Revealing Private Information in a Patent Race*

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November 2023

Abstract

We investigate the role of private information in a patent race. Since firms often do their research in secrecy, the standard assumption in the patent race literature that firms know each other's position in the race is questionable. We analyze how the dynamics of the game changes when a firm's progress is its private information. Further, we address the question whether revealing its private information might be to the firm's advantage, even though it does not have any direct payoff consequences. We find that the firm has an incentive to reveal its breakthrough only if its rival has not done so, and only if R&D is inefficient.

1 Introduction

Patent races have been studied extensively since the seminal paper of Loury (1979). Many studies analyze how firms' R&D investments vary as firms' positions in the race evolve. However, that type of analysis assumes that a firm's progress in the race is publicly observable. In this paper we lift this extreme assumption and analyze a patent race, where the firm's progress is its private information. Our main goal is to investigate whether the firms have the incentives to reveal their progress voluntarily.

Dropping the assumption of complete information in a patent race significantly affects the dynamics of the game. Lacking the information about the rival's progress, the firms

*We are grateful to Dilip Abreu, Corina Boar, Jaroslav Borovička, Laurent Cavenaile, Sylvain Chassang, Francisco Espinosa, Shrijita Ghosh, Boyan Jovanovic, Rumen Konstadinov, Natalie Lee, Alessandro Lizzeri, Erik Madsen, Laurent Mathevet, David Pearce, Michael Richter, Francisco Roldán, and Nikhil Vellodi for helpful comments. A special thank is dedicated to Ennio Stacchetti for his invaluable guidance and support throughout this project. P. Kocourek acknowledges the financial support by the ERC grant 770652 (J. Steiner) and the DFG grant 450175676 (E. Kováč).

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can only act based on their beliefs. As the beliefs change continuously, the firms continuously adjust the intensity of their R&D investments. In this respect several questions arise. How does a firm's R&D effort evolve? Do firms invest most intensely early on in the game or do they invest increasingly aggressively over time? How does a firm's R&D effort change with the arrival of a breakthrough? And most importantly, can a firm discourage its rival by revealing its progress towards a patent?

Although this study focuses on the patent race, it can be seen as an example of a broader class of dynamic games, like contests or tournaments. Within those strands, most of literature assumes that the underlying game is either static or it is dynamic with publicly observed states, while there is very little known about the games in which each player's state is observed privately (Seel and Strack, 2016). In series of papers, P. Strack and various coauthors explore the role of private information in games in which each player's only action is the choice of a stopping time (Kruse and Strack, 2015; Fudenberg et al., 2018). In this paper, we investigate the role of private information in games in which players have the choice of effort at any time. Whilst optimal stopping is an irreversible decision to stop a certain process once and for all, the choice of effort level allows players to adapt their level of participation to the momentary situation. The effect on rival's investment then adds another layer of strategic motives, where a firm can benefit from discouraging the rival's investment. In this setup, it is natural to ask which of the information settings (public or private) is better for the players, and what would happen if the players themselves had the control over the information setting by having the option to reveal their progress.

More specifically, we study a private information version of the patent race introduced in Grossman and Shapiro (1987). There are two firms (or players) that compete in making a patentable discovery. At any point in time each firm continuously chooses its research effort, which translates into the hazard rate of making a breakthrough and involves flow costs. In order to obtain the final discovery, each firm needs to attain two consecutive breakthroughs. Thus, there is an interim state on the way to the patent that is reached after making the first breakthrough. We think about the interim stage as a prototype of a new technology or results of first trials in pharmaceutical research. We assume that attaining the interim stage has no direct payoff consequences. Only winning the patent race by attaining two breakthroughs yields a positive payoff and also ends the game.

The novel feature of our model is that reaching the interim state is the firm's private information unless the firm discloses it. The disclosure is verifiable, while there is no credible way to reveal not having made a breakthrough. In addition, we assume that the disclosure is costless and has purely informational effect without a any technological spillovers. Thus, our model involves neither cheap talk nor signaling. As an example, a pharmaceutical firm can publish results of audited randomized trials, without publishing details about the drug. A technological firm can publish a video demonstrating the function of its prototype, as Samsung did with its folding smartphone. Another example could be the rocket launches of Space-X, which publicly demonstrate certain capabilities of their rockets without disclosing the technology behind.

In this paper we analyze four different information settings with increasing complexity in order to address different types of questions. As a benchmark we first we solve the complete information version of the patent race, where the intermediate success is observed by the rival. Under the assumption that innovation follows the memoryless Poisson process, each players' effort is constant over time as long as there is no breakthrough. We show that a breakthrough encourages a successful rival, but discourages an unsuccessful rival. This is in line with standard findings in studies of complete information patent races, where players exert the highest effort when the race is neck and neck.

Second, we study the private information setting, where each firm only observes its own progress. Neither firm can reveal its breakthrough and it updates its belief about the rival having one as time proceeds. Updating the belief over time involves two opposing effects. On the one hand, there is a positive effect on the belief, since the rival is increasingly likely to have completed the first stage of R&D over time. On the other hand, there is a negative effect, due to not observing the rival patenting the final discovery. Indeed, if the rival would have already completed the first stage of R&D, the longer time has passed, the more likely he would have completed the second stage as well. We show that the the first effect dominates the second one so that the posterior probability that the rival has completed the first stage keeps increasing over time and it converges to a steady-state value that is, due to the second effect, strictly smaller than 1. As a player becomes increasingly convinced that his rival has already made the first breakthrough, he becomes increasingly pessimistic about his own situation. We show that if a player still has not completed the first stage of R&D, the increasing pessimism makes him decrease effort over time. However, once a player completes the first stage of R&D, his effort increases and continues increasing until either of the players makes a patent.

Third, we consider an asymmetric version of the private information setting is the case where one of the two firms (say, firm A) is known to be successful, i.e., having made the first breakthrough. While such a situation is interesting on its own, it is essential for further analysis of players' decisions to reveal breakthroughs. Since player A does not observe the rival's (player B's) state, he updates his posterior belief about player B's progress and adjusts his effort accordingly. As a result, player A increases his effort over time. In contrast, player B's belief does not change as he knows that his rival A is just one step away from patenting. Nevertheless, player B's effort changes continuously as he reacts to player A's increasing rivalry. We show that if player B has attained a breakthrough as well, then his effort is higher than player A's effort, because he knows that the players are neck and neck. As a result, player A is worse off than player B.

Fourth, we study the patent race with the option to disclose breakthroughs. Our first result is that a player never discloses a breakthrough knowing that his rival is successful. In fact, a successful player becomes only encouraged to work harder by learning about the rival's success. Thus, if one player has revealed, the other player will keep his progress secret until attaining the second breakthrough. Knowing that, we analyze firms' strategies before any of them has revealed success. This is a symmetric situation. Revelation of a player's (say player A's) success involves three effects on the rival's (player B's) effort: The desirable effect of the revelation is that it discourages the rival if the rival is unsuccessful. However, there are two undesirable effects that come with the revelation. First, if the rival is already successful, or once he becomes successful, having the information about player A's success causes him to increase effort. Second, player A's revelation gives player A an informational disadvantage, as player B will be informed about player A's success, but will never reveal his own success afterwards. It turns out that the total effect is ambiguous. Therefore, we make use of numerical simulations in order to compare those effects and to determine players' revelation behavior in equilibrium. We conclude that a player reveals instantly the first breakthrough only if research is difficult (or inefficient) in the sense that making a breakthrough is a long-term project. Indeed, if research is difficult, the rival is expected to remain unsuccessful for a long time, and so the desirable effect of discouraging the unsuccessful rival dominates the two undesirable effects. In contrast, if research is easy, then players never reveal the first breakthrough, as each of them expects the rival to catch up promptly. Finally, in the case of a moderate research difficulty, a successful player has a tendency to reveal, yet he prefers to wait for his rival to do so. Players randomize over their revelation decision and the equilibrium resembles the one known from the war of attrition games.

The structure of this paper is as follows. After a literature review, we introduce the model of patent race in Section 2. Section 3 then analyzes the complete information version of the game as a benchmark. Section 4 analyzes a firm's R&D effort over time in the private information game with no option to reveal. Section 5 analyzes an asymmetric version of the model, in which one of the players is known to be successful. Finally, using the insights from the previous sections, Section 6 studies the case in which players have the option to reveal their success. We provide conditions for the existence of no-revelation equilibrium and instant-revelation equilibrium and characterize the mixed-revelation equilibrium. Section 7 concludes. The proofs of all propositions and lemmas from the main text are relegated to the Appendix.

Literature Review

The nature of R&D investments in a competitive environment was first studied by Loury (1979), Lee and Wilde (1980), and Dasgupta and Stiglitz (1980). In those studies, the patent race is static in the sense that the firms choose their R&D effort once and for all at the beginning. Shortly after, the analysis was extended into dynamic environments by Grossman and Shapiro (1987) and Harris and Vickers (1987). Their models feature a specific finishing line: a firm wins the patent race once it completes a given number of R&D stages. One of their main results is that the firms invest in R&D most intensively when they are even and close to the finishing line. On the other hand, Hörner (2004) studies a perpetual race in which the firm that is ahead of the other receives flow pay-off. Contrary to previous results, he shows that in such a case the competition is not necessarily fiercest when the firms are closest.

The paper by Dosis et al. (2013) considers a patent race with two stages, called the research phase and the development phase. In either phase, the breakthroughs arrive in a random fashion, and in addition to that, in the research phase there is uncertainty about the innovation being feasible. One of their main findings is that under-investment is the dominant effect in the initial stage of the race (research phase), while over-investment is the dominant effect in the more advanced stages (development phase).[1]

This paper is also related to various studies of races featuring some type of private information about a player's success. Akcigit and Liu (2015) assume that firms have private information about dead-end projects, and they show that a firm is silent about abandoning a bad project in order to let its rival misallocate R&D investments. Private information about a breakthroughs is present in Hopenhayn and Squintani (2015), in which a firm makes a discovery at a random time, and its value subsequently grows during the developmental phase until the firm patents it. The firm faces a tradeoff between increasing the value of its potential patent and risking being pre-empted by its rival. A race with private information is also studied in Moscarini and Squintani (2010) and Hopenhayn and Squintani (2011), where players choose optimal stopping instead of effort.

The closest related study is the working paper of Gordon (2011), who also studies a two stage patent race with private information, but only allows for two effort levels of the firms: high and low. This restriction dramatically simplifies the analysis of the model, however, it also has two crucial downsides. First, unlike in this study, the game might

[1] Other papers have also studied over- and underinvestment in patent races. Judd (2003) and Moscarini and Smith (2007) study continuous state-space versions of Harris and Vickers (1987) and find that innovators over-invest on risky projects, and that leaders invest more than followers. Hopenhayn and Squintani (2021) investigates firms' allocation of R&D investments across research areas and concludes that firms overinvest in high return areas.

have no or multiple equilibria, depending on the choice of the parameters, as a result of the effort choice being discrete. Second, the revelation has an impact only in the rare situation where the optimal effort level flips. In contrast, in this paper players adapt their effort level to any minor changes in their incentives.

Although Bonatti and Hörner (2011) study collaboration in place of competition, it shares some features with this study. There are two players collectively engaged in a project. The players dynamically choose their effort that determines the hazard rate of completion of the project. There is uncertainty about the quality of the project and over time the players become increasingly pessimistic about the project being good. Most of their work assumes that effort cost is linear, which allows for the use of the bang-bang solution method and simplifies the analysis. In contrast, in this paper we cannot use the bang-bang solution method and thus a completely different solution technique is needed.

Another strand of literature relevant to our paper is the literature on disclosure. Jovanovic (1982) studies disclosure of product quality, and, in contrast to Akerlof (1970), he assumes that truthful disclosure is feasible. Jovanovic (1982) finds that competition in the free market drives the amounts of disclosure beyond the socially-optimal level. Dye (1985) and Milgrom (2008) study a firm's disclosure of information as a form of persuasion of potential investors. An extensive overview of literature on quality disclosure can be found in Dranove and Jin (2010). In all of the above literature the firms' disclosure is studied in relation to a consumer or an investor. In contrast, in this study the strategic disclosure occurs between two rivals.

Furthermore, there are studies that analyze disclosure of information in a patent races. Lichtman et al. (2000) and Baker and Mezzetti (2005) study disclosure as a way to increase the prior art in order to prevent or delay the rival from patenting a new technology. Gill (2008) studies a firm's tradeoff between disclosing its progress in order to discourage the rival from investments and the potential technological spillover. Aoki and Spiegel (2009) study the impact of the pre-grant publications of patents that are mandatory within 18 month of patent application in most industrial countries except for the U.S. The study by Kultti et al. (2007) investigates whether secrecy or patenting is a better way of protecting intellectual property, when being concerned about the technological spillovers.

2 Model

We study an infinite horizon continuous time model of a patent race with two risk-neutral firms (players) A and B. The firms invest in R&D with the aim to win a specific patent. In order to patent the discovery, a firm needs to attain two consecutive breakthroughs. We define the state of the firm $j \in \{A, B\}$ as the number of breakthroughs the firm has made by time t and denote it $x_t^j \in \{0, 1, 2\}$. Initially, both firms start in the state 0. Once a firm makes the first breakthrough it reaches state 1. When making the second breakthrough, the firm attains state 2, patents the discovery and wins the value of the patent $v > 0$.

In order to obtain the innovation, the firms conduct R&D. At any time $t \geq 0$, each firm $j \in \{A, B\}$ chooses its research effort $e_t^j \geq 0$, which translates into the probability of attaining a breakthrough. We identify the research effort $e_t^j \geq 0$ as the hazard rate of making a breakthrough at time t. This means that the knowledge is not accumulated unless a breakthrough is made and it can be interpreted, as if the research consisted of independent trials. Obtaining a breakthrough within the time interval $[t, t + \Delta t]$ corresponds to a change in the state from x_t^j at time t to $x_{t+\Delta t}^j = x_t^j + 1$ at time $t + \Delta t$, which happens with the probability.[2,3]

$$P[x_{t+\Delta t}^j = x_t^j + 1] = e_t^j \Delta t + o(\Delta t).$$

The research effort is a result of R&D investments, and thus it is costly. Player j incurs flow cost $c(e_t^j)$ that is a function only of the current effort. We assume that the marginal cost of effort is increasing; in particular, we consider a quadratic cost function of the form

$$c(e) = \frac{1}{2}\alpha e^2$$

where $\alpha > 0$. The assumption of quadratic costs is restrictive. We use this specification, because it simplifies the algebra considerably.

The firm that patents first receives the prize v, while v is the only positive payoff that a firm can obtain. All future payoffs are discounted at rate $r > 0$. The expected utility of player j then is

$$EU^j = \mathbb{E}\left[\underbrace{-\int_0^s \exp(-rt) \cdot c(e_t^j) dt}_{\text{effort cost}} + \underbrace{\exp(-rs) \cdot v \cdot \mathbf{1}_{s=s^j}}_{\text{value of the patent}}\right],$$

where s^j is the time at which player j files a patent (or infinity) and $s = \min\{s^A, s^B\}$. The first term (effort cost) represents the accumulated flow costs of research. Each firm need to bear those costs, even if it does not win the race. The second term (the value of the patent) is equal to the value v discounted at time s, when firm j wins the race, while it is equal to 0 otherwise.[4]

[2] We assume that each realization of the trajectory $t \mapsto x_t^j$ as well as $t \mapsto e_t^j$ is right-continuous.

[3] $o(\cdot)$ represents any function such that $o(\Delta t)/\Delta t \to 0$ as $\Delta t \searrow 0$

[4] We use the variable e for effort, whereas $\exp(\cdot)$ denotes the exponential function. The function $\mathbf{1}_{s=s^j}$

We would like to point out that the model involves only three parameters: the value of the patent v, the effort cost multiplier α, and the discount rate r. All these parameters are commonly known to the firms. Moreover, the generality of the problem will not be compromised by setting $v = 1$ and $\alpha = 1$. Apart from choosing a unit of value such that $v = 1$, it is also possible to choose the units of time such that $\alpha = 1$. In that respect, all results can be formulated in terms of a single parameter $\hat{r} = \alpha r/v$ that we refer to as *research difficulty*. Its inverse $1/\hat{r} = v/(\alpha r)$ can be then interpreted are the *R&D efficiency*.[5]

3 Complete Information Case

As a benchmark, we first analyze the version of the patent race in which players' progress is common knowledge, i.e., each player knows the state his rival is in.[6] We restrict our attention to symmetric Markov perfect equilibria.[7] The state of the game is given by the combination of the state $k, l \in \{0, 1, 2\}$ of player A and B, respectively. Since there is no private information, under the Markov assumption, the firms' efforts as well as their continuation values remain constant over time as long as there is no new breakthrough. Let us denote v^{kl} and e^{kl} the continuation value and effort of a firm (A or B) in state k, while the rival is in state l. Since, the patent race is over, once a firm obtains its second breakthrough, we have $v^{2l} = v$ for the winner of the patent race and $v^{k2} = 0$ for its rival (where $k, l \in \{0, 1\}$).

While we relegate the details of the analysis to the Appendix (Section A.1), let us point out that the optimal effort in state kl satisfies

$$\alpha e^{kl} = v^{k+1,l} - v^{kl}, \tag{1}$$

which means that it is proportional to the potential gain from making a breakthrough. The above condition can also be interpreted as an equation between marginal costs and marginal benefit. The left-hand side represents the (instantaneous) marginal costs of

is the indicator function determining whether player j wins the race. Note that the case of both players patenting simultaneously occurs with zero probability and can thus be neglected.

[5]While we keep the parameters v and α in the main text, we will use the normalization $v = 1$ and $\alpha = 1$ in the proofs. Furthermore, we will use it in numerical computations, where it reduces the dimension of the relevant parameter space to one.

[6]This model has been analyzed in Harris and Vickers (1987) under the assumption $r = 0$, or in Grossman and Shapiro (1987), in which a more general class of cost function is considered, and hence, a less clear conclusions can be made.

[7]In fact, the game has only the symmetric equilibrium, as can be shown by making a minor modification to the proof of uniqueness of symmetric equilibrium. We assume symmetry here for the sake of notation simplicity.

effort. The right-hand side represents the expected marginal benefit of effort, i.e., the expected gain from a breakthrough. We will obtain analogous results in other versions of the model.

Under the optimal efforts, we obtain a system of for equations (one for each state) with the remaining continuation values (v^{00}, v^{10}, v^{01}, and v^{11}) as unknowns. The following proposition postulates that the system has a unique solution, which allows us to compare players' research efforts among different scenarios.[8]

Proposition 1. *The patent race with complete information has a unique symmetric Markov perfect equilibrium. In this equilibrium,*

$$\text{(i)} \quad e^{01} < e^{00} \qquad \text{and} \qquad \text{(ii)} \quad e^{10} < e^{11}.$$

Both inequalities provide comparisons of efforts when the rival obtains a breakthrough. The first inequality means that an unsuccessful player is discouraged from exerting effort by the success of its rival. The second inequality means that a successful player is encouraged to exert effort by the success of its rival. This is indeed consistent with earlier literature see (Grossman and Shapiro, 1987; Harris and Vickers, 1987), which concludes that the players exert a higher effort when the race is tight.

Based on the above results, we can expect that in the private information setting, where a player does not observe a rival's state and only updates the posterior belief about him, an unsuccessful player will continuously decrease effort as a result of increasingly being convinced about the rival having a lead. Conversely, a successful player will continue to increase effort over time as he is increasingly likely to be in the neck and neck situation. Moreover, these results also shed some light on the question whether players want to reveal their success when having the option to do so credibly without leaking any technological secrets. In particular, they suggest that a player would reveal his success only if he expects to discourage the rival's R&D effort. This is the case, when the player beliefs the rival to be still unsuccessful and not to catch up any soon.

4 Private Information Case

We proceed with modeling the private information about firms' progress in the patent race. We assume that they do not observe each other's research progress on the way to the patent. Since attaining the second breakthrough ends the game, in this section we

[8]Grossman and Shapiro (1987) show that $e^{11} > e^{10} > e^{01}$. However, since they allow for a more general class of convex effort cost functions, they conclude that the relationship between e^{00}, e^{01} and e^{10} is ambiguous.

assume that it is firm j's private information whether it is in state 0 or in state 1 (i.e., whether $x_t^j = 0$ or $x_t^j = 1$). Each firm knows its own progress towards the patent, but it does not observe the progress of its rival. There is also no way to reveal the progress to the rival. In addition, we assume that firm j's effort is not observed by the rival, otherwise the rival might be able to infer the state x_t^j from the effort.

Given that a player does not know the rival's state, he forms a belief about it. Define p_t^j as player $-j$'s posterior belief about player j being in state 1:[9]

$$p_t^j = P[x_t^j = 1 \mid x_t^j < 2].$$

This specification of beliefs yields a non-trivial dynamics. While the unconditional probability of having a success is simply increasing over time, the dynamics of beliefs is now more complex. Since the game ends once a player attains state 2, player $-j$ knows with certainty that $x_t^j < 2$, if the race is still ongoing. Thus, player $-j$ also conditions his belief on the fact that player j has not attained his second success (i.e., that $x_t^j < 2$).

We are interested in *Markov perfect Bayesian Nash equilibria (MPBNE)* of the game. Thus, players condition their actions on the payoff relevant state, which (for player j) consists of his own state x_t^j and the profile of mutual posterior beliefs (p_t^A, p_t^B). Player j's strategy is then the choice of the effort $e \geq 0$ as a function of the profile (x_t^j, p_t^A, p_t^B). To simplify the notation, we will denote player j's effort when in state $x \in \{0, 1\}$ at time t as $e_t^{x,j}$, since the pair of beliefs (p_t^A, p_t^B) is only a function of the time.[10] However, we need to keep in mind that the effort choice does not depend on the calendar time, and thus $e_{t_1}^{x,j} = e_{t_2}^{x,j}$, whenever $(p_{t_1}^A, p_{t_1}^B) = (p_{t_2}^A, p_{t_2}^B)$.

At time $t = 0$ the posterior belief is $p_0^j = 0$ as players start from the state 0 with certainty. As the time proceeds, both players update their beliefs using Bayes' law. The posterior belief is then governed by the well known law of motion specified in the following lemma.

Lemma 1. *The posterior belief p_t^j follows the law of motion $\dot{p}_t^j = (1 - p_t^j)(e_t^{0j} - p_t^j e_t^{1j})$.*

Intuitively, a higher effort in state 0 makes the first breakthrough more likely. At the same time, a higher effort in state 1 makes the second breakthrough more likely. More specifically, the dynamics of the belief is governed by the hazard rate of reaching state 1 and the hazard rate of leaving state 1, i.e., filing the patent. The former is equal to the effort in state 0, which is e_t^{0j}. The latter is equal to $\phi_t^j := p_t^j e_t^{1j}$, since from the perspective of the rival, firm j has made the first breakthrough with probability p_t^j, and if that is the case, then he patents with the hazard rate e_t^{1j}.

[9] Here we utilize the standard notation, where $-j$ denotes the rival of firm j.

[10] Note that firm j's effort only depends on its own state, as opposed to the complete information case, where the effort depends on both firms' states.

Note that the belief p_t^j remains constant once it attains the value 1. One might expect that p_t^j will eventually approach 1, but this is not the case. Although the rival is increasingly likely have made a breakthrough, he is also increasingly likely to have patented already. Thus, conditioned on the fact that the rival has not patented, his probability of being in state 1 asymptotically converges to a value below 1.

In the next step, we focus on the continuation values. Every state has an associated continuation value; we will use the same system of notation $v_t^{k,j}$ for the continuation value as for the effort. In the Appendix (Section B.2) we derive for each value function $v_t^{k,j}$ (where $k \in \{0, 1\}$ and $j \in \{A, B\}$) a corresponding ordinary differential equation (ODE). Jointly with the law of motion for players' beliefs we obtain a system of six ODEs that govern the dynamics of the model.

In the remainder of this section we focus on symmetric *Markov perfect Bayesian Nash equilibrium*.[11] We show the existence and uniqueness of such equilibrium, and we study its properties such as the dynamics of firms' R&D efforts over time. The symmetry reduces the system of six ODEs (see (24)–(26) in the Appendix, Section B.2) into the following system of three ODEs (omitting the superscript j denoting the player):

$$-\dot{v}_t^1 = \frac{\alpha}{2}(e_t^1)^2 - (r + p_t e_t^1)v_t^1 \tag{2}$$

$$-\dot{v}_t^0 = \frac{\alpha}{2}(e_t^0)^2 - (r + p_t e_t^1)v_t^0 \tag{3}$$

$$\dot{p}_t = (1 - p_t)(e_t^0 - p_t e_t^1), \tag{4}$$

along with identities

$$\alpha e_t^1 = v - v_t^1, \qquad \alpha e_t^0 = v_t^1 - v_t^0, \tag{5}$$

the initial condition $p_0 = \hat{p}$, and inequalities $v_t^0, v_t^1 \in [0, v)$, $v_t^0 \leq v_t^1$ and $p_t \in [0, 1]$, for all $t \geq 0$. To satisfy the Markov condition, the effort e_t^x can only depend on the belief p_t, i.e., $e_{t_1}^x = e_{t_2}^x$ whenever $p_{t_1} = p_{t_2}$, for any $x \in \{0, 1\}$ and $t_1, t_2 \in \mathbb{R}^+$.

Note that the ODEs (2)–(3) feature the term $r + p_t e_t^1$, representing the risk-adjusted discount rate. This is composed from the discount rate r and the perceived hazard rate of rival's patenting, which is $\phi_t = p_t e_t^1$.

Solving the system of ODEs (2)–(4) is complicated by the fact that it is not an initial value problem: whilst we know $p_0 = \hat{p}$, we do not know the initial values of v_0^1 and v_0^0, and an error of the initial guess grows exponentially going forward in time. On the other hand, solving backwards in time, the error of the guess of p_t would be a problem. We can, nevertheless, show the uniqueness of the solution.

[11] By symmetric we mean that both players have the same strategies.

Proposition 2. *The patent race with private information has a unique symmetric Markov perfect Bayesian Nash equilibrium.*

The proof of the result is provided in the Appendix (Section B.4). The proof proceeds in three steps, each being formulated as a a separate lemma. First, we show that the system of ODEs has a unique critical point (Lemma B.2). Second, we show that every solution converges to this point (Lemma B.4). Third, we show that there is a unique direction in which it can occur (Lemma B.3).

Knowing that there is a unique equilibrium, we can discuss its properties. The following proposition provides results about the dynamics of players' efforts (see Section B.9 for a proof).

Proposition 3. *In a patent race with private information, each player's effort is decreasing over time until he makes the first breakthrough; then his effort jumps up and keeps increasing.*

The proposition consists of three statements about the equilibrium effort levels. First, a successful player gets increasingly rivalrous, i.e., the effort e_t^1 increases over time. This is quite intuitive as the rival is increasingly likely to be successful over time, which means that the race is more likely neck and neck. Second, an unsuccessful player gives up over time, i.e., e_t^0 decreases over time. As time proceeds, the player without any success beliefs he is more likely to be behind. As a response, he decreases its effort. Third, a successful player is always more rivalrous than an unsuccessful one, i.e., $e_t^1 > e_t^0$. Intuitively, the player's marginal benefit from effort increases after a breakthrough, while the belief about the rival remains the same. Thus, it pays off to increase the effort. The results are illustrated in Figure 1.

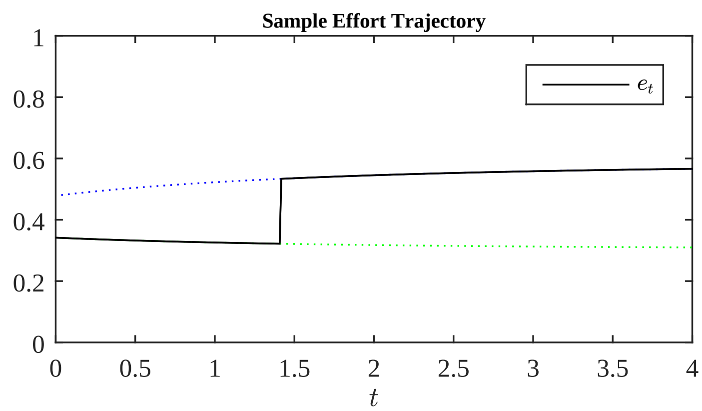

Figure 1: An example of a player's effort over time; the rise occurs as the player makes the first breakthrough.

5 One Player Known to be Successful

This section deals with the asymmetric case, in which one player is known to be successful. Without loss of generality, assume that the player known to be successful is player A. In other words, player A is in state 1 and it is common knowledge, while player B's state is his private information. In this setting, we can analyze which of the two players is better off. The results obtained in this section have significance on their own, but their main importance is to prepare the basis for the analysis of the game in which players have the option to reveal their success. In that respect, let us assume that the initial belief (i.e., belief at time $t = 0$) about player B being in state 1 is equal to $\hat{p}$. Due to the memoryless nature of the innovation process, the value functions obtained in this section will serve as continuation values in the subgame, where some player has revealed the success, while the belief about the other player being in state 1 is equal to $\hat{p}$.

We analyze *Markov perfect Bayesian Nash equilibria (MPBNE)* of the game, defined in an analogous fashion as in the previous section. The dynamics of value functions and firms efforts is characterized by the following system of ODEs

$$-\dot{v}_t^{1A} = \frac{\alpha}{2}(e_t^{1A})^2 - (r + p_t^B e_t^{1B})v_t^{1A}, \tag{6}$$

$$-\dot{v}_t^{1B} = \frac{\alpha}{2}(e_t^{1B})^2 - (r + e_t^{1A})v_t^{1B}, \tag{7}$$

$$-\dot{v}_t^{0B} = \frac{\alpha}{2}(e_t^{0B})^2 - (r + e_t^{1A})v_t^{0B}, \tag{8}$$

$$\dot{p}_t^B = (1 - p_t^B)(e_t^{0B} - p_t^B e_t^{1B}), \tag{9}$$

along with the identities

$$\alpha e_t^{1A} = v - v_t^{1A}, \qquad \alpha e_t^{1B} = v - v_t^{1B}, \qquad \alpha e_t^{0B} = v_t^{1B} - v_t^{0B}, \tag{10}$$

the initial condition $p_0^B = \hat{p} \in [0, 1)$, and inequalities $v_t^{0B}, v_t^{1A}, v_t^{1B} \in [0, v)$, $v_t^{0B} \leq v_t^{1B}$, and $p_t^B \in [0, 1]$, for all $t \geq 0$. This system follows from the system of six ODEs obtained in the private information case (see (24)–(26) in the Appendix, Section B.2), under the assumption that player A has already attained state 1, i.e., $p^A = 1$.

Note that from the perspective of the informed player player B, the perceived hazard rate of rival's patenting becomes e_t^{1A}. Therefore, he faces the risk-adjusted discount factor $r + e_t^{1A}$, which appears in ODEs (7)–(8). On the other hand, from the perspective of the uninformed player B, the perceived hazard rate of rival's patenting still includes the belief p_t^B, yielding risk-adjusted discount factor $r + p_t^{1B} e_t^{1B}$ in ODE (6).

Following a similar line of reasoning as in the previous section, we prove the existence of a unique equilibrium in this game and obtain the dynamics of equilibrium efforts. The proofs are, however, more involved due to asymmetry.

Proposition 4. *The patent race with private information and one player known to be successful has a unique Markov perfect Bayesian Nash equilibrium.*

Proposition 5. *Suppose player A is known to be successful, while player B's state is unknown. Then player A increases his effort e_t^{1A} over time, and so does player B after making the first breakthrough.*[12]

This result is analogous to the one from the symmetric version of the game with both players in an unknown state. However, in this case the result can be found surprising, because player B changes his effort over time even though his belief about the state of his rival is fixed. The result is driven by the second order beliefs. Player B knows that he is increasingly being expected to be already in state 1, and thus player A becomes increasingly rivalrous over time.

The information asymmetry gives rise to a number of additional questions. In particular, when both players have made a breakthrough, but only one of them is known for that, then we can ask which of them invests in R&D more aggressively, and who is better off.

Proposition 6. *Suppose that both players A and B are successful, but only player A has revealed his success. The informed player B exerts a higher effort (i.e., $e_t^{1A} < e_t^{1B}$) but has a lower continuation value (i.e., $v_t^{1A} > v_t^{1B}$) than the uninformed player A. However, from the perspective of a fully informed third party, the informed player B is better off than the uninformed player A.*

The first statement claims that the uninformed A player invests less into R&D, due to uncertainty about the rival's state. At the same time player A has a higher continuation value. However, this claim is a bit misleading, since player A is not aware of player B's success and thus his continuation value is computed based on the belief p_t^B. A better comparison is provided by the last statement, where the continuation values are compared from a perspective of an informed third party (or equivalently, from the perspective of the informed player B). Note that this result is not as trivial as it might seem at first glance. On the one hand, having extra information gives a player the option (not obligation) to use the knowledge. On the other hand, being known to have extra knowledge can be to the player's disadvantage.

[12] It can be shown numerically that in contrast player B drops effort before making the first breakthrough, i.e., that e_t^{0B} decreases over time, as in the symmetric version of the game. Furthermore, it can be shown numerically that player B's effort jumps up upon the arrival of his first breakthrough, i.e., $e_t^{0B} < e_t^{1B}$.

6 Patent Race with Optional Revelation

In this section, we address the main question of this paper, whether the players want to reveal their success. In other words, after a player has completed the first stage of the R&D, is it to his advantage when his rival knows about it? Recall that revealing a success does not have any payoff consequences. The way a firm can benefit from revelation can come from discouraging the rival's R&D effort.

In contrast to the private information game, in which players only silently choose their research efforts, the information revelation introduces additional strategic motives. In fact, by revealing own success, a player changes the incentives of his rival to reveal. In the subsequent subgame, where one player is known to be successful, revelation deters an unsuccessful rival, but it accelerates the rival that is successful (as shown by Propositions 5 and 6 in the previous section). As a consequence, the player faces a trade-off that depends on how likely he expects the rival to be unsuccessful, for how long, and whether the rival plans to reveal breakthroughs.

In the analysis we again focus on *Markov perfect Bayesian Nash equilibria (MPBNE).* Analogously as in the previous sections, the players can condition their actions on the payoff relevant state, which (for player j) consists of the state x_t^j and the belief profile (p_t^A, p_t^B). This restriction allows us to use the solution method of backward induction.[13]However, compared to the previous sections, the action space is extended by the action of *revealing* own success, once the player is *successful* (i.e., has reached state 1).

Solving the game using backward induction involves the following four steps.[14] First, suppose that both players have revealed already. The subgame is then equivalent to the complete information game with both players being in state 1, as analyzed in Section 3. Therefore, the first step, does not require any additional analysis.

Second, suppose that both players are in state 1, but only j has revealed it. We will show that his rival never wants to reveal success. Consequently, the continuation values in this game are the same as in the private information game with player j being known

[13]This assumption simplifies the analysis, but it does not necessarily rule out any of the Nash equilibria that this game might have. The analysis of the MPBNE is simpler than the one of Nash equilibria, because it excludes all the strategies in which a player would respond to the rival's actions by punishing him by acting sub-optimally. In the patent race game with revelation, the only observable action of the rival is the revelation. In a Nash equilibrium, a player could potentially choose an unreasonably high effort after his rival's revelation, to dissuade the rival from revealing at the first place. In contrast, in a Markov perfect equilibrium, players always take optimal actions regardless of the past and even in scenarios that never occur. However, we conjecture that a player benefits from the revelation of his rival, and thus he would have no reason to punish his rival in any Nash equilibrium anyways.

[14]We would like to point out that, unfortunately, due to the complex structure of equations, the explicit characterization of equilibria is not possible. Therefore, we accompany the implicit characterization by explicit results obtained by the use of numerical methods. Any result that uses numerical methods in its proof is clearly marked as a numerical result.

to be successful. We have analyzed this game in Section 5.

Third, knowing what players do in the described suggames, we study the incentives to reveal before anyone has revealed. We identify two types of symmetric equilibria in pure strategies: a *no-revelation equilibrium*, where players never reveal their success and an *instant-revelation equilibrium*, where the first player to get success reveals it immediately. In addition, we identify a *mixed-revelation equilibrium*, where the players use mixed strategies and reveal their successes only with a certain probability.

Fourth, we show that the equilibrium is always unique and it is of one of the three above types. In addition, we characterize parameter values so that each type of the equillibrium occurs.

Throughout this section, we refer to the value function $v^{k,j}(p^A, p^B)$, $k \in \{0, 1\}$, $j \in \{A, B\}$, and $p^A, p^B \in [0, 1]$ defined previously as the continuation value in the patent race without revelation of player j who is in state k, while the beliefs are given by the profile of posteriors (p^A, p^B). Note that since we only consider symmetric games, $v^{k,j}(p^A, p^B) = v^{k,-j}(p^B, p^A)$. We often use this fact and directly write $v^{kB}(1, p)$ instead of $v^{kA}(p, 1)$, since the case of player A being known to be successful was closely studied in Section 5.

6.1 Never Reveal Second

We now proceed with the second step. Consider the situation in the patent race with revelation in which one player (say, player A) has already revealed success. If the rival (player B) has an success as well, he has the option to reveal it. After revelation, both players are in the state 1, yielding a continuation value v^{11} obtained in the complete information version of the game. On the other hand, by not revealing, player B's continuation value depends on player A's belief p and is equal to $v^{1B}(1, p)$. The comparison of those continuation values yields the following proposition.

Proposition 7. *In every equilibrium of the game with optional revelation, a player never reveals success after observing the rival's revelation.*

In the proof we first show that indeed $v^{1B}(1, p) > v^{1B}(1, 1) = v^{11}$ for any $p \in [0, 1)$. This means, that not revealing is indeed an equilibrium strategy for player B. Then we argue that it is the only equilibrium strategy. This step is, however, more elaborate.

The intuition for this result has been already indicated in the public information case, where player B's revelation encourages the effort of player A (Proposition 1). This effect is also present, albeit weaker, under optional revelation. Since revealing the success has no direct payoff consequence, the incentives to reveal are determined only via the effect on the rival's effort. Thus, player B prefers to keep his success secret in order to prevent player A from increasing the effort in response to his revelation.

Proposition 7 implies that after player A reveals success, then player B will never reveal, and so the game continues as in the private information version of the game with player j known to be successful. Therefore, if player A reveals his success before player B does, he obtains the continuation value $v^{1A}(1, p_t^B)$, while player B obtains the continuation value $v^{kB}(1, p_t^B)$, where k is player B's actual state and p_t^B is player A's current belief about player B being successful.

Knowing how the game continues after either of the players reveals success allows us to discuss the incentives for revelation that a player has before any of them have revealed. It can be observed numerically that for any choice of the parameters the inequalities $e^{0B}(1, p) < e^0(p)$ and $e^{1B}(1, p) > e^1(p)$ hold for all $p \in [0, 1)$. The comparison of efforts is also illustrated in Figure 2. The solid line represents the rival's effort if A does not reveal, while the dashed line represents the rival's effort if A reveals. This means that player A's revelation discourages the rival's effort so long as he is in state 0. However, once the rival makes a breakthrough, the information about player A's success only makes him choose a higher effort.

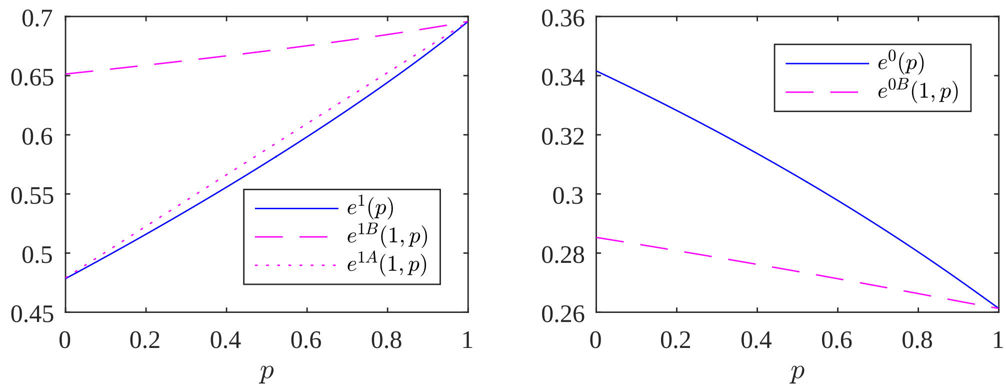

Figure 2: Illustration of the effect of player A's revelation on the rival's effort (parameter values $r = 0.1$, $\alpha = v = 1$).

6.2 Pure-strategy Equilibria

In this subsection we focus on the two pure-strategy equilibria of the patent race with optional revelation, in which players either never reveal or they reveal breakthroughs instantly. The first type of equilibrium is a *no-revelation equilibrium*, which we define as a symmetric equilibrium where both players have the strategy to never reveal their first success. In such a case, none of them reveals and the game evolves as in the private information case described in Section 4. Therefore, each player's continuation value is equal to $v^1(p)$. Should either of the players (say player A) deviate and reveal its success, the subsequent subgame is equivalent to the case where one player is known to be successful, as described in Section 5. The continuation value from the deviation is then equal to $v^{1A}(1,p)$. The comparison of the continuation values then yields the following proposition.

Proposition 8. *There is a no-revelation equilibrium if and only if $v^{1A}(1,p) \leq v^1(p)$, for all $p \in [0,p_*]$, where p_* is the steady-state value of p in the private information version of the game. In that case, such equilibrium is equivalent to the equilibrium of the patent race with private information.*

Proposition 8 characterizes the condition for the existence of a no-revelation equilibrium by referring to value functions for which we do not have explicit formulas, due to the complexity of the problem. Nevertheless, we can compute the value functions numerically. For each value of the research difficulty $\hat{r} = \alpha r/v$ we can solve for the steady-state and then compute the corresponding value functions by proceeding backwards in time.[15] The comparison of those value functions (see Lemma D.1 in the Appendix) yields a simple condition for the existence of a no-revelation equilibrium, namely $\hat{r} = \alpha r/v < \hat{r}_N$, where $\hat{r}_N \approx 0.1113$.

The second type of equilibrium is an *instant-revelation equilibrium*, which we define as a symmetric equilibrium in the patent race with optional revelation in which both players have the strategy to reveal instantly unless the rival has already revealed. Once either of the players reveals, the game continues as in the private information version of the game with one player known to be successful, as analyzed in Section 5. Interestingly, in an instant-revelation equilibrium the game is static until either of the players reveals. This is because until either of the players reveals, they are certain to be both at the starting point. Due to the Markov property and symmetry, both players exert the same constant effort until the first success. Let us denote e_I^0 the equilibrium effort and v_I^0 the equilibrium continuation value.

An instant-revelation equilibrium exists whenever a player will not be tempted to deviate by delaying revelation. That condition gives us the following implicit characterization of the existence of an instant-revelation equilibrium.

Proposition 9. *There is an instant-revelation equilibrium if and only if*

$$0 \geq \frac{\alpha}{2}[v - v^{1A}(1,0)]^2 + e_I^0 v^{1B}(1,0) - (r + e_I^0)v^{1A}(1,0), \tag{11}$$

[15] Recall that, as argued in footnote 5, we can reduce the parameter space to one dimension by setting $\hat{r} = \alpha r/v$.

where $e_I^0 \in (0, v^{1A}(1,0))$ *is the unique positive solution of the quadratic equation*

$$0 = \frac{\alpha}{2}\left(e_I^0\right)^2 + e_I^0 v^{0B}(1,0) - (r + e_I^0)[v^{1A}(1,0) - e_I^0]. \tag{12}$$

In a similar way as described above, it can be shown numerically that the inequality (11) holds if and only if the research difficulty $\hat{r}$ is sufficiently large, namely $\hat{r} = \alpha r/v > \hat{r}_I$, where $\hat{r}_I \approx 0.1707$.

The intuition indeed confirms that no-revelation equilibrium exists when the research difficulty $\hat{r}$ is small, while instant-revelation equilibrium exists then the research difficulty $\hat{r}$ is large. Recall that revealing success discourages the effort of an unsuccessful rival, but encourages the effort of a successful rival. If the research is difficult, the rival is expected to remain unsuccessful for a long time, and so the desirable effect of discouraging the unsuccessful rival dominates the undesirable effect of encouraging the successful rival. Thus, each player reveals their first success immediately (provided the rival has not revealed yet). In contrast, if the research is easy, then each player expects the rival to catch up promptly. Thus, the undesirable effect dominates the desirable one and the players never reveal their success.

Notice that, according to the numerical results, the no-revelation and the instant-revelation equilibria cannot exist simultaneously (for a given value of $\hat{r}$). The intuition for the result is that revealing is less attractive when the rival has the strategy to reveal. When $\hat{r} > \hat{r}_N \approx 0.1113$, a player would have an incentive to deviate by revealing in the no-revelation equilibrium, and yet he would not have an incentive to reveal in the instant-revelation equilibrium so long as $\hat{r} < \hat{r}_I \approx 0.1707$.

The situation is depicted in Figure 3, which shows the incentives to reveal. The blue curve shows the difference between the continuation value from revealing and the equilibrium continuation value (of not revealing) in the no revelation equilibrium, i.e., $v^{1A}(1,0) - v^1(0)$. The red curve shows the difference between the equilibrium continuation value and the continuation value from not revealing in the instant revelation equilibrium, i.e., $v^{1A}(1,0) - \tilde{v}^1$.[16] When both curves are below zero (i.e., $\hat{r}$ is low), then no-revelation equilibrium is present. When both curves are above zero (i.e., $\hat{r}$ is high), then the instant-revelation equilibrium is present. In between, neither of the pure strategy equilibria exists.[17] In the following subsection we show that a mixed-revelation equilibrium is

[16] The value $\tilde{v}^1$ is formally introduced in the proof of Proposition 9 in the appendix.

[17] One might expect that there could also be an asymmetric equilibrium in which one player reveals instantly (say player A) and the other never reveals (player B). However, this is not possible in the symmetric game (with both players facing the same research costs and starting in state 1 with the same probability). To see this, let us analyze the game before any of the players has revealed. Since player A is known to reveal instantly, he must be in state 0 with certainty. However, as player B does not reveal, he is increasingly likely to be already successful (in state 1). As a result, player A has less incentive to reveal than player B does, as he is more likely to face a rival that is already successful. This is a

exists when $\hat{r}$ lies between the two thresholds, i.e., $\hat{r} \in (\hat{r}_N, \hat{r}_I)$.

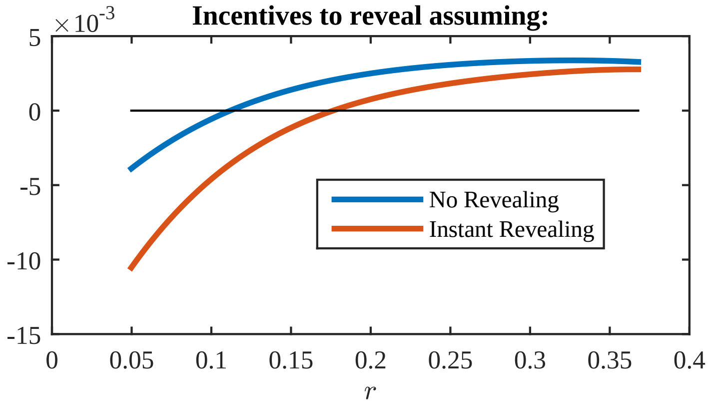

Figure 3: The incentives to reveal as a function of the research difficulty $\hat{r} = \alpha r/v$.

6.3 Mixed-strategy Equlibria

So far, we have discussed two extreme types of equilibria in pure strategies. In this subsection we consider equilibria in mixed strategies. Instead of instant revelation and no revelation, players may also randomize over revelation within the next time interval $[t, t+\Delta t]$. In this sense, the revelation also follows a random process with a certain hazard rate. Formally, a mixed strategy (before anyone has revealed) for player j is represented by a profile of non-negative right-continuous functions $(e^{1j}_{Mt}, e^{0j}_{Mt}, \theta^j_{Mt})$ defined for $t \geq 0$, where e^{1j}_{Mt} and e^{0j}_{Mt} are player j's efforts in states 1 and 0, respectively, and θ^j_{Mt} is the hazard rate with which player j is expected to reveal by his rival $-j$.[18]

Let us point out three important remarks about this definition. First, note that the definition specifies with what hazard rate player j reveals from the perspective of his rival, but it does not specify whether he reveals a breakthrough that he has just made or one that he made before.[19] Second, while our definition of θ^j_{Mt} is not standard, it will turn out to be useful for our analysis, as it reflect the nature of mixed strategy equilibria. Recall that also in simple simultaneous moves games a player's mixing probability is determined by the rival's indifference condition. Here, from the rival's perspective, the indifference condition will indeed depend on the rival's belief about the arrival of revelation, which encompasses both mixing probability as well as the belief about success. Third, this

contradiction with player A revealing instantly and player B never revealing being an equilibrium.

[18] The rival has no information about player j's state, and thus the hazard rate does not depend on it.

[19] One potential way to interpret player j's strategy is that he reveals a breakthrough upon its arrival with the probability $\Theta^j_{Mt} \in [0,1]$. Then, $\theta^j_{Mt} = (1-p^j_t)e^{0j}_{Mt}\Theta^j_{Mt}$, because from $-j$'s perspective player j is with probability $1-p^j_t$ in the state 0, in which case he makes a breakthrough with the hazard rate e^{0j}_{Mt}, and he consequently reveals it with probability Θ^j_{Mt}.

definition covers both no revelation as well as instant revelation strategies. Setting $\theta^j_{Mt} = 0$ means that player j does not reveal a success at time t. On the other hand, $\theta^j_{Mt} = e^{0j}_{Mt}$ corresponds to revealing with certainty, since the hazard rate of revealing is equal to the arrival rate of success.

The strategy has to satisfy the Markov property, which in the case of a symmetric equilibrium, means that there exist functions $e^1_M(\cdot)$, $e^0_M(\cdot)$ and $\theta_M(\cdot)$, such that

$$e^{1j}_{Mt} = e^1_M(p_t), \quad e^{0j}_{Mt} = e^0_M(p_t), \quad \theta^j_{Mt} = \theta_M(p_t), \quad \text{for all } t \geq 0.$$

In other words, the strategy depends on time t only via the belief p_t. For regularity, we require the functions e^1_M, e^0_M and θ_M to be piece-wise continuous. Furthermore, we denote the associated continuation values as $v^{0j}_{Mt} = v^0_M(p_t)$ and $v^{1j}_{Mt} = v^1_M(p_t)$.

Similarly, as in the private information case, we can compute the belief of player $-j$ about player j having a success. This belief is now conditioned on the fact that player j has not obtained his second success, but also on the fact that he did not reveal his first success. Compared to Lemma 1, the law of motion now needs to be adjusted by taking into account the additional probability θ^j_{Mt} of player j revealing the success.

Lemma 2. *Whenever the hazard rate θ^j_{Mt} is finite, the posterior belief before either of the players reveals p^j_t follows the law of motion $\dot{p}^j_t = (1 - p^j_t)(e^{0j}_{Mt} - p_t e^{1j}_{Mt} - \theta^j_{Mt})$.*

The dynamics of the belief is now governed by the relation of the hazard rates e^{0j}_t and $\phi^j_t + \theta^j_{Mt} = p^j_t e^{1j}_t + \theta^j_{Mt}$. Much like in the private information case, the former represents the hazard rate of reaching the the state 1. The latter now includes the hazard rate of leaving the state 1 as well as the hazard rate of revelation, since player $-j$ conditions his actions on the fact that player j has neither patented his second success nor has he revealed his first success.

The next lemma then claims that the hazard rate θ^j_{Mt} is indeed finite and states some dynamic properties of the equilibrium.

Lemma 3. *In any symmetric Markov perfect Bayesian Nash equilibrium, the hazard rate θ^j_{Mt} is finite and the belief p^j_t is non-decreasing over time. Moreover, in a symmetric equilibrium other than the instant-revelation equilibrium, players never reveal with certainty.*

The law of motion together with p_t being non-decreasing imply that $\theta^j_{Mt} \leq e^{0j}_{Mt} - p_t e^{1j}_{Mt}$, for any $t \geq 0$. This imposes an upper bound on the hazard rate θ^j_{Mt}. Note also that it implies that instant revelation (i.e., $\theta^j_{Mt} = e^{0j}_{Mt}$), can occur only when $p^j_t = 0$. Conversely, whenever the rival faces uncertainty about player j having a success (i.e., $p^j_t > 0$), then player j delays revelation with a positive probability ($\theta^j_{Mt} < e^{0j}_{Mt}$).

Now consider an equilibrium other than the instant-revelation equilibrium. Much like in the previous sections, the dynamics when players use mixed strategies is governed by the system of ODEs (see the Appendix, Section D.4 for details):

$$-\dot{v}^1_{Mt} = \frac{\alpha}{2}(e^1_{Mt})^2 + \theta_{Mt} v^{1B}(1, p_t) - (r + \theta_{Mt} + p_t e^1_{Mt}) v^1_{Mt}, \tag{13}$$

$$-\dot{v}^0_{Mt} = \frac{\alpha}{2}(e^0_{Mt})^2 + \theta_{Mt} v^{0B}(1, p_t) - (r + \theta_{Mt} + p_t e^1_{Mt}) v^0_{Mt}, \tag{14}$$

$$\dot{p}_t = (1 - p_t)(e^0_{Mt} - p_t e^1_{Mt} - \theta_{Mt}), \tag{15}$$

together with the identities

$$\alpha e^1_{Mt} = v - v^1_{Mt}, \qquad \alpha e^0_{Mt} = v^1_{Mt} - v^0_{Mt}, \tag{16}$$

the initial condition $p_0 = 0$, and inequalities $v^0_{Mt}, v^1_{Mt} \in [0, v)$, $v^0_{Mt} \leq v^1_{Mt}$ and $p_t \in [0, 1]$, $\theta_{Mt} \in [0, e^0_{Mt} - p_t e^1_{Mt}]$, for all $t \geq 0$. Compared to the ODEs in the private information case, the equations (13)–(14) involve an additional term $\theta_{Mt} v^{kB}(1, p_t)$ that corresponds to the case where the rival reveals its first success, which occurs with a hazard rate θ_{Mt}, yielding the continuation value $v^{kB}(1, p_t)$. In addition, the risk-adjusted discount rate now also includes the hazard rate θ_{Mt} of the rival revealing his success.

The above system of ODEs determines the dynamics of the continuation values, efforts and beliefs for a given trajectory of the hazard rate θ_{Mt}. In addition, since the player has always the option to reveal, in equilibrium the value function after achieving a success needs to be is weakly larger than the value function from revealing, i.e.,

$$v^1_{Mt} \geq v^{1A}(1, p_t).$$

If this inequality is strict, the player will not reveal his success at time t, and so $\theta_{Mt} = 0$. Conversely, whenever a player is mixing at time t (i.e., $\theta_{Mt} > 0$), then his continuation value is the same as under no revelation and we obtain the indifference condition

$$v^1_{Mt} = v^{1A}(1, p_t), \qquad \text{whenever } \theta_{Mt} > 0. \tag{17}$$

In equilibrium (other than instant revelation), this indifference condition can be used to determine the this hazard rate θ_{Mt}.

Now let us point out that the characterization of equilibrium (other than instant revelation) is still not complete. In principle, it is possible that a player switches back and forth between playing a mixed strategy and no revelation. Let us thus now focus on a special type of symmetric equilibrium in mixed strategies in which players mix over revelation (assuming none of them has revealed yet) until a certain deadline T, after which they do not reveal at all, i.e., $\theta_{Mt} > 0$ for all $t < T$ and $\theta_{Mt} = 0$ for all $t \geq T$. We call such an equilibrium a *mixed-revelation equilibrium.*

Proposition 10. *Mixed-revelation equilibrium is uniquely characterized as follows. There is some $T > 0$ such that:*

(i) $v^{1A}(1, p_T) = v^1(p_T)$.

(ii) For all $t \in [0, T)$,

$$\theta_{Mt} = \frac{\frac{\alpha}{2}[e^{1A}(1,p_t)]^2 - (r + e^0_{Mt})v^{1A}(1,p_t) + [e^0_{Mt} - p_t e^{1A}(1,p_t)]\tilde{v}^{1A}_t(p_t)}{\tilde{v}^{1A}_t(p_t) - v^{1B}(1,p_t)}, \tag{18}$$

where $\tilde{v}^{1A}_t(p_t) = v^{1A}(1,p_t) + (1-p_t)\frac{\partial}{\partial p}v^{1A}(1,p_t)$. The value function v^0_{Mt} follows the ODE (14) *and the belief p_t satisfies the law of motion* (15).

(iii) For all $t \geq T$, then $\theta_{Mt} = 0$ and the game continues as the private information game (i.e., without the option to reveal). The value functions are then the same as in the private information case, i.e., $v^1_{Mt} = v^1(p_t)$, $v^0_{Mt} = v^0(p_t)$, and the belief p_t satisfies the law of motion (4).

Statement (i) claims that the players stop revealing at the point when revealing yields the same continuation value as not revealing, assuming none of the players will reveal later on. Statement (ii) is a dynamic version of the indifference condition (17), according to which the value function $v^{1A}(1, p_t)$ needs to follow the ODE (13) before the deadline T. Statement (iii) then specifies the continuation game after the deadline T no player reveals his success and the continuation game is equivalent to the private information game.

The characterization of the mixed-revelation equilibrium in Proposition 10 also provides the prescription for calculating it: Consider $\bar{p} \in (0, 1)$ such that $v^{1A}(1, \bar{p}) = v^1(\bar{p})$, and let $v^0_M(\bar{p}) = v^0(\bar{p})$.[20] Then we solve the ODE $(v^0_M)'(p) = \dot{v}^0_M(p)/\dot{p}_M(p)$ for p from $\bar{p}$ down to 0, where $\dot{p}_M(p) = (1-p)[e^0_M(1,p) - pe^1_M(1,p) - \theta_M(p)]$ is the right hand-side of the law of motion (15) as a function of the belief p. Having solved for $v^0_M(p)$, we start from time $t = 0$, set $p_0 = 0$ and compute p_t using the law of motion (15) until p_t reaches the value $\bar{p}$. We define this time as T, i.e., $p_T = \bar{p}$.

It can be shown numerically that for any value of the parameter $\hat{r} = \alpha r/v$ (research difficulty), both the numerator and the denominator of (18) are positive, and thus θ_{Mt} is always well defined. A potential trouble could arise if the derivative $\dot{p}_M(p)$ would reach zero at some point. However, as we show numerically, this is the case if and only

[20] See also Lemma D.1 in the appendix.

if the instant-revelation equilibrium exists. Now define $(\underline{p}, \underline{v}_M^0)$, where $\underline{p} \in [0,1)$ and $0 \leq \underline{v}_M^0 \leq v^{1A}(1, \underline{p})$, to be the steady-state of the system of ODEs (14)–(15). It can be shown numerically that such a steady-state exists (and is unique) if and only if the instant-revelation equilibrium exists.

Figure 4 shows how $\bar{p}$ and $\underline{p}$ vary with the research difficulty $\hat{r}$. Consider the values of $\hat{r}$ such that $\underline{p} > 0$. Since we assume $p_0 = 0$, players can only reveal with certainty and p_t remains at zero. However, the situation would be different if we considered that players initially had made a breakthrough with a positive probability. If $p_0 \in (\underline{p}, \bar{p})$, then players would randomize over revelation until they stop revealing when p_t reaches the value $\bar{p}$. On the other hand, if $p_0 \in (0, \underline{p})$, then players would randomize over revelation until p_t reaches 0, after which they reveal instantly.

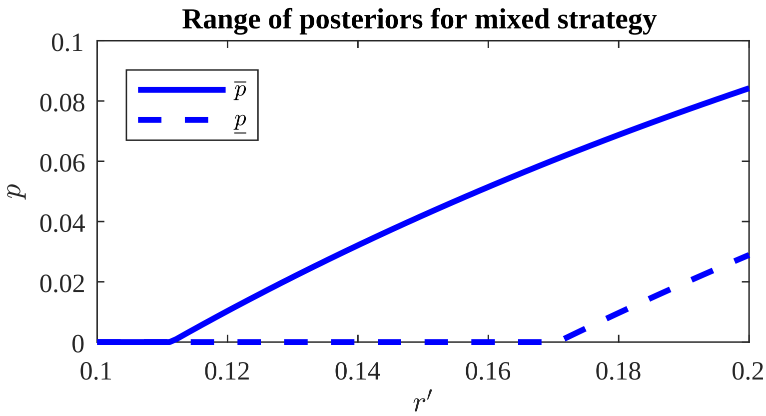

Figure 4: The range $[\underline{p}, \bar{p}]$ of p at which players might be randomizing over revelation, as a function of the research difficulty $\hat{r} = \alpha r / v$.

6.4 Equilibrium Characterization

Summing up, in the previous subsections, we have characterized three types of equilibria. For small values of the research difficulty we obtain the *no-revelation equilibrium*, where the players do not reveal success. For large values of the research difficulty we obtain the *instant-revelation equilibrium*, where the player reveal their first success immediately. However, there is a range of values of research difficulty $\hat{r} = \alpha r / v$ for which neither of those two pure equilibria exists: In a no-revelation equilibrium, a player would be tempted to reveal, yet he will not have sufficient incentives to reveal in an instant-revelation equilibrium. Then, an equilibrium can be found only in mixed strategies, so that each player reveals exactly with the probability that keeps his rival indifferent between revealing and not revealing. We have characterized one type of equilibria in mixed strategies, the *mixed-revelation equilibrium*, where the players delay the revelation of their first success until some deadline, after which they do not reveal anymore. It turns out that this type of equilibrium indeed exists for intermediate values of the research difficulty. The following proposition provides a complete characterization of symmetric equilibria.

Proposition 11 (partially numerical)**.** *The patent race with optional revelation has a unique symmetric Markov perfect equilibrium. Depending on the research difficulty* $\hat{r} = \alpha r/v$*, the type of the equilibrium is:*

(a) no-revelation equilibrium for $\hat{r} \in [0, \hat{r}_N]$*;*

(b) mixed-revelation equilibrium for $\hat{r} \in (\hat{r}_N, \hat{r}_I)$*;*

(c) instant-revelation equilibrium for $\hat{r} \in [\hat{r}_I, +\infty)$*,*

where the thresholds are approximately $\hat{r}_N \approx 0.1113$ *and* $\hat{r}_I \approx 0.1707$*.*

The proposition contains two important results. First, the only symmetric equilibrium involving randomizing over revelation is the mixed-revelation equilibrium. The intuition for this result is the following. Due to Lemma 3, such an equilibrium does not involve instant revelation. Whenever a player starts to randomize at some point in time, he would prefer to preempt and start revealing earlier. On the other hand, asymptotic properties of the system of ODEs imply that it is not possible to sustain randomization forever. Thus, a player can randomize only over a finite time interval starting at time 0.

The second result is that equilibria of different types cannot coexist for the same parameter values. In order to provide an intuition for this result, consider the mixed revelation equilibrium for some $\hat{r} \in (\hat{r}_N, \hat{r}_I)$. Using numerical simulations, we conclude that the deadline T at which the player stops revealing is increasing in the research difficulty $\hat{r}$. Indeed, as $\hat{r}$ increases, the player expects the rival to remain unsuccessful for a longer time and is thus more willing to reveal his own success in order to discourage the rival's effort. When $\hat{r}$ converges to the lower threshold $\hat{r}_N$, the deadline T converges to zero and the equilibrium converges to the *no-revelation equilibrium*. Below the threshold, the player considers it likely that the rival already has a success, implying only a small benefit from revelation. On the other hand, when $\hat{r}$ converges to the upper threshold $\hat{r}_I$, the deadline T converges to infinity and the the hazard rate of revealing is large. Beyond the threshold, it is not sustainable anymore to preserve the indifference condition and the equilibrium becomes the one of *instant revelation*.

7 Conclusion

This paper explores the role of private information about firms' progress in a patent race. The two primary objectives are (i) to understand how the race evolves when firms do not observe each others progress towards making a patent, and (ii) to investigate when firms have the incentive to reveal their success. The main takeaways are that a rival's success discourages the effort of an unsuccessful firm, but it encourages effort of a successful firm. Accordingly, a firm wants to reveal its success only if it expects its rival to be and to remain behind.

We implicitly characterize three types of equilibria in the patent race with the option to reveal breakthroughs. However, to determine which of the equilibria exists for a given choice of parameters, it is necessary to solve the given system of equations numerically. Our additional analysis indicatas that the results extend even to more general settings, yet the necessity to use numerical methods is a shortcoming of the complexity of the model used.

There are various ways to simplify the model, one of which would be to rewrite the model into discrete time and consider a small number of time periods. This certainly simplifies the numerical solution as the system of ODE's simplifies into a system of multivariate cubic equations. However, even if there were only three periods (the absolute minimum for the revelation of the breakthrough to matter), we would not be able to obtain a closed form solution.[21] Note that such a system of equations cannot be solved recursively, because while the continuation values are given by the boundary conditions in the last period, the posterior belief is given only in the first period. An alternative simplification is to allow only for low and high effort, as in Gordon (2011). However, in that case the game might have multiplicity or no equilibria, and thus it is difficult to draw any conclusions about player's behavior. In fact, such a simplification completely changes the structure of the incentives, as then information influences a player's decision only when he is almost indifferent between low and high effort. For example, if an unsuccessful player learns that his rival is ahead, he decreases his effort in the setting with continuous effort choice, but the information would perhaps not matter to him in the setting with a binary effort choice, as he would exert low effort when being unsuccessful anyway.

In practice, there are many other factors relevant for a firm's decision whether to reveal a breakthrough. The revelation might help to raise further investments, but it might also lead to technological leakage, or it might show rivals that a solution to a certain technological problem exists. We abstract from these factors and focus on the single aspect of the patent race, where private information about a firm's progress and its revelation discourages the rival's R&D effort. Nevertheless, such additional aspects represent interesting routes for future research.

[21] Alternatively, we would obtain a system of multivariate quadratic equations if we considered a linear effort cost function, but that still does not allow a closed form solution.

A Appendix: Proofs for Section 3 (Complete Information Case)

A.1 Derivation of Optimal Effort (1)

The player's continuation value in state kl can be characterized recursively as

$$v^{kl} = \max_{e \geq 0} \left\{ v^{k+1,l} e\Delta t - \frac{\alpha}{2} e^2 \Delta t + v^{k,l+1} e^{lk} \Delta t + [1 - (e + e^{lk})\Delta t](1 - r\Delta t) v^{kl} + o(\Delta t) \right\}. \tag{19}$$

The first, third, and fourth term represent the values from making a breakthrough, rival making a breakthrough, and nobody making a breakthrough, respectively, within the time interval $[t, t + \Delta t]$, multiplied by the corresponding probabilities. The second term is the cost of effort e.

Subtracting v^{kl} from both sides of the equation, dividing by $\Delta t > 0$, and taking the limit $\Delta t \searrow 0$, we obtain

$$0 = \max_{e \geq 0} \left\{ v^{k+1,l} e - \frac{\alpha}{2} e^2 + v^{k,l+1} e^{lk} - (e + e^{lk} + r) v^{kl} \right\} \tag{20}$$

The expression on the right-hand side is quadratic and concave in effort e. The first order condition for optimal effort then indeed yields $\alpha e^{kl} = v^{k+1,l} - v^{kl}$, which is indeed equation (1).

A.2 Proof of Proposition 1

Thourough this section we use the normalization $v = 1$ and $\alpha = 1$. Substituting the optimal effort $e^{kl} = v^{k+1,l} - v^{kl}$ into (20), we obtain a the system of equations

$$0 = \tfrac{1}{2}(v^{k+1,l} - v^{kl})^2 + e^{lk}(v^{k,l+1} - v^{kl}) - r v^{kl}, \qquad k, l \in \{0, 1\} \tag{21}$$

along with the boundary conditions $v^{2,l} = 1$ and $v^{k,2} = 0$. Moreover, $e^{lk} = v^{l+1,k} - v^{lk} > 0$ represents the effort of the rival. Note this system contains four equations with four unknowns: v^{00}, v^{10}, v^{01}, and v^{11}.

The proof of Proposition 1 is based on four lemmas. Lemma A.1 establishes monotonicity of values in the state. Lemma A.2 establishes uniqueness of the solution. Lemmas A.3 and A.4 provide inequalities and bounds that yield Proposition 1. We first state and prove the lemmas and then proceed with the actual proof of Proposition 1.

Lemma A.1. *The inequalities $v^{k,l+1} < v^{kl} < v^{k+1,l}$ hold for any $k, l \in \{0, 1\}$.*

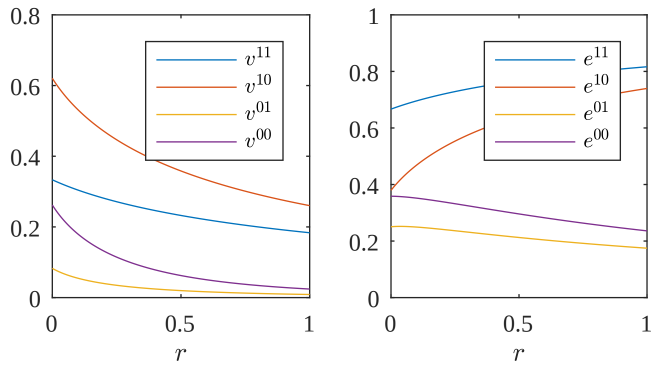

Figure 5: Continuation values and efforts in the four different states of the public information version of the game as a function of the research difficulty $\hat{r}$.

Proof. First, $v^{k+1,l} > v^{kl}$ holds trivially as $e^{kl} > 0$ by assumption. The inequality $v^{k,l+1} < v^{kl}$ holds trivially for $l = 1$ as $v^{k,2} = 0$, it remains to prove it for $l = 0$. We will use mathematical induction, in which we show that weak inequality $v^{k+1,1} \leq v^{k+1,0}$ implies the strong inequality $v^{k,1} < v^{k,0}$. We have $v^{21} \leq v^{20}$ as both values are equal 1. Consider $k \in \{0, 1\}$, and assume that the inequality $v^{k+1,1} \leq v^{k+1,0}$ holds. Then

$$\begin{aligned}
0 &= \max_{e\in[0,1]} \left\{e \cdot v^{k+1,0} - \tfrac{1}{2}e^2 + e^{0k} \cdot (v^{k1} - v^{k0}) - (r + e) \cdot v^{k0}\right\} \\
&\geq e^{k1} \cdot v^{k+1,0} - \tfrac{1}{2}(e^{k1})^2 + e^{0k} \cdot (v^{k1} - v^{k0}) - (r + e^{k1}) \cdot v^{k0} \\
&\geq e^{k1} \cdot v^{k+1,1} - \tfrac{1}{2}(e^{k1})^2 + e^{0k} \cdot (v^{k1} - v^{k0}) - (r + e^{k1}) \cdot v^{k0} \\
&= e^{k1} \cdot v^{k+1,1} - \tfrac{1}{2}(e^{k1})^2 - (r + e^{k1}) \cdot v^{k1} + (r + e^{k1} + e^{0k}) \cdot (v^{k1} - v^{k0}) \\
&> \underbrace{e^{k1} \cdot v^{k+1,1} - \tfrac{1}{2}(e^{k1})^2 + e^{1k} \cdot (v^{k2} - v^{k1}) - (r + e^{k1}) \cdot v^{k1}}_{=0} + (r + e^{k1} + e^{0k}) \cdot (v^{k1} - v^{k0}) \\
&= (r + e^{k1} + e^{0k}) \cdot (v^{k1} - v^{k0}),
\end{aligned}$$

and so $0 > v^{k1} - v^{k0}$. We conclude that $v^{k1} < v^{k0}$ for $k \in \{0, 1\}$. □

Lemma A.2. *The system of four equations* (21) *has a unique solution.*

Proof. We prove it recursively by decreasing k and l. For $k = 2$ or $l = 2$ the uniqueness is trivial. Take any $k, l \in \{0, 1\}$ for which the uniqueness of $v^{k+1,l}, v^{k,l+1}, v^{l+1,k}, v^{z,k+1}$ has been proven already (initially it is the case for $k = l = 1$). Separating $e^{lk} = v^{l+1,k} - v^{lk}$

in the equation (21),

$$v^{lk} = v^{l+1,k} - e^{lk} = v^{l+1,k} - \frac{\frac{1}{2}(v^{k+1,l} - v^{kl})^2 - rv^{kl}}{v^{kl} - v^{k,l+1}} = g^{lk}(v^{kl}),$$
$$\text{where} \quad g^{lk}(z) := v^{l+1,k} - \frac{\frac{1}{2}(v^{k+1,l} - z)^2 - rz}{z - v^{k,l+1}} \tag{22}$$

so that v^{lk} is expressed as a function of v^{kl} and other variable which are already known to be uniquely defined. Notice that since $v^{k,l+1} \leq v^{k+1,l+1} < v^{k+1,l}$ (Lemma A.1), we have

$$\tfrac{1}{2}(v^{k+1,l} - v^{k,l+1})^2 - rv^{k,l+1} > \tfrac{1}{2}(v^{k+1,l+1} - v^{k,l+1})^2 - rv^{k,l+1} \geq 0,$$

and so the strictly decreasing function $z \mapsto \frac{1}{2}(v^{k+1,l} - z)^2 - rz$ has unique root on the interval $(v^{k,l+1}, v^{k+1,l})$; denote it $\bar{v}^{kl}$.

It follows that $g^{lk}(z)$ is a strictly increasing function on the interval $(v^{k,l+1}, \bar{v}^{kl}]$. Moreover,

$$\begin{aligned} g^{lk}(z) &= v^{l+1,k} - \frac{\frac{1}{2}z^2 - v^{k+1,l}z + \frac{1}{2}(v^{k+1,l})^2 - rz}{z - v^{k,l+1}} \\ &= v^{l+1,k} - \tfrac{1}{2}v^{k,l+1} + v^{k+1,l} + r - \tfrac{1}{2}z - \frac{\frac{1}{2}(v^{k+1,l} - v^{k,l+1})^2 - rv^{k,l+1}}{z - v^{k,l+1}}. \end{aligned}$$

Since the term in the numerator is positive, function $g^{lk}(z)$ is concave. In summary, $g^{lk}(z)$ is a continuous, concave, strictly increasing function on the interval $(v^{k,l+1}, \bar{v}^{kl}]$ with range from $-\infty$ to $v^{l+1,k}$.

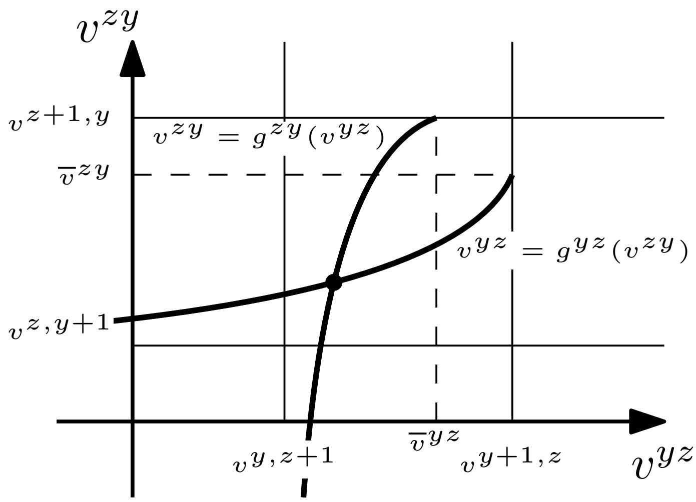

Figure 6: Illustration of the uniqueness of v^{kl} and v^{lk} as an intersection of the graphs of reaction functions.

By symmetry, there is continuous, concave, strictly increasing function $g^{kl}(z)$ defined on the interval $(v^{l,k+1}, \overline{v}^{lk}]$ with range from $-\infty$ to $v^{k+1,l}$, such that $v^{kl} = g^{kl}(v^{lk})$. As illustrated in Figure 6 it should be clear that there is unique point $(v^{kl}, v^{lk}) \in (v^{k,l+1}, \overline{v}^{kl}] \times (v^{z,k+1}, \overline{v}^{lk}]$ that satisfies both $v^{lk} = g^{lk}(v^{kl})$ and $v^{kl} = g^{kl}(v^{lk})$. $\square$

Lemma A.3. *Let*

$$\begin{aligned} h^{11}(z) &:= \tfrac{3}{2}z^2 + (r-1)z - r, \\ h^{10}(z) &:= \tfrac{1}{2}z^2 + (r+e^{01})z - e^{01}e^{11} - r, \\ h^{01}(z) &:= \tfrac{1}{2}z^2 + (r+e^{10})z - (r+e^{10})(1-e^{11}) \\ h^{00}(z) &:= \tfrac{3}{2}z^2 + (r-e^{11}+e^{10}-e^{01})z - r(1-e^{10}). \end{aligned}$$

Then e^{kl} satisfies $h^{kl}(e^{kl}) = 0$ for all $k, l \in \{0,1\}$. In addition, $h^{kl}(z) < 0$ when $0 \le z < e^{kl}$, and $h^{kl}(z) > 0$ when $z > e^{kl}$, for all $k, l \in \{0,1\}$.

Proof. Using the optimality condition $e^{kl} = v^{k+1,l} - v^{kl}$, we can express the values in terms of efforts: $v^{11} = 1 - e^{11}$, $v^{10} = 1 - e^{10}$, $v^{01} = v^{11} - e^{01} = 1 - e^{11} - e^{01}$, and $v^{00} = v^{10} - e^{00} = 1 - e^{10} - e^{00}$. Substituting those values into (21), we obtain that equation (21) can be written as $h^{kl}(e^{kl}) = 0$, for all $k, l \in \{0,1\}$. Observe that for any $k, l \in \{0,1\}$, the quadratic polynomial $h^{kl}(z)$ has a positive leading coefficient and a negative intercept, and so it has a unique positive root. Hence, e^{kl} is the root, and $h^{kl}(z)$ is negative on the left from the root, and is positive the right from it. $\square$

Lemma A.4. *The following inequalities hold:*

(i) $\underline{e}^{11} := 1 - \dfrac{1}{3+2r} < e^{11} < 1 - \dfrac{1}{4+2r} =: \overline{e}^{11}$;

(ii) $e^{01} < 1 - e^{11} = \dfrac{1}{3+2r} =: \overline{e}^{01}$;

(iii) $e^{10} < 1 - \dfrac{1}{2+2r} =: \overline{e}^{10}$;

(iv) $\underline{e}^{00} := \dfrac{1}{3+2r} < e^{00}$.

Proof. (i) Evaluating h^{11} at the lower and upper estimate of e^{11},

$$h^{11}(\underline{e}^{11}) = -\frac{r}{(2r+3)^2} < 0, \quad \text{and} \quad h^{11}(\overline{e}^{11}) = \frac{3}{8(r+2)^2} > 0.$$

This former yields $\underline{e}^{11} < e^{11}$, whereas the latter yields $e^{11} < \overline{e}^{11}$.

(ii) Applying the result of (i), $e^{01} = v^{11} - v^{01} < v^{11} = 1 - e^{11} < 1 - \underline{e}^{11} = \frac{1}{3+2r}$.

(iii) We have

$$\begin{aligned} h^{10}(\overline{e}^{10}) &= \tfrac{1}{2}(\overline{e}^{10})^2 + r\overline{e}^{10} + e^{01}(\overline{e}^{10} - e^{11}) - r \\ &> \tfrac{1}{2}(\overline{e}^{10})^2 + r\overline{e}^{10} + e^{01}(\overline{e}^{10} - \overline{e}^{11}) - r \\ &> \tfrac{1}{2}(\overline{e}^{10})^2 + r\overline{e}^{10} + \overline{e}^{01}(\overline{e}^{10} - \overline{e}^{11}) - r \\ &= \frac{2+3r+2r^2}{2(2+2r)^2(2r^2+7r+6)} > 0, \end{aligned}$$

where the first inequality follows from (i) and the second inequality from (ii) and $\overline{e}^{10} - \overline{e}^{11} = 1/[(2+2r)(2+r)] < 0$. Thus, indeed $\overline{e}^{10} > e^{10}$.

(iv) We have

$$\begin{aligned} h^{00}(\underline{e}^{00}) &= \tfrac{1}{2}(\underline{e}^{00})^2 + \underline{e}^{00}(r - e^{11} + e^{10}) - r(1 - e^{10}) \\ &< \tfrac{1}{2}(\underline{e}^{00})^2 + \underline{e}^{00}(r - \underline{e}^{11} + \overline{e}^{10}) - r(1 - \overline{e}^{10}) \\ &= -\frac{r}{(3+2r)^2} < 0, \end{aligned}$$

where the first inequality follows from (i) and (iii). This implies $\underline{e}^{00} < e^{00}$. □

Proof of Proposition 1. To begin with, by Lemma A.2 the system of equations (21) has unique solution, which allows us to analyze it. The inequality $e^{10} < e^{11}$ follows from the fact that $e^{1l} = v^{2l} - v^{1l} = 1 - v^{1l}$, for $l \in \{0, 1\}$, and the inequality $v^{11} < v^{10}$ (Lemma A.1). The inequality $e^{01} < e^{00}$ follows from the estimates in Lemma A.4 as $e^{01} < \overline{e}^{01} = \underline{e}^{00} < e^{00}$. □

B Appendix: Proofs for Section 4 (Private Information Case)

B.1 Law of motion for the posterior belief

Proof of Lemma 1. The posterior belief follows the Bayes Law. Take the conditioned probability p_t^j as given and assume that the game has not ended by time t. Then with probability $(1 - p_t^j)$ the state is $x_t^j = 0$, and with hazard rate e_t^{0j} it proceeds to the state $x_{t+\Delta t}^j = 1$; with probability p_t^j the state is $x_t^j = 1$ and with the hazard rate e_t^{1j} it proceeds and the game ends. Then

$$p^j_{t+\Delta t} = P[x^j_{t+\Delta t} = 1 \mid x^j_{t+\Delta t} < 2] = \frac{P\left[x^j_{t+\Delta t} = 1 \mid x^j_t < 2\right]}{P\left[x^j_{t+\Delta t} < 2 \mid x^j_t < 2\right]}$$
$$= \frac{(1-p^j_t)e^{0j}_t \Delta t + p^j_t(1-e^{1j}_t \Delta t)}{1 - p^j_t e^{1j}_t \Delta t},$$

and so

$$\dot{p}^j_t = \frac{\partial}{\partial \Delta t} p^j_{t+\Delta t}\Big|_{\Delta t=0} = (1-p^j_t)(e^{0j}_t - p^j_t e^{1j}_t).$$

This completes the proof. □

B.2 Optimal Effort and Value Functions

The continuation value of player j in state k at time t is

$$v^{k,j}_t = \max_{e \geq 0} \left\{ v^{k+1,j}_t e \Delta t - \frac{\alpha}{2} e^2 \Delta t + [1 - (e + \phi^{-j}_t)\Delta t](1 - r\Delta t) v^{k,j}_{t+\Delta t} + o(\Delta t) \right\},$$

The first represents value from making a breakthrough within the time interval $[t, t+\Delta t]$, multiplied by the probability. The second term is the cost of effort e. The third term represents the value when the state (from the perspective of firm j) does not change within the time interval $[t, t+\Delta t]$, multiplied the corresponding probability. Note that this probability involves the hazard rate ϕ^{-j}_t with which the rival patents at time t.

Subtracting $v^{k,j}_{t+\Delta t}$ from both sides of the equation, dividing them by $\Delta t > 0$, and taking the limit $\Delta t \searrow 0$, we obtain

$$-\dot{v}^{k,j}_t = \max_{e \geq 0} \left\{ (v^{k+1,j}_t - v^{k,j}_t)e - \frac{\alpha}{2}(e)^2 - (r + \phi^{-j}_t) v^{k,j}_t \right\}.$$

The first order condition for e implies

$$\alpha e^{k,j}_t = v^{k+1,j}_t - v^{k,j}_t \tag{23}$$

in other words, player j's optimal effort is equal to the potential gain from instant completion of the current stage of R&D. This result is analogous to condition (1) derived in the complete information case. Consequently, for each state $k \in \{0, 1\}$ and player $j \in \{A, B\}$, we obtain the following differential equation for the value function:

$$-\dot{v}^{k,j}_t = \frac{\alpha}{2}(e^{k,j}_t)^2 - (r + \phi^{-j}_t) v^{k,j}_t.$$

The game can be summarized by the following system of six ODEs (three for each with

$j \in \{A, B\}$):

$$-\dot{v}_t^{1j} = \frac{\alpha}{2}(e_t^{1j})^2 - (r + p_t^{-j} e_t^{1,-j}) v_t^{1j}, \tag{24}$$

$$-\dot{v}_t^{0j} = \frac{\alpha}{2}(e_t^{0j})^2 - (r + p_t^{-j} e_t^{1,-j}) v_t^{0j}, \tag{25}$$

$$\dot{p}_t^j = (1 - p_t^j)(e_t^{0j} - p_t^j e_t^{1j}), \tag{26}$$

together with conditions (23).

B.3 Normalization of Parameters

Now note that the presented model of the patent race involves three parameters: the value of the patent v, the effort cost multiplier α, and the discount rate r. However, the generality of the problem will not be compromised if we set $v = 1$ and $\alpha = 1$. Intuitively, apart from choosing a unit of value such that $v = 1$, it is also possible to choose the units of time such that $\alpha = 1$. The normalization allows us to simplify the notation of the proofs. It also has the advantage that if some property can only be shown numerically, then the property can be tested in a one-dimensional parameter space. The following lemma provides a formal statement.

Lemma B.1. *Any equilibrium in the patent race with private information corresponds to an equilibrium of the game with $\hat{v} = 1$, $\hat{\alpha} = 1$, and $\hat{r} = \frac{\alpha r}{v}$.*

Proof. Multiplying each of the equations in the system of ODEs (24)–(26) by α/v and dividing the first two by v we obtain the following system of ODEs:

$$-\alpha \frac{\dot{v}_t^{1j}}{v^2} = \frac{1}{2}\left(\frac{\alpha e_t^{1j}}{v}\right)^2 - \left(\frac{\alpha r}{v} + p_t^{-j} \frac{\alpha e_t^{1,-j}}{v}\right) \frac{v_t^{1j}}{v}$$

$$-\alpha \frac{\dot{v}_t^{0j}}{v^2} = \frac{1}{2}\left(\frac{\alpha e_t^{0j}}{v}\right)^2 - \left(\frac{\alpha r}{v} + p_t^{-j} \frac{\alpha e_t^{1,-j}}{v}\right) \frac{v_t^{0j}}{v}$$

$$\frac{\alpha \dot{p}_t^j}{v} = (1 - p_t^j)\left(\frac{\alpha e_t^{0j}}{v} - p_t^j \frac{\alpha e_t^{1j}}{v}\right).$$

This system of equations is identical to the system of ODEs (24)–(26) with parameters $\hat{v} = 1$, $\hat{\alpha} = 1$, and $\hat{r} = \alpha r / v$, and variables

$$\hat{v}_t^{k,j} = \frac{v_{t\alpha/v}^{k,j}}{v}, \quad \hat{e}_t^{k,j} = \frac{\alpha e_{t\alpha/v}^{k,j}}{v}, \quad \text{and} \quad \hat{p}_t^j = p_{t\alpha/v}^j,$$

where $k \in \{0, 1\}$, $j \in \{A, B\}$. □

B.4 Existence and Uniqueness of Equilibrium

Using the conditions $e_t^1 = 1 - v_t^1$ and $e_t^0 = v_t^1 - v_t^0$, the system of ODEs (2)–(4) can be expressed in terms of optimal effort levels and belief (e_t^1, e_t^0, p_t) as follows

$$\dot{e}_t^1 = \tfrac{1}{2}(e_t^1)^2 - (r + p_t e_t^1)(1 - e_t^1) \tag{27}$$

$$\dot{e}_t^0 = \tfrac{1}{2}(e_t^0)^2 - \tfrac{1}{2}(e_t^1)^2 + (r + p_t e_t^1)e_t^0 \tag{28}$$

$$\dot{p}_t = (1 - p_t)(e_t^0 - p_t e_t^1), \tag{29}$$

with the initial condition $p_0 = \hat{p} \in [0, 1)$, and restrictions $e_t^1, p_t \in [0, 1]$, $e_t^0 \in [0, 1 - e_t^1]$ for all $t \geq 0$.[22] Although we only need to study the problem with the initial condition containing $\hat{p} = 0$, considering any $\hat{p} \in [0, 1)$ we allows us to define us variables $e^0(p)$ and $e^1(p)$ as functions of the posterior belief $p \in [0, 1)$.

Note that the vector $(\dot{e}_t^1, \dot{e}_t^0, \dot{p}_t)$ is a continuous function of the vector (e_t^1, e_t^0, p_t), and thus the solution vector is an analytic function of time.

The proof of Proposition 2 relies on the following three lemmas. The proofs of the lemmas follow below.

Lemma B.2. *The system of ODEs* (27)–(29) *has a unique critical point* (e_*^1, e_*^0, p_*) *with* $p_* < 1$.

Lemma B.3. *The Jacobian at the critical point of the system of ODEs* (27)–(29) *has unique eigenvalue with non-positive real part. This eigenvalue is a negative real number.*

Lemma B.4. *Any solution of the system of ODEs* (27)–(29) *converges to the steady-state* (e_*^1, e_*^0, p_*) *as* $t \to \infty$. *Moreover, the belief* p_t *is increasing over time.*

B.5 Proof of Lemma B.2 (Unique Critical Point)

Before proceeding with the proof of Lemma B.2 we formally state and prove some elementary properties of a function that will be used in numerous proofs.

Lemma B.5. *For any given* $r > 0$, *the function*

$$\psi(z) := \frac{z^2}{2(1 - z)} - r \tag{30}$$

is strictly increasing and strictly convex on $[0, 1)$ *and it has a well defined inverse function* ψ^{-1} *on* $[0, 1]$ *which is strictly increasing and strictly concave. In addition,* ψ *has unique positive fixed point* z^*. *The fixed point is in the interval* $z^* \in (\frac{2}{3}, 1)$, *and* $\psi(z) < z$ *for all* $z \in [0, z^*)$ *and* $\psi(z) > z$ *for all* $z \in (z^*, 1)$.

[22] The condition $e_t^0 \in [0, 1 - e_t^1]$ follow from the fact that $1 - e_t^1 - e_t^0 = v_t^0 \geq 0$.

Proof. Since both of the functions $z \mapsto z^2$ and $z \mapsto 1/(1-z)$ are positive, strictly increasing and strictly convex on the interval $[0,1)$, so is their product. Consequently also the function ψ is strictly increasing and strictly convex. Further, $\psi(0) = -r < 0$ and $\psi(1_-) = +\infty$.[23]

It follows from continuity that the inverse function ψ^{-1} is well defined on $[0,1]$, and it is strictly increasing and strictly concave. Since $\psi(0) = -r$, $\psi'(0) = 0$, $\psi(1_-) = +\infty$, and ψ is strictly convex, the function ψ has to intersect the identity function at a unique point z^*, and $\psi(z) < z$ if only if $z < z^*$. Since $\psi(\frac{2}{3}) = \frac{2}{3} - r < \frac{2}{3}$, necessarily $z^* > \frac{2}{3}$. $\square$

Proof of Lemma B.2. We prove an equivalent statement that the system (27)–(29) has a unique critical point with $p < 1$. A critical point is characterized by the condition $(\dot{e}_t^1, \dot{e}_t^0, \dot{p}_t) = (0,0,0)$. Dropping the time index, we obtain $e^0 = pe^1$ and

$$0 = \frac{1}{2}(e^1)^2 - (r + e^0)(1 - e^1), \tag{31}$$

$$0 = \frac{1}{2}(e^0)^2 - \frac{1}{2}(e^1)^2 + (r + e^0)e^0. \tag{32}$$

Now note that $e^1 \neq 1$ and $e^0, e^1, p \neq 0$.[24] The above system is equivalent to[25]

$$e^0 = \frac{(e^1)^2}{2(1 - e^1)} - r, \tag{33}$$

$$1 = \left(\frac{e^0}{e^1}\right)^2 + \frac{e^0}{1 - e^1}. \tag{34}$$

Define the function g on the interval $(0,1)$ by

$$g(z) := \left(\frac{\psi(z)}{z}\right)^2 + \frac{\psi(z)}{1 - z},$$

where the function ψ is defined by (30). Then the initial system of equations is equivalent to $e^0 = \psi(e^1)$, $1 = g(e^1)$, and $p = e^0/e^1$.

Recall that by Lemma B.5 the function ψ is increasing. The function g is strictly increasing on the interval $[\psi^{-1}(0), 1)$, as both $\psi(z)/(1-z)$ and $\psi(z)/z$ are strictly increasing

[23] The notation $\psi(1_-)$ stands for $\lim_{z \nearrow 1} \psi(z)$.

[24] Since $e^0 = pe^1$, it is suffient to show that $e^0 \neq 0$. If to the contrary $e^0 = 0$, then it follows from (32) that $e^1 = 0$, which contradicts (31).

[25] The equation (34) is obtained by substituting for $(r + e^0)$ in (32) from (31), and dividing the equation by $\frac{1}{2}(e^1)^2 > 0$.

functions of z as long as $\psi(z) \geq 0$. Moreover, $g(\psi^{-1}(0)) = 0$, and

$$g(e^{11}) > \left(\frac{\psi(e^{11})}{e^{11}}\right)^2 = 1,$$

where e^{11} is the equilibrium effort in state 11 in the complete information case. Recall that e^{11} satisfies the condition $h^{11}(e^{11}) = 0$ from Lemma A.3, which is equivalent to $\psi(e^{11}) = e^{11}$.

We conclude that there exists unique $e^1_* \in (\phi^{-1}(0), e^{11})$ such that $g(e^1_*) = 1$. Put $e^0_* = \psi(e^1_*)$ and $p_* = e^0_*/e^1_*$. Applying the function ψ to the inequality $e^1_* < e^{11}$, we obtain $e^0_* < e^1_*$, which implies $p_* < 1$. The tuple (e^1_*, e^0_*, p_*) is the unique critical point of the system of ODEs (27)–(29) with $p_* < 1$. □

B.6 Proof of Lemma B.3 (Jacobian)

Before proceeding with the proof of Lemma B.3, we establish some useful inequalities among the variables at the critical point (Lemma B.6).

Lemma B.6. *At the unique critical point $(e^1, e^0, p) = (e^1_*, e^0_*, p_*)$ the following statements hold:*

(i) $e^1 > \frac{1}{2} > v^1$;

(ii) $e^0 > v^1 e^1$;

(iii) $r + e^0 > \frac{1}{2} e^1$.

Proof. (i) Notice that

$$e^0 = \psi(e^1) < \frac{(e^1)^2}{2(1-e^1)},$$

and so (34) gives us

$$1 = \left(\frac{e^0}{e^1}\right)^2 + \frac{e^0}{1-e^1} < \left(\frac{\frac{(e^1)^2}{2(1-e^1)}}{e^1}\right)^2 + \frac{\frac{(e^1)^2}{2(1-e^1)}}{1-e^1} = \frac{3}{4}\left(\frac{e^1}{1-e^1}\right)^2.$$

Thus, $e^1/(1-e^1) > \frac{2}{\sqrt{3}} > 1$ and $e^1 > \frac{1}{2}$. Since $e^1 + v^1 = 1$, this implies $e^1 > \frac{1}{2} > v^1$.

(ii) If $e^0 \leq e^1 v^1 = e^1(1-e^1)$, then the equation (34) would lead to a contradiction

$$1 = \left(\frac{e^0}{e^1}\right)^2 + \frac{e^0}{1-e^1} \leq (1-e^1)^2 + e^1 < (1-e^1) + e^1 = 1.$$

(iii) Applying the result of (i),

$$r + e^0 = \frac{(e^1)^2}{2(1-e^1)} = \frac{e^1}{1-e^1} \cdot \frac{e^1}{2} > \frac{e^1}{2}.$$

This completes the proof. $\square$

Proof of Lemma B.3. Consider the critical point $(e^1, e^0, p) = (e^1_*, e^0_*, p_*)$ and define $R := r + 2e^0$ and $d := e^1 - e^0$. The Jacobian of the system (27)–(29) is equal to (recall that $e^0 = pe^1$ at the critical point)

$$J = \begin{bmatrix} R + e^1 - p & 0 & -v^1 e^1 \\ -e^1 + pe^0 & R & e^0 e^1 \\ -p(1-p) & 1-p & -d \end{bmatrix}.$$

Eigenvalues λ of J are the roots of the polynomial $P(\lambda) := |J - \lambda I|$, where I is the identity matrix. We obtain

$$P(\lambda) = (R + e^1 - p - \lambda)[(R-\lambda)(-d-\lambda) - e^0 d] + v^1 d[e^1 - p(R + e^0 - \lambda)]. \tag{35}$$

We can express the polynomial in terms of its coefficients as $P(\lambda) = -\lambda^3 + a_2\lambda^2 - a_1\lambda + a_0$. Then, by Lemma B.6 (i), $p = e^0/e^1 < 2e^0$, and so

$$a_2 = R + e^1 - p + R - d = 2r + 5e^0 - p > 2r + 3e^0 > 0.$$

Next, using inequalities from Lemma B.6 (i) and (ii),

$$\begin{aligned} a_0/d = P(0)/d &= -(R + e^1 - p)(R + e^0) + e^1 v^1 - p(R + e^0)v^1 \\ &= -[R + e^1 - p(1 - v^1)](R + e^0) + e^1 v^1 \\ &= -(r + e^0 + e^1)(r + 3e^0) + e^1 v^1 \\ &< -\tfrac{1}{2}(r + 3e^0) + e^0 < 0. \end{aligned}$$

Since $P(0) < 0$ and $P(\lambda) \to +\infty$ for $\lambda \to -\infty$, the polynomial $P(\lambda)$ has at least one negative root; denote it λ_1. It remains to prove that the other two (complex) roots λ_2, λ_3 have positive real parts. According to *Viète's formulas*, $a_2 = \lambda_1 + \lambda_2 + \lambda_3$ and $a_0 = \lambda_1\lambda_2\lambda_3$. Then $\lambda_2 + \lambda_3 = a_2 - \lambda_1 > 0$ and $\lambda_2\lambda_3 = a_0/\lambda_1 > 0$. If the roots λ_2 and λ_3 are real numbers, then they are necessarily positive. Finally, if the roots have nonzero imaginary parts, then they must be complex conjugates of each other, and thus have positive real parts. $\square$

B.7 Proof of Lemma B.4 (Convergence to the Critical Point)

The proof of Lemma B.4 is based on three additional lemmas. First we prove a general auxiliary lemma about convergence of a solution to a critical point (Lemma B.7). Second, we prove a lemma that establishes some useful properties of the eigenvector corresponding to the negative eigenvalue (Lemma B.8). Third, we apply Lemma B.7 and show that if p_t converges monotonically, then the whole solution vector converges (Lemma B.9).

Lemma B.7. *Let $F_t : [0,1] \to \mathbb{R}$, $t \in \mathbb{R}_+$ be a system of continuously differentiable functions that uniformly converge to some continuous function $F_\infty : [0,1] \to \mathbb{R}$ as $t \to \infty$. Assume that F_∞ has unique root z_∞ and that $F'_\infty(z_\infty) > 0$. If $\dot{z}_t = F_t(z_t)$, where $z_t \in [0,1]$, for all $t \geq 0$, then $z_t \to z_\infty$ as $t \to \infty$.*

Proof. Consider a fixed $\varepsilon > 0$. Define $L_\varepsilon = \frac{1}{2} \cdot \min\{|F_\infty(z)| : z \in [0,1], |z - z_\infty| \geq \varepsilon\}$. Since F_∞ is continuous with unique root z_∞, L_ε is well defined and positive.

Since the functions F_t converge uniformly to F_∞, there is $\tau_\varepsilon \geq 0$ such that $|F_t(z) - F_\infty(z)| < L_\varepsilon$ for all $z \in [0,1]$ and $t \geq \tau_\varepsilon$. Using a triangle inequality, we conclude that if $|z_t - z_\infty| \geq \varepsilon$ and $t \geq \tau_\varepsilon$, then

$$|F_t(z_t)| \geq |F_\infty(z_t)| - |F_\infty(z_t) - F_t(z_t)| \geq 2L_\varepsilon - L_\varepsilon = L_\varepsilon.$$

Since $\dot{z}_t = F_t(z_t)$, it follows that if $z_{t_0} \leq z_\infty - \varepsilon$ (resp. $z_{t_0} \geq z_\infty + \varepsilon$) for some $t_0 \geq \tau_\varepsilon$, then $z_t \leq z_\infty - \varepsilon - L_\varepsilon(t - t_0)$ (resp. $z_{t_0} \geq z_\infty + \varepsilon + L_\varepsilon(t - t_0)$) for all $t \geq t_0$, and eventually z_t goes out of bounds. We conclude that for every $\varepsilon > 0$ and every $t \geq \tau_\varepsilon$, $|z_t - z_\infty| < \varepsilon$, and so indeed $z_t \to z_\infty$ as $t \to \infty$. □

Lemma B.8. *The eigenvector $\mu = (\mu_1, \mu_0, \mu_p)$ of the Jacobian matrix J associated with the negative eigenvalue λ_1 satisfies $\mu_1/\mu_p > 0$ and $\mu_0/\mu_p < 0$.*

Proof. As the first step, we show that $\lambda_1 < -d = -(e^1 - e^0)$. It follows from (35) and $R = r + 2e^0$ that

$$P(-d) = -(r + e^0 + 2e^1 - p)e^0 d + v^1 d[e^1 - p(r + 2e^0 + e^1)].$$

By Lemma B.6 (i), $p = e^0/e^1 < 2e^0 < 2e^1$ and $e^0 = pe^1 > pv^1$. Thus, $(r + e^0 + 2e^1 - p)e^0 >$

rpv^1 and we obtain

$$
\begin{aligned}
P(-d)\cdot\frac{e^1}{dv^1} &< -rpe^1 + e^1[e^1 - p(r+2e^0+e^1)] \\
&= (e^1)^2 - p(e^1)^2 - 2pe^1(r+e^0) \\
&= (e^1)^2 - e^0e^1 - 2e^0(r+e^0) \\
&< (e^1)^2 - (e^0)^2 - 2(r+e^0)e^0 = 0,
\end{aligned}
$$

where the last equality follows from (32). Since the polynomial $P(\lambda)$ has a unique negative root, $P(-d) < 0$, and $P(\lambda) \to +\infty$ for $\lambda \to -\infty$, thus indeed $\lambda_1 < -d = -(e^1 - e^0)$.

Now, the eigenvector associated with λ_1 is characterized by the equation $(J - \lambda_1 I)\mu = 0$, which is equivalent to

$$
\begin{aligned}
(R + e^1 - p - \lambda_1)\mu_1 - v^1e^1\mu_p &= 0, \\
-(e^1 - pe^0)\mu_1 + (R - \lambda_1)\mu_0 + e^0e^1\mu_p &= 0.
\end{aligned}
$$

Clearly, $\mu_p \neq 0$, as otherwise the whole eigenvector μ would be zero. Since the coefficient of μ_1 in the first equation is positive, $\mu_1/\mu_p > 0$. Substituting for μ_1 from the first into the second equation and using the inequality $\lambda_1 < -(e^1 - e^0)$ together with the inequality $e^0 > e^1v^1$ and $e^1 > \frac{1}{2}$ (Lemma B.6 (ii) and (i)),

$$
\begin{aligned}
\frac{1}{e^1}(R + e^1 - p - \lambda_1)(R - \lambda_1)\frac{\mu_0}{\mu_p} &= (e^1 - pe^0)v^1 - e^0(R + e^1 - p - \lambda_1) \\
&= e^1v^1 - e^0(R + e^1 - pe^1 - \lambda_1) \\
&= e^1v^1 - e^0(r + e^0 + e^1 - \lambda_1) \\
&< e^0 - e^0(r + 2e^1) < 0.
\end{aligned}
$$

In conclusion, $\mu_0/\mu_p < 0$. □

Lemma B.9. *Assume that $p_t < 1$ for all $t \geq 0$ and $p_t \to p_\infty$, where $p_\infty \in [0,1]$. Then $(e_t^1, e_t^0, p_t) \to (e_*^1, e_*^0, p_*)$ as $t \to \infty$.*

Proof. The ODEs (27)–(29) for e_t^1 and e_t^0 can be written as

$$
\begin{aligned}
\dot{e}_t^1 &= F_t^1(e_t^1), \qquad \text{where} \qquad F_t^1(z) := \tfrac{1}{2}z^2 - (r + p_t z)(1 - z), \\
\dot{e}_t^0 &= F_t^0(e_t^0), \qquad \text{where} \qquad F_t^0(z) := \tfrac{1}{2}z^2 - \tfrac{1}{2}(e_t^1)^2 + (r + p_t e_t^1)z.
\end{aligned}
$$

The functions $F_t^1(z)$ are continuously differentiable and converge uniformly to $F_\infty^1(z) = \frac{1}{2}z^2 - (r + p_\infty z)(1 - z)$ as $t \to \infty$. Since F_∞^1 is a quadratic function with a positive leading coefficient, negative intercept, and $F^1_\infty(1) > 0$, it has unique positive root $e^1_\infty \in (0,1)$ and $(F^1_\infty)'(e^1_\infty) > 0$. Applying Lemma B.7, we conclude that $e^1_t \to e^1_\infty$.

Similarly, the functions $F^0_t(z)$ are continuously differentiable and converge uniformly to $F^0_\infty(z) = \frac{1}{2}z^2 - \frac{1}{2}(e^1_\infty)^2 + (r + p_\infty e^1_\infty)z$ as $t \to \infty$. Since F^0_∞ is a quadratic polynomial with a positive leading coefficient, negative intercept, and $F^0_\infty(e^1_\infty) > 0$, it has a unique positive root $e^0_\infty \in (0, e^1_\infty)$, and $(F^0_\infty)'(e^0_\infty) > 0$. Applying Lemma B.7, we conclude that $e^0_t \to e^0_\infty$.

Since $e^0_\infty < e^1_\infty$, it follows from (29) that $p_\infty \leq 1$, because otherwise $\dot{p}_t$ would necessarily be negative for t large (which would prevent it from exceeding 1). We conclude that $(e^1_\infty, e^0_\infty, p_\infty)$ is a critical point of the system (27)–(29) with $p_\infty < 1$, and thus, by Lemma B.2, $(e^1_\infty, e^0_\infty, p_\infty) = (e^1_*, e^0_*, p_*)$. □

Proof of Lemma B.4. In any Markov perfect Bayesian Nash equilibrium, p_t is monotone, as otherwise there would be $0 < t_1 < t_2$ such that $p_{t_1} = p_{t_2}$, but $\dot{p}_{t_1} \neq \dot{p}_{t_2}$, which is not consistent with the Markov property.[26]

Since p_t is monotone on a bounded range, it converges to some value p_∞. The convergence then follows from Lemma B.9. Finally, since p_t is monotone, $p_0 = 0$ and $p_* > 0$ (see the proof of Lemma B.2), then p_t is increasing. □

B.8 Proof of Proposition 2 (Existence of a Unique Solution)

Proof of Proposition 2. From Lemma B.2, B.4, and B.3 we know that every solution of the system of ODEs (27)–(4) converges to a unique critical point from a uniquely given direction.

As discussed in Section B.9, let us consider the efforts as functions of the belief, as $e^1(p)$ and $e^0(p)$. It follows from (27)–(3) that the functions $e^1(p)$ and $e^0(p)$, where $p \in [0,1)$, are characterized by the following equations

$$(e^1)'(p) = \frac{\frac{1}{2}(e^1)^2 - (r + pe^1)(1 - e^1)}{(1-p)(e^0 - pe^1)}, \tag{36}$$

$$(e^0)'(p) = \frac{\frac{1}{2}(e^0)^2 - \frac{1}{2}(e^1)^2 + (r + pe^1)e^0}{(1-p)(e^0 - pe^1)}, \tag{37}$$

for $p \in [0,1) \setminus \{p_*\}$, and conditions $(e^1)'(p_*) = \mu_1/\mu_p$, $(e^0)'(p_*) = \mu_0/\mu_p$, $e^1(p_*) = e^1_*$, and $e^0(p_*) = e^0_*$. This system of equations is an initial value problem, so the existence and uniqueness of its solution is guaranteed as long as the derivatives are bounded.

[26]Indeed, p_t is the only state in the game, so if p_t is the same at the two times, then also v^1_t, v^0_t has to be the same. But that implies that also $\dot{p}_t$ is the same at the two times.

We now show that the solution of the system of ODEs (36)–(37) exists on the interval $[0, p_*]$. Suppose to the contrary that this is not the case. Let $\underline{p} \in [0, p_*]$ be minimal such that a solution exists on the interval $[\underline{p}, p_*]$ and assume that $\underline{p} > 0$.[27] The numerators of the right-hand sides of equations (36) and (37) are both bounded, and their common denominator is

$$D(p) := (1-p)[e^0(p) - pe^1(p)].$$

From Lemma B.8, it follows that

$$\begin{aligned} D'(p_*) &= (1-p_*)\left[(e^0)'(p_*) - e^1(p_*) - p_*(e^1)'(p_*)\right] \\ &= (1-p_*)\left(\frac{\mu_0}{\mu_p} - e^1_* - p_*\frac{\mu_1}{\mu_p}\right) < 0. \end{aligned}$$

Thus, $D(p) > 0$ for $p < p_*$ close enough to p_*, and the initial value problem has unique solution close to p_*. This yields $\underline{p} < p_*$. What is more, $D(\underline{p}) > 0$. Indeed, otherwise it would have to be that $e^0(\underline{p}) = \underline{p} \cdot e^1(\underline{p})$. Consequently, the nominator of equation (37) turns into (after putting $p = \underline{p}$)

$$\tfrac{1}{2}(pe^1)^2 - \tfrac{1}{2}(e^1)^2 + (r + pe^1)pe^1 = \tfrac{3}{2}p^2(e^1)^2 - \tfrac{1}{2}(e^1)^2 + rpe^1.$$

Now we argue that the inequalities $0 < e^1(\underline{p})$, $0 < e^0(\underline{p})$, and $e^0(\underline{p}) + e^1(\underline{p}) < 1$ hold. If to the contrary $0 = e^1(\underline{p})$, then $(e^1)'(\underline{p}) < 0$, which would mean that the inequality would already be violated for $p > \underline{p}$. The argument for other inequalities is analogous.

Summing up, the right-hand sides of the system (36)–(37) are well defined for $p = \underline{p}$ and we can extend the solution even below $\underline{p}$, which is contradicts the assumption $\underline{p} > 0$. Thus, the initial problem has indeed a unique solution on the interval $[0, p_*]$. Showing that there is unique solution on the interval $[p_*, 1)$ is analogous. □

B.9 Proof of Proposition 3 (Effort Over Time)

The proof of Proposition 3 follows from the three lemmas below.[28]

Lemma B.10. *The effort $e^1(p)$ is increasing in the belief p, i.e., $(e^1)'(p) > 0$ for all $p \in [0, p^*)$.*

Lemma B.11. *The effort $e^0(p)$ is decreasing in the belief p, i.e., $(e^0)'(p) < 0$ for all $p \in [0, p^*)$.*

[27] The existence of the minimum is guaranteed. Let $\underline{p}$ be the infimum. Then the solution exists on the interval $(\underline{p}, p_*]$, and so it can be continuously extended to the interval $[\underline{p}, p_*]$.

[28] The statements of these lemmas hold of all $p \in [0, 1)$ in fact, however, considering only $p \in [0, p_*)$ simplifies the proofs of the lemmas and it is sufficient taking into consideration that $p_* > p^B_*$ (Lemma C.9).

Lemma B.12. *The effort of a player is higher if he is successful, i.e.,* $e^1(p) > e^0(p)$ *for any* $p \in [0, p^*)$.

Proof of Lemma B.10. By Lemma B.8, the direction (ν^1, ν^0, ν^p) in which the solution has to converge to the steady-state is such that $\nu^1/\nu^p > 0$, and so the claim holds at $p = p_*$. Suppose to the contrary that the claim is violated at some $p'' \in [0, p_*)$. Then there exists $p' \in [0, p_*)$ such that $(e^1)'(p') = 0$. Consider such p' that is closest to p_* and let t' be such that $p_{t'} = p'$. At such point $\ddot{e}^1_{t'} \geq 0$, because $0 = \dot{e}^1_{t'} < \dot{e}^1_t$ for all $t > t'$.

Now recall that the dynamics of e^1_t follows the ODE $\dot{e}^1_t = \frac{1}{2}(e^1_t)^2 - (r + p_t e^1_t)(1 - e^1_t)$ (formula (27)). Taking the time derivative and setting $t = t'$, for which $\dot{e}^1_{t'} = 0$, we obtain

$$\ddot{e}^1_{t'} = -e^1_{t'}(1 - e^1_{t'})\dot{p}_{t'} < 0,$$

(Lemma B.4). This yields a contradiction. □

Proof of Lemma B.11. The proof of this result uses similar technique as the proof of Lemma B.10. By Lemma B.8, the direction (ν^1, ν^0, ν^p) in which the solution has to converge to the steady-state is such that $\nu^0/\nu^p < 0$, and so the claim holds at $p = p_*$. Suppose to the contrary that the claim is violated at some $p'' \in [0, p_*)$. Then there exists $p' \in [0, p^*]$ such that $(e^0)'(p') = 0$. Consider such p' that is closest to p_* and let t' be such that $p_{t'} = p'$. At such point $\ddot{e}^1_{t'} \leq 0$ because $0 = \dot{e}^0_{t'} > \dot{e}^0_t$ for all $t > t'$.

Now recall that the dynamics of e^0_t follows the ODE $\dot{e}^0_t = \frac{1}{2}(e^0_t)^2 - \frac{1}{2}(e^1_t)^2 + (r + p_t e^1_t)e^0_t$ (formula (27)). Taking the time derivative and setting $t = t'$, for which $\dot{e}^0_{t'} = 0$, we obtain

$$\ddot{e}^0_{t'} = e^1_{t'}(-\dot{e}^1_{t'} + e^0_t \dot{p}_{t'}) + p_{t'} e^0_t \dot{e}^1_{t'},$$

which is strictly positive assuming that $(e^1)'(p') < e^0(p')$. This yields a contradiction. The inequality $(e^1)'(p') < e^0(p')$ can be verified numerically, or it can be proved analytically as is shown in the online appendix.

□

Proof of Lemma B.12. Fix any $p \in [0, p^*)$ and let $t \geq 0$ be such that $p_t = p$, the claim then is that $e^1_t > e^0_t$. By Lemma B.11, $0 > (e^0)'(p) = \dot{e}^0_t / \dot{p}_t$. By Lemma B.4, $\dot{p}_t > 0$. Consequently,

$$0 > \dot{e}^0_t = \frac{1}{2}(e^0_t)^2 - \frac{1}{2}(e^1_t)^2 + (r + p_t e^1_t)e^0_t.$$

Thus necessarily $e^1_t > e^0_t$. □

C Appendix: Proofs for Section 5 (One Player Known to be Successful)

Instead of characterizing the trajectory of the vector $(v_t^{1A}, v_t^{1B}, v_t^{0A}, p_t^B)$ by the ODEs (6)–(9), substituting the optimal efforts from (10), we can equivalently characterize the trajectory of the vector $(e_t^{1A}, e_t^{1B}, e_t^{0A}, p_t^B)$ by the following ODEs:

$$\dot{e}_t^{1A} = \tfrac{1}{2}(e_t^{1A})^2 - (r + p_t^B e_t^{1B})(1 - e_t^{1A}) \tag{38}$$

$$\dot{e}_t^{1B} = \tfrac{1}{2}(e_t^{1B})^2 - (r + e_t^{1A})(1 - e_t^{1B}) \tag{39}$$

$$\dot{e}_t^{0B} = \tfrac{1}{2}(e_t^{0B})^2 - \tfrac{1}{2}(e_t^{1B})^2 + (r + e_t^{1A})e_t^{0B} \tag{40}$$

$$\dot{p}_t^B = (1 - p_t^B)(e_t^{0B} - p_t^B e_t^{1B}), \tag{41}$$

where $p_0^B = \hat{p}^B$, and $e_t^{1A}, e_t^{1B} \in (0, 1]$, $e_t^{0B} \in [0, 1 - e_t^{1B}]$, and $p_t \in [0, 1]$.

Similarly as in the case with both players starting from the state 0, the proof of existence and uniqueness of the solution consists of three steps, making sure that the ODE has unique critical point, that every solution has to converge to it, and that there is unique direction in which it can happen.

Lemma C.1. *The system of ODEs* (38)–(41) *has unique critical point* $(e_*^{1A}, e_*^{1B}, e_*^{0B}, p_*^B)$ *with* $p_*^B < 1$.

Lemma C.2. *The Jacobian at the critical point of the system of ODEs* (38)–(41) *has unique eigenvalue with non-positive real part. This eigenvalue is a negative real number.*

Lemma C.3. *Any solution vector* $(e_t^{1A}, e_t^{1B}, e_t^{0B}, p_t^B)$ *of the system of ODEs* (38)–(41) *with* $p_0^B < 1$ *converges to the critical point* $(e_*^{1A}, e_*^{1B}, e_*^{0B}, p_*^B)$ *as* $t \to \infty$.

C.1 Proof of Lemma C.1 (Unique Critical Point)

Proof of Lemma C.1. We proceed similarly as in the proof of Lemma B.2. A critical point of the system (38)–(41) is characterized by the condition $(\dot{e}_t^{1A}, \dot{e}_t^{1B}, \dot{e}_t^{0A}, \dot{p}_t^B) = (0, 0, 0, 0)$. Dropping the time index, we obtain that $e^{0B} = p^B e^{1B}$ and

$$0 = \tfrac{1}{2}(e^{1A})^2 - (r + e^{0B})(1 - e^{1A})$$

$$0 = \tfrac{1}{2}(e^{1B})^2 - (r + e_t^{1A})(1 - e_t^{1B})$$

$$0 = \tfrac{1}{2}(e^{0B})^2 - \tfrac{1}{2}(e^{1B})^2 + (r + e^{1A})e^{0B}.$$

This system is equivalent to

$$e^{0B} = \frac{(e^{1A})^2}{2(1-e^{1A})} - r \tag{42}$$

$$e^{1A} = \frac{(e^{1B})^2}{2(1-e^{1B})} - r \tag{43}$$

$$1 = \left(\frac{e^{0B}}{e^{1B}}\right)^2 + \frac{e^{0B}}{1-e^{1B}}. \tag{44}$$

Let us define the function $\tilde{g}(z)$ on the interval $[0,1)$ by

$$\tilde{g}(z) := \left(\frac{\psi(\psi(z))}{z}\right)^2 + \frac{\psi(\psi(z))}{1-z},$$

where the function $\psi(z) = \dfrac{z^2}{2(1-z)} - r$ was introduced in Lemma B.5. A quadruple $(e^{1A}, e^{1B}, e^{0B}, p^B)$ is then a critical point of the system of ODEs (38)–(41) if and only if $e^{0B} = \psi(e^{1A})$, $e^{1A} = \psi(e^{1B})$, $1 = \tilde{g}(e^{1B})$, and $p^B = e^{0B}/e^{1B}$.

Consider some $z \in [\psi^{-1}(\psi^{-1}(0)), 1)$. Clearly, the functions $\psi(z)/[1-\psi(z)]$ and $\psi(z)/z$ are strictly increasing and positive. Thus, their product as well as the function

$$\frac{1}{2} \cdot \frac{\psi(z)}{1-\psi(z)} \cdot \frac{\psi(z)}{z} - \frac{r}{z} = \left(\frac{(\psi(z))^2}{2(1-\psi(z))} - r\right) \cdot \frac{1}{z} = \frac{\psi(\psi(z))}{z}.$$

are strictly increasing and positive. Consequently, the function $\tilde{g}(z)$ is a sum of two strictly increasing functions, and so it is strictly increasing itself. Finally, $\tilde{g}(\psi^{-1}(\psi^{-1}(0))) = 0$ and

$$\tilde{g}(e^{11}) > \left(\frac{\psi(\psi(e^{11}))}{e^{11}}\right)^2 = \left(\frac{e^{11}}{e^{11}}\right)^2 = 1,$$

where e^{11} is the equilibrium effort in state 11 in the complete information case, which satisfies $\psi(e^{11}) = e^{11}$ (see the proof of Lemma B.2). Therefore, $e_*^{1B} = \tilde{g}^{-1}(1) \in (\psi^{-1}(\psi^{-1}(0)), e^{11})$ is unique. By Lemma B.5, the inequality $e_*^{1B} < e^{11}$ implies $e_*^{1A} = \psi(e_*^{1B}) < e_*^{1B}$, which, in turn, implies $e_*^{0B} = \psi(e_*^{1A}) < e_*^{1A}$. Thus, $p_*^B = e_*^{0B}/e_*^{1B} < 1$. The quadruple $(e_*^{1A}, e_*^{1B}, e_*^{0B}, p_*^B)$ is, thus, indeed the unique critical point of the system of ODEs (38)–(41). Note that the condition $e_*^{0B} < 1 - e_*^{1B}$ follows from the equation (44). □

C.2 Proof of Lemma C.2 (Jacobian)

Before proceeding with the proof of Lemma C.2, we establish some useful inequalities among the variables at the critical point (Lemma C.4).

Lemma C.4. *At the unique critical point* $(e^{1A}, e^{1B}, e^{0B}, p^B) = (e^{1A}_*, e^{1B}_*, e^{0B}_*, p^B_*)$*, the following statements hold:*

(i) $v^{1B} < v^{1A} < \frac{1}{2} < e^{1A} < e^{1B}$*;*

(ii) $e^{0B} > e^{1B}v^{1B}$*;*

(iii) $r + e^{1A} + e^{0B} > e^{1B}$*.*

Proof. (i) The inequality $e^{1A} < e^{1B}$ has already been proved in the proof of Lemma C.1. Since $v^{1A} = 1 - e^{1A}$ and $v^{1B} = 1 - e^{1B}$, it only remains to show that $\frac{1}{2} < e^{1A}$. Recall that $1 = \tilde{g}(e^{1B})$, $e^{1A} = \psi(e^{1B})$, and the functions ψ and $\tilde{g}$ are strictly increasing. Thus, the claim is equivalent to $\frac{1}{2} < \psi(\tilde{g}^{-1}(1))$. Let $\tilde{z} := \psi^{-1}(\frac{1}{2})$. We now show that $\tilde{g}(\tilde{z}) < 1$. Observe that $\frac{1}{2} = \psi(\tilde{z}) = \frac{1}{2}\tilde{z}^2/(1-\tilde{z}) - r$ and $\psi(\psi(\tilde{z})) = \psi(\frac{1}{2}) = \frac{1}{4} - r$. Moreover, the former equality yields $1 - \tilde{z} = \tilde{z}^2/(1+2r)$. We then obtain

$$\begin{aligned}\tilde{g}(\tilde{z}) &= \frac{(\frac{1}{4}-r)^2}{\tilde{z}^2} + \frac{\frac{1}{4}-r}{1-\tilde{z}} = \frac{(\frac{1}{4}-r)^2}{\tilde{z}^2} + \frac{(\frac{1}{4}-r)(1+2r)}{\tilde{z}^2} \\ &= \frac{\frac{5}{16}-r-r^2}{\tilde{z}^2} < \frac{\frac{5}{16}}{\tilde{z}^2} < \frac{5}{8(3-\sqrt{5})} < 1,\end{aligned}$$

where the last inequality follows from $1 < \tilde{z}^2/(1-\tilde{z})$, which yields $\tilde{z}^2 > \frac{1}{2}(3-\sqrt{5})$.

(ii) If $e^{0B} \le e^{1B}v^{1B} = e^{1B}(1-e^{1B})$, then equation (44) leads to a contradiction, as

$$1 = \left(\frac{e^{0B}}{e^{1B}}\right)^2 + \frac{e^{0B}}{1-e^{1B}} \le \left(1-e^{1B}\right)^2 + e^{1B} < (1-e^{1B}) + e^{1B} = 1.$$

(iii) Recall that $e^{1A} = \psi(e^{1B})$, and so

$$r + e^{1A} = \frac{(e^{1B})^2}{2(1-e^{1B})} > (e^{1B})^2 > e^{1B} - e^{0B},$$

where the former inequality follows from $e^{1B} > 1/2$ (due to (i)), and the later inequality follows from $e^{0B} > e^{1B}(1-e^{1B})$ (due to (ii)). □

Proof of Lemma C.2. Consider the critical point $(e^{1A}, e^{1B}, e^{0B}, p^B) = (e^{1A}_*, e^{1B}_*, e^{0B}_*, p^B_*)$ and define

$$R^A := r + e^{1A} + e^{0B} \quad \text{and} \quad d^B := e^{1B} - e^{0B}.$$

The Jacobian matrix of the system of ODEs (38)–(41) at the critical point is

$$J^A = \begin{bmatrix} R^A & -p^B v^{1A} & 0 & -e^{1B}v^{1A} \\ -v^{1B} & R^A + d^B & 0 & 0 \\ e^{0B} & -e^{1B} & R^A & 0 \\ 0 & -p^B(1-p^B) & (1-p^B) & -e^{1B}(1-p^B) \end{bmatrix}$$

The eigenvalues of J^A are the complex roots of the polynomial $P^A(\lambda) := |J^A - \lambda I|$, where I is the identity matrix. We have

$$P^A(\lambda) = \begin{vmatrix} R^A - \lambda & -p^B v^{1A} & 0 & -e^{1B}v^{1A} \\ -v^{1B} & R^A + d^B - \lambda & 0 & 0 \\ e^{0B} & -e^{1B} & R^A - \lambda & 0 \\ 0 & -p^B(1-p^B) & (1-p^B) & -e^{1B}(1-p^B) - \lambda \end{vmatrix}.$$

Subtracting p^B/e^{1B} times the last column of the determinant from its second column, and using the identity $e^{1B}(1-p^B) = d^B$, we obtain

$$P^A(\lambda) = \begin{vmatrix} R^A - \lambda & 0 & 0 & -e^{1B}v^{1A} \\ -v^{1B} & R^A + d^B - \lambda & 0 & 0 \\ e^{0B} & -e^{1B} & R^A - \lambda & 0 \\ 0 & \frac{p^B}{e^{1B}}\lambda & 1-p^B & -d^B - \lambda \end{vmatrix}.$$

Expanding the determinant by the first row yields

$$\begin{aligned} P^A(\lambda) &= (R^A-\lambda)^2(R^A+d^B-\lambda)(-d^B-\lambda) + e^{1B}v^{1A} \begin{vmatrix} -v^{1B} & R^A + d - \lambda & 0 \\ e^{0B} & -e^{1B} & R^A - \lambda \\ 0 & \frac{p^B}{e^{1B}}\lambda & 1-p^B \end{vmatrix} \\ &= (R^A-\lambda)^2(R^A+d^B-\lambda)(-d^B-\lambda) \\ &\quad + e^{1B}v^{1A}[(-v^{1B})(-e^{1B})(1-p^B) - (-v^{1B})(R^A-\lambda)\tfrac{p^B}{e^{1B}}\lambda \\ &\quad - (R^A+d-\lambda)e^{0B}(1-p^B)] \\ &= (R^A-\lambda)^2(R^A+d^B-\lambda)(-d^B-\lambda) \\ &\quad + v^{1A}v^{1B}[e^{1B}d^B + p^B(R^A-\lambda)\lambda] - v^{1A}e^{0B}d^B(R^A+d^B-\lambda). \end{aligned}$$

We can express the polynomial in terms of its coefficients as $P^A(\lambda) = \lambda^4 - b_3\lambda^3 + b_2\lambda^2 - b_1\lambda + b_0$. Then, taking into account the inequalities $R^A = r + e^{1A} + e^{0B} > e^{1A}$, $R^A + d^B = r + e^{1A} + e^{1B} > e^{1B}$, and $v^{1B} < v^{1A} < \frac{1}{2} < e^{1A} < e^{1B}$ (by Lemma C.4 (i)), we have

$$\begin{aligned}
b_0/d^B = P(0)/d^B &= -(R^A)^2(R^A+d^B) + v^{1A}v^{1B}e^{1B} - v^{1A}e^{0B}(R^A+d^B) \\
&< -(R^A)^2(R^A+d^B) + v^{1A}v^{1B}e^{1B} \\
&< -(R^A)^2(R^A+d^B) + (R^A)^2(R^A+d^B) = 0.
\end{aligned}$$

Thus P^A has at least one positive root and at least one negative root (recall that $P^A(\lambda)$ goes to infinity as $\lambda \to +\infty$ or $\lambda \to -\infty$).

We now show that P^A has a unique negative root; denote it λ_1^A. We have

$$b_3 = 2R^A + (R^A + d^B) - d^B = 3R^A > 0.$$

Since $(P^A)'''(\lambda) = 24\lambda - 6b_3 < 0$ for all $\lambda \leq 0$, $(P^A)''(\lambda)$ is decreasing. In addition,

$$(P^A)''(0)/2 = b_2 = 3(R^A)^2 - (d^B)^2 - d^B R^A - v^{1A}v^{1B}p^B > 0,$$

where the inequality follows from $R^A > v^{1A} > v^{1B}$ (by Lemma C.4 (i)) and $R^A > d^B$ (by Lemma C.4 (iii)). Therefore, $(P^A)''(\lambda) > 0$ for all $\lambda \leq 0$, and so the polynomial P^A is convex on $(-\infty, 0]$. Thus, P^A has indeed a unique negative root.

It remains to show that P^A has no root with non-positive real part and nonzero imaginary part. Suppose to the contrary that λ_2^A was such a root. Then its complex conjugate λ_3^A would be also a root. Denote λ_4^A the positive root of P^A. By *Viète's formulas*, $b_3 = \lambda_1^A + \lambda_2^A + \lambda_3^A + \lambda_4^A$. Since all the roots except for λ_4^A have non-positive real part $\lambda_4^A > b_3 = 3R^A > 0$. Because λ_4^A is the unique positive root, it follows that $P^A(3R^A) < 0$. However,

$$\begin{aligned}
P^A(3R^A) &= (2R^A)^2(2R^A - d^B)(3R^A + d^B) \\
&\quad + v^{1A}v^{1B}[e^{1B}d^B - 6p^B(R^A)^2] + v^{1A}e^{0B}d^B(2R^A - d^B) \\
&> (2R^A)^2 R^A(3R^A) + (R^A)^2[-6(R^A)^2] = 6(R^A)^4 > 0,
\end{aligned}$$

where the former inequality follows from $R^A > v^{1A} > v^{1B}$ (by Lemma C.4 (i)) and $R^A > d^B$ (by Lemma C.4 (iii)). This is a contradiction. □

C.3 Proof of Lemma C.3 (Convergence to the Critical Point)

We proceed similarly as in the proof of Lemma B.4 (Section B.7). First, we prove a lemma establishes some useful properties of the negative eigenvalue and the corresponding eigenvector (Lemma C.5). Second, we provide bounds for the dynamics of efforts

(Lemma C.6). Third, we apply Lemma B.7 and show that if p_t converges monotonically, then the whole solution vector converges (Lemma B.9).

Lemma C.5. *The eigenvector $\mu^A = (\mu_{1A}, \mu_{1B}, \mu_{0B}, \mu_p)$ of the Jacobian matrix J^A associated with the negative eigenvalue λ_1^A is such that $\mu_{1A}/\mu_p > 0$, $\mu_{1B}/\mu_p > 0$ and $\mu_{0B}/\mu_p < 0$.*

Proof. The eigenvector μ^A is characterized by the vector equation $(J^A - \lambda_1^A I)\mu^A = 0$, which gives us

$$\begin{aligned}(R^A - \lambda_1^A)\mu_{1A} - p^B v^{1A}\mu_{1B} - e^{1B}v^{1A}\mu_p &= 0\\ -v^{1B}\mu_{1A} + (R^A + d^B - \lambda_1^A)\mu_{1B} &= 0\\ e^{0B}\mu_{1A} - e^{1B}\mu_{1B} + (R^A - \lambda_1^A)\mu_{0B} &= 0.\end{aligned}$$

Substituting for μ_{1A} from the second equation into the others,

$$\begin{aligned}\left[(R^A - \lambda_1^A)(R^A + d^B - \lambda_1^A) - p^B v^{1A}v^{1B}\right]\mu_{1B} - e^{1B}v^{1A}v^{1B}\mu_p &= 0\\ \left[e^{0B}(R^A + d^B - \lambda_1^A) - e^{1B}v^{1B}\right]\mu_{1B} + (R^A - \lambda_1^A)v^{1B}\mu_{0B} &= 0.\end{aligned}$$

Since $R^A > e^{1A} > \frac{1}{2} > v^{1A} > v^{1B}$, $(R^A)^2 > v^{1A}v^{1B}$, and so the coefficient of μ_{1B} in the first equation is positive. Consequently, $\mu_{1B}/\mu_p > 0$, and thus also $\mu_{1A}/\mu_p > 0$ (clearly, $\mu_p \neq 0$, as otherwise the whole vector μ^A would be zero). Finally, the coefficient of μ_{1B} in the second equation is positive as $e^{0B} > e^{1B}v^{1B}$ (Lemma C.4 (ii)), and $R^A + d^B - \lambda_1^A > R^A + d^B = r + e^{1A} + e^{1B} > 1$ (by Lemma C.4 (i)). Consequently, $\mu_{0B}/\mu_p < 0$. □

Lemma C.6. *There are strictly increasing functions $E^{1A}, E^{1B} : [0,1] \to [0,1]$ such that:*

(i) If $p_t^B \geq \underline{p}^B$ for all $t \geq t_0$, then $e_t^{1A} \geq E^{1A}(\underline{p}^B)$ and $e_t^{1B} \geq E^{1B}(\underline{p}^B)$ for all $t \geq t_0$.

(ii) If $p_t^B \leq \bar{p}^B$ for all $t \geq t_0$, then $e_t^{1A} \leq E^{1A}(\bar{p}^B)$ and $e_t^{1B} \leq E^{1B}(\bar{p}^B)$ for all $t \geq t_0$.

Proof of Lemma C.6. For any $p^B \in [0,1]$, let us define the functions E^{1A} and E^{1B} so that $e^{1A} = E^{1A}(p^B)$ and $e^{1B} = E^{1B}(p^B)$ solves the following system of equations (critical point conditions for (38) and (39)):

$$0 = \tfrac{1}{2}(e^{1A})^2 - (r + p^B e^{1B})(1 - e^{1A}), \tag{45}$$

$$0 = \tfrac{1}{2}(e^{1B})^2 - (r + e^{1A})(1 - e^{1B}). \tag{46}$$

In other words, the functions are chosen so that $e_t^{1A} = E^{1A}(p_t^B)$ and $e_t^{1B} = E^{1B}(p_t^B)$ would imply $\dot{e}_t^{1A} = 0$ and $\dot{e}_t^{1B} = 0$.

We now argue that the functions E^{1A} and E^{1B} are well defined. Using the function $\psi(z) = \dfrac{z^2}{2(1-z)} - r$ from Lemma B.5, the above system of equations can be equivalently written as $p^B e^{1B} = \psi(e^{1A})$ and $e^{1A} = \psi(e^{1B})$. Recall that the function $\psi^{-1} : [0,1] \to [0,1]$ is continuous, strictly increasing and strictly concave (Lemma B.5), and so is the function $z \mapsto \psi^{-1}(\psi^{-1}(p^B z))$ then, for any given $p^B \in (0,1]$. Since $\psi^{-1}(\psi^{-1}(p^B \cdot 0)) > 0$ and $\psi^{-1}(\psi^{-1}(p^B \cdot 1)) < 1$, the strictly concave function $z \mapsto \psi^{-1}(\psi^{-1}(p^B z))$ has a unique fixed point on the interval $[0,1]$. Let $E^{1B}(p^B)$ be equal to this fixed point, i.e., $E^{1B}(p^B) = \psi^{-1}(\psi^{-1}(p^B E^{1B}(p^B)))$, and let $E^{1A}(p^B) := \psi(E^{1B}(p^B))$. It follows that the functions E^{1B} and E^{1A} are continuous. Moreover,

$$z > \psi^{-1}(\psi^{-1}(p^B z)), \qquad \text{if and only if} \qquad z > E^{1B}(p^B). \tag{47}$$

In the next step we show that the functions E^{1A} and E^{1B} are strictly increasing. Let $p_1^B, p_2^B \in [0,1)$ be such that $p_1^B < p_2^B$. For $z_2 = E^{1B}(p_2^B)$ it follows from the monotonicity of ψ^{-1} that $z_2 = \psi^{-1}(\psi^{-1}(p_2^B z_2)) > \psi^{-1}(\psi^{-1}(p_1^B z_2))$. Then (47) yields $E^{1B}(p_2^B) = z_2 > E^{1B}(p_1^B)$. Thus, $E^{1B}(p^B)$ is indeed strictly increasing and $E^{1A}(p^B) = \psi(E^{1B}(p^B))$ is strictly increasing as well.

Now we proceed with the proof of the lemma. We prove the part (i), the proof of part (ii) is analogous. Let $\underline{p}^B \in [0,1]$ and $t_0 \geq 0$ be such that $p_t^B \geq \underline{p}^B$ for all $t \geq t_0$. Let us define

$$\underline{e}^{1A} = \inf_{t \geq t_0} e_t^{1A} \qquad \text{and} \qquad \underline{e}^{1B} = \inf_{t \geq t_0} e_t^{1B}.$$

Morever, let us denote

$$G(z_1, z_2) := \tfrac{1}{2} z_1^2 - (r + z_2)(1 - z_1).$$

Clearly, $G(z_1, z_2)$ is increasing in z_1 and decreasing in z_2 for $z_1, z_2 \in [0,1)$. Moreover $G(z_1, z_2) \geq 0$ if and only if $z_2 \leq \psi(z_1)$.

We now argue that

$$0 \leq G(\underline{e}^{1A}, \underline{p}^B \underline{e}^{1B}) \qquad \text{and} \qquad 0 \leq G(\underline{e}^{1B}, \underline{e}^{1A}). \tag{48}$$

Suppose to the contrary that the first inequality is violated, i.e., $G(\underline{e}^{1A}, \underline{p}^B \underline{e}^{1B}) = -2\delta < 0$. Define the set

$$\mathcal{T} = \{t \geq t_0 : G(e_t^{1A}, \underline{p}^B \underline{e}^{1B}) \leq -\delta\}.$$

By definition of $\underline{e}^{1A}$ the set is nonempty. Moreover, the set is closed as the function $t \mapsto G(e_t^{1A}, \underline{p}^B \underline{e}^{1B})$ is continuous. Note that $\dot{e}_t^{1A} = G(e_t^{1A}, p_t^B e_t^{1B}) \leq G(e_t^{1A}, \underline{p}^B \underline{e}^{1B}) \leq -\delta$ for all $t \in \mathcal{T}$. As $\mathcal{T}$ is closed, every t' from the boundary of $\mathcal{T}$ belongs to $\mathcal{T}$ itself, and so $\dot{e}_{t'}^{1A} \le -\delta$, meaning that $[t', t'+\varepsilon] \in \mathcal{T}$ for some $\varepsilon > 0$. However, this means that the only boundary of $\mathcal{T}$ is a point t_1 such that $\mathcal{T} = [t_1, \infty)$. Consequently, e_t^{1A} is decreasing at rate at least δ for any $t \ge t_1$, which, however, contradicts the fact that it is bounded from below by $\underline{e}^{1A}$. The second inequality in (48) can be proved analogously.

Next we show that $\underline{e}^{1A} \ge E^{1A}(\underline{p}^B)$ and $\underline{e}^{1B} \ge E^{1B}(\underline{p}^B)$. It follows from (48) that $\underline{p}^B \underline{e}^{1B} \le \psi(\underline{e}^{1A})$ and $\underline{e}^{1A} \le \psi(\underline{e}^{1B})$. Since the function ψ is increasing, we obtain $\underline{p}^B \underline{e}^{1B} \le \psi(\psi(\underline{e}^{1B}))$, and, consequently, $\psi^{-1}(\psi^{-1}(\underline{p}^B \underline{e}^{1B})) \le \underline{e}^{1B}$. It follows from (47) that indeed $\underline{e}^{1B} \ge E^{1B}(\underline{p}^B)$. The proof of the inequality $\underline{e}^{1B} \ge E^{1B}(\underline{p}^B)$ is analogous.

Finally, by definition of $\underline{e}^{1A}$ and $\underline{e}^{1B}$, we obtain $e_t^{1A} \ge E^{1A}(\underline{p}^B)$ and $e_t^{1B} \ge E^{1B}(\underline{p}^B)$ for all $t \ge t_0$. □

Lemma C.7. *Assume that $p_t^B < 1$ for all $t \ge 0$ and $p_t^B \to p_\infty^B$, where $p_\infty \in [0,1]$. Then $(e_t^{1A}, e_t^{1B}, e_t^{0B}, p_t^B) \to (e_*^{1A}, e_*^{1B}, e_*^{0B}, p_*^B)$ as $t \to \infty$.*

Proof ot Lemma C.7. Since $p_t \to p_\infty$, p_t^B can be estimated from below and above arbitrarily narrowly for t large. Then by Lemma C.6 we obtain $e_t^{1A} \to e_\infty^{1A} := E^{1A}(p_\infty^B)$ and $e_t^{1B} \to e_\infty^{1B} := E^{1B}(p_\infty^B)$ as $t \to \infty$. The ODE (40) can be written as

$$\dot{e}_t^{0B} = F_t^{0B}(e_t^{0B}), \qquad \text{where} \qquad F_t^{0B}(z) := \tfrac{1}{2}z^2 - \tfrac{1}{2}(e_t^{1B})^2 + (r + e_t^{1A})z$$

are continuously differentiable functions converging uniformly to the function $F_\infty^{0B}(z) = \frac{1}{2}z^2 - \frac{1}{2}(e_\infty^{1B})^2 + (r + e_\infty^{1A})z$ as $t \to \infty$. Since F_∞^{0B} is a quadratic polynomial with a positive leading coefficient, and since $F^{0B}(0) < 0$ and $F_\infty^{0B}(e_\infty^{1B}) > 0$, it has a unique positive root; denote it e_∞^{0B}. Then $e_\infty^{0B} \in (0, e_\infty^{1B})$ and $(F_\infty^{0B})'(e_\infty^{0B}) > 0$. Applying Lemma B.7, we conclude that $e_t^{0B} \to e_\infty^{0B}$.

Since $e_\infty^{0B} < e_\infty^{1B}$, it follows from the ODE (41) that $p_\infty^B \le 1$, because otherwise $\dot{p}_t^B$ would necessarily be negative for t large (which would prevent it from exceeding 1). We conclude that $(e_\infty^{1A}, e_\infty^{1B}, e_\infty^{0B}, p_\infty^B)$ is a critical point of the system of ODEs (38)–(41) with $p_\infty^B < 1$, and thus, by Lemma C.1, $(e_\infty^{1A}, e_\infty^{1B}, e_\infty^{0B}, p_\infty^B) = (e_*^{1A}, e_*^{1B}, e_*^{0B}, p_*^B)$. □

Proof of Lemma C.3. In any Markov perfect Bayesian Nash equilibrium, p_t^B must be monotone as otherwise there would be $0 < t_1 < t_2$ such that $p_{t_1}^B = p_{t_2}^B$, but $\dot{p}_{t_1}^B \ne \dot{p}_{t_2}^B$, which is not consistent with the Markov property.[29]

Since p_t^B is monotone on a bounded range, it converges. The rest follows from Lemma C.7. □

[29] Indeed, p_t^B is the only state in the game, so if p_t^B is the same at the two times, then also e_t^{1A}, e_t^{1B}, and e_t^{0B} has to be the same. But that implies that also $\dot{p}_t^B$ is the same at the two times.

C.4 Proofs of Propositions 5 (Effort) and 4 (Uniqueness)

The proof of Proposition 5 follows from the following lemmas:

Lemma C.8. *Both efforts e_t^{1A} and e_t^{1B} increase over time, i.e., $\dot{e}_t^{1A} > 0$ and $\dot{e}_t^{1B} > 0$ for all $t \geq 0$.*

Proof of Lemma C.8. Define $\tau = \inf\{t \geq 0 : \dot{e}_t^{1A} > 0 \text{ and } \dot{e}_t^{1B} > 0\}$. Since the direction the direction (ν^A, ν^B, ν^p) in which the solution has to converge to the steady state satisfies $\nu^A/\nu^B > 0$ and $\nu^B/\nu^p > 0$, both of the efforts are increasing for t large, and so τ is finite. Suppose to the contrary that at least one of the inequalities $\dot{e}_t^{1A} > 0$ and $\dot{e}_t^{1B} > 0$ is violated at some time $t \geq 0$. Then $\dot{e}_\tau^{1A} \geq 0$ and $\dot{e}_\tau^{1B} \geq 0$, and at least of one of the inequalities is binding.

We discuss two cases. First, assume that $\dot{e}_\tau^{1A} = 0$. Then taking derivative of the formula for $\dot{e}_t^{1A}$ we obtain

$$\ddot{e}_\tau^{1A} = -(\dot{p}_\tau^B e_\tau^{1B} + p_\tau^B \dot{e}_\tau^{1B})(1 - e_\tau^{1A}) < 0,$$

which is, however, a contradiction with the fact that $\dot{e}_t^{1A} > 0$ for all $t > \tau$.

Second, assume that $\dot{e}_\tau^{1A} > 0$ and $\dot{e}_t^{1B} = 0$. In that case taking derivative of the formula for $\dot{e}_t^{1B}$ we obtain

$$\ddot{e}_\tau^{1B} = -\dot{e}_\tau^{1A}(1 - e_\tau^{1B}) < 0,$$

which is a contradiction with the fact that $\dot{e}_t^{1B} > 0$ for all $t > \tau$. □

C.5 Proof of Proposition 6 (Comparison of Efforts)

Proof of Proposition 6. We now prove the inequality between efforts, $e_t^{1A} < e_t^{1B}$. The statement for continuation values, $v_t^{1A} > v_t^{1B}$, follows from identities (10). First we argue that the inequality between efforts holds at the steady-state, i.e., $e_*^{1A} < e_*^{1B}$. Indeed, recall (from Lemma C.1) that the steady-state efforts are given by the equations $p_*^B e_*^{1B} = \psi(e_*^{1A})$ and $e_*^{1A} = \psi(e_*^{1B})$, where $\psi(z) = \dfrac{z^2}{2(1-z) - r}$ is the function analyzed in Lemma B.5.

Suppose to the contrary that $e_*^{1B} \leq e_*^{1A}$. Then $p_*^B e_*^{1B} < e_*^{1A}$, and since the function ψ^{-1} is strictly increasing, the inequality is preserved by applying the function ψ^{-1} to it, so $e_*^{1A} < e_*^{1B}$, which is a contradiction.

Next, suppose to the contrary that $e_t^{1B} > e_t^{1A}$ does not hold for all $t \geq 0$. Let $\tau \geq 0$ be than the smallest real number such that $e_t^{1B} > e_t^{1A}$ holds for all $t > \tau$. Then necessarily

$e_\tau^{1B} = e_\tau^{1A}$ and $\dot{e}_\tau^{1B} \geq \dot{e}_\tau^{1A}$. However, due to (38)–(39),

$$\begin{aligned}\dot{e}_\tau^{1A} &= \tfrac{1}{2}(e^{1A})^2 - (r + p_\tau^B e^{1B})(1 - e^{1A}) = \tfrac{1}{2}(e^{1B})^2 - (r + p_\tau^B e^{1A})(1 - e^{1B}) \\ &> \tfrac{1}{2}(e^{1B})^2 - (r + e^{1A})(1 - e^{1B}) = \dot{e}_\tau^{1B},\end{aligned}$$

which is a contradiction.

Now it remains to prove the comparison of continuation values from the perspective of an informed party. Since, player B is informed, such a continuation value is simply equal to v_t^{1B}. Let us denote $v_t^{1A/B}$ the corresponding continuation value of player A. Those continuation values are given by the following ODEs

$$\begin{aligned}-\dot{v}_t^{1A/B} &= \tfrac{1}{2}\left(e_t^{1A}\right)^2 - (r + e_t^{1B})v_t^{1A/B}, \\ -\dot{v}_t^{1B} &= \tfrac{1}{2}(e_t^{1B})^2 - (r + e_t^{1A})v_t^{1B}.\end{aligned}$$

The latter ODE is identical to (7). On the other hand, the former ODE differs from the ODE (6) for v_t^{1A} in the missing term p_t^B, since the informed party is aware of player B's success. Thus, the perceived hazard rate of B's patenting is simply equal to his true hazard rate of patenting, which is e_t^{1B}.

Again, we first look at the steady-state. It follows from the inequality $e_*^{1A} < e_*^{1B}$ proved above that

$$v_*^{1A/B} = \frac{\frac{1}{2}\left(e_*^{1A}\right)^2}{r + e_*^{1B}} < \frac{\frac{1}{2}(e_*^{1B})^2}{r + e_*^{1A}} = v_*^{1B}.$$

Thus, the inequality $v_t^{1A/B} < v_t^{1B}$ necessarily holds for t large. Suppose that the inequality does not hold for all $t \geq 0$. Let $\tau \geq 0$ be the smallest real number such that $v_t^{1A/B} < v_t^{1B}$ holds for all $t > \tau$. Then necessarily $v_\tau^{1A/B} = v_\tau^{1B}$ and $\dot{v}_t^{1A/B} \leq \dot{v}_t^{1B}$. This, together with the inequality $e_\tau^{1A} < e_\tau^{1B}$ implies

$$\dot{v}_\tau^{1A/B} = -\tfrac{1}{2}\left(e_\tau^{1A}\right)^2 + (r + e_\tau^{1B})v_\tau^{1A/B} > -\tfrac{1}{2}(e_\tau^{1B})^2 + (r + e_\tau^{1A})v_\tau^{1B} = \dot{v}_t^{1B},$$

which is a contradiction. □

C.6 Comparison of Steady-States

Lemma C.9. *The following comparison of the steady-state values holds:*

(i) $e_*^{1B} > e_*^1$;

(ii) $p_*^B < p_*$;

(iii) $e_*^{0B} < e_*^0$;

(iv) $e_*^{1A} < e_*^1$.

Proof. Throughout this proof we will omit the "$*$" symbol in the variables representing steady-state values.

(i) Recall that $p = e^0/e^1$, $e^0 = \psi(e^1)$, and so the equation (34) can be written as

$$1 = H(p, e^1) := p^2 + p \cdot \frac{e^1}{1-e^1},$$

where H is a function strictly increasing in both of its arguments. What is more,

$$p = \frac{e^0}{e^1} = \frac{\psi(e^1)}{e^1} = \frac{e^1}{2(1-e^1)} - \frac{r}{e^1}$$

is strictly increasing with e^1, and so the function $z \mapsto H(\psi(z)/z, z)$ is strictly increasing. Likewise, $p^B = e^{0B}/e^{1B}$, $e^{0B} = \psi(e^{1A})$, and the equation (44) can be represented as $1 = H(p^B, e^{1B})$. Consequently, using the inequality $e^{1A} < e^{1B}$ (Lemma C.4 (i)), we obtain

$$H\left(\frac{\psi(e^1)}{e^1}, e^1\right) = 1 = H\left(\frac{\psi(e^{1A})}{e^{1B}}, e^{1B}\right) < H\left(\frac{\psi(e^{1B})}{e^{1B}}, e^{1B}\right).$$

This gives us the inequality $e^1 < e^{1B}$.

(ii) Using the just obtained result and the fact that H is strictly increasing in both of its arguments, $H(p, e^1) = 1 = H(p^B, e^{1B})$ gives us that $p > p^B$.

(iii) The equation (34) can be alternatively written as

$$1 = \tilde{H}(e^0, e^1) := \left(\frac{e^0}{e^1}\right)^2 + \frac{e^0}{1-e^1},$$

where $\tilde{H}$ is increasing in e^0. What is more, for any $z \in [e^1, 1)$,

$$\frac{\partial}{\partial z}\tilde{H}(e^0, z) = -\frac{2(e^0)^2}{z^3} + \frac{e^0}{(1-z)^2} > 0,$$

because

$$\frac{z^3}{2(1-z)^2} = [\psi(z) + r] \cdot \frac{z}{1-z} > e^0,$$

where we used the fact that $z > e^1 > \frac{1}{2}$ and that $\psi(z) \geq \psi(e^1) = e^0$. Consequently, $e^{1B} > e^1$ implies $\tilde{H}(e^0, e^{1B}) > \tilde{H}(e^0, e^1) = 1$. However, the equation (44) gives us that $\tilde{H}(e^{0B}, e^{1B}) = 1$, and so necessarily $e^{0B} < e^0$.

(iv) Using the fact that the function ψ is strictly increasing,

$$e^{1A} = \psi^{-1}(e^{0B}) < \psi^{-1}(e^0) = e^1.$$

This completes the proof. □

D Appendix: Proofs for Section 6 (Patent Race with Optional Revelation)

D.1 Proof of Proposition 7 (Never Reveal Second)

Proof of Proposition 7. Suppose one player has revealed success already, without loss of generality, let it be player A. First, the strategy to never reveal is an equilibrium strategy of player B. Indeed, player B's continuation value implied by the strategy to never reveal is $v^{1B}(1, p_t)$, while his continuation value of revealing is $v^{11} = v^{1B}(1, 1)$. Applying Lemma C.6, we conclude that $e^{1B}(1, p_t) < E^{1B}(\sup_{s \geq t} p_s) < E^{1B}(1) = e^{1B}(1, 1)$, and so $v^{1B}(1, p_t) > v^{1B}(1, 1)$. Thus, a player has indeed no incentive to reveal.

To show that not revealing second is the only equilibrium, we need to consider any strategy of player B over revealing second, because player B's strategy over revealing impacts his rival's effort, and so it impacts his own incentive to reveal. The efforts and continuation values of the two players follow the same differential equations as those in private information game with one player being known to be successful, except for that the dynamics of p_t^B is influenced by player B's strategy over revelation. If player B is expected to reveal with a positive probability once being successful, then his rival's posterior belief p_t^B grows slower (or even falls), than it would in the game without revelation; and in the event of player B revealing it jumps to 1 and stays there. We do not need to describe the exact process of p_t, what is relevant is that p_t^B is less than 1 with a positive probability for a while. We can follow the reasoning from the proof of Lemma C.6 and generalize its results for a stochastic process p_t^B, and obtain the estimate that player B's continuation value while being in state 1 is strictly more than $1 - E^{1B}(1)$, which is the continuation value he would get after revealing.

In conclusion, regardless of what player B's strategy over revealing as second is, he has a strict incentive not to reveal. Thus, the only equilibrium strategy for player B can have is not to reveal second. □

D.2 Pure-strategy Equilibria

Proof of Proposition 8. First, the condition $v^{1A}(1, p) \leq v^1(p)$ for all $p \in [0, p_*)$ is a necessary for a no-revelation equilibrium to exist. Indeed, suppose that a no-revelation equilibrium exists and that $v^{1A}(1, p) > v^1(p)$ for some $p \in [0, p_*)$. Then player A has a strict incentive to reveal arrival of a breakthrough at time t such that $p_t^A = p$. Contradiction.

Assume that $v^{1A}(1,p) \leq v^1(1,p)$ for all $p \in [0,p_*)$, and suppose that both players have the strategy to never reveal. We check that none of the players has an incentive to deviate. Given that players do not reveal, their efforts and continuation values are identical to those from the private information version of the game (without revelation). In particular, the continuation value of a successful player at time t is $v^1(p_t)$. In contrast, if a successful player deviated and revealed, his continuation value would be $v^{1A}(1,p_t)$, which is no more than $v^1(p_t)$ by the assumption.

The no-revelation equilibrium is unique, as the effort levels have to correspond to the unique solution of the private information version of the game (Proposition 2). □

Lemma D.1 (partially numerical)**.** *The following statements hold in a no-revelation equilibrium:*

(i) If $v^{1A}(1,0) \leq v^1(0)$, then the inequality $v^{1A}(1,p) \leq v^1(p)$ holds for all $p \in [0,p_*)$.

(ii) If $v^{1A}(1,0) > v^1(0)$, then there exists $\bar{p} \in (0,p^*)$ such that $v^{1A}(1,\bar{p}) = v^1(\bar{p})$ and $v^{1A}(1,p) < v^1(p)$ if and only if $p < \bar{p}$.

(iii) The inequality $v^{1A}(1,0) \leq v^1(0)$ holds if and only if $\hat{r} = \alpha r/v < \hat{r}_N$, where $\hat{r}_N \approx 0.1113$.

Statements (i) and (ii) together with Proposition 8 imply that a sufficient and necessary condition for a no-revelation equilibrium to exist is $v^{1A}(1,0) \leq v^1(0)$. Statement (iii) then provides an equivalent condition in terms of the parameters.

The validity of this lemma can be tested numerically. As noted in the main text, due to the complexity of the problem it is not possible to find an explicit solution of the system of ODEs. Proceeding backwards in time from the steady-state, we can can solve the system numerically. The statements are then obtained by comparing the value functions for different values of the research difficulty $\hat{r}$.

Intuitively, the lemma holds for the following reason: The functions $v^{1A}(1,p)$ and $v^1(p)$ attain the same value (namely v^{11}) for $p = 1$. However, the function $v^1(p)$ has a higher curvature, as it corresponds to the value function based on the the posterior of both players changing simultaneously, while the function $v^{1A}(p)$, only reflects the posterior about player B is changing with p.

Proof of Proposition 9. Suppose that both players have the strategy to reveal the arrival of a breakthrough instantly (unless the rival has revealed already), and none has done so by time $t \geq 0$. Until either of the players reveals, the game is static in the sense that each player is certain that his rival is unsuccessful ($p_t = 0$). As argued in the main text, the effort as well as the value function are constant over time. Moreover, as follows from (19),

$$v_I^0 = \max_{e\geq 0} \left\{ v^{1A}(1,0)e\Delta t - \tfrac{1}{2}e^2\Delta t + v^{0B}(1,0)e_I^0\Delta t + [1-(e+e_I^0)\Delta t](1-r\Delta t)v_I^0 + o(\Delta t) \right\}.$$

After subtracting v_I^0 from both sides, dividing by $\Delta t > 0$ and taking the limit $\Delta t \searrow 0$ we obtain

$$0 = \max_{e\geq 0} \left\{ v^{1A}(1,0)e - \tfrac{1}{2}e^2 + v^{0B}(1,0)e_I^0 - (r+e+e_I^0)v_I^0 \right\}.$$

The first order condition becomes $e_I^0 = v^{1A}(1,0) - v_I^0$, which after substituting yields

$$0 = \tfrac{1}{2}(e_I^0)^2 + e_I^0 v^{0B}(1,0) - (r+e_I^0)[v^{1A}(1,0) - e_I^0],$$

which is the equation (12). Its right-hand side is a convex quadratic polynomial of e_I^0, that is negative at $e_I^0 = 0$ and positive at $e_I^0 = v^{1A}(1,0)$. As a result, the equation has one root e_I^0 in the interval $(0, v_I^{1A}(1,0))$ and one negative root.

Now we show that no player wants to deviate if and only if inequality (11) is satisfied. Assume that a player (say, player A) deviates and does not reveal. Let us denote $\tilde{e}^1$ his corresponding optimal effort and $\tilde{v}^1$ his continuation value. By a similar argument as above, after taking the limit $\Delta t \searrow 0$,

$$0 = \max_{e\geq 0} \left\{ 1\cdot e - \tfrac{1}{2}e^2 + v^{1B}(1,0)e_I^0 - (r+e+e_I^0)\tilde{v}_I^1 \right\}.$$

The first order condition becomes $\tilde{e}^1 = 1 - \tilde{v}^1$, , which after substituting yields

$$0 = \tfrac{1}{2}(1-\tilde{v}^1)^2 + e_I^0 v^{1B}(1,0) - (r+e_I^0)\tilde{v}^1. \tag{49}$$

The right-hand side of this equation is a quadratic polynomial of $\tilde{v}^1$, that is positive at $\tilde{v}^1 = 0$ and negative at $\tilde{v}^1 = 1$. Thus, there is unique $\tilde{v}^1 \in (0,1)$ that solves the equation (49).

Now, the equilibrium condition so that firm A does not reveal its success is $\tilde{v}^1 \leq v^{1A}(1,0)$. This condition holds, if and only if the right-hand side of (49) evaluated at $\tilde{v}^1 = v^{1A}(1,0)$ is non-positive. That gives us the inequality (11). □

D.3 Mixed-strategy Equlibria: Law of Motion

Proof of Lemma 2. Second, we derive the law of motion. Recall that p_t^j is the posterior probability of player j being in state 1 at time t. Unlike in case of the game without the option to reveal, the posterior here is conditioned not only on that player j has not patented, but also on the fact that he has not revealed by time t, which we denote as event N_t^j.[30] Accordingly,

$$\begin{aligned} p_{t+\Delta t}^j &= P[x_{t+\Delta t}^j = 1 \mid x_{t+\Delta t}^j < 2, N_{t+\Delta t}^j] = \frac{P[x_{t+\Delta}^j = 1, N_{t+\Delta t}^j \mid x_t^j < 2, N_t^j]}{P[x_{t+\Delta t}^j < 2, N_{t+\Delta t}^j \mid x_t^j < 2, N_t^j]} \\ &= \frac{(1-p_t^j)e_{Mt}^{0j}\Delta t + p_t^j(1-e_{Mt}^{1j}\Delta t) - \theta_{Mt}^j \Delta t}{1 - p_t^j e_{Mt}^{1j}\Delta t - \theta_{Mt}^j \Delta t} + o(\Delta t). \end{aligned}$$

Taking derivative with respect to Δt and evaluating at $\Delta t = 0$, we conclude

$$\begin{aligned} \dot{p}_t^j &= [(1-p_t^j)e_{Mt}^{0j} - p_t^j e_{Mt}^{1j} - \theta_{Mt}^j] \cdot 1 + p_t^j \cdot \left(p_t^j e_{Mt}^{1j} + \theta_{Mt}^j\right) \\ &= (1-p_t^j)(e_{Mt}^{0j} - p_t^j e_{Mt}^{1j} - \theta_{Mt}^j). \end{aligned}$$

This concludes the proof. □

Proof of Lemma 3. First we show that the hazard rate is finite. Consider a symmetric equilibrium, and let us analyze the situation from the perspective of player A. If $p_t = 0$, then the claim is trivial as there is nothing to be revealed. In the rest of the proof, consider $p_t > 0$. Let us distinguish three cases based on how the continuation value $v^{1A}(1, p_t)$ that player A (when being successful) obtains by revealing compares with the continuation value $v_M^{1A}(p_t)$, that he obtains by not revealing.

Case 1. If $v^{1A}(1, p_t) < v_M^{1A}(p_t)$, then it does not pay off to reveal, and thus $\theta_M(p_t) = 0$.

Case 2. If $v^{1A}(1, p_t) > v_M^{1A}(p_t)$, then player A, if successful, would reveal already before time t, and p_t would have to be 0.

Case 3. If $v^{1A}(1, p_t) = v_M^{1A}(p_t)$, suppose to the contrary that $\theta_{Mt} = +\infty$. Hence, the chance with which player B reveals in the time interval $[t, t+\Delta t]$ is an arbitrarily large multiple of Δt as Δt goes to 0. If player A reveals, he gets the continuation value $v^{1A}(1, p_t)$, whilst if he waits an arbitrarily short time Δt, he likely ends up with the continuation value $v^{1B}(1, p_t)$, which is larger than $v^{1A}(1, p_t)$ due to Proposition 6. Thus, player A prefers to postpone revelation in that case, implying $\theta_M(p_t) = 0$.

Second, we show that p_t is non-decreasing. Since $\theta_M(p_t)$ is finite, p_t is a continuous function of time (it cannot drop discretely). However, then p_t can never be decreasing in a Markov perfect Bayesian Nash equilibrium, as otherwise there would be times $t_1 < t_2$ such that $p_{t_1} = p_{t_2}$, but $\dot{p}_t = e_M^0(p_t) - p_t e_M^1(p_t) - \theta_M(p_t)$ is positive at t_1 and negative at t_2, impossible. As a result, $\dot{p}_t \geq 0$ and so $\theta_M(p_t) \leq e_M^0(p_t) - p_t e_M^1(p_t)$.

[30] In fact, the probability is conditioned on that neither of the players have patented or revealed, but it has not impact on the calculation.

Finally, consider an equilibrium. If a player has the strategy to reveal making a breakthrough with certainty at time $t = 0$ ($\theta_{M0} = e^0_{Mt}$), then p_t stays constantly at zero. Indeed, the Markov property then implies that until one of the players reveals, the players have to choose the same action at all times because the payoff relevant state p_t does not change. So if the player reveals with certainty at time $t = 0$, then the equilibrium has to be the *instant-revelation equilibrium*.

Reversely, if the player does not have the strategy to reveal with certainty at time $t = 0$, then $\dot{p}_0 > 0$, and since p_t is non-decreasing, it follows that $p_t > 0$ for all $t > 0$, and thus he does not reveal with certainty, as $\theta_{Mt} \leq e^0_{Mt} - p_t e^1_{Mt} < e^0_{Mt}$. □

D.4 Mixed-strategy Equlibria: Optimal Effort and Value Functions

The continuation value of a successful player (before anyone has revealed) v^1_{Mt} is given by the following recursive formula:

$$v^1_{Mt} = \max_{e \geq 0} \Big\{ e\Delta t - \tfrac{1}{2}e^2\Delta t + v^{1B}(1, p_t)\theta_{Mt}\Delta t + [1 - (r + e + \theta_{Mt} + p_t e^1_{Mt})\Delta t] v^1_{M,t+\Delta t} + o(\Delta t) \Big\}.$$

Compared to the formulas in previous sections, this one contains the term $v^{1B}(1, p_t)\theta_{Mt}\Delta t$ that corresponds to the case where rival reveals in within the time interval $[t, t+\Delta t]$ with probability $\theta_{Mt}\Delta t$ and yields continuation value $v^{1B}(1, p_t)$ (see Section 6.1). Note that this formula is evaluated as if the player whose value function is being calculated did not reveal, because if he reveals, then he is indifferent from revealing, and so it has no impact on the utility.

After subtracting $v^1_{M,t+\Delta t}$, dividing by Δt, and taking the limit $\Delta t \searrow 0$, we obtain

$$-\dot{v}^1_{Mt} = \max_{e \geq 0} \left\{ e - \tfrac{1}{2}e^2 + v^{1B}(1, p_t)\theta_{Mt} - (r + e + \theta_{Mt} + p_t e^1_{Mt}) v^1_{Mt} \right\}.$$

The first order condition yields $e^1_{Mt} = 1 - v^1_{Mt}$. Plugging it back yields the equation (13). The derivation of the equation (14) for $\dot{v}^0_{Mt}$ is analogous, and the equation (15) for $\dot{p}_t$ follows from Lemma 2.

Proof of Proposition 10. (i) Given that players stop revealing at time T, the continuation game is identical to the game without the option to reveal. Accordingly, the continuation value of player A at time T is $v^1(p_T)$. Clearly, $v^1(p_T) \geq v^{1A}(1, p_T)$, because otherwise player A would be tempted to reveal at time T. At the same time, $v^1(p_T) \leq v^{1A}(1, p_T)$, because otherwise player A would have strict incentive not to reveal at some time $t < T$.

This follows from the continuity of the functions $v^{1A}(1,p_t)$ and $v^1_M(p_t)$. Consequently, $v^{1A}(1,p_T) = v^1(p_T)$.

(ii) As the next step we derive a necessary condition for a player to be randomizing over revelation in equilibrium, i.e., for $0 < \theta_{Mt} < e^0_{Mt}$. As argued above, this implies $v^1_{Mt} = v^{1A}(1,p_t)$. As the function $t \mapsto \theta_{Mt}$ is by definition right-continuous, $\theta_{M,t+\Delta t} > 0$ for all $\Delta t \geq 0$ small enough, and thus also $v^1_{M,t+\Delta t} = v^{1A}(1,p_{t+\Delta t})$ for all $\Delta t \geq 0$ small enough. Thus, $\dot{v}^1_{Mt} = \frac{\partial}{\partial p} v^{1A}(1,p_t) \cdot \dot{p}_t$. Substituting this into the equation (13) yields a necessary condition

$$-\frac{\partial}{\partial p} v^{1A}(1,p_t) \cdot \dot{p}_t = \tfrac{1}{2}[e^{1A}(1,p_t)]^2 + \theta_{Mt} v^{1B}(1,p_t) - [r + \theta_{Mt} + p_t e^{1A}(1,p_t)] v^{1A}(1,p_t). \quad (50)$$

Note that this condition necessary, but does not need to be sufficient, as the player might not want to reveal at all. Substituting the law of motion (15) for $\dot{p}_t$ and denoting $\tilde{v}^{1A}_t(p_t) = v^{1A}(1,p_t) + (1-p_t)\frac{\partial}{\partial p} v^{1A}(1,p_t)$ then gives

$$0 = \tfrac{1}{2}[e^{1A}(1,p_t)]^2 + \theta_{Mt} v^{1B}(1,p_t) - (r + e^0_{Mt}) v^{1A}(1,p_t) + [e^0_{Mt} - p_t e^{1A}(1,p_t) - \theta_{Mt}] \tilde{v}^{1A}_t(p_t).$$

Solving for θ_{Mt} yields the condition (18).

(iii) When the player stops revealing, the continuation game is indeed equivalent to the private information game after time T. □

D.5 Equilibrium Characterization

The characterization provided in Proposition 11 is based on two lemmas below. Lemma D.2 postulates that any mixed-strategy equilibrium is a mixed-revelation equilibrium. According to Lemma D.3 a mixed-revelation equilibrium cannot coexist with any pure-startegy equilibrium for the same parameter values.

Lemma D.2 (partially numerical)**.** *If players ever randomize over revelation, then they do so on a time interval $t \in [0, T)$ for some $T > 0$.*

Proof of Lemma D.2 (partially numerical). First we show that players must stop revealing definitively at some $T > 0$. Suppose to the contrary that players never stop revealing definitively. Then there exists p to which p_t converges and $\theta_M(p) > 0$. This p then corresponds to a steady-state of the system of ODEs (14)–(15). However, analyzing the ODE, it can be shown that this steady-state can be classified as a source (the eigen-values of the Jacobean at the steady-state have positive real part), which means that no solution of the ODE can converge towards it. Thus, the players are not mixing over revelation indefinitely.

□

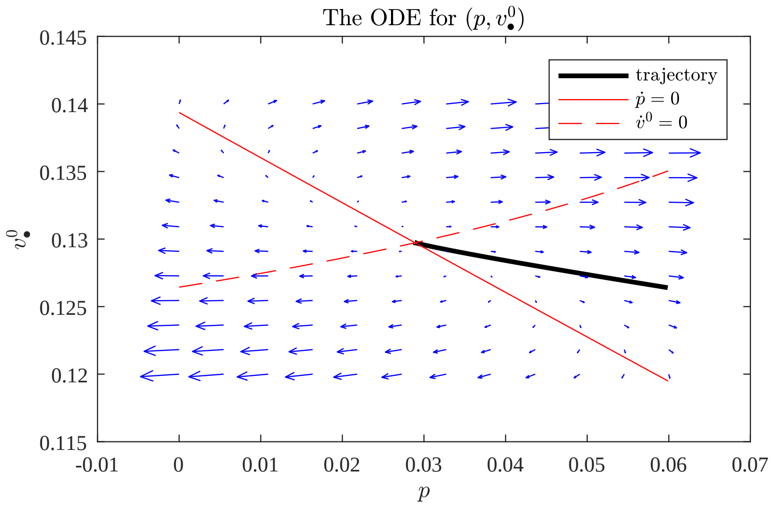

Figure 7: The neighborhood of the steady-state of the system of ODEs (14)–(15) for $\hat{r} = 0.2$.

Lemma D.3 (partially numerical)**.** *A mixed-revelation equilibrium exists if and only if there is no pure-strategy equilibrium (no-revelation equilibrium or instant-revelation equilibrium).*

The intuition behind this lemma is as follows: For $\hat{r} < \hat{r}_N \approx 0.1113$ players never mix over revelation because they are always better off by not revealing. However, for $\hat{r} > \hat{r}_I \approx 0.1707$ players would randomize only if $p_0 \in (\underline{p}, \bar{p})$, which is not consistent with the assumption that both players start in state 0 with certainty. Only for $\hat{r}$ in between the two bounds does a mixed-revelation equilibrium exit.

Proof of Lemma D.3 (partially numerical). We first show that a mixed-revelation and no-revelation equilibrium cannot coexist. Assume that no-revelation equilibrium exists. Then by Lemma D.1 $v^{1A}(1,p) < v^1(p)$ for all $p \in (0,1)$. This implies that the condition (i) from Proposition 10 is never satisfied and, thus, a mixed-revelation equilibrium does not exist.

Second, we show that a mixed-revelation and an instant-revelation equilibrium cannot coexist. Assume that instant-revelation equilibrium exists. It can be shown numerically that in that case there exists $\underline{p} \in [0, \overline{p})$ at which the system of ODEs (14)–(15) has a steady-state, as shown in Figure 4. When going backwards in time from $p_T = \overline{p}$, the belief p_t will never drop below $\underline{p}$, and thus no mixed-revelation equilibrium exists. The situation is illustrated in Figure 7. The thick solid line shows the trajectory of the solution ending in the point $(\bar{p}, v^0(\bar{p}))$; when going back in time, the trajectory converges towards the steady-state. The derivative $\dot{p}$ is positive above the solid line, and $\dot{v}_M^0$ is positive above the dashed line. The two lines cross if and only if an instant revelation equilibrium exists ($\hat{r} > \hat{r}_I \approx 0.1707$).

Third, if none of the pure-stretegy equilibria exists, then there is unique mixed-revelation equilibrium. Indeed, the equilibrium is characterized by Proposition 10. It can be shown numerically by starting from the posterior $\bar{p}$ and solving for $v_M(p)$, $e_M^1(p)$, and $\theta_M(p)$, as p goes down from $\bar{p}$ to 0, verifying that the solution never gets out of bounds. Once the functions of p are calculated, p_t can be solved for. Starting from $p_0 = 0$ and following the ODE (15) until hitting the value $\bar{p}$, which defines the time T such that $p_T = \bar{p}$. □

Proof of Proposition 11 (partially numerical). By Lemma D.2 the mixed-revelation equilibrium is the unique candidate for equilibrium involving mixed strategies. By Lemma D.3 the mixed-revelation equilibrium exists if and only if any of the pure-strategy equilibria does not. As a result, there always exists unique equilibrium. Its type depends on the parameters. The thresholds $\hat{r}_N$ and $\hat{r}_I$ are found numerically. □

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