, once participation starts, it continues until success or the end of the game. The success probability of the group is determined by the order of this initial participant n^*. The earlier the initial participant (i.e., lower n^*), the more potential participants there are and the higher is the ex-ante probability of success. Note that with sufficiently high cost c, there is no initial participant and all N players pass on the equilibrium path.

The second observation states a sufficient condition for a given Player i to be a potential participant on the equilibrium path. If Player i knows that all of the later $N - i$ movers are potential participants and all of the earlier movers have passed, then the return to her participation is given by $p(N - i + 1)$. If this return is greater than the cost, then Player i is a potential participant in equilibrium. In other words, if Player i is willing to be the initial participant when those who move later are potential participants regardless of her action, then she herself is a potential participant in equilibrium. Note that this is not a necessary condition. The cascades described in the remainder of the paper occur because when the decision of a player influences later movers, she may participate even though the return to her individual participation does not cover the cost.

Finally, note that the two observations together mean there is at least $\max\{n \in \{1, ..., N\} : p(n) \geq c\}$ potential participants in equilibrium. That is, if the probability that a given participation level n is the threshold for success is greater than the cost, then that participation level will be met in equilibrium (unless success is already achieved before it is met). To see this, note that if $p(n) \geq c$ for some $n \in \{1, ..., N\}$, then observation (b) implies that Player $N - n + 1$ is a potential participant, which by observation (a) means that all n players in $\{N - n + 1, ..., N\}$ are potential participants. It is worth noting that lower bound $\max\{n \in \{1, ..., N\} : p(n) \geq c\}$ is the number of potential participants that would arise without the information externality caused by observing earlier players participate without success. In particular, it is the highest number of participants across all Nash Equilibria if the game is played with simultaneous decisions. Furthermore, it is the number of participants in the unique equilibrium of the game if the decisions are sequential but the threshold is revealed after all players have made their moves instead of being revealed as soon as it is reached. This level will be used as a benchmark for the analysis of the participation cascades in our setting.

3.2 Participation Cascades: A Sufficient Condition

Denote by s^* the equilibrium group size (i.e., number of potential participants on the equilibrium path). By Proposition 1, we have $s^* := N - n^* + 1$ and the ex-ante success probability in equilibrium is given by $F(s^*) = F(N - n^* + 1)$. The following result states a sufficient condition on threshold distribution F such that s^* makes an upward jump as a result of an increase in cost c.

Theorem 1. *If $p(N-2) > p(N-1) > p(N)$ and $\frac{p(N)}{p(N-1)} > 1 - p(1)$, then s^* is non-monotonic with respect to cost c.*

The main observation of this result is that if the ex-ante marginal return $p(n)$ to one additional participant is decreasing above $N-2$ participants, but not too fast, then a higher cost c can yield a larger group size in equilibrium. In particular, when cost c is just below $p(N-1)$, Player 1 passes and Player 2 is the initial participant on the equilibrium path. This leads to equilibrium group size $s^* = N-1$. If the cost is raised to just above $p(N-1)$, however, Player 1 becomes the initial participant, and the equilibrium group size increases to N.

For the argument behind this increase, consider the equilibrium action of the first two movers under two cases $c = p(N-1) - \epsilon$ and $c = p(N-1) + \epsilon$ with ϵ small. Theorem 1 requires that we have $c < p(N-2)$ in both of these cases. Thus, if Player 1 and Player 2 pass, which means the move of Player 3 is reached with zero past participants, all of the remaining $N-2$ players are potential participants in the continuation game by Proposition 1. Now suppose $c = p(N-1) - \epsilon$. If Player 1 has passed, then Player 2 decides based on her prior belief and concludes that the return to her participation is $p(N-1)$ (since the remaining $N-2$ players are potential participants), which is greater than the cost. Anticipating this, Player 1 then knows that her own participation would yield return $p(N)$. This return is smaller than the cost for small ϵ, so she passes and Player 2 is the initial participant in equilibrium. This yields group size $N-1$.

If the cost is raised to $c = p(N-1) + \epsilon$, the return $p(N-1)$ is not sufficient to convince

Player 2 to participate. Thus, Player 1 knows that if she passes, Player 2 will pass as well. If Player 1 participates and the threshold is not reached, however, then Player 2 rules out the possibility that the threshold is equal to 1 and she updates her belief accordingly. Since Player 1 has already participated and there are $N-2$ more potential participants after her, Player 2 knows that her own participation is pivotal to success if and only if the threshold is N. The increase in success probability resulting from her participation is then given by $\frac{p(N)}{1-p(1)}$, which is greater than the cost (by the condition of Theorem 1). Hence, Player 2 is willing to participate if and only if she has observed Player 1 participate without success before her. Anticipating this, Player 1 knows that the participation of Player 2 depends on hers. She is effectively adding not 1 but 2 potential participants to the group: Herself and Player 2. Hence, Player 1 becomes the initial participant and the equilibrium group size increases to N.

To summarize, a player can be incentivized to participate herself by observing others participate without success. Through this observation, she rules out low group sizes as threshold candidates, which makes her believe that her own participation is more likely to be pivotal for reaching the threshold. When the cost is raised to a level where she needs this additional incentive, her participation becomes contingent on a certain number of past participants. Anticipating this, earlier movers are prompted to participate as well, knowing that this will induce the contingent player to do so. As a result, the group size is higher and success is more likely in equilibrium.

The sufficient condition in Theorem 1 only uses the contingency of one player (Player 2) on one past participant (Player 1). As such, it corresponds to a cascade of size 1. The examples in the next section show that more drastic upward jumps are possible. In particular, the first example demonstrates that multiple players can be contingent on a past participant for a given cost. The second example shows that a player can be contingent on a large number of past participants. This includes the the extreme case where a player who moves late in the game is contingent on all players up to her turn having participated.

3.3 Interpreting the Condition in Large Populations

Suppose F has full support over positive integers. In this case, if the convergence rate $\lim_{n\to\infty}\frac{p(n+1)}{p(n)}$ of p to 0 is sufficiently high (greater than $1-p(1)$), then there is a number $\underline{N}$ such that the condition presented in Theorem 1 is satisfied for all $N > \underline{N}$. In other words, if the threshold distribution decays sufficiently slowly and the game is played by sufficiently many players, then Theorem 1 applies.

The condition holds for large N regardless of initial value $p(1)$ if p converges to zero sublinearly. That is, if $\lim_{n\to\infty}\frac{p(n+1)}{p(n)} = 1$. For example, if $p(n) = \int_{n-1}^{n} f(x)dx$ where f is the density of the log-normal distribution, then the property $\lim_{n\to\infty}\frac{p(n+1)}{p(n)} = 1$ is satisfied and the upward jumps in group size will occur for sufficiently large N. Furthermore, any power law distribution (i.e., $p(n)$ follows a power law $n^{-\alpha}$ after some value of n), converges to zero at a sublinear rate. An example is the case where the threshold is Pareto distributed. Hence one implication of Theorem 1 is that in large games, participation cascades occur at certain cost intervals when the threshold distribution is sufficiently heavy tailed. Note that sublinear convergence of p is not a necessary condition. Example 2 in the following section analyses a case where $\lim_{n\to\infty}\frac{p(n+1)}{p(n)} < 1$ but participation cascades are still observed.

4 Examples

This section analyzes the equilibrium of the model under two threshold distributions to illustrate the extent of the participation cascades introduced for the general case above. First, I discuss the case where there the threshold can only take one of two possible values. Second, I look at the case where the threshold follows a geometric distribution with an uncertain parameter.

4.1 Two Possible Thresholds

Suppose $supp(F) = \{\underline{t}, \bar{t}\}$ with $\underline{t}, \bar{t} \in \mathbb{N}_+$ and $\underline{t} < \bar{t} < N$. Assume $p(\underline{t}) = 1 - p(\bar{t}) > 0.5$. The following result states the equilibrium group size as a function of cost c.

Theorem 2. *The equilibrium group size in the model with two possible thresholds is as follows.*

$$s^* = \begin{cases} \bar{t}; & c \in (0, p(\bar{t})] \\ \underline{t}; & c \in (p(\bar{t}), p(\underline{t})] \\ \bar{t}; & c \in (p(\underline{t}), 1) \end{cases}$$

Early movers $\{1, ..., N-\bar{t}\}$ free ride off the later $\bar{t}$ movers and never participate in equilibrium. If cost c is lower than the probability of both possible thresholds, then the group size is $\bar{t}$, which guarantees that the public good is produced. For interior levels where the cost c is lower than the probability of low threshold $\underline{t}$ but higher than that of high threshold $\bar{t}$, only the low threshold is met in equilibrium. This leads to ex-ante probability $p(\underline{t})$ of producing the public good. The main observation of this result is the third case. If the cost is above $p(\underline{t})$, then once again just enough players participate in equilibrium to guarantee the production of the public good. Hence, raising the cost above the level $p(\underline{t})$ increases the group size and the probability that the public good is produced. In short, the reason is that with high cost, participation by late movers becomes contingent on sufficiently many past participants. This leads to cascades of participation among earlier movers.

The equilibrium group size in the first two cases $c \leq p(\bar{t})$ and $c \in (p(\bar{t}), p(\underline{t})]$ are unsurprising: A candidate group size is met in equilibrium if and only if the likelihood that it is the necessary threshold for success is greater than the cost. That is, $s^* = \max\{n \in \{1, ..., N\} : p(n) \geq c\}$

If $c > p(\underline{t})$, then no individual participation yields high enough returns to cover the cost under the prior threshold distribution. If, however, a player follows a history where at least $\underline{t}$ earlier players have participated without success, she rules out the possibility that the threshold is $\underline{t}$ and is certain that it is $\bar{t}$. If she is in a position where she is pivotal to the group size reaching $\bar{t}$ after such a history, then her decision is between producing the public good for certain or with zero probability, which prompts her to participate at any cost. Thus, if the move of Player $N - (\bar{t} - \underline{t}) + 1$ is reached with $\underline{t}$ past participants and no success, the remaining $\bar{t} - \underline{t}$ players are all potential participants since they know for certain that the threshold is $\bar{t}$. Otherwise, they all pass since their decision is based on their prior belief. The previous $\underline{t}$ movers anticipate this. Starting from Player $N - \bar{t} + 1$, each of them participates and puts the next mover in a position where she must also do so in order the continue the chain of $\underline{t}$ participations that will change the belief of the last $\bar{t} - \underline{t}$ movers. Thus, every player from Player $N - \bar{t} + 1$ onwards is in a position where the potential participation of all later movers is contingent on hers. This makes it worth paying cost c and the number of potential participants is $\bar{t}$. Success is guaranteed.

4.2 Geometric Threshold

Suppose the threshold follows a geometric distribution with a parameter p that is unknown (i.e., F is a compound geometric distribution). In particular, p can take one of two possible values p_1 or p_2 with $p_1 > p_2$. The common prior belief of the players is that $p = p_1$ with probability q and $p = p_2$ with probability $1 - q$. Then the prior probability that threshold t is equal to $n \in \mathbb{N}_+$ is given by

$$p(n) = q((1 - p_1)^{n-1} p_1) + (1 - q)((1 - p_2)^{n-1} p_2)$$

It is possible to interpret this threshold distribution in two ways. One interpretation is that as the current group size becomes larger, the marginal contribution of a further member decreases at a constant but unknown rate. A second interpretation is that when there is a new participant, the group makes a new (independent) attempt at producing the public good.[6] However, the difficulty of the attempts is unknown to the players. Then, $p = p_1$ and $p = p_2$ correspond to the easy and the difficult states of the world respectively.

In the remainder of this section, I state the necessary and sufficient condition under which participation cascades occur in this setting. Then I discuss the equilibrium group size in cases where this condition is violated and satisfied, respectively.

[6] With this interpretation, it can be argued that these attempts should not be independent but instead become more likely to succeed with more participants. Here I take parameter p as constant across number of participants for tractability of the equilibrium characterization. In general, it is confirmed by Theorem 1 that the cascades discussed here occur whenever parameter p is a non-decreasing function of the group size.

Proposition 2. *Equilibrium group size s^* is monotonic with respect to cost c if and only if $p_1(1-p_1)^{N-2} \geq p_2(1-p_2)^{N-2}$. If s^* is monotonic, then $s^* = max\{n \in \{1, ..., N\} : p(n) \geq c\}$*

This first implication of this result is that for any given $(p_1, p_2) \in (0,1)^2$, there exists a value $\underline{N}$ such that s^* is non-monotonic for all $N > \underline{N}$. Thus, if the game is played by sufficiently many players, upward jumps in group size with respect to cost c will be observed. The second implication is with respect to the possible parameters p_1 and p_2. Fixing N, we can conclude that if s^* is non-monotonic under some pair (p_1, p_2), then it is also non-monotonic under any (p'_1, p'_2) with $(p'_1, p'_2) \geq (p_1, p_2)$.[7] This can be interpreted as for a given N, cascades occurring if and only if the possible parameters of the distribution are sufficiently high. In other words, for a given population of players, cascades occur if the prior belief is that producing the public good is "sufficiently easy".

When s^* is monotonic with respect to c, a number $n \in \{1, ..., N\}$ of potential participants is met in equilibrium if and only if the likelihood $p(n)$ that n is the threshold for success is greater than the cost. As a result, we have $s^* = \max\{n \in \{1, ..., N\} : p(n) \geq c\}$ for the equilibrium group size. Clearly, s^* is decreasing with respect to cost c in this case.

The following result states the equilibrium group size for a range of parameters in the non-monotonic case.

Theorem 3. *Assume $p_1 + p_2 > 1$ and $p_2 \in (q, 1-q)$. For all $n \in \{2, ..., N-1\}$ there is a $k_n \in \mathbb{N}$ such that an interval*

$$C_n^k := \Big(\frac{p(n+k-1)}{1-F(k-1)}, \min\Big\{\frac{p(n+k)}{1-F(k)}, \sum_{m=n}^{n+k} p(m)\Big\}\Big]$$

exists for all $k \in \{1\} \cup \{k_n, k_n+1, ...\}$. If $c \in C_n^k$ for some pair (n, k) with $n + k \leq N$, then $s^* = n + k$. Otherwise $s^* = max\{n \in \{1, ..., N\} : p(n) \geq c\}$.

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[7]The condition stated in Proposition 2 is under the assumption $p_1 > p_2$ and reversed for $p_2 > p_1$. The observation that the pairs (p_1, p_2) which violate the monotonicity condition constitutes an upper set of the allowed pairs holds in both cases.

Theorem 3 means that for $n \in \{2, ..., N-1\}$ and for k either equal to 1 or sufficiently high, there is a cost interval C_n^k in which s^* makes an upward jump (of size $k+1$) with respect to the monotonic case group size $max\{n \in \{1, ..., N\} : p(n) \geq c\}$. Outside these intervals, s^* is as in the monotonic case. Therefore, a small increase in cost c can cause a large surge in success probability, when it leads to c entering an interval C_n^k. This is in contrast to the monotonic case, where s^* decreases gradually with respect to c. There is a straightforward interpretation for these intervals: If cost c lies in interval C_n^k, then Player $N-n+1$ is willing to participate if and only if she follows a history where at least k earlier players have participated without success. Furthermore, for this cost interval the k players preceding her are willing to provide that history. Figure 1 is a visual representation of the equilibrium group size with respect to c in a 4 player game.

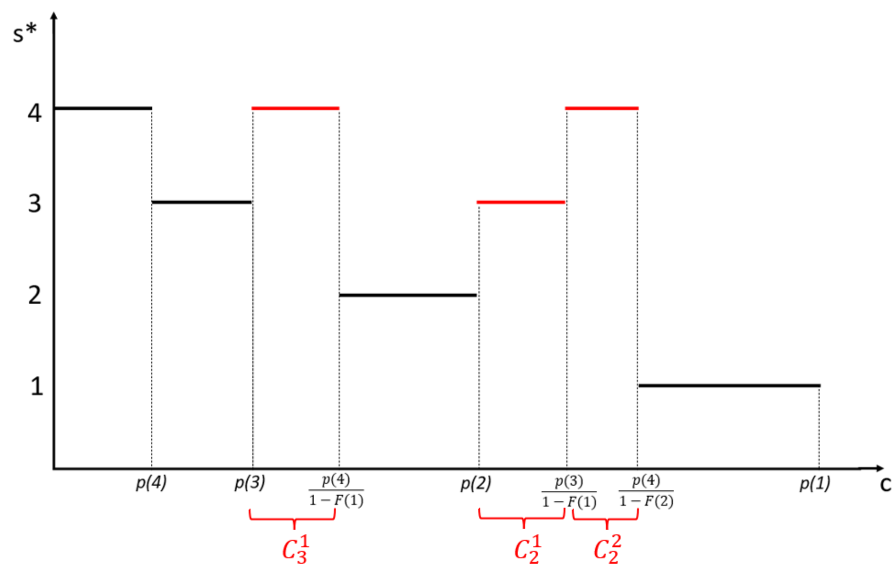

Figure 1: The equilibrium group size w.r.t. c under $(q, p_1, p_2) = (0.4, 0.8, 0.5)$ in a 4 player game. The red regions represent the "upward jump" intervals (from left to right: Player 2 contingent on 1, Player 3 contingent on 1 and Player 3 contingent on 2 past participants). The group size in the black regions are identical to the monotonic case. Note that depending on the parameters, regions C_2^1 and C_2^2 may or may not be adjacent.

To see the why a cascade occurs for $c \in C_n^k$, consider the decision of the contingent Player $N-n+1$. Given that all $n-1$ players after her are potential participants and there has been $k-1$ participants without success before her move, she knows that her participation is pivotal to success only if the threshold is $n+k-1$. Furthermore, she knows that the threshold is greater than $k-1$. In this case, the return to her participation is $\frac{p(n+k-1)}{1-F(k-1)}$. Similarly, if she follows k past participants, then this return is $\frac{p(n+k)}{1-F(k)}$. Since $c \in C_n^k$, the return is higher than the cost if and only if she is moving after at least k past participants. The player who moves k stages before the contingent player $N-n+1$ is responsible for starting the chain that makes her a potential participant. In effect, by participating, this player increases the number of potential participants from $n-1$ to $n+k$. Then the increase in success probability caused by the participation of this player is $\sum_{m=n}^{n+k} p(m)$, which is higher than the cost for $c \in C_n^k$. She starts the cascade of k participants aimed at encouraging the contingent player $N-n+1$ and the resulting group size is $n+k$.

Note that a cascade of size $k > 1$ occurs only if there is a pair (n, k) such that cost interval C_n^k exists and there are at least $n+k$ players in the game. The former condition is satisfied when k is sufficiently large (i.e., greater than lower bound k_n for given n). For the latter condition, we need N to be sufficiently large (i.e., greater than $n+k$). Combining these two observations, we see that drastic upward jumps in group size occur when the game is played by many players. For instance if N is sufficiently large, then for each $n \in \{2, ..., N-1\}$ there is an interval C_n^{N-n}. When $c \in C_n^{N-n}$ this means Player $N-n+1$ participates if and only if every player before her did so, and Player 1 is indeed willing to start this cascade of size $N-n$. So when the cost is raised to this interval, the equilibrium group size can jump from $n-1$ (which can be as low as 1) to N (which means all players are potential participants). This demonstrates the extent to which the probability of success in large groups can be influenced by the common understanding that when the goal is reached, the players will know it.

Finally, turning to the parameter restrictions, it can be seen from Proposition 2 that $p_2 \in (q, 1-q)$ and $p_1 + p_2 > 1$ are not necessary conditions for cascades to occur. In particular, $p_1 + p_2 > 1$ implies that probability $\frac{p(n+k)}{1-F(k)}$ is increasing with k for all $n \in \{2, ..., N-1\}$. This means that for any player in $\{2, ..., N-1\}$, the return to participating is increasing with respect to the number of past participants without success. In general, observing one more past participant before her move has two opposing effects on a player's incentive to participate: First, a larger group size means higher chance of success without her participation and provides additional incentive to free ride.[8] Second, observing one more past participant without success allows her to rule out one more threshold value, increasing the probability she assigns to higher threshold values. This makes it more likely that the group will fail without her, making her participation more likely to be necessary. In the example here, if p_1, p_2 are sufficiently high, the second (positive) information effect outweighs the first (negative) free riding effect. As a result, there are intervals of cost where the participation of a player is contingent on sufficiently many past participants. Condition $p_1 + p_2 > 1$ guarantees that this is the case for all players in $\{2, ..., N-1\}$. [9]

5 Concluding Remarks

Successful grassroots movements are often characterized by sudden surges in mobilization, triggered by developments that make life more difficult for participants. I set out to examine this seemingly counter-intuitive phenomenon by focusing on the incentives of "ordinary people" who support the cause during the formation of the movement. To this end, I account for two natural aspects of movement formation. First, achieving the common goal benefits not only those who played an active role, but everyone who values it. Second, when individuals observe the movement's current degree of success, they use this observation to make inferences about the difficulty of its task.

I find that an arbitrarily small change in the circumstances surrounding the movement

[8] Note that in our geometric example, the marginal return to a new participant is strictly decreasing with group size. In other words, there is decreasing returns to scale. This is the case whenever the threshold threshold probability $p(n)$ is a decreasing function. This means if distribution F has infinite support, decreasing returns to scale will always hold for a sufficiently large number of players.

[9] The other condition $p_2 \in (q, 1-q)$ guarantees that at most one player is contingent for a given cost, and is assumed for tractability.

can lead to a drastic rise in its membership. This rise can be to such an extent that a small increase in participation cost allows a virtually non-existent movement to mobilize all of its supporters. Furthermore, I describe a mechanism where by participating herself, an individual with no special characteristics or private information can induce others to do so. This mechanism suggests an informational foundation for conditional participation (described as "I will go if you go" by Chwe (2000)), a pattern commonly observed in grassroots movements and often imposed a priori through strategic complementarity in theoretical literature.

While the focus of this paper is grassroots movements aimed at political change, the mechanism I describe can be extended to other games of regime change where the benefits of leaving the status quo are not entirely restricted to the ones who contribute to it (i.e., where regime change has a public good component).[10] Examples of such cases include crowdfunding of an entrepreneurial venture whose success is enjoyed by a large group and investment in R&D for curing a widespread disease. A potentially useful direction of further analysis is to study the incentive effects of the information externality I describe in situations where regime change has a "public bad" component that needs to be prevented at a personal cost. Among possible instances are the consideration of adverse macroeconomic consequences of a large bank's insolvency during a bank run and firms investing in measures against climate change.

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[10] Games of regime change are described as "[...] coordination games in which a status quo is abandoned, causing a discrete change in payoffs, once a sufficiently large number of agents take an action against it" (Angeletos et al. (2007)).

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Appendix A: General Case

In Appendix A.1., I introduce additional notation and show that for generic cost c (or alternatively under the assumption to participate when indifferent), there exists a unique equilibrium. Then, I derive the condition such that a player's equilibrium path action is to participate. The additional notation and the participation condition will be used in all proofs. In Appendix A.2., I prove the results from Section 3.

A.1. Equilibrium Uniqueness and Optimality of Participation

Denote by $x_i \in \{0, ..., i-1\}$ the number of participants prior to period $i \in \{1, .., N\}$. That is, $x_i := \sum_{j=1}^{i-1} a_j$. Recall that when the threshold is reached, it is immediately revealed to all players. Since there is no gain from participating after the threshold is reached, all strategies that do so are strictly dominated. Henceforth, unless otherwise noted, the term "history x_i" will refer to histories of play where x_i previous players have chosen to participate and the threshold has not yet been revealed.

Consider the decision of Player N (final mover) following any history where $x_N \in \{0, ..., N-1\}$ previous players participated and the threshold has not been revealed (i.e., no success). No success means that the threshold is greater than x_N, so Player N conditions her belief on $t > x_N$. If Player N plays $a_N = 1$ herself, then the game ends with $x_N + 1$ participants. This yields success probability $\frac{p(x_N+1)}{1-F(x_N)}$ with participation cost c. If Player N plays $a_N = 0$, she pays no cost and success is impossible. Therefore, given history x_N, participating is optimal for Player N if and only if $\frac{p(x_N+1)}{1-F(x_N)} \geq c$. Since x_N and therefore the left hand side can only take finitely many values, this condition is either violated or holds with strict inequality for generic cost. In either case, Player N has a unique best response. In other words, the following is true for all $x_N \in \{0, ..., N-1\}$: Any subgame that starts at period N with x_N previous participants and no success yields the same unique equilibrium.

Next, take some period $j \in \{1, ..., N-1\}$ and suppose by induction that the subgame starting in period $j+1$ has a unique equilibrium for each history x_{j+1}. Define function $s_i : \{0, ..., i-1\} \mapsto \{0, ..., N-i+1\}$ where $s_i(x_i)$ denotes the number of potential participants on the equilibrium path of the subgame that starts at period i under history x_i. [11] We can describe the decision of Player j under history x_j as follows. Since there has been no success with x_j participants, Player j conditions her belief on $t > x_j$. If Player j plays $a_j = 1$, she pays cost c. If the threshold is not reached with her participation, the continuation game inherits history $x_{j+1} = x_j + 1$ and yields $s_{j+1}(x_j + 1)$ potential participants. This results in a total success probability of $\frac{F(x_j+s_{j+1}(x_j+1)+1)-F(x_j)}{1-F(x_j)}$. If Player j plays $a_j = 0$, she pays no cost and the continuation game inherits history

[11]Recall that potential participant is defined as a player who participates if the threshold has not been reached by their turn. For instance, given x_N previous participants and no success, we have $s_N(x_N) = 1$ if participating is optimal for Player N and $s_N(x_N) = 0$ otherwise.

$x_{j+1} = x_j$, yielding $s_{j+1}(x_j)$ potential participants. This results in total success probability of $\frac{F(x_j+s_{j+1}(x_j))-F(x_j)}{1-F(x_j)}$. Therefore, participating is optimal for Player j if and only if

$$\frac{F(x_j + s_{j+1}(x_j + 1) + 1) - F(x_j + s_{j+1}(x_j))}{1 - F(x_j)} \geq c$$

where the left hand side corresponds to the difference in success probability caused by her participation, given her posterior belief. Once again, this left hand side can only take finitely many values, so Player j has a unique best response for generic c. Since the induction hypothesis has been shown above to hold for $j = N - 1$, we can set $j = 1$ and conclude that the supergame yields a unique equilibrium for generic cost. The equilibrium number of potential participants in the supergame is $s_1(0)$.

Finally, define function $a_i^* : \{i, ..., N\} \times \{0, ..., i-1\} \mapsto \{1, 0\}$ where $a_i^*(j, x_i)$ denotes the equilibrium path action of Player $j \in \{i, ..., N\}$ in the subgame that starts from period i under history x_i, conditional on the threshold not having been reached by period j. This means $s_i(x_i) = \sum_{j=i}^{N} a_i^*(j, x_i)$ for the number of potential participants in each subgame. Using this notation, we restate the optimality condition for Player i to participate under history $x_i < t$ (given equilibrium continuation) as

$$a_i^*(i, x_i) = 1 \iff \frac{F(x_i + s_{i+1}(x_i + 1) + 1) - F(x_i + s_{i+1}(x_i))}{1 - F(x_i)} \geq c \qquad (IC_{i,x_i})$$

where (IC_{i,x_i}) stands for the "incentive constraint" of Player i under history x_i. This is the condition that we will consider throughout the equilibrium analysis. The equilibrium path action of Player i in the supergame is given by $a_1^*(i, 0)$.

A.2. Proofs for Section 3

The following Lemma will be used in all proofs.

Lemma 1. *For any* $i \in \{1, ..., N-1\}$ *and* $x_i \in \{0, ..., i-1\}$, $\frac{p(x_i+N-i+1)}{1-F(x_i)} \geq c$ *implies* $s_i(x_i) = N - i + 1$.

Proof. Consider Player N under history $x_N = x_i + N - i$. Then IC_{N,x_N} yields

$$\frac{p(x_i + N - i + 1)}{1 - F(x_i + N - i)} \geq \frac{p(x_i + N - i + 1)}{1 - F(x_i)} \geq c$$

where the first inequality holds by $N \geq i$ and the second inequality is the condition of the Lemma. This means $a_N^*(N, x_i + N - i) = 1$ and $s_N(x_i + N - i) = 1$.

Now suppose by induction for some $j \in \{i, ..., N-1\}$ that $s_{j+1}(x_i + j - i + 1) = N - j$. Then under history $x_j = x_i + j - i$, IC_{j,x_j} yields

$$\begin{aligned}
&\frac{F\big(x_i + j - i + s_{j+1}(x_i + j - i + 1) + 1\big) - F\big(x_i + j - i + s_{j+1}(x_i + j - i)\big)}{1 - F(x_i + j - i)} \\
&= \frac{F\big(x_i + N - i + 1\big) - F\big(x_i + j - i + s_{j+1}(x_i + j - i)\big)}{1 - F(x_i + j - i)} \\
&\geq \frac{F\big(x_i + N - i + 1\big) - F\big(x_i + N - i\big)}{1 - F(x_i + j - i)} \\
&= \frac{p(x_i + N - i + 1)}{1 - F(x_i + j - i)} \\
&\geq \frac{p(x_i + N - i + 1)}{1 - F(x_i)} \geq c
\end{aligned}$$

where the first equality is by the induction hypothesis, the first inequality holds since $s_{j+1}(x_i + j - i) \leq N - j$ by definition of s (i.e., there are only $N - j$ remaining movers after Player j), the second inequality holds from $j \geq i$ and the third inequality is the condition of the Lemma. This means $a_j^*(j, x_i + j - i) = 1$ which together with the induction hypothesis yields $s_j(x_i + j - i) = N - j + 1$. The induction hypothesis is shown above to hold for $j = N - 1$. Plugging in i for j in $s_j(x_i + j - i) = N - j + 1$ yields $s_i(x_i) = N - i + 1$, completing the statement of the Lemma. □

Proposition 1. *This game yields a unique equilibrium. The following hold on the equilibrium path.*

(a) *If there are any potential participants, then there exists an initial participant* $n^* \in \{1, ..., N\}$ *such that Player* i *is a potential participant if and only if* $i \in \{n^*, ..., N\}$.

(b) *If* $c \leq p(N - i + 1)$*, then Player* i *is a potential participant.*

Proof. Existence of a unique equilibrium is shown in Appendix A.1.

Observation (a): I prove this by showing that for all $i \in \{1, ..., N-1\}$ and $j \in \{i, .., N\}$, $a_i^*(i, x_i) = 1$ implies $a_i^*(j, x_i) = 1$. That is, if the optimal action of Player i is to participate after history x_i, then in the continuation game the optimal action of all later movers must be to participate as well.

If Player i participates under history x_i, then we have $x_{i+1} = x_i + 1$. Thus the observation is equivalent to $a_i^*(i, x_i) = 1 \Rightarrow a_{i+1}^*(i+1, x_i+1) = 1$. I show the counterpositive of this. First note that $a_{i+1}^*(i+1, x_i+1) = 0$ implies $s_{i+1}(x_i+1) = s_{i+2}(x_i+1)$. Then we have two cases:

Case (i) $a_{i+1}^*(i+1, x_i) = 1$: This implies $s_{i+1}(x_i) = s_{i+2}(x_i+1) + 1$. Then we have

$$\frac{F(x_i + s_{i+1}(x_i+1)+1) - F(x_i + s_{i+1}(x_i))}{1 - F(x_i)}$$
$$= \frac{F(x_i + s_{i+2}(x_i+1)+1) - F(x_i + s_{i+2}(x_i+1)+1)}{1 - F(x_i)}$$
$$= 0 < c$$

which by (IC_{i,x_i}) means $a_i^*(i, x_i) = 0$.

Case (ii) $a_{i+1}^*(i+1, x_i) = 0$: this implies $s_{i+1}(x_i) = x_{i+2}(x_i)$. Then we have

$$\frac{F(x_i + s_{i+1}(x_i+1)+1) - F(x_i + s_{i+1}(x_i))}{1 - F(x_i)}$$
$$= \frac{F(x_i + s_{i+2}(x_i+1)+1) - F(x_i + s_{i+2}(x_i))}{1 - F(x_i)} < c$$

which once again by (IC_{i,x_i}) means $a_i^*(i, x_i) = 0$. The inequality holds because if it is violated, by (IC_{i+1,x_i}) we must have $a_{i+1}^*(i+1, x_i) = 1$, which contradicts Case (ii).

From the two cases we conclude that $a_{i+1}^*(i+1, x_i+1) = 0 \Rightarrow a_i^*(i, x_i) = 0$, and thus $a_i^*(i, x_i) = 1 \Rightarrow a_i^*(i+1, x_i) = a_{i+1}^*(i+1, x_i+1) = 1$. Iterating this statement forward yields $a_i^*(i, x_i) = 1 \Rightarrow a_i^*(j, x_i) = 1$ for all (i, j) such that $i \in \{1, .., N-1\}$ and $j \in \{i, ..., N\}$.

Observation (b): Setting $x_i = 0$ in the statement of Lemma 1, we obtain that $p(N - i + 1) \geq c$ implies $s_i(0) = N - i + 1$. Since there are only $N - i$ movers after Player i, this means $a_i^*(i, 0) = 1$. If in equilibrium we have $\sum_{j=1}^{i-1} a_1^*(j, 0) = 0$ then Player i is indeed reached with history $x_i = 0$ and $a_i^*(i,0) = a_1^*(i,0) = 1$. If $\sum_{j=1}^{i-1} a_1^*(j,0) > 0$, then it must be the case that $a_1^*(j,0) = 1$ for some $j \in \{1, ..., i-1\}$ which by Observation (a) means $a_1^*(i,0) = 1$ □

Theorem 1. *If* $p(N-2) > p(N-1) > p(N)$ *and* $\frac{p(N)}{p(N-1)} > 1 - p(1)$*, then* s^* *is non-monotonic with respect to cost* c*.*

Proof. First suppose $c \leq p(N)$. By Lemma 1, this implies $s^* = s_1(0) = N$.

Second, let $c = p(N) + \epsilon$ for ϵ small. Since Theorem 1 requires $p(N) < p(N-1)$, this implies $p(N) < c < p(N-1)$. By Lemma 1, we then have $s_2(0) = N - 1$ and $(IC_{1,0})$ reads:

$$\begin{aligned} F(s_2(1)+1) - F(s_2(0)) &= F(s_2(1)+1) - F(N-1) \\ &\leq p(N) < p(N) + \epsilon = c \end{aligned}$$

where the first inequality holds because $s_2(x_2) \leq N - 1$ for all x_2 by definition. Thus, the optimal action of Player 1 is $a_1^*(1,0) = 0$ and we have $s^* = s_1(0) = s_2(0) = N - 1$ when $c = p(N) + \epsilon$ with ϵ small.

Finally, let $c = p(N-1) + \epsilon$. From the condition $p(N)/p(N-1) > 1 - p(1)$ of Theorem 1, we have

$$\frac{p(N)}{1-p(1)} > p(N-1) + \epsilon = c$$

for ϵ sufficiently small. This means by Lemma 1 that $s_2(1) = N - 1$.

Furthermore, since Theorem 1 requires $p(N-2) > p(N-1)$, we have $c = p(N-1) + \epsilon < p(N-2)$. Then, again by Lemma 1, we have $s_3(0) = N - 2$. This yields the following for $(IC_{2,0})$.

$$F(s_3(1)+1) - F(s_3(0)) = F(s_3(1)+1) - F(N-2) \leq p(N-1) < c = p(N-1) + \epsilon$$

The first inequality holds because $s_3(1) \leq N - 2$ by definition of s. Hence, $a_2^*(2,0) = 0$ and $s_2(0) = s_3(0) = N - 2$.

Given the above continuation game, Player 1 faces the following $(IC_{1,0})$.

$$F(s_2(1)+1) - F(s_2(0)) = F(N) - F(N-2) = p(N) + p(N-1)$$
$$> p(N-1) + \epsilon = c$$

This means $a_1^*(1,0) = 1$ and thus $s^* = s_1^*(0) = N$.

To summarize, we have shown that for sufficiently small ϵ, the equilibrium group size is given by

$$s^* = \begin{cases} N; & c \leq p(N) \\ N-1; & c = p(N) + \epsilon \\ N; & c = p(N-1) + \epsilon \end{cases}$$

Hence, s^* is non-monotonic with respect to c.

□

Appendix B: Examples

In Appendix B.1., I prove the results for Section 4.1. In Appendix B.2., I prove the results for Section 4.2.

B.1. Two Possible Thresholds

Theorem 2. *The equilibrium group size in the model with two possible thresholds is as follows.*

$$s^* = \begin{cases} \bar{t}; & c \in (0, p(\bar{t})] \\ \underline{t}; & c \in (p(\bar{t}), p(\underline{t})] \\ \bar{t}; & c \in (p(\underline{t}), 1) \end{cases}$$

Proof. First note that for all $n \in \{1, ..., N\}$ and $k \in \{1, ..., N-n\}$ we have

$$\frac{p(k+n)}{1-F(k)} = \begin{cases} q; & n+k = \underline{t} \\ 1-q; & n+k = \bar{t} \ \wedge \ k < \underline{t} \\ 1; & n+k = \bar{t} \ \wedge \ k \geq \underline{t} \\ 0; & otherwise \end{cases} \tag{1}$$

These probabilities will be used throughout the proof.

Observation (i): Equilibrium action $a_1^*(i,0) = 0$ for all $\{1,...,N-\bar{t}\}$. That is, all players $\{1,...,N-\bar{t}\}$ pass in equilibrium. Thus, $s^* = s_{N-\bar{t}+1}(0)$.

To see this, suppose $a_1^*(i,0) = 1$ for some $i \leq N-\bar{t}$. By Proposition 1(a), this implies $a_1^*(j,0) = 1$ for all $j > i$ and thus, on the equilibrium path we have $x_N > \bar{t}$ (i.e., the move of Player N is reached with more than $\bar{t}$ past participants). This yields the following decision (IC_{N,x_N}) for Player N.

$$\frac{p(x_N+1)}{1-F(x_N)} = 0 < c$$

where the equality holds by Equation (1). So Player N has incentive to deviate to $a_N = 0$.

Regarding the equilibrium action of the remaining players $\{N-\bar{t}+1,...,N\}$, there are 3 possible cases.

Case 1 (**$c \leq 1-q$**)**:**. Since $p(\bar{t}) = 1-q \geq c$, Lemma 1 directly implies $s^* = s_{N-\bar{t}+1}(0) = \bar{t}$.

Case 2 (**$c \in (1-q, q]$**)**:** First note that since $p(\underline{t}) = q \geq c$, Lemma 1 implies $s_{N-\underline{t}+1}(0) = \underline{t}$. Consider the decision of Player $N-\underline{t}$ under history $x_{N-\underline{t}} = 0$. Her decision $(IC_{N-\underline{t},0})$ is given by

$$\begin{aligned}F(s_{N-\underline{t}+1}(1)+1) - F(s_{N-\underline{t}+1}(0)) &= F(s_{N-\underline{t}+1}(1)+1) - F(\underline{t})\\ &\leq p(\underline{t}+1) \leq 1-q < c\end{aligned}$$

where the first inequality holds since $s_{N-\underline{t}+1}(1) \leq \underline{t}$ by definition of s and the second inequality holds since $p(\underline{t}+1) = 1-q$ if $\bar{t} = \underline{t}+1$ and $p(\underline{t}+1) = 0$ otherwise. Then $(IC_{N-\underline{t},0})$ yields $a_{N-\underline{t}}^*(N-\underline{t},0) = 0$ and $s_{N-\underline{t}}(0) = s_{N-\underline{t}+1}(0) = \underline{t}$.

By induction, assume for some $i \in \{N-\bar{t}+1,...,N-\underline{t}\}$ that $s_{i+1}(0) = \underline{t}$. Under history $x_i = 0$, she faces the following $(IC_{i,0})$.

$$F(s_{i+1}(1)+1) - F(s_{i+1}(0)) \leq F(N-i+1) - F(\underline{t}) \leq 1-q < c$$

where the first inequality holds by the induction hypothesis and the definition of s (i.e., there are only $N-i$ movers after Player i). Thus we have $a_i^*(i,0) = 0$ and $s_i(0) = s_{i+1}(0) = \underline{t}$.

The induction hypothesis is shown to hold for $i = N - \underline{t} - 1$ above. Replacing i by $N - \bar{t} + 1$ yields $s^* = s_{N-\bar{t}+1}(0) = \underline{t}$

Case 3 ($c > q$): Observation (i) implies $x_i \in \{0, ..., i - (N - \bar{t} + 1)\}$ for all $i \in \{N - \bar{t} + 1, ..., N\}$ on the equilibrium path: That is, since all players$\{1, ..., N - \bar{t}\}$ pass, there can be at most $\{0, ..., i - (N - \bar{t} + 1)\}$ participants before Player i. First consider some player $i \in \{N - \bar{t} + 1, ..., N\}$ under some history $x_i < i - (N - \bar{t} + 1)$. Her decision is given by (IC_{i,x_i}), which is as follows.

$$\begin{aligned}
&\frac{F(x_i + s_{i+1}(x_i + 1) + 1) - F(x_i + s_{i+1}(x_i))}{1 - F(x_i)} \\
&\quad \leq \frac{F(x_i + N - i + 1) - F(x_i)}{1 - F(x_i)} \\
&\quad = \frac{\sum_{k=1}^{N-i+1} p(x_i + k)}{1 - F(x_i)} \\
&= \begin{cases} q; & \underline{t} - (N - i + 1) < x_i < \underline{t} \\ 0; & otherwise \end{cases}
\end{aligned}$$

where the inequality holds since $s_{i+1}(x_i + 1) \leq N - i$ by definition of s and the second equality holds by the $x_i < i - (N - \bar{t} + 1)$ assumption (i.e., there are not enough players remaining to reach $\bar{t}$ after such a history x_i so the added success probability by the participation of Player i is no higher than q). Since $c > q$, this inequality tells us that $a_i^*(i, x_i) = 0$ for all $x_i < i - (N - \bar{t} + 1)$.

As $a_i = 0$ means $x_{i+1} = x_i < i - (N - \bar{t} + 1) < i + 1 - (N - \bar{t} + 1)$, the same holds for the decision of Player $i + 1$ in the continuation game, so she passes as well. Thus, $a_i^*(i, x_i) = 0$ implies $a_i^*(i + 1, x_i) = 0$. Iterating this argument forward, we obtain $a_i^*(j, x_i) = 0$ for all $j \in \{i, ..., N\}$. Hence, for any $x_i < i - (N - \bar{t} + 1)$ we have $s_i(x_i) = 0$. That is, all remaining players pass if the move of Player i is reached with a history of less than $i - (N - \bar{t} + 1)$ participants.

Next, note that since $\frac{p(\bar{t} - \underline{t})}{1 - F(\underline{t})} = 1 > c$, Lemma 1 implies $s_{N-(\bar{t}-\underline{t})+1}(\underline{t}) = \bar{t} - \underline{t}$. That is, if the move of Player $N - (\bar{t} - \underline{t}) + 1$ is reached with history $\underline{t}$, all of the remaining

$\bar{t} - \underline{t}$ players are potential participants. Summarizing these two observations, we have the following history dependent continuation after the move of Player $N - (\bar{t} - \underline{t})$.

$$s_{N-(\bar{t}-\underline{t})+1}\left(x_{N-(\bar{t}-\underline{t})+1}\right) = \begin{cases} \bar{t} - \underline{t}; & x_{N-(\bar{t}-\underline{t})+1} = \underline{t} \\ 0; & otherwise \end{cases} \tag{2}$$

Now assume by induction for some $i \in \{N - \bar{t} + 1, ..., N - (\bar{t} - \underline{t})\}$ that we have $s_{i+1}(i - N + \bar{t}) = N - i$ (this is shown to hold for $i = N - (\bar{t} - \underline{t})$ in Equation (2)). Then under history $x_i = i - N + \bar{t} - 1$, decision of Player i is given by (IC_{i,x_i}), which is as follows.

$$\begin{aligned} &\frac{F(x_i + s_{i+1}(x_i + 1) + 1) - F(x_i + s_{i+1}(x_i))}{1 - F(x_i)} \\ &= \frac{F(\bar{t}) - F\left(i - N + \bar{t} - 1 + s_{i+1}(i - N + \bar{t} - 1)\right)}{1 - F(i - N + \bar{t} - 1)} \\ &= \frac{F(\bar{t}) - F(i - N + \bar{t} - 1)}{1 - F(i - N + \bar{t} - 1)} \\ &= F(\bar{t}) = 1 > c \end{aligned}$$

where the first equality is the induction hypothesis and the second equality is the application of the above observation that $s_i(x_i) = 0$ for all $x_i < i - (N - \bar{t} + 1)$. The third equality is because our restriction on i implies $i - N + \bar{t} - 1 < \underline{t} - 1$, which means $F(i - N + \bar{t} - 1) = 0$.

Hence, it is optimal for Player i to participate and we have $a_i^*(i, i - N + \bar{t} - 1) = 1$ and thus $s_i(i - N + \bar{t} - 1) = N - i + 1$. This verifies that the induction hypotheses also holds for $i - 1$. Setting $i = N - \bar{t} + 1$ yields $s_{N-\bar{t}+1}(0) = \bar{t}$. By Observation (i), this means $s^* = \bar{t}$, the equilibrium group size. □

B.2. Geometric Threshold

For the results in Section 4.2. (i.e., Proposition 2 and Theorem 3), the following notation will be used. Define function $g_k : \mathbb{N}^2 \mapsto [-1, 1]$ as the difference in public good provision probability between having n and m potential participants under a history of k (past) participants where the threshold has not been reached. That is,

$$g_k(n,m) := \frac{F(k+n) - F(k+m)}{1 - F(k)}$$

With this notation, the optimality condition of Player i to participate under history x_i is given by

$$a_i^*(i, x_i) = T \iff g_{x_i}(s_{i+1}(x_i + 1) + 1, s_{i+1}(x_i)) \geq c \qquad (IC'_{i,x_i})$$

Under the compound geometric distribution $p(n) = q((1-p_1)^{n-1}p_1) + (1-q)((1-p_2)^{n-1}p_2)$, it is useful to underline three properties of function g.

Property 1:

$$g_k(n+m, n) > g_k(n+1+m, n+1), \ \ \forall n, m, k \in \mathbb{N}$$

That is, the the probability gain from having m additional participants is decreasing in the number of potential participants they are added to.

Property 2:

$$g_{k+1}(n, m) > g_k(n+1, m+1), \ \ \forall n, m, k \in \mathbb{N} \ \text{ s.t. } \ n > m$$

That is, the probability gain from a given number of additional participants increases if we increase the number of past participants and decrease the number of potential participants by one.

Property 3: For any $k, n \in \mathbb{N}$, we have

$$g_k(n+1, n) \geq g_{k+1}(n+1, n) \iff (1-p_1)^n p_1 \geq (1-p_2)^n p_2$$

That is, the probability gain from an $n+1^{st}$ potential participant is decreasing with the number of past participants if and only if $(1-p_1)^n p_1 \geq (1-p_2)^n p_2$.

Proposition 2. *Equilibrium group size s^* is monotonic with respect to cost c if and only if $p_1(1-p_1)^{N-2} \geq p_2(1-p_2)^{N-2}$. If s^* is monotonic, then $s^* = max\{n \in \{1, ..., N\} : p(n) \geq c\}$*

Proof. **First Direction of Proposition 2:** If $(1-p_2)^{N-2}p_1 \geq (1-p_2)^{N-2}p_2$, then the equilibrium group size is $s^* = max\{n \in \{1,...,N\} : p(n) \geq c\}$, which is monotonically decreasing w.r.t. c.

It will be shown by induction that under this condition, the equilibirium action of Player i is $a_1^*(i,0) = 1$ if and only if $p(N-i+1) = g_0(N-i+1, N-i) \geq c$. Since $p(n)$ is strictly decreasing over all $n \in \mathbb{N}_+$, this implies that $s^* = \sum_{i=1}^N a_1^*(i,0) = max\{n \in \{1,...,N\} : p(n) \geq c\}$.

Base Case: Consider Player N. Since she is the final mover, (IC'_{N,x_N}) yields

$$a_N^*(N, x_N) = 1 \iff g_{x_N}(1,0) = g_{x_N}(N-i+1, N-i) \geq c$$

for all $x_N \in \{0,...,N-1\}$.

Inductive Step: Consider some player $i \in \{1,...,N-1\}$. Suppose for any player $j \in \{i+1,...,N\}$ and under any history $x_{i+1} \in \{0,...,i\}$, we have $a_{i+1}^*(j, x_{i+1}) = 1$ if and only if $g_{x_{i+1}}(N-j+1, N-j) \geq c$. We want to show that this hypothesis implies for any $x_i \in \{0,...,i-1\}$ that we have $a_i^*(i, x_i) = 1$ if and only if $g_{x_i}(N-i+1, N-i) \geq c$. That is, we need to show the induction hypothesis implies

$$g_{x_i}(s_{i+1}(x_i+1)+1, s_{i+1}(x_i)) \geq c \iff g_{x_i}(N-i+1, N-i) \geq c \tag{3}$$

for all $x_i \in \{0,...,i-1\}$.

Direction $\Rightarrow$ of (3): By the induction hypothesis, we have:

$$s_{i+1}(x_{i+1}) = \left|\{j \in \{i+1,...,N\} : g_{x_{i+1}}(N-j+1, N-j) \geq c\}\right|$$

Note that $(1-p_1)^{N-2}p_1 \geq (1-p_2)^{N-2}p_2$ implies $(1-p_1)^n p_1 \geq (1-p_2)^n p_2$ for all $n \leq N-2$. By property 3, this means that $g_{x_i+1}(N-j+1, N-j) \leq g_{x_i}(N-j+1, N-j)$ for all $j \in \{2,...,N\}$. Then under the induction hypothesis we have $s_{i+1}(x_i+1) \leq s_{i+1}(x_i)$. In particular, we have $s_{i+1}(x_i+1) < s_{i+1}(x_i)$ if $c \in (g_{x_i+1}(N-j+1, N-j), g_{x_i}(N-j+1, N-j)]$ for some $j \in \{i+1,...,N\}$ and we have $s_{i+1}(x_i+1) = s_{i+1}(x_i)$ otherwise.

Now suppose c is such that $s_{i+1}(x_i+1) < s_{i+1}(x_i)$. Since the number of participants takes values in natural numbers this means $s_{i+1}(x_i+1)+1 \leq s_{i+1}(x_i)$ and we have

$$g_{x_i}(s_{i+1}(x_i+1)+1, s_{i+1}(x_i)) \leq g_{x_i}(s_{i+1}(x_i), s_{i+1}(x_i)) = 0 < c$$

which contradicts the left-hand side of (3). Thus, if $s_{i+1}(x_i+1) < s_{i+1}(x_i)$, participating cannot be optimal for Player i. If participating is optimal for Player i, we must have $s_{i+1}(x_i+1) = s_{i+1}(x_i)$. In that case, we can write the optimality condition as

$$g_{x_i}(s_{i+1}(x_i)+1, s_{i+1}(x_i)) \geq c \tag{4}$$

By definition, we know that $s_{i+1}(x_i) \in \{0, ..., N-i\}$. Now suppose $s_{i+1}(x_i) < N-i$. Then the induction hypothesis and property 1 together imply that $a^*_{i+1}(j, x_i) = 1$ for any $j \in \{N - s_{i+1}(x_i)+1, ..., N\}$ and $a^*_{i+1}(j', x_i) = 0$ for any $j' \in \{i+1, ..., N - s_{i+1}(x_i)\}$. Under the induction hypothesis, the latter observation can be written as

$$\begin{aligned} c &> g_{x_i}\big(N - (N - s_{i+1}(x_i)) + 1, N - (N - s_{i+1}(x_i)\big) \\ &= g_{x_i}(s_{i+1}(x_i)+1, s_{i+1}(x_i)) \end{aligned}$$

which contradicts (4). Thus, if participating is optimal for Player i, we cannot have $s_i(x_i+1) < N-i$. The only remaining case for $a^*_i(i, x_i) = 1$ is $s_i(x_i) = s_i(x_i+1) = N-i$. Thus, we can write

$$g_{x_i}(s_{i+1}(x_i+1)+1, s_{i+1}(x_i)) \geq c \Rightarrow g_{x_i}(N-i+1, N-i) \geq c$$

which is the first direction of (3)

Direction $\Leftarrow$ of (3): If $g_{x_i}(N-i+1, N-i) = \frac{p(N-i+1)}{1-F(x_i)} \geq c$, then Lemma 1 implies $s_i(x_i) = N-i+1$. Since there are only $N-i$ movers after Player i, it must be the case that $a^*_i(i, x_i) = 1$, which is equivalent by (IC'_{i,x_i}) to $g_{x_i}(s_{i+1}(x_i+1)+1, s_{i+1}(x_i)) \geq c$. Thus, we shown that (3) holds for Player i under the induction hypothesis.

Note that setting $i = N-1$ in the induction hypothesis yields the base case. Setting $i = 1$ (for whom the only possible history is $x_1 = 0$), shows that for all $i \in \{1, ..., N\}$, we have $a_1^*(i,0) = 1$ if and only if $g_0(N-i+1, N-i) \geq c$. Since $g_0(n, n-1)$ is strictly decreasing in n by property 1, only the final $\max\{n \in \{1,...,N\} : g_0(n,n-1) \geq c\}$ movers are potential participants in equilibrium.

Second Direction of Proposition 2: If $(1-p_2)^{N-2}p_2 < (1-p_1)^{N-2}p_1$, then s^* is non-monotonic w.r.t. c.

This statement is obtained immediately by plugging in $p(n) = q((1-p_1)^{n-1}p_1) + (1-q)((1-p_2)^{n-1}p_2)$ into the non-monotonicity condition of Theorem 1. $\square$

For the proof, Theorem 3 will be stated equivalently to the text using function g.

Theorem 3. *Assume $p_1 + p_2 > 1$ and $p_2 \in (q, 1-q)$. For all $n \in \{2, ..., N-1\}$ there is a $k_n \in \mathbb{N}$ such that an interval*

$$C_n^k := \big(g_{k-1}(n, n-1), \min\{g_k(n, n-1), g_0(n+k, n-1)\}\big]$$

exists for all $k \in \{1\} \cup \{k_n, k_n+1, ...\}$. If $c \in C_n^k$ for some pair (n,k) with $k+n \leq N$, then $s^* = n+k$. Otherwise $s^* = max\{n \in \{1,...,N\} : g_0(n,n-1) \geq c\}$.

Proof. First, I partition the possible values of c into two cases under the parameter restrictions of Theorem 3, using Lemma 2 below. Second, I separately derive the equilibrium group size s^* for these two cases. Finally, I show that interval C_n^k exists for given n when $k = 1$ or k sufficiently large.

Lemma 2. *If $p_2 > \max\{1-p_1, q\}$, then the values of $g_k(n, n-1)$ at $n \geq 2$ are lexigraphically ordered; first strictly decreasing with n, then strictly increasing with k. That is for all $n \geq 2$ and $k, k' \in \mathbb{N}$, we have $g_k(n, n-1) < g_{k+1}(n, n-1)$ and $g_{k'}(n+1, n) < g_k(n, n-1)$.*

Proof. First note that from $p_1 > p_2$, we have that $p_1 + p_2 > 1$ if and only if $p_1(1-p_1) < p_2(1-p_2)$. This implies $p_1(1-p_1)^n < p_2(1-p_2)^n$ for all $n \geq 2$. Therefore, by property 3, we know that $g_k(n, n-1)$ is strictly increasing in k for any $n \geq 2$.

Second, additionally assuming $p_2 > q$ implies $g_k(n, n-1) > g_{k'}(n+1, n)$ for all $n \geq 2$ and $k, k' \in \mathbb{N}$. The reason is as follows. Since $g_k(n, n-1)$ is strictly increasing in k as shown above, we have $g_{k'}(n+1,n) < \lim_{k\to\infty} g_k(n+1,n)$ and $g_k(n,n-1) \geq g_0(n,n-1)$ for all $k,k' \in \mathbb{N}$. Furthermore

$$\lim_{k\to\infty} g_k(n+1,n) < g_0(n,n-1)$$
$$\Leftrightarrow p_2(1-p_2)^n < qp_1(1-p_1)^{n-1} + (1-q)p_2(1-p_2)^{n-1}$$
$$\Leftrightarrow (q-p_2)p_2(1-p_2)^{n-1} < qp_1(1-p_1)^{n-1}$$

A sufficient condition for this inequality to hold is $p_2 > q$. So for all $n \geq 2$ and $k,k' \in \mathbb{N}$, we have $g_k(n,n-1) \geq g_0(n,n-1) > \lim_{k\to\infty} g_k(n+1,n) > g_{k'}(n+1,n)$. We thus obtain the second statement of the lemma. □

Lemma 2 leaves us with two possible cases regarding cost c.[12]

Case 1: There is a unique $\tilde{n} \in \{1,...N-1\}$ such that $c \in \big(g_{N-(\tilde{n}+1)}(\tilde{n}+1,\tilde{n}), g_0(\tilde{n},\tilde{n}-1)\big]$.

Case 2: There is a unique pair $(\tilde{n},k)$ with $\tilde{n} \in \{2,...N-1\}$ and $k \in \{1,...,N-\tilde{n}\}$ such that $c \in (g_{k-1}(\tilde{n},\tilde{n}-1), g_k(\tilde{n},\tilde{n}-1)]$

Next, I separately derive s^* under these two cases.

Equilibrium in Case 1: First note that $c \leq g_0(\tilde{n},\tilde{n}-1)$ implies $c \leq g_k(n,n-1) = \frac{p(k+n)}{1-F(k)}$ for all $n \leq \tilde{n}$ and all k by Lemma 2. By Lemma 1, this means $s_{N-\tilde{n}+1}(x_{N-\tilde{n}+1}) = \tilde{n}$ for all $x_{N-\tilde{n}+1} \in \{0,...,N-\tilde{n}\}$. That is, the final $\tilde{n}$ movers participate in equilibrium regardless of the history.

Next, I show that all the earlier movers $\{1,...,N-\tilde{n}\}$ pass regardless of history. That is, $s_i(x_i) = \tilde{n}$ for all $i \in \{1,...,N-\tilde{n}\}$ and all $x_i \in \{0,...,i-1\}$. As shown above, $s_{N-\tilde{n}+1}(x_{N-\tilde{n}}) = s_{N-\tilde{n}+1}(x_{N-\tilde{n}}+1) = \tilde{n}$. Since Case 1 dictates that $c > g_{x_{N-\tilde{n}}}(\tilde{n}+1,\tilde{n})$ for all $x_{N-\tilde{n}}$, we have by $(IC'_{N-\tilde{n},x_{N-\tilde{n}}})$ that Player $N-\tilde{n}$ passes under any history. That is, $a^*_{N-\tilde{n}}(N-\tilde{n},x_{N-\tilde{n}}) = 0$ for all $x_{N-\tilde{n}}$. Since this is true for any history, $a^*_1(N-\tilde{n},0) = 0$ must hold on the equilibrium path. By Proposition 1(a), this means all previous players

[12] There are two additional cases that are trivial and not discussed here: It is straightforward to see that $c > g_0(1,0)$ and $c \leq g_0(N,N-1)$ yield equilibrium group size $s^* = 0$ and $s^* = N$ respectively.

$i \in \{1, ..., N - \tilde{n} - 1\}$ pass as well, which yields

$$s^* = s_1(0) = s_{N-\tilde{n}+1}(0) = \tilde{n} = max\{n \in \{1, ..., N\} : g_0(n, n-1) \geq c\}$$

where the last equality holds by Case 1.

<u>**Equilibrium in Case 2:**</u> For players $i \in \{N-\tilde{n}+2, ..., N\}$ we have $c \leq g_{x_i}(i, i-1) = \frac{p(x_i+i)}{1-F(x_i)}$ for all x_i by Lemma 2. Then once again by Lemma 1, we have $s_{N-\tilde{n}+2}(x_{N-\tilde{n}+2}) = \tilde{n} - 1$ for all $x_{N-\tilde{n}+2}$.

For Player $N-\tilde{n}+1$, the case $c \in (g_{k-1}(\tilde{n}, \tilde{n}-1), g_k(\tilde{n}, \tilde{n}-1)]$ implies the following: In the continuation game, she knows $s_{N-\tilde{n}+2}(x_{N-\tilde{n}+1}) = s_{N-\tilde{n}+2}(x_{N-\tilde{n}+1}+1) = \tilde{n}-1$. By Lemma 2 and Case 2, we have $g_{x_{N-\tilde{n}+1}}(\tilde{n}, \tilde{n}-1)$ is greater than c if $x_{N-\tilde{n}+1} \geq k$ and smaller than c otherwise. Then by $(IC'_{N-\tilde{n}+1, x_{N-\tilde{n}+1}})$ we have $a^*_{N-\tilde{n}+1}(N-\tilde{n}+1, x_{N-\tilde{n}+1}) = 1$ if and only if $x_{N-\tilde{n}+1} \geq k$. That is, participating is optimal of Player $N - \tilde{n} + 1$ if and only if she follows a history of at least k past participants. This means

$$s_{N-\tilde{n}+1}(x_{N-\tilde{n}+1}) = \begin{cases} \tilde{n}; & x_{N-\tilde{n}+1} \geq k \\ \tilde{n}-1; & otherwise \end{cases}$$

I derive the equilibrium actions of earlier players $\{1, ..., N - \tilde{n}\}$ by induction.

Base Case: Consider player $N - \tilde{n}$ under different histories:

If $x_{N-\tilde{n}} < k - 1$, then $s_{N-\tilde{n}+1}(x_{N-\tilde{n}} + 1) = s_{N-\tilde{n}+1}(x_{N-\tilde{n}}) = \tilde{n} - 1$. Case 2 yields $c > g_{x_{N-\tilde{n}}}(\tilde{n}, \tilde{n}-1)$ for any $x_{\tilde{n}+1} < k$. Therefore by $(IC'_{N-\tilde{n}, x_{N-\tilde{n}}})$, Player $N - \tilde{n}$ passes. That is, $a^*_{N-\tilde{n}}(N-\tilde{n}, x_{N-\tilde{n}}) = 0$ for all $x_{N-\tilde{n}} < k - 1$.

If $x_{N-\tilde{n}} > k - 1$, then $s_{N-\tilde{n}+1}(x_{N-\tilde{n}} + 1) = s_{N-\tilde{n}+1}(x_{N-\tilde{n}}) = \tilde{n}$. Case 2 yields $c > g_{x_{N-\tilde{n}}}(\tilde{n}+1, \tilde{n})$ for all $x_{N-\tilde{n}}$. Therefore by $(IC'_{N-\tilde{n}, x_{N-\tilde{n}}})$, Player $N - \tilde{n}$ passes. That is, $a^*_{N-\tilde{n}}(N-\tilde{n}, x_{N-\tilde{n}}) = 0$ for all $x_{N-\tilde{n}} > k - 1$.

If $x_{N-\tilde{n}} = k-1$, then $s_{N-\tilde{n}+1}(x_{N-\tilde{n}}) = \tilde{n}-1$ and $s_{N-\tilde{n}+1}(x_{N-\tilde{n}}+1) = \tilde{n}$. Therefore, Player $N-\tilde{n}$ participates if and only if $c \leq g_{k-1}(\tilde{n}+1, \tilde{n}-1)$. Since $g_{k-1}(\tilde{n}+1, \tilde{n}-1) > g_{k-1}(\tilde{n}, \tilde{n}-1)$, there is always a range of c within Case 2 where this holds.

We can summarize the optimal action of Player $N - \tilde{n}$ as follows.

$$a^*_{N-\tilde{n}}(N-\tilde{n}, x_{N-\tilde{n}}) = 1 \Leftrightarrow x_{N-\tilde{n}} = k-1 \ \wedge \ c \leq g_{k-1}(\tilde{n}+1, \tilde{n}-1) \tag{5}$$

Inductive Step: Fix a player $i \in \{N-(\tilde{n}+k)+1,...,N-\tilde{n}-1\}$. Suppose for all $j \in \{i+1,...,N-\tilde{n}\}$ we have

$$a_j^*(j,x_j)=1 \Leftrightarrow x_j = k+\tilde{n}-(N-j+1) \ \ \wedge \ \ c \leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^{N-j+1-\tilde{n}}$$

which is shown to hold for $i = N-\tilde{n}-1$ in Equation (5). Our goal is to show that this induction hypothesis implies

$$a_i^*(i,x_i)=1 \Leftrightarrow x_i = k+\tilde{n}-(N-i+1) \ \ \wedge \ \ c \leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^{N-i+1-\tilde{n}} \quad (6)$$

The hypothesis yields that unless $x_i = \tilde{n}+k-(N-i+1)$ and $c \leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^{N-i-\tilde{n}}$ both hold, we have $a_{i+1}^*(i+1,x_i+1)=0$. By Proposition 1(a), this implies $a_i^*(i,x_i)=0$.

If $x_i = \tilde{n}+k-(N-i+1)$ and $c \leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^{N-i-\tilde{n}}$ both hold, we can make the following observations.

(i) If Player i participates, we have $x_{i+1} = x_i+1$. Furthermore, we know by the induction hypothesis that $a_{i+1}^*(i+1,x_i+1)=1$. By Proposition 1(a), this implies all later movers participate as well and we have $s_{i+1}(x_i+1) = N-i$.

(ii) If Player i passes, then we have $x_{i+1} = x_i = \tilde{n}+k-(N-i+1)$. This means $x_j < \tilde{n}+k-(N-j+1)$ for all $j \in \{i+1,...,N-\tilde{n}\}$. Therefore, by the induction hypothesis, these players all pass and we have $s_{i+1}(x_i) = \tilde{n}-1$.

From observations **(i)** and **(ii)**, we can see that when $x_i = \tilde{n}+k-(N-i+1)$ and $c \leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^{N-i-\tilde{n}}$ both hold, the optimality condition (IC_{i,x_i}) of Player i is given by $c \leq g_{\tilde{n}+k-(N-i+1)}(N-i+1,\tilde{n}-1)$. This cost threshold can be obtained by substituting $(N-i+1)-\tilde{n}$ for l in $g_{k-l}(\tilde{n}+l,\tilde{n}-1)$.

Hence, we can summarize the necessary and sufficient conditions for $a_i^*(i,x_i)=1$ as $x_i = \tilde{n}+k-(N-i+1)$ and $c \leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^{N-i+1-\tilde{n}}$. This confirms that the induction hypothesis implies equation (6).

Setting $i = N-(\tilde{n}+k)+1$, we get

$$a_{N-(\tilde{n}+k)+1}^*(N-(\tilde{n}+k)+1, x_{N-(\tilde{n}+k)+1}) = 1$$
$$\Leftrightarrow x_{N-(\tilde{n}+k)+1} = 0 \ \ \wedge \ \ c \leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^{k}$$

Note that Player $N-(\tilde{n}+k)+1$ passes under any positive history. So if any of the previous players participate, Player $N-(\tilde{n}+k)+1$ will pass. By Proposition 1(a), this implies that all previous players $\{1,...,N-(\tilde{n}+k)\}$ pass and we indeed have $x_{N-(\tilde{n}+k)+1}=0$ on the equilibrium path. This, together with the optimality condition we derived for players $\{N-(\tilde{n}+k)+1,...,N-\tilde{n}\}$ in the above induction yields

$$s^*=s_1(0)=\begin{cases}\tilde{n}+k; & c\leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^k\\ \tilde{n}-1; & otherwise\end{cases}$$

where $\tilde{n}-1=\max\{n\in\{1,...,N\}:g_0(n,n-1)\geq c\}$ by Case 2. Finally, since $g_{k-l}(\tilde{n}+l,\tilde{n}-1)$ is quasi-concave in l and $c\leq g_k(\tilde{n},\tilde{n}-1)$ by Case 2, the condition $c\leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^k$ is equivalent to just $c\leq g_0(\tilde{n}+k,\tilde{n}-1)$. The restriction of c satisfying Case 2 and $c\leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^k$can be summarized together as $c\in C_{\tilde{n}}^k$, where $C_{\tilde{n}}^k$ is given by

$$C_{\tilde{n}}^k:=\left(g_{k-1}(\tilde{n},\tilde{n}-1),\min\{g_k(\tilde{n},\tilde{n}-1),g_0(\tilde{n}+k,\tilde{n}-1)\}\right]$$

When this restriction is violated, we have shown that $s^*=\max\{n\in\{1,...,N\}:g_0(n,n-1)\geq c\}$.

The final step of the proof is to show that for given $n\in\{2,...,N-1\}$, C_n^k exists when $k=1$ or k is sufficiently large. To show that C_n^k exists is to show that $g_{k-1}(n,n-1)<\min\{g_k(n,n-1),g_0(n+k,n-1)\}$. We know by Lemma 2 that $g_{k-1}(n,n-1)<g_k(n,n-1)$. Thus, given n, the conditions for $g_{k-1}(n,n-1)<g_0(n+k,n-1)$ will be derived.

First note that this clearly holds for $k=1$, since

$$g_0(n,n-1)=p(n)<p(n)+p(n+1)=g_0(n+1,n-1)$$

Thus, C_n^1 exists for all $n\in\{2,...,N-1\}$. For $k>1$, we are looking for k such that the inequality below holds

$$g_{k-1}(n, n-1)) = \frac{p(n+k-1)}{1-F(k-1)}$$
$$= \frac{q(1-p_1)^{n+k-2}p_1}{q(1-p_1)^{k-1} + (1-q)(1-p_2)^{k-1}} + \frac{(1-q)(1-p_2)^{n+k-2}p_2}{q(1-p_1)^{k-1} + (1-q)(1-p_2)^{k-1}}$$
$$< g_0(n+k, n-1)$$
$$= F(n+k) - F(n-1)$$
$$= q[(1-p_1)^{n-1} - (1-p_1)^{n+k}] + (1-q)[(1-p_2)^{n-1} - (1-p_2)^{n+k})]$$
$$= q(1-p_1)^{n-1}[1-(1-p_1)^{k+1}] + (1-q)(1-p_2)^{n-1}[1-(1-p_2)^{k+1}]$$

Since $p_1 + p_2 > 1$ implies $p_1(1-p_1)^{n-1} < p_2(1-p_2)^{n-1}$, we have $g_k(n, n-1) < p_2(1-p_2)^{n-1}$. That means $g_{k-1}(n, n-1) < g_0(n+k, n-1)$ is satisfied for any k that satisfies the following inequality

$$p_2(1-p_2)^{n-1} - g_0(n+k, n-1)$$
$$= p_2(1-p_2)^{n-1} - \left[q(1-p_1)^{n-1}[1-(1-p_1)^{k+1}] + (1-q)(1-p_2)^{n-1}[1-(1-p_2)^{k+1}]\right] < 0$$

Note that the left-hand side of this inequality is continuous and strictly decreasing with respect to k. Furthermore,

$$\lim_{k\to\infty} p_2(1-p_2)^{n-1} - \left[q(1-p_1)^{n-1}[1-(1-p_1)^{k+1}] + (1-q)(1-p_2)^{n-1}[1-(1-p_2)^{k+1}]\right]$$
$$= (1-p_2)^{n-1}\left[p_2 - (1-q)\right] - q(1-p_1)^{n-1} < 0$$

where the inequality holds from the restriction $p_2 < 1-q$ of Theorem 3. Thus, we know that for sufficiently large k, we have $g_{k-1}(n, n-1) < g_0(n+k, n-1)$. Hence, for any given $n \in \{2, ..., N-1\}$, interval C_n^k exists for sufficiently large k.

□