Aggregation vs. Distribution: Banking sector capitalisation and systemic liquidity crises*

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Abstract

This paper revisits the conventional wisdom that a better capitalized banking system is more resilient by examining the link between the capitalisation of the banking sector and its resilience to systemic liquidity shocks. We build a model that endogenizes the amount of liquid assets that banks hold ex-ante and the extent of deleveraging that ensues the realisation of liquidity shocks. We uncover a novel inverted-U shaped relationship between the aggregate capital of the banking sector and its system-wide vulnerability. This finding implies that a higher aggregate capital ratio does not necessarily result in greater resilience of the banking system to systemic liquidity shocks. Moreover, we find that the stability of the banking system against such shocks also depends on how the aggregate amount of capital is distributed across banks.

JEL Classification: D82, G21.

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1 Introduction

A conventional wisdom in banking is that a better capitalized banking system is more resilient. This hypothesis is grounded in an extensive literature that sheds light on the impacts of banks' capital on their risk-taking behaviour and resilience. Fundamentally, higher capital equips banks with a bigger buffer to absorb losses, while simultaneously curbing their propensity for excessive risk-taking (ref some paper). These insights are however predominantly demonstrated in the context of individual banks. There are still very limited works that examine the impact of the banking sector's capitalisation on its resilience at the system-wide level. In practice, when macro-prudential authorities monitor the financial stability, they often emphasize the aggregate capital ratio with apparently no attention to an important aspect: the distribution of that aggregate capital across banks.

In this paper, we challenge such conventional wisdom and real-world monitoring practice by arguing that a higher aggregate capital ratio does not necessarily result in greater resilience of the banking system to systemic liquidity shocks. Moreover, the stability of the banking system against such shocks also depends on the cross-sectional distribution of capital in the system. We develop these arguments in a model that endogenizes the amount of liquidity that banks hold ex-ante to protect themselves from liquidity shocks and the extent of deleveraging asset sales that ensue the realisation of such shocks. At a given expected price of assets, a higher aggregate capital, which implies a better capitalised banking system in the first-order stochastic dominance sense, increases the number of banks that hold liquidity for precautionary reasons. This however leads to a second effect. That is, the higher number of banks with precautionary liquidity holdings implies lower selling pressures and so higher asset price. The boost of price in turn induces more banks to rely on the spare liquidity in the system to deal with liquidity shocks. If this second effect is strong enough, higher aggregate capital can actually cause more banks to fail. The relative strength of the two effects depends on both the aggregate capital level and the distribution of that aggregate capital across banks.

Our model has a banking system composed of a continuum of heterogenous banks that are capitalised at different degrees. The rest of their funding is raised through wholesale unsecured short-term debts maturing after one period. Banks can invest at least part of their funding in long-term assets that take two periods to yield a cash flow. The maturity mismatch between the payoff of these long-term assets and debt repayments gives rise to a need for banks to arrange for some liquidity at the interim date when their short-term debt repayments are due.

In our setup, banks have rich sources of liquidity. We first assume that banks can raise liquidity at the interim date by pledging future cash flows of their long-term assets âĂŞ funding liquidity. However, the banks' capacity to generate liquidity in this way can be restricted by the moral hazard problem. Specifically, we assume that at the interim date, the returns of long-term assets may be hit by a negative shock, which requires banks to exert effort to monitor them. As banks' monitoring effort is unobservable and costly, banks' ability to raise new funding will be limited following such bad news. We refer to this situation as banks being hit by a liquidity shock. We also assume that the returns of the long-term assets are perfectly correlated across banks. This implies that the liquidity shock in our model takes the form of systemic shocks as it will affect all banks simultaneously.

To protect themselves from the liquidity shock, banks can choose to hold ex-ante some liquid assets. These liquid assets are less profitable than the long-term assets, but they can be monetised one by one at the interim date. We also allow banks to rely on market liquidity to meet their debt obligations by accounting for the existence, at the interim date, of a secondary market for long-term assets where banks can deleverage and sell these assets to raise liquidity if necessary. We model asset sales Ãă la Acharya and Viswanathan (2011). Precisely, we assume that the long-term assets are specific and can only be acquired by banks that survive liquidity shocks and have spare liquidity. Therefore, the price of the long-term assets, which is determined by the market-clearing condition, is of the "cash-in-the-market" type proposed by Allen and Gale (1994).

The richness of banks' liquidity sources enables us to capture the linkage between not only funding liquidity and market liquidity (as, e.g., in Brunnermeier and Pedersen (2009) and Acharya and Viswanathan (2011)) but also market liquidity and banks' ex-ante liquidity holdings. As clarified below, the second interaction bears an important implication for the link between the capitalisation of the banking sector and its resilience to liquidity shocks.

In our setting, the competitive equilibrium features fire sales where the price discount is negatively correlated with the funding liquidity and the amount of spare liquidity in the system. The latter is determined by the ex-ante liquidity holdings of banks. We show that two factors affect the incentives of each individual banks in the system to hold liquidity.

The first one is the capital level: a bank with higher capital level will have better incentives to hold liquidity to insure themselves from the liquidity shock - we dub this the *precautionary liquidity holding effect*. This effect is similar to the well-understood effect of capital on the risk-taking incentives of individual banks in the literature. It implies that in the equilibrium, there exists a cut-off level of capital such that only banks with capital ratio above that level will hold liquidity ex-ante to survive the liquidity shock.

The second factor is market liquidity: as holding liquidity is costly (the cost is the foregone return of investing in the long-term assets) and banks can sell long-term assets ex-post to meet liquidity needs, if banks expect that the price of those assets is high, that will reduce their incentives to secure any liquidity in advance. This effect means that the cut-off capital level will depend on the expected equilibrium price of long-term assets.

We uncover a novel inverted-U shaped relationship between the aggregate capital of the banking sector and its system-wide vulnerability measured by the fraction of banks that fail following the liquidity shock. As implications, a higher aggregate capital ratio that leads to a better capitalised banking system in the first-order stochastic dominance sense does not result in greater stability if the system is initially poorly capitalised.

To understand the intuition underlying this counter-intuitive non-monotonic relationship, note that higher aggregate capital has two opposite effects on the system-wide resilience of the banking sector to systemic liquidity shocks. First, higher aggregate capital entails higher fraction of banks that are well capitalised. Due the precautionary liquidity holding effect, there are more banks with incentives to hold liquidity to insure themselves from the liquidity shock. As a consequence, a higher aggregate capital will lead to a lower proportion of banks that fail when the shock hits. This effect is the extension of the impact of capital on individual banks' resilience to the system-wide level.

However, higher aggregate capital also has another effect that works through the price of the long-term assets which is overlooked by the conventional wisdom. When higher aggregate capital leads to more banks that are well-capitalised and so hold liquidity to protect themselves, there will be less long-term assets put for sale in the market and more banks able to buy them. This results in an increase in the equilibrium price. The increase in the asset price undermines banks' incentive to hold liquidity leading to an increase of the cut-off capital level and so a higher fraction of banks that fail. We refer to this effect as the liquidity free riding effect as it captures the incentives of a banks to free ride on other banks' liquidity holding.

The overall impact of higher aggregate capital in the system on its resilience therefore depends on which of the two effects - the precautionary liquidity holding effect vs. the liquidity free riding effect âĂŞ is stronger. To shed additional insight on this impact, we rely on numerical analysis and find that the overall relationship between the aggregate capital of the banking system and the fractions of failed banks is of inverted-U shaped.

The numerical example also provides another interesting insight when we investigate if the dispersion of the banking sector's capital distribution matters for the system-wide resilience. We find that the higher that dispersion is, the bigger is the fraction of failed banks. This positive relationship suggests that to improve the stability of the banking sector, it is useful to reduce the difference in the capitalisation between banks.

The organization of the paper is as follows. After discussing the related literature in the next section, we present the model in Section 3. In Sections 4 and 5, we derive the competitive equilibrium and analyze how it changes due to changes in the capital distribution within the banking system. Finally, we conclude in Section 6. All proofs are provided in the Appendix.

2 Related Literature

The relationship between the liability structure and banks' asset composition is linked to an extensive literature. One branch of this literature examines how banks' incentives to take excessive risk can be curbed by requiring them to maintain an adequate capital ratio. See, among others, Rochet (1992), Besanko and Kanatas (1996), Blum (1999), and

Repullo (2004).[1] Another branch of this literature focuses on the determinants of banks' leverage choice (Acharya and Viswanhatan (2011), Castiglionesi, Feriozzi and Pelizzon (2014), and Song and Thakor (2023)). Acharya and Viswanthan (2011) show that banks' optimal leverage choice hinges on future economic prospects, while Castiglionesi et al. (2014) show that banks use capital to insure themselves against the undiversifiable liquidity risk. In both papers, banks' decisions are homogeneous in equilibrium, leading to the absence of high- and low-capitalized banks in equilibrium. Interestingly, Song and Thakor (2023) show that an equilibrium, where high-capitalized banks purchase loans from low-capitalized banks, emerges after a liquidity shock. Whereas in these papers, the capital structure is endogenous, we take it as exogenously given. The reason is that our focus lies in understanding how heterogeneous capital structures impact banks' incentives to manage their liquidity risk.

This paper also contributes to the literature on bank liquidity hoarding. As such, it is directly related to several papers that study banks' choice of investment between liquid and illiquid assets (e.g., Acharya, Shin, and Yorulmazer (2011), Malherbe (2014), Heider et al. (2015), Acharya, Iyer and Sundaram (2015), Farhi and Tirole (2012), Kahn and Wagner (2021), Anhert (2016) and Eisenbach (2017), Gale and Yorulmazer (2013)). Among these papers, the one closest to our model is Gale and Yorulmazer (2013). As in their model, banks hold liquidity for precautionary and speculative motives. However, meanwhile they show that banks hoard inefficiently high levels of liquidity, generating, therefore, a role for policy intervention. By contrast, we abstract from a normative analysis. Our paper considers the effects of banks' liability structures on their choices of liquidity holdings, taking into account other banks' liquidity decisions as well.

The current paper is also connected to several contributions that use the "cash-in-the-market-pricing" mechanism proposed by Allen and Gale (1994, 2004, 2005) to understand financial fragility (see, e.g., Bolton, Santos and Scheinkman (2011); Acharya and Viswanathan (2011); Freixas, Martin and Skeie (2011); Diamond and Rajan (2011); Gale and Yorulmazer (2013); Carletti and Leonello (2019), Dow and Han (2018), Kurlat (2016), and Lorenzoni (2008)). Our design of the loan sale market is directly inspired by Acharya and Viswanathan's (2011) framework, although with a crucial distinction.

[1] For an excellent review of this literature, see Freixas and Rochet (2023).

In our model, we allow banks to proactively hold liquidity ex-ante to self-insure against liquidity shocks. This innovation sheds light on the dynamics of how banks' incentives to manage liquidity risk are are influenced by their specific liability structures.

Finally, a number of recent papers have focused on the optimal design of bank liquidity regulation (see, e.g., Calomiris et al. (2014), Walther (2016), Santos and Suarez (2016), Diamond and Kashyap (2016), Kashyap et al. (2017) and Kara and Ozsoy (2019)). All of these papers present different rationales for introducing liquidity requirements. The closest paper to ours is Kashyap et al. (2017), as they consider the interaction of banks' liquidity and leverage decisions. They provide a rationale for capital and liquidity requirements by showing that there is a wedge between the optimal liquidity and leverage choices made by a bank and a social planner. In their setting, there is no room for speculative liquidity holdings as there is no interbank market for long-term assets. In our paper, we take the leverage structure as given but analyze how it shapes banks' liquidity holdings considering both the precautionary and speculative motives for liquidity holdings. This allows us to draw conclusions on how banks' leverage has an impact on fire-sale prices.

3 The model

We consider an economy that lasts for 3 dates, $t = 0, 1, 2$. It consists of a unit mass of competitive and heterogeneous banks, each being indexed by i and having a balance sheet of size normalized to 1. Banks differ in their capital ratio, with bank i financed by a fraction E_i of equity at date 0. The remaining fraction is uninsured and unsecured short-term debt repaid at $t = 1$ that is held by risk-neutral investors.[2] The repayment that bank i has to make to its short-term debtholders at $t = 1$ is endogenously determined in the model and denoted by D_1^i.

We assume that the banks' capital ratio $\{E_i\}_{i\in[0,1]}$ is distributed in the system accord-

[2] The academic literature has offered two explanations about why banks use short-term debt. The first one comes from the beneficial effects of short-term debt on disciplining banks' managers. The second explanation focuses on the role of banks as liquidity providers: banks issue short-term debt to provide flexibility to creditors who may be hit by a liquidity shock. In the current paper, we do not explicitly model the reason for which the bank uses short-term funding, but assume it exogenously. In line with the second explanation, we justify such use as a bank's response to investors' demand for liquid investments.

ing to a family of continuous distributions $F_h(E)$ on $[0, 1]$ with the density $f_h(E)$. The parameter h characterises the shape of the distribution. We formally define h in Section 5, where we analyze its effects on the equilibrium outcomes.

Investment opportunities. Each bank i has access to two investment opportunities. The first one is a short-term asset, referred to as a liquid asset, which produces a gross deterministic return of 1 per period. The second investment opportunity is a constant-return-to-scale project, referred to as a long-term asset, with two main features. First, it generates a random cash flow $\tilde{y}$ per unit of investment only at $t = 2$. Second, its returns are exposed to a shock that is realized at $t = 1$, as described below.

Systemic liquidity shock. At $t = 1$, a new piece of publicly observable but not verifiable information regarding the returns of the long-term asset becomes available. The non verifiability means that the short-term debt repayment cannot be contingent upon this new information.[3] We assume, for simplicity, two states of nature, i.e., the new information represents either bad news or good news. Bad news is revealed with the probability $1 - \alpha$ and good news happens with the complementary probability. In the case of good news, the long-term asset yields at date 2 a payoff equal to $y_H > 0$ per unit of investment when it succeeds, which occurs with the probability θ, and zero when it fails. The revelation of bad news has two implications for long-term asset returns. First, the unit cash flow y_L generated by this asset in the case of success is lower (i.e., $y_L < y_H$). Second, the success probability depends on banks' monitoring, which is not observable by outsiders. For simplicity, we assume that banks can choose either to exert monitoring effort or to shirk. If a bank does exert effort, the probability of success is equal to $\tilde{\theta} = \theta$, as in the case of good news. However, if it shirks, the probability of success is reduced to $\tilde{\theta} = \theta - \Delta$. Monitoring is costly for banks, and we capture it by assuming that banks obtain some private benefit B per unit of the long-term asset if they shirk. Figure 1 summarizes the payoff structure of the long-term asset.

We will, hereafter, refer to the revelation of bad news as the materialisation of a liquidity shock since, as made clear below, it will limit the extent to which banks can pledge the future cash flow of their long-term assets. We also assume that returns of

[3] Professional forecasts about the state of the economy and projected income-statements of the firms could be examples of piece of information that is observable but not verifiable.

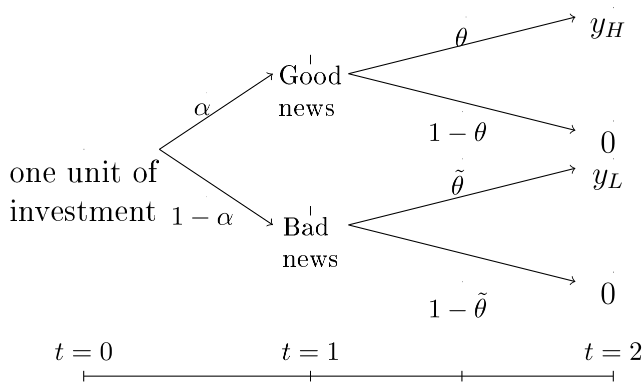


Figure 1: Payoff structure of the long-term asset

long-term assets are perfectly correlated across banks, which implies that the new piece of information will reveal the quality of the long-term assets held by all banks. Therefore, in our setup, the liquidity shock is of a systemic nature because it will hit all of the banks simultaneously.

Secondary market for long-term assets. At $t = 1$, a secondary market for long-term assets is opened, which allows banks in shortage of liquidity to sell their long-term asset holdings to raise additional liquidity. We assume that, due to some sort of asset specificity, potential purchasers of a bank's long-term assets are other banks. Moreover, purchaser banks can raise financing against the assets that they buy. Hence, following Allen and Gale (1994, 2004, 2005), the price of long-term assets will depend on the amount of liquidity available in the banking system.

Timing. The sequence of events, which is summarized in Figure 2, is as follows. At $t = 0$, each bank decides how much to invest in each type of asset. At $t = 1$, information regarding the quality of all banks' long-term assets is revealed, and short-term debt contracts mature. If the amount of liquid assets held by a bank is not enough to repay its debtholders, that bank can sell part of its long-term asset holdings as well as issue new debt by pledging the future payoff of the remaining fraction of its long-term assets. In the case where a bank cannot raise enough liquidity to repay debtholders even after selling all of its long-term assets, it will be closed. At date $\frac{3}{2}$, between $t = 1$ and $t = 2$, if necessary, banks decide whether to exert effort to monitor the long-term assets. At $t = 2$, long-term asset returns are realized, and all payments are settled.

Note that in our framework, it does not matter whether recipients of the new short-

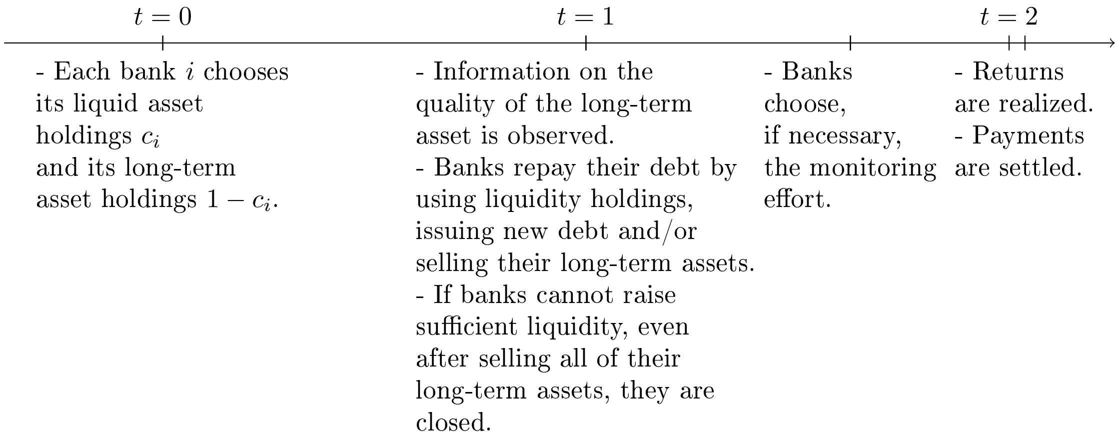


Figure 2: Timeline

term debt issued at $t = 1$ are the current bank's debtholders or new investors. What is relevant is that the price of this new short-term debt hinges on the information revealed. In the scenario where new debt is offered to new investors, we assume that these investors will incorporate the new information into their evaluations of debt repayments. On the other hand, if the new debt is issued to the current banks' debtholders, this action can be viewed as the current debtholders agreeing to refinance the debt and, critically, to reevaluate the debt's pricing in light of the new information.[4]

From now on, for notional convenience, we denote the net expected return of the long-term asset as NPV, i.e., $NPV = \alpha\theta y_H + (1-\alpha)\theta y_L - 1$. We also make the following assumptions on the parameters of the model.

Assumption 1.

$$\theta y_L \geq 1 \geq (\theta - \Delta) y_L + B$$

The main implication of Assumption 1 is that in the case of bad news at date 1, investors will lend to a bank in the case of bad news only if they are ensured that that bank will exert monitoring effort.

Assumption 2.

$$\theta \left(y_L - \frac{B}{\Delta} \right) < 1$$

As shown later, the left-hand side in Assumption 2 represents the maximum amount that a bank could borrow per unit of the long-term asset if bad news is revealed at date

[4] This repricing possibility is different from other contributions in the literature, which assume that debt is rolled over with the same repayment. We believe that our repricing assumption is more consistent with unsecured wholesale debts, which are usually held by sophisticated investors.

1. Hence, the inequality states that the amount of liquidity raised against one unit of the long-term asset in the case of bad news is lower than the amount of liquidity provided by one unit of liquid assets. Notice that this assumption ensures the role of liquid asset holdings in our setting.

4 The competitive equilibrium

Our objective is to investigate how the allocation of capital within the banking system impacts both the magnitude of fire-sale discounts and the severity of liquidity crises within a competitive equilibrium. In the next section, we derive the competitive equilibrium of the present economy by following a series of four sequential steps. First, we examine banks' liquidity needs at $t = 1$. Second, we analyze the demand and supply of long-term assets in the secondary market. Third, we determine banks' liquidity holdings at $t = 0$. Finally, we characterize the competitive equilibrium.

4.1 Banks' liquidity needs

At $t = 1$, bank i has to repay D_1^i to its short-term debtholders. Let c_i denote the amount of liquid assets held by bank i. Consequently, bank i's liquidity needs at $t = 1$ are $D_1^i - c_i$. The bank can meet this liquidity demand by issuing new debt repaid at $t = 2$. The extent to which banks can borrow at $t = 1$ depends on the realized state of nature at that time. We denote the face value of the new debt issued by bank i in state $s \in (g, b)$ as $D_{2,s}^i$.

When favorable information is disclosed at $t = 1$, $s = g$, banks can pledge the entire cash flow generated by their long-term assets to investors, meaning that $D_{2,g}^i \leq y_H(1-c_i)$. Consequently, they can borrow up to θy_H per unit of long-term assets, and there are no issues in meeting their repayment obligations.

However, in the event of adverse news, $s = b$, the face value of the new debt $D_{2,b}^i$ needs to satisfy the following incentive compatibility condition to ensure that bank i does exert monitoring effort:

$$\theta \left(y_L(1 - c_i) - D_{2,b}^i\right) \geq (\theta - \Delta) \left(y_L(1 - c_i) - D_{2,b}^i\right) + B(1 - c_i) \tag{1}$$

Condition (1) puts an upper bound on the funding liquidity that bank i could get:

$$D_{2,b}^{i} \leq \left(y_L - \frac{B}{\Delta}\right)(1 - c_i)$$

Hence, in the case of adverse news, a bank's ability to pledge cash flow per unit of long-term assets is capped at $y_L - \frac{B}{\Delta}$. This implies that the maximum borrowing capacity of each bank per unit of long-term assets is limited to $\theta(y_L - \frac{B}{\Delta})$, which falls strictly below the expected cash flow θy_L.

Define ρ_i and ρ^* as follows:

$$\rho_i = \frac{D_1^i - c_i}{1 - c_i} \quad \text{and} \quad \rho^* = \theta\left(y_L - \frac{B}{\Delta}\right)$$

ρ_i thus represents the liquidity need per unit of long-term asset for bank i, meanwhile ρ^* signifies the maximum funding capacity per unit of long-term asset at $t = 1$, when a liquidity shock occurs.

The following lemma summarizes the banks' liquidity profile at date 1:

Lemma 1. *For any bank i, at $t = 1$:*

(i) *If $\rho_i \leq \rho^*$, bank i can repay its short-term debt without selling its long-term assets in both states of nature.*

(ii) *If $\rho_i > \rho^*$, in case of bad news, bank i is in liquidity shortage and has to sell at least part of its long-term asset holdings to repay its short-term debt.*

We refer to the first situation as the one in which the bank is liquid. The second situation is referred to as the case where the bank is illiquid. Note that unlike in Acharya and Viswanathan (2011), the liquidity needs ρ_i is determined not only by its debt level, D_1^i, but also by the banks' liquidity holdings, c_i. This is crucial because, as we will show below, the optimal banks' liquidity holdings will depend on their own level of capital as well as the distribution of capital within the system.

4.2 Asset sales: demand and supply of long-term assets

Next, we examine the secondary market for banks' long-term assets. As per Lemma 1, at date 1, if good news prevails, all banks can repay their debts. Conversely, in the event of bad news, banks with ρ_i exceeding ρ^* face a liquidity shortfall and must sell their long-term assets to generate additional liquidity. Let p denote the unit price of this asset.

Individual banks' supply. Denote by β_i the fraction of assets that the illiquid bank i needs to sell. It is then determined as follows:

$$\beta_i (1 - c_i) p + (1 - c_i) (1 - \beta_i)\rho^* \geq D_1^i - c_i \tag{2}$$

In Inequality (2), the left-hand side (LHS) represents the total liquidity that bank i could raise. It comprises the proceeds from selling the fraction β_i of assets and the liquidity obtained by issuing new debts against the remaining fraction $1 - \beta_i$. After simplification, we obtain:

$$\beta_i = \min\left(1, \frac{\rho_i - \rho^*}{p - \rho^*}\right) \tag{3}$$

Observe that the funding capacity expands with asset sales if and only if the unit price p exceeds ρ^*. We assume for now that $p \geq \rho^*$, and we will subsequently demonstrate this to be the case. The extent of asset sales is decreasing with the asset's price. Bank i will have to sell all of its existing long-term assets when the price p falls below its liquidity demand ρ_i.

Individual banks' demand. Denote by γ_i the volume of assets that the liquid bank i could buy per unit of the long-term assets it has. Note that no bank would acquire assets at a price higher than their expected payoff. Hence, if p exceeds θy_L, γ_i should be equal to zero for all banks. When $\rho^* < p < \theta y_L$, we determine γ_i as follows:

$$(1 - c_i) (1 + \gamma_i) \rho^* - (D_1^i - c_i) \geq \gamma_i(1 - c_i)p \tag{4}$$

In Inequality (4), the LHS represents the total liquidity available to bank i for asset purchases. It consists of its spare debt capacity from existing assets, $(1 - c_i) \rho^* - (D^i 1 - ci)$, and the liquidity that can be raised against assets to be acquired, $(1 - c_i)\gamma_i\rho^*$. After some rearrangements, we get:

$$\gamma_i = \frac{\rho^* - \rho_i}{p - \rho^*}$$

Note that when $p = \rho^*$ - indicating that the liquidity raised against assets to be acquired is sufficient to pay for the assets, the demand for assets becomes infinitely high. In summary, for each bank i with ρ_i less than or equal to ρ^*, the demand for long-term assets is as follows:

$$\gamma_i(\rho_i, p) = \begin{cases} 0 & \text{if} \quad p > \theta y_L \\ \text{any value between 0 and } \frac{\rho^* - \rho_i}{p - \rho^*} & \text{if} \quad p = \theta y_L \\ \frac{\rho^* - \rho_i}{p - \rho^*} & \text{if} \quad \rho^* < p < \theta y_L \\ \infty & \text{if} \quad p = \rho^* \end{cases} \tag{5}$$

4.3 Banks' ex-ante liquidity holdings

At $t = 0$, bank i chooses how much liquidity to hold, c_i. One primary benefit of holding liquidity for banks is to help them withstand the liquidity shock. Note that at date $t = 1$, following bad news, bank i will have enough liquidity to meet its debt obligations if the following condition is satisfied:

$$D_1^i - c_i \leq (1 - c_i)\rho^* \tag{6}$$

where D_1^i is determined by the condition ensuring that the date 0 debtholders are break-even as follows:

$$\alpha D_1^i + (1 - \alpha) D_1^i = 1 - E_i \tag{7}$$

Equation (6) states that the total debt repayments net of available liquidity is not higher than the amount of liquidity that bank i can raise by pledging the future cash flow of its long-term assets in case of bad news.

Combining Equations (6) and (7), we get the amount of liquidity that each bank needs to hold to overcome the liquidity shock without the need to sell its long-term asset as follows:

$$c_{precau} = max\left(\frac{1 - \rho^* - E_i}{1 - \rho^*}, 0\right) \tag{8}$$

We then can express the liquid asset holdings of each bank i as follows:

$$c_i = max\left(\frac{1-\rho^*-E_i}{1-\rho^*},0\right)+\varepsilon_i \tag{9}$$

As $0 \leq c_i \leq 1$ for all i, ε_i must satisfy the following conditions:

$$min\left(-\frac{1-\rho^*-E_i}{1-\rho^*},0\right) \leq \varepsilon_i \leq min\left(\frac{E_i}{1-\rho^*},1\right) \quad \text{for} \quad \text{all} \quad i \tag{10}$$

The second term in the RHS of Expression (9) reflects bank i's liquidity strategy. It can be summarized as follows:

(i) If $\varepsilon_i = 0$, bank i holds liquidity only to withstand the liquidity shock.

(ii) If $\varepsilon_i > 0$, bank i holds excess liquidity to benefit from fire-sales.

(iii) If $\varepsilon_i < 0$, bank i will be an illiquid bank and must sell its long-term assets if the liquidity shock occurs.

The first situation represents a *precautionary* strategy, in which bank i holds $c_i = max\left(\frac{1-\rho^*-E_i}{1-\rho^*},0\right)$, which is the minimum amount needed to withstand the liquidity shock without selling long-term assets. The second situation corresponds to a *speculative* strategy, in which bank i holds liquidity with the intention of acquiring assets from illiquid banks. The last situation corresponds to a *gambling* strategy, where the bank survives only if the liquidity shock does not occur.

The problem that determines the optimal liquidity holdings of a bank i can be written as follows:[5]

Program $\wp$

$$\begin{aligned}\underset{\varepsilon_i}{\text{Max}}\{NPV + E_i - \left(max\left(\frac{1-\rho^*-E_i}{1-\rho^*},0\right)+\varepsilon_i\right)NPV \\ +(1-\alpha)(\theta y_L - p)[\frac{\varepsilon_i(1-\rho^*)}{p-\rho^*} - min(\frac{1-\rho^*-E_i}{p-\rho^*},0)]\mathbb{1}_{\rho_i<p} \\ -(1-\alpha)(\theta y_L - p)\left(1-\varepsilon_i - max\left(\frac{1-\rho^*-E_i}{1-\rho^*},0\right)\right)\mathbb{1}_{\rho_i\geq p}\Big\}\end{aligned} \tag{11}$$

subject to Condition (10)

[5]See the Appendix for a detailed derivation of this problem.

The first two terms of Expression (11) represent bank i's net expected returns from the long-term asset investment. The third term captures the opportunity cost of holding cash. The fourth and fifth terms account for the expected gains or losses from trading. Specifically, the fourth term in the bracket of Expression (11) denotes the additional profit that bank i gets when acting as a buyer in the market, or the loss it incurs if it must sell a portion of its long-term assets to overcome the liquidity shock. Meanwhile, the last term in the bracket represents the loss bank i faces if it is liquidated at $t = 1$. It's important to note that the absolute value of the last two terms is strictly positive only when the price of the long-term assets falls below their fundamental value θy_L.

For notation convenience, we define δ as follows:

$$\delta = \frac{(1-\alpha)(1-\rho^*)(\theta y_L - \rho^*)}{NPV + (1-\alpha)(1-\rho^*)} \tag{12}$$

The following lemma summarizes relationship between banks' liquidity strategies and the liquidity of the long-term asset.

Lemma 2. *For any bank i, the optimal liquidity strategy is:*

(i) *If $\rho^* < p < \rho^* + \delta$:*

$$\varepsilon_i = min\left(\frac{E_i}{1-\rho^*}, 1\right)\mathbb{1}_{\rho_i \le p} + min\left(-\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right)\mathbb{1}_{\rho_i > p}$$

(ii) *If $\rho^* + \delta < p \le \theta y_L$:*

$$\varepsilon_i = min\left(-\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right)$$

where $\mathbb{1}_A$ is indicator function.

The first part of Lemma 2 establishes that when banks anticipate low asset prices, some of them, specifically those with $\rho_i \le p$, will choose to hold speculative liquidity.[6] The rationale behind this choice is that when asset prices are low, the potential gains from acquiring assets at a lower cost become substantial. Consequently, banks are incentivized to maintain ample liquid assets to withstand liquidity shocks and capitalize

[6] In our setup, these banks will invest all of their funding in liquid assets. This extreme result is due to our assumption concerning the constant-return-to-scale feature of the long-term assets, which implies that banks' expected profit is linear in their liquidity holdings.

on opportunities presented by fire sales. The remainder of the banks, those with $\rho_i > p$, will choose to be illiquid and hold no cash ($c_i = 0$). The second part of the lemma complements this by stating that if banks expect asset prices to be sufficiently high, none of them will find it necessary to secure pre-existing sources of liquidity. This is because in a high-price scenario, banks can rely on the liquidity of the long-term asset to address liquidity shocks and avoid the costs associated with holding liquidity.

4.4 Characterisation of competitive equilibrium

We are now equipped to examine the existence and main features of the competitive equilibrium, which is defined by a set of liquidity holdings and the long-term asset equilibrium price.

Definition of the ex-ante competitive equilibrium: A competitive equilibrium in our setup is (1) a set of banks' liquidity holdings $\{c_i^*\}_{i\in[0,1]}$; and (2) the equilibrium price p^e of the long-term assets at $t = 1$, such that:

(1) c_i^* is the optimal amount of liquid assets that each bank i holds, given p^e.

(2) p^e is the equilibrium price induced by the choices $\{c_i^*\}_{i\in[0,1]}$ after the occurrence of the liquidity shock.

We present a property of the competitive equilibrium in the following proposition:

Proposition 1. *Only a competitive equilibrium where $p^e \leq \rho^* + \delta$ can exist.*

The intuition underlying Proposition 1 comes directly from Lemma 2. In fact, as underscored in this lemma, if banks anticipate a price than higher than $\rho^* + \delta$, there would be no motivation for banks to hold liquidity ex-ante. Consequently, in such a scenario, there would be an insufficient supply of liquidity available ex-post to support this price. In simpler terms, a price exceeding $\rho^* + \delta$ cannot be sustained in equilibrium due to the absence of ex-ante incentives for liquidity hoarding.

A direct implication of Proposition 1 is that in equilibrium, the price is lower than the fundamental value of the long-term assets. This characteristic pertains to the *systemic* nature of our liquidity shock. In a scenario with *idiosyncratic* liquidity shocks, an equilibrium where the price is equal to the fundamental value can exist if only a small fraction of banks are affected.[7] However, this equilibrium is not possible when the shock is systemic. No bank would have an incentive to maintain excess liquidity ex- ante to absorb assets if they anticipate that these assets will be traded at their fundamental value.

To determine the competitive equilibrium, we explore two possible situations: one in which the equilibrium price equals $\rho^* + \delta$ and another where it is strictly lower.

Lemma 3. *In the equilibrium where the price equals $\rho^* + \delta$, if such an equilibrium exists, it is characterized by a threshold capital ratio of $1 - \rho - \delta$. Under this equilibrium:*

- *Banks with a capital ratio lower than $1 - \rho^* - \delta$ hold no liquidity and face closure at $t = 1$ following the realization of the liquidity shock.*
- *Banks with a capital ratio greater than or equal to $1 - \rho^* - \delta$ are indifferent to any liquidity holdings between max $\left(\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right)$ and 1 and will survive the shock.*

Lemma 4. *In the equilibrium where the price is below $\rho^* + \delta$, if it exists, it is defined by a threshold capital ratio denoted as $\overline{E}$. Under this equilibrium:*

- *Banks with a capital ratio lower than $\overline{E}$ hold no liquidity and face closure at $t = 1$ following the realization of the liquidity shock.*
- *Banks with a capital ratio greater than or equal to $\overline{E}$ invest all funds in liquid assets and will survive the shock.*

The cutoff level $\overline{E}$ and the price p^e in this equilibrium are determined by the following conditions:

$$\int_{\overline{E}}^{1} E f_h(E) dE = (p^e - \rho^*) \int_0^{\overline{E}} f_h(E) dE \tag{13}$$

$$\frac{\overline{E}}{p^e - \rho^*} + 1 - \frac{NPV}{(1-\alpha)(\theta y_L - p^e)} = 0 \tag{14}$$

$$p^e < \rho^* + \delta \tag{15}$$

Lemmas 3 and Lemma 4 delineate the characteristics of the two possible equilibria. A direct implication of these results is that, in any banking system, poorly-capitalized banks will not survive the liquidity shock, while well-capitalized banks have an incentive

[7] See, for example, Acharya et al. (2011).

to maintain liquidity and withstand such shocks. This conclusion aligns with established findings in the risk-capital literature, as observed in Rochet (1992), Bensako and Kantas (1996), Blum (1999), and Repullo (2004), among others. The novelty here is that the cutoff level, determining whether a bank is under-capitalized or well-capitalized, is contingent on the distribution of capital within the system, as it is shown in Lemma 4.

It is worth elaborating on the two Equations (13) - (14), as they jointly determine the equilibrium price and the cutoff level. Equation (13) represents the balance between the total available liquidity to absorb the long-term assets at $t = 1$ (on the LHS) and the total market value of all assets offered for sale (on the RHS). Therefore, the resulting equilibrium price is of *cash-in-the-market pricing* as in Allen and Gale (1994). As such, the equilibrium price depends on the overall amount of liquidity in the system for purchasing the long-term asset. This available amount of liquidity depends on the capital distribution of well-capitalized banks.

Equation (14) defines the threshold for the marginal bank that will be illiquid at $t = 1$. By working out this threshold and taking derivative with respect to p, we get the following result.

Proposition 2. *The threshold $\overline{E}$ increases with the equilibrium price p^e.*

The result in Proposition 2 arises directly from the dynamics of the cash-in-the-market equilibrium price. Notably, due to fire-sale prices, liquidity holdings emerge as strategic substitutes among banks. In this strategic interaction, banks find themselves in a position to free-ride on the liquidity holdings of others, particularly those willing to acquire the long-term asset at a sufficiently high price. A crucial observation arises from this dynamic: the marginal bank, opting not to hold liquidity, emerges as better capitalized precisely when the equilibrium price is high. The important message conveyed in this result is that banks' liquidity hoarding decision depends on the capital distribution of potential buyers and on the liquidation of assets of low capitalized banks through the equilibrium price.

The next questions to address are how this threshold and the equilibrium price varies with the distribution parameter h and what implications it holds for the proportion of failed banks.

5 Fire-sale prices and severity of liquidity crises

Having characterized the equilibrium, our focus shifts to analyzing how changes in the capital distribution of the banking system affect the extent of the fire-sale problem and the severity of liquidity crises. We begin this analysis with characterising which of the two equilibria characterised in Lemmas 3 and 4 prevails. We assume that the parameter h is a positive scalar such that $F_h(E) \geq F_{h'}(E)$ for $h < h'$. In other words, we assume that $F_{h'}(E)$ first-order stochastic dominates $F_h(E)$. We complement the results with a numerical example where the parameter h represents the mean of the distribution. We also use the numerical analysis to investigate how the severity of liquidity crises varies with the variance of the capital distribution of the banking system.

5.1 Analytical decomposition

The result in the following proposition helps to determine which of the equilibria described in Lemmas 3 and 4 prevails, and it hinges on the parameter of the capital distribution h. To emphasize this dependency, the solutions to the system of equations in Lemma 4 will be denoted as as $\overline{E}(h)$ and $p(\overline{E}(h), h)$.

Proposition 3. *There exists a unique threshold $\hat{h}$ defined by $p(\overline{E}(\hat{h}), \hat{h}) = \rho^* + \delta$ such that:*

- *For $h \geq \hat{h}$, the prevailing equilibrium corresponds to the description in Lemma 3.*
- *For $h < \hat{h}$, the prevailing equilibrium corresponds to the description in Lemma 4.*

The cutoff capital ratio and the unit price of the long-term assets in the equilibrium can therefore be summarized respectively as follows:

$$\hat{E}^e(h) = \begin{cases} 1 - \rho^* - \delta & \text{if} \quad h \geq \hat{h} \\ \overline{E}(h) & \text{if} \quad h < \hat{h} \end{cases} \tag{16}$$

and

$$p^e\left(\hat{E}^e(h), h\right) = \begin{cases} \rho^* + \delta & \text{if} \quad h \geq \hat{h} \\ p(\overline{E}(h), h) & \text{if} \quad h < \hat{h} \end{cases} \tag{17}$$

To compare how the equilibrium is affected by the distribution of capital in the economy, we compute the total derivatives of $p^e(\hat{E}^e(h), h)$ and $F_h\left(\hat{E}^e(h)\right)$ with respect to the parameter h. Here on, we indicate the distribution function as $[F(E,h), f(E,h)]$, in order to pin down the effect of a change in the shape parameter h.[8]

$$\frac{dp^e(\hat{E}^e(h),h)}{dh} = \underbrace{\overbrace{\frac{\partial p^e(\hat{E}^e(h),h)}{\partial h}}^{\geq 0}}_{\text{Effect of precautionary motive}} + \underbrace{\overbrace{\frac{\partial p^e(\hat{E}^e(h),h)}{\partial \hat{E}^e}}^{<0}\overbrace{\frac{\partial \hat{E}^e(h)}{\partial h}}^{\geq 0}}_{\text{Effect of liquidity free-riding}} \tag{18}$$

and

$$\frac{dF(\hat{E}^e,h)}{dh} = \underbrace{\overbrace{\frac{\partial F(\hat{E}^e(h),h)}{\partial h}}^{\leq 0}}_{\text{Effect of precautionary motive}} + \underbrace{\overbrace{\frac{\partial F(\hat{E}^e(h),h)}{\partial \hat{E}^e}}^{\geq 0}\overbrace{\frac{\partial \hat{E}^e(h)}{\partial h}}^{\geq 0}}_{\text{Effect of liquidity free-riding}} \tag{19}$$

We observe that a variation in the parameter h exerts two distinct effects on the equilibrium price and the fraction of banks that experience failure following a liquidity shock. The first effect, as represented by the initial term on the right-hand side of Equations (18) - (19), operates through h's impact on the shape of the banks' capital distribution, $F(E,h)$. An increase in h shifts the distribution to the right, implying fewer banks falling below the cutoff capital ratio, as seen in the first term of Equation (19). In simpler terms, more banks are incentivized to hold cash for precautionary purposes. Consequently, there are fewer long-term assets in the market, and more banks are willing to purchase them. This results in an increase in the equilibrium price of long-term assets, as indicated in Equation (18).

The increase in the equilibrium price of long-term assets gives rise to the second effect, accounted for by the latter term on the right-hand side of Equations (18) - (19). This effect comes from h's influence on the identity of the marginal banks that may fail. Essentially, the higher expected price reduces the loss arising from fire-sale prices, consequently diminishing the incentives for banks to hold liquidity. Formally, the cutoff capital ratio increases. Consequently, the group of banks that does not hold liquidity and benefits from others' liquidity holding becomes larger compared to the scenario without

[8] Since the capital ratio in the banking system is distributed according to the distribution $F(E,h)$, the fraction of the banks that fail in the equilibrium can be computed as $F(\hat{E}^e(h),h)$.

adjustments in the capital ratio threshold.

We refer to the first effect as the *the precautionary liquidity holding effect*, as it reflects the direct impact of banks' capital on their incentives to hold liquidity. The second effect is referred to as the *liquidity free riding effect* because it measures the influence of changes in the banks' capital distribution on the losses from selling assets at fire-sale prices. Therefore, when the banking system becomes more capitalized, due to the precautionary motive for liquidity holdings, there is a weak increase in the equilibrium price and a decrease in the fraction of the banks that fail following the liquidity shock. Conversely, the free-riding effect of liquidity holdings has the opposite impact. The overall result thus depends on which one of the two effects is stronger, and they are summarized in the following propositions.

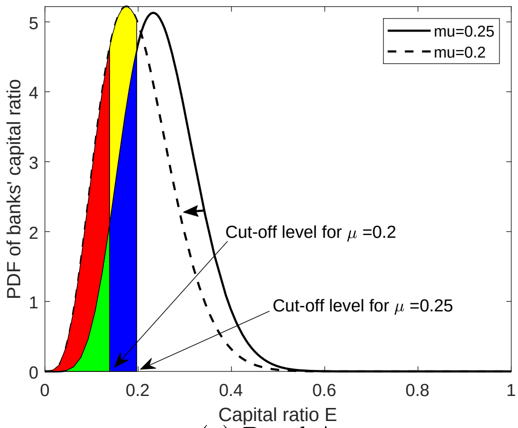
Proposition 4. *The equilibrium price of long-term assets is weakly increasing in the degree of capital in the banking system.*

In the case of the equilibrium price, we prove that the free-riding effect is never more substantial than the impact of the precautionary motive for liquidity holding. This is intuitive, as banks would never increase their liquidity holdings to the extent that it saturates the gains from buying assets. Doing so would prevent them from actually realizing any additional profits.

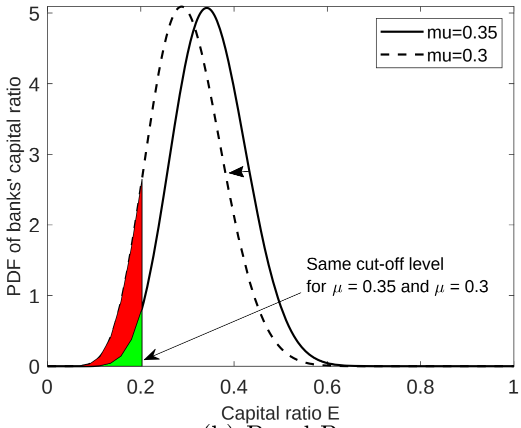
For the case of the proportion of failed banks after a liquidity shock, we know from Expressions (16) and (17) that neither the cutoff capital ratio nor the equilibrium price change for high enough values of h. This neutralizes the liquidity free riding effect, leaving only the precautionary liquidity holding effect, which in this case is negative. However, for low values of h which of the two effects dominates remains an empirical question.

Figure 3 illustrates, in a numerical example where the banks' leverage distribution is the beta distribution, the impact of a shift in the banks' leverage distribution. When the probability density function (pdf) of banks' capital ratio is the solid line, the fraction of the banks that will fail at date 1 is represented by the blue and green areas in Panel A, and by the green area in Panel B.

- In Panel A, when the banks' leverage distribution shifts from the solid line to the dashed line, as explained above, two countervailing effects come into play. First,



(a) Panel A



(b) Panel B

Figure 3: **Impact of a shift in the banks' leverage distribution**
The figure shows the numerical results for the case in which the capital ratio of the banks in the system is distributed according to the beta distribution, parametrized in terms of its mean $\mu \in (0, 1)$ and its variance σ. Since the lower the value of μ, the bigger the mass on the left of the beta distribution is, we adjust the degree of leverage in the banking system by varying μ while fixing $\sigma = 0.006$. The numerical values for other parameters are as follows: $\alpha = 0.6, \theta = 0.7, y_H = 1.8, y_L = 1.5, \Delta = 0.4$ and $B = 0.3$.

holding $\hat{E}^e$ fixed, this shift of the distribution increases the number of banks that would be illiquid, which is now represented by a combination of 4 areas, namely blue, green, yellow, and red. This drives down the expected price and moves the threshold $\hat{E}^e$ to the left. Hence, in the new equilibrium, the fraction of banks that will be closed at date 1 is composed of the green and red areas.

- In Panel B, the second effect is mute. The reason is that when the banking system is well capitalised, the spare capacity of the liquidity holdings among well-capitalised banks is still high. Therefore, a small shift in the leverage distribution does not lead to a decrease in the equilibrium price. Formally, Formally, as per Proposition 3, in high h scenario, banks with a capital ratio greater than $1 - \rho^* - \delta$ are indifferent to any liquidity holdings between $max\left(\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right)$ and 1. Following a small shift to the left of the capital distribution, these banks can increase their liquidity holdings, and thus, the pre-shift equilibrium price can still be supportable.

5.2 Numerical analysis

We conclude this section with a numerical implementation of our model. It illustrates the above comparative statistics results and sheds light on some interesting results.

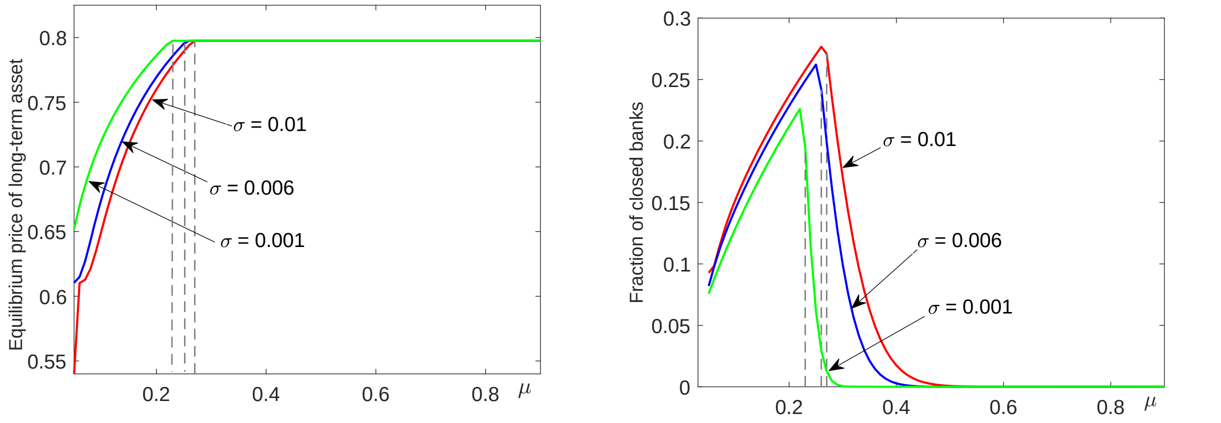
Our baseline parameter values are as follows: good news is revealed with a probability of 0.6. In this scenario, long-term assets yield a payoff of 1.8 with a probability, denoted as θ, equal to 0.7. In the case of bad news, the successful cash flow of the long-term assets is reduced to 1.5. Without banks' monitoring, the probability of success is decreased by $\delta = 0.4$. Banks' private benefits in the case of shirking are assumed to be 0.3. The capital of banks in the system follows a beta distribution, characterized by its mean, μ, and its variance, σ. For a beta distribution, an increase in the mean implies that the new distribution first-order stochastically dominates the initial one.

In terms of our model, we consider the mean of capital in the system as the parameter h, which changes the mass of banks on the left of the distribution.

To perform a numerical analysis, we vary the mean μ while keeping the variance σ fixed. We numerically solve the system of two equations (Equations (14) and (13)) for each combination of (μ, σ), and we determine the competitive equilibrium corresponding to each value of this couple by checking whether the resulting price from solving the two equations satisfies Condition (15). If it does, the corresponding equilibrium aligns with the characteristics described in Lemma 4. Otherwise, the corresponding equilibrium is the one in Lemma 3.

Figure 4(a)illustrates the equilibrium price for various values of μ, while Figure 4(b) represents the fraction of banks that will be closed after the liquidity shock, also for different μ values. For the sake of completeness, we also depict in both figures different curves for different values of σ, while keeping μ fixed.

Consistent with Proposition 4, we find that the more capitalized the banking system is, i.e., the larger the mean, the higher the equilibrium price is. An additional insight from Figure 4(a) is that the equilibrium is also larger when the dispersion of the capital distribution decreases. More interestingly, we see from Figure 4(b) that the proportion of the banks that will fail when the liquidity shock is materialized is not monotonic with respect to the *mean* of capital in the banking system but is monotonically increasing with



(a) Equilibrium price for different values of μ (b) Fraction of closed banks for different values of μ

Figure 4: Characteristics of the competitive equilibrium

the *dispersion* of the capital in the system. Several interesting insights emerge from these results.

First, the inverted-U shaped relationship between the fraction of failed banks and the mean of capital in the banking system implies that improving the banking system's capitalization is beneficial for financial stability. However, an exception arises when the system is poorly capitalized. In such cases, substantial recapitalization efforts are necessary to enhance financial stability. Furthermore, the monotonic relationship of the fraction of closed banks with respect to the dispersion of capital suggests that recapitalization actions should focus not only on improving overall capitalization but also on reducing the dispersion of capitalization across banks.

Second, the fact that the link between the fraction of failed bank and the mean of capital in the system has different monotonicity properties than the link between the former and the dispersion of the capital could highlight an interesting dissimilarity between solvency crises and liquidity crises. While increasing the level of capitalization in the system would definitely benefit the solvency crises, for liquidity crises, the dispersion of capitalization across banks would play an important role.

Finally, when comparing the impact of the capital distribution of the banking system on the fire-sale discount and the fraction of failed banks, we can observe that a severe fire-sale problem and a high proportion of bank failures in the system do not necessarily occur simultaneously. This finding highlights the complex interplay between capital distribution, fire-sale prices, and systemic stability, underlining the need for a multifaceted approach to addressing these challenges.

6 Conclusion

This paper investigates the interplay between capital distribution in the banking system, banks' liquidity decisions, and their consequences on fire-sale problems and liquidity crises. We show that well-capitalized banks exhibit a stronger motivation to maintain sufficient liquidity buffers. Moreover, our findings indicate that in better-capitalized banking systems with a higher proportion of well-capitalized banks, fire-sale discounts tend to be lower, contributing to overall system stability. However, the relationship between the capitalization of the banking system and the proportion of failing banks during a liquidity shock is not linear; enhancing capitalization is generally stabilizing, except in poorly capitalized systems.

These findings have implications for regulators and policymakers aiming to boost financial stability. In poorly capitalized banking systems, our results emphasize the necessity for substantial recapitalization efforts to fortify the system against liquidity shocks effectively. Furthermore, the reduction of capital dispersion among banks emerges as an alternative strategy for mitigating the severity of liquidity crises.

We believe that the present framework provides a useful springboard for future research that helps deepen our understanding of the impact of banks' leverage on their incentives for liquidity management. Promising extensions include endogenizing banks' leverage choices to analyze their effects on reducing the likelihood of a liquidity shock. Additionally, exploring the relationship between holding liquid assets and funding through long-term debt as substitutes from a liquidity risk perspective presents an intriguing avenue for further study.

Appendix

A Derivation of Program $\wp$.

Note that at date 1, after the realization of the liquidity shock, a bank i either has to sell all of its long-term assets and be closed, or it survives the shock after selling a fraction of its long-term assets, or it is liquid and can buy the long-term assets sold by illiquid banks. Therefore, its expected profit can be written as follows:

$$\Pi_i = NPV + E_i - c_i NPV + (1-\alpha)(1-c_i)\gamma_i(\theta y_L - p)\mathbb{1}_{\rho_i \leq \rho^*} \\ - (1-\alpha)(1-c_i)\beta_i(\theta y_L - p)\mathbb{1}_{\rho_i > \rho^*} \quad \text{(A1)}$$

The problem that determines the optimal liquidity holdings of bank i is:

$$\underset{c_i \in [0,1]}{\text{Max}}\, \Pi_i \quad \text{(A2)}$$

subject to

$$\alpha D_1^i + (1-\alpha) D_1^i \mathbb{1}_{\rho_i \leq \rho^*} + (1-\alpha) min\left[D_1^i, (1-c_i)\beta_i p + (1-c_i)\left(1-\beta_i\right)\rho^* + c_i\right] \mathbb{1}_{\rho_i > \rho^*} \\ = 1 - E_i \quad \text{(A3)}$$

$$\frac{D_1^i - c_i}{1-c_i} = \rho_i \quad \text{(A4)}$$

$$\beta_i = \min\left(1, \frac{\rho_i - \rho^*}{p - \rho^*}\right) \quad \text{(A5)}$$

$$\gamma_i = \frac{\rho^* - \rho_i}{p - \rho^*} \quad \text{for} \quad \rho^* < p < \theta y_L \quad \text{(A6)}$$

To express the above program in terms of ε_i, let us first rewrite the fourth term in the RHS of Expression (A1). We have:

$$(1-c_i)\gamma_i = \frac{(1-c_i)\rho^* - D_1^i + c_i}{p - \rho^*} = \frac{(1-c_i)\rho^* - 1 + E_i + c_i}{p - \rho^*} \quad \text{(A7)}$$

where the second equality comes from Condition (A3) under the case $\rho_i \leq \rho^*$. Replacing

c_i as defined in Expression (**??**) into Expression (A7), we obtain:

$$(1-c_i)\gamma_i = \frac{\varepsilon_i(1-\rho^*)}{p-\rho^*} - min\left(\frac{1-\rho^*-E_i}{p-\rho^*}, 0\right) \quad \text{for} \quad \rho^* < p < \theta y_L \tag{A8}$$

We proceed similarly with the last term in the RHS of Expression (A1) and obtain:

$$(1-c_i)\beta_i = \begin{cases} -\frac{\varepsilon_i(1-\rho^*)}{p-\rho^*} + min\left(\frac{1-\rho^*-E_i}{p-\rho^*}, 0\right) & \text{if} \quad \rho^* < \rho_i < p \\ 1-\varepsilon_i - max\left(\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right) & \text{if} \quad \rho_i \geq p \end{cases} \tag{A9}$$

Then by plugging Expressions (A8) - (A9) into Expression (A1), we get the objective function of Program $\wp$.

B Proofs

Proof of Lemma 2. To determine the optimal liquidity holdings, let us compute the first derivative of Expression (11) as follows:

$$-NPV + \frac{(1-\alpha)(1-\rho^*)(\theta y_L - p)}{p-\rho^*}\mathbb{1}_{\rho_i<p} + (1-\alpha)(\theta y_L - p)\mathbb{1}_{\rho_i \geq p} \tag{A10}$$

It is obvious that for banks with $\rho_i \geq p$, Expression (A10) is strictly negative. For banks with $\rho_i < p$, after some arrangements, we see that Expression (A10) is strictly positive if and only if the following condition is satisfied:

$$p < \rho^* + \frac{(1-\alpha)(1-\rho^*)(\theta y_L - \rho^*)}{NPV + (1-\alpha)(1-\rho^*)} = \rho^* + \delta \tag{A11}$$

Hence, as stated in Lemma 2, we have:

- If $p > \rho^* + \delta$, Expression (11) is strictly decreasing with ε_i for all i, which means that at the optimum $\varepsilon_i = min\left(-\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right)$.
- If $p < \rho^* + \delta$, Expression (11) is strictly increasing with ε_i for banks with $\rho_i < p$, and strictly decreasing with ε_i for banks with $\rho_i \geq p$. Therefore, at the optimum:

$$\varepsilon_i = min\left(\frac{E_i}{1-\rho^*}, 1\right)\mathbb{1}_{\rho_i<p} + min\left(-\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right)\mathbb{1}_{\rho_i \geq p} \tag{A12}$$

Proof of Proposition 1. Note that from Lemma 2, we see that if banks expect $p > \rho^* + \delta$, they will all hold zero liquidity. Hence, there is no liquidity available ex-post to support this price. In other words, an equilibrium where $p > \rho^* + \delta$ cannot exist.

Proof of Lemma 3. This Lemma characterizes the equilibrium in which the price is equal to $\rho^* + \delta$. From Program $\wp$, we see that banks with $\rho_i > p$ will hold zero liquidity, and, consequently, will have to sell all of their long-term assets. Their profits will thus be equal:

$$\Pi_i^{illi} = NPV + E_i - (1-\alpha)(\theta y_L - \rho^* - \delta) \tag{A13}$$

For banks that have $\rho_i < p$, we also see that they will be indifferent to any liquidity holdings between $max\left(\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right)$ and 1. Their expected profit is as follows:

$$\Pi_i^{li} = E_i + \frac{E_i}{1-\rho^*} NPV \tag{A14}$$

Note that the condition $\rho_i > p = \rho^* + \delta$ implies that $\frac{D_1^i - c_i}{1-c_i} > \rho^* + \delta$. Since a bank i that has $\rho_i > p$ will be closed at date 1, D_1^i is determined as follows

$$\alpha D_1^i + (1-\alpha)p = 1 - E_i \tag{A15}$$

Therefore, condition $\rho_i > p = \rho^* + \delta$ is equivalent to $E_i < 1 - \rho^* - \delta$. Notice also that if $E_i < 1 - \rho^* - \delta$, then we have

$$\Pi_i^{li} < \Pi_i^{illi} \tag{A16}$$

which means that banks with capital ratio lower than $1 - \rho^* - \delta$ will indeed prefer to hold zero liquidity. They will thus be closed at date 1 following the realization of the liquidity shock.

Proof of Lemma 4. This Lemma characterizes the equilibrium in which the price is strictly lower than $\rho^* + \delta$.

We know from Lemma 2 that if banks expect the price to be strictly lower than $\rho^* + \delta$, the banks' optimal liquidity holdings are as follows:

$$\varepsilon_i = min\left(\frac{E_i}{1-\rho^*}, 1\right) \mathbb{1}_{\rho_i \leq p} + min\left(-\frac{1-\rho^*-E_i}{1-\rho^*}, 0\right) \mathbb{1}_{\rho_i > p}$$

Therefore, we need to determine when a bank i will choose their liquidity holding such that $\rho_i \leq p$ or $\rho_i > p$ by comparing its expected profits in the case it is liquid and in the case it is not. If bank i chooses to be illiquid, its expected profit is as follows:

$$\Pi_i^{illi} = NPV + E_i - (1-\alpha)(\theta y_L - p) \tag{A17}$$

If bank i chooses to be liquid, its expected profit is as follows:

$$\Pi_i^{li} = E_i + (1-\alpha)(\theta y_L - p)\frac{E_i}{p - \rho^*} \tag{A18}$$

Hence, the cutoff capital ratio $\overline{E}$ is determined by the following conditions:

$$\overline{E} + (1-\alpha)(\theta y_L - p)\frac{\overline{E}}{p - \rho^*} = NPV + \overline{E} - (1-\alpha)(\theta y_L - p) \tag{A19}$$

which is equivalent to:

$$\frac{\overline{E}}{p - \rho^*} + 1 = \frac{NPV}{(1-\alpha)(\theta y_L - p)} \tag{A20}$$

Note that all banks with $E_i \geq \overline{E}$ will choose to invest all of their funds in the liquid assets, which implies that the spare liquidity of each bank i is E_i. Therefore, the total spare liquidity is equal to $\int_{\overline{E}}^1 Ef(E,h)dE$. Since all banks with $E_i < \overline{E}$ will hold zero liquidity and will sell all of their long-term assets, if the liquidity shock is realized, the total supply of the long-term assets in the secondary market is $\int_0^{\overline{E}} f(E,h)dE$. Hence, the market clearing condition implies that the equilibrium price is determined by the following equation:

$$\int_{\overline{E}}^1 Ef(E,h)dE = p^e \int_0^{\overline{E}} f(E,h)dE \tag{A21}$$

To summarize, in the equilibrium where $p < \rho^* + \delta$, the cutoff capital level and the equilibrium price are jointly determined by the following equations:

$$\frac{\overline{E}}{p - \rho^*} + 1 = \frac{NPV}{(1-\alpha)(\theta y_L - p)} \tag{A22}$$

$$\int_{\overline{E}}^1 Ef(E,h)dE = p^e \int_0^{\overline{E}} f(E,h)dE \tag{A23}$$

Next, we prove that the system of the two Equations (14) - (13) has unique solutions. Indeed, from Equation (13), we can derive p^e as a function of $\overline{E}$ and h as follows:

$$p^e = \frac{\int_{\overline{E}}^{1} Ef(E,h)dE}{\int_{0}^{\overline{E}} f(E,h)dE} \tag{A24}$$

With this expression of the equilibrium price at hand, we can derive the following interim result.

Lemma 5. *The equilibrium price, p^e, monotonically decreases with $\overline{E}$.*

Proof of Lemma 5. Computing the partial derivative of p^e with respect to $\overline{E}$, we have:

$$\frac{\partial p^e}{\partial \overline{E}} = -\frac{f(\overline{E},h)\left[\overline{E}\int_{0}^{\overline{E}} f(E,h)dE + \int_{\overline{E}}^{1} Ef(E,h)dE\right]}{\left(\int_{0}^{\overline{E}} f(E,h)dE\right)^2} < 0 \tag{A25}$$

Hence, p^e is a decreasing function of $\overline{E}$, which proves the Lemma.

Now, define $\overline{E}_1$ and $\overline{E}_2$ as follows:

$$p^e\left(\overline{E}_1\right) = \theta y_L \quad \text{and} \quad p^e\left(\overline{E}_2\right) = \rho^* \tag{A26}$$

From Lemma 5 we know that p^e is a decreasing function of $\overline{E}$, then $\overline{E}_1 < \overline{E}_2$. Given that the natural boundaries for p^e are ρ^* and θy_L (i.e, $\rho^* \leq p^e \leq \theta y_L$), we are only interested in the solution where $\overline{E}_1 \leq \overline{E} \leq \overline{E}_2$.

Define

$$g(\overline{E}) = \frac{\overline{E}}{p^e - \rho^*} \tag{A27}$$

and $G(\overline{E})$ as the left-hand side of Equation(14), i.e,

$$G(\overline{E}) = g\left(\overline{E}\right) + 1 - \frac{NPV}{(1-\alpha)(\theta y_L - p^e)} \tag{A28}$$

where p^e is computed using Expression (A24). Hence, to show that the system of the two Equations (14) - (13) has unique solutions, we need to show that the following equation has unique solution in the interval $\left[\overline{E}_1, \overline{E}_2\right]$:

$$G\left(\overline{E}\right) = 0 \tag{A29}$$

We have:

$$\frac{\partial g(\overline{E})}{\partial \overline{E}} = \frac{p^e - \rho^* - \overline{E}\frac{\partial p^e}{\partial \overline{E}}}{(p^e - \rho^*)^2} \tag{A30}$$

Since $\frac{\partial p^e}{\partial \overline{E}} < 0$ and $p^e > \rho^*$ for $\overline{E}_1 \leq \overline{E} \leq \overline{E}_2$, we have:

$$\frac{\partial g(\overline{E})}{\partial \overline{E}} > 0 \quad \forall \overline{E}_1 \leq \overline{E} \leq \overline{E}_2 \quad \text{and} \quad \underset{\overline{E}\longrightarrow \overline{E}_2}{lim} g(\overline{E}) = +\infty \tag{A31}$$

Hence,

$$\underset{\overline{E}\longrightarrow \overline{E}_2^-}{lim} G(\overline{E}) = +\infty \quad \text{and} \quad \underset{\overline{E}\longrightarrow \overline{E}_1^+}{lim} G(\overline{E}) = -\infty \tag{A32}$$

Moreover, it is easy to check that $G(\overline{E})$ is a monotonically increasing function of $\overline{E}$ for $\overline{E}_1 \leq \overline{E} \leq \overline{E}_2$. This, together with Result (A32), implies that Equation (A29) has unique solution $\overline{E}(h)$, satisfying $\overline{E}_1 \leq \overline{E} \leq \overline{E}_2$.

Proof of Proposition 3. We will prove that the equilibrium price, $p(\overline{E}(h), h)$ monotonically increases with h, hence, there exists a unique value $\hat{h}$, for which $p(\overline{E}(\hat{h}), \hat{h}) = \rho^* + \delta$. For this purpose, we need the following interim result.

Lemma 6. *Properties of the equilibrium*

- $\overline{E}(h)$ *is an increasing function of* h*.*
- $p(\overline{E}(h), h)$ *is an increasing function of* h*.*

Proof of Lemma 6.

First, we show that $\overline{E}(h)$ is an increasing function of h. From Equation (14), using implicit differentiation, we can compute the total derivative of $\overline{E}(h)$ with respect to h as follows:

$$\frac{d\overline{E}}{dh} = -\frac{-\frac{\partial p^e}{\partial h}\overline{E}(1-\alpha) + \frac{\partial p^e}{\partial h}(1-\alpha)(\theta y_L - p^e)}{(1-\alpha)(\theta y_L - p^e) - \frac{p^e}{\partial h}\overline{E}(1-\alpha) + \frac{\partial p^e}{\partial \overline{E}}(1-\alpha)(\theta y_L - p)} \tag{A33}$$

After some arrangements, we obtain:

$$\frac{d\overline{E}}{dh}=\frac{\partial p^e}{\partial h}\frac{NPV-(1-\alpha)\theta y_L+(1-\alpha)(\overline{E}+2p^e-\rho^*)}{(1-\alpha)(\theta y_L-p^e)-\frac{\partial p^e}{\partial \overline{E}}(NPV-(1-\alpha)\theta y_L+(1-\alpha)(\overline{E}+2p^e-\rho^*))} \tag{A34}$$

Since $p^e\geq\rho^*$ and we know from Lemma 5 that $\frac{\partial p^e}{\partial \overline{E}}\leq 0$, from Expression (A34), we see that $\frac{d\overline{E}}{dh}$ has the same sign as $\frac{\partial p^e}{\partial h}$. Since h measures the mass on the right side of the banks' capital distribution, we have $\frac{\partial p^e}{\partial h}$ as positive. Therefore, $\overline{E}(h)$ is increasing with h.

Next, we show that $p\left(\overline{E}(h),h\right)$ is increases with h. We know from Lemma 5 that the partial derivative of p^e with respect to $\overline{E}$ is negative, which means that $p\left(\overline{E}(h),h\right)$ is decreasing with $\overline{E}$. Compute the total derivative of p^e with respect to h:

$$\frac{dp^e}{dh}=\frac{\partial p^e}{\partial \overline{E}}\frac{d\overline{E}}{dh}+\frac{\partial p^e}{\partial h} \tag{A35}$$

Using Expression (A34), we obtain:

$$\frac{dp^e}{dh}=\frac{\partial p^e}{\partial h}\frac{(1-\alpha)(\theta y_L-p^e)}{(1-\alpha)(\theta y_L-p^e)-\frac{\partial p^e}{\partial \overline{E}}\left[NPV-(1-\alpha)\theta y_L+(1-\alpha)(\overline{E}+2p^e-\rho^*)\right]} \tag{A36}$$

Hence, $\frac{dp^e}{dh}$ has the same sign as $\frac{\partial p^e}{\partial h}$, which means that $p\left(\overline{E}(h),h\right)$ is increasing with h.

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