Tacit Collusion and Dynamic Reference Prices*

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Abstract

This paper examines the extent of tacit collusion in an oligopoly market where consumers are affected by past prices. In particular, we study an infinite horizon Bertrand competition between two identical firms where today's demand for the good at a given price is higher if it is a discount relative to past prices and lower if the price has been raised. This history dependence has an ambiguous effect on collusion: On one hand, colluding on a higher price makes long run coordination more fruitful, as it yields higher demand at any given price tomorrow. On the other, higher prices make it more attractive to undercut the other firm today and obtain the entire demand in the short run. First, we find that history dependent demand leads to overpricing (relative to the myopic profit maximizing price), as the firms consider it an investment in future demand which they can take advantage of through discounts. Second, the firms are able to coordinate on monopoly behavior as long as an upper bound is not crossed. Prices that are too high are followed by very large discounts, after which the firms gradually raise it until a steady state is reached. Above this upper bound, a higher price today leads to a larger discount tomorrow and lower lifetime profits for the firms.

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1 Introduction

A well established observation in the empirical marketing literature is the reference-pricing bias: When consumers make a purchasing decision, they are heavily influenced by some reference price which they use to contextualize the prices at which a good is offered to them. Furthermore, this reference price is formed to a large extent by the consumer's past observations. In their work, which surveys and generalizes the findings on the reference-price bias, Kalyanaram and Winer (1995) summarize as follows: "First, there is ample evidence that consumers use reference prices in making brand choices. Second, [...] consumers rely on past prices as part of the reference price formation process". In particular, buyers experience an additional sensation of gain (loss) when purchasing the good at below (above) the prices observed in the past. For instance, a consumer who makes a one-shot decision of how many units of a product to purchase at $50 would buy more if she has observed it being sold for $70 in the past than if she observed it being sold for $30, even when she is certain of the intrinsic value of the good. We refer to this aspect of consumer behavior as "dynamic reference pricing".

This paper looks at the influence of such consumer behavior on the outcomes of oligopolistic price setting competition with repeated interactions. In this environment, dynamic reference pricing by consumers affects the competitiveness of the market, as forward looking firms have to take into account the future demand shifts caused by the current prices. This affects the sustainability and profitability of tacit collusion. For instance, if the firms coordinate on a very high price today, there is the additional benefit that the demand will be higher at any given price tomorrow, as tomorrow's consumer will compare the prices she faces to a high reference. On the other hand, the same dynamic reference pricing behavior can provide the firms additional incentive to undercut each other, which reduces the ability of the firms to collude. In this setting, our main focus is to study how the ability of firms to coordinate on collusive pricing patterns is determined by the current reference price. We check if and when a higher demand due to higher reference prices translates into higher profits for competing firms.

To this end, we present an infinite horizon Bertrand duopoly model with linear demand. In each round, two forward looking firms selling an identical good produced at zero cost set prices simultaneously and the firm that sets the lower price takes the entire demand. If they set the same price, they share the demand equally. The objective of the firms is to maximize their discounted lifetime profit. The addition in this paper is that the demand for the good in each period is not only determined by the price they set in the current round, but also by how it compares to a reference point that is elicited from the past prices. For any given current price, the demand is increasing in the reference point.

The consumer reference price in each period is given by the smaller of the two prices set by the firms in the previous period. First note that since they are selling identical goods, the same reference price applies for both firms. As a result, the price that one firm sets creates an additional externality on the other firm. Not only does the competitor's price determine the share of the market that a firm gets today, but it also determines the demand function that she will face tomorrow due to a change in consumer reference. The idea behind taking the smaller of the two previous prices as the reference is that in each period, the only firm who makes any sales is the one that sets the lower price. As a result, if a consumer is observing realized transactions, the only price she can base her reference price on is the lowest one.

For this model, we characterize the set of lifetime profits that a firm can obtain as a symmetric strategy subgame perfect equilibrium (sSPE) outcome. That is, we pin down the profits that the firms can obtain from colluding on a price path where they always set the same price and have no incentive to deviate following any pricing history. Allowing for public randomization, this set corresponds to all the values between the lowest and the highest sSPE payoffs. We show that as in the standard Bertrand model without reference effects, the lowest sSPE payoff is always zero, obtained when both firms set zero price after any history.

The highest payoff that the firms can obtain by colluding on a sSPE pricing strategy depends on the level of patience that firms have (i.e. how much they discount future profits), and the consumer reference price at the start of the game. We show that for low levels of patience, there is no sSPE that yields positive payoff for the firms, so the highest sSPE profit is zero. For intermediate and high levels of patience, the analysis takes the highest feasible payoff as a benchmark. The highest feasible payoff is sharing the lifetime profit of a monopolist who faces the same demand function and reference evolution rule as in the two firm competition. We call this the "long run monopolist" profit. Whenever the two firms can imitate the optimal pricing policy of the long run monopolist in a sSPE, the highest sSPE payoff is half of the long run monopolist profit. For sufficiently high levels of patience, the firms can collude on long run monopolist pricing with any initial reference price.

For intermediate values of firm patience, we show that collusion on long run monopolist pricing is possible (in a sSPE) if and only if the initial reference price is low enough. If the initial reference is above a certain threshold, the firms have incentive to deviate under the long run monopolist policy. In that case, the best they can coordinate on is setting a price that is low enough today, and then following the long run monopolist policy tomorrow onward.

The reason that high initial reference prices do not allow collusion on long run monopolist behavior can be described by two mechanisms. First, if the initial reference is very high, the long run monopolist policy follows a decreasing price path. That means the future demand will be decreasing over time as the reference price becomes lower. So the competing firms have an incentive to undercut today and take the entire demand before the market becomes small. This effect is similar to the incentive to undercut in the boom periods under exogenous business cycles. Another reason why high initial reference leads to deviation is that the long run monopolist always overprices relative to a myopic monopolist who would like to maximize the profit today. This can be seen as an "investment" in future reference points as it means making less profits in the current period. Then for the competing firms, coordinating on the long run monopolist policy requires forgoing immediate demand that they could extract by undercutting (and setting the myopic monopoly price). The overpricing, and thus the forgone profit is increasing with the initial reference price. After some level, the immediate profit the firms need to sacrifice by colluding becomes too large.

Finally, we look at the comparative statics of the highest sSPE payoff with respect to the initial reference. We show that for interior levels of patience, the highest sSPE payoff obtains a unique maximum at the highest initial reference that allows for coordination on long run monopolist pricing. Below this threshold, the highest sSPE profit is increasing in initial reference. If, on the other hand, the initial reference price is such that the firms are not able to coordinate on the long run monopolist policy, the highest sSPE payoff is strictly decreasing.

To the best of our knowledge, there is no study that looks at the effect of dynamic reference prices on the sustainability and profitability of tacit collusion in an infinite horizon setting. The effects of reference dependence on monopoly pricing (Heidhues & Koszegi (2004, 2014), Spiegler (2012)) and competition (Heidhues & Koszegi (2008, 2014), Karle & Peitz (2014), Hahn et. al. (2018)) in static environments have been explored by several studies. These papers focus on the effect of loss aversion on market outcomes. Piccolo & Pignataro (2018) model reference dependent consumers in an infinite horizon price competition setting as we do, but they impose a time-invariant reference point, and study the effects of loss aversion when the consumer is uncertain about the intrinsic value of the good. Note that loss aversion (i.e., asymmetry between the utility effect of a higher and lower price than the reference) is an empirically well documented aspect of consumer behavior and commonly included in theoretical modelling. Here, however, our focus is the effects of the change in the reference point itself over time. Therefore, we abstract away from loss aversion and consider instead a symmetric effect for tractability. A discussion on the potential implications of loss aversion for our results and possible extensions can be found in Section 6.

Dynamic pricing with history dependent reference prices has been studied predominantly in the operations research literature. Notable papers in the area include Kopalle et. al. (1996), Fibich et.al. (2003), and Popescu & Wu (2007), who study optimal monopoly pricing when reference points are determined by past prices. Our result that the long run monopolist price policy converges to a steady state and overprices relative to a myopic monopolist is in line with the findings of Kopalle et.al. (1996), who postulate this behavior numerically, and Fibich et.al. (2003), who formally derive it for continuous time. Popescu and Wu (2007) study the problem for a more general class of demand functions (as opposed to linear) and derive a wide range of conditions such that this behavior holds. Furthermore, Kopalle et. al. (1996) look at an infinite horizon duopoly problem with product differentiation where the firms have two possible pricing strategies: constant or cyclical pricing. They show that with loss neutral consumers, the equilibrium given these two alternatives is constant pricing. Also note that they use a setting where the reference price is completely firm specific. Thus, their analytical results do not take into account the "prisoner's dilemma" caused by the reference externality firms can impose on each other. Other papers that look at competition with dynamic reference prices include Anderson et. al. (2005), Yang et. al. (2012) and Coulter et. al. (2014). However, they work in a finite horizon framework with some product differentiation. As such, they are unable to make observations regarding tacit collusion.

The rest of the paper is structured as follows. Section 2 introduces the model. Section 3 states the set of sSPE payoffs as a function of firm patience and the initial reference price. Section 4 outlines the argument underlying the highest sSPE payoffs. Section 5 presents comparative statics with respect to initial reference. Section 6 underlines some potentially useful further steps in the analysis and summarizes the findings.

2 Model Setup

Two identical firms ($i = 1, 2$) play an infinite horizon price setting game where they sell an identical good. In each stage $t \in \{1, 2, ..., \infty\}$, they set prices simultaneously. The firm that sets the lower price takes the entire market. If the prices are equal, demand is split equally. Let the period t demand for the good at price p be given by

$$d(p|r_t) := v - p + \lambda(r_t - p)$$

where $v \in \mathbb{R}_{++}$ is a constant and r_t is the consumer's period t reference price for the good, formed upon observing past prices. Then $\lambda \in \mathbb{R}_+$ corresponds to the relative weight of the reference effect. This demand function can be interpreted as the optimal purchase amount of one individual who lives for a single period, has a known intrinsic value of the good, suffers disutility from the price paid and experiences an additional loss (gain) for how much the transaction price exceeds (is lower than) the reference price. The fact that we use the same parameter λ for $p > r_t$ and $p < r_t$ implies loss neutrality.[1]

[1]This linear demand function is used for tractability. In general, one can derive the demand from a consumer utility function that satisfies these properties. For example, if we maximize utility $u(p, q|r) = vlnq - q(p + \lambda(p - r_t))$ where q denotes the quantity, we obtain a demand function that is decreasing and convex in p and increasing in r_t.

We take the initial period reference price r_1 as given. Denote the price set by firm $i \in \{1, 2\}$ in period t by p_t^i. For $t \geq 2$, we set $r_t = min\{p_{t-1}^1, p_{t-1}^2\}$.[2] That is, we take the reference price equal to the smallest price in the previous period. One motivation for taking the minimum as the reference is that in our model, the firm that sets the lower price makes all the sales. Thus, the smallest of the two prices is the only one at which a transaction occurs. A consumer who only observes realized transactions in earlier stages would therefore take only the lowest price into account when forming reference prices. In other settings, it is possible to think of other measures of past prices as the reference price, such as the average of the two prices set in the previous period.

For the initial reference price, assume $r_1 \in [0, v]$. Let $p^c(r_t) := \frac{v+\lambda r_t}{1+\lambda}$ denote the choke price as a function of the reference price (i.e. the price at which the demand equals zero: $d(p^c(r_t)|r_t) = 0$). For both firms $i \in \{1, 2\}$, we restrict the set of possible prices at time t given reference r_t to $p_t^i \in [0, p^c(r_t)]$. That is, given r_t, we only allow for prices that yield non-negative demand. Note that $r_1 \in [0, v]$ and $p_t^i \in [0, p^c(r_t)]$ together imply that $p_t^i, r_t \in [0, v]$ for all $t \in \mathbb{N}_+$ and $i \in \{1, 2\}$.

Assume that both firms produce at zero marginal cost. The profit from selling $d(p|r_t)$ units at price p is thus given by $\pi(p|r_t) := pd(p|r_t)$. Denote the vector of prices set by the two firms at time t by $p^t := (p_t^1, p_t^2)$. The period t profit of firm i under price vector p^t and reference price r_t is as follows.

$$\pi^i(p^t|r_t) = \begin{cases} \pi(p_t^i|r_t); & p_t^i < p_t^j \\ \frac{1}{2}\pi(p_t^i|r_t); & p_t^i = p_t^j \\ 0; & p_t^i > p_t^j \end{cases}, \qquad i \neq j \in \{1, 2\}$$

In the dynamic game, the payoff of firm i under the price vector path $\{p^t\}_{t=1}^{\infty}$ is then given by the discounted lifetime profit

$$\sum_{t=1}^{\infty} \delta^{t-1} \pi^i(p^t|r_t)$$

[2]This process for the reference price can be interpreted as the limit case of the exponential smoothing process $r_t = \alpha r_{t-1} + (1-\alpha) \min\{p_{t-1}^1, p_{t-1}^2\}$ as the memory parameter α converges to zero. Note that exponential smoothing is the most commonly used rule in the dynamic reference literature discussed in the previous section.

where discount factor $\delta \in (0,1)$ and initial reference $r_1 \in [0.v]$ are given and $r_t = \min\{p^1_{t-1}, p^2_{t-1}\}$ for all $t \geq 2$.

Let $s^i : H \to [0, v]$ describe a pure pricing strategy of firm i, where H is the set of all possible pricing histories with a time $t \geq 1$ history $h_t \in H$ of the form $h_t = \{p^\tau\}_{\tau=1}^t$ and h_0 the initial node. Then given strategy s^i, $s^i(h_t) \in [0, p^c(r_t)]$ denotes the price set at time $t+1$ under history h_t and $s^i(h_0) \in [0, p^c(r_1)]$ denotes the initial price at $t = 1$. Denote by S the set of all such strategies. Since the firms are identical, S denotes the set of all pure strategies for both firms.

3 Equilibrium Payoffs

Given the above environment, this section states the set of lifetime profits that can be obtained as the outcome of a subgame perfect equilibrium with symmetric strategies ($s^1(h_t) = s^2(h_t)$, for all $h_t \in H$). The set of such payoff vectors are determined by the initial reference price $r_1 \in [0, v]$ and the discount factor $\delta \in (0, 1)$.

Since we are only looking at symmetric strategy subgame perfect equilibria (henceforth sSPE), any payoff vector will yield the same value for both firms. That is, a payoff vector resulting from symmetric strategies is of the form $(w, w) \in \mathbb{R}^2_+$.

Allowing for public randomization over such equilibria, the set of sSPE payoffs corresponds to all convex combinations of the lowest and the highest sSPE payoff. Let $\underline{w} : [0, v] \times (0, 1) \to \mathbb{R}_+$ denote the lowest sSPE payoff as a function of the initial reference r_1 and discount factor δ. Similarly, define $\bar{w} : [0, v] \times (0, 1) \to \mathbb{R}_+$ as the highest sSPE payoff. The set of all sSPE payoffs as a correspondence is then given by:

$$W(r_1, \delta) = \{w \in \mathbb{R}_+ : w = \alpha \underline{w}(r_1, \delta) + (1 - \alpha)\bar{w}(r_1, \delta), \text{ for some } \alpha \in [0, 1]\}$$

Propositions 1 and 2 characterize the functions $\underline{w}(r_1, \delta)$ and $\bar{w}(r_1, \delta)$ respectively. Complete proofs of all results can be found in the Appendix.

Proposition 1. $\underline{w}(r_1, \delta) = 0$ *for all* $r_1 \in [0, v]$ *and* $\delta \in (0, 1)$.

That is, the lowest sSPE payoff is zero for any level of firm patience δ and any initial reference r_1, obtained by both firms setting zero price following any history. This result follows directly from the observation that $p_t^1 = p_t^2 = 0$ is the unique Nash Equilibrium (NE) of the one shot game under any reference r_t: Denote by $p^m(r_t) := argmax_p \pi(p|r_t) = \frac{v+\lambda r_t}{2(1+\lambda)}$ the optimal price of a myopic short-run (henceforth SR) monopolist that maximizes the profit in the current round under reference r_t. Then in the stage game, if the opponent is setting a price higher than $p^m(r_t)$, the best response of a firm is to set price $p^m(r_t)$. If the opponent is setting a price lower than $p^m(r_t)$, a firm has incentive to undercut the opponent by an arbitrarily small amount. Since zero is the lowest feasible lifetime payoff in the dynamic game and it is obtained by repeating the unique stage NE, it is the lowest sSPE payoff.

In identifying the highest sSPE payoff $\bar{w}(r_1, \delta)$, we take the highest feasible symmetric payoff as a benchmark. For given δ, the highest feasible payoff is a function of the initial reference price r_1 and corresponds to the firms equally splitting the lifetime profit of a monopolist with cost zero who faces the same demand function $d(p|r_t)$ and the reference evolution rule $r_t = p_{t-1}$ in an infinite horizon price setting problem. From now on we call this the long-run (henceforth LR) monopolist problem. [3]

As in the two firm case,the LR monopolist constrained to non-negative prices below the choke price at each stage. Formally, the lifetime profit of the LR monopolist under initial reference r_1 is given by:

$$V(r_1) = \max_{\{p_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \delta^{t-1} \pi(p_t|r_t)$$
$$\text{s.t.} \quad p_t \in [0, p^c(r_t)], \ \ \forall t \in \mathbb{N}_+$$

where $r_t = p_{t-1}$ for all $t \geq 2$. The highest feasible symmetric payoff in the two firm game is then $\frac{V(r_1)}{2}$, obtained when both firms follow the (unique) optimal pricing policy of the LR monopolist. Using this benchmark, the next result states the highest symmetric sSPE payoff $\bar{w}(r_1, \delta)$ for

[3]The long-run monopolist problem presented here is similar to the one analyzed in Kopalle et al. (1996) and Popescu and Wu (2007) for a more general class of demand functions. The findings on the optimal LR monopolist policy are in line with the authors' analysis of the loss-neutral demand case.

different values of $r_1 \in [0, v]$ and $\delta \in (0, 1)$.

Proposition 2. *There exist $\underline{\delta}$, $\bar{\delta}$ with $1 > \bar{\delta} > \underline{\delta} > \frac{1}{2}$ such that:*

(i) *If $\delta \in (0, \frac{1}{2})$, then $\bar{w}(r_1, \delta) = 0$ for all $r_1 \in [0, v]$.*

(ii) *For each $\delta \in [\underline{\delta}, \bar{\delta})$, there exists a unique $\bar{r}(\delta) \in (0, v)$ that yields:*

$$\bar{w}(r_1, \delta) = \begin{cases} \frac{V(r_1)}{2}; & r_1 \in [0, \bar{r}(\delta)] \\ \delta V(p^*(r_1)); & r_1 \in (\bar{r}(\delta), v] \end{cases}$$

Where $p^*(r_1)$ is the unique value in the interval $(0, \min\{p^m(r_1), \bar{r}(\delta)\})$ that solves $\pi(p^*(r_1)|r_1) = \delta V(p^*(r_1))$.

(iii) *If $\delta \in [\bar{\delta}, 1]$ then $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ for all $r_1 \in [0, v]$*

Proposition $2(i)$ states that if the firms have little patience (i.e. $\delta \in (0, \frac{1}{2})$), the firms cannot collude on any positive price sequence regardless of the initial reference r_1. As a result, the highest and the only sSPE outcome of the game is zero lifetime profit.

Proposition $2(ii)$ and (iii) state that there is an interval of firm patience levels $[\underline{\delta}, \bar{\delta})$ within $(\frac{1}{2}, 1)$ that we can use to describe the highest sSPE payoff as a function of the initial reference r_1. Given a δ in this interval, the firms can coordinate on LR monopoly pricing if and only if the initial reference is low enough. That is, there is an interior threshold $\bar{r}(\delta)$ such that if the initial reference r_1 is (weakly) lower than $\bar{r}(\delta)$, the LR monopoly profit $\frac{V(r_1)}{2}$ is indeed a sSPE outcome.

If the initial reference r_1 is above this threshold, then the firms cannot coordinate on LR monopoly pricing, and thus the payoff $\frac{V(r_1)}{2}$ is not sustainable in a sSPE. In that case, the best that the firms can coordinate on is initially setting a price that is sufficiently low (lower than the SR monopoly price $p^m(r_1)$), and following the LR monopolist path tomorrow onward. Starting with the highest initial price $p^*(r_1)$ that yields no incentive to deviate, the lifetime payoff from following such a price sequence is given by $\delta V(p^*(r_1))$. Finally, if the firms are patient enough ($\delta \geq \bar{\delta}$), sharing the LR monopoly profit can be sustained as a sSPE payoff with any initial reference $r_1 \in [0, v]$.

4 Discussion of Highest sSPE Payoffs

This section outlines the arguments underlying Proposition 2. That is, we discuss how the highest sSPE payoff is determined as a function of δ and r_1. First, we specify the class of strategy profiles that are relevant when determining whether a lifetime profit $w \in \mathbb{R}_+$ is a sSPE payoff. Using these strategy profiles, we derive the highest sSPE payoff $\bar{w}(r_1, \delta)$ for $\delta < 1/2$ and $\delta \geq 1/2$ respectively.

4.1 Strategies

As shown in Proposition 1, $p_t^1 = p_t^2 = 0$ is the unique NE of the stage game under any reference r_t and 0 is the lowest feasible lifetime payoff under any initial r_1 and δ. Now suppose we want to sustain a price sequence $\{p_t\}_{t=1}^{\infty}$ on the equilibrium path of a sSPE. Since 0 is the lowest feasible lifetime payoff, setting price 0 forever (grim trigger) upon deviation from $\{p_t\}_{t=1}^{\infty}$ is the harshest feasible punishment. Since both firms setting $p_t^1 = p_t^2 = 0$ is a NE of the stage game under any r_t, both firms setting price zero forever is a sSPE of any subgame following a deviation.

From these two observations we can conclude that given r_1 and δ, a sequence of symmetric price vectors $\{p^t\}_{t=1}^{\infty}$ is a sSPE equilibrium path if and only if the firms have no incentive to deviate at any stage under grim trigger punishment with price zero. A lifetime profit resulting from such an equilibrium path is a sSPE payoff.

Formally, $w \in \mathbb{R}_+$ is a sSPE payoff if and only if there exists a strategy $s^* \in S$ of the form $s^*(h_0) = p_1^*$ and for $t \geq 1$

$$s^*(h_t) = \begin{cases} p_{t+1}^*; & p^\tau = (p_\tau^*, p_\tau^*), \forall p^\tau \in h_t \\ 0; & otherwise \end{cases},$$

that satisfies the following. Given r_1, the price sequence $\{p_t^*\}_{t=1}^{\infty}$ yields

$$w = \sum_{t=1}^{\infty} \delta^{t-1} \frac{\pi(p_t^* | r_t)}{2}$$

where $r_t = p_{t-1}^*$ for all $t \geq 2$. Furthermore, sequence $\{p_t^*\}_{t=1}^{\infty}$ satisfies the incentive constraint

$$\sum_{t=\tau}^{\infty} \delta^{t-\tau} \frac{\pi(p_t^*|r_t)}{2} \geq \begin{cases} \pi(p_\tau^*|r_\tau); & p_\tau^* < p^m(r_\tau) \\ \pi(p^m(r_\tau)|r_\tau); & p_\tau^* \geq p^m(r_\tau) \end{cases} \tag{IC}$$

for all $\tau \in \{1, 2, ..., \infty\}$. The right hand side of the inequality corresponds to the lifetime payoff obtained by the optimal deviation from s^* at stage τ. If at τ the price p_τ^* is greater than the SR monopoly price $p^m(r_\tau)$, the optimal deviation is to $p^m(r_\tau)$. If $p_\tau^* < p^m(r_\tau)$, the optimal deviation is to undercut p_τ^* by an arbitrarily small amount. In both cases the deviator initially obtains the entire demand at the deviation price and makes zero profits forever after.

Thus, $\bar{w}(r_1, \delta)$ is the highest value in $\mathbb{R}_+$ that is the lifetime profit from a symmetric price vector sequence that satisfies the above incentive constraint at every stage. Note that r_1 plays a role in determining $\bar{w}$ in two ways. First, it affects the lifetime payoff obtained from a price sequence. Second, it determines whether the incentive constraint is violated in the initial round.

4.2 Highest Payoff under Low Patience

If $\delta \in (0, \frac{1}{2})$, then the highest sSPE payoff is zero and the firms cannot coordinate on any nonzero price sequence (Proposition $2(i)$). The argument behind this result is as follows. First recall that since we restricted the set of prices under reference r_t to $p_t^i \in [0, p^c(r_t)]$ and initial reference r_1 to $[0, v]$, any possible price sequence (as well as the sequence of reference points implied by the previous price) only takes values in $[0, v]$. Since a price sequence $\{p_t\}_{t=1}^{\infty}$ can only take values in the compact set $[0, v]$, the resulting sequence of stage profits $\{\frac{\pi(p_t|r_t)}{2}\}_{t=1}^{\infty}$ has a finite upper bound (supremum).

Second, note that if $\delta < \frac{1}{2}$, a price sequence that yields a constant stream of stage profits does not satisfy (IC) (i.e does not deter deviation).

Now take any candidate price sequence $\{\tilde{p}_t\}_{t=1}^{\infty}$ and consider the period τ where it yields the highest stage profit across all periods. Clearly, the continuation payoff from pursuing $\tilde{p}_t$ at time τ onward yields a (weakly) lower lifetime payoff than a constant stream of the stage profit obtained at time τ. However, even a constant stream of the stage profit obtained at time τ is not a high enough deter the deviation available. Thus, with the continuation payoff from $\{\tilde{p}_t\}_{t=\tau}^{\infty}$, the time τ

incentive constraint will certainly be violated.

When the sequence of stage profits converges to the upper bound instead of obtaining a maximum, we can say that for any $\delta < \frac{1}{2}$ there is a period that yields a stage profit close enough to the upper bound that (IC) is violated, using the same argument as above.

To summarize, we can make the following observation for firms with low ($\delta < \frac{1}{2}$) patience. While at some periods the firms might have no immediate incentive to deviate from colluding on a given sequence of prices, they anticipate a period where the stage profit will be "as high as it ever will". At that period, $\delta < \frac{1}{2}$ guarantees that they have incentive to undercut. The anticipation of the future deviation leads to the unraveling of the collusion. This "highest stage profit" period exists because firms cannot set prices above the choke price, which confines them below a finite reference point. As a result, the only sSPE is setting price zero after any history.

4.3 Highest Payoff under High Patience

To determine the highest sSPE payoff when firms have patience $\delta \in [\frac{1}{2}, 1)$ (Proposition $2(ii)$ and (iii)), we first derive some important properties of the highest feasible payoff. Recall that the highest feasible payoff in the two firm game with initial reference price r_1 is given by half of the LR monopolist profit $V(r_1)$. Since for any given price path the future profit of the LR monopolist depends only on the current reference price, we can reformulate the problem in recursive form:

$$V(r) = \max_{r' \in [0, p^c(r)]} \{\pi(r'|r) + \delta V(r')\}$$

Here, state variable r corresponds to the initial reference r_1. The price set at a period with reference r is denoted by r' and is equal to the state in the next period. Let the optimal policy of the LR monopolist be given by the function $f : [0, v] \to [0, v]$. That is, the optimal price in any period with reference r is given by $f(r) \in [0, p^c(r)]$. There is a unique optimal policy function in the recursive problem and it is derived in the proof of Lemma 1 below.

Clearly, payoff $\frac{V(r_1)}{2}$ can be obtained by firms with symmetric strategies only if they both set prices according to the unique monopolist policy (starting from the given r_1). Denote by $f_n(r)$ the n$^{\text{th}}$ iterate of function f on state r (e.g. $f_2(r) = f(f(r))$). Then, the only symmetric price sequence that yields payoff $\frac{V(r_1)}{2}$ is $p_t^1 = p_t^2 = f_t(r_1)$ for all $t \in \mathbb{N}_+$. This means $\frac{V(r_1)}{2}$ is a sSPE payoff of the game starting from r_1 if and only if the price sequence $\{f_t(r_1)\}_{t=1}^{\infty}$ is a sSPE equilibrium path. Since it is the highest feasible payoff, we have that $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ whenever $\frac{V(r_1)}{2}$ is a sSPE payoff.

The next result presents some properties of the recursive LR monopolist problem that are useful in the analysis of the sSPE payoffs in the two firm game.

Lemma 1. *For the solution of the recursive LR monopolist problem, the following properties hold.*

(a) The unique optimal policy function $f(r)$ satisfies $f(r) \in (p^m(r), p^c(r)]$ for all $r \in [0, v]$

(b) For any initial $r \in [0, v]$, $f_n(r)$ monotonically converges to unique steady state $r^{ss} := \frac{v}{2+\lambda(1-\delta)}$ as n tends to infinity.

(c) If the choke price constraint $r' \leq p^c(r)$ is binding at $r = r_1 \in [0, v]$, then it is binding at $r = r_2$ for any $r_2 \in [0, r_1]$.

(d) $V(r)$ is continuously differentiable and strictly increasing.

Here it is helpful to discuss the intuition behind some of the above properties. Recall that we defined $p^m(r) = argmax_p \pi(p|r)$ as the price that a short-run (SR) monopolist would set in a one-period problem with reference r. Then, (a) states that under any reference point, the LR monopolist sets a strictly higher price than the SR monopolist who maximizes the stage profit. The intuition behind this result is simple. The stage profit $\pi(r'|r)$ is a strictly concave function that obtains its maximum at $r' = p^m(r)$. On the other hand, the continuation payoff $V(r')$ of the LR monopolist is strictly increasing in the price r' that she currently sets. Clearly, the LR monopolist has no incentive to set a price below the SR monopolist. When comparing prices $r' \in [p^m(r), p^c(r)]$, the LR monopolist faces a trade-off. In this region, setting a higher current price decreases the current profit but increases the future value through a higher reference point tomorrow. As a result, for any positive δ, it is optimal to set a strictly higher price than the SR monopolist price.

In other words, it is always optimal for the LR monopolist to forgo some current profit as an "investment" in future reference prices. This creates a wedge $f(r) - p^m(r)$ between the optimal LR and SR monopolist prices, respectively. This wedge plays an important role in determining the competing firms' ability to coordinate on the LR monopolist policy. The wedge implies that in addition to taking the current demand in its entirety, the optimal deviation (to $p^m(r)$) also leads to an increase in the current demand.

Result (b) states that that under the optimal LR monopolist policy, the reference point converges to the unique state $r^{ss} = \frac{v}{2+\lambda(1-\delta)}$, regardless of the initial reference r. Recall that since the price of today is the reference point of tomorrow, the n^{th} iterate $f_n(r)$ of the policy function corresponds to the optimal price in the n^{th} period and the reference price in the $(n+1)^{\text{th}}$ period. The convergence is monotonic in the sense that if the LR monopolist is facing a reference $r < r^{ss}$, she sets a price r' strictly higher than the current reference price and if she faces a reference with $r > r^{ss}$, she sets a price strictly lower than the current reference price. Thus, the LR monopolist chooses a decreasing price path if the initial reference is large (greater than r^{ss}) and increasing if it is small (smaller than r^{ss}). In the two firm game, this property affects the deviation decision of the firms through the comparison of current versus future profits when coordinating on the LR monopoly policy. Point (c) and the differentiability of $V(r)$ are more technical results that we use when deriving the sSPE payoffs of the two firm game.

Using the above properties of the LR monopoly problem and bearing in mind that the highest feasible symmetric lifetime payoff under initial reference r_1 is $\frac{V(r_1)}{2}$, we now determine $\bar{w}(r_1, \delta)$ under $\delta \in [\frac{1}{2}, 1)$ in two steps. First, we describe the cases when coordination on LR monopoly pricing is possible. Then, we pin down the highest sSPE payoff when it is not. Recall that a payoff is a sSPE outcome given initial reference r_1 if and only if it is the lifetime profit from a symmetric price sequence that that yields no incentive to deviate at any stage under grim punishment.

When is LR monopoly pricing sSPE?

Suppose that given r_1, we would like to support the highest feasible payoff $\frac{V(r_1)}{2}$ as a sSPE outcome. As stated above, the only way that the firms can obtain this payoff is if they coordinate on the optimal policy $\{f_t(r_1)\}_{t=1}^{\infty}$ of the LR monopolist. Lemma 1(a) tells us that the LR monopolist always prices above the SR monopolist price $p^m(r)$. As a result, the optimal deviation at a stage with reference r is to set $p^m(r)$. Thus, the incentive constraint at each stage states that the payoff from sharing the LR monopolist profit equally is higher than taking the entire SR monopolist profit under the current reference point (and zero tomorrow onward). This inequality can be expressed in terms of "positive net gain from cooperating" as $g(r) := V(r) - 2\pi(p^m(r)|r) \geq 0$.[4] For subgame perfection, the constraint $g(r) \geq 0$ has to hold for all reference points $r \in \{r_1\} \cup \{f_t(r_1)\}_{t=1}^{\infty}$ on the LR monopolist path starting from r_1.

In the Appendix, it is shown that the continuous function $g(r)$ cuts zero at no more than one value of $r \in [0, v]$. Furthermore, if it does cut zero, then this intersection must be from above. Finally, for any $\delta \geq \frac{1}{2}$, we have that $g(0) > 0$. Then depending on δ, we are in one of two possible cases: Either $g(r) > 0$ for all $r \in [0, v]$, or there is a unique "incentive threshold" $\bar{r}$ with $g(\bar{r}) = 0$ such that we have $g(r) > 0$ for all $r < \bar{r}$ and $g(r) < 0$ for all $r > \bar{r}$. In the first case, the firms never have incentive to deviate from LR monopolist policy. In the second, they have incentive to deviate if only if the reference point is above a certain interior threshold $\bar{r}$.

By Lemma 1(b), we know that the LR monopolist policy monotonically converges to the unique steady state r^{ss}. Then starting from initial reference r_1, any reference price that the firms encounter on the LR monopolist path will be between r_1 and r^{ss}. Further note that the above observation on $g(r)$ implies convexity of the set of references that satisfy $g(r) \geq 0$. Thus, we can conclude that LR monopolist policy is a sSPE if and only if $g(r_1) \geq 0$ and $g(r^{ss}) \geq 0$. That is, if the incentive constraint holds at both the initial stage and the steady state, then we know that the firms have no incentive to deviate at any stage.

Clearly the incentive constraint becomes more slack as firms become more patient. Thus, the incentive threshold $\bar{r}$ is increasing with δ. It is shown that there is a value $\underline{\delta} \in (\frac{1}{2}, 1)$ that yields $\bar{r} = r^{ss}$. If the firms' patience is strictly below $\underline{\delta}$, we have $\bar{r} < r^{ss}$ and the incentive constraint does not hold at the steady state. In that case, the firms anticipate the eventual incentive to deviate (as the reference gets close enough to steady state), which leads to the unravelling of the collusion. Hence we know that for $\delta < \underline{\delta}$ it is impossible for the firms to coordinate on LR monopolist pricing, regardless of the initial reference.

Note that if the firms are infinitely patient ($\delta \to 1$), they will never want to deviate. As a result, we can say that there is an interior patience level $\bar{\delta} \in (\underline{\delta}, 1)$ that yields $\bar{r} = v$. Above this level of

[4] Cooperation is assumed in case of indifference.

patience, the incentive constraint holds for all reference points in $[0, v]$, and thus, LR monopolist policy is a sSPE outcome from any initial reference r_1. That is, $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ for any $r_1 \in [0, v]$ whenever $\delta \geq \bar{\delta}$.

When $\delta \in [\underline{\delta}, \bar{\delta})$, the threshold $\bar{r}$ is in $[r^{ss}, v)$. At the steady state, there is no incentive to deviate. This means that whenever the incentive constraint holds in the initial round, then LR monopoly pricing is a sSPE outcome. So if the initial reference price r_1 is smaller than $\bar{r}$, $\frac{V(r_1)}{2}$ is a sSPE payoff. If $r_1 > \bar{r}$, the firms would like to deviate in the initial round and the LR monopoly behavior cannot be sustained as a sSPE. Hence the result "LR monopoly pricing is a sSPE outcome if and only if the initial reference price is low enough".

Before describing the highest payoff when LR monopoly pricing is not a sSPE, it is helpful to briefly discuss the driving force behind the result that coordination on LR monopoly behavior is only possible if the firms start with a low reference price. There are two main channels through which the incentive constraint gets stricter as the initial reference price increases (i.e. $g(r)$ cuts zero from above).

First, the LR monopolist policy converges to the same steady state regardless of where it starts. Then, the higher the initial demand, the more sharply decreasing a path it follows. So for the two firms who want to coordinate on this path, a higher initial reference point makes today's market more valuable relative to the markets in future stages. This decreases the net gain from coordinating in the initial stage. Above a certain level, it becomes optimal to undercut in the initial period and take the whole market before the reference decreases. We can view this channel as the "dynamic effect" of the initial reference, since it determines how the firms compare today's market with the continuation value.

The second channel follows from the observation that the LR monopolist always prices strictly above the SR monopoly level. So if the firms want to coordinate on the LR monopolist policy, this also includes coordination on overpricing as an investment in tomorrow's reference point. If they deviate, however, the future markets do not matter under grim punishment. As a result, the optimal deviation is to the SR monopolist price, which maximizes the stage profit in the current round. Then, the deviator not only takes the entire demand instead of just half, but she also faces a higher total demand at the SR monopoly price. The wedge $f(r) - p^m(r)$ between LR monopoly price and the optimal deviation price is increasing in the reference point r. So if the initial reference r_1 is too high, cooperation means forgoing too much demand in the initial stage. Thus, collusion on the LR monopolist policy becomes harder when the initial reference is higher. We can think of this as the "static effect" of a high initial reference price: The increasing wedge would make deviation more attractive even if we kept the size of today's demand relative to future demand constant on the equilibrium path.

What is $\bar{w}(r_1, \delta)$ when LR monopoly pricing is not sSPE?

Now consider the case when $\delta \in [\underline{\delta}, \delta)$ and $r_1 > \bar{r}(\delta)$. As discussed above, in this case the initial reference is too high to coordinate on the LR monopolist policy. That is, $\frac{V(r_1)}{2}$ is not high enough to deter a deviation to $p^m(r_1)$ in the initial round. If even the highest feasible lifetime payoff cannot deter a downward deviation to $p^m(r_1)$, then no feasible payoff can. So we know that a price sequence that allows downward deviation to $p^m(r_1)$ in the initial round cannot be a sSPE outcome. As a result, any sSPE path should initially set a price $p < p^m(r)$.

Consider an initial reference r_1 with $r_1 > \bar{r}(\delta) \geq p^m(r_1)$. That is, LR monopolist pricing is not a sSPE starting from r_1, but it is a sSPE starting from any initial reference below the SR monopolist price $p^m(r_1)$. Then if the firms are setting some $p < p^m(r_1)$ in the initial round, the highest sSPE payoff of the game starting tomorrow is the LR monopoly payoff $\frac{V(p)}{2}$. Thus, we know that the highest sSPE payoff starting from r_1 is obtained by setting some $p < p^m(r_1)$ today and following the LR monopolist policy tomorrow onward. What remains to be found is the initial price p that yields the highest payoff among those that create no incentive to deviate in the first period.

Since $p < p^m(r_1)$, the optimal deviation in the first round is to undercut by an arbitrarily small amount, in which case the deviator takes the entire demand under reference r_1 and makes zero profit tomorrow onward. The payoff from cooperating is sharing the demand today at price p, and sharing the LR monopoly profit starting from initial reference p tomorrow onward. The initial period incentive constraint is that cooperation is weakly better than the optimal deviation. Again, this incentive constraint can be expressed in terms of "net gain from coordination" as $h(p|r_1) := \delta V(p) - \pi(p|r_1) \geq 0$.

The lifetime payoff from a price sequence that initially sets p and then follows the LR monopolist policy is given by $\frac{1}{2}(\pi(p|r_1) + \delta V(p))$ and is strictly increasing in p as long as $p < p^m(r_1)$. So the highest sSPE payoff from such a strategy is obtained by setting the highest initial price p that satisfies the incentive constraint. It is shown in the Appendix that $h(p|r_1)$ cuts zero exactly once for $p \in [0, p^m(r_1)]$ and it is from above. To see the intuition as to why the incentive constraint holds only for low initial prices, consider the two extremes. If the firms are setting initial price p very close to zero, then payoff from undercutting is essentially zero. On the other hand, the continuation value from collusion, which is the lifetime profit of a LR monopolist who starts at initial reference zero, is clearly positive. So the incentive constraint definitely holds. On the high extreme ($p = p^m(r_1)$), the incentive constraint is definitely violated by the above argument that even the highest feasible payoff is not good enough to prevent a deviation that yields $\pi(p^m(r_1)|r_1)$.

The highest sSPE profit is thus obtained by setting the unique $p^*(r_1) \in [0, p^m(r_1))$ that satisfies $h(p^*(r_1)|r_1) = 0$. If the firms set a higher initial price than $p^*(r_1)$, they have incentive to deviate in the first round. If they set a lower initial price, they can do better while still maintaining the incentive constraint by setting $p^*(r_1)$. When we plug in $h(p^*(r_1)|r_1) = 0$ into the lifetime payoff, we obtain $\bar{w}(r_1) = \delta V(p^*(r_1))$.

The result can be extended to initial references r_1 such that there is a range of values below $p^m(r_1)$ where LR monopoly pricing is not sSPE (i.e. $r_1 > p^m(r_1) > \bar{r}(\delta)$). If the initial price is in this range, it is shown that the first period incentive constraint can never hold. Therefore, the initial price has to be in the range $(0, \bar{r}(\delta))$, where the LR monopoly pricing can be sustained from the second period onward. Hence, we can generalize the result as $\bar{w}(r_1, \delta) = \delta V(p^*(r_1))$ with $p^*(r_1) \in (0, \min\{p^m(r_1), \bar{r}(\delta)\})$ whenever $r > \bar{r}(\delta)$. To summarize, if the initial reference r_1 is too high to collude on the LR monopolist behavior today, then the best that the firms can collude on is setting a price that is low enough to sustain LR monopoly behavior starting tomorrow.

5 Comparative Statics

We now look at the behavior of $\bar{w}(r_1,\delta)$ with respect to r_1. From Proposition $2(i)$ we know that if $\delta < \frac{1}{2}$, then $\bar{w}(r_1,\delta)$ is constant at zero. If $\delta \geq \bar{\delta}$, we have $\bar{w}(r_1,\delta) = \frac{V(r_1)}{2}$ which is strictly increasing in r_1. For the intermediate region of patience $\delta \in [\underline{\delta},\bar{\delta})$, the behavior of $\bar{w}(r_1,\delta)$ is ambiguous and depends on the level of initial reference r_1. In this case, we can make the following observation from Proposition $2(ii)$.

Corollary to Proposition 2. *If $\delta \in [\underline{\delta},\bar{\delta})$, then $\bar{w}(r_1,\delta)$ obtains a unique maximum at $r_1 = \bar{r}(\delta)$.*

That is, if the firms' patience level is in the region $[\underline{\delta},\bar{\delta})$, the highest sSPE payoff is obtained when starting from the largest initial reference that allows coordination on LR monopoly pricing. Note that $\bar{r}(\delta)$ is interior. So for $r_1 \in (\bar{r}(\delta), v)$, the highest sSPE payoff is strictly decreasing in the size of the first period demand.

For any $r_1 \leq \bar{r}(\delta)$, we have $\bar{w}(r_1,\delta) = \frac{V(r_1)}{2}$. This means that as long as the firms are able to coordinate on LR monopolist pricing, the best they can do improves with a higher initial reference. On the other side of $\bar{r}(\delta)$, recall that if we have $r_1 > \bar{r}(\delta)$, then $\bar{w}(r_1,\delta) = \delta V(p^*(r_1))$. The reason why $\bar{w}(r_1,\delta)$ is decreasing in this region is that $p^*(r_1)$ is itself a strictly decreasing function. In other words, suppose the firms want to sustain a price sequence that initially sets a low price and follows the LR monopolist path tomorrow onward. Then the highest price that does not create incentive to deviate in the first round is decreasing in the size of the demand. The reason is that the gain from cooperation $h(p|r_1)$ is strictly decreasing in r_1 for any given $p < (0, \min\{p^m(r_1), \bar{r}(\delta)\})$.

For illustration, suppose the firms have incentive to deviate from setting a price $p < p^m(r_1)$ today and following LR monopolist pricing tomorrow onward if their initial reference is r_1. With a higher reference price $r_1' > r_1$, gain from deviation increases because the demand in the first period is higher. However, the gain from cooperation (which is the LR monopolist profit starting from p tomorrow) is the same. So if setting p is not incentive compatible in the first round under r_1, then it is also not incentive compatible under r_1'. Therefore, the highest price that is incentive compatible is decreasing in the initial reference.

Finally, note that $\bar{w}(\bar{r}(\delta),\delta) = \frac{V(\bar{r}(\delta))}{2}$ is the highest feasible payoff when starting from $\bar{r}(\delta)$.

Then clearly, it is greater than $\lim_{r_1 \searrow \bar{r}(\delta)} \bar{w}(r_1, \delta)$. This observation, together with $\bar{w}(r_1, \delta)$ increasing in r_1 for $r_1 < \bar{r}(\delta)$ and decreasing for $r_1 > \bar{r}(\delta)$ allows us to conclude that the highest sSPE payoff $\bar{w}(r_1, \delta)$ is maximized at $r_1 = \bar{r}(\delta)$. Hence, for interior values of patience, the highest lifetime profit that the firms can collude on does not necessarily increase with the initial size of the market. It is increasing if the initial market is such that they are able to collude on the optimal pricing policy of the LR monopolist and decreasing otherwise.

6 Comments

In the above setting, we can think of other features to include in the model that can be helpful in understanding the effect of dynamic reference pricing on collusion. The most natural of such considerations is consumer loss aversion. It is a common modelling assumption based on substantial empirical evidence that the utility effect of a price increase (relative to the reference point) is greater than that of a price decrease for the consumers. In the our setting, this can be represented by a demand function which is kinked at the reference price. A rigorous discussion of how the collusion behavior is affected by this consideration can improve the robustness of our predictions, and therefore seems warranted. However, using findings from existing literature, we can make some conjectures as to why the main results will continue to hold qualitatively under loss aversion.

Fibich et. al. (2003) show that in the case of a kinked (linear) demand, the behaviour described Lemma $1(a)$ and $1(b)$ holds.[5] That is, from any initial reference, the LR monopolist policy monotonically converges to a steady state and overprices with respect to the SR monopolist level. One difference is that it is possible for the set of steady states to be a closed interval instead of a singleton. Even in that case, however, the LR monopolist policy converges to the upper (lower) bound of this set when the initial refence is above (below) this interval. This means the price path that provides the highest feasible payoff for the competing firms has the same form under loss

[5] They do so in a continuous time framework. However, it can be seen from the monotonicity conditions of Popescu and Wu (2007) that this also holds in discrete time.

aversion.[6]

Given these observations about loss aversion, consider for example our result that collusion on LR monopoly pricing is not possible when the initial reference is too high. [7] Since for higher initial references the reference still decreases more steeply over time, the dynamic channel we describe (i.e., deviate before the market becomes small) continues to apply. Since the optimal deviation under a high initial reference price is still a discrete downward jump to the SR monopolist price, the static channel (i.e., the "wedge" is too large to ignore) continues to apply as well. These effects are qualitatively unaffected when we include asymmetry around the reference price. Overall, as the strategic considerations by the competing firms that drive our main results do not rely on symmetry around the reference point, we can reasonably predict their robustness to loss aversion. Nevertheless, the formal analysis of these incentives when allowing for asymmetry is a useful further step.

A second consideration is that the current model works with a very short consumer memory. While forming reference prices, the consumers only use the prices from the period immediately preceding the current one. An interesting observation would be the change in the possible profits from collusion with respect to the length of consumer memory. As stated above, we can interpret the results here as the limit case of the commonly used exponential smoothing process $r_t = \alpha r_{t-1} + (1-\alpha)p_{t-1}$ as α tends to zero.

Some intuitive predictions can be made about the consequences of increasing the memory parameter α. Recall the intuition behind the result "the highest sSPE payoff is decreasing if LR monopoly pricing is not sustainable" is that if the best collusion is to initially set a low price and follow the LR monopolist path tomorrow onward, then for firms who are setting a given initial price, a higher initial reference increases only the current demand and does not affect the continuation payoff. As a result, the net gain from deviation is increasing with the initial reference. If

[6]Linear demand is sufficient but not necessary for monotonic convergence of the LR monopolist policy to a steady state under loss aversion. In their analysis of monopoly pricing for a more general class of demand functions, Popescu and Wu (2007) provide a wide range of relatively weak conditions such that this holds. From their findings, for instance, it follows that $\alpha = 0$ or the convexity of $V(r)$ both imply this behaviour.

[7]Recall that such an initial reference is higher than the steady state

the consumers have longer memory, however, a higher initial reference will directly mean a higher reference tomorrow. So with a longer consumer memory, we can expect the highest sSPE payoff to be “less decreasing” in initial reference. In general, we would expect the effects of dynamic reference pricing to become weaker as the consumer memory becomes longer. Note that we obtain the standard Bertrand model without reference effects if consumers have infinite memory ($\alpha = 1$).

Other promising extensions include the behavior of the sustainable collusion profits with respect to the number of firms as well as varying levels of product differentiation and cross-reference effect between the firms. Under product differentiation, we would expect some degree of independence between the reference price of the two firms to reduce the deterrence of punishment for deviating from a collusive price pattern. These, among other further steps in the analysis, may help deepen our understanding of the dynamics described in the model above.

In sum, the analysis of the price competition model presented in this paper gives us the following insights about the extent of tacit collusion in a market with dynamically reference pricing consumers. If the firms have very little patience, collusion is not possible. If they are sufficiently patient, they can always imitate a long run monopolist, which gives them the highest feasible profit in the market. If the firms are moderately patient, then the current reference of the consumers plays a critical role in determining the extent to which they can collude. If the they are starting with consumers with low enough reference, the firms can coordinate on setting long run monopolist prices. Otherwise, the best they can do is to lower the reference price by setting a low price initially, and imitate the long run monopolist thereafter. From this case, we also observe that a high reference price does not always translate to a higher collusion profit. If the current reference is high, increasing it further leads to a decrease in the highest profit that the firms can obtain through collusion.

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Appendix

We first prove Proposition 1, Proposition $2(i)$ and Lemma 1 respectively. The results of Lemma 1 are then used to prove Proposition $2(ii)$ and (iii) jointly.

Proposition 1. $\underline{w}(r_1, \delta) = 0$ *for all* $r_1 \in [0, v]$ *and* $\delta \in (0, 1)$.

Proof. First, note that the restriction $p^i_t \in [0, p^c(r_t)]$ implies that the stage profit $\pi^i(p^t|r_t)$ of firm i is non-negative. Thus, for any r_1 and δ, the lowest feasible lifetime profit is zero.

As in the standard Bertrand model, the unique Nash Equilibrium (NE) of the stage game is $p^1_t = p^2_t = 0$ under any reference price r_t: Denote the short-run monopoly price induced by r_t by $p^m(r_t) := argmax_p \pi(p|r_t) = \frac{v+\lambda r_t}{2(1+\lambda)}$. If the opponent is setting price $p^j_t \in (0, p^m(r_t)]$, firm i has incentive to undercut j by an arbitrarily small amount and obtain $\pi(p^j_t|r_t) > \frac{1}{2}\pi(p^j_t|r_t)$. If j is setting $p^j_t > p^m(r_t)$, then it is optimal for firm i to set $p^m(r_t)$ and obtain $\pi(p^m(r_t)|r_t) > \frac{1}{2}\pi(p^j_t|r_t)$.

Since both firms setting price zero is the unique stage NE, the strategy profile $s^1(h_t) = s^2(h_t) = 0$ for all $h_t \in H$ is a sSPE for any r_1 and δ. Since this sSPE yields the lowest feasible payoff 0, we have $\underline{w}(r_1, \delta) = 0$ for all $r_1 \in [0, v]$ and $\delta \in (0, 1)$.

$\square$

Proposition 2 (i). *If* $\delta \in (0, \frac{1}{2})$*, then* $\bar{w}(r_1, \delta) = 0$ *for all* $r_1 \in [0, v]$.

Proof. Given initial reference $r_1 \in [0, v]$, consider a symmetric price sequence $\{\tilde{p}_t\}_{t=1}^{\infty}$ with $\tilde{p}_t \in [0, p^c(r_t)]$ for all $t \in \mathbb{N}_+$ under the evolution rule $r_t = \tilde{p}_{t-1}$ for $t \geq 2$. Suppose $\tilde{p}_t > 0$ for some t. Recall that $r_1 \in [0, v]$ and $\tilde{p}_t \in [0, p^c(r_t)]$ together imply $\tilde{p}_t, r_t \in [0, v]$ for all t. Then the sequence of stage profits $\{\pi(\tilde{p}_t|r_t)\}_{t=1}^{\infty}$ resulting from r_1 and $\{\tilde{p}_t\}_{t=1}^{\infty}$ takes values in the compact set $[0, \pi(p^m(v)|v)]$ for all t. Let $\bar{\pi}(\tilde{p}|r_1) := \sup_{t \in \mathbb{N}_+} \{\pi(\tilde{p}_t|r_t)\}_{t=1}^{\infty}$.

Since 0 is the lowest feasible lifetime profit and $p^1_t = p^2_t = 0$ for all $t \in \mathbb{N}_+$ is a sSPE of any subgame following a deviation, $\{\tilde{p}_t\}_{t=1}^{\infty}$ is a sSPE path if and only if it yields no incentive to deviate under grim trigger strategies with price zero punishment. Under grim trigger strategies with price zero punishment, the period τ incentive constraint to sustain $\{\tilde{p}_t\}_{t=1}^{\infty}$ is

$$\sum_{t=\tau}^{\infty} \delta^{t-\tau} \frac{\pi(\tilde{p}_t|r_t)}{2} \geq \begin{cases} \pi(\tilde{p}_\tau|r_\tau); & \tilde{p}_\tau < p^m(r_\tau) \\ \pi(p^m(r_\tau)|r_\tau); & \tilde{p}_\tau \geq p^m(r_\tau) \end{cases} \tag{1}$$

which has to hold for all $\tau \in \mathbb{N}_+$ for subgame perfection. Since the sequence $\{\pi(\tilde{p}_t|r_t)\}_{t=1}^{\infty}$ takes values from a compact set in $\mathbb{R}_+$, we are in one of two cases. Either $\{\pi(\tilde{p}_t|r_t)\}_{t=1}^{\infty}$ obtains a maximum at some finite period $\tau \in \mathbb{N}_+$, or it converges to $\bar{\pi}(\tilde{p}|r_1)$ as t tends to infinity. If the sequence of stage profits obtains a maximum at $t = \tau$, we have

$$\pi(p^m(r_\tau)|r_\tau) \geq \pi(\tilde{p}_\tau|r_\tau) = \bar{\pi}(\tilde{p}|r_1) > \frac{\bar{\pi}(\tilde{p}|r_1)}{2(1-\delta)} \geq \sum_{t=\tau}^{\infty} \delta^{t-\tau} \frac{\pi(\tilde{p}_t|r_t)}{2}$$

which contradicts (1). The first inequality holds as $p^m(r_\tau)$ is by definition the price that maximizes stage profit under reference r_τ. The second inequality holds by $\delta < \frac{1}{2}$. The third inequality holds by the definition of the supremum.

Now suppose that $\pi(\tilde{p}_t|r_t)$ converges to $\bar{\pi}(\tilde{p}, r_1)$. Then for any $\delta < \frac{1}{2}$ there exists a finite $\tau \in \mathbb{N}_+$ with $\pi(\tilde{p}_\tau|r_\tau) = \bar{\pi}(\tilde{p}|r_1) - \epsilon$ where $\epsilon > 0$ is small enough such that

$$\pi(p^m(r_\tau)|r_\tau) \geq \pi(\tilde{p}_\tau|r_\tau) = \bar{\pi}(\tilde{p}|r_1) - \epsilon > \frac{\bar{\pi}(\tilde{p}|r_1)}{2(1-\delta)} \geq \sum_{t=\tau}^{\infty} \delta^{t-\tau} \frac{\pi(\tilde{p}_t|r_t)}{2}$$

which contradicts (1). Thus, if $\delta < \frac{1}{2}$, there is no symmetric price sequence that sets a positive price in some period and satisfies the incentive constraint in every period. As a result, the only sSPE payoff is zero, obtained when both firms play the unique stage NE in each period.

□

Lemma 1. *For the solution of the recursive LR monopolist problem, the following properties hold.*

(a) The unique optimal policy function $f(r)$ satisfies $f(r) \in (p^m(r), p^c(r)]$ for all $r \in [0, v]$

(b) For any initial $r \in [0, v]$, $f_n(r)$ monotonically converges to unique steady state $r^{ss} := \frac{v}{2+\lambda(1-\delta)}$ as n tends to infinity.

(c) If the choke price constraint $r' \leq p^c(r)$ is binding at $r = r_1 \in [0, v]$, then it is binding at $r = r_2$ for any $r_2 \in [0, r_1]$.

(d) $V(r)$ is continuously differentiable and strictly increasing.

Proof. First, we derive the unique optimal pricing policy function of the problem. Denote this function by $f : [0, v] \to [0, v]$. That is, $f(r)$ is the optimal price by the LR monopolist under state r. We then use function f to prove Lemma 1. In deriving the optimal policy of the original problem, we use the optimal policy of the same problem without imposing the the constraints $0 \leq r' \leq p^c(r)$ (henceforth "unconstrained LR monopolist problem").

We can formulate the unconstrained recursive LR monopolist problem as

$$\begin{aligned} V^u(r) : &= \max_{r' \in \mathbb{R}} \pi(r'|r) + \delta V^u(r') \\ &= \max_{r' \in \mathbb{R}} r'(v - r' + \lambda(r - r')) + \delta V^u(r') \end{aligned}$$

Recall that $\pi(r'|r)$ is strictly concave and obtains a unique maximum at $r' = p^m(r)$. Denoting the optimal policy function of the unconstrained problem by $f^u : [0, v] \to \mathbb{R}$, the resulting Euler equation is:

$$f^u(r) = \frac{v + \lambda r + \delta \lambda f^u(f^u(r))}{2(1 + \lambda)} \tag{2}$$

This functional equation yields 2 roots for $f^u(r)$, namely:

$$\begin{aligned} f^{u1}(r) &:= \frac{v}{(1+\lambda) - \lambda\delta + \sqrt{(1+\lambda)^2 - \lambda^2\delta}} + \frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - \lambda^2\delta}}{\lambda\delta} r \\ f^{u2}(r) &:= \frac{v}{(1+\lambda) - \lambda\delta - \sqrt{(1+\lambda)^2 - \lambda^2\delta}} + \frac{(1+\lambda) + \sqrt{(1+\lambda)^2 - \lambda^2\delta}}{\lambda\delta} r \end{aligned}$$

However, we have $f^{u2}(0) < 0$ for any $\lambda \geq 0$ and $\delta \in (0, 1)$. The optimal policy under any state (including $r = 0$) cannot be negative. To see this, consider the sequential formulation $V(r_1) = \max_{\{r_t\}_{t=2}^{\infty}} \sum_{t=1}^{\infty} \delta^{t-1} \pi(r_{t+1}|r_t)$ of the problem. The function $\pi(r_{t+1}|r_t)$ is strictly increasing in r_{t+1} at any $r_{t+1} < 0$ and strictly increasing in r_t at any $r_t \in \mathbb{R}$. Then for any given r_1 and continuation sequence $\{r_t\}_{t=3}^{\infty}$, setting $r_2 = 0$ yields a strictly higher lifetime value than setting any $r_2 < 0$. As a result the optimal policy cannot set a negative price, and $f^{u2}(r)$ cannot be an optimal policy in the unconstrained problem.

Note that we have $f^{u1}(0) > 0$ and $f^{u1}(r)$ strictly increasing. Thus, the unique optimal policy function of the unconstrained problem is $f^{u1}(r)$. That is, we know that $f^u(r) = a + br$ where

$$a := \frac{v}{(1+\lambda) - \lambda\delta + \sqrt{(1+\lambda)^2 - \lambda^2\delta}} \text{ and } b := \frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - \lambda^2\delta}}{\lambda\delta}$$

Then in the constrained problem, the optimal policy is $f(r) = \min\{p^c(r), f^u(r)\} = \min\{\frac{v+\lambda r}{1+\lambda}, a + br\}$. The optimal policy is equal to the choke price if the upper bound is binding and to the unconstrained policy otherwise.

Lemma 1(a)**:** Note that $p^c(r) = \frac{v+\lambda r}{1+\lambda} > \frac{v+\lambda r}{2(1+\lambda)} = p^m(r)$ for all $r \in [0, v]$. Furthermore, we have $a > \frac{v}{2(1+\lambda)}$ and $b > \frac{\lambda}{2(1+\lambda)}$. This means $f^u(r) = a + br > \frac{v+\lambda r}{2(1+\lambda)} = p^m(r)$ for all $r \in [0, v]$. These two observations allow us to conclude that $f(r) = \min\{p^c(r), f^u(r)\} \in (p^m(r), p^c(r)]$.

Lemma 1(b)**:** First note that as $f(r)$ is the minimum of two continuous functions, it is continuous. Furthermore, we have $f(0) > 0$ and $f(v) < v$. These observations imply that $f(r)$ obtains at least one fixed point r^{ss} with $f(r^{ss}) = r^{ss}$ in the interval $(0, v)$.

Whenever the choke price constraint is binding, we have $f(r) = \frac{v+\lambda r}{1+\lambda} > r$ for any $r < v$. Thus, at any fixed point, the choke price constraint is non-binding and we have $f(r) = a + br^{ss} = r^{ss}$. The unique solution the this equality is $r^{ss} = \frac{v}{2+\lambda(1-\delta)}$, which we can conclude is the unique fixed point of the policy function $f(r)$ for $r \in [0, v]$.

As $f(0) > 0$, $f(v) < v$, and r^{ss} is the unique fixed point, we know that $f(r)$ cuts the 45° line from above at r^{ss}. That is, $f(r) > r$ for all $r < r^{ss}$ and $f(r) < r$ for all $r > r^{ss}$. Since $f(r)$ is strictly increasing, we have $f(r) < f(r^{ss}) = r^{ss}$ for all $r < r^{ss}$ and $f(r) > f(r^{ss}) = r^{ss}$ for all $r > r^{ss}$. Combining these two observations, we have $f(r) \in (r, r^{ss})$ for all $r < r^{ss}$ and $f(r) \in (r^{ss}, r)$ for all $r > r^{ss}$. Denoting the n^{th} iterate of the policy function on state r by $f_n(r)$ (e.g. $f_2(r) = f(f(r))$), this (together with the continuity of f) implies that for all $r \in [0, v]$, $f_n(r)$ monotonically converges to r^{ss} as n tends to infinity.

Lemma 1(c)**:** Suppose the upper bound constraint $r' \leq p^c(r)$ is binding at some $r = r_1 \in [0, v]$. This means $f^u(r_1) = a + br_1 \geq \frac{v+\lambda r_1}{1+\lambda}$. Note that $b < \frac{\lambda}{1+\lambda}$. That is, the slope of the unconstrained policy function is lower than the slope of the choke price. This implies

$$a + br_1 \geq \frac{v + \lambda r_1}{1+\lambda} \Rightarrow a + br_2 > \frac{v + \lambda r_2}{1+\lambda}, \quad \forall r_2 \leq r_1$$

Thus for any $r_2 \in [0, r_1]$, we have that $f^u(r_2) > p^c(r_2)$ and the upper bound constraint is binding.

Lemma 1(d)**:** First we show that $V(r)$ is continuous at any $r \in (0, v)$. Then we show that the first difference $\frac{dV(r)}{dr}$ is also continuous.

Continuity of $V(r)$**:** Clearly, the unconstrained value function $V^u(r)$ is continuously differentiable. Furthermore, we know from Lemma $1(c)$ that the choke price constraint is non-binding only for a convex interval $(\tilde{r}, v]$ with $\tilde{r} \in [0, r^{ss})$ given by $p^c(\tilde{r}) = f^u(\tilde{r})$. In this region, we have $V(r) = V^u(r)$. Thus, the value function $V(r)$ in the constrained problem is continuously differentiable in the region $(\tilde{r}, v)$ where the optimal policy $f(r)$ is interior. We additionally need to show that $V(r)$ is continuous for $r \in (0, \tilde{r}]$, where the choke price constraint is binding and we have $f(r) = p^c(r)$.

Let $p^c_n(r)$ denote the n^{th} iterate of $p^c(r)$ on state r. Since $lim_{n\to\infty} p^c_n(r) = v > \tilde{r}$ for all $r \in [0, v]$, we know that for any $r \in [0, v]$ there is a finite $n \in \mathbb{N}_+$, such that $p^c_n(r) > \tilde{r}$. That is, starting from any state $r \in [0, v]$, a state high enough that the constraint is not binding can be reached through a finite number iterations of the choke price. Until such a state is reached, the optimal policy is to set the choke price which yields zero stage profit. Then for any r where the choke price is binding, we can say:

$$p^c_n(r) > \tilde{r} > p^c_{n-1}(r) \quad \text{for some } n \in \mathbb{N}_+ \Rightarrow V(r) = \delta^n V(p^c_n(r)) = \delta^n V^u(p^c_n(r)) \tag{3}$$

which is continuous since $V(r)$ is continuous at any $r > \tilde{r}$.

Next, we show the continuity of $V(r)$ at $r = \tilde{r}$ and at states $r \in [0, \tilde{r})$ with $p^c_n(r) = \tilde{r}$ for some $n \in \mathbb{N}_+$. Note that $p^c(r) > r$ for all $r \in [0, v]$ implies that there is an interval $(\tilde{r} - \epsilon, \tilde{r})$ that yields $p^c(r) > \tilde{r}$ for all $r \in (\tilde{r} - \epsilon, \tilde{r})$. From Equation (3) we know that $V(r)$ is continuous for this interval. Then we have

$$\lim_{r \nearrow \tilde{r}} V(r) = \lim_{r \nearrow \tilde{r}} \delta V(p^c(r)) = \delta V(p^c(\tilde{r})) = \pi(p^c(\tilde{r})|\tilde{r}) + \delta V(p^c(\tilde{r}))$$
$$= \pi(f^u(\tilde{r})|\tilde{r}) + \delta V(f^u(\tilde{r})) = \lim_{r \searrow \tilde{r}} V(r) = V(\tilde{r}) \tag{4}$$

which implies that $V(r)$ is continuous at $\tilde{r} = r$. The first equality follows from Equation (3). The second equality follows from continuity of $V(r)$ at $r = p^c(\tilde{r}) > \tilde{r}$. The third equality holds by the definition of choke price ($\pi(p^c(r)|r) = 0$). The fourth equality holds by the definition of $\tilde{r}$ ($f^u(\tilde{r}) = p^c(\tilde{r})$). The fifth equality holds by the continuity of $V^u(r)$ at $r = \tilde{r}$.

Finally, consider a $\tilde{\tilde{r}} \in [0, \tilde{r})$ with $p_n^c(\tilde{\tilde{r}}) = \tilde{r}$ for some $n \in \mathbb{N}_+$. There we have

$$\lim_{r \nearrow \tilde{\tilde{r}}} V(r) = \lim_{r \nearrow \tilde{\tilde{r}}} \delta^{n+1} V(p_{n+1}^c(r)) = \delta^{n+1} V(p^c(\tilde{r}))$$
$$\lim_{r \searrow \tilde{\tilde{r}}} V(r) = \lim_{r \searrow \tilde{\tilde{r}}} \delta^{n} V(p_{n}^c(r)) = \delta^{n} V(\tilde{r})$$

The two limits are equal if and only if $\delta V(p^c(\tilde{r})) = V(\tilde{r})$, which is shown in equation (4). Thus, $V(r)$ is continuous at any such point $\tilde{\tilde{r}}$. Function $V(r)$ is now shown to be continuous at $r = \tilde{r}$, for r such that $\tilde{r} \in (p_n^c(r), p_{n+1}^c(r))$ for some $n \in \mathbb{N}_+$ and r such that $p_n^c(r) = \tilde{r}$ for some $n \in \mathbb{N}_+$. These values span the interval $(0, \tilde{r}]$. Therefore we can conclude that $V(r)$ is continuous at any $r \in (0, v)$.

Continuity of $\frac{dV(r)}{dr}$: Omitting the non-negativity constraint (which is shown above to never bind), the Lagrangian of the LR monopolist problem is given by

$$L = \pi(r'|r) + \delta V(r') + \mu\left(\frac{v + \lambda r}{1 + \lambda} - r'\right)$$

where μ is the multiplier for the choke price constraint. Then if we denote by $\mu(r)$ the marginal (shadow) value of relaxing the choke price constraint under state r and the optimal policy $f(r)$, the derivative of $V(r)$ with respect to r is given by

$$\frac{dV(r)}{dr} = \lambda\left(f(r) + \frac{\mu(r)}{1 + \lambda}\right)$$

If $r > \tilde{r}$ the choke price is not binding ($\mu(r) = 0$) and we have $\frac{dV(r)}{dr} = \lambda f^u(r) = \lambda(a + br)$ which is continuous. If we have state $r = \tilde{\tilde{r}}$ such that $p^c_{n-1}(\tilde{\tilde{r}}) < \tilde{r} < p^c_n(\tilde{\tilde{r}})$ for some $n \in \mathbb{N}_+$, then $V(\tilde{\tilde{r}}) = \delta^n V(p^c_n(\tilde{\tilde{r}}))$ (Equation (3)). In that case:

$$\frac{dV(r)}{dr}\Big|_{r=\tilde{\tilde{r}}} = \delta^n \frac{dV(p^c_n(r))}{dr}\Big|_{r=\tilde{\tilde{r}}} = \delta^n \left(\frac{dV(r)}{dr}\Big|_{r=p^c_n(\tilde{\tilde{r}})} \cdot \frac{dp^c_n(r)}{dr} \right)$$

$$= \left(\frac{\delta\lambda}{1+\lambda} \right)^n \cdot \frac{dV(r)}{dr}\Big|_{r=p^c_n(\tilde{\tilde{r}})} = \frac{\delta^n \lambda^{n+1}}{(1+\lambda)^n}(a + bp^c_n(\tilde{\tilde{r}})) \tag{5}$$

which is continuous since $p^c_n(r)$ is continuous for all n. For state $r = \tilde{r}$ recall that there is an interval $(\tilde{r} - \epsilon, \tilde{r})$ with $p^c(r) > \tilde{r}$. Then for the one-sided limits of $\frac{dV(r)}{dr}$ at $r = \tilde{r}$, we have

$$\lim_{r \nearrow \tilde{r}} \frac{dV(r)}{dr} = \frac{\delta\lambda^2}{1+\lambda}(a + bp^c(\tilde{r})) = \frac{\delta\lambda^2}{1+\lambda}\left(a + b\frac{v + \lambda\tilde{r}}{1+\lambda} \right) \tag{6}$$

$$\lim_{r \searrow \tilde{r}} \frac{dV(r)}{dr} = \lambda(a + b\tilde{r}) \tag{7}$$

We would like to show that these two limits are equal. Setting Equation (6) equal to Equation (7), and using the definition of $\tilde{r}$ which implies $f^u(\tilde{r}) = a + b\tilde{r} = \frac{v+\lambda\tilde{r}}{1+\lambda} = p^c(\tilde{r})$ we get

$$\frac{\delta\lambda^2}{1+\lambda}\left(a + b\frac{v + \lambda\tilde{r}}{1+\lambda} \right) = \lambda(a + b\tilde{r})$$

$$\Leftrightarrow \quad \delta\lambda(a + b(a + b\tilde{r})) = v + \lambda\tilde{r}$$

$$= 2(1+\lambda)(a + b\tilde{r}) - (v + \lambda\tilde{r})$$

$$\Leftrightarrow a + b\tilde{r} = \frac{v + \lambda\tilde{r} + \delta\lambda(a + b(a + b\tilde{r}))}{2(1+\lambda)}$$

which is exactly the unconstrained Euler equation (Equation (2)), and holds by the definition of optimal unconstrained policy $f^u(r) = a + br$. Thus we know that $\lim_{r \nearrow \tilde{r}} \frac{dV(r)}{dr} = \lim_{r \searrow \tilde{r}} \frac{dV(r)}{dr} = \frac{dV(r)}{dr}\Big|_{r=\tilde{r}}$. So $\frac{dV(r)}{dr}$ is continuous at $r = \tilde{r}$. Finally, consider a state $\tilde{\tilde{r}}$ such that $p^c_n(\tilde{\tilde{r}}) = \tilde{r}$ for some $n \in \mathbb{N}_+$. In that case, by Equation (5), the one-sided limits are

$$\lim_{r \nearrow \tilde{\tilde{r}}} \frac{dV(r)}{dr} = \lim_{r \nearrow \tilde{\tilde{r}}} \frac{\delta^{n+1}\lambda^{n+2}}{(1+\lambda)^{n+1}}(a + bp^c_{n+1}(r)) = \frac{\delta^{n+1}\lambda^{n+2}}{(1+\lambda)^{n+1}}(a + bp^c(\tilde{r})) \tag{8}$$

$$\lim_{r \searrow \tilde{\tilde{r}}} \frac{dV(r)}{dr} = \lim_{r \searrow \tilde{\tilde{r}}} \frac{\delta^n\lambda^{n+1}}{(1+\lambda)^n}(a + bp^c_n(r)) = \frac{\delta^n\lambda^{n+1}}{(1+\lambda)^n}(a + b\tilde{r}) \tag{9}$$

The two limits are equal if and only if $\frac{\delta\lambda}{(1+\lambda)}(a+bp^c(\tilde{r})) = a+b\tilde{r}$. This condition is identical to the necessary and sufficient condition for the equality of the expressions (6) and (7), which holds as shown above. Thus, $\frac{dV(r)}{dr}$ is continuous at any r such that $p_n^c(r) = \tilde{r}$ for some $n \in \mathbb{N}$.

We have shown continuity of $\frac{dV(r)}{dr}$ for $r = \tilde{r}$, all r such that $p_n^c(r) > \tilde{r} > p_{n-1}^c(r)$ for some $n \in \mathbb{N}_+$ and all r such that $p_n^c(r) = \tilde{r}$ for some $n \in \mathbb{N}_+$. These values span the interval $(0, \tilde{r})$. As a result, we can conclude that $\frac{dV(r)}{dr}$ is continuous at any $r \in (0, v)$.

Since $\frac{dV(r)}{dr} = \lambda\left(f(r) + \frac{\mu(r)}{1+\lambda}\right) > 0$ for all $r \in (0, v)$, $V(r)$ is increasing.

□

Proposition 2 (ii) and (iii). *There exist $\underline{\delta}$, $\bar{\delta}$ with $1 > \bar{\delta} > \underline{\delta} > \frac{1}{2}$ such that:*

***(ii)** For each $\delta \in [\underline{\delta}, \bar{\delta})$, there exists a unique $\bar{r}(\delta) \in (0, v)$ that yields:*

$$\bar{w}(r_1, \delta) = \begin{cases} \frac{V(r_1)}{2}; & r_1 \in [0, \bar{r}(\delta)] \\ \delta V(p^*(r_1)); & r_1 \in (\bar{r}(\delta), v] \end{cases}$$

Where $p^*(r_1)$ is the unique value in the interval $(0, \min\{p^m(r_1), \bar{r}(\delta)\})$ that solves $\pi(p^*(r_1)|r_1) = \delta V(p^*(r_1))$.

***(iii)** If $\delta \in [\bar{\delta}, 1]$ then $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ for all $r_1 \in [0, v]$*

Proof. This proof follows two steps. First we show the existence of $\underline{\delta}, \bar{\delta}$ with $1 > \bar{\delta} > \underline{\delta} > \frac{1}{2}$ that yield $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ when either $\delta > \bar{\delta}$ or $\delta \in [\underline{\delta}, \bar{\delta})$ with r_1 smaller than a threshold $\bar{r}(\delta) \in [r^{ss}, v)$. Second, we show that if $\delta \in [\underline{\delta}, \bar{\delta})$ and $r > \bar{r}(\delta)$, then $\bar{w}(r_1, \delta) = \delta V(p^*(r_1))$ with $p^*(r_1)$ as defined in the Proposition.

Conditions for $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$:

First note that by construction, the highest feasible lifetime payoff a firm can obtain under a symmetric strategy profile (given initial reference r_1) is $\frac{V(r_1)}{2}$. So whenever $\frac{V(r_1)}{2}$ is a sSPE payoff, we have $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$.

It is shown above that in the recursive LR monopolist problem with initial state r, profit $V(r)$ is obtained by following the unique optimal policy $f(r)$. This implies that in the two firm game given initial reference r_1, the only symmetric price sequence that yields payoff $\frac{V(r_1)}{2}$ for the firms is $\{f_t(r_1)\}_{t=1}^{\infty}$ (as before, $f_t(r_1)$ denotes the t^{th} iterate of function f on state r_1). Therefore, we have $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ if and only if symmetric price sequence $\{f_t(r_1)\}_{t=1}^{\infty}$ is a sSPE path.

Recall that a symmetric price sequence is a sSPE path if and only if it yields no incentive to deviate at any stage under grim trigger strategies with price zero punishment. From Lemma1(a), we know that $f(r) > p^m(r)$ for all $r \in [0, v]$. Then at a period with reference r, the optimal deviation from the LR monopolist policy is to set $p^m(r)$. Assuming no deviation in case of indifference, this yields the following incentive constraint under reference r:

$$\frac{V(r)}{2} \geq \pi(p^m(r)|r) \Leftrightarrow V(r) - 2\pi(p^m(r)|r) \geq 0$$

Let $g(r) := V(r) - 2\pi(p^m(r)|r)$. For subgame perfection, we need $g(r) \geq 0$ to hold for all reference points $r \in \{r_1\} \cup \{f_t(r_1)\}_{t=1}^{\infty}$ on the equilibrium path. Since both $V(r)$ and $\pi(p^m(r)|r)$ are continuous, $g(r)$ is also continuous. Next we show that for any $\delta \geq \frac{1}{2}$, $g(r)$ cuts zero for at most one $r \in [0, v]$ and if it does, the intersection is from above. Together with the continuity of g, this implies that the set of references r that satisfy $g(r) \geq 0$ is convex.

First note that $\delta \geq \frac{1}{2}$ implies $g(0) > 0$:

$$\begin{aligned} V(0) = \max_{r' \in [0, p^c(0)]} \{\pi(r'|0) + \delta V(r')\} &\geq \pi(p^m(0)|0) + \delta V(p^m(0)) \\ &> \pi(p^m(0)|0) + \delta V(0) \end{aligned}$$

The first inequality holds as the left hand side is the highest feasible payoff and the right hand side is feasible. The second inequality holds as $V(r)$ is increasing and $p^m(0) = \frac{v}{2(1+\lambda)} > 0$. Then if $\delta \geq \frac{1}{2}$, we have:

$$V(0) > \frac{\pi(p^m(0)|0)}{1-\delta} \geq 2\pi(p^m(0)|0) \Rightarrow g(0) > 0$$

Differentiating $g(r)$, we obtain

$$g'(r) = \frac{dV(r)}{dr} - 2\frac{d\pi(p^m(r)|r)}{dr} = \lambda\left(f(r) + \frac{\mu(r)}{1+\lambda}\right) - \lambda\frac{v + \lambda r}{1+\lambda}$$

where function $\frac{dV(r)}{dr}$ is as shown in the proof of Lemma $1(d)$. Again, $\mu(r)$ denotes the shadow value of relaxing the choke price constraint $r' \leq p^c(r) = \frac{v+\lambda r}{1+\lambda}$ under state r and optimal policy $f(r)$. If the choke price constraint is binding, we have $f(r) = \frac{v+\lambda r}{1+\lambda}$ and $\mu(r) \geq 0$, which yields $g'(r) \geq 0$. If the constraint is non-binding, we have $f(r) < \frac{v+\lambda r}{1+\lambda}$ and $\mu(r) = 0$, which yields $g'(r) < 0$. That is, $g(r)$ is increasing if the choke price constraint is binding and strictly decreasing if it is non-binding. Then by Lemma $1(c)$, if $g(r)$ is strictly decreasing at some reference $r = r_1$ then it is strictly decreasing at any greater value $r = r_2 \in (r_1, v)$. This, together with $g(0) > 0$ implies that for any $\delta \in [\frac{1}{2}, 1)$, if $g(r_1) \leq 0$, then $g(r_2) < 0$ for all $r_2 \in (r_1, v]$. Therefore, we are in one of two cases: either $g(r) > 0$ for all $r \in [0, v]$, or there exists a unique threshold $\bar{r} \in (0, v]$ such that $g(\bar{r}) = 0$, $g(r) > 0$ for all $r < \bar{r}$ and $g(r) < 0$ for all $r > \bar{r}$. In either case, the set of references r that satisfy $g(r) \geq 0$ is convex.

By Lemma $1(b)$, we know that $f_t(r_1) \in [r_1, r^{ss})$ for all t if $r_1 < r^{ss}$ and $f_t(r_1) \in (r^{ss}, r_1]$ for all t if $r_1 > r^{ss}$. Furthermore, $f_t(r_1)$ converges to r^{ss} as t tends to infinity. Since the set of references r that satisfy $g(r) \geq 0$ is convex, a necessary and sufficient condition for $g(r) \geq 0$ for all $r \in \{r_1\} \cup \{f_t(r_1)\}_{t=1}^{\infty}$ is $g(r_1) \geq 0$ and $g(r^{ss}) \geq 0$. That is, the LR monopolist profit is a sSPE outcome if and only if there is no incentive to deviate from the optimal LR monopolist policy at the initial reference price and the steady state. Then if a threshold $\bar{r} \in [r^{ss}, v)$ with $g(\bar{r}) = 0$ exists, we have $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ for all $r_1 \in [0, \bar{r}]$.

To see when this interior threshold exists, first note $g(r)$ is continuous and strictly increasing with respect to δ for any $r \in [0, v]$. This implies that the value $\bar{r}$ that yields $g(\bar{r}) = 0$ is strictly increasing in δ. Next, observe that $\delta \leq \frac{1}{2}$ implies $g(r^{ss}) < 0$:

$$V(r^{ss}) = \pi(r^{ss}|r^{ss}) + \delta V(r^{ss}) < \pi(p^m(r^{ss})|r^{ss}) + \delta V(r^{ss})$$

The equality follows the definition of the steady state. The inequality holds by the definition of the SR monopoly price $p^m(r^{ss})$. So if $\delta \leq \frac{1}{2}$:

$$V(r^{ss}) < \frac{\pi(p^m(r^{ss})|r^{ss})}{1-\delta} \leq 2\pi(p^m(r^{ss})|r^{ss}) \Rightarrow g(r^{ss}) < 0$$

Furthermore, we have $\lim_{\delta \to 1} g(r) = \infty$ for all $r \in [0, v]$. This, together with $g(r^{ss}) < 0$ for all $\delta \leq \frac{1}{2}$ implies that there exists a unique $\underline{\delta} \in (\frac{1}{2}, 1)$ such that $g(r^{ss}) \geq 0$ if and only if $\delta \geq \underline{\delta}$. If

$\delta < \underline{\delta}$, then $g(r^{ss}) < 0$ and the LR monopolist policy is not sSPE from any initial reference r_1.

Since $g(r)$ is strictly increasing in δ and tends to infinity for all r as δ tends to one, we know that there is a unique $\bar{\delta} \in (\underline{\delta}, 1)$ that yields $g(v) = 0$ when $\delta = \bar{\delta}$. If $\delta \geq \bar{\delta}$, we have $g(r) \geq 0$ for all $r \in [0, v]$ and thus $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ for all $r_1 \in [0, v]$.

Finally if $\delta \in [\underline{\delta}, \bar{\delta})$, there is a unique $\bar{r} \in [r^{ss}, v)$ that satisfies $g(\bar{r}) = 0$. In that case, since $g(r_1) \geq 0$ if and only if $r_1 \leq \bar{r}$, we have $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ for all $r_1 \in [0, \bar{r}]$.

Highest sSPE Payoff when $\bar{w}(r_1, \delta) < \frac{V(r_1)}{2}$:

As shown above, the LR monopolist policy is not sSPE when $\delta \in [\underline{\delta}, \bar{\delta})$ and $r_1 > \bar{r}(\delta)$[8]. Here, we pin down $\bar{w}(r_1, \delta)$ under such parameter values.

First, recall that when $r_1 > \bar{r}$, we have $\frac{V(r_1)}{2} < \pi(p^m(r_1)|r_1)$. By construction, $\frac{V(r_1)}{2}$ is the highest per firm payoff that is feasible under symmetric strategies. Then for any feasible payoff $w \in [0, \frac{V(r_1)}{2}]$, we have $w < \pi(p^m(r_1)|r_1)$. So if a price sequence is such that it allows deviation to $p^m(r_1)$ in the initial round, it cannot be a sSPE outcome. In Period 1, any price sequence that is a sSPE path must set a price p lower the SR monopoly price $p^m(r_1)$.

We determine $\bar{w}(r_1, \delta)$ separately for two regions of $r_1 > \bar{r}$. First we look at r_1 with $r_1 > \bar{r} \geq p^m(r_1)$. That is, the LR monopolist policy cannot be sustained starting from r_1, but it can be sustained starting from any reference lower than the SR monopolist price induced by reference r_1. Second, we extend the result for this region to values of r_1 with $r_1 > p^m(r_1) > \bar{r}$.

Note that we have $r > p^m(r) = \frac{v+\lambda r}{2(1+\lambda)}$ for any $r > \frac{v}{2+\lambda}$ and $\bar{r} \geq r^{ss} = \frac{v}{2+\lambda(1-\delta)} > \frac{v}{2+\lambda}$. So if r_1 is greater than $\bar{r}$, it is also greater than $p^m(r_1)$. Thus, the two regions above (r_1 such that $r_1 > \bar{r} \geq p^m(r)$ and r_1 such that $r_1 > p^m(r_1) > \bar{r}$) span $(\bar{r}, v]$.

Region 1: r_1 such that $r_1 > \bar{r} \geq p^m(r_1)$

Under any sSPE price path, the price p in the initial round has to be below the SR monopoly price, which in this region is lower than $\bar{r}$. Thus, a necessary condition for a price path to be sSPE path is $p < \bar{r}$ in the initial round. Since $r_2 = p < \bar{r}$, the highest sSPE continuation payoff (in the

[8]In the remainder of this proof, we suppress the input value of the function $\bar{r}(\delta)$. That is, $\bar{r}$ corresponds to the value of the function $\bar{r}(\delta)$ evaluated at a fixed $\delta \in [\underline{\delta}, \bar{\delta})$

game starting from Period 2) upon setting p is $\frac{V(p)}{2}$. As $p < p^m(r_1)$, the optimal Period 1 deviation is to undercut p by an arbitrarily small amount. So under grim trigger strategies, the Period 1 incentive constraint for a price path that initially sets price p and yields the highest sSPE payoff in the game starting in Period 2 is given by

$$\frac{\pi(p|r_1) + \delta V(p)}{2} \geq \pi(p|r_1) \Leftrightarrow h(p|r_1) := \delta V(p) - \pi(p|r) \geq 0$$

where $p \in [0, p^m(r_1))$. Since the left-hand side of the first inequality is the equilibrium path payoff, and is strictly increasing in p at any $p \in [0, p^m(r_1))$, the highest symmetric sSPE payoff is obtained by setting $p^*(r_1) := \max\{p \in [0, p^m(r_1)) : h(p|r_1) \geq 0\}$ in the initial stage, and following the LR monopolist policy $\{f_t(p^*(r_1))\}_{t=1}^{\infty}$ from the second period onward.

That is, the highest sSPE payoff under $\delta \in [\underline{\delta}, \bar{\delta})$ and $r_1 > \bar{r}(\delta) \geq p^m(r_1)$ is given by:

$$\bar{w}(r_1, \delta) = \frac{\pi(p^*(r_1)|r_1) + \delta V(p^*(r_1))}{2}$$

with $p^*(r_1)$ as defined above. Next, we show that in the region $p \in [0, p^m(r_1))$, function $h(p|r_1)$ crosses zero exactly once, and it is from above. That is, for each r_1 with $r_1 > \bar{r} \geq p^m(r_1)$, there exists a unique $p^*(r_1) \in [0, p^m(r_1))$ that satisfies $h(p^*(r_1)|r_1) = 0$, $h(p|r_1) > 0$ for all $p \in [0, p^*(r_1))$ and $h(p|r_1) < 0$ for all $p \in (p^*(r_1), p^m(r_1))$.

Since both $V(p)$ and $\pi(p|r_1)$ are continuously differentaible (Lemma $1(d)$), $h(p|r_1)$ is continuously differentiable in p at any $p \in (0, p^m(r_1))$. Furthermore, for all $r_1 \in [0, v]$, we have:

$$\delta V(0) = \delta\pi(f(0)|0) + \delta^2 V(f(0)) > 0 = \pi(0|r_1) \Rightarrow h(0|r_1) > 0$$

Next, note that for all $r_1 > \bar{r}$:

$$\begin{aligned} \frac{\pi(p^m(r_1)|r_1) + \delta V(p^m(r_1))}{2} &\leq \frac{V(r_1)}{2} < \pi(p^m(r_1)|r_1) \\ &\Rightarrow \delta V(p^m(r_1)) < \pi(p^m(r_1)|r_1) \\ &\Rightarrow h(p^m(r_1)|r_1) < 0 \end{aligned} \tag{10}$$

The first inequality holds because under r_1, both sides are feasible and the right-hand side is the highest feasible payoff by construction. The second inequality holds because $r_1 > \bar{r}$ implies

$g(r_1) = V(r_1) - 2\pi(p^m(r_1)|r_1) < 0$ by definition of $\bar{r}$. Since $h(p|r_1)$ is continuous and we have $h(0|r_1) > 0$ and $h(p^m(r_1)|r_1) < 0$, we know that there is at least one $p \in (0, p^m(r_1))$ that yields $h(p|r_1) = 0$.

Differentiating $h(p|r_1)$ with respect to p, we obtain

$$\frac{dh(p|r_1)}{dp} = \delta\frac{dV(p)}{dp} - \frac{d\pi(p|r_1)}{dp} = \delta\lambda\left(f(p) + \frac{\mu(p)}{1+\lambda}\right) - \frac{d\pi(p|r_1)}{dp}$$

where as before, $\mu(p)$ is the shadow value of the choke price constraint ($r' \leq p^c(r)$) under state p and optimal policy $f(p)$. Recall that $\pi(p|r_1)$ is strictly concave in p at all $p \in (0, p^m(r_1))$. If the choke price constraint is non-binding in the LR problem under state p, we have $\mu(p) = 0$ and $\delta\frac{dV(p)}{dp} = \delta\lambda f(p)$, which is strictly increasing. As a result, $h(p|r_1)$ is strictly convex whenever the choke price constraint $r' \leq p^c(r)$ is non-binding under state $r = p$ in the LR monopolist problem. By Lemma 1(c), this implies that if the continuously differentiable function $h(p|r_1)$ is convex at some $p = p_1$, it is also convex at any greater value $p = p_2 > p_1$. As a result, $h(p|r_1)$ can cut zero from above for at most one p. This, together with the two observations $h(0|r_1) > 0$ and $h(p^m(r_1)|r_1) < 0$ implies that there exists exactly one value $p^*(r_1)$ in $(0, p^m(r_1))$ that satisfies $h(p^*(r_1)|r_1) = 0$, and it yields $h(p|r_1) > 0$ for all $p \in [0, p^*(r_1))$ and $h(p|r_1) < 0$ for all $p \in (p^*(r_1), p^m(r_1)]$.

Thus, the highest Period 1 price p in $[0, p^m(r_1))$ that satisfies incentive constraint $h(p|r_1) \geq 0$ is the unique value $p^*(r_1)$ that solves $\pi(p^*(r_1)|r_1) = \delta V(p^*(r_1))$. Plugging this equality into the lifetime profit, we obtain

$$\bar{w}(r_1, \delta) = \frac{\pi(p^*(r_1)|r_1) + \delta V(p^*(r_1))}{2} = \delta V(p^*(r_1))$$

Region 2: r_1 such that $r_1 > p^m(r_1) > \bar{r}$

We extend the above result to all $r_1 > \bar{r}$. As shown above, this corresponds to deriving $\bar{w}(r_1, \delta)$ for r_1 such that $r_1 > p^m(r_1) > \bar{r}$. We show by induction that if $p^m(r_1) > \bar{r}$, then $\bar{w}(r_1, \delta) = \delta V(p^*(r_1))$ where $p^*(r_1)$ is the unique value in $(0, \bar{r})$ that solves $\delta V(p^*(r_1)) = \pi(p^*(r_1)|r_1)$

Denote by $p_n^m(r_1)$ the n^{th} iterate of the SR monopolist price function on reference r_1. Note that $\lim_{n\to\infty} p_n^m(r_1) = \frac{v}{2+\lambda} < r^{ss} \leq \bar{r}$ for all $r_1 \in [0, v]$. Then for any $r_1 \in (\bar{r}, v]$, there exists a lowest $\tilde{n} \in \mathbb{N}$ such that $p_n^m(r) \leq \bar{r}$ for all $n > \tilde{n}$.

Base Case: Let $\tilde{n} = 1$. That is, $p^m(r_1) > \bar{r} \geq p^m(p^m(r_1))$. As before, since no symmetric initial price above $p^m(r_1)$ can deter deviation, we only need to look at initial prices $p \in [0, p^m(r_1))$. Given $p^m(r_1) > \bar{r}$ we can divide this interval in two: $p \in [0, \bar{r}]$ and $p \in (\bar{r}, p^m(r_1))$. The first period incentive constraint for setting $p \in [0, \bar{r}]$ is as before. When initially setting $p \in (\bar{r}, p^m(r_1))$, the highest sSPE continuation payoff in the game starting tomorrow is $\delta V(p^*(p))$ with $p^*(p) \in (0, p^m(p))$. This is because we have $p^m(p) < p^m(p^m(r_1)) \leq \bar{r}$ and thus, any $p \in (\bar{r}, p^m(r_1))$ as an initial reference is in "Region 1" discussed above.

We can write the Period 1 incentive constraints under the highest sSPE continuation payoffs when setting initial price $p \in [0, p^m(r_1))$ as:

$$h(p|r_1) = \delta V(p) - \pi(p|r_1) \geq 0 \ \text{ if } \ p \in [0, \bar{r}]$$

$$\frac{\pi(p|r_1)}{2} + \delta \bar{w}(p, \delta) \geq \pi(p|r_1) \Leftrightarrow \delta^2 V(p^*(p)) - \frac{\pi(p|r_1)}{2} \geq 0 \ \text{ if } \ p \in (\bar{r}, p^m(r_1)) \tag{11}$$

Recall from the proof for Region 1 that for all $r_1 > \bar{r}$, there is a unique $p^*(r_1) \in (0, p^m(r_1))$ that satisfies $h(p^*(r_1)|r_1) = 0$ with $h(p|r_1) > 0$ for all $p \in [0, p^*(r_1))$ and $h(p|r_1) < 0$ for all $p \in (p^*(r_1), p^m(r_1)]$.

Next, we show that if $\bar{r} < p^m(r_1)$, we have $h(\bar{r}|r_1) < 0$: Let $\tilde{r} \in (\bar{r}, v)$ be given by $p^m(\tilde{r}) = \bar{r}$. By Inequality (10), we know that $h(\bar{r}|\tilde{r}) < 0$. Then since $\pi(\bar{r}|r)$ is strictly increasing in r, we have $\delta V(\bar{r}) < \pi(\bar{r}|\tilde{r}) < \pi(\bar{r}|r_1)$ for all $r_1 > \tilde{r}$. Thus, $h(\bar{r}|r_1) = \delta V(\bar{r}) - \pi(\bar{r}|r_1) < 0$ for all r_1 such that $p^m(r_1) > \bar{r}$.

For any r_1 with $p^m(r_1) > \bar{r}$, we now know that $p^*(r_1) \in (0, \bar{r})$ and $h(p|r_1) < 0$ for all $p \in (\bar{r}, p^m(r_1))$. We can use this observation to show that the incentive constraint for setting initial price $p \in (\bar{r}, p^m(r_1))$ (i.e. Inequality (11)) can never hold: For any $p \in (\bar{r}, p^m(r_1))$, we have:

$$\frac{\pi(p|r_1)}{2} > \frac{\delta V(p)}{2} > \delta^2 V(p^*(p))$$

which contradicts Inequality (11). The first inequality follows from $h(p|r_1) = \delta V(p) - \pi(p|r_1) < 0$ for all $p \in (\bar{r}, p^m(r_1))$. The second inequality holds because under initial reference p, $\frac{V(p)}{2}$ is the highest feasible payoff and $\delta V(p^*(p))$ is the highest sSPE payoff (since p is in Region 1).

So if the firms are initially setting $p \in (\bar{r}, p^m(r_1))$ the Period 1 incentive constraint is violated even under the highest sSPE continuation payoff. Thus, there is no symmetric sSPE following $p \in (\bar{r}, p^m(r_1))$ that is good enough to deter the first period deviation available. Then any sSPE strategy must initially set price $p \in [0, \bar{r}]$. In this interval, the highest price that satisfies the incentive constraint $h(p|r_1) \geq 0$ is the unique value $p^*(r_1)$ that solves $\delta V(p^*(r_1)) = \pi(p^*(r_1)|r_1)$. Thus, we have $w(r_1, \delta) = \delta V(p^*(r_1))$ with $p^*(r_1) \in (0, \bar{r})$ for all r_1 with $p^m(r_1) > \bar{r} \geq p^m(p^m(r_1))$.

Inductive Step: Suppose we have some $\tilde{n} \in \mathbb{N}_+$ such that for any $n \in \{1, ..., \tilde{n}\}$, $\bar{r} \in [p^m_{n+1}(r_1), p^m_n(r_1))$ implies $\bar{w}(r_1, \delta) = \delta V(p^*(r_1))$ where $p^*(r_1) \in (0, \bar{r})$ is defined as before.

Now let initial reference $r_1 \in (\bar{r}, v]$ be such that $\bar{r} \in [p^m_{\tilde{n}+2}(r_1), p^m_{\tilde{n}+1}(r_1))$. We need to show that this implies $\bar{w}(r_1, \delta) = \delta V(p^*(r_1))$. The first period incentive constraint under the highest sSPE continuation payoff for initial price $p \in [0, \bar{r}]$ is once again $h(p|r_1) \geq 0$. For any $p \in (\bar{r}, p^m(r_1))$ we have $\bar{r} \in [p^m_{n+1}(p), p^m_n(p))$ for some $n \in \{1, ..., \tilde{n}\}$. Therefore by our induction hypothesis we have $\bar{w}(p, \delta) = \delta V(p^*(p))$. The Period 1 incentive constraint for setting initial price $p \in (\bar{r}, p^m(r_1))$ under the highest sSPE continuation payoff is then given by

$$\delta^2 V(p^*(p)) \geq \frac{\pi(p|r_1)}{2}$$

which is identical to Inequality (11) and is shown in the base case to be violated for all $p \in (\bar{r}, p^m(r_1))$. Therefore in any sSPE, the initial price must be in the interval $[0, \bar{r}]$. As we have shown above, the highest sSPE payoff in this interval is obtained by setting the unique $p^*(r_1) \in (0, \bar{r})$ that satisfies $\pi(p^*(r_1)|r_1) = \delta V(p^*(r_1))$ which yields $\bar{w}(r_1, \delta) = \delta V(p^*(r_1))$.

We have shown the induction hypothesis to be true for $\tilde{n} = 1$. Thus we have $\bar{w}(r_1, \delta) = \delta V(p^*(r_1))$ with $p^*(r_1) \in (0, \bar{r})$ for any initial reference r_1 with $\bar{r} \in [p^m_{n+1}(r_1), p^m_n(r_1))$ for some $n \in \mathbb{N}_+$. The values of r_1 that satisfy $\bar{r} \in [p^m_{n+1}(r_1), p^m_n(r_1))$ for some $n \in \mathbb{N}_+$ span the entire Region 2 (r_1 such that $r_1 > p^m(r_1) > \bar{r}$).

Combining the results for Region 1 and Region 2, we can summarize the highest sSPE payoffs as follows: For $\delta \in [\underline{\delta}, \bar{\delta})$ and $r_1 > r(\bar{\delta})$, we have $\bar{w}(r_1, \delta) = \delta V(p^*(r_1))$ with $p^*(r_1)$ the unique value in the interval $(0, \min\{p^m(r_1), \bar{r}(\delta)\})$ that solves $\delta V(p^*(r_1)) = \pi(p^*(r_1)|r_1)$.

$\square$