Instability of Factor Strength in Asset Returns*

Daniele Massacci

King's College London, University of Naples Federico II & CSEF

April 10, 2024

Abstract

We study the problem of detecting structural instability of factor strength in asset pricing models for financial returns with observable factors. We allow for strong and weaker factors, in which the sum of squared betas grows at a rate equal to and slower than the number of test assets, respectively: this growth rate determines the strength of the corresponding factor. We propose LM and Wald statistics for the null hypothesis of stability and derive their asymptotic distribution when the break fraction is known, as well as when it is unknown and has to be estimated. We corroborate our theoretical results through a comprehensive series of Monte Carlo experiments. An extensive empirical analysis uncovers the dynamics of instability of factor strength in financial returns from equity portfolios.

JEL classification: C12, C33, C58, G10, G12.

Keywords: Factor strength, structural break, hypothesis testing, stock portfolios.

*I really am highly indebted to Hashem Pesaran for several invaluable comments and suggestions. The paper also greatly benefits from conversations with George Kapetanios. Comments from participants at the IAAE 2022 Annual Conference, the 14th Annual SoFiE Conference, and the 33rd EC^2 Conference are gratefully acknowledged. Errors and omissions are my own responsibility only. Please address correspondence to Daniele Massacci, King's Business School, Banking and Finance Department, Bush House, 30 Aldwych, London, WC2B 4BG, United Kingdom. Email: daniele.massacci@kcl.ac.uk.

1 Introduction

Financial asset returns exhibit a factor structure, as a handful of common factors drives their cross-sectional dependence.[1] This empirical evidence has generated a large number of contributions on factor models in asset pricing: see Giglio et al. (2021) for an overview of the literature. In estimating asset pricing models, it has been common to assume that all factors are strong, meaning that they are pervasive and influence almost all securities: see Fama and MacBeth (1973), and Shanken (1992). The assumption of strong factor structure may be restrictive in practice, as some of the factors may not be strong and do not actually drive the cross-section of all securities: Kan and Zhang (1999), Kleibergen (2009), Bryzgalova (2016), Burnside (2016), Gospodinov et al. (2017), and Anatolyev and Mikusheva (2021), study this scenario when factors are known and observable; in the spirit of Connor and Korajczyk (1986), Lettau and Pelger (2020), Bai and Ng (2021), Freyaldenhoven (2021), Giglio et al. (2021), and Uematsu and Yamagata (2023a,b) consider specifications in which all factors are latent and estimated.

We focus on observable factors. We follow Chudik et al. (2011) and define the strength of a factor based on how the sum of squared betas grows with the number of test assets N. We classify a factor as being strong, semi-strong or weak, depending on whether the sum of squared betas grows at a rate equal to N, between $N^{1/2}$ (excluded) and N (excluded), or less than or equal to $N^{1/2}$, respectively. Bailey et al. (2021), Connor and Korajczyk (2022), and Pesaran and Smith (2021a,b), employ the same classification scheme. Bailey et al. (2021) develops an estimator for the factor strength that is based on the fraction of statistically significant betas and takes into account the associated multiple testing problem. Pesaran and Smith (2021b) show that the convergence rate of the Fama and MacBeth (1973) two-pass estimator depends on pricing errors and factors strength, and thus an estimation of the latter is required.

To the very best of our knowledge, existing studies that allow for semi-strong and weak factors assume that the factor strength is stable over the estimation period. This assumption may not be supported by the data. Bailey et al. (2021), and Pesaran and Smith (2021b), document time-variation in factor strength in large cross-sections of equity returns over rolling estimation windows. In particular, as discussed in Bailey et al. (2021), changes in factor strength may be

[1] For example, see Litterman and Scheinkman (1991), Fama and French (1993), and Lustig et al. (2011), in relation to U.S. government bonds, equity returns, and exchange rates, respectively

associated to sizeable financial events such as crisis periods, examples being the burst of the dot-com bubble or the Global Financial Crisis. Based on the results in Pesaran and Smith (2021b), detecting breaks in factor strength is important as these breaks may affect the convergence rate of two-pass estimators.

This paper fills a gap in the literature by addressing the problem of instability of factor strength in asset pricing models. It introduces a general testing strategy for the null hypothesis of strength stability. We build LM and Wald-type test statistics based on the difference between the estimator for the factor strength before and after the break. They differ in their variance being estimated under the null and under the alternative, respectively. We derive their asymptotic distribution under the null and show that it is normal. Under the alternative, both statistics asymptotically diverge. Our results are corroborated by an extensive set of Monte Carlo simulations, which shows the good performance of our tests in finite samples.

We stress that we focus on instability in the factor *strength* and not in the factor *betas*. Strength instability can only occur if the corresponding betas experience a break, and betas instability is a *necessary* condition for strength instability. This has implications for deriving the asymptotic distribution of our test statistics under the null. In particular, our proposed test statistics do not suffer from the problem of a nuisance parameter being identified only under the alternative: see Davies (1977, 1987). Stability of factor strength is tested after a break in the betas is detected and the break fraction is identified both under the null and the alternative.

Finally, we illustrate the usefulness of our procedure for empirical work through an analysis of equity portfolios.[2] We consider a large set of 739 portfolios from Chen and Zimmermann (2021). We set up a factor model for the cross-sectional variation of returns and apply our testing procedure using rolling estimation windows of suitable length. Our results shed light on the dynamics of local instability of factor strength over time for the set of test assets and the factor model specification we consider. From an asset pricing perspective, they imply that stability of factor strength may not be a realistic assumption for empirical purposes. Strength instability should be accounted for when running inference on risk premia in order to avoid potentially misleading inferential results.

The rest of the paper is organized as follows. Section 2 sets up the problem. Section 3

[2]The data used in the empirical analysis are described in details in Section 5.1

introduces the tests. Section 4 runs a set of Monte Carlo experiments. Section 5 performs the empirical analysis. Section 6 concludes. Mathematical proofs are provided in Appendix A.

Notation: $\mathbb{I}(\cdot)$ denotes the indicator function; $\lfloor \cdot \rfloor$ is the integer part of the argument; given a positive integer A, $\boldsymbol{\iota}_A$ is the $A \times 1$ vector of ones; $|\cdot|$ is the absolute value of the argument; $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution, and $\Phi^{-1}(\cdot)$ is its inverse; $\xrightarrow{d}$ denotes convergence in distribution; $\text{vec}(\mathbf{A})$ denotes the vectorization of the matrix $\mathbf{A}$; the norm of a generic matrix $\mathbf{A}$ is $\|\mathbf{A}\| = [\text{tr}(\mathbf{A}'\mathbf{A})]^{1/2}$, where $\text{tr}(\mathbf{B})$ denotes the trace of a square matrix $\mathbf{B}$; $\xrightarrow{a.s.}$ denotes almost sure convergence.

2 Set up

2.1 Econometric model

We assume that asset (excess) returns are generated according to

$$R_{it} = \mathbb{I}(t/T \leq \tau)(\alpha_{1i} + \boldsymbol{\beta}'_{1i}\mathbf{f}_t) + \mathbb{I}(t/T > \tau)(\alpha_{2i} + \boldsymbol{\beta}'_{2i}\mathbf{f}_t) + e_{it}, \tag{1}$$

for $i = 1, \ldots, N$, and $t = 1, \ldots, T$, where N is the total number of assets, and T is the time series dimension: R_{it} is the return on asset i at time t; $0 < \tau < 1$ is the break fraction, which can be either known or unknown; α_{ji} is the asset-specific intercept, for $j = 1, 2$; $\boldsymbol{\beta}_{ji} = (\beta_{ji1}, \ldots, \beta_{jiK})'$ is the $K \times 1$ vector of regression betas, for $j = 1, 2$; $\mathbf{f}_t = (f_{1t}, \ldots, f_{Kt})'$ is the $K \times 1$ vector of observable traded factors; e_{it} is the idiosyncratic component for return i at time t.[3] We further assume that the cross-sectional dispersion of regression betas evolves according to

$$\begin{array}{ll} \beta_{jik} \neq 0, & i = 1, \ldots, \lfloor N^{\lambda_{jk}} \rfloor, \\ \beta_{jik} = 0, & i = \lfloor N^{\lambda_{jk}} \rfloor + 1, \ldots, N, \end{array}, \quad 0 \leq \lambda_{jk} \leq 1, \quad j = 1, 2, \quad k = 1, \ldots, K, \tag{2}$$

where the ordering of the betas is for ease of exposition only and it is not required for the validity of our results, as it becomes clear in the condition in (4) below.

[3] We focus on the case in which the factors in $\mathbf{f}_t$ are all traded. If some of the factors in $\mathbf{f}_t$ are not returns themselves, following Breeden (1979) we conjecture that our results can be extended using a "mimicking-portfolio" approach. A similar idea underlies the model comparison tests of Barillas et al. (2020). We aim at formalising this interesting extension in future work.

We are interested in the null hypothesis $\mathcal{H}_{0k}$ against the alternative $\mathcal{H}_{1k}$ defined as

$$\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k}, \qquad \mathcal{H}_{1k} : \lambda_{1k} \neq \lambda_{2k}, \qquad k \in \{1, \ldots, K\} : \tag{3}$$

for any k such that $k \in \{1, \ldots, K\}$, λ_{1k} is equal to λ_{2k} under the null hypothesis, whereas λ_{1k} and λ_{2k} are different from each other under the alternative hypothesis. From (3), we can see that our framework is analogous to Bai and Perron (1998), and Qu and Perron (2007), in that we model a break as a discrete change in the parameters of interest.

Although formal assumptions are introduced in Sections 3.2.2 and 3.3, it is worth discussing some of the features of the model in (1). For $j = 1, 2$ an $i = 1, \ldots, N$, we do not impose any restriction on the structure of α_{ji}, which, in the spirit of Pesaran and Smith (2023), is allowed to depend on a spanning error related to some common factors and on a security-specific idiosyncratic pricing error. The factor structure is described in (2), and the strength of factor k in regime j is determined by λ_{jk}. Finally, for ease of tractability, the idiosyncratic component e_{it} is assumed to be distributed independently along the cross-section. This implies that missing factors that display weak cross-sectional dependence are not allowed, as instead advocated in Pesaran and Smith (2023): missing factors could be allowed for at the expense of higher mathematical complexity.

It is also worth discussing the panel dimensions N and T. In particular, we assume that $N \to \infty$ and $T \to \infty$, as imposed in Theorem 3.1 and Theorem 3.2 below: $T \to \infty$ is required to consistently estimate the betas; $N \to \infty$ is needed to then conduct inference on the factor strength. This is the same set up as in Theorem 1 in Bailey et al. (2021) and does not require any restriction on the relative speed of convergence of N and T to infinity.

From an econometric perspective, (1) describes a factor model subject to structural instability occurring at the break fraction τ. The evolution of regression betas in (2) determines the strength of the factors before and after the break. In particular, the strength of the $k - th$ factor within regime j is determined by the parameter λ_{jk}, for $j = 1, 2$, and $k = 1, \ldots, K$. Following Chudik et al. (2011), and Pesaran and Smith (2021a,b), we classify the $k - th$ factor within regime j as strong, semi-strong, and weak, depending on whether $\lambda_{jk} = 1$, $0.5 < \lambda_{jk} < 1$, and $0 \leq \lambda_{jk} \leq 0.5$, respectively. Connor and Korajczyk (2022) use a similar classification. The role played by the factor strength within our testing procedure is discussed in details in Section 3.2.1. Finally, the condition on the cross-sectional dispersion of the betas in (2) may be written more generally as

$$N^{-\lambda_{jk}} \sum_{i=1}^{N} \beta_{jik}^2 \to C_{jk}, \quad 0 < C_{jk} < \infty, \quad j = 1, 2, \quad k = 1, \ldots, K, \tag{4}$$

as $N \to \infty$, which states that the sum of squared betas for factor k within regime j grows at rate $N^{\lambda_{jk}}$: this extends the analogous condition given for linear asset pricing models in Pesaran and Smith (2021a,b) and employed in Connor and Korajczyk (2022).

2.2 Interpretation of instability in factor strength

The null and alternative hypotheses $\mathcal{H}_{0k}$ and $\mathcal{H}_{1k}$, respectively, in (3) deserve further considerations. In particular, they do not refer to the regression betas in (1), but to the parameters λ_{jk} that govern the strength of the factors within each regime. In other words, the null and the alternative hypotheses in (3) relate to the stability of the factor strength and not to the stability of the regression betas. The two concepts are distinct although related. In a system of equations with observable factors such as (1), the stability of the regression betas may be assessed through the procedure developed in Qu and Perron (2007) for systems of equations, which suitably extends the seminal work by Bai and Perron (1998) for single equation models. Clearly, instability in the betas is a *necessary* condition for instability in the factor strength. Therefore, a break in the factor strength can occur only conditional upon a break in the factor betas: we explore this intuition in Section 3.3, where we let the break fraction τ be unknown. On the other hand, instability in the factor strength is a *sufficient* condition for instability in the factor betas. Therefore, this paper effectively proposes a two-step procedure: the first step requires inference on the stability of the betas and estimation of the break fraction τ, as studied in Qu and Perron (2007); the second step is the actual inference on the factor strength stability, which is the focus of this paper.

Structural instability in the betas in (1) relates our set up to latent factor models with structural breaks and, more generally, with discrete shifts in the loadings: see Barigozzi and Massacci (2022), and Massacci (2017, 2023), and references therein. To the very best of our knowledge, this literature has mainly worked under the maintained assumption that all latent factors are strong. We explicitly focus on the model in (1) with observable factors.

Finally, the model in (1) has one break fraction. We handle the case of multiple breaks in two ways. We let T be the whole time series dimension and estimate the multiple break fractions using a sequential algorithm, as outlined in Section 3.4 below. Alternatively, (1) may be seen as a local model that applies to a window of length T strictly shorter than the whole available time series. This second approach allows to test the null hypothesis of *local stability*: under this null hypothesis, the factor strength is stable within the time interval T; however, this does not imply that it is globally stable within the whole available time period, which is strictly greater than T. The notion of local (as opposed to global) stability is not new. For example, in an out-of-sample framework, Timmermann (2008) documents the existence of short spells of time in which stock returns are predictable; and Giacomini and Rossi (2010) develop a measure of local relative forecasting performance between two competing predictive models, and assess the stability of this measure through a suitable inferential procedure. As further discussed in Section 5.2, inference on local stability is consistent with existing studies, which document a high degree of time-variation in the factor strength by using rolling window estimation strategies: see Bailey et al. (2021), and Pesaran and Smith (2021a). Therefore, the results in Theorem 3.1 and Theorem 3.2 below, which refer to the model with one break in (1), can be interpreted in light of the null hypothesis of local stability.

2.3 Asset pricing implications

Define $\mathbf{R}_t = (R_{1t}, \ldots, R_{Nt})'$, $\boldsymbol{\alpha}_j = (\alpha_{j1}, \ldots, \alpha_{jN})'$, $\mathbf{B}_j = (\boldsymbol{\beta}_{j1}, \ldots, \boldsymbol{\beta}_{jN})'$, and $\mathbf{e}_t = (e_{1t}, \ldots, e_{Nt})'$, for $j = 1, 2$. The model in (1) can then be written as

$$\mathbf{R}_t = \mathbb{I}\left(t/T \leq \tau\right)\left(\boldsymbol{\alpha}_1 + \mathbf{B}_1 \mathbf{f}_t\right) + \mathbb{I}\left(t/T > \tau\right)\left(\boldsymbol{\alpha}_2 + \mathbf{B}_2 \mathbf{f}_t\right) + \mathbf{e}_t. \tag{5}$$

Let $\boldsymbol{\Gamma}_j = \left(\gamma_{j0}, \boldsymbol{\gamma}_{j1}'\right)'$, where γ_{j0} is the zero-beta rate, and $\boldsymbol{\gamma}_{j1}$ is the $K \times 1$ vector of factor risk premia, for $j = 1, 2$. Define as $\mathbf{X}_j = (\boldsymbol{\iota}_N, \mathbf{B}_j)$ the $N \times (K+1)$ beta matrix augmented by the $N \times 1$ vector of ones $\boldsymbol{\iota}_N$.

Consider first the case when the number of test assets N is finite. Since $\mathbf{f}_t$ is a vector of traded factors, then $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are vectors of pricing errors. Therefore, under the assumption of exact pricing (correct model specification), it holds that $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 = \mathbf{0}$. In this case, the vector of asset expected returns $\boldsymbol{\mu}_t$ is state-dependent and defined as

$$\boldsymbol{\mu}_t = \mathrm{E}\left(\mathbf{R}_t\right) = \mathbb{I}\left(t\,/T \leq \tau\right)\mathbf{X}_1\boldsymbol{\Gamma}_1 + \mathbb{I}\left(t\,/T > \tau\right)\mathbf{X}_2\boldsymbol{\Gamma}_2. \tag{6}$$

Under correct model specification, the model in (1) allows for structural instability in the quantity and in the price of risk, as measured by $\mathbf{X}_j$ and $\boldsymbol{\Gamma}_j$, respectively, for $j = 1, 2$. The model in (6) can be estimated from the multivariate regression model in (5): inference on the structural break and estimation of the break fraction τ first has to be conducted following for example the procedure of Qu and Perron (2007); $\mathbf{B}_j$ and $\boldsymbol{\Gamma}_j$ can then be estimated using the standard Fama and MacBeth (1973) two step procedure within each regime $j = 1, 2$. This implies that a piecewise linear asset pricing model has to be estimated whenever a break in the betas occurs. The piecewise specification for the risk premia in (6) is also consistent with a model with time-varying betas, such as the one implicitly employed for the empirical analysis in Section (5). The model in (6) can then be thought as being valid for a time window of length T that is a fraction of the whole available time series: in this case, the null hypothesis $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k}$ in (3) is consistent with the idea of local stability discussed in Section 2.2.

In linear asset pricing models, Pesaran and Smith (2021b) show that it is still possible to estimate the risk premia in the presence of non-zero pricing errors as $N \to \infty$. For a given factor, Pesaran and Smith (2021b) show that the rate of convergence as $N \to \infty$ of the Fama and MacBeth (1973) two-pass estimator for the risk premium monotonically increases and decreases in the strength of the factor and of the pricing errors, respectively. The estimator for the risk premia is consistent if the strength of the factor is greater than the strength of the pricing errors, and the convergence rate is slower the smaller the difference between the two.

For a given sample split induced by structural instability at the break fraction τ, our setting can be cast within the framework of Pesaran and Smith (2021b). Therefore, the result shown for linear asset pricing models in Pesaran and Smith (2021b) holds within each regime of the piecewise linear model in (1) in relation to the risk premia $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ in (6). This is true regardless of whether the break fraction τ is known, or it is unknown and has to be estimated. Following Qu and Perron (2007), and as also discussed in details in Section 3.3, this is because the convergence rate of the least squares estimator for τ is faster than that of the remaining set of parameters in (1). Following Pesaran and Smith (2021b), under the alternative hypothesis $\mathcal{H}_{1k} : \lambda_{1k} \neq \lambda_{2k}$ the convergence rate of the Fama and MacBeth (1973) two-pass estimator for the risk premium of the $k-th$ factor within regime j also experiences a break and becomes equal to $N^{-(\lambda_{jk}-\lambda_{\alpha j})/2}$, where $\lambda_{\alpha j}$ regulates the strength of the pricing errors in regime j. Formally, $\lambda_{\alpha j}$ satisfies

$$N^{-\lambda_{\alpha j}} \sum_{i=1}^{N} \alpha_{ji}^2 \to C_{\alpha j}, \quad 0 < C_{\alpha j} < \infty, \quad j = 1, 2,$$

which means that, within regime j, the sum of squared pricing errors grows at rate $N^{\lambda_{\alpha j}}$ as $N \to \infty$. If the strength of the pricing errors is stable over time (i.e., $\lambda_{\alpha 1} = \lambda_{\alpha 2}$), testing for stability in factor strength gives valuable information about the stability of the convergence rate of the Fama and MacBeth (1973) two-pass estimator. However, the assumption of constant strength of the pricing errors is unlikely to hold in practice, as it implies that misspecification of the factor model is stable even when the strength of the factors is not. For example, if λ_{jk} and $\lambda_{\alpha j}$ vary by the same amount, the convergence rate $N^{-(\lambda_{jk}-\lambda_{\alpha j})/2}$ of the estimator for the corresponding risk premium is unaffected. However, if λ_{jk} and $\lambda_{\alpha j}$ change by different amounts between the two regimes, the break in the factor strength impacts $N^{-(\lambda_{jk}-\lambda_{\alpha j})/2}$. Therefore, detecting breaks in factor strength is generally informative about changes in the convergence rate of the estimator for the corresponding risk premium.

Existing contributions have studied whether cross-sectional risk premia are stable over time. Fama and MacBeth (2021) assess the stability of the value premium by splitting the sample between the period July 1963 – June 1991 and the period July 1991 – June 2019. This is analogous to considering a model like (1) with a known value of the break fraction τ, which corresponds to a break occurring in June 1991. In a Bayesian setting, Smith and Timmermann (2021) study the more general problem of stability in risk premia by allowing for multiple unknown breaks in the data generating process of asset returns. This setting in analogous to the one discussed in Section 3.4 below. To the best of our knowledge, no existing contribution accounts for the role of factor strength in assessing the stability of risk premia. We make a contribution on this respect by formally studying whether factor strength is constant over time. We do so in the empirical analysis in Section 5 by testing for local stability as discussed in Section 2.2.

3 Detecting instability in factor strength

3.1 Estimation of factor strength under structural instability

In order to estimate the factor strength before and after the break, we extend the estimator developed in Bailey et al. (2021) to allow for the piecewise linear setting of our framework. For ease of exposition, we start by assuming that the break fraction τ in (1) is known. Section 3.3 deals with the case in which τ is unknown and has to be estimated.

We consider the multi-factor model in (1), and the null and the alternative hypothesis in (3). Let $\mathbb{I}_{1t}(\tau) = \mathbb{I}(t/T \leq \tau)$, $\mathbb{I}_{2t}(\tau) = \mathbb{I}(t/T > \tau)$, and the matrix $\mathbf{I}_{jT}(\tau)$ be

$$\mathbf{I}_{jT}(\tau) = \begin{bmatrix} \mathbb{I}_{j1}(\tau) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & \mathbb{I}_{jT}(\tau) \end{bmatrix}, \quad j = 1, 2,$$

which is the $T \times T$ diagonal matrix with $t-th$ diagonal element equal to $\mathbb{I}_{jt}(\tau)$. For $k = 1, \ldots, K$, define the $T \times K$ matrix $\mathbf{F}_{j,-k}(\tau)$ as

$$\mathbf{F}_{j,-k}(\tau) = \mathbf{I}_{jT}(\tau)\left(\boldsymbol{\iota}_T, \underline{\mathbf{f}}_1, \ldots, \underline{\mathbf{f}}_{k-1}, \underline{\mathbf{f}}_{k+1}, \underline{\mathbf{f}}_K\right),$$

where $\underline{\mathbf{f}}_k = (f_{k1}, \ldots, f_{kT})'$: the matrix $\mathbf{F}_{j,-k}(\tau)$ collects all but the $k-th$ factor and it is augmented by the $T \times 1$ vector of ones $\boldsymbol{\iota}_T$. Let the $T \times T$ matrix $\mathbf{M}_{jT,-k}(\tau)$ be

$$\mathbf{M}_{jT,-k}(\tau) = \mathbf{I}_{jT}(\tau) - \mathbf{F}_{j,-k}(\tau)\left[\mathbf{F}_{j,-k}(\tau)' \mathbf{F}_{j,-k}(\tau)\right]^{-1} \mathbf{F}_{j,-k}(\tau)',$$

and the $T \times 1$ vector $\underline{\mathbf{f}}_{jkT}(\tau)$ as

$$\underline{\mathbf{f}}_{jkT}(\tau) = \mathbf{M}_{jT,-k}(\tau)\underline{\mathbf{f}}_k = \left[f_{jk1}(\tau), \ldots, f_{jkT}(\tau)\right]'.$$

Given the estimator $\hat{\beta}_{jikT}(\tau)$ for β_{jik} defined as

$$\begin{aligned}\hat{\beta}_{jikT}(\tau) &= \left[\underline{\mathbf{f}}_k' \mathbf{M}_{jT,-k}(\tau)\underline{\mathbf{f}}_k\right]^{-1}\left[\underline{\mathbf{f}}_k' \mathbf{M}_{jT,-k}(\tau)\underline{\mathbf{R}}_i\right] \\ &= \left[\underline{\mathbf{f}}_{jkT}(\tau)'\underline{\mathbf{f}}_{jkT}(\tau)\right]^{-1}\left[\underline{\mathbf{f}}_{jkT}(\tau)'\underline{\mathbf{R}}_i\right],\end{aligned} \tag{7}$$

with $\underline{\mathbf{R}}_i = (R_{i1}, \ldots, R_{iT})'$, the relevant test statistic for the significance of β_{jik} is

$$\hat{t}_{jikT}(\tau) = \frac{\hat{\beta}_{jikT}(\tau)}{\sqrt{\hat{\omega}_{jiT}(\tau)}} = \frac{\left[\underline{\mathbf{f}}_{jkT}(\tau)'\underline{\mathbf{f}}_{jkT}(\tau)\right]^{-1}\left[\underline{\mathbf{f}}_{jkT}(\tau)'\underline{\mathbf{R}}_i\right]}{\sqrt{\hat{\omega}_{jiT}(\tau)}},$$

where

$$\hat{\omega}_{jiT}(\tau) = \frac{\sum_{t=1}^{T_j(\tau)} f_{jkt}(\tau)\hat{e}_{jit}(\tau) f_{jkt}(\tau)\hat{e}_{jit}(\tau)}{T_j(\tau)},$$

$T_j(\tau) = \sum_{t=1}^{T} \mathbb{I}_{jt}(\tau)$ is the number of time series observations within regime j, and

$$\hat{\mathbf{e}}_{jiT}(\tau) = \left[\hat{e}_{ji1}(\tau), \ldots, \hat{e}_{jiT}(\tau)\right]' = \mathbf{M}_{jT,-k}(\tau)\left[\underline{\mathbf{R}}_i - \underline{\mathbf{f}}_k \hat{\beta}_{jikT}(\tau)\right].$$

Therefore, $\hat{\omega}_{jiT}(\tau)$ is the White (1980) estimator, which allows for conditional heteroskedasticity in the error terms e_{it} in line with Assumption 1 below.

For given nominal size of the individual tests p and critical value exponent $\delta > 0$, from Chudik et al. (2018) define the critical value function $c_p(N)$ as

$$c_p(N) = \Phi^{-1}\left(1 - \frac{p}{2N^{\delta}}\right). \tag{8}$$

Following Bailey et al. (2021), the factor strength λ_{jk} is estimated as

$$\hat{\lambda}_{jkNT}(\tau) = \mathbb{I}\left[\hat{\pi}_{jkNT}(\tau) > 0\right]\tilde{\lambda}_{jkNT}(\tau), \tag{9}$$

where

$$\hat{d}_{jikT}(\tau) = \mathbb{I}\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N)\right], \quad \hat{\pi}_{jkNT}(\tau) = \frac{1}{N}\sum_{i=1}^{N}\hat{d}_{jikT}(\tau), \tag{10}$$

with

$$\tilde{\lambda}_{jkNT}(\tau) = 1 + \frac{\ln \hat{\pi}_{jkNT}(\tau)}{\ln N}, \quad \hat{\pi}_{jkNT}(\tau) > 0. \tag{11}$$

From (9), $\hat{\lambda}_{jkNT}(\tau) = 0$ if $\hat{\pi}_{jkNT}(\tau) = 0$, and $\hat{\lambda}_{jkNT}(\tau) = \tilde{\lambda}_{jkNT}(\tau)$ if $\hat{\pi}_{jkNT}(\tau) > 0$, with $\hat{\pi}_{jkNT}(\tau)$ and $\tilde{\lambda}_{jkNT}(\tau)$ defined in (10) and (11), respectively. By construction, $0 \leq \hat{\pi}_{jkNT}(\tau) \leq 1$, since $\hat{\pi}_{jkNT}(\tau)$ is the proportion of cross-sectional units with non-zero beta on the factor within regime j. Also, $\hat{\lambda}_{jkNT}(\tau)$ and $\tilde{\lambda}_{jkNT}(\tau)$ are asymptotically equivalent since the probability of the event $\hat{\pi}_{jkNT}(\tau) = 0$ is equal to zero as $N \to \infty$.

3.2 Testing for strength instability

3.2.1 Test statistics

Our inferential procedure tests for stability of the strength using the estimators obtained before and after the break as described in Section 3.1. In doing so, we assume the break fraction τ in (1) is known. We relax this assumption in Section 3.3, in which we let the break fraction τ be unknown so that it has to be estimated.

Given $\hat{d}_{jikT}(\tau)$ and $\hat{\pi}_{jkNT}(\tau)$ as in (10), define

$$\hat{D}_{jkNT}(\tau) = \sum_{i=1}^{N} \hat{d}_{jikT}(\tau) = N^{\hat{\lambda}_{jkNT}(\tau)}, \quad D_{jkN} = \sum_{i=1}^{N} d_{jik} = N^{\lambda_{jk}}, \quad d_{jik} = \mathbb{I}(\beta_{jik} \neq 0),$$

so that

$$\frac{\hat{D}_{jkNT}(\tau)}{D_{jkN}} = \frac{N^{\hat{\lambda}_{jkNT}(\tau)}}{N^{\lambda_{jk}}} = N^{\hat{\lambda}_{jkNT}(\tau) - \lambda_{jk}}. \tag{12}$$

Given

$$\hat{A}_{jkNT}(\tau) = \frac{\sum_{i=1}^{N} \left\{ \hat{d}_{jikT}(\tau) - \mathrm{E}\left[\hat{d}_{jikT}(\tau)\right] \right\}}{N^{\lambda_{jk}}}, \qquad B_{jkNT}(\tau) = \frac{\sum_{i=1}^{N} \mathrm{E}\left[\hat{d}_{jikT}(\tau)\right] - N^{\lambda_{jk}}}{N^{\lambda_{jk}}},$$

the approximate equality

$$[\ln(N)] \left[\hat{\lambda}_{jkNT}(\tau) - \lambda_{jk}\right] = \hat{A}_{jkNT}(\tau) + B_{jkNT}(\tau), \tag{13}$$

holds.[4] Given (13), interest lies in the difference

$$\begin{aligned}&[\ln(N)]\left\{\left[\hat{\lambda}_{1kNT}(\tau)-\lambda_{1k}\right]-\left[\hat{\lambda}_{2kNT}(\tau)-\lambda_{2k}\right]\right\}\\=&\left[\hat{A}_{1kNT}(\tau)+B_{1kNT}(\tau)\right]-\left[\hat{A}_{2kNT}(\tau)+B_{2kNT}(\tau)\right].\end{aligned}\tag{14}$$

Under $\mathcal{H}_{0k}:\lambda_{1k}=\lambda_{2k}$, (14) simplifies to

$$[\ln(N)]\left[\hat{\lambda}_{1kNT}(\tau)-\hat{\lambda}_{2kNT}(\tau)\right]=\left[\hat{A}_{1kNT}(\tau)+B_{1kNT}(\tau)\right]-\left[\hat{A}_{2kNT}(\tau)+B_{2kNT}(\tau)\right].$$

Under Assumptions 1 - 3 in Section 3.2.2 below, for $0<C_1,C_2,C_3,C_4<\infty$,

$$\operatorname{Var}\left[\hat{A}_{jkNT}(\tau)\right]=\frac{N-N^{\lambda_{jk}}}{N^{2\lambda_{jk}}}C_T\frac{p}{N^{\delta}}\left(1-C_T\frac{p}{N^{\delta}}\right)+O\left[\frac{\exp\left(-C_1T^{C_2}\right)}{N^{\lambda_{jk}}}\right]\tag{15}$$

and

$$B_{jkNT}(\tau)=\frac{N-N^{\lambda_{jk}}}{N^{\lambda_{jk}}}C_T\frac{p}{N^{\delta}}+O\left[\exp\left(-C_3T^{C_4}\right)\right],\tag{16}$$

for some $0<C_T<\infty$ such that $C_T\to 1$ as $T\to\infty$.[5]

The terms $B_{1kNT}(\tau)$ and $B_{2kNT}(\tau)$ defined according to (16) account for the bias induced by the multiple testing strategy that underlies our inferential procedure. This bias is due to the number of non-zero betas being the outcome of a test with a type one and a type two errors. Therefore, there is a positive probability that the outcome of the N individual tests run in the first step within each regime is incorrect. From (16), $B_{1kNT}(\tau)$ and $B_{2kNT}(\tau)$ are *both* asymptotically negligible if $\delta>1-\min\{\lambda_{1k},\lambda_{2k}\}$. More importantly, $[B_{1kNT}(\tau)-B_{2kNT}(\tau)]$ converges to zero exponentially fast as $T\to\infty$ under $\mathcal{H}_{0k}:\lambda_{1k}=\lambda_{2k}$.

From (15), both $\hat{A}_{1kNT}(\tau)=o_p(1)$ and $\hat{A}_{2kNT}(\tau)=o_p(1)$ if $\delta>1-2\min\{\lambda_{1k},\lambda_{2k}\}$. This general condition links the factor strength to the critical value exponent δ: in particular, for $\delta>0$, it is satisfied for $0.5<\min\{\lambda_{1k},\lambda_{2k}\}\leq 1$, and the $k-th$ factor is at least semi-strong before and after the break. This is consistent with the empirical findings in Section 5.2, which show that the factors are always either strong of semi-strong given the empirical model we employ. Also, $\hat{A}_{jkNT}(\tau)=O_p\left(N^{1/2-\delta/2-\lambda_{jk}}\right)$ if $0\leq\lambda_{jk}<1$, for $j=1,2$: as noted in Bailey et al. (2021), when

[4]See equation (A.1) in Appendix A.
[5]See the proof of Theorem 3.1 in Appendix A.

$\lambda_{jk} = 1$ the distribution of $\hat{A}_{jkNT}(\tau)$ is degenerate as the convergence rate is exponential. This implies that a test for the null hypothesis $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k}$ against the alternative $\mathcal{H}_{1k} : \lambda_{1k} \neq \lambda_{2k}$ can be implemented only if $\delta > 1 - 2\min\{\lambda_{1k}, \lambda_{2k}\}$ and $0 \leq \min\{\lambda_{1k}, \lambda_{2k}\} < 1$: in particular, either $\hat{A}_{1kNT}(\tau)$ or $\hat{A}_{2kNT}(\tau)$ (or both) need to have a non-degenerate asymptotic distribution under $\mathcal{H}_{1k}$. It is important to note that the test can be implemented if either $0 \leq \lambda_{1k} < 1$ *or* $0 \leq \lambda_{2k} < 1$: therefore, the test can be implemented if the factor strength were to change from unity to a lower value. From the empirical results in Section 5.2, the more restrictive case $\delta > 1 - 2\min\{\lambda_{1k}, \lambda_{2k}\}$ and $0.5 < \min\{\lambda_{1k}, \lambda_{2k}\} < 1$ is relevant in practice.

The above sufficient condition $\delta > 1 - 2\min\{\lambda_{1k}, \lambda_{2k}\}$ and $0.5 < \min\{\lambda_{1k}, \lambda_{2k}\} < 1$ has implications for the vector of factors $\mathbf{f}_t$. In particular, for inference on the factor strength, $\mathbf{f}_t$ enters the model both before and after the break, and no factor is irrelevant in either regime. Also, following the classification introduced in Chudik et al. (2011), the factors have to be at least semi-strong to conduct valid inference on their strength stability.

Consider the quantity

$$\varphi_N(\lambda_{jk}) = \frac{N - N^{\lambda_{j1k}}}{N^{2\lambda_{jk}}} \frac{p}{N^\delta}\left(1 - \frac{p}{N^\delta}\right), \tag{17}$$

defined in Bailey et al. (2021): $\varphi_N(\lambda_{jk}) = O\left(N^{1-\delta-2\lambda_{jk}}\right)$ for $0 \leq \lambda_{jk} < 1$ and $\varphi_N(\lambda_{jk}) = 0$ for $\lambda_{jk} = 1$. Therefore, $\varphi_N(\lambda_{jk})$ is a consistent estimator for $\text{Var}\left[\hat{A}_{jkNT}(\tau)\right]$ in (15) as $N, T \to \infty$. In order to test the null hypothesis $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k}$ against the alternative $\mathcal{H}_{1k} : \lambda_{1k} \neq \lambda_{2k}$, we propose the test statistics $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\tau)$ respectively defined as

$$\widehat{\mathcal{LM}}_{kNT}(\tau) = \frac{[\ln(N)]\left[\hat{\lambda}_{1kNT}(\tau) - \hat{\lambda}_{2kNT}(\tau)\right]}{\left[2\max\left\{\varphi_N\left[\hat{\lambda}_{1kNT}(\tau)\right], \varphi_N\left[\hat{\lambda}_{2kNT}(\tau)\right]\right\}\right]^{1/2}}. \tag{18}$$

and

$$\widehat{\mathcal{W}}_{kNT}(\tau) = \frac{[\ln(N)]\left[\hat{\lambda}_{1kNT}(\tau) - \hat{\lambda}_{2kNT}(\tau)\right]}{\left\{\varphi_N\left[\hat{\lambda}_{1kNT}(\tau)\right] + \varphi_N\left[\hat{\lambda}_{2kNT}(\tau)\right]\right\}^{1/2}}. \tag{19}$$

The statistics $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\tau)$ in (18) and (19), respectively, differ in the estimator for the asymptotic variance of $[\ln(N)]\left[\hat{\lambda}_{1kNT}(\tau) - \hat{\lambda}_{2kNT}(\tau)\right]$. The Wald statistic $\widehat{\mathcal{W}}_{kNT}(\tau)$ employs the *unrestricted* estimator $\left\{\varphi_N\left[\hat{\lambda}_{1kNT}(\tau)\right] + \varphi_N\left[\hat{\lambda}_{2kNT}(\tau)\right]\right\}$. The LM statistic $\widehat{\mathcal{LM}}_{kNT}(\tau)$ uses the *restricted* estimator $\left\{2\max\left\{\varphi_N\left[\hat{\lambda}_{1kNT}(\tau)\right],\varphi_N\left[\hat{\lambda}_{2kNT}(\tau)\right]\right\}\right\}$, which deserves some attention. The null hypothesis $\mathcal{H}_{0k}:\lambda_{1k}=\lambda_{2k}$ does not rule out a break in the factor loadings, as this may occur even if the factor strength is constant over time. Therefore, the factor strength cannot be estimated over the full sample period under the null hypothesis, as the loadings may still experience a break. The denominator of $\widehat{\mathcal{LM}}_{kNT}(\tau)$ accounts for this by taking the maximum between $\varphi_N\left[\hat{\lambda}_{1kNT}(\tau)\right]$ and $\varphi_N\left[\hat{\lambda}_{2kNT}(\tau)\right]$: $\hat{\lambda}_{1kNT}(\tau)$ and $\hat{\lambda}_{2kNT}(\tau)$ converge to the same probability limit under $\mathcal{H}_{0k}:\lambda_{1k}=\lambda_{2k}$; $\max\left\{\varphi_N\left[\hat{\lambda}_{1kNT}(\tau)\right],\varphi_N\left[\hat{\lambda}_{2kNT}(\tau)\right]\right\}$ accounts for the small sample discrepancy between $\hat{\lambda}_{1kNT}(\tau)$ and $\hat{\lambda}_{2kNT}(\tau)$ by making $\widehat{\mathcal{LM}}_{kNT}(\tau)$ more conservative in finite samples. Finally, the convergence rate $\ln(N)$ of the test statistics $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\tau)$ comes from (13), which in turn follows from (12), and it is formally derived in equation (A.1) in Appendix A using relevant results in Bailey et al. (2021).

3.2.2 Asymptotic properties of test statistics

In order to study the asymptotic distribution of the test statistics $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\tau)$ in (18) and (19), respectively, we consider the following set of assumptions.

Assumption 1 *The error terms e_{it}, and the demeaned factors $\mathbf{f}_t-\mathrm{E}(\mathbf{f}_t)$, are martingale difference processes with respect to $\mathcal{F}_t^{u_i}=\sigma(u_{it-s},s\leq t)$ and $\mathcal{F}_t^{f}=\sigma(\mathbf{f}_{t-s},s\leq t)$, respectively. The error terms e_{it} are independent over i and of $\mathbf{f}_t$.*

Assumption 2 $\mathrm{E}\left\{[\mathbf{f}_t-\mathrm{E}(\mathbf{f}_t)][\mathbf{f}_t-\mathrm{E}(\mathbf{f}_t)]'\right\}=\mathbf{\Sigma_f}$, *where $\mathbf{\Sigma_f}$ is a positive definite matrix.*

Assumption 3 *There exist sufficiently large positive constants C_1, $C_2>0$, and $q>0$ such that*

$$\sup_{i,t}\Pr(|e_{it}|>\nu)\leq C_1\exp(-C_2\nu^q),\quad\forall\nu>0,$$

and

$$\sup_{k,t}\Pr(|f_{kt}|>\nu)\leq C_1\exp(-C_2\nu^q),\quad\forall\nu>0.$$

Assumption 4 *The breaks in the regression betas satisfy $\mathbf{B}_2-\mathbf{B}_1=\mathbf{\Delta}$, where $\mathbf{\Delta}\neq\mathbf{0}$ is independent of the time series dimension T.*

Assumption 5 *The break fraction τ satisfies $0<\tau<1$.*

Assumptions 1 - 3 are the same as the homologous Assumptions 1 - 3 in Bailey et al. (2021) and allow to use results in Lemma A.10 in Chudik et al. (2018). According to Assumption 1, the error terms e_{it} are cross-sectionally independent, which ensures that the central limit theorem that underlies Theorem 3.1 below still holds. On this respect, Assumption 1 could be weakened by assuming some suitable spatial mixing condition, as discussed in Bailey et al. (2021). Alternatively, once could assume that e_{it} follows a spatial martingale difference process as defined in Definition 1 in the online appendix of Kapetanios et al. (2023). As it is, Assumption 1 implies that the tradable factors $\mathbf{f}_t$ are serially uncorrelated, a reasonable approximation for financial returns as argued in Barillas et al. (2020). The important point is that the degree of cross-sectional dependence in e_{it} is such that an underlying central limit theorem holds both before and after the break fraction τ: in particular, the break can still impact the degree of cross-sectional dependence in e_{it} provided that such a central limit theorem remains valid. Assumption 1 also restricts the demeaned factors $\mathbf{f}_t$ to be a martingale difference sequence, as in Chudik et al. (2018): weaker mixing conditions could be employed at the expense of higher mathematical complexity, as discussed in Bailey et al. (2021). Assumption 2 imposes a standard regularity condition on the covariance matrix of the factors, which ensures that the estimator in (7) is well defined. Note that Assumption 2 accommodates a break in the covariance matrix of the factors $\mathbf{f}_t$, as it does not rule out regime-specific covariance matrices: this is important in modelling financial returns, as discussed in Baele et al. (2010). Assumption 3 imposes thin probability tail conditions used for the asymptotic distribution of the test statistics in (18) and (19) stated in Theorem 3.1 below. Assumption 4 is analogous to Assumption A6 in Qu and Perron (2007) and captures a large shift in the betas: this is required because a break in the factor strength occurs only if a break in the betas takes place, as discussed in Section 2.2. From Assumption 4, for a given factor, the break happens at the same time in all cross-sectional units that are affected. However, it does not require that all units or all factors experience a break, and it is satisfied if structural instability affects at least one cross-sectional unit and one factor. Also, Assumption 4 does not require $N \to \infty$ since the break is identified along the time series dimension. Note that Assumption 4 is not tested in the paper as this can be done using the procedure developed in Qu and Perron (2007). If Assumption 4 fails to hold, the factor strength is stable over time because betas instability is a necessary condition for strength instability. However, Assumption 4 is likely to hold empirically because factor models for asset returns experience structural instability, as shown in Smith and Timmermann (2021). Finally, Assumption 5 is standard in the literature and allows to identify the model before and after the break: see Assumption A8 in Qu and Perron (2007).

Our aim is to test for stability in the factor strength, and not to estimate or conduct inference on risk premia. Therefore, for $j = 1, 2$, Assumptions 1 - 5 do not impose any restriction on the asset specific intercept α_{ji}: as discussed in Section 2.1, α_{ji} could depend on a spanning error generated by some common factors and on a security-specific idiosyncratic pricing error, as advocated in Pesaran and Smith (2023).

Theorem 3.1 *Consider the model in* (1)*, and let Assumptions 1 - 5 hold. Further, assume that the break fraction τ is known. For $k \in \{1, \ldots, K\}$, if $0 \leq \lambda_{1k} < 1$ or $0 \leq \lambda_{2k} < 1$ (or both), with $\delta > 1 - 2\min\{\lambda_{1k}, \lambda_{2k}\}$, then the test statistics $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\tau)$ defined in* (18) *and* (19)*, respectively, are such that for $N, T \to \infty$: (a) $\widehat{\mathcal{LM}}_{kNT}(\tau) \xrightarrow{d} \mathcal{N}(0, 1)$ and $\widehat{\mathcal{W}}_{kNT}(\tau) \xrightarrow{d} \mathcal{N}(0, 1)$ under the null $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k}$; (b) $\Pr\left(\left|\widehat{\mathcal{LM}}_{kNT}(\tau)\right| > C_1\right) \to 1$ and $\Pr\left(\left|\widehat{\mathcal{W}}_{kNT}(\tau)\right| > C_2\right) \to 1$ for any positive constants C_1 and C_2 under the alternative $\mathcal{H}_{1k} : \lambda_{1k} \neq \lambda_{2k}$.*

For $k \in \{1, \ldots, K\}$, Theorem 3.1 shows the validity of the test statistics defined in (18) and (19) for the null hypothesis $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k}$ against the alternative $\mathcal{H}_{1k} : \lambda_{1k} \neq \lambda_{2k}$. Intuitively, the numerator $[\ln(N)]\left[\hat{\lambda}_{1kNT}(\tau) - \hat{\lambda}_{2kNT}(\tau)\right]$ of the test statistics converges to a normal distribution under the null, whereas it diverges under the alternative. The formal proof of the theorem is provided in Appendix A. The results in the theorem are valid provided that either $0 \leq \lambda_{1k} < 1$ or $0 \leq \lambda_{2k} < 1$ (or both): if $\lambda_{jk} = 1$, from (15) it follows that $\hat{A}_{jkNT}(\tau) \xrightarrow{p} 0$ exponentially fast as $T \to \infty$; therefore, the asymptotic distribution of the test statistics no longer holds under the null hypothesis $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k} = 1$. This implies that we can still test the null hypothesis $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k}$ even if $\lambda_{jk^\star} = 1$, for $j = 1$ or $j = 2$ (or both), $k^\star \in \{1, \ldots, K\}$ and $k^\star \neq k$. It also implies that we can test if the factor strength changes from unity to a lower value. The results in Theorem 3.1 hold when the break fraction τ is known: Section 3.3 deals with the scenario in which τ is treated as unknown and has to be estimated.

3.3 Unknown change point

Theorem 3.1 holds if the break fraction τ is known. We now relax this assumption and consider the case in which the break fraction τ is unknown and needs to be estimated. The multi-factor model in (1) can be cast within the general framework considered in equation (1) in Qu and Perron (2007). We thus employ relevant findings obtained therein to show that the results stated in Theorem 3.1 apply also when τ no longer is known and needs to be estimated.

Recall the formulation in (5), which we repeat for ease of exposition,

$$\mathbf{R}_t = \mathbb{I}_{1t}(\tau)(\boldsymbol{\alpha}_1 + \mathbf{B}_1\mathbf{f}_t) + \mathbb{I}_{2t}(\tau)(\boldsymbol{\alpha}_2 + \mathbf{B}_2\mathbf{f}_t) + \mathbf{e}_t.$$

Let $\hat{\tau}$, $\hat{\boldsymbol{\alpha}}_j$ and $\hat{\mathbf{B}}_j$ be the least squares estimators for τ, $\boldsymbol{\alpha}_j$ and $\mathbf{B}_j$, respectively, for $j = 1, 2$. Denote by $\hat{\boldsymbol{\theta}} = \left[\hat{\tau}, \hat{\boldsymbol{\alpha}}_1', \text{vec}\left(\hat{\mathbf{B}}_1'\right)', \hat{\boldsymbol{\alpha}}_2', \text{vec}\left(\hat{\mathbf{B}}_2'\right)'\right]'$ the estimator for $\boldsymbol{\theta} = \left[\tau, \boldsymbol{\alpha}_1', \text{vec}(\mathbf{B}_1')', \boldsymbol{\alpha}_2', \text{vec}(\mathbf{B}_2')'\right]'$: $\hat{\boldsymbol{\theta}}$ solves

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \frac{1}{NT}\sum_{t=1}^{T} \left\|\mathbf{R}_t - \mathbb{I}_{1t}(\tau)(\boldsymbol{\alpha}_1 + \mathbf{B}_1\mathbf{f}_t) - \mathbb{I}_{2t}(\tau)(\boldsymbol{\alpha}_2 + \mathbf{B}_2\mathbf{f}_t)\right\|^2.$$

In practice, $\hat{\boldsymbol{\theta}}$ is obtained in two steps. First, the estimator $\hat{\tau}$ is computed. Second, given $\hat{\tau}$, $\hat{\boldsymbol{\alpha}}_j$ and $\hat{\mathbf{B}}_j$ are obtained for $j = 1, 2$. Formally, for given τ, the estimators $\hat{\boldsymbol{\alpha}}_j(\tau)$ and $\hat{\mathbf{B}}_j(\tau)$ for $\boldsymbol{\alpha}_j$ and $\mathbf{B}_j$, respectively, are obtained by concentrating out τ as

$$\left[\hat{\boldsymbol{\alpha}}_j(\tau), \hat{\mathbf{B}}_j(\tau)\right] = \left[\sum_{t=1}^{T} \mathbb{I}_{jt}(\tau)\mathbf{R}_t\mathbf{g}_t'\right]\left[\sum_{t=1}^{T} \mathbb{I}_{jt}(\tau)\mathbf{g}_t\mathbf{g}_t'\right]^{-1},$$

for $j = 1, 2$, where $\mathbf{g}_t = (1, \mathbf{f}_t')'$. The estimator $\hat{\tau}$ for τ is then obtained as

$$\hat{\tau} = \arg\min_{\tau} \frac{1}{NT}\sum_{t=1}^{T} \left\|\mathbf{R}_t - \mathbb{I}_{1t}(\tau)\left[\hat{\boldsymbol{\alpha}}_1(\tau) + \hat{\mathbf{B}}_1(\tau)\mathbf{f}_t\right] - \mathbb{I}_{2t}(\tau)\left[\hat{\boldsymbol{\alpha}}_2(\tau) + \hat{\mathbf{B}}_2(\tau)\mathbf{f}_t\right]\right\|^2.$$

Given $\hat{\tau}$, the estimators $\hat{\boldsymbol{\alpha}}_j$ and $\hat{\mathbf{B}}_j$ are obtained as $\hat{\boldsymbol{\alpha}}_j(\hat{\tau})$ and $\hat{\mathbf{B}}_j(\hat{\tau})$, respectively, for $j = 1, 2$.

Once $\hat{\tau}$ is computed, the test statistics $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\tau)$ in (18) and (19) can be modified as

$$\widehat{\mathcal{LM}}_{kNT}(\hat{\tau}) = \frac{[\ln(N)]\left[\hat{\lambda}_{1kNT}(\hat{\tau}) - \hat{\lambda}_{2kNT}(\hat{\tau})\right]}{\left[2\max\left\{\varphi_N\left[\hat{\lambda}_{1kNT}(\hat{\tau})\right], \varphi_N\left[\hat{\lambda}_{2kNT}(\hat{\tau})\right]\right\}\right]^{1/2}}, \tag{20}$$

and

$$\widehat{\mathcal{W}}_{kNT}\left(\hat{\tau}\right)=\frac{\left[\ln\left(N\right)\right]\left[\hat{\lambda}_{1kNT}\left(\hat{\tau}\right)-\hat{\lambda}_{2kNT}\left(\hat{\tau}\right)\right]}{\left\{\varphi_{N}\left[\hat{\lambda}_{1kNT}\left(\hat{\tau}\right)\right]+\varphi_{N}\left[\hat{\lambda}_{2kNT}\left(\hat{\tau}\right)\right]\right\}^{1/2}}, \tag{21}$$

respectively. In order to derive the asymptotic properties of $\widehat{\mathcal{LM}}_{kNT}\left(\hat{\tau}\right)$ and $\widehat{\mathcal{W}}_{kNT}\left(\hat{\tau}\right)$, we consider the following additional set of assumptions.

Assumption 6 *For* $l_1 \leq \lfloor \tau T \rfloor$ *and* $l_2 \leq T - \lfloor \tau T \rfloor$, $\left(1/l_1\right)\sum_{t=1}^{l_1}\mathbf{f}_t\mathbf{f}_t' \overset{a.s.}{\rightarrow} \mathbf{Q}_1$ *as* $l_1 \rightarrow \infty$, *and* $\left(1/l_2\right)\sum_{t=\lfloor \tau T \rfloor+1}^{\lfloor \tau T \rfloor+l_2}\mathbf{f}_t\mathbf{f}_t' \overset{a.s.}{\rightarrow} \mathbf{Q}_2$ *as* $l_2 \rightarrow \infty$, *where* $\mathbf{Q}_1$ *and* $\mathbf{Q}_2$ *are nonrandom positive definite matrices non necessarily equal to each other.*

Assumption 7 *There exists a* $l_0 > 0$ *such that for all* $l > l_0$ *the minimum eigenvalues of* $\left(1/l\right)\sum_{t=\lfloor \tau T \rfloor-l}^{\lfloor \tau T \rfloor}\mathbf{f}_t\mathbf{f}_t'$ *and of* $\left(1/l\right)\sum_{t=\lfloor \tau T \rfloor+1}^{\lfloor \tau T \rfloor+l}\mathbf{f}_t\mathbf{f}_t'$ *are bound away from zero.*

Assumption 8 $\sum_{t=q}^{l}\mathbf{f}_t\mathbf{f}_t'$ *is invertible for* $l-q \geq q_0$ *for some* $0 < q_0 < \infty$.

Assumptions 6, 7 and 8 are analogous to Assumptions A.1, A.2 and A.3, respectively, in Qu and Perron (2007), and impose restrictions on a local neighbourhood of the break fraction τ, which allow for consistent estimation of τ itself. Assumption 6 is stronger than Assumption 2 and still allows the factors to have different distributions before and after the break. Assumption 7 rules out local collinearity. Assumption 8 is an invertibility requirement. The remaining relevant conditions in Assumptions A.4 through A.8 in Qu and Perron (2007) are implied by Assumptions 1, 3, 4, and 5. The asymptotic properties of $\widehat{\mathcal{LM}}_{kNT}\left(\hat{\tau}\right)$ and $\widehat{\mathcal{W}}_{kNT}\left(\hat{\tau}\right)$ defined in (20) and (21), respectively, are stated in Theorem 3.2 below.

Theorem 3.2 *Consider the model in* (1). *Let Assumptions 1, and 3 - 8 hold. For* $k \in \{1,\ldots,K\}$, *if* $0 \leq \lambda_{1k} < 1$ *or* $0 \leq \lambda_{2k} < 1$ *(or both), with* $\delta > 1 - 2\min\{\lambda_{1k},\lambda_{2k}\}$, *then the results in* (a) *and* (b) *of Theorem 3.1, and stated for* $\widehat{\mathcal{LM}}_{kNT}\left(\tau\right)$ *and* $\widehat{\mathcal{W}}_{kNT}\left(\tau\right)$, *remain valid for* $\widehat{\mathcal{LM}}_{kNT}\left(\hat{\tau}\right)$ *and* $\widehat{\mathcal{W}}_{kNT}\left(\hat{\tau}\right)$, *respectively, as defined in* (20) *and* (21), *as* $N,T \rightarrow \infty$.

Theorem 3.2 shows that the asymptotic distribution of $\widehat{\mathcal{LM}}_{kNT}\left(\hat{\tau}\right)$ and $\widehat{\mathcal{W}}_{kNT}\left(\hat{\tau}\right)$ under the null hypothesis is the same as it would be if τ was known and did not have to be estimated by $\hat{\tau}$: intuitively, following from Corollary 1 in Qu and Perron (2007), the limiting distribution of the estimator for the betas is the same as it would be if τ was known; the result in Theorem

3.2 then naturally follows from Theorem 3.1 since $[\ln(N)]\left[\hat{\lambda}_{1kNT}(\hat{\tau})-\hat{\lambda}_{2kNT}(\hat{\tau})\right]$ converges to a normal distribution under the null, whereas it diverges under the alternative. This is because the break fraction τ is estimated at rate T, which is fast enough not to affect the asymptotic distribution of the estimator $\hat{\mathbf{B}}_j = \hat{\mathbf{B}}_j(\hat{\tau})$, for $j = 1, 2$: on this, see also Bai (1997) and Bai and Perron (1998). Although the test statistics $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$ are based on the outcome of N tests of significance on the regression betas, they are unaffected by $\hat{\tau}$: their asymptotic distribution depends only on the asymptotic distribution of $\hat{\mathbf{B}}_1$ and $\hat{\mathbf{B}}_2$, which is unaffected by $\hat{\tau}$ due to its fast convergence rate T. Theorem 3.2 is formally proved in Appendix A. Also, neither $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ nor $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$ suffer from the problem of having one parameter being identified only under the alternative originally addressed in Davies (1977, 1987), since both statistics are constructed under the maintained assumption that τ is identified also under the null hypothesis: this is because a *necessary* condition for a break in factor strength is the occurrence of a break in the betas, as stated in Assumption 4; this allows to identify τ regardless of whether the factor strength remains stable over time. As in Section 3.2.2, we take Assumption 4 as given and we do not test for it: this could be done following the procedure developed in Qu and Perron (2007).

3.4 Multiple change points

So far, we have worked under the maintained assumption of a single structural break. In the case of multiple breaks, the specification in (1) generalizes to the following model with J break fractions τ_j such that $0 < \tau_j < 1$, for $j = 1, \ldots, J$, and $J + 1$ regimes

$$R_{it} = \begin{cases} \alpha_{1i} + \boldsymbol{\beta}'_{1i}\mathbf{f}_t + e_{it}, & t/T \leq \tau_1, \\ \alpha_{2i} + \boldsymbol{\beta}'_{2i}\mathbf{f}_t + e_{it}, & \tau_1 < t/T \leq \tau_2, \\ \vdots & \vdots \\ \alpha_{J+1,i} + \boldsymbol{\beta}'_{J+1,i}\mathbf{f}_t + e_{it}, & t/T > \tau_J, \end{cases}, \tag{22}$$

where α_{ji} is the asset-specific intercept, and $\boldsymbol{\beta}_{ji} = (\beta_{ji1}, \ldots, \beta_{jiK})'$, for $j = 1, \ldots, J+1$. Note that the specification in (22) means that Assumption 4 holds for each break fraction τ_j, for $j = 1, \ldots, J$. Formally, given $\mathbf{B}_j = (\boldsymbol{\beta}_{j1}, \ldots, \boldsymbol{\beta}_{jN})'$, it must hold that $\mathbf{B}_{j+1} - \mathbf{B}_j = \boldsymbol{\Delta}_j$, where

$\boldsymbol{\Delta}_j \neq \mathbf{0}$, for $j = 1, \ldots, J$. In this case, the cross-sectional dispersion of betas in (2) becomes

$$\begin{array}{l} \beta_{jik} \neq 0, \quad i = 1, \ldots, \lfloor N^{\lambda_{jk}} \rfloor, \\ \beta_{jik} = 0, \quad i = \lfloor N^{\lambda_{jk}} \rfloor + 1, \ldots, N, \end{array}, \quad 0 \leq \lambda_{jk} \leq 1, \quad j = 1, \ldots, J+1, \quad k = 1, \ldots, K, \tag{23}$$

where the ordering of the betas is for ease of exposition only. We then consider the following null and alternative hypotheses $\mathcal{H}_{0j_1j_2k}$ and $\mathcal{H}_{1j_1j_2k}$, respectively,

$$\mathcal{H}_{0j_1j_2k} : \lambda_{j_1k} = \lambda_{j_2k}, \qquad \mathcal{H}_{1j_1j_2k} : \lambda_{j_1k} \neq \lambda_{j_2k}, \qquad j_1, j_2 = 1, \ldots, J+1, \qquad j_1 \neq j_2, \qquad k \in \{1, \ldots, K\} :$$

we can then test for factor strength equality over any two regimes even if they are not consecutive. In particular, a factor strength that is equal between two non-consecutive regimes is evidence of a cyclical component in the strength itself.

Let $\hat{\tau}_j$ be the estimator for τ_j, for $j = 0, \ldots, J+1$, where $\hat{\tau}_0 = \tau_0 = 0$ and $\hat{\tau}_{J+1} = \tau_{J+1} = 1$: $\hat{\tau}_j$ can be estimated using the procedure in Qu and Perron (2007), for $j = 1, \ldots, J$. From (22) and (23), λ_{jk} is the strength of factor k in regime $j = 1, \ldots, J+1$, which occurs for $\tau_{j-1} < t/T \leq \tau_j$. Given the estimators $\hat{\tau}_{j-1}$ and $\hat{\tau}_j$ for τ_{j-1} and τ_j, respectively, we can estimate λ_{jk} following steps analogous to those detailed in Section 3.1. Let $\hat{\lambda}_{jkNT}(\hat{\tau}_{j-1}, \hat{\tau}_j)$ denote the estimator for λ_{jk} obtained within the interval $\hat{\tau}_{j-1} < t/T \leq \hat{\tau}_j$, for $j = 1, \ldots, J+1$. For $j_1, j_2 = 1, \ldots, J+1$, with $j_1 \neq j_2$, and $k \in \{1, \ldots, K\}$, the test statistics $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$ defined in (20) and (21), respectively, generalize to

$$\widehat{\mathcal{LM}}_{kNT}(\hat{\tau}_{j_1-1}, \hat{\tau}_{j_1}, \hat{\tau}_{j_2-1}, \hat{\tau}_{j_2}) = \frac{[\ln(N)] \left[\hat{\lambda}_{j_1kNT}(\hat{\tau}_{j_1-1}, \hat{\tau}_{j_1}) - \hat{\lambda}_{j_2kNT}(\hat{\tau}_{j_2-1}, \hat{\tau}_{j_2})\right]}{\left[2 \max\left\{\varphi_N\left[\hat{\lambda}_{j_1kNT}(\hat{\tau}_{j_1-1}, \hat{\tau}_{j_1})\right], \varphi_N\left[\hat{\lambda}_{j_2kNT}(\hat{\tau}_{j_2-1}, \hat{\tau}_{j_2})\right]\right\}\right]^{1/2}},$$

and

$$\widehat{\mathcal{W}}_{kNT}(\hat{\tau}_{j_1-1}, \hat{\tau}_{j_1}, \hat{\tau}_{j_2-1}, \hat{\tau}_{j_2}) = \frac{[\ln(N)] \left[\hat{\lambda}_{j_1kNT}(\hat{\tau}_{j_1-1}, \hat{\tau}_{j_1}) - \hat{\lambda}_{j_2kNT}(\hat{\tau}_{j_2-1}, \hat{\tau}_{j_2})\right]}{\left\{\varphi_N\left[\hat{\lambda}_{j_1kNT}(\hat{\tau}_{j_1-1}, \hat{\tau}_{j_1})\right] + \varphi_N\left[\hat{\lambda}_{j_2kNT}(\hat{\tau}_{j_2-1}, \hat{\tau}_{j_2})\right]\right\}^{1/2}},$$

respectively. Under conditions analogous to those in Theorem 3.2, $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau}_{j_1-1}, \hat{\tau}_{j_1}, \hat{\tau}_{j_2-1}, \hat{\tau}_{j_2})$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau}_{j_1-1}, \hat{\tau}_{j_1}, \hat{\tau}_{j_2-1}, \hat{\tau}_{j_2})$ inherit the properties of the asymptotic distribution of $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$

and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$, respectively, as stated in Theorem 3.2. This result follows directly from Corollary 1 in Qu and Perron (2007), which states that, in the model with multiple change points in (22), the estimator $\hat{\tau}_j$ for the break fraction τ_j is such that the estimator for the betas is the same as it would be if τ_j was known, for $j = 1, \ldots, J$. Therefore, the explanation provided in Section 3.2 for the model with a single change point also holds in the case of multiple breaks.

In the model with multiple change points, the true number of breaks J (as defined by suitably extending Assumption 4) does not need to be known. Inference on the number of breaks can be run using the procedure in Qu and Perron (2007), provided that the factors $\mathbf{f}_t$ are stationary. Should this assumption fail to hold, suitable bootstrap procedures may be employed: see Hansen (2000), and Cavaliere and Georgiev (2020). The number of breaks could also be fixed *a priori*. If the true number of breaks is chosen, the model is correctly specified. However, if the wrong number of breaks is imposed, the model becomes misspecified. The consequences of this kind of misspecification on the performance of the test statistics we propose are currently unknown and will be studied in future research.

4 Monte Carlo study

4.1 Data generating process

For $s = 1, \ldots, S$, $i = 1, \ldots, N$, and $t = 1, \ldots, T$, we consider the DGP

$$R_{it}^s = \mathbb{I}(t/T \leq \tau)\left(\alpha_{1i} + \beta_{1i1} f_{1t}^s + \beta_{1i2} f_{2t}^s\right) + \mathbb{I}(t/T > \tau)\left(\alpha_{2i} + \beta_{2i1} f_{1t}^s + \beta_{2i2} f_{2t}^s\right) + e_{it},$$

where s is the replication index and S is the total number of replications, with $S = 2000$. We consider combinations of N and T such that $N \in \{100, 200, 500, 1000\}$ and $T \in \{500, 1000\}$. We look at two values for the break fraction τ, namely $\tau = 1/2$ and $\tau = 1/3$. We generate the intercept α_{1i} as $\alpha_{1i} \sim \text{IID}\mathcal{N}(0, 1)$ fixed in repeated samples and we set $\alpha_{2i} = \alpha_{1i}$, for $i = 1, \ldots, N$.

The factors f_{1t}^s and f_{2t}^s are generated as

$$f_{kt}^s = \rho_{f_k} f_{k,t-1}^s + \sqrt{1 - \rho_{f_k}^2}\varepsilon_{kt}^s, \quad k = 1, 2, \quad t = -99, \ldots, T, \quad f_{k,-100}^s = 0,$$

with $\rho_{f1} = \rho_{f2} = 0.5$ and $\varepsilon_{kt}^{s} \sim \text{IID}\mathcal{N}(0,1)$, so that $\text{Var}(f_{kt}^{s}) = \text{Var}(\varepsilon_{kt}^{s}) = 1$. We minimize the effect of the starting value $f_{k,-100}^{s} = 0$ by discarding the first 100 observations in the DGPs for f_{kt}^{s}, for $k = 1,2$. We generate the idiosyncratic components as $e_{it} = \sigma_i [(u_{it} - 2)/2]$, with $\sigma_i^2 \sim \chi^2(1)$ fixed in repeated samples, and $u_{it} \sim \text{IID}\chi^2(2)$: this set up, which is analogous to the one used in Chudik et al. (2018), implies that e_{it} has a non-Gaussian distribution such that $\text{E}(e_{it}) = 0$, $\text{Var}(e_{it}) = \sigma_i^2$ and $\lim_{N\to\infty}\left[N^{-1}\sum_{i=1}^{N}\text{Var}(e_{it})\right] = 1$.

As for the factor loadings, we first consider those on f_{1t}^{s}. We begin by generating $v_i \sim \text{IID}\mathcal{U}(\mu_v - d_v, \mu_v + d_v)$ fixed in repeated samples, with $\mu_v = 1.00$ and $d_v = 0.2$. We then randomly assign $\lfloor N^{\lambda_{11}}\rfloor$ elements of v_i to $\lfloor N^{\lambda_{11}}\rfloor$ elements of the sequence $\{\beta_{1i1}\}_{i=1}^{N}$ and set to zero the remaining elements of $\{\beta_{1i1}\}_{i=1}^{N}$. In a similar way, we randomly assign $\lfloor N^{\lambda_{21}}\rfloor$ elements of v_i to $\lfloor N^{\lambda_{21}}\rfloor$ elements of the sequence $\{\beta_{2i1}\}_{i=1}^{N}$ and set to zero the remaining elements of $\{\beta_{2i1}\}_{i=1}^{N}$. In this way, under the null hypothesis $\mathcal{H}_{01} : \lambda_{11} = \lambda_{21} = \lambda_1$, the sequences $\{\beta_{1i1}\}_{i=1}^{N}$ and $\{\beta_{2i1}\}_{i=1}^{N}$ have the same number of non-zero elements, although those elements may be different since they are obtained from independent draws from $\{v_i\}_{i=1}^{N}$. Under the alternative hypothesis $\mathcal{H}_{11} : \lambda_{11} \neq \lambda_{21}$, the sequences $\{\beta_{1i1}\}_{i=1}^{N}$ and $\{\beta_{2i1}\}_{i=1}^{N}$ have a different number of non-zero elements: in this case, we define $\kappa_1 = \lambda_{21} - \lambda_{11}$, so that if $\kappa_1 < 0$ the factor strength decreases, whereas f_{1t}^{s} becomes stronger if $\kappa_1 > 0$. Both under the null and under the alternative, we set $\lambda_{11} = 0.55, 0.75, 0.80, 0.85, 0.90, 0.95, 0.99$.

As for the loadings of f_{2t}^{s}, for $j = 1,2$ we randomly assign $\lfloor N^{\lambda_{j2}}\rfloor$ elements of v_i generated as previously described to as many elements of the sequence $\{\beta_{ji2}\}_{i=1}^{N}$ and set to zero the remaining elements of $\{\beta_{ji2}\}_{i=1}^{N}$, with $\lambda_{12} = \lambda_{22} = 0.85$: therefore, the strength of f_{2t}^{s} is kept fixed, although its betas may experience a break.

4.2 Results

We group our results based on the underlying Monte Carlo experiment and consider four scenarios given by as many experiments: consistently with Theorem 3.1, Experiments 1 and 2 treat the break fraction τ as known and study $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\tau)$ in (18) and (19), respectively; as in Theorem 3.2, Experiments 3 and 4 let τ be unknown and look at $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$ in (20) and (21), respectively. In all experiments, we follow Bailey et al. (2021) and implement the critical value function in (8) by setting $p = 0.10$ and $\delta = 1/4$. We consider the size of

$\widehat{\mathcal{LM}}_{kNT}(\tau)$, $\widehat{\mathcal{W}}_{kNT}(\tau)$, $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$ as being equal to 0.05.

4.2.1 Experiment 1

Table 1 about here

The focus is on the size when τ is known. The results in Table 1 show that $\widehat{\mathcal{LM}}_{kNT}(\tau)$ generally has good properties irrespective of N, λ_1 and τ (see Panel A). On the other hand, $\widehat{\mathcal{W}}_{kNT}(\tau)$ overreject slightly more often than $\widehat{\mathcal{LM}}_{kNT}(\tau)$ (see Panel B): this is likely to be due to the different estimator for the asymptotic variance of $[\ln(N)]\left[\hat{\lambda}_{1kNT}(\tau) - \hat{\lambda}_{2kNT}(\tau)\right]$, which we discuss extensively in Section 3.2.1. In particular, the denominator of $\widehat{\mathcal{LM}}_{kNT}(\tau)$ is greater than that of $\widehat{\mathcal{W}}_{kNT}(\tau)$, which makes $\widehat{\mathcal{LM}}_{kNT}(\tau)$ a slightly more conservative test statistics. Note that $\widehat{\mathcal{W}}_{kNT}(\tau)$ overrejects when $\lambda_1 = 0.99$ for $N = 1000$, when instead $\widehat{\mathcal{LM}}_{kNT}(\tau)$ performs well. We conclude that $\widehat{\mathcal{LM}}_{kNT}(\tau)$ has a hedge over $\widehat{\mathcal{W}}_{kNT}(\tau)$.

4.2.2 Experiment 2

Table 2 about here

We study the power when τ is known. We fix $T = 500$ and consider $\kappa_1 = -0.02, -0.01, 0.01, 0.02$ with the exception of $\lambda_{11} = 0.99$, in which case we consider $\kappa_1 = -0.02, -0.01, 0.01$ only. The results collected in Table 2 show that $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\tau)$ have similar power properties: it increases in the cross-sectional dimension N, in the factor strength λ_{11}, and in the magnitude of the break κ_1. Both $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\tau)$ thus have good empirical power properties.

4.2.3 Experiment 3

Table 3 about here

The focus is on the size when the break fraction τ no longer is known. We obtain the estimator $\hat{\tau}$ using the algorithm detailed in Section 3.3 through the grid $\{0.05, 0.10, 0.15, \ldots, 0.85, 0.90, 0.95\}$. We also compute the average (over the replications) bias and root-mean square error (RMSE) of $\hat{\tau}$. We consider $T = 500$ and $\tau = 1/2$. The findings in Table 3 support the results in Theorem 3.2: given the very low values of bias and RMSE for $\hat{\tau}$, τ is precisely estimated; the size of both

$\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$ is unaffected by the estimation noise induced by $\hat{\tau}$, as the values in Table 3 are almost identical to their counterparts in Table 1. Note that $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ is correctly sized, whereas $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$ is more often oversized, especially when $N = 1000$ and $\lambda_{11} = 0.99$, thus confirming the conclusions drawn from Experiment 1.

4.2.4 Experiment 4

Table 4 about here

Finally, we study the power when τ is unknown. We use the same framework as in Experiment 3, and we set κ_1 as in Experiment 2. The findings in Table 4 confirm the results in Theorem 3.2: the power of both $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$ increases in N, λ_{11} and κ_1, thus mirroring the conclusions drawn in Experiment 2.

4.3 Discussion

The Monte Carlo results in Section 4.2 support the theoretical findings in Theorems 3.1 and 3.2. In particular, $\widehat{\mathcal{LM}}_{kNT}(\tau)$ and $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ have an edge over $\widehat{\mathcal{W}}_{kNT}(\tau)$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$, respectively, in terms of better empirical size properties when the factor strength is very close to unity. For this reason, in the empirical analysis in Section 5 we employ the LM-type statistic.

5 Empirical analysis

5.1 Data and empirical specification

We study the Chen and Zimmermann (2021) large dataset of equity portfolios and use the April 2021 version of it. Given 205 characteristics, Chen and Zimmermann (2021) build a number of portfolios whose returns are then provided; we then obtain the excess returns of those portfolios by subtracting the risk-free rate measured as the one-month Treasury bill rate.[6] The sample period of interest runs from July 1967 through December 2020, a total of $T = 690$ time series observations. To ensure that inference on the factor strength is not affected by the time-varying

[6] The Chen and Zimmermann (2021) dataset is available at `https://www.openassetpricing.com/` .

dimension and nature of the cross-section, we balance the dataset and retain only those portfolios that are available over the entire sample period. This results in $N = 739$ portfolios.

We consider the six factor model proposed in Fama and French (2016).[7] This is made of the following factors: the market return in excess of the risk-free rate as measured by the one-month Treasury bill rate ($RmRf$), size (SMB), value (HML), operating profitability (RMW), investment (CMA), and momentum (MOM).

We estimate our empirical model using rolling windows of length equal to 240 months. Given the discussion in Section 4.3, we then test for the stability of the factor strength over two consecutive non-overlapping windows using the $\widehat{\mathcal{LM}}_{kNT}(\tau)$ test defined in (18) and discussed in Section 3.2. From a methodological standpoint, this is equivalent to estimating the model over $T = 480$ time series observations and testing for a break in factor strength at a known break fraction $\tau = 0.50$. This set up is consistent with the Monte Carlo results in Section 4, which show the good finite sample properties of the $\widehat{\mathcal{LM}}_{kNT}(\tau)$ statistic for T approximately equal to 500, as stressed in Section 4.3. This strategy therefore is informative about *local stability* of factor strength, which we further motivate in Section 5.2 below. Note also that pre-break and post-break estimation windows of 240 month are aligned with the set up in Fama and MacBeth (2021), who consider the first and the second half of the July 1963–June 2019 period to test for the stability of the value premium. As in the Monte Carlo experiments in Section 4, we set $p = 0.10$ and $\delta = 1/4$ in (8). We consider the size of $\widehat{\mathcal{LM}}_{kNT}(\hat{\tau})$ and $\widehat{\mathcal{W}}_{kNT}(\hat{\tau})$ equal to 0.05.

5.2 Results

We first empirically motivate the detection of local instability as discussed in Section 5.1. Following the strategy adopted in Bailey et al. (2021), and Pesaran and Smith (2021a), we document substantial time-variation in the strength of the six factors included in our specification: this is a first empirical contribution of our paper. As discussed in Section 5.1, we estimate the model using rolling windows of length equal to 240 months.

Figure 1 and Figure 2 about here

[7] The data for the pricing factors are available from Kenneth French website at `https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`.

The sequences of estimated strength for the six factors are displayed in Figure 1. The factor $RmRf$ is strong over the whole sample period, since its estimated strength is always equal to unity: as such, given Theorem 3.1, in this case we cannot run inference on the strength stability. Turning to SMB, it is a semi-strong factor, although its estimated strength always lies in the proximity of unity: in particular, the estimated values fall between 0.991 and 0.993. The remaining factors displays a higher degree of strength variation over time: HML is characterized by a cyclical behaviour around an average value of 0.921; RMW displays a clear upward trend, starting from 0.808 at the beginning of the sample, and reaching an average value of approximately 0.941 from early 2000s onwards; CMA has a very pronounced cyclical behaviour, with a peak of 0.945 in January 2000 and a trough of 0.748 in October 1990; MOM reaches an average value approximately equal to 0.986 from January 2000 onwards.

We conduct inference on the local stability as discussed in Section 5.1. Figure 2 displays the evolution over time of the $\widehat{\mathcal{LM}}_{kNT}(\tau)$ test statistic together with the 95% confidence band. The SMB factor is stable over the whole sample period. To a different degree, the remaining factors display evidence of strength instability: HML is locally unstable at the beginning of the sample and during a short spell between January 1996 and September 1998; RMW exhibit significant local increases until January 2000, whereas this behaviour is somehow reverted after June of the same year; CMA is unstable from April 1995 onwards; MOM has dynamics similar to those of RMW, in that local increases in factor strength take place almost until the end of the sample.

6 Conclusions

This paper studies the detection of structural instability in factor strength in asset pricing models for financial returns. We distinguish between strong and weaker factors. We construct LM and Wald statistics and show that they are asymptotically normally distributed under the null hypothesis of factor strength stability. The empirical analysis conducted over a rolling estimation window uncovers the dynamics of factor strength instability in empirical models for equity portfolio returns. Given the tools we have developed, future work will focus upon the consequences of structural instability in factor strength for asset pricing and portfolio choice.

References

Anatolyev, S. and A. Mikusheva (2021). Factor models with many assets: Strong factors, weak factors, and the two-pass procedure. *Journal of Econometrics Forthcoming.*

Baele, L., G. Bekaert, and K. Inghelbrecht (2010). The determinants of stock and bond return comovements. *Review of Financial Studies 23*(6), 2374–2428.

Bai, J. (1997). Estimating multiple breaks one at a time. *Econometric Theory 13*(3), 315–352.

Bai, J. and S. Ng (2021). Approximate factor models with weaker loadings. *Working Paper, Columbia University.*

Bai, J. and P. Perron (1998). Estimating and testing linear models with multiple structural changes. *Econometrica 66*, 47–78.

Bailey, N., G. Kapetanios, and H. Pesaran (2021). Measurement of factor strength: Theory and practice. *Journal of Applied Econometrics Forthcoming.*

Barigozzi, M. and D. Massacci (2022). Modelling large dimensional datasets with markov switching factor models. *Working Paper, University of Bologna and King's College London.*

Barillas, F., R. Kan, C. Robotti, and J. Shanken (2020). Model comparison with sharpe ratios. *Journal of Financial and Quantitative Analysis 55*, 1840–1874.

Breeden, D. T. (1979). An intertemporal asset pricing model with stochastic consumption and investment opportunities. *Journal of Financial Economics 7*, 265–296.

Bryzgalova, S. (2016). Spurious factors in linear asset pricing models. *Working Paper, Stanford Graduate School of Business.*

Burnside, C. (2016). Identification and inference in linear stochastic discount factor models with excess returns. *Journal of Financial Econometrics 14*(2), 295–330.

Cavaliere, G. and I. Georgiev (2020). Inference under random limit bootstrap measures. *Econometrica 88*(6), pp. 2547–2574.

Chen, A. Y. and T. Zimmermann (2021). Open source cross-sectional asset pricing. *Critical Finance Review Forthcoming.*

Chudik, A., G. Kapetanios, and M. H. Pesaran (2018). A one covariate at a time, multiple testing approach to variable selection in high-dimensional linear regression models. *Econometrica 86*, 1479–1512.

Chudik, A., M. H. Pesaran, and E. Tosetti (2011). Weak and strong cross-section dependence and estimation of large panels. *Econometrics Journal 14*, C45–C90.

Connor, G. and R. A. Korajczyk (1986). Performance measurement with the arbitrage pricing theory: A new framework for analysis. *Journal of Financial Economics 15*, 373–394.

Connor, G. and R. A. Korajczyk (2022). Semi-strong factors in asset returns. *Journal of Financial Econometrics available at https://doi.org/10.1093/jjfinec/nbac028*.

Davies, R. B. (1977). Hypothesis testing when a nuisance parameter is present only under the alternative. *Biometrika 64*(2), 247–254.

Davies, R. B. (1987). Hypothesis testing when a nuisance parameter is present only under the alternative. *Biometrika 74*(1), 33–43.

Fama, E. F. and K. R. French (1993). Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics 33*(1), 3–56.

Fama, E. F. and K. R. French (2016). Dissecting anomalies with a five-factor model. *Review of Financial Studies 29*(1), 69–103.

Fama, E. F. and J. D. MacBeth (1973). Risk, return, and equilibrium: Empirical tests. *Journal of Political Economy 81*(3), 607–636.

Fama, E. F. and J. D. MacBeth (2021). The value premium. *Review of Asset Pricing Studies 11*(1), 105–121.

Freyaldenhoven, S. (2021). Factor models with local factors - determining the number of relevant factors. *Journal of Econometrics Forthcoming.*

Giacomini, R. and B. Rossi (2010). Forecast comparisons in unstable environments. *Journal of Applied Econometrics 25*(4), 595–620.

Giglio, S., B. Kelly, and D. Xiu (2021). Factor models, machine learning, and asset pricing. *Annual Review of Financial Economics Forthcoming.*

Giglio, S., D. Xiu, and D. Zhang (2021). Test assets and weak factors. *Working Paper, Yale School of Management and Booth School of Business.*

Gospodinov, N., R. Kan, and C. Robotti (2017). Spurious inference in reduced-rank asset-pricing models. *Econometrica 85*(5), 1613–1628.

Hansen, B. E. (2000). Testing for structural change in conditional models. *Journal of Econometrics 97*(1), 93–115.

Kan, R. and C. Zhang (1999). Two-pass tests of asset pricing models with useless factors. *Journal of Finance 54*, 203–235.

Kapetanios, G., L. Serlenga, and Y. Shin (2023). An lm test for the conditional independence between regressors and factor loadings in panel data models with interactive effects. *Journal of Business & Economic Statistics 0*(0), 1–19.

Kleibergen, F. (2009). Tests of risk premia in linear factor models. *Journal of Econometrics 149*, 149–173.

Lettau, M. and M. Pelger (2020). Factors that fit the time series and cross-section of stock returns. *Review of Financial Studies 33*(5), 2274–2325.

Litterman, R. B. and J. Scheinkman (1991). Common factors affecting bond returns. *Journal of Fixed Income 1*, 54–61.

Lustig, H., N. Roussanov, and A. Verdelhan (2011). Common risk factors in currency markets. *Review of Financial Studies 24*, 3731—-3777.

Massacci, D. (2017). Least squares estimation of large dimensional threshold factor models. *Journal of Econometrics 197*(1), 101–129.

Massacci, D. (2023). Testing for regime changes in portfolios with a large number of assets: A robust approach to factor heteroskedasticity. *Journal of Financial Econometrics 21*(2), 316–367.

Pesaran, M. H. and R. P. Smith (2021a). Arbitrage pricing theory, the stochastic discount factor and estimation of risk premia from portfolios. *Econometrics and Statistics Forthcoming.*

Pesaran, M. H. and R. P. Smith (2021b). Factor strengths, pricing errors, and estimation of risk premia. *CESifo Working Paper 8947*.

Pesaran, M. H. and R. P. Smith (2023). The role of pricing errors in linear asset pricing models with strong, semi-strong, and latent factors. *Cambridge Working Papers in Economics 2317*.

Qu, Z. and P. Perron (2007). Estimating and testing structural changes in multivariate regressions. *Econometrica 75*, 459–502.

Shanken, J. (1992). On the estimation of beta-pricing models. *Review of Financial Studies 5*, 1–33.

Smith, S. C. and A. Timmermann (2021). Have risk premia vanished? *Journal of Financial Economics 145*(2), 553–576.

Timmermann, A. (2008). Elusive return predictability. *International Journal of Forecasting 24*(1), 1–18.

Uematsu, Y. and T. Yamagata (2023a). Estimation of sparsity-induced weak factor models. *Journal of Business & Economic Statistics 41*(1), 213–227.

Uematsu, Y. and T. Yamagata (2023b). Inference in sparsity-induced weak factor models. *Journal of Business & Economic Statistics 41*(1), 126–139.

White, H. (1980). A heteroskedasticity-consistent covariance matrix estimator and a direct test for heteroskedasticity. *Econometrica 48*(4), 817–838.

Table 1: Experiment 1: $\lambda_{11} = \lambda_{21} = \lambda_1$, $\lambda_{12} = \lambda_{22} = 0.85$

Panel A: LM test

(a) $\tau = 1/2$

T	500							1000						
λ_1	0.55	0.75	0.80	0.85	0.90	0.95	0.99	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N														
100	0.0225	0.0365	0.0765	0.0580	0.0500	0.0350	0.0255	0.0230	0.0555	0.0755	0.0555	0.0510	0.0365	0.0170
200	0.0345	0.0490	0.0375	0.0675	0.0325	0.0295	0.0550	0.0295	0.0670	0.0375	0.0670	0.0325	0.0220	0.0535
500	0.0335	0.0410	0.0540	0.0470	0.0395	0.0685	0.0390	0.0335	0.0425	0.0585	0.0390	0.0395	0.0530	0.0305
1000	0.0400	0.0630	0.0475	0.0500	0.0410	0.0490	0.0595	0.0320	0.0550	0.0450	0.0505	0.0585	0.0445	0.0350

(b) $\tau = 1/3$

T	500							1000						
λ_1	0.55	0.75	0.80	0.85	0.90	0.95	0.99	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N														
100	0.0210	0.0310	0.0645	0.0510	0.0525	0.0460	0.0505	0.0180	0.0375	0.0770	0.0515	0.0475	0.0295	0.0200
200	0.0330	0.0510	0.0355	0.0585	0.0315	0.0310	0.0725	0.0225	0.0425	0.0305	0.0510	0.0240	0.0245	0.0505
500	0.0470	0.0560	0.0670	0.0465	0.0465	0.0760	0.0635	0.0385	0.0455	0.0615	0.0435	0.0425	0.0650	0.0380
1000	0.0350	0.0640	0.0525	0.0620	0.0575	0.0535	0.0715	0.0355	0.0575	0.0440	0.0475	0.0365	0.0465	0.0485

Panel B: Wald test

(a) $\tau = 1/2$

T	500							1000						
λ_1	0.55	0.75	0.80	0.85	0.90	0.95	0.99	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N														
100	0.0540	0.1000	0.0765	0.0580	0.1070	0.0350	0.0255	0.0600	0.0555	0.0755	0.0555	0.1070	0.0365	0.0170
200	0.0440	0.0490	0.0415	0.0675	0.0750	0.0525	0.0550	0.0395	0.0670	0.0405	0.0670	0.0730	0.0500	0.0535
500	0.0580	0.0780	0.0540	0.0790	0.0640	0.0685	0.0390	0.0610	0.0760	0.0585	0.0725	0.0600	0.0530	0.0305
1000	0.0540	0.0630	0.0715	0.0500	0.0410	0.0490	0.1185	0.0480	0.0550	0.0740	0.0505	0.0585	0.0445	0.0870

(b) $\tau = 1/3$

T	500							1000						
λ_1	0.55	0.75	0.80	0.85	0.90	0.95	0.99	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N														
100	0.0530	0.0810	0.0645	0.0510	0.1130	0.0460	0.0505	0.0565	0.0915	0.0770	0.0515	0.1000	0.0295	0.0200
200	0.0430	0.0510	0.0390	0.0585	0.0755	0.0640	0.0725	0.0310	0.0425	0.0355	0.0510	0.0635	0.0475	0.0505
500	0.0815	0.0860	0.0670	0.0770	0.0660	0.0760	0.0635	0.0670	0.0720	0.0615	0.0800	0.0600	0.0650	0.0380
1000	0.0540	0.0640	0.0750	0.0620	0.0575	0.0535	0.1425	0.0540	0.0575	0.0675	0.0475	0.0365	0.0465	0.1035

Table 2: Experiment 2: $\lambda_{21} = \lambda_{11} + \kappa_1$, $\lambda_{12} = \lambda_{22} = 0.85$, $T = 500$

Panel A: LM test														
(a) $\tau = 1/2$														
κ_1				-0.02							-0.01			
λ_{11}	0.55	0.75	0.80	0.85	0.90	0.95	0.99	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N														
100	0.0300	0.2230	0.2335	0.7655	0.9135	0.9995	1.0000	0.0225	0.0565	0.0865	0.3480	0.3190	0.8540	0.9975
200	0.0575	0.5510	0.6760	0.9530	0.9995	1.0000	1.0000	0.0380	0.1855	0.2850	0.5740	0.7405	0.9920	1.0000
500	0.1190	0.7720	0.9890	1.0000	1.0000	1.0000	1.0000	0.0520	0.2750	0.6200	0.8655	0.9970	1.0000	1.0000
1000	0.1455	0.9870	1.0000	1.0000	1.0000	1.0000	1.0000	0.0610	0.5885	0.8870	0.9975	1.0000	1.0000	1.0000
κ_1				0.01							0.02			
λ_{11}	0.55	0.75	0.80	0.85	0.90	0.95	0.99	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N														
100	0.0295	0.1105	0.2075	0.1785	0.5435	0.9105	1.0000	0.0295	0.2060	0.5960	0.6030	0.9870	1.0000	-
200	0.0455	0.1950	0.2815	0.5470	0.8710	0.9995	1.0000	0.0765	0.5425	0.7940	0.9835	1.0000	1.0000	-
500	0.0545	0.3575	0.6280	0.9575	1.0000	1.0000	1.0000	0.1105	0.8845	0.9990	1.0000	1.0000	1.0000	-
1000	0.0550	0.6430	0.9020	0.9990	1.0000	1.0000	1.0000	0.1715	0.9970	1.0000	1.0000	1.0000	1.0000	-
(b) $\tau = 1/3$														
κ_1				-0.02							-0.01			
λ_{11}	0.55	0.75	0.80	0.85	0.90	0.95	0.99	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N														
100	0.0345	0.2460	0.2510	0.7875	0.9075	0.9975	1.0000	0.0210	0.0525	0.0890	0.3845	0.3145	0.8275	0.9895
200	0.0655	0.5685	0.6935	0.9645	0.9990	1.0000	1.0000	0.0420	0.2030	0.2955	0.5860	0.7430	0.9955	1.0000
500	0.1405	0.8150	0.9835	1.0000	1.0000	1.0000	1.0000	0.0810	0.3135	0.6560	0.8800	0.9950	1.0000	1.0000
1000	0.1745	0.9875	1.0000	1.0000	1.0000	1.0000	1.0000	0.0785	0.6425	0.8945	0.9960	1.0000	1.0000	1.0000
κ_1				0.01							0.02			
λ_{11}	0.55	0.75	0.80	0.85	0.90	0.95	0.99	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N														
100	0.0235	0.0875	0.1945	0.1720	0.5560	0.9245	1.0000	0.0235	0.1935	0.5540	0.5700	0.9870	1.0000	-
200	0.0350	0.1580	0.2590	0.5255	0.8655	0.9990	1.0000	0.0625	0.5350	0.7870	0.9870	1.0000	1.0000	-
500	0.0535	0.2950	0.5870	0.9540	0.9995	1.0000	1.0000	0.0860	0.8680	0.9970	1.0000	1.0000	1.0000	-
1000	0.0540	0.5965	0.8920	0.9985	1.0000	1.0000	1.0000	0.1460	0.9965	1.0000	1.0000	1.0000	1.0000	-

Panel B: Wald test														
(a) $\tau = 1/2$														
κ_1				-0.02							-0.01			
λ_{11}	0.55	0.75	0.80	0.85	0.90	0.95	0.99	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N														
100	0.0760	0.2645	0.3650	0.7655	0.9135	0.9995	1.0000	0.0540	0.1105	0.1085	0.3480	0.4995	0.8540	0.9975
200	0.0795	0.5510	0.6760	0.9530	0.9995	1.0000	1.0000	0.0515	0.1855	0.2855	0.5740	0.7420	0.9920	1.0000
500	0.1710	0.8330	0.9890	1.0000	1.0000	1.0000	1.0000	0.0875	0.3615	0.6200	0.9180	0.9970	1.0000	1.0000
1000	0.1890	0.9870	1.0000	1.0000	1.0000	1.0000	1.0000	0.0885	0.5885	0.9095	0.9975	1.0000	1.0000	1.0000
κ_1				0.01							0.02			
λ_{11}	0.55	0.75	0.80	0.85	0.90	0.95	0.99	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N														
100	0.0750	0.2145	0.2075	0.1785	0.6395	0.9110	1.0000	0.0750	0.3805	0.5960	0.6030	0.9870	1.0000	-
200	0.0570	0.1950	0.3145	0.5470	0.9450	0.9995	1.0000	0.0960	0.5425	0.8840	0.9835	1.0000	1.0000	-
500	0.0860	0.4395	0.6280	0.9575	1.0000	1.0000	1.0000	0.1630	0.9275	0.9990	1.0000	1.0000	1.0000	-
1000	0.0775	0.6430	0.9355	0.9990	1.0000	1.0000	1.0000	0.2185	0.9970	1.0000	1.0000	1.0000	1.0000	-
(b) $\tau = 1/3$														
κ_1				-0.02							-0.01			
λ_{11}	0.55	0.75	0.80	0.85	0.90	0.95	0.99	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N														
100	0.0895	0.2840	0.4020	0.7875	0.9225	0.9975	1.0000	0.0530	0.1075	0.1115	0.3845	0.4915	0.8275	0.9895
200	0.0905	0.5685	0.6935	0.9645	0.9990	1.0000	1.0000	0.0620	0.2030	0.2955	0.5860	0.7430	0.9955	1.0000
500	0.1940	0.8680	0.9835	1.0000	1.0000	1.0000	1.0000	0.1115	0.4110	0.6560	0.9195	0.9950	1.0000	1.0000
1000	0.2260	0.9875	1.0000	1.0000	1.0000	1.0000	1.0000	0.1165	0.6425	0.9150	0.9960	1.0000	1.0000	1.0000
κ_1				0.01							0.02			
λ_{11}	0.55	0.75	0.80	0.85	0.90	0.95	0.99	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N														
100	0.0715	0.2055	0.1945	0.1720	0.6555	0.9245	1.0000	0.0715	0.3575	0.5540	0.5700	0.9870	1.0000	-
200	0.0470	0.1580	0.3000	0.5255	0.9455	0.9990	1.0000	0.0825	0.5350	0.8725	0.9870	1.0000	1.0000	-
500	0.0805	0.3865	0.5870	0.9550	0.9995	1.0000	1.0000	0.1310	0.9130	0.9970	1.0000	1.0000	1.0000	-
1000	0.0705	0.5965	0.9270	0.9985	1.0000	1.0000	1.0000	0.1850	0.9965	1.0000	1.0000	1.0000	1.0000	-

Table 3: Experiment 3: $\lambda_{11} = \lambda_{21} = \lambda_1$, $\lambda_{12} = \lambda_{22} = 0.85$, $\tau = 1/2$, $T = 500$

Panel A: $\hat{\tau}$							
			(a) Bias				
λ_1	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N							
100	-0.0011	-0.0007	-0.0001	0.0000	-0.0004	-0.0003	0.0000
200	-0.0016	-0.0010	-0.0003	-0.0003	0.0000	0.0000	0.0000
500	-0.0002	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
1000	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
			(b) RMSE				
λ_1	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N							
100	0.0600	0.0342	0.0215	0.0186	0.0131	0.0127	0.0117
200	0.0723	0.0295	0.0249	0.0161	0.0098	0.0056	0.0032
500	0.0158	0.0071	0.0049	0.0035	0.0022	0.0000	0.0000
1000	0.0081	0.0030	0.0019	0.0000	0.0000	0.0000	0.0000

Panel B: LM test							
λ_1	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N							
100	0.0580	0.0485	0.0800	0.0590	0.0515	0.0360	0.0260
200	0.1200	0.0720	0.0495	0.0755	0.0365	0.0295	0.0550
500	0.0505	0.0440	0.0560	0.0475	0.0395	0.0685	0.0390
1000	0.0475	0.0650	0.0490	0.0500	0.0410	0.0490	0.0595

Panel C: Wald test							
λ_1	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N							
100	0.0975	0.1150	0.0800	0.0590	0.1080	0.0360	0.0260
200	0.1335	0.0720	0.0540	0.0755	0.0790	0.0540	0.0550
500	0.0790	0.0820	0.0560	0.0800	0.0640	0.0685	0.0390
1000	0.0645	0.0650	0.0730	0.0500	0.0410	0.0490	0.1185

Table 4: Experiment 4: $\lambda_{21} = \lambda_{11} + \kappa_1$, $\lambda_{12} = \lambda_{22} = 0.85$, $\tau = 1/2$, $T = 500$

Panel A: LM test														
κ_1	-0.02							-0.01						
λ_{11}	0.55	0.75	0.80	0.85	0.90	0.95	0.99	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N														
100	0.0365	0.2230	0.2325	0.7655	0.9135	0.9995	1.0000	0.0580	0.0590	0.0880	0.3480	0.3190	0.8540	0.9975
200	0.0605	0.5510	0.6755	0.9530	0.9995	1.0000	1.0000	0.0495	0.1860	0.2850	0.5735	0.7405	0.9920	1.0000
500	0.1190	0.7720	0.9890	1.0000	1.0000	1.0000	1.0000	0.0540	0.2750	0.6200	0.8655	0.9970	1.0000	1.0000
1000	0.1460	0.9870	1.0000	1.0000	1.0000	1.0000	1.0000	0.0615	0.5885	0.8870	0.9975	1.0000	1.0000	1.0000
κ_1	0.01							0.02						
λ_{11}	0.55	0.75	0.80	0.85	0.90	0.95	0.99	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N														
100	0.0305	0.1115	0.2080	0.1790	0.5430	0.9105	1.0000	0.0305	0.2060	0.5960	0.6030	0.9870	1.0000	-
200	0.0740	0.1980	0.2815	0.5470	0.8710	0.9995	1.0000	0.0845	0.5420	0.7940	0.9835	1.0000	1.0000	-
500	0.0575	0.3575	0.6280	0.9575	1.0000	1.0000	1.0000	0.1110	0.8845	0.9990	1.0000	1.0000	1.0000	-
1000	0.0575	0.6430	0.9020	0.9990	1.0000	1.0000	1.0000	0.1715	0.9970	1.0000	1.0000	1.0000	1.0000	-

Panel B: Wald test														
κ_1	-0.02							-0.01						
λ_{11}	0.55	0.75	0.80	0.85	0.90	0.95	0.99	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N														
100	0.0830	0.2650	0.3640	0.7655	0.9315	0.9995	1.0000	0.0975	0.1135	0.1100	0.3480	0.4990	0.8540	0.9975
200	0.0815	0.5510	0.6755	0.9530	0.9995	1.0000	1.0000	0.0640	0.1860	0.2855	0.5735	0.7420	0.9920	1.0000
500	0.1710	0.8330	0.9890	1.0000	1.0000	1.0000	1.0000	0.0910	0.3615	0.6200	0.9180	0.9970	1.0000	1.0000
1000	0.1895	0.9870	1.0000	1.0000	1.0000	1.0000	1.0000	0.0895	0.5885	0.9095	0.9975	1.0000	1.0000	1.0000
κ_1	0.01							0.02						
λ_{11}	0.55	0.75	0.80	0.85	0.90	0.95	0.99	0.55	0.75	0.80	0.85	0.90	0.95	0.99
N														
100	0.0775	0.2160	0.2080	0.1790	0.6390	0.9110	1.0000	0.0775	0.3810	0.5960	0.6030	0.9870	1.0000	-
200	0.0870	0.1980	0.3145	0.5470	0.9450	0.9995	1.0000	0.1035	0.5420	0.8840	0.9835	1.0000	1.0000	-
500	0.0900	0.4395	0.6280	0.9575	1.0000	1.0000	1.0000	0.1635	0.9275	0.9990	1.0000	1.0000	1.0000	-
1000	0.0805	0.6430	0.9355	0.9990	1.0000	1.0000	1.0000	0.2185	0.9970	1.0000	1.0000	1.0000	1.0000	-

Figure 1: Factor strength, equity portfolios, six-factor model

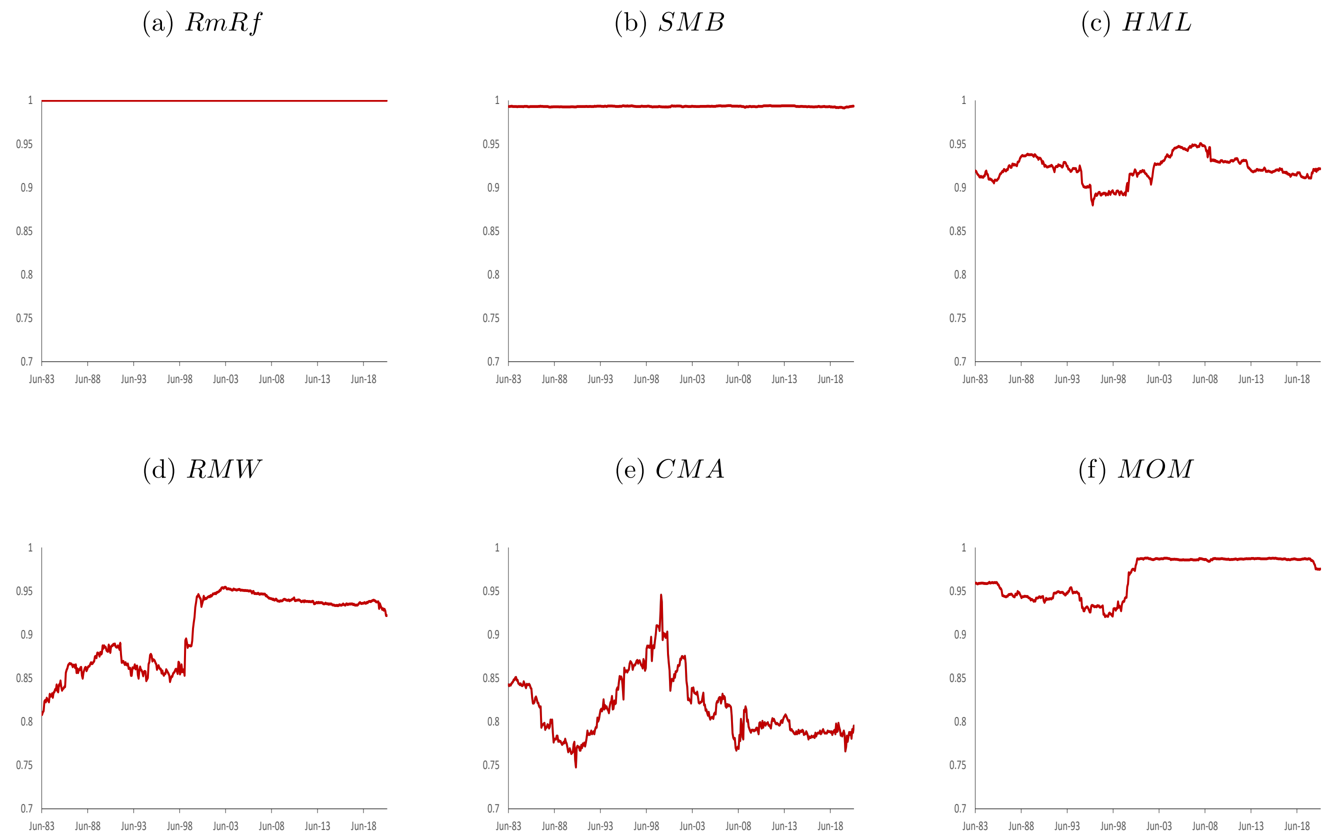
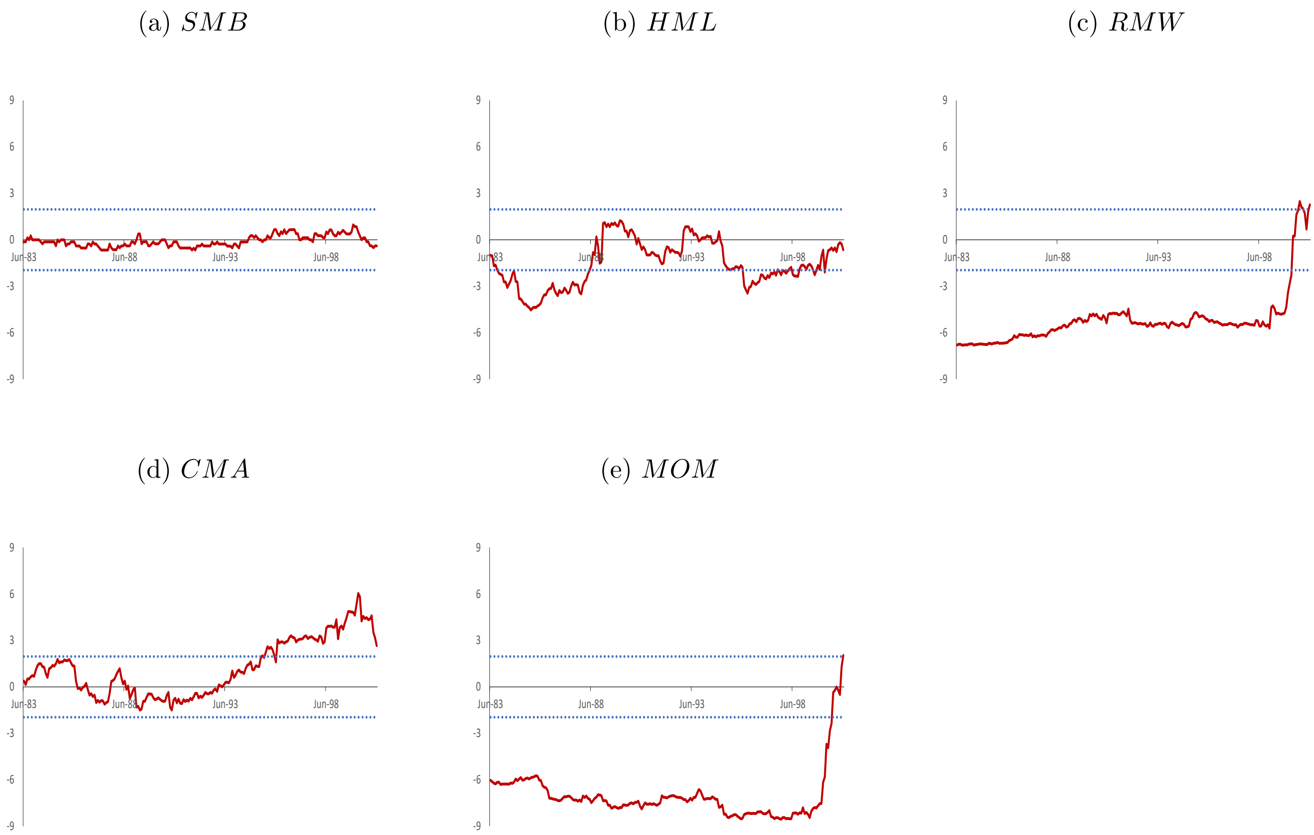


Figure 2: LM statistic, equity portfolios, six-factor model



Supplementary material to "Instability of Factors Strength in Asset Returns"

A Appendix: proofs of theorems

Proof of Theorem 3.1. For $k \in \{1, \ldots, K\}$, consider $\hat{d}_{jikT}(\tau)$ defined in (10), and let

$$\hat{D}_{jkNT}(\tau) = \sum_{i=1}^{N} \hat{d}_{jikT}(\tau) = N^{\hat{\lambda}_{jkNT}(\tau)}, \quad D_{jkN} = \sum_{i=1}^{N} d_{jik} = N^{\lambda_{jk}}, \quad d_{jik} = \mathbb{I}(\beta_{jik} \neq 0).$$

We have

$$\begin{aligned}
[\ln(N)]\left[\hat{\lambda}_{jkNT}(\tau) - \lambda_{jk}\right] &= \ln\left[\frac{\hat{D}_{jkNT}(\tau)}{D_{jkN}}\right] \\
&= \ln\left[\frac{\hat{D}_{jkNT}(\tau) + N^{\lambda_{jk}} - N^{\lambda_{jk}}}{N^{\lambda_{jk}}}\right] \\
&= \ln\left[1 + \frac{\hat{D}_{jkNT}(\tau) - N^{\lambda_{jk}}}{N^{\lambda_{jk}}}\right] \\
&\simeq \frac{\hat{D}_{jkNT}(\tau) - N^{\lambda_{jk}}}{N^{\lambda_{jk}}} \\
&= \frac{\sum_{i=1}^{N} \hat{d}_{jikT}(\tau) - N^{\lambda_{jk}}}{N^{\lambda_{jk}}} \\
&= \frac{\sum_{i=1}^{N} \left\{\hat{d}_{jikT}(\tau) - \mathrm{E}\left[\hat{d}_{jikT}(\tau)\right]\right\}}{N^{\lambda_{jk}}} + \frac{\sum_{i=1}^{N} \mathrm{E}\left[\hat{d}_{jikT}(\tau)\right] - N^{\lambda_{jk}}}{N^{\lambda_{jk}}} \\
&= \hat{A}_{jkNT}(\tau) + B_{jkNT}(\tau),
\end{aligned} \tag{A.1}$$

where

$$\hat{A}_{jkNT}(\tau) = \frac{\sum_{i=1}^{N} \left\{\hat{d}_{jikT}(\tau) - \mathrm{E}\left[\hat{d}_{jikT}(\tau)\right]\right\}}{N^{\lambda_{jk}}}, \quad B_{jkNT}(\tau) = \frac{\sum_{i=1}^{N} \mathrm{E}\left[\hat{d}_{jikT}(\tau)\right] - N^{\lambda_{jk}}}{N^{\lambda_{jk}}}.$$

Since $\mathrm{E}\left[\hat{d}_{jikT}(\tau)\right] = \pi_{jik} = \Pr\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N)\right]$, then

$$\begin{aligned}
B_{jkNT}(\tau) &= \frac{\sum_{i=1}^{N} \Pr\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N)\right] - N^{\lambda_{jk}}}{N^{\lambda_{jk}}} \\
&= \frac{\sum_{i=1}^{\lfloor N^{\lambda_{jk}} \rfloor} \Pr\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N) \,|\beta_{jik} \neq 0\right] - N^{\lambda_{jk}}}{N^{\lambda_{jk}}} \\
&\quad + \frac{\sum_{i=\lfloor N^{\lambda_{jk}} \rfloor + 1}^{N} \Pr\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N) \,|\beta_{jik} = 0\right]}{N^{\lambda_{jk}}}.
\end{aligned} \tag{A.2}$$

Following steps analogous to those in the proof of Theorem 1 in Bailey et al. (2021),

$$\Pr\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N)\,|\beta_{jik} \neq 0\right] = 1 - \exp\left(-C_1 T^{C_2}\right), \tag{A.3}$$

for some $0 < C_1, C_2 < \infty$, so that

$$\frac{\sum_{i=1}^{\lfloor N^{\lambda_{jk}} \rfloor} \Pr\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N)\,|\beta_{jik} \neq 0\right] - N^{\lambda_{jk}}}{N^{\lambda_{jk}}} = \exp\left(-C_1 T^{C_2}\right). \tag{A.4}$$

Further,

$$\Pr\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N)\,|\beta_{jik} = 0\right] \leq C_T \frac{p}{N^{\delta}}, \tag{A.5}$$

for some $0 < C_T < \infty$ such that $C_T \to 1$ as $T \to \infty$, since the distribution of $\hat{t}_{jikT}(\tau)$ converges to a standard normal for $T \to \infty$. This implies that

$$\frac{\sum_{i=\lfloor N^{\lambda_{jk}} \rfloor+1}^{N} \Pr\left[\left|\hat{t}_{jikT}(\tau)\right| > c_p(N)\,|\beta_{jik} = 0\right]}{N^{\lambda_{jk}}} = C_T \frac{p\left(N - N^{\lambda_{jk}}\right)}{N^{\delta+\lambda_{jk}}}. \tag{A.6}$$

Therefore, taking into account (A.2), (A.4) and (A.6), it follows that

$$B_{jkNT}(\tau) = C_T \frac{p\left(N - N^{\lambda_{jk}}\right)}{N^{\delta+\lambda_{jk}}} + O\left[\exp\left(-C_1 T^{C_2}\right)\right].$$

Under Assumption (1), the error terms e_{it} are cross-sectionally independent and

$$\begin{aligned}
\mathrm{Var}\left[\hat{A}_{jkNT}(\tau)\right] &= \mathrm{Var}\left\{\frac{\left\{\sum_{i=1}^{N} \hat{d}_{jikT}(\tau) - \mathrm{E}\left[\hat{d}_{jikT}(\tau)\right]\right\}}{N^{\lambda_{jk}}}\right\} \\
&= \frac{1}{N^{2\lambda_{jk}}} \sum_{i=1}^{N} \mathrm{Var}\left\{\hat{d}_{jikT}(\tau) - \mathrm{E}\left[\hat{d}_{jikT}(\tau)\right]\right\} \\
&= \frac{1}{N^{2\lambda_{jk}}} \sum_{i=1}^{N} \pi_{jikT}(\tau)\left[1 - \pi_{jikT}(\tau)\right] \\
&= \frac{1}{N^{2\lambda_{jk}}} \sum_{i=1}^{N} \pi_{jikT}(\tau) - \frac{1}{N^{2\lambda_{jk}}} \sum_{i=1}^{N} \left[\pi_{jikT}(\tau)\right]^2 \\
&= \frac{1}{N^{2\lambda_{jk}}} \sum_{i=1}^{\lfloor N^{\lambda_{jk}} \rfloor} \pi_{jikT}(\tau) + \frac{1}{N^{2\lambda_{jk}}} \sum_{i=\lfloor N^{\lambda_{jk}} \rfloor+1}^{N} \pi_{jikT}(\tau) \\
&\quad - \frac{1}{N^{2\lambda_{jk}}} \sum_{i=1}^{\lfloor N^{\lambda_{jk}} \rfloor} \left[\pi_{jikT}(\tau)\right]^2 - \frac{1}{N^{2\lambda_{jk}}} \sum_{i=\lfloor N^{\lambda_{jk}} \rfloor+1}^{N} \left[\pi_{jikT}(\tau)\right]^2.
\end{aligned}$$

Therefore, taking into account (A.3) and (A.5), we have

$$\begin{aligned}
\operatorname{Var}\left[\hat{A}_{jkNT}(\tau)\right] &= \frac{1}{N^{2\lambda_{jk}}}\sum_{i=1}^{\lfloor N^{\lambda_{jk}}\rfloor}\left[1-\exp\left(-C_1T^{C_2}\right)\right]+\frac{1}{N^{2\lambda_{jk}}}\sum_{i=\lfloor N^{\lambda_{jk}}\rfloor+1}^{N}C_T\frac{p}{N^{\delta}} \\
&\quad -\frac{1}{N^{2\lambda_{jk}}}\sum_{i=1}^{\lfloor N^{\lambda_{jk}}\rfloor}\left[1-\exp\left(-C_1T^{C_2}\right)\right]^2-\frac{1}{N^{2\lambda_{jk}}}\sum_{i=\lfloor N^{\lambda_{jk}}\rfloor+1}^{N}\left(C_T\frac{p}{N^{\delta}}\right)^2 \\
&= \frac{1-\exp\left(-C_1T^{C_2}\right)}{N^{\lambda_{jk}}}+C_T\frac{p\left(N-N^{\lambda_{jk}}\right)}{N^{\delta+2\lambda_{jk}}} \\
&\quad -\frac{\left[1-\exp\left(-C_1T^{C_2}\right)\right]^2}{N^{\lambda_{jk}}}+\frac{N-N^{\lambda_{jk}}}{N^{2\lambda_{jk}}}\left(C_T\frac{p}{N^{\delta}}\right)^2 \\
&= \frac{1}{N^{\lambda_{jk}}}\left[1-\exp\left(-C_1T^{C_2}\right)\right]\left\{1-\left[1-\exp\left(-C_1T^{C_2}\right)\right]\right\} \\
&\quad +\frac{N-N^{\lambda_{jk}}}{N^{2\lambda_{jk}}}C_T\frac{p}{N^{\delta}}\left(1-C_T\frac{p}{N^{\delta}}\right) \\
&= \frac{N-N^{\lambda_{jk}}}{N^{2\lambda_{jk}}}C_T\frac{p}{N^{\delta}}\left(1-C_T\frac{p}{N^{\delta}}\right)+O\left[\frac{\exp\left(-C_1T^{C_2}\right)}{N^{\lambda_{jk}}}\right].
\end{aligned}$$

This implies that, for $0\le\lambda_{jk}<1$ we have

$$\hat{A}_{jkNT}(\tau)=O_p\left(N^{1/2-\delta/2-\lambda_{jk}}\right).$$

whereas for $\lambda_{jk}=1$

$$\hat{A}_{jkNT}(\tau)=O_p\left[\exp\left(-C_1T^{C_2}\right)/N^{0.5\lambda_{jk}}\right].$$

Recall $\varphi_N(\lambda_{jk})$ defined in (17) and define $\zeta_N(\lambda_{jk})$ as

$$\zeta_N(\lambda_{jk})=\frac{p\left(N-N^{\lambda_{jk}}\right)}{N^{\delta+\lambda_{jk}}}.$$

Consider the case $0\le\lambda_{jk}<1$, for $j=1,2$. For some $0<C_3,C_4<\infty$, we then have

$$\begin{aligned}
&\frac{\varphi_N(\lambda_{1k})^{-1/2}\left\{[\ln(N)]\left[\hat{\lambda}_{1kNT}(\tau)-\lambda_{1k}\right]\right\}}{2^{1/2}}-\frac{\varphi_N(\lambda_{2k})^{-1/2}\left\{[\ln(N)]\left[\hat{\lambda}_{2kNT}(\tau)-\lambda_{2k}\right]\right\}}{2^{1/2}} \\
=\;&\frac{\varphi_N(\lambda_{1k})^{-1/2}\left\{O_p\left(N^{1/2-\delta/2-\lambda_{1k}}\right)+O(1)\,\zeta_N(\lambda_{1k})+O\left[\exp\left(-C_1T^{C_2}\right)\right]\right\}}{2^{1/2}} \\
&-\frac{\varphi_N(\lambda_{2k})^{-1/2}\left\{O_p\left(N^{1/2-\delta/2-\lambda_{2k}}\right)+O(1)\,\zeta_N(\lambda_{2k})+O\left[\exp\left(-C_3T^{C_4}\right)\right]\right\}}{2^{1/2}}.
\end{aligned}$$

Under $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k} = \lambda_1$, it follows that $\varphi_N(\lambda_{1k}) = \varphi_N(\lambda_{2k}) = \varphi_N(\lambda_k)$, and $\zeta_N(\lambda_{1k}) = \zeta_N(\lambda_{2k}) = \zeta_N(\lambda_k)$, and

$$\frac{[\ln(N)]\left[\hat{\lambda}_{1kNT}(\tau) - \hat{\lambda}_{2kNT}(\tau)\right]}{[2\varphi_N(\lambda_k)]^{1/2}} \xrightarrow{d} \mathcal{N}(0,1):$$

the result in (a) follows since

$$\left\{2\max\left\{\varphi_N\left(\hat{\lambda}_{1k}\right), \varphi_N\left(\hat{\lambda}_{2k}\right)\right\}\right\} \xrightarrow{p} 2\varphi_N(\lambda_k)$$

and

$$\left[\varphi_N\left(\hat{\lambda}_{1k}\right) + \varphi_N\left(\hat{\lambda}_{2k}\right)\right] \xrightarrow{p} 2\varphi_N(\lambda_k)$$

as $N \to \infty$ under $\mathcal{H}_{0k} : \lambda_{1k} = \lambda_{2k} = \lambda_k$. Under $\mathcal{H}_{1k} : \lambda_{1k} \neq \lambda_{2k}$ it follows that

$$\begin{aligned}
& \varphi_N(\lambda_{1k})^{-1/2}[\ln(N)]\hat{\lambda}_{1kNT}(\tau) - \varphi_N(\lambda_{2k})^{-1/2}[\ln(N)]\hat{\lambda}_{2kNT}(\tau) \\
= \;& \left\{\varphi_N(\lambda_{1k})^{-1/2}\{[\ln(N)]\lambda_{1k} + O(1)\zeta_N(\lambda_{1k})\} - \varphi_N(\lambda_{2k})^{-1/2}\{[\ln(N)]\lambda_{2k} + O(1)\zeta_N(\lambda_{2k})\}\right\} \\
& + O_p(1) + \left\{\varphi_N(\lambda_{1k})^{-1/2} O\left[\exp\left(-C_1 T^{C_2}\right)\right] - \varphi_N(\lambda_{2k})^{-1/2} O\left[\exp\left(-C_3 T^{C_4}\right)\right]\right\},
\end{aligned}$$

and

$$\left|\varphi_N(\lambda_{1k})^{-1/2}\{[\ln(N)]\lambda_{1k} + O(1)\zeta_N(\lambda_{1k})\} - \varphi_N(\lambda_{2k})^{-1/2}\{[\ln(N)]\lambda_{2k} + O(1)\zeta_N(\lambda_{2k})\}\right| \to \infty$$

as $N \to \infty$, which is sufficient to prove (b). This completes the proof of the theorem. ■

Proof of Theorem 3.2. By Corollary 1 in Qu and Perron (2007), the estimator $\hat{\tau}$ for τ is such that the limiting distribution of the betas is the same as it would be if τ was known. This implies that $[\ln(N)]\left[\hat{\lambda}_{jkNT}(\hat{\tau}) - \lambda_{jk}\right]$ and $[\ln(N)]\left[\hat{\lambda}_{jkNT}(\tau) - \lambda_{jk}\right]$ in (A.1) are asymptotically equivalent. The result in the theorem then follows from the same steps as in the Proof of Theorem 3.1. ■

References

Bailey, N., G. Kapetanios, and H. Pesaran (2021). Measurement of factor strength: Theory and practice. *Journal of Applied Econometrics Forthcoming.*

Qu, Z. and P. Perron (2007). Estimating and testing structural changes in multivariate regressions. *Econometrica 75*, 459–502