Optimal Fiscal Policy in Debt and Unemployment Crisis *

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Abstract

This paper examines the optimal fiscal policy in the economy plagued by high unemployment and sovereign debt. Employing a model with a frictional labor market and sovereign default risk, we apply analytical and numerical methods to study the reaction of the fiscal authority that uses government spending, taxes and debt issuance to mitigate the crisis. Higher expenditures reduce unemployment and have a positive and permanent impact on the tax base, but, at the same time, they increase the exposure to sovereign default risk. We calibrate the model to match the moments characterizing the Spanish economy and quantify those two opposite forces. We find that high debt is a critical concern the government faces, even if a substantial amount of unemployed resources gives rise to high spending multipliers. Consequently, fiscal austerity is the policy recommendation for the economy with high unemployment and debt.

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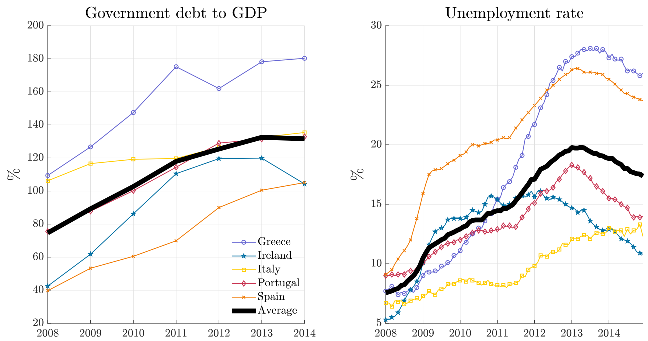
1 Introduction

During the European debt crisis, high unemployment was accompanied by soaring sovereign debt levels (see Figure 1), which sparked a heated debate about the desired fiscal policy response. The adherents of fiscal austerity argued that the return to a balanced budget (or budget surplus) would help build up trust in the financial markets and lower debt service costs. Moreover, reducing government expenditures was supposed to crowd in private spending and, consequently, spur a recovery. However, there has been substantial criticism of this view: Paul Krugman, together with over 9.000 signatories of *A Manifesto for Economic Sense*, claimed that drastic austerity policies might be contractionary and have devastating consequences for economic growth and thus prolong the recession.

The advocates and the opponents of fiscal austerity articulated two different aspects determining the desired policy reaction to the crisis. The former highlighted the role of high government debt: fiscal authority should cut expenditures to avoid the cliff, i.e., to lower the sovereign default risk and thus prevent rising debt service costs. Instead, the latter focused on high unemployment: a substantial amount of economic slack increased the effectiveness of fiscal stimulus: the macroeconomic adjustment to higher fiscal spending under such circumstances was supposed to feature a more significant positive adjustment in employment and output than in prices.

Two important papers have formalized the intuitions underlying those two views: Michaillat and Saez (2019) and Bianchi et al. (2021). Michaillat and Saez (2019) consider the economy with a frictional labor market that gives rise to inefficient unemployment and find that optimal stimulus spending is positive and increasing in the unemployment gap (the difference between current and efficient unemployment

Figure 1: Government debt and unemployment in the EA periphery during the European debt crisis



rates). Importantly, Michaillat and Saez (2019) assume that the government runs a balanced budget and thus abstract from sovereign debt management and the associated default risk. Bianchi et al. (2021) study the model with endogenous sovereign default and nominal rigidities and conclude that higher government expenditures may not be desirable during debt crisis even when the stabilization gains driven by downward nominal wage rigidity are considerable. Their analysis, however, abstracts from the role of economic slack in shaping the optimal fiscal policy response or, more formally, employment is not an element of the set of endogenous state variables affecting the conduct of fiscal policy.

The above-mentioned conclusions based on two complementary perspectives - the unemployment-oriented one by Michaillat and Saez (2019) and the debt-oriented one by Bianchi et al. (2021) - do not provide a clear policy recommendation for the case of the economy simultaneously affected by two problems (i.e., by high debt and unemployment). Indeed, while the suggestion motivated by the approach by Michaillat and Saez (2019) would call for higher expenditures, the solution based on Bianchi et al. (2021) would be just the opposite. Thus solving the stimulus-austerity dilemma in the economy plagued by high unemployment and high debt requires studying jointly both motives (slack/unemployment and debt). Our paper is intended to achieve this goal.

To this end, we embed the Diamond-Mortensen-Pissarides model of the frictional labor market into a version of the model of sovereign default by Eaton and Gersovitz (1981). This allows us to capture the trade-offs relevant from the point of the policy debate originating from the European debt crisis. First, the sovereign default component of the model gives rise to sovereign risk, and, like in Bianchi et al. (2021), it imposes an endogenous borrowing limit on the government. This, in turn, limits the desirability of fiscal expansions when debt is high. Second, the component associated with the frictional labor market engenders inefficient unemployment in equilibrium and, similarly to Michaillat and Saez (2019), provides a rationale for active aggregate demand management when the level of economic slack is sizable. Third, the interaction between both components gives rise to interesting mechanisms shaping the optimal fiscal policy. On the one hand, the forward-looking job creation in the Diamond-Mortensen-Pissarides model interacts with the future sovereign default risk. Specifically, if the government issues more debt in the current period, it increases future sovereign default risk. As sharp drops in productivity usually accompany the default (see Arellano (2008)), higher default risk in the future lowers the expected values of jobs today and hampers job creation in the current period. On the other hand, if the resources from debt issuance are used for stimulating aggregate demand in the current period, then it leads to additional job creation and, given that the changes to employment in the Diamond-Mortensen-Pissarides feature some persistence, it positively affects current and future employment. This, in turn, increases the tax base and mechanically lowers the debt to GDP ratio in present and future periods. By construction, the mechanisms resulting from the interaction between the Diamond-Mortensen-Pissarides and Eaton-Gersovitz components are absent in Michaillat and Saez (2019) and Bianchi et al. (2021).

We use the model to study the optimal fiscal policy in debt and unemployment crises. First, analogously to Michaillat and Saez (2019) and Bianchi et al. (2021), we analytically derive a version of the Samuelson rule characterizing the optimal level of government spending. We find that the efficient level of the provision of public goods is, in addition to the marginal rate of substitution between public and private consumption and their marginal rate of transformation (as in Samuelson (1954)), affected by the component associated with job-creation spurred by government spending and the current account considerations. Second, we calibrate the model to match the moments characterizing the Spanish economy. Third, we use the quantitative model to assess the desirability of fiscal austerity implemented in Spain during the European debt crisis. In doing so, we highlight the role of labor market frictions articulated by comparing the dynamics of the model to a hypothetical case of constrained-efficient allocation in which the government internalizes the possible inefficiencies related to the wage-setting and vacancy posting processes. We find that the motives related to sovereign default risk outweigh those related to gains from job creation when unemployment is high. As a result, austerity turns out to be a desired policy solution in debt and unemployment crisis.

The remaining sections of the paper are organized as follows. We review the related literature in the next section. Section 3 presents the model. Section 4 describes analytical exercises. In Section 5 we calibrate the model. Section 6 describes the quantitative exercises. Section 7 concludes.

2 Literature

Our paper is intended to bridge two essential strands of the literature. The first of them studies the dependence of the stimulus's effectiveness on the public debt level. From the normative perspective, this issue is addressed by Bianchi et al. (2021). Romer and Romer (2019) discuss the empirical perspective and find that the relationship between the debt-to-GDP ratio and the fiscal policy response is driven partly by problems with access to sovereign markets. The second strand is related to the relationship between the effectiveness of fiscal policy and economic slack. Michaillat and Saez (2019) address this problem using normative analysis. A theoretical paper by Michaillat (2014) studies this dependence from a positive angle and argues that the multiplier's value increases with unemployment. Moreover, numerous works focused on the differences in government spending multiplier's size in recessions and expansions. For instance, Auerbach and Gorodnichenko (2012) find that fiscal policy in the US is considerably more effective in recessions. Somewhat contrarily, Ramey and Zubairy (2018) find that the size of the multiplier in the US is below unity, irrespective of the amount of slack in the economy. Moreover, Owyang et al. (2013) is in line with the result of Ramey and Zubairy (2018) for the US economy but also document evidence of the countercyclical multiplier's size in Canada.

From the technical point of view, the closest works to ours are papers studying various versions of the Eaton and Gersovitz (1981) model. Balke and Ravn (2016), Bianchi et al. (2021), and Anzoategui (2022) study optimal government spending in that framework. The main novelty of our approach is the use of the dynamic model of the frictional labor market that gives rise to transmission channels absent in those papers. Prein (2019) and Balke (2022) combine Eaton and Gersovitz (1981) with the

Diamond-Mortensen-Pissarides framework but do not include government spending as a policy option, which is the main focus of our paper. Froemel and Paczos (2023) explore the link between default risk and the cyclicality of fiscal transfers.

Moreover, our paper contributes to the literature discussing the sources of unemployment fluctuations. Shimer (2005), Hall (2005), and Hagedorn and Manovskii (2008) analyze the role of productivity shocks in that context. The influence of demand shocks is discussed in Michaillat and Saez (2015), and the impact of discount factor shocks on unemployment fluctuations is explored in Hall (2017). Similarly to Balke (2022), we study the role of sovereign default risk in unemployment dynamics.

3 Model

3.1 Overview of the model

Time is infinite and divided into discrete periods. We consider a small open economy without independent monetary authority and with a fixed nominal exchange rate standardized to unity (the economy can be thought of as a part of a monetary union). The economy is populated by three types of agents: households, firms (owned by households) and the government (fiscal authority). As in Eaton and Gersovitz (1981), the government is benevolent and uses the available fiscal tools (debt issuance, default, expenditures on public goods and taxes) to maximize the household's utility derived from a composite consumption good and public good. The former is a consumption index composed of domestic and imported goods (also called home and foreign, respectively). The latter is produced from domestic goods using a one-for-one technology. The production technology of home goods is linear and uses labor as the only input. The amount of home goods that are exported is a decreasing function of the price of home goods. International financial intermediaries trade government bonds and thus lend resources to the economy. The two key frictions in the model are: the government's inability to commit to repay debt in the future and the decentralized labor market featuring search and matching fricitons.

3.2 Matching technology

Matching technology T combines vacancies v_t posted by producers with workers that are jobless at the beginning period of period t. Measure of the latter is given by $1 - (1-\delta) \cdot l_{t-1}$ where $\delta \in (0,1)$ is the parameter describing the exogenous job-separation rate. By l_{t-1} we denote employment level in period $t-1$. More specifically, T is the mass of jobs created, which is given by:

$$T\left(1-(1-\delta)\cdot l_{t-1}, v_t\right) = \left(\left[1-(1-\delta)\cdot l_{t-1}\right]^{-\gamma} + v_t^{-\gamma}\right)^{-\frac{1}{\gamma}}.$$

This formulation was introduced by den Haan et al. (2000), where $\gamma > 0$ governs the elasticity of substitution of matching inputs.

Labor market tightness x_t is defined as:

$$x_t \equiv \frac{v_t}{1-(1-\delta)\cdot l_{t-1}} \tag{1}$$

and the vacancy-filling rate θ_t is given by:

$$\theta_t \equiv \theta\left(x_t\right) = \frac{T\left(1-(1-\delta)\cdot l_{t-1}, v_t\right)}{v_t}, \tag{2}$$

The economy-wide law of motion for employment l_t is:

$$l_t = (1-\delta)\cdot l_{t-1} + T\left(1-(1-\delta)\cdot l_{t-1}, v_t\right). \tag{3}$$

The timeline of the events on the labor market within each period is the following: proportion δ of employed households are separated from their jobs at the end of period $t-1$. At the beginning of next period (i.e. in period t) they are pooled with measure $1-l_{t-1}$ of households that were unemployed at time $t-1$ and new matchings are formed using technology T. Subsequently, the employed l_t produce goods and unemployed households $1-l_t$ remains idle.

3.3 Households

The economy is populated by a mass one of infinitely-lived, identical households who can be employed or unemployed. There is perfect insurance against unemployment risk across them. In period t, households derive the utility from a composite consumption good C_t and public good g_t. They discount future utility streams with factor $\beta \in (0,1)$ and supply labor inelastically (i.e. there is no disutility from work). This implies that, from the perspective of period 0, their lifetime utility can be formulated as follows:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \cdot U\left(C_t,\, g_t\right)$$

where C_t is defined as follows:

$$C_t = \left[(1-\omega)\cdot\left(c_t^H\right)^{\frac{\epsilon-1}{\epsilon}} + \omega\cdot\left(c_t^F\right)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} \tag{4}$$

and where $U\left(C_t,\, g_t\right)$ is given by:

$$U\left(C_t,\, g_t\right) = (1-\psi) \cdot \frac{C_t^{1-\sigma}}{1-\sigma} + \psi \cdot \frac{g_t^{1-\sigma}}{1-\sigma}.$$

Parameter $\psi \in (0,1)$ is the weight attached to public goods, $\omega \in (0,1)$ is inversely related to the degree of home bias in preferences, $\epsilon > 0$ measures the substitutability between domestic and foreign goods, $\sigma > 0$ is relative risk aversion (inverse elasticity of intertemporal substitution), c_t^H and c_t^F are the amounts of home and foreign goods, respectively.

Employed households earn nominal wage w_t measured, without loss of generality, in the common currency units (recall that nominal exchange rate is normalized to one). The same convention applies to profits π_t received by households. The mass of households employed in period t is l_t, and the price of home goods is denoted by p_t. All this implies that the household's budget constraint can be written as:

$$p_t \cdot c_t^H + c_t^F + p_t \cdot \tau_t = w_t \cdot l_t + \pi_t \tag{5}$$

Note that in the competitive equilibrium, price p_t has to satisfy the following intratemporal optimality condition:

$$p_t = \frac{1-\omega}{\omega} \cdot \left(\frac{c_t^H}{c_t^F}\right)^{-\frac{1}{\epsilon}} \tag{6}$$

which becomes an implementability condition for the benevolent government's allocation. As in Eaton and Gersovitz (1981), the intertemporal choice is delegated to fiscal authority.

3.4 Firms

Each firm consists of a single job and produces a_t of domestic goods. The value of a_t can be thought of as the productivity level in the economy and it is governed by the standard Markovian process denoted by Z_t (the relationship between a_t and Z_t is specified later). The value of the job J_t is given by:

$$J_t \equiv a_t \cdot \frac{p_t}{P_t} - \frac{w_t}{P_t} + \Omega_t \tag{7}$$

where P_t is the price index that satisfies:[1]

$$P_t = \frac{1}{\omega^{\omega} \cdot (1-\omega)^{1-\omega}} \cdot p_t^{1-\omega} \tag{8}$$

and where Ω_t is the expected discounted value of the job:

$$\Omega_t \equiv (1-\delta) \cdot \mathbb{E}_t \left[M_{t,\,t+1} \cdot J_{t+1} \right]$$

where $M_{t,t+1}$ is stochastic discount factor:

$$M_{t,\,t+1} = \beta \cdot \frac{U_{C,\,t+1}}{U_{C,\,t}}.$$

To enter the market, firms post vacancies. If a vacancy is filled successfully (which occurs with probability θ_t), firm starts producing home goods.

The aggregate technology converting consumption goods into vacancies is potentially concave:

$$v_t = \frac{1}{\kappa} \cdot (i_t)^{\nu} \tag{9}$$

[1] It can be derived from the identity $P_t \cdot C_t = p_t \cdot c_t^H + c_t^F$ using equations (4) and (6).

where $\nu \in (0, 1]$ is set to match the empirical labor market dynamics and i_t is the total amount of the composite good "invested" in vacancies. In particular, the amounts of domestic and foreign goods used for generating i_t are given by:

$$i_t^H = \frac{1-\omega}{\omega^\omega \cdot (1-\omega)^{1-\omega}} \cdot \frac{i_t}{p_t^\omega}, \tag{10}$$

$$i_t^F = \frac{\omega}{\omega^\omega \cdot (1-\omega)^{1-\omega}} \cdot p_t^{1-\omega} \cdot i_t, \tag{11}$$

respectively.[2]

Note that the cost of a single vacancy is $P_t \cdot \kappa \cdot v_t^{\frac{1}{\nu}-1}$ so the free-entry condition formulated for the labor market reads:

$$\kappa^{\frac{1}{\nu}} \cdot v_t^{\frac{1}{\nu}-1} \geq \theta_t \cdot J_t. \tag{12}$$

The law of motion describing the employment/firm dynamics that is consistent with equation (3) is:

$$l_t = (1-\delta) \cdot l_{t-1} + \theta_t \cdot v_t. \tag{13}$$

Finally, the aggregate nominal profits π_t satisfy:

$$\pi_t = \left(a_t \cdot p_t - w_t - P_t \cdot (\kappa \cdot v_t)^{\frac{1}{\nu}}\right) \cdot l_t \tag{14}$$

[2]This disaggregation of i_t follows from the fact that the sum of invested foreign goods and the product of price p_t and invested domestic goods add up to $P_t \cdot i_t$. Moreover, these amounts satisfy the maximization problem associated with the "packing" technology (see equation (4) summarized with condition (6) (and where c_t^F and c_t^H are replaced with invested foreign goods and invested domestic goods, respectively).

3.5 Wage-setting protocol

In the benchmark simulation, we assume perfect nominal wage rigidity:

$$w_t = \bar{w} \tag{15}$$

where $\bar{w} > 0$ is a parameter. In addition, we consider alternative wage-setting protocols: Nash-bargaining, real wage rigidities and wages that decentralize the constrained-efficient allocation (where the government sets the number of vacancies in an optimal way).

3.6 Market clearing constraints and the government budget constraint

The market clearing condition for domestic goods reads:

$$a_t \cdot l_t = c_t^H + g_t + exp_t + i_t^H \tag{16}$$

where exp_t is exports function given by:

$$exp_t = z \cdot (p_t)^{-\epsilon_{exp}} \tag{17}$$

where $\epsilon_{exp} > 0$ governs the price elasticity of export and $z > 0$ is a constant.

If the government has an access to international financial markets, then the budget constraint reads:

$$p_t \cdot \tau_t - p_t \cdot g_t = \lambda \cdot b_t - q_t \cdot (b_{t+1} - (1 - \lambda) \cdot b_t) \tag{18}$$

where b_t is debt accumulated in previous periods, b_{t+1} is the issuance of new debt, $\lambda \in (0,1)$ is the fraction of the outstanding long-term debt that matures every period and q_t is price of debt (specified later).

If the government is excluded from international financial markets, which is a punishment for defaulting on public debt in the past, its budget constraint is:

$$p_t \cdot \tau_t - p_t \cdot g_t = 0. \tag{19}$$

Combining (5), (14), (16) and (18) yields the balance of payments identity:

$$p_t \cdot exp_t - c_t^F - i_t^F = \lambda \cdot b_t - q_t \cdot \left(b_{t+1} - (1-\lambda) \cdot b_t\right), \tag{20}$$

which takes the following form:

$$p_t \cdot exp_t - c_t^F - i_t^F = 0 \tag{21}$$

if the government is excluded from international financial markets.

3.7 The government maximization problem

It is assumed that at the beginning of each period and prior to the matching on the labor market, the government that has an access to financial markets decides whether to default on debt. The decisions concerning the level of taxes, government spending and debt issuance are made right after the default decisions and prior to all the private sector decisions (this pertains also to the government excluded from financial markets).

It is assumed that default is costly. First, as in Arellano (2008), we assume that the government is excluded from financial markets for a stochastic number periods (with the probability of re-entering the markets equal to $\zeta \in (0, 1)$). Second, during the period of exclusion, the economy suffers from productivity loss. Specifically, in the periods when the government has an access to financial markets, the productivity in the economy equals $a_t = Z_t$ while in the periods of exclusion we have:

$$a_t = \phi\left(Z_t\right) \leq Z_t.$$

Third, to simplify numerical computations (and to apply the solution method by Kiiashko and Maliar (2021)), we assume that there is a stochastic utility cost $\chi_t \sim N(\mu^{\chi},\, \sigma^{\chi})$ related to a positive default decision that is independent from the process $\{Z_t\}$.

Given the Markovian structure of stochastic process Z_t, we are now in a position to present the recursive formulations of the government's problem. To this end, the future values of variables are denoted with prime symbols. Given that both l_{t-1} and b_t are pre-determined, the problem of the government that decides to repay its debt is described by the following Bellman equation:

$$V^{R}\left(Z,\, b,\, l\right) = \max_{X} \left\{U(C,\, g) + \beta \cdot \mathbb{E}\left[W\left(Z',\, b',\, l'\right)|Z\right]\right\} \tag{22}$$

s.t.

$$X = \left\{C, c^{H}, c^{F}, p, \tau, w, l', \pi, P, \theta, v, x, g, exp, i, b', J, i^{H}, i^{F}\right\}$$

X satisfies 1, 2 and 4-18

and given $\Omega = \Omega^{R}(Z,\, b',\, l'),\; q = q(Z,\, b',\, l'),\; a = Z.$

where W is defined later. When the government decides to default, its utility (net off the utility cost of default) equals:

$$V^D(Z,l) = \max_X \left\{ U(C,\, g) + \beta \cdot \mathbb{E}\left[\zeta \cdot W(Z',0,l') + (1-\zeta) \cdot V^D(Z',l')\,|Z \right] \right\} \quad (23)$$

s.t.

$$X = \left\{ C, c^H, c^F, p, \tau, w, l', \pi, P, \theta, v, x, g, exp, i, J, i^H, i^F \right\}$$

$$X \text{ satisfies 1, 2 and 4-17, 19}$$

$$\text{and given } \Omega = \Omega^D(Z,\, l'),\; a = \phi(Z)$$

where:

$$W(Z,\, b,\, l) = \int_{-\infty}^{\infty} \max\left\{ V^R(Z,\, b,\, l)\,,\, V^D(Z,\, l) - \chi \right\} dF(\chi)$$

where F is the c.d.f. of the normal distribution $N(\mu^{\chi},\, \sigma^{\chi})$.

Note that the default decision $D(\chi, Z, b, l)$ is positive and equals to one if:

$$V^R(Z,\, b,\, l) < V^D(Z,\, l) - \chi.$$

If the opposite holds then $D(\chi, Z, b, l) = 0$.

3.8 Financial intermediaries

Risk neutral financial intermediaries trade sovereign bonds in the international financial markets featuring the risk-free rate r. This implies the following price of the bond contract:

$$q(Z,\, b',\, l') = \frac{1}{1+r} \times$$

$$\mathbb{E}\left[\mathbb{P}(Z',\, b',\, l') \cdot (\lambda + (1-\lambda) \cdot q\left(Z',\, b(Z',\, b',\, l'),\, l(Z',\, b',\, l')\right)) \,|Z\right]. \tag{24}$$

where $\mathbb{P}(Z,\, b,\, l)$ is the repayment probability prior to the realization of the utility shock:

$$\mathbb{P}(Z,\, b,\, l) = \text{Prob}\left(V^R\left(Z,\, b,\, l\right) > V^D\left(Z,\, l\right) - \chi\right).$$

3.9 Equilibrium

We can now define the equilibrium in the model:

Definition. The *Markov Perfect Competitive Equilibrium* consists of:

1. Value functions of the government: $V^R(Z,\, b,\, l)$, $V^D(Z,\, l)$, $W(Z,\, b,\, l)$ and the associated policies X satisfying the corresponding implementability constraints,

2. Bond price schedule $q(Z,\, b',\, l')$ satisfying condition (24).

3.10 Constrained-efficient allocation

To analyze the mechanisms related to a malfuctioning labor market, it is instructive to characterize an allocation for which this market works in an optimal way. To this end, we consider a modified version of the problem (22) and (23) in which the benevolent government is able to control the process of job creation.

The problem of the government that decides to repay its debt is now given by the following Bellman equation:

$$\hat{V}^R\left(Z,\, b,\, l\right) = \max_X \left\{U(C,\, g) + \beta \cdot \mathbb{E}\left[\hat{W}\left(Z',\, b',\, l'\right) |Z\right]\right\} \tag{25}$$

s.t.

$$X=\left\{C,c^{H},c^{F},p,l',v,g,exp,i,i^{H},i^{F},b'\right\}$$

X satisfies 3, 4, 6, 9, 10, 11, 16, 17, 20

and given $q=q(Z,\,b',\,l'),\ a=Z.$

The problem of the government that decides to default on debt is characterized by:

$$\hat{V}^{D}\left(Z,l\right)=\max_{X}\left\{U(C,\,g)+\beta\cdot\mathbb{E}\left[\zeta\cdot\hat{W}\left(Z',0,l'\right)+(1-\zeta)\cdot\hat{V}^{D}\left(Z',l'\right)|Z\right]\right\} \tag{26}$$

s.t.

$$X=\left\{C,c^{H},c^{F},p,\tau,w,l',\pi,P,\theta,v,x,g,exp,i,J,i^{H},i^{F}\right\}$$

X satisfies 3, 4, 6, 9, 10, 11, 16, 17, 21

and given $a=\phi\left(Z\right)$

where by $\hat{V}^{R}$ and $\hat{V}^{D}$ we denote the value functions associated with the constrained efficient outcome and where $\hat{W}$ is defined analogously to W.

Note that, from the technical perspective, the differences between problems (25)-(26) and (22)-(23) follow from the replacement of the job-creation based on free entry (and characterized by conditions (7), (12), (13)) and the wage-setting protocol (described by (15)) with an optimal vacancy posting that assumes the knowledge of the mechanisms underlying the aggregate law of motion (3) by the government. Moreover, the budget constraints implied by the existence of free entry and wage-setting protocol are replaced with balance of payment identities (20) and (21) so that neither wages nor firm profits are present in (25)-(26). Finally, as the job creation based on free entry is removed from the model, the government problem ceases to be dependent on the future value of firm Ω that was taken as given by the fiscal authority in (22)-(23).

4 Analytical exploration

Under the competitive equilibrium with a decentralized job creation, the government problem after the repayment decision can be described by the following Bellman equation:

Proposition 1. *Under the repayment scenario in the competitive equilibrium, the maximization problem of the government in a country featuring exogenous shock Z, previous period's employment l and debt b is described as follows:*

$$V^R(Z,b,l) = \max_{c^F,l',g,b'} \left\{ U\left(\left(c^H\left(c^F,l',g\right)\right)^{1-\omega} \cdot \left(c^F\right)^{\omega}, g\right) + \beta \cdot \mathbb{E}\left[W\left(Z',b',l'\right)|Z\right]\right\} \tag{27}$$

subject to:

$$p\left(c^F,l',g\right) \cdot exp\left(p\left(c^F,l',g\right)\right) - c^F - \omega \cdot P\left(c^F,l',g\right) \cdot i\left(l'\right)$$

$$= \lambda \cdot b - q\left(Z,b',l'\right) \cdot \left(b' - (1-\lambda) \cdot b\right) \tag{28}$$

$$\theta\left(l'\right) \cdot \left(a \cdot \omega^{\omega} \cdot (1-\omega)^{1-\omega} \cdot \left(p\left(c^F,l',g\right)^{\omega}\right) - \frac{w}{P\left(c^F,l',g\right)} + \Omega^J\left(Z',b',l'\right)\right) \tag{29}$$

$$\leq \kappa^{\frac{1}{\nu}} \cdot v\left(l'\right)^{\frac{1}{\nu}-1}.$$

Functions $c^H\left(c^F, l', g\right)$, $p\left(c^F, l', g\right)$, $P\left(c^F, l', g\right)$ are defined in the proof of Proposition 1 (which is delegated to the Appendix) and are analogous to function $\mathcal{P}$ in the Bianchi et al. (2021) (note that Bianchi et al. (2021) define only one such function and there are three such functions in this paper - this is because our problem is more complicated and it does not admit for the closed-form characterizations of $c^H\left(c^F, l', g\right), p\left(c^F, l', g\right), P\left(c^F, l', g\right)$).

Corollary 2. *(Modified Samuelson Rule) The optimal level of government spending in the repayment scenario is characterized by the following condition:*

$$\overbrace{\frac{\partial U}{\partial C} \cdot \frac{\partial U}{\partial c^H} \cdot \frac{\partial c^H}{\partial g} + \frac{\partial U}{\partial g}}^{\textit{Samuelson}} + \overbrace{\eta \cdot \left[-\frac{\partial p}{\partial g} \cdot exp - p \cdot \frac{dexp}{dp} \cdot \frac{\partial p}{\partial g} + \omega \cdot \frac{\partial P}{\partial g} \cdot i\right]}^{\textit{current account effects=fiscal deficit effects}}$$

$$+ \underbrace{\mu \cdot \theta \cdot \left[-a \cdot \omega^{\omega+1} \cdot (1-\omega)^{1-\omega} \cdot p^{\omega-1} \cdot \frac{\partial p}{\partial g} + w \cdot \frac{1}{P^2} \cdot \frac{\partial P}{\partial g}\right]}_{\textit{job creation: current profits}} = 0 \qquad (30)$$

where η and μ are the Lagrange multipliers associated with constraints (28) and (29) in the maximization problem (27), respectively.

Corollary 2 is the first order condition corresponding to variable g that is derived from the problem (27).[3] It has the following interpretation: the choice of the optimal

[3]To see that the Lagrange multiplier's (and the Kuhn-Tucker conditions) are applied correctly in Corollary 2, notice that although the constraint (16) is binding in the optimum, it can be expressed as an inequality:

$$a_t \cdot l_t \geq c_t^H + g_t + exp_t + \frac{1-\omega}{\omega^\omega \cdot (1-\omega)^{1-\omega}} \cdot \frac{i_t}{p_t^\omega}$$

which after combining with the household budget constraint, definition of firm's profits and the government budget constraint yields:

$$p\left(c^F, l', g\right) \cdot exp\left(p\left(c^F, l', g\right)\right) - c^F - \omega \cdot P\left(c^F, l', g\right) \cdot i\left(l'\right)$$
$$\leq \lambda \cdot b - q\left(s, b', l'\right) \cdot \left(b' - (1-\lambda) \cdot b\right).$$

value of government spending is influenced by three motives. The first of them is associated with the standard Samuelson rule that equates the marginal rate of substitution between public and private consumption equals with the marginal rate of transformation between those goods. The current account effects/public debt motive reflect the fact that changes to public spending affect domestic prices p. This, in turn, implies that the terms-off-trade is affected (see element $-\frac{\partial p}{\partial g} \cdot exp$), demand for exported goods changes (element $-p \cdot \frac{dexp}{dp} \cdot \frac{\partial p}{\partial g}$) and the effective price of foreign resources needed to finance domestic job creation (i.e., $\omega \cdot \frac{\partial P}{\partial g} \cdot i$) is higher. Finally, the job creation motive captures the fact that changes to government spending affect firm's profits which affects the job value and influences job creation.

Another dimension of the optimal fiscal policy in the competitve equilibrium, i.e. debt issuance, is described by the Euler equation of the government:

Corollary 3. *The Euler equation associated with the problem in Proposition (1) is:*

$$\eta \cdot \left(\frac{\partial q}{\partial b'} \cdot b' + q \right) + \mu \cdot \theta \cdot \frac{\partial \Omega^J}{\partial b'} = \beta \cdot \mathbb{E} \left(1 - D'\right) \cdot \eta' \cdot \left(\lambda + q' \cdot (1 - \lambda)\right) \tag{31}$$

The proof of Corollary 3 is delegated to the Appendix.[4] This condition shows that, in the optimum, the marginal benefit from issuing additional debt today equals the marginal cost of servicing and repaying it in the future. The benefit is composed of two terms. The first follows because additional debt allows the government to relax the current constraint associated with current account and to import additional consumption and investment good. The second term lowers the marginal benefit because, by borrowing debt today, the government increases default probability in

[4]The only non-trivial part of that proof is to show that:

$$\frac{\partial}{\partial b'} \mathbb{E} W = \mathbb{E} \frac{\partial V^R}{\partial b'} \cdot \left(1 - D'\right).$$

the future. This, in turn, lowers the expected productivity (as $\phi(Z) < Z$ becomes more probable) and thus hampers job creation today and tightens the resource constraint for domestic goods.

To understand the role of the frictional labor market in shaping the optimal fiscal policy, it is instructive to rewrite the constrained-efficient problem (25)-(26) analogously to Proposition 1 that summarizes the problem (22)-(23) associated with the competitive equilibirum. We have the following:[5]

Proposition 4. *Under the repayment scenario, the maximization problem of the government that is consistent with the constrained-efficient allocation in a country featuring exogenous shock Z, previous period's employment l and debt b is described as follows:*

$$\hat{V}^R(Z,b,l) = \max_{c^F,l',g,b'} \left\{ U\left(\left(c^H\left(c^F,l',g\right)\right)^{1-\omega} \cdot \left(c^F\right)^{\omega}, g\right) + \beta \cdot \mathbb{E}\left[\hat{W}\left(Z',b',l'\right)|Z\right]\right\}$$

subject to:

$$p\left(c^F,l',g\right) \cdot exp\left(p\left(c^F,l',g\right)\right) - c^F - i^F\left(c^F,l',g\right)$$

$$= \lambda \cdot b - q\left(Z,b',l'\right) \cdot \left(b' - (1-\lambda) \cdot b\right)$$

From the comparison of Propositions 1 and 4 it is immediate, that the versions of the MSR and Euler Equation that correspond to the problem in Proposition 4 can be obtained from (30) and (31) by setting $\mu = 0$. To put it differently, the decentralized process of job-creation leads to the emergence of the job-creation motive for the government spending that works through changes to current profits

[5] To derive this result it is sufficient to repeat the reasoning underlying the proof of Proposition 1, i.e., to use conditions 3, 4,6, 9, 10, 11, 17, 20 to derive functions $c^H\left(c^F,l',g\right)$, $p\left(c^F,l',g\right)$, and $i^F\left(c^F,l',g\right)$ that are identical to those in Proposition 1.

Table 1: Calibrated parameter values

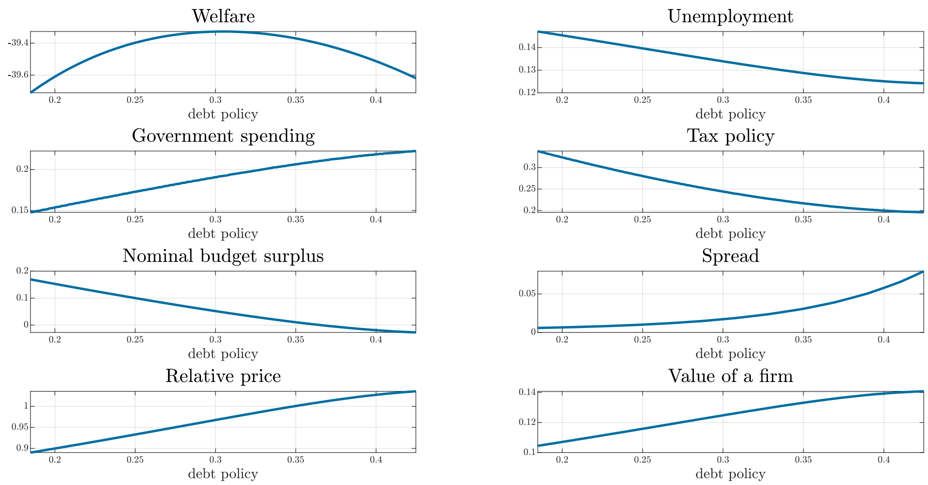
Parameter	Description	Value	Source / Target
r	risk-free rate	0.01	standard value
β	discount rate	0.93	external debt to GDP
κ	vacancy cost multiplier	0.5	unemployment rate
ν	vacancy production cost curvature	1	standard value
γ	elasticity of matching technology	1.5	vacancy-filling rate
$\bar{w}$	nominal wage	0.9	labor share
λ	coupon rate	0.16	average maturity
ω	inverse of the home bias	0.35	imports to GDP
ϵ_{exp}	price elasticity of export demand	4	standard value
ϵ	elasticity of substitution between H and F	1	simplifying computations
σ	coefficient of risk aversion	2	standard value
ψ	public good weight in utility function	0.15	government consumption to GDP
z	export scaling coefficient	0.22	exports to GDP
ζ	probability of the re-entrance	0.12	average autarky spell
μ^{χ}	mean of the utility shock	1.75	average default frequency
σ^{χ}	standard deviation of the utility shock	0.75	average spread

(see condition (30)). At the same time, the job-creation based on the forward-looking entry decision gives rise to the motive that constrains debt issuance as additional debt lowers current output by lowering job creation (see term $\eta \cdot \theta \cdot \frac{\partial \Omega^J}{\partial b'}$ in (31)).

5 Calibration

The period in the model corresponds to one year. Our calibration targets are moments characterizing the Spanish economy - a country that was severely affected by the European debt crisis. Parameter values are reported in Table 1.

Figure 2: Responses of main economic aggregates to deviations of debt issuance from the optimal level



6 Quantitative analysis

6.1 Trade-offs underlying fiscal austerity

To analyze the trade-offs underlying the government's decision about optimal level of debt issuance, we propose the following counterfactual experiment. We analyze the impact of changes to debt issuance (deviations from the optimal level) and, at the same time, we keep future value functions unchanged. Moreover, it is assumed that the remaining variables satisfy implementability constraints.

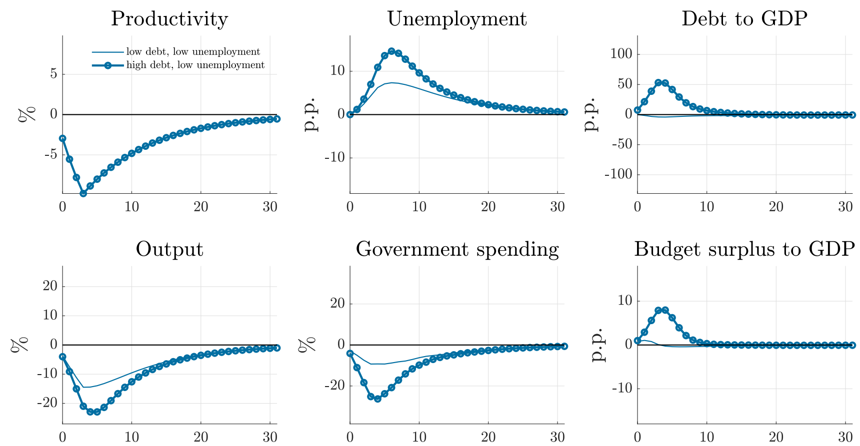
Figure 2 displays the results. It can be inferred that additional debt allows for more government spending, which stimulates aggregate demand and thus increases the price level. This, in turn, raises firm profits, increases its value and boosts job creation. As a result, unemployment level lowers. At the same time, additional debt issuance increases debt service costs which mitigates the desirability of more expansionary fiscal policy.

6.2 Determinants of the optimal fiscal policy response: sovereign debt

This section considers the role of sovereign debt level as a determinant of the optimal fiscal policy response to an adverse macroeconomic shock. To this end, we compare the impulse responses of the economy to a contractionary productivity shock in two cases: first, the economy in which debt level is low and second, in which it is high. Specifically, low debt is defined as the value for which 10% of simulated debt values (when computing the moments characterizing the economy) are lower. High debt is defined as the value for which 90% of simulated debt values are lower. Moreover, it is assumed that the initial unemployment level is low, i.e. it is the value for which 10% of simulated unemployment values are lower. Impulse responses are defined as deviations of macroeconomic aggregates from the transition paths associated with the scenarios undistorted by the productivity shock that start from the low unemployment - low debt and low unemployment - high debt points.

Results are displayed in Figure 3. It can be seen that negative productivity shock implies a more aggressive contraction to government spending in the high-debt economy, which suffers from an insufficient fiscal space needed to absorb the shock. Lower government spending decrease aggregate demand which translate into higher unemployment and, as a result, leads to a collapse in output. This, in turn, implies a large increase in debt-to-GDP in the high debt economy that occurs despite the undertaken austerity measures. The difference between impulse responses between high debt and low debt scenarios (both characterized by low unemployment) is significant. Thus, the debt level seems to be an important determinant of the optimal fiscal policy response to adverse macroeconomic conditions, which corroborates the findings by Bianchi et al. (2021).

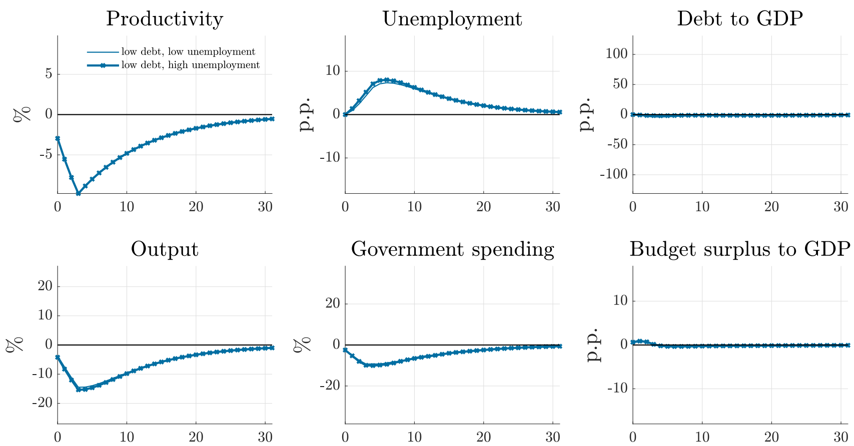
Figure 3: Negative productivity shock and the fiscal policy adjustment: the role of the sovereign debt level



6.3 Determinants of the optimal fiscal policy response: unemployment

This section considers the role of unemployment (economic slack) level as a determinant of the optimal fiscal policy response to an adverse macroeconomic shock. To this end, we compare the impulse responses of the economy to a contractionary productivity shock in two cases: first, the economy in which unemployment is low and second, in which it is high. Specifically, low unemployment is defined as the value for which 10% of simulated unemployment values (when computing the moments characterizing the economy) are lower. High unemployment is defined as the value for which 90% of simulated unemployment values are lower. Moreover, it is assumed that the initial debt level is low, i.e. it is the value for which 10% of simulated debt values are lower. Impulse responses are defined as deviations of macroeconomic aggregates from the transition paths associated with the scenarios undistorted by the productivity shock that start from the low unemployment - low debt and high unemployment - low debt points.

Figure 4: Negative productivity shock and the fiscal policy adjustment: the role of unemployment level

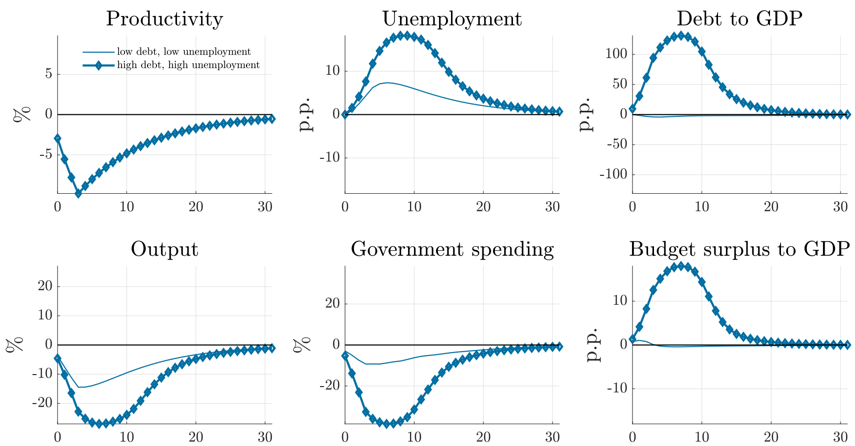


Results are displayed in Figure 4. It can be seen that, unlike debt level, unemployment barely affects the response of fiscal policy to an adverse productivity shock. This implies that, from the quantitative perspective, economic slack analyzed by Michaillat and Saez (2019) does not play a substantial role in shaping the optimal fiscal policy response to a recession.

6.4 Determinants of the optimal fiscal policy response: the joint role of debt and unemployment

This section considers the joint role of unemployment and debt as determinant of the optimal fiscal policy response to an adverse macroeconomic shock. To this end, we compare the impulse responses of the economy to a contractionary productivity shock in two cases: first, the economy in which both debt and unemployment is low and second, in which both values are high. Low and high values are defined as in the previous subsections. Impulse responses are defined as deviations of macroeconomic aggregates from the transition paths associated with the scenarios undistorted by

Figure 5: Negative productivity shock and the fiscal policy adjustment: the role of the interaction between unemployment and sovereign debt



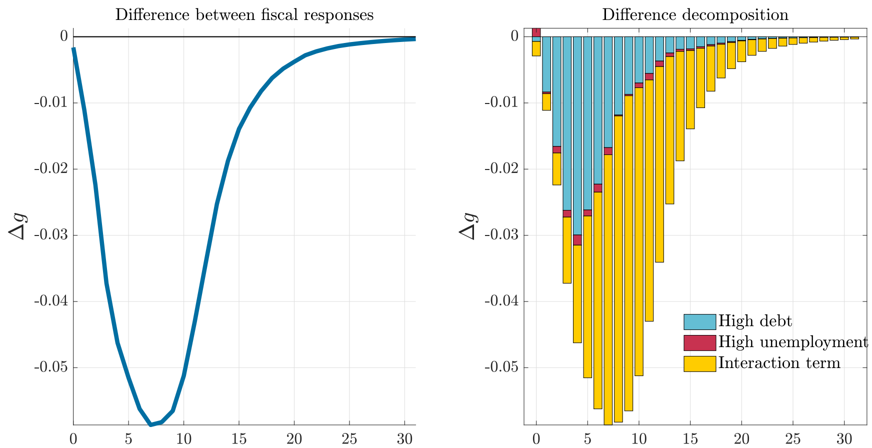
the productivity shock that start from the low unemployment - low debt and high unemployment - high debt points.

Figure 5 shows the results. We can see that the interaction between high unemployment and high debt has a dire consequences for the response of the economy to a negative productivity shock. This results from the inability of fiscal policy to counteract the recession when both unemployment and debt are high.

6.5 Relative importance of debt, unemployment and their interaction in shaping the optimal response of fiscal policy

In this section we summarize the experiments conducted in subsections 6.2-6.4. In particular, we decompose the difference in reactions of government spending to the crisis between the scenario featuring high initial levels of debt and unemployment and the scenario when those levels are low. Specifically, this change in fiscal reaction is divided into three parts. The first measures the contribution of the difference in the

Figure 6: Decomposition of the difference in reactions of government spending to an adverse productivity shock between the scenario featuring high initial levels of debt and unemployment and the scenario featuring low initial levels of debt and unemployment



initial debt levels and is obtained by subtracting the two impulse response functions displayed in Figure 3. The second measures the contribution of the difference in the initial unemployment levels and is obtained by subtracting the two impulse response functions displayed in Figure 4. The third measures the impact of the interaction between high unemployment and high debt on the fiscal policy response to the crisis and is given by the difference between the impulse response associated with the high debt - high unemployment scenario displayed in Figure 5 and the sum of the first and second component.

Figure 6 shows the results. It turns out that while the role of economic slack in shaping the optimal fiscal policy response to the crisis is rather negligible, the impact of the sovereign debt level and the interaction between high unemployment and high debt give rise to strong motives that drive the austerity as an optimal response to adverse macroeconomic conditions.

7 Conclusions

This paper examines the optimal fiscal policy in the economy plagued by high unemployment and sovereign debt. Employing a model with a frictional labor market and sovereign default risk, we apply analytical and numerical methods to study the reaction of the fiscal authority that uses government spending, taxes and debt issuance to mitigate the crisis. Higher expenditures reduce unemployment and have a positive and permanent impact on the tax base, but, at the same time, they increase the exposure to sovereign default risk. We calibrate the model to match the moments characterizing the Spanish economy and quantify those two opposite forces. We find that high debt is a critical concern the government faces, even if a substantial amount of unemployed resources gives rise to high spending multipliers. Consequently, fiscal austerity is the policy recommendation for the economy with high unemployment and debt.

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Appendix

Proof of Proposition 1

Proof. Let us order all the relevant equations (implementability constraints in the maximization problem associated with the repayment scenario) that constrain the government under the repayment scenario in period t:

$$C_t = \left(c_t^H\right)^{1-\omega} \cdot \left(c_t^F\right)^{\omega} \tag{32}$$

$$p_t \cdot c_t^H + c_t^F + p_t \cdot \tau_t = w_t \cdot l_t + \pi_t \tag{33}$$

$$p_t = \frac{1-\omega}{\omega} \cdot \frac{c_t^F}{c_t^H} \tag{34}$$

$$P_t = \frac{1}{\omega^{\omega} \cdot (1-\omega)^{1-\omega}} \cdot p_t^{1-\omega} \tag{35}$$

$$l_t = (1-\delta) \cdot l_{t-1} + \theta_t \cdot v_t \tag{36}$$

$$x_t = \frac{v_t}{1-(1-\delta) \cdot l_{t-1}} \tag{37}$$

$$\theta_t = \frac{1}{(1+x_t^{\gamma})^{\frac{1}{\gamma}}} \tag{38}$$

$$\kappa^{\frac{1}{\nu}} \cdot v_t^{\frac{1}{\nu}-1} \geq \theta_t \cdot J_t \tag{39}$$

$$\pi_t = \left(a_t \cdot p_t - w_t - P_t \cdot (\kappa \cdot v_t)^{\frac{1}{\nu}}\right) \cdot l_t \tag{40}$$

$$a_t \cdot l_t = c_t^H + g_t + exp_t + i_t^H \tag{41}$$

$$exp_t = z_t \cdot (p_t)^{-\epsilon_{exp}} \tag{42}$$

$$p_t \cdot exp_t - c_t^F - i_t^F = \lambda \cdot b_t - q_t \cdot (b_{t+1} - (1-\lambda) \cdot b_t) \tag{43}$$

$$w_t = \bar{w} \tag{44}$$

$$i_t^H = \frac{1-\omega}{\omega^\omega \cdot (1-\omega)^{1-\omega}} \cdot \frac{i_t}{p_t^\omega}, \tag{45}$$

$$i_t^F = \frac{\omega}{\omega^\omega \cdot (1-\omega)^{1-\omega}} \cdot p_t^{1-\omega} \cdot i_t, \tag{46}$$

$$J_t \equiv a_t \cdot \frac{p_t}{P_t} - \frac{w_t}{P_t} + \Omega_t \tag{47}$$

$$i_t = \kappa^{\frac{1}{\nu}} \cdot v_t^{\frac{1}{\nu}} \tag{48}$$

In what follows, we show that the information contained in equations (32)-(48) can be summarized with the constraints in Proposition 1 and functions $c^H\left(c^F, l', g\right)$, $p\left(c^F, l', g\right)$, $P\left(c^F, l', g\right)$.

First, observe that equation (18) was replaced with (20) in the system of implementability constraints (32)-(48). This is because (18) can be derived from equations (33), (40), (41), (43). This leaves us with a system of 17 equations with 19 variables (C_t,c_t^H,c_t^F, p_t, τ_t, w_t, l_t, π_t, P_t, θ_t, v_t, x_t, g_t, exp_t, i_t, b_{t+1}, J_t, i_t^H, i_t^F).

Second, note that the values of variables π_t and τ_t can be obtained from equations (33) and (40) if we know the remaining variables' values. This implies that the knowledge of π_t and τ_t is not essential for calculating C_t,c_t^H,c_t^F, p_t, w_t, l_t, P_t, θ_t, v_t, x_t, g_t, exp_t, i_t, b_{t+1}, J_t, i_t^H, i_t^F. Thus, we can eliminate both (33) and (40) and π_t and τ_t which leaves us with 15 equations and 17 variables.

Third, equations (36)-(38) can be combined to get:

$$l_t = (1-\delta)\cdot l_{t-1} + \frac{1}{\left(1+\left(\frac{v_t}{1-(1-\delta)\cdot l_{t-1}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\cdot v_t$$

which establishes an implicit relationship $v\left(l_t\right)$. Then, we use: (37) and $v\left(l_t\right)$ to define $x\left(l_t\right)$, (38) and $x\left(l_t\right)$ to define function $\theta\left(l_t\right)$. All this means that we used 3 equations (i.e., (36)-(38)) to eliminate 3 variables (i.e. x_t, v_t and θ_t).

Fourth, we use condition (48) and the fact that $v_t = v\left(l_t\right)$ to define $i\left(l_t\right)$ which implies that the information described by (48) is exploited, which reduces the set of the relevant equations and variables by one.

Fifth, note that combining (34), (35), (41), (42), equation (45) and the observation that $i_t = i\left(l_t\right)$ allows for obtaining the following equation:

$$p_t = \frac{1-\omega}{\omega}\cdot\frac{c_t^F}{a_t\cdot l_t - g_t - z_t\cdot\left(p_t\right)^{-\epsilon_{exp}} - \frac{1-\omega}{\omega^{\omega}\cdot(1-\omega)^{1-\omega}}\cdot\frac{i(l_t)}{p_t^{\omega}}}$$

which implicitly defines a mapping $p\left(c^F, l', g\right)$. For convenience, we also define:

$$exp\left(p_t\right) = z_t\cdot\left(p_t\right)^{-\epsilon_{exp}}$$

$$\Longrightarrow exp\left(p\left(c^F, l', g\right)\right) = z_t\cdot\left(p\left(c^F, l', g\right)\right)^{-\epsilon_{exp}}$$

Then, using $p\left(c^F, l', g\right)$ and $exp\left(p\left(c^F, l', g\right)\right)$ we obtain $P\left(c^F, l', g\right)$ and $c^H\left(c^F, l', g\right)$ from (35) and (41), respectively. This implies that we exploited (34), (35), (41), (42), (45) to get functions $p\left(c^F, l', g\right)$, $exp\left(p\left(c^F, l', g\right)\right)$, $P\left(c^F, l', g\right)$, $c^H\left(c^F, l', g\right)$, $i^H\left(c^F, l', g\right)$ and from now those equations can be ignored.

Sixth, we use condition (44) to substitute for w_t/P_t in equation (39) which, effectively, after using the above-mentioned observations and by denoting $l' = l_t$ and $b' = b_{t+1}$, and using equation (46) leaves us with two conditions:

$$p\left(c^F, l', g\right) \cdot exp\left(p\left(c^F, l', g\right)\right) - c^F - \omega \cdot P\left(c^F, l', g\right) \cdot i\left(l'\right)$$

$$= \lambda \cdot b - q\left(Z, b', l'\right) \cdot \left(b' - (1-\lambda) \cdot b\right)$$

$$\theta\left(l'\right) \cdot \left(a \cdot \omega^{\omega} \cdot (1-\omega)^{1-\omega} \cdot \left(p\left(c^F, l', g\right)^{\omega}\right) - \frac{w}{P\left(c^F, l', g\right)} + \Omega^J\left(Z', b', l'\right)\right) \leq \kappa^{\frac{1}{\nu}} \cdot v\left(l'\right)^{\frac{1}{\nu}-1},$$

which we wanted to show. □

Proof of Corrollary 3

Proof. Equation (31) is the first order condition associated with b' for the problem in Proposition 1 combined with the envelope condition related to b. The only non-trivial part of the proof is to show that $\frac{\partial}{\partial b'}\mathbb{E}\left[W\left(Z', b', l'\right)|Z\right] = \mathbb{E}\frac{\partial V^R}{\partial b'} \cdot (1 - D')$ (term $\frac{\partial V^R}{\partial b'}$ is the reformulated using the envelope condition). To simplify the exposition, let us assume that the space for Z' is continuous and that densities $f_Z\left(Z'|Z\right)$ and $f_\chi\left(\chi'\right)$ exist. We have:

$$\frac{\partial}{\partial b'}\mathbb{E}\left[W\left(Z', b', l'\right)|Z\right]$$

$$= \frac{\partial}{\partial b'}\left[\int_{\mathcal{R}(b', l')} V^R\left(Z',\, b',\, l'\right) f_Z\left(Z'|Z\right) f_\chi\left(\chi'\right) dZ' d\chi'\right]$$
$$+ \frac{\partial}{\partial b'}\left[\int_{\mathcal{D}(b', l')} \left\{V^D\left(Z',\, l'\right) - \chi'\right\} f_Z\left(Z'|Z\right) f_\chi\left(\chi'\right) dZ' d\chi'\right] \tag{49}$$

where $\mathcal{R}\left(b', l'\right)$ is the set of elements $\left\{Z', \chi'\right\}$ for which:

$$V^R\left(Z',\, b',\, l'\right) > V^D\left(Z',\, l'\right) - \chi'$$

given the values of b' and l', i.e. it is a repayment region. Term $\mathcal{D}(b', l')$ denotes the complementary (default) region composed of elements $\{Z', \chi'\}$ for which:

$$V^R(Z', b', l') \leq V^D(Z', l') - \chi'.$$

To differentiate the terms in square bracketts in (49) we use the multidimensional version of the Leibnitz integration rule (see Flanders (1973)) to rewrite (49) as:

$$\begin{aligned}
&\int_{\mathcal{R}(b',l')} \frac{\partial}{\partial b'} V^R(Z', b', l') f_Z(Z'|Z) f_\chi(\chi') dZ' d\chi' \\
&+ \int_{\partial\mathcal{R}(b',l')} V^R(Z', b', l') f_Z(Z'|Z) f_\chi(\chi') \left[\mathcal{R}^\chi dZ' - \mathcal{R}^Z d\chi'\right] \\
&+ \int_{\mathcal{D}(b',l')} \frac{\partial}{\partial b'} \left\{V^D(Z', l') - \chi'\right\} f_Z(Z'|Z) f_\chi(\chi') dZ' d\chi' \\
&+ \int_{\partial\mathcal{D}(b',l')} \left\{V^D(Z', l') - \chi'\right\} f_Z(Z'|Z) f_\chi(\chi') \left[\mathcal{D}^\chi dZ' - \mathcal{D}^Z d\chi'\right].
\end{aligned} \tag{50}$$

where $\partial\mathcal{R}(b', l')$ is the boundary of region $\mathcal{R}(b', l')$ and $\mathcal{R}^\chi dZ' - \mathcal{R}^Z d\chi'$ is a differential one-form describing the "speed" at which region $\mathcal{R}(b', l')$ expands as b' increases (the "speed" is projected on the coordinates Z' and χ' using factors $\mathcal{R}^\chi$ and $\mathcal{R}^Z$). Objects $\partial\mathcal{D}(b', l')$, $\mathcal{D}^\chi$, and $\mathcal{D}^Z$ are defined analogously.

Now, given that the expansion of region $\mathcal{R}(b', l')$ is equivalent to a contraction of region $\mathcal{D}(b', l')$, we have: $\mathcal{R}^\chi = -\mathcal{D}^\chi$ and $\mathcal{R}^Z = -\mathcal{D}^Z$ which, coupled with the fact that on the boundary $\partial\mathcal{R}(b', l') = \partial\mathcal{D}(b', l')$ the terms under the integral sign are equal:

$$V^R(Z', b', l') = V^D(Z', l') - \chi',$$

implies that terms two and four in the sum (50) cancel out. Term three is zero because:

$$\frac{\partial}{\partial b'}\left\{V^{D}\left(Z',\, l'\right)-\chi'\right\}=0.$$

Given that $D\left(\chi', Z', b', l'\right)=0$ for the elements of $\mathcal{R}\left(b', l'\right)$, and $D\left(\chi', Z', b', l'\right)=1$ outside that region, then the first term in sum (50) can be rewritten as:

$$\int_{\mathcal{R}(b',l')}\frac{\partial}{\partial b'}V^{R}\left(Z',\, b',\, l'\right)f_{Z}\left(Z'|Z\right)f_{\chi}\left(\chi'\right)dZ'd\chi'$$

$$=\int_{\mathbb{R}^2}\frac{\partial}{\partial b'}V^{R}\left(Z',\, b',\, l'\right)\cdot\left(1-D\left(\chi', Z', b', l'\right)\right)dZ'd\chi'$$

$$=\mathbb{E}\frac{\partial V^{R}}{\partial b'}\cdot\left(1-D'\right)$$

which we wanted to show. □