EXPERIMENTATION AND LEARNING UNDER COMPETITIVE SEARCH*

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Abstract. We examine contracts in performance-centric professions, such as entrepreneurship, executive leadership, and scientific research, where matches between workers and firms are an experience good and parties face friction in the matching process. Both parties learn about match productivity through experimentation. We show that if the firms can offer fully flexible wages, the competitive search equilibrium is efficient, i.e. the equilibrium number of vacancies and the experimentation duration are the same as what a planner chooses to maximize output net of search cost. Under fixed-wage contracts, if the workers serve their entire contract in equilibrium, the competitive search equilibrium is efficient. However, if the search costs are sufficiently high, then the equilibrium wages decline, the workers quit before the contract ends, and the experimentation is sub-optimal. Finally, we highlight the importance of firms' commitment to contract duration and common priors for efficiency.

1. Introduction

In many economic environments, finding a suitable partner for an economic relationship is subject to search frictions. Models that capture search frictions explain various empirical regularities of labor, financial, and other markets. An important feature of bilateral matching markets is that there is often a lot of uncertainty about bilateral match productivity and parties learn about it through experimentation. If the relationship is not productive either party can walk away and end the relationship. Do the parties persist in experimentation long

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enough or do they dissolve their partnership too soon? How does the answer to this question depend on the contracts between the parties? In this paper, we address these questions in an environment with competitive search.

One prominent example of the model we study is performance-driven professions, such as entrepreneurship, executive leadership, and scientific research. Such professions are inherently risky for the workers and the principals (hereafter, firms) employing them. Both sides are uncertain whether the association will be successful. They learn about match productivity through ongoing experimentation during the initial period of employment. A string of bad signals may make them pessimistic about the likelihood of success until either one or both parties dissolve the match. Once a match is dissolved, finding a new match may be costly due to search frictions in the labor market. The duration of experimentation and who dissolves the match, the worker or the firm, depends crucially on the contract between the parties which depends on contractual and search frictions. Hence, our model can explain changes in endogenous tenure length based on observable features of the environment.

For example, consider the case of CEO tenure. By 2022, the median tenure of CEOs in the S&P500 companies declined by 20% compared to 2013, with the modal tenure being 4-5 years (39% of CEOs).[1] Scientific careers have also shortened. As of 2010, it takes 5 years for half of the cohort of young scientists to drop out, as opposed to 35 years in the 1960s, Milojević et al. (2018). Our model provides a framework to analyze such endogenous tenure and how it may be affected by contract and matching frictions.

Our model has applications beyond labor markets, for example to patent technology licensing. Companies developing a new product are often looking for suitable technologies and commonly license patents from other patent owners for a fixed fee. These licenses are extended or bought only if the match is successful, otherwise, the company may search for

[1]See `https://corpgov.law.harvard.edu`

other suitable technologies. Either party could choose to break the relationship at any time. Such licensing contracts are common in pharmaceutical industry and in the development of cutting-edge software technology, such as in machine learning and artificial intelligence. Another potential application is to patient-physician relationships. Both the patient and the physician must decide through experimentation whether to persist with the current match or explore alternative treatment options from another physician. Other applications include venture capitalist firm matching with entreprenuers, companies matching with consultancy firms, and consumers matching with subscription service.

In our model, workers (or matches) can be of two types: productive or unproductive. Only productive workers are capable of producing stellar output (or succeed). There is symmetric imperfect information about the worker's type, i.e. workers and firms share a common prior over worker's productivity. The output is publicly observable. If the worker produces stellar output, then all market participants know the worker's type. If the worker does not produce stellar output, then both parties update their beliefs about the likelihood of success, and the expected payoff from continuation declines. The parties may cease experimentation and dissolve the match if there is no success. We are interested in the initial learning period when the uncertainty about match productivity is still unresolved. Therefore, we model the surplus-sharing arrangements after the productivity revelation in a stylized manner. We assume that a portion of the future match surplus accrues to the worker depending on their exogenous bargaining power.

We account for the matching frictions through a matching function and search costs. Workers earn a safe outside option wage when not employed by the firm. Firms incur search costs when they are searching for a match. Firms post publicly observable contracts that specify the transfers to the workers, and workers direct their search effort to the most lucrative contracts. We assume free entry by firms. That is, if firms make positive profit by posting a certain contract, more firms post this contract, leading to a shorter queue length (or higher search cost for the firms) and leads to zero profit in equilibrium. The contract, the experimentation duration, and the measure of searching workers and firms are determined in equilibrium.

One of our goals is to compare equilibrium outcomes with the efficient benchmark which we define as one where a planner chooses the number of vacancies and the experimentation duration to maximize the total output net of dead-weight search cost. First, note that as the experimentation continues without a breakthrough, the likelihood of success declines. Therefore, there is an optimal experimentation duration beyond which continuing experimentation reduces the expected value of the match. The optimal experimentation duration is longer if the ex-ante likelihood of success is higher, or the total output from a successful match is higher. Second, given the expected match value under optimal tenure, the planner chooses the number of vacancies to maximize the expected flow output net of the search cost. If there are too few vacancies relative to the searching workers (long queue), then the workers spend most of their time being unemployed and there is potential loss of output as the expected output from the match is higher than their unemployment income. On the other hand, if there are too many vacancies, then the firms end up searching longer increasing the search cost. The number of vacancies (or equivalently, the queue length) and the experimentation duration chosen by the planner constitute the efficient outcome.

Suppose first that firms can post contracts with a fully flexible wage schedule.[2] We show that such contracts can be fully summarized by the total income promised to the worker, and an expiration time at which the relationship ends if the worker does not succeed by that time. Each contract is associated with a queue of workers who apply to that contract. Workers prefer contracts with a higher income promise and a shorter queue length, leading to indifference curves for the workers in the income/queue length space. Firms prefer contracts

[2]Such a compensation scheme eliminates concerns about the worker quitting because the firm can just adjust the compensation so that the worker earns just slightly more than his outside option payoff. Moreover, it subsumes compensation schemes involving startup and exit bonuses.

with a lower income promise and a longer queue of workers leading to isoprofit curves for the firms in the same space. Because of free entry, firms must make zero profit. In equilibrium, workers indifference curve is tangent to the isoprofit curve of the firms that gives zero profit. Using this construction, we show that a decentralized equilibrium with fully flexible wages exists and the equilibrium is efficient. That is, equilibrium experimentation duration and the queue length are the same as the planner's solution. The intuition for this result is similar to the intuition of the efficiency of competitive search in Moen (1997), which is closely related to the efficiency of competitive equilibrium.

Firms may find it costly to administer such flexible wage contracts, or there may be institutional limitations on the rate at which wages can grow during employment. We account for such contractual frictions by analyzing fixed-wage contracts. For fixed wage contracts whether the firm can commit to the duration of the contract or not becomes important and we consider both cases.[3]

With commitment, the contract specifies the constant wage and the duration for which the worker earns this wage if she does not produce stellar output. Note that the contract duration is relevant only if the worker does not quit before the contract term ends. The worker quits if the continuation payoff from employment falls below his outside option payoff. The worker's continuation payoff and hence the preferred quitting time is increasing in the wage. For any contract duration, there is a wage threshold below which the worker quits before the contract expires. The firm internalizes this quitting decision and posts contracts guaranteeing a fixed wage for a set duration. The wage, experimentation duration, and queue length are determined in equilibrium.

[3]We assume that workers do not commit to the duration of the contract. If they did, then the analysis is the same as the fully flexible wage case, where the contracts offer a promised income, and the experimentation duration is optimal. We ignore this case because such contracts are rarely observed in practice and workers are generally free to opt out of employment.

Next, we prove the existence of competitive search equilibrium with fixed-wage contracts. There are two kinds of equilibria. When search costs are below a threshold, firms offer contracts such that at the equilibrium wage workers prefer to continue beyond the contract duration. We refer to this type of equilibrium as a *tenure equilibrium.* When search costs are sufficiently high, firms offer contracts such that at the equilibrium wage workers quit, and at the time of quitting, the firms would have preferred to retain the worker. We refer to this type of equilibrium as a *quitting equilibrium.*

Our next result comprises of two parts. The first part proves that under a tenure equilibrium, the queue length and the duration of experimentation are efficient. The result rests on the fact that the firms can commit to the contract duration. If the contract duration is shorter than the efficient one, firms can deviate and commit to a slightly longer contract duration and offer a slightly lower wage and both the deviating firms and the workers would be better off. Therefore, in equilibrium the offered contract duration is optimal. With the optimal contract duration, the indifference curve of the worker, and the iso-profit curve of the firm in the wage, queue length space must be tangent to each other. The argument for tangency is the same as discussed earlier. Therefore, the tenure equilibrium is efficient. This result provides a rationale for the tenure contracts commonly offered to academic researchers and professional athletes.[4]

Second, the quitting equilibrium always results in an inefficient number of vacancies, and sub-optimal experimentation. The inefficiency rests on the fact that the worker is unable to commit to staying with the firm. The workers will quit employment before the contract expires if the offered wage is below the threshold wage for the contract duration. Therefore, there is a threshold wage corresponding to the optimal experimentation duration such that

[4]The rookie contracts in the NBA offer a fixed salary schedule based on the position of the player in the draft lottery. These contracts are continued without negotiation for a period of 4-5 years. Successful rookies generally turn free agents at the end of this period to negotiate a larger compensation for themselves.

for any wage below this threshold, the worker's preferred quitting time precedes the optimal duration. The firms internalize this and post contracts. If the wage at the tangency between the worker's indifference curve and the firm's iso-profit curve is below the threshold wage, then the equilibrium features workers quitting employment. Since the equilibrium experimentation duration is shorter than the optimal, the quitting equilibrium is not efficient.

Next, we consider the case in which firms cannot commit to the contract duration or such a commitment cannot be enforced legally. In this case, there is a threshold wage at which both parties prefer to dissolve the match simultaneously. Below this threshold wage, the worker's payoff is so low that he prefers to quit before the firm can fire him. Similarly, if the wage is above the threshold, then the worker is willing to continue experimentation, however, the firm prefers to fire the worker because its continuation payoff is lower. It turns out the equilibrium wage can be above or below the threshold and is inefficient except in the knife edge case where it is equal to the threshold.

To gain some intuition for this result, we can draw an analogy with a two-player version of the two-arm bandit problem. The firm and the worker each pull either a risky or a safe arm. If they both pull the risky arm then they continue experimentation with the chance of a windfall gain. If either one pulls the safe arm the worker takes the outside option, and the firm goes back to recruitment. If the fixed wage is below a threshold, the proportion of the expected surplus that accrues to the worker is too low, and the worker pulls the same arm (i.e.quits) sooner than optimal. Similarly, when the fixed wage is above the threshold, the proportion of the expected surplus that accrues to the firm is too low, and the firm pulls the safe arm (i.e. fires the worker) sooner than optimal. At the unique threshold wage, the firm and the worker split the surplus in such a way that the experimentation duration is optimal. The equilibrium features sub-optimal experimentation as long as the equilibrium wage does not coincide with this threshold wage.

We can rationalize the observed declining trend in scientific career duration along with the increasing proportion of early-career researchers who never graduate to chief investigators, Milojević et al. (2018). If in our model the windfall gain from a successful experiment declines, then there are fewer vacancies, however, the proportion of junior researchers increases, and a larger proportion of the cohort ends up quitting before achieving success for a given fixed wage. If the equilibrium wages do not increase sufficiently, then the average duration of a scientific career declines with more researchers stuck in the early phase of their careers.

Finally, we discuss the implications of mandated minimum wage, or worker liquidity constraints that may prevent the firm from offering very low initial wages. We show that the workers are weakly better off, as long as the minimum wage is below the threshold wage. If the minimum wage is above the threshold, the experimentation duration is shorter than optimal as firms, unable to commit to the tenure, prefer to fire workers sooner. As a result, the workers may be worse off as their post-match payoff declines because they get a shorter trial period to prove themselves. We also point out that efficiency also rests on the common prior assumption. We show that if the workers and firms have different priors, with only one of them having the correct belief, the equilibrium experimentation may no longer be optimal even if the firms can offer fully flexible wages. If either of the parties is excessively optimistic or pessimistic about match productivity, even fully flexible wage schedules cannot lead to optimal experimentation and efficiency in equilibrium.

Related Literature: This paper builds upon the search and matching framework for wage determination as in Pissarides (1984). Rogerson et al. (2005) provide an excellent survey of the search and matching literature. We employ a directed competitive search model as in Moen (1997), however, the match productivity in our case is not deterministic. We show that the efficiency result in Moen (1997) extends to dynamic environments where match productivity is uncertain and there is learning only if firms can offer fully flexible wage schedules. If there are contractual frictions, the competitive search equilibrium is not necessarily efficient due to sub-optimal experimentation and learning.

Jovanovic (1979) also studies matches as an experience good. He focuses on fully flexible wages and provides results on separation probabilities over the tenure of a match. His model does not have search and contractual frictions and does not address optimality of experimentation and excessive turnover. Harris and Holmstrom (1982) focus on the growth in wages observed during employment due to more precise information about the productivity of risk-averse workers. The wages may be increasing in our model albeit with risk-neutral workers. In our model, as the workers become more pessimistic the firms compensate them with a higher wage so as to continue experimentation.

Our model is also related to the literature that deals with self-selection in labor markets as in Salop and Salop (1976) and Guasch and Weiss (1981) where the high-type workers may signal their type by either paying for a screening test, or agreeing to a lower pay initially that gradually increases over the tenure of the job. As mentioned earlier, firms in our model would also pay a lower wage to workers when the match productivity is more likely to be high and this wage increases over time.

The optimal experimentation in our model naturally lends itself to applications in innovation financing. The literature on entrepreneurial financing has shown that a common structure of such financing involves a period of unconditional financing despite low productivity, followed by a period of conditional financing, Ewens et al. (2018), Ewens et al. (2020). Our model where the firm continues the experimentation even as the worker remains unproductive seems to fit this stage of entrepreneurial financing where the monitoring and governance costs are high. In our model, a planner would prefer a longer match duration than what may result under a fixed-wage competitive equilibrium. There is empirical evidence provided by Ederer and Manso (2011) where extending this initial period of unconditional financing increases innovation, Acharya et al. (2014) who show that increasing dismissal costs increase match duration, and hence innovation, Tian and Wang (2014) show that innovation increases as the tolerance of VC firms for this initial period of low productivity increases. In our model such tolerance increases if the initial likelihood of success is higher or firms face high search costs.

Lastly, our model is related to the literature on strategic experimentation with multiple agents (e.g. Keller et al. (2005) and Keller and Rady (2010)). In those papers, the focus is on free-riding in experimentation with multiple agents, when experimentation is costly and agents observe the results of others' actions. Gieczewski and Kosterina (2023) also study experimentation with multiple agents. Their model features a learning process similar to ours. If agents do not observe success, they become pessimistic over time and can exit. There are several differences between their model and ours. They restrict attention to voting as a decision rule for experimentation and do not allow transfers between agents. In our model, experimentation continues only if both agents choose to do so and they can enter into contracts stipulating how to split the surplus. In addition, their model does not have search frictions. Hoppe-Wewetzer et al. (2019) also looks at learning and experimentation with multiple agents. Unlike us, their game has a winner-take-all structure and they focus on preemption motives in experimentation.

2. Model

Time is continuous. There are two kinds of agents: workers and firms. The workers and the firms discount their future income at the rate ρ.

2.1. **Workers.** The measure of the continuum of workers is normalized to 1. The workers earn b_0 flow payoff per unit of time when they are not working for a firm. The workers may either be productive (p) or unproductive (u). Productive workers working for a firm produce a stellar output $\overline{y}$ at a Poisson rate λ and produce regular output $\underline{y}$ ($< \overline{y}$) otherwise.[5] Unproductive workers working for a firm always produce $\underline{y}$. We can also interpret the types

[5]In the text, we sometimes refer to stellar and regular outputs as high and low.

as match quality instead of inherent worker types. A successful match would produce a high output at the rate λ and an unsuccessful one always produces a low output. We assume that it is socially efficient for an unproductive worker to accept the outside option and the productive worker to be employed by a firm:

$$\underline{y} < b_0 < \underline{y} + \lambda \overline{y}. \tag{1}$$

Workers are born and die at a rate δ. If the workers quit employment or are fired, they continue to earn b_0 flow payoff. Hence a worker's continuation value after leaving employment is $v_0 = b_0/(\rho + \delta)$.

2.2. **Information and Learning.** The probability that a worker is productive is $\mu \in (0, 1)$ and this belief is common to workers and firms. When a worker is matched with a firm, his output is observable. A worker who produces $\overline{y}$ learns that he is productive and so does the firm. Moreover, the common belief that the worker is productive after working at the firm for time t having never produced $\overline{y}$ is given by

$$\gamma(\mu, t) = \frac{\mu e^{-\lambda t}}{(1 - \mu) + \mu e^{-\lambda t}} = \frac{\mu}{\mu + (1 - \mu) e^{\lambda t}}$$

Since $\gamma(\mu, t)$ is decreasing in t, over time, both the worker, and the firm employing the worker, become less confident that the worker is of type p if the worker does not produce $\overline{y}$.

2.3. **Contracts.** There is a continuum of ex-ante homogeneous firms. A subset of these firms enter the recruitment market and incur flow cost κ while searching. Firms post vacancies with contracts $\boldsymbol{\omega} \equiv \{w(t)\}_{t=0}^{\infty}$ that specify the transfer from the matched firm to the worker at each time t. The expected payoff of a firm from matching with a worker who is productive with probability μ under contract $\boldsymbol{\omega}$ is $V(\mu, \boldsymbol{\omega})$ and that of the worker is $v(\mu, \boldsymbol{\omega})$.

2.4. **Bargaining.** Under any contract, if at any time t, the worker produces $\overline{y}$, then both the firms and the worker update their beliefs about the worker and learn that the worker is productive with probability 1. At this point, the worker's wage is determined through Nash bargaining with the worker's bargaining power β. We assume that if the negotiation breaks down without an agreement, the productive worker's disagreement payoff is v_p ($\geq v_0$). We take v_p as exogenously specified in the main paper. In Appendix B.1 we provide micro-foundations for endogenizing v_p where we assume that in case of disagreement, workers' continuation value is determined through competitive search in a market where all workers are productive. Throughout the paper we assume free-entry, therefore the firms' disagreement payoff is 0. The total surplus from a productive match is $(\lambda\overline{y} + \underline{y})/(\rho + \delta)$. Hence, the worker's continuation payoff after the stellar output is,

$$v_h = v_p + \beta \left(\frac{\lambda\overline{y} + \underline{y}}{\rho + \delta} - v_p \right) \tag{2}$$

and the firm's continuation payoff is,

$$V_h = (1 - \beta) \left(\frac{\lambda\overline{y} + \underline{y}}{\rho + \delta} - v_p \right) \tag{3}$$

2.5. **Directed Search.** When a firm enters the recruitment market, it posts a contract $\boldsymbol{\omega}$ and commits to the transfers specified in the contract. All workers can see the contracts posted by all the firms. We assume that workers direct their search to one of the posted contracts.

Matching is bilateral and therefore an agent contacts at most a single firm. We capture the search frictions in forming bilateral matches using a matching function. For u unemployed workers looking for a job and v firms searching for workers, the matching rate is given by a matching function $m(u, v)$ which is concave and homogeneous of degree 1. The queue length is $q = u/v$ and the market tightness is $1/q$. Under directed search, the queue length, $q(\boldsymbol{\omega})$ is endogenous and depends on all posted contracts. Define the transition rate from unemployed to employed by

$$\alpha_u(q) = \frac{m(u, v)}{u} = m\left(1, \frac{v}{u}\right) = m(1, q^{-1})$$

$\alpha_u(q)$ is decreasing in q, i.e. if the queue is longer, workers transition less rapidly to employment. Similarly define the arrival rate of workers to a vacancy by

$$\alpha_v(q) = \frac{m(u,v)}{v} = m\left(\frac{u}{v}, 1\right) = m(q, 1)$$

$\alpha_v(q)$ is increasing in q, i.e. if the queue is longer, the firm fills its vacancy faster. Note that, $\alpha_v(q) = q\alpha_u(q)$. The continuation payoff of a firm, $V^u(\mu, \boldsymbol{\omega}, q)$, from posting a contract $\boldsymbol{\omega}$ and facing a queue length q is

$$\rho V^u(\mu, \boldsymbol{\omega}, q) = -\kappa + \alpha_v(q)(V(\mu, \boldsymbol{\omega}) - V^u(\mu, \boldsymbol{\omega}, q)) \tag{4}$$

where $V(\mu, \boldsymbol{\omega})$ is the firm's payoff at the instant it is matched with a worker. Similarly, the continuation payoff of a worker, $v^u(\mu, \boldsymbol{\omega}, q)$ from applying to a contract $\boldsymbol{\omega}$ and facing a queue length q is,

$$(\rho + \delta)v^u(\mu, \boldsymbol{\omega}, q) = v_0(\rho + \delta) + \alpha_u(q)(v(\mu, \boldsymbol{\omega}) - v^u(\mu, \boldsymbol{\omega}, q)) \tag{5}$$

where $v(\mu, \boldsymbol{\omega})$ is the worker's payoff at the instant he is matched with a firm.

2.6. **Equilibrium.** Suppose measure v_m of firms post contracts $\boldsymbol{\omega}_m \in \boldsymbol{\Omega}$ with $m \in \{1, 2, \ldots, M\}$. Each worker directs their search based on the posted contracts and the search behavior of other workers. The measure v_m firms with contract $\boldsymbol{\omega}_m$ and the measure u_m workers applying for this contract constitute the sub-market m with $q_m = u_m/v_m$. In equilibrium, the workers must receive the same payoff, $\overline{v}$, from applying to any of the posted contracts. Hence, if $\boldsymbol{\omega}_m$ is a posted contract then $v^u(\mu, \boldsymbol{\omega}_m, q_m) = \overline{v}$. Equation (5) gives us a relationship that governs the queue length $q_m = q(\mu, \boldsymbol{\omega}_m; \overline{v})$. A contract that gives a higher expected payoff to the workers, attracts more workers increasing the queue length until the expected payoff of the workers from any posted contract is equalized across sub-markets.

Three conditions pin down a competitive search equilibrium. First condition requires the firms to post profit-maximizing contracts taking into account the workers' search behavior, i.e. the firms believe that the queue length is $q(\mu, \boldsymbol{\omega}; \overline{v})$. Since firms are homogeneous, in equilibrium they all post the same contract $\boldsymbol{\omega}^*$, and workers apply to this contract giving us a single sub-market. There is free-entry, so the second condition requires that the firms enter the recruitment market until the expected value of the vacancy is driven to 0. The third condition gives us the steady state number of unemployed workers. At each instance fraction $\alpha_u(q(\mu, \boldsymbol{\omega}; \overline{v}))$ of the unemployed workers transition to employment and fraction δ of the unemployed workers die. At the same time, mass δ of new unemployed workers are born. Equating the outflow to inflow gives us the third condition.

We summarize the equilibrium conditions in the next definition.

Definition 1. *A competitive search equilibrium is the vector* $(\boldsymbol{\omega}^*, q, \overline{v}, u)$ *that satisfies,*

(1) Firms' Profit Maximization:

$$V^*(\mu; \overline{v}) = \max_{\{\boldsymbol{\omega}, q\}} \left\{ \frac{-\kappa + \alpha_v(q) V(\mu, \boldsymbol{\omega})}{\rho + \alpha_v(q)} \right\}$$

subject to the worker's optimal search behavior,

$$v^u(\mu, \boldsymbol{\omega}, q) = \overline{v}$$

(2) Free Entry:

$$V^*(\mu; \overline{v}) = 0$$

(3) Worker population:

$$u(\boldsymbol{\omega}, \overline{v}) = \frac{\delta}{\alpha_u(q(\mu, \boldsymbol{\omega}; \overline{v})) + \delta}$$

3. Flexible Wage

First, we consider the benchmark of fully-flexible wages. When the wage schedule can be chosen flexibly, the firm retains full control over the duration of experimentation and can effectively choose the time, T, at which the worker's employment at the firm is terminated if the worker has not produced the high output until that time. The firm can choose the termination time because it can prevent the worker from quitting by offering a high enough wage and can induce the dissolution of the match by offering a low enough wage. Hence, without loss of generality, we can think of a flexible contract as a choice of duration T and a flow wage transfer to the worker at each instant over the duration of the contract, $\boldsymbol{\omega} := \{w(t)\}_{t=0}^{T}$. In what follows, to save from notation we continue to use $\boldsymbol{\omega}$ to refer to the contract, but note that $\boldsymbol{\omega}$, and therefore all the functions that depend on it, all depend on the contract duration T.

Let us examine the worker and firm's post-match expected payoffs under the variable wage contract. We denote the worker's post-match continuation payoff after an unproductive spell of length t by $v(\mu, \boldsymbol{\omega}, t)$. The differential equation governing $v(\mu, \boldsymbol{\omega}, t)$ is given by

$$-\frac{dv(\mu, \boldsymbol{\omega}, t)}{dt} = w(t) + \lambda\gamma(\mu, t)v_h - (\rho + \delta + \lambda\gamma(\mu, t))v(\mu, \boldsymbol{\omega}, t) \tag{6}$$

The first two terms on the right hand side of this equation capture the instantaneous payoff to the worker. The third term captures the impact of discounting, the death rate and the change in the belief that the worker is the productive type on the rate at which the continuation value changes over time. See Appendix B.3 for a detailed derivation of (6). Solving the differential equation, and imposing the boundary condition that the value to the worker for all $t \geq T$ is v_0 we obtain:

$$\begin{aligned} v(\mu, \boldsymbol{\omega}) = &\int_0^T (w(z) + \lambda\gamma(\mu, z)v_h)(\mu + (1-\mu)e^{\lambda z})e^{-(\rho+\delta+\lambda)z}dz \\ &+ v_0 e^{-(\rho+\delta+\lambda)T}\left(\mu + (1-\mu)e^{\lambda T}\right) \end{aligned} \tag{7}$$

We denote the firm's post-match continuation payoff after an unproductive spell of length t by $V(\mu, \boldsymbol{\omega}, t)$. The differential equation governing t by $V(\mu, \boldsymbol{\omega}, t)$ is given by

$$\frac{dV(\mu, \boldsymbol{\omega}, t)}{dt} - (\rho + \delta + \lambda\gamma(\mu, t))V(\mu, \boldsymbol{\omega}, t) = \underline{y} - w + \lambda\gamma(\mu, t)(V_h + \overline{y}) \tag{8}$$

Solving the differential equation for the firm,

$$V(\mu,\boldsymbol{\omega}) = \int_0^T (\underline{y} - w(z) + \lambda\gamma(\mu,z)(V_h + \overline{y})) \cdot (\mu + (1-\mu)e^{\lambda z}) \cdot e^{-(\rho+\delta+\lambda)z} dz \tag{9}$$

We define the expected surplus from a match which dissolves after T, $V^p(\mu,T)$ as

$$V^p(\mu,T) = \mu\left(\frac{\lambda\overline{y} + \underline{y}}{\rho+\delta} - v_0\right)\left(1 - e^{-(\rho+\delta+\lambda)T}\right) + (1-\mu)\left(\frac{\underline{y}}{\rho+\delta} - v_0\right)\left(1 - e^{-(\rho+\delta)T}\right) + v_0 \tag{10}$$

Substituting $V_h = (\lambda\overline{y} + \underline{y})/(\rho+\delta) - v_h$, integrating the first expression, using (7), and rearranging the terms we obtain:

$$V(\mu,\boldsymbol{\omega}) = V^p(\mu,T) - v(\mu,\boldsymbol{\omega}) \tag{11}$$

The expected post-match payoff for the firm is the total expected surplus generated through the match, less the compensation to the worker. We construct a wage path $\boldsymbol{\omega}$ such that the worker does not quit employment. Consider the following wage,

$$w(t) = b + v_0(\rho+\delta) - \lambda\gamma(\mu,t)(v_h - v_0) \tag{$\boldsymbol{\omega}$}$$

where $b \geq 0$ is some constant. Substituting the wage path into the equation (7) and imposing the boundary conditions, the value to the worker at any time t is,

$$v(\mu,\boldsymbol{\omega},t) = v_0 + \gamma(\mu,t)\frac{b}{\rho+\delta+\lambda}\left(1 - e^{-(\rho+\delta+\lambda))(T-t)}\right) \quad + (1-\gamma(\mu,t))\frac{b}{\rho+\delta}\left(1 - e^{-(\rho+\delta)(T-t)}\right) \tag{12}$$

The second part of the above expression is increasing in b and $v(\mu,\boldsymbol{\omega},t,T) \geq v_0$ for all $b \geq 0$ and $t \in [0,T]$. The expected value of the match for the worker and the firm at $t=0$ is,

$$v(\mu,\boldsymbol{\omega}) = v_0 + \mu\frac{b}{\rho+\delta+\lambda}\left(1 - e^{-(\rho+\delta+\lambda))T}\right) + (1-\mu)\frac{b}{\rho+\delta}\left(1 - e^{-(\rho+\delta)T}\right) \tag{13}$$

$$V(\mu,\boldsymbol{\omega}) = V^p(\mu,T) - v(\mu,\boldsymbol{\omega}) \tag{14}$$

Note that there could be multiple wage paths $\{w(t)\}_{t=0}^{T}$ for a given T that give the same expected payoff to the worker. We remain agnostic about the wage path offered, as long as the expected continuation payoff of the worker during employment $v(\mu, \boldsymbol{\omega}, t) \geq v_0$ for all $t \in [0, T]$. Therefore, $\boldsymbol{\omega} = (\hat{v}, T)$. Firms take the equilibrium value of the expected post-match income of the worker, which we denote by $\hat{v}$, as given. Thus,

$$V(\mu, \hat{v}, T) = V^p(\mu, T) - \hat{v} \tag{15}$$

The contract duration that maximizes firms' post-match value for a given $\hat{v}$,

$$\overline{T} = \arg\max_T V(\mu, \hat{v}, T) = \arg\max_T V^p(\mu, T)$$

Note that the optimal contract duration does not depend on the offered expected value to the worker $\hat{v}$. Examining the expression for $V^p(\mu, T)$ in Equation (11), we can see that it is hump-shaped in T. The first expression is increasing and the second expression is decreasing in T. As the duration of the contract (or experimentation) is increased incrementally, the expected marginal benefit comes from the worker producing the stellar output, and the marginal loss due to the worker failing to do so while forgoing the higher outside option wage. For lower values of T the marginal benefit is greater than the loss and the difference between them declines as the firm becomes more pessimistic. The firm will choose the contract duration so that the marginal benefit equals the marginal loss.

$$\frac{\partial V^p(\mu, T)}{\partial T} = e^{-(\rho+\delta+\lambda)T} \left[\underbrace{\mu(\rho+\delta+\lambda)\left(\frac{\lambda\overline{y} + \underline{y}}{\rho+\delta} - v_0\right)}_{\text{Marginal Benefit}} - \underbrace{(1-\mu)e^{\lambda T}\left(v_0(\rho+\delta) - \underline{y}\right)}_{\text{Marginal Loss}} \right] \tag{16}$$

Therefore, the optimal $\overline{T}$ has the following form.

$$\overline{T} = \frac{1}{\lambda}\left[\ln\left(\frac{\mu}{1-\mu}\right) - \ln\left(\frac{\underline{\mu}_p}{1-\underline{\mu}_p}\right)\right] \tag{17}$$

where the threshold belief $\underline{\mu}_p$ is given by

$$\lambda\underline{\mu}_p\left(\overline{y}+\frac{\lambda\overline{y}+\underline{y}}{\rho+\delta}-v_0\right)=v_0(\rho+\delta)-\underline{y}.$$

In words, $\underline{\mu}_p$ is the belief at which the marginal benefit from increasing the experimentation duration is equal to the marginal loss when $T=0$ (or immediate match dissolution).

The firm chooses a longer contract duration if the difference between the log-odds ratio of the belief that the worker is productive and the threshold belief is higher. Observe that for a given difference between the log-odds ratio, the experimentation duration is shorter if the speed of learning, λ is higher. The firm's value-maximizing experimentation duration is increasing in the worker's type μ, the output values $(\overline{y},\underline{y})$, and decreasing in v_0. The following proposition specifies the variable wage contract offered by the firm $\boldsymbol{\omega}$ which is characterized by the offered contract duration $\overline{T}$ and the offered expected income $\hat{v}$.

Proposition 1. *The firm offers a variable wage contract $\boldsymbol{\omega}$ such that the worker's expected income upon matching is $\hat{v}\geq v_0$ and the contract duration is $\overline{T}$ given by* (17)*. The continuation value $v(\mu,\boldsymbol{\omega},t)\geq v_0$ for all $t\in[0,\overline{T}]$.*

3.1. **Competitive Search.** With competitive search and flexible wages, the firms offer contracts, $\boldsymbol{\omega}\equiv(\hat{v},\overline{T})$ to the workers. The value to the worker when starting the job $(t=0)$ is,

$$v(\mu,\boldsymbol{\omega})=\hat{v}$$

The value to the firm is

$$V(\mu,\boldsymbol{\omega})=V^p(\mu,\overline{T})-\hat{v}$$

Hereafter, we use the contract $\boldsymbol{\omega}$ and the summary statistics $(\hat{v},\overline{T})$ that fully characterize it, interchangeably. The firms' expected income from posting a vacancy with contract $(\hat{v},\overline{T})$ is

$$\rho V^u(\mu,\boldsymbol{\omega},q)=-\kappa+\alpha_v(q)(V(\mu,\boldsymbol{\omega})-V^u(\mu,\boldsymbol{\omega},q))$$

The workers' expected income in the same sub-market,

$$(\rho + \delta)v^u(\mu, \boldsymbol{\omega}, q) = v_0 + \alpha_u(q)(\hat{v} - v^u(\mu, \boldsymbol{\omega}))$$

As discussed earlier, the following equation determines the queue length in the sub-market $q(\mu, \boldsymbol{\omega}; \overline{v})$ so that

$$v^u(\mu, \boldsymbol{\omega}, q(\mu, \boldsymbol{\omega}; \overline{v})) = \overline{v}$$

Proposition 2. *The competitive search equilibrium under the variable wage contract is the tuple* $(\hat{v}^*, \overline{T}, q^*, \overline{v}^*, u^*)$ *and the competitive search equilibrium with variable wage contracts exists.*

Proof. The firms maximize,

$$V^*(\mu; \overline{v}) = \max_{(\hat{v}, T)} \left\{ \frac{-\kappa + \alpha_v(q)(V^p(\mu, T) - \hat{v})}{\rho + \alpha_v(q)} \right\}$$

subject to

$$\frac{v_0(\rho + \delta) + \alpha_u(q)\hat{v}}{\rho + \delta + \alpha_u(q)} = \overline{v}$$

Note that for a given $\hat{v}$, the firm maximization with respect to T is equivalent to maximizing $V^p(\mu, T)$. Therefore, $T = \overline{T}$. Next, we show that $V^*(\mu; \overline{v})$ is a well-defined function. The expected payoff to the workers that keeps them indifferent across sub-markets is $\overline{v} \in [v_0, V^p(\mu, \overline{T})]$. The contract is $\boldsymbol{\omega} = (\hat{v}, \overline{T})$ and the queue length corresponding to the workers' indifference, $q(\mu, \boldsymbol{\omega}; \overline{v})$. If the firm offers $\hat{v} = \overline{v}$, the workers must get matched without waiting, so that $q(\mu, \boldsymbol{\omega}; \overline{v}) = 0$, and the expected income from a vacancy is $V^u(\mu, \boldsymbol{\omega}, q(\mu, \boldsymbol{\omega}; \overline{v})) = -\kappa/\rho < 0$. If the firm offers all of the post-match surplus to the workers, $\hat{v} = V^p(\mu, \overline{T})$ the expected income from a vacancy is $V^u(\mu, \boldsymbol{\omega}, q(\mu, \boldsymbol{\omega}; \overline{v})) = -\kappa/(\rho + \alpha_v(q(\mu, \boldsymbol{\omega}; \overline{v})) < 0$. The firms expected income $V^u(\mu, \boldsymbol{\omega}, q(\mu, \boldsymbol{\omega}; \overline{v}))$ is continuous in $\hat{v} \in [\overline{v}, V^p(\mu, \overline{T})]$. A continuous function on a compact domain attains a maximum. Therefore, $V^*(\mu; \overline{v})$ is well-defined. Because the firms choose $\hat{v}$, subject to the constraint on $(\hat{v}, q)$ that keeps the workers indifferent, the firms' value maximizing $\hat{v}$ is the point of tangency between the indifference curve of the workers corresponding to their expected payoff $\overline{v}$ and the iso-profit curve of the firm in the $\hat{v}$-q space.

By the application of the theorem of maximum, $V^*(\mu;\overline{v})$ is continuous, and applying the envelope theorem implies that $V^*(\mu;\overline{v})$ is decreasing in $\overline{v}$. Finally, free-entry condition pins down the equilibrium $\overline{v}^*$. When $\overline{v} \to v_0$, $V^*(\mu;\overline{v}) = V^p(\mu,\overline{T}) - v_0 > 0$ and when $\overline{v} \to V^p(\mu,\overline{T})$, $V^*(\mu;\overline{v}) = -\kappa/\rho < 0$. Therefore, by the intermediate value theorem, there must exist a $\overline{v}^* \in [v_0, V^p(\mu,\overline{T})]$ such that the free-entry condition $V^*(\mu;\overline{v}^*) = 0$ holds. Note that the equilibrium $(\hat{v}^*, q^*)$ is the point of tangency between the iso-profit curve of the firm corresponding to 0 profit and the workers' indifference curve corresponding to the expected payoff $\overline{v}^*$. The equilibrium contract is $\boldsymbol{\omega}^* = (\hat{v}^*, \overline{T})$, and $q^* = q(\mu, \boldsymbol{\omega}^*; \overline{v}^*)$. The workers' entry condition pins down the equilibrium unemployment u^*. □

4. Fixed Wage Contracts with Commitment to Contract Duration

Firms often face contractual frictions that prevent them from offering a fully flexible wage schedule. For example, firms may find it costly to monitor and administer contracts with wages that vary over the contract term, or there may be limits on how fast the wage can grow. To account for such contractual limitations, we assume that firms can only use contracts where the wage is constant throughout the contract duration. With fixed wage contracts it becomes important whether the firm can commit to retaining the worker for the duration of the contract. In this section, we assume that firms have this commitment power and relax this assumption in the next section. In both sections, we assume that workers can quit at any point during the duration of the contract if they prefer their outside option. Observe that the contract term is irrelevant if the worker quits before the contract term expires. Both, the wage and the contract duration are determined in equilibrium.

4.1. **Worker's Quitting Decision.** We denote a fixed wage contract by $\boldsymbol{\omega} = (w, T)$ where w is the wage and T is the contract duration. To solve for the quitting time, in this subsection we temporarily assume that the firms offer lifetime employment ($T \to \infty$). We then solve for profit maximizing contracts taking the worker's quitting time as a constraint on the contract duration.

Denote the worker's quitting time by t_q. The differential equation governing the worker's post-match continuation payoff $v(\mu, w, t, t_q)$ when the worker is employed is,

$$-\frac{dv(\mu, w, t, t_q)}{dt} = w + \lambda\gamma(\mu, t)v_h - (\rho + \delta + \lambda\gamma(\mu, t))v(\mu, w, t, t_q) \tag{18}$$

Solving the differential equation, and imposing the boundary condition that the value to the worker for all $t \geq t_q$ is v_0 we obtain:

$$\begin{aligned} v(\mu, w, t, t_q) - v_0 = \gamma(\mu, t)\left(\frac{w + \lambda v_h}{\rho + \delta + \lambda} - v_0\right)\left(1 - e^{-(\rho+\delta+\lambda)(t_q - t)}\right) \\ + (1 - \gamma(\mu, t))\left(\frac{w}{\rho + \delta} - v_0\right)\left(1 - e^{-(\rho+\delta)(t_q - t)}\right) \end{aligned} \tag{19}$$

By setting $t = 0$ in the previous expression, we get the expected post-match payoff as a function of the quitting time,

$$\begin{aligned} v(\mu, w, t_q) - v_0 = \mu\left(\frac{w + \lambda v_h}{\rho + \delta + \lambda} - v_0\right)\left(1 - e^{-(\rho+\delta+\lambda)t_q}\right) \\ + (1 - \mu)\left(\frac{w}{\rho + \delta} - v_0\right)\left(1 - e^{-(\rho+\delta)t_q}\right) \end{aligned} \tag{20}$$

The expected payoff comprises two terms. The first term captures the payoff if the worker is productive and the second term captures the payoff if the worker is unproductive. The worker chooses t_q to maximize $v(\mu, w, t_q)$. We denote,

$$\overline{T}_q(\mu, w) = \arg\max_{t_q} v(\mu, w, t_q)$$

Observe from (20) that, if the offered wage is above $v_0(\rho + \delta)$, then $\overline{T}_q(\mu, w) \to \infty$. This is because for all $w \geq v_0(\rho + \delta)$, regardless of the belief μ, the worker is better off working for the firm, and therefore never quits. On the other hand, if the offered wage is below $v_0(\rho+\delta+\lambda) - \lambda v_h$ then $\overline{T}_q(w, \mu) = 0$. This is because for all $w < v_0(\rho+\delta+\lambda) - \lambda v_h$, even if the match is productive for sure, the worker is not getting compensated adequately and he is better off quitting right away. Combining these observations, without loss of generality, we restrict attention to $w \in (v_0(\rho+\delta+\lambda) - \lambda v_h, v_0(\rho+\delta))$. If the offered wage w is in this range, increasing the quitting time provides a marginal benefit if the worker is productive, however, if the worker is unproductive, then the worker incurs a marginal loss. As the quitting time increases, the marginal benefit declines faster than the marginal loss as the worker puts less weight on being productive. So the value function is hump-shaped in the quitting time. For the worker to accept employment, the marginal benefit $\lambda\mu(v_h - v_0)$ must exceed the marginal loss $(v_0(\rho+\delta) - w)$ at $t_q = 0$ leading to the following lemma.

Lemma 1. *Let $\underline{\mu}(w)$ be defined as:*

$$\underline{\mu}(w) := \frac{1}{\lambda}\left(\frac{v_0(\rho+\delta) - w}{v_h - v_0}\right).$$

A worker with prior belief μ accepts employment, i.e. $\overline{T}_q(\mu, w) > 0$, if and only if $\mu > \underline{\mu}(w)$.

Intuitively, the threshold belief is the ratio of the flow marginal loss and the marginal benefit. Workers are more likely to join if the post-success payoff is much higher relative to the loss in earnings due to the low flow wage during the experimentation phase. The following proposition characterizes the worker's quitting time.

Proposition 3. *If the firm does not fire the worker, the workers endogenously quit employment at time $\overline{T}_q(\mu, w)$,*

$$\overline{T}_q(\mu, w) = \begin{cases} 0 & w \leq v_0(\rho+\delta) - \lambda\mu(v_h - v_0) \\ \frac{1}{\lambda}\cdot\left[\ln\left(\frac{\mu}{1-\mu}\right) - \ln\left(\frac{\underline{\mu}(w)}{1-\underline{\mu}(w)}\right)\right] & w \in (v_0(\rho+\delta) - \lambda\mu(v_h - v_0), v_0(\rho+\delta)) \\ \infty & w \geq v_0(\rho+\delta) \end{cases}$$

The quitting time is increasing in the belief of the worker, μ and w so that workers who have stronger beliefs about being productive quit later for a given wage and workers with a given belief quit later when offered a higher wage.

The workers' quitting time is the difference between the log odds ratio of the worker being productive $\left(\frac{\mu}{1-\mu}\right)$ and the threshold belief $\left(\frac{\underline{\mu}(w)}{1-\underline{\mu}(w)}\right)$ at a given wage, weighted by the inverse of the speed of learning λ. The higher the difference between the worker's type and the threshold belief, the worker works longer before quitting. Moreover, for a given difference between the log odds ratios, a higher speed of learning would reduce the quitting time because the worker learns about her type, or the match productivity faster.

4.2. **Threshold Wage.** Recall from Section 3 that the contract duration, $\overline{T}$, chosen by a planner who maximizes the expected payoff from a match is

$$\overline{T} = \frac{1}{\lambda}\left[\log\left(\frac{\mu}{1-\mu}\right) - \log\left(\frac{\underline{\mu}_p}{1-\underline{\mu}_p}\right)\right]$$

where $\underline{\mu}_p$ solves,

$$\underline{\mu}_p\left(\frac{\lambda}{\rho+\delta}\right)\left(\overline{y} + \frac{\lambda\overline{y}+\underline{y}}{\rho+\delta} - v_0\right) = v_0 - \frac{\underline{y}}{\rho+\delta}$$

Whether the worker quits earlier or later than the optimal experimentation duration depends on his wage, since the quitting time $\overline{T}_q(\mu, w)$ increases with w. Let $\tilde{w}$ be the threshold wage such that the quitting time is exactly the optimal experimentation duration, i.e. $\overline{T}_q(\mu, \tilde{w}) = \overline{T}$. At $\tilde{w}$ the threshold beliefs must be equalized, $\underline{\mu}_p = \underline{\mu}(\tilde{w})$, hence

$$\tilde{w} = v_0(\rho+\delta)\left(1 - \frac{\underline{\mu}_p}{\underline{\mu}(0)}\right)$$

This implies that for a given value of the total post-success match surplus, if the post-success payoff to the worker is lower, then the threshold belief corresponding to 0 wage is higher, which implies that the worker must be paid a higher wage to experiment optimally before quitting. If the equilibrium wage is below this threshold, then the worker quits sooner than the optimal experimentation duration. Note that this threshold wage is independent of the initial belief μ because the initial belief increases the optimal experimentation time and the quitting time by the same amount.

4.3. **Competitive Search Equilibrium.** The competitive search equilibrium is the tuple $(w^*, T^*, q^*, \overline{v}^*, u^*)$ as in Definition 1, where the firms must maximize their value by choosing the contract features (w, T) taking into account the search behavior of the workers,

$q(\mu, w, T; \overline{v})$. The free-entry condition pins down $\overline{v}^*$, and the workers' entry condition u^*, the equilibrium level of unemployment.

Proposition 4. *The competitive search equilibrium,* $(w^*, T^*, q^*, \overline{v}^*, u^*)$ *when the firms offer fixed-wage contracts* $\boldsymbol{\omega} = (w, T)$ *exists.*

The proof of existence follows similar arguments as in Proposition 2 with the modification that the contracts are fixed-wage contracts.[6]

Note that without loss of generality, we can restrict attention to contracts (w, T) such that $T \leq \overline{T}_q(\mu, w)$. When $T = \overline{T}_q(\mu, w^*)$, the firm prefers (weakly) a longer contract duration but the worker quits at T. When $T < \overline{T}_q(\mu, w^*)$, the worker prefers a longer contract duration but the firm fires the worker at T. Hence, there are two types of equilibria.

(1) **Quitting Equilibrium** where $T^* = \overline{T}_q(\mu, w^*)$.
(2) **Tenure Equilibrium** where $T^* < \overline{T}_q(\mu, w^*)$.

Lemma 2 shows that the type of equilibrium that arises depends on whether the equilibrium wage is above or below the threshold wage.

Lemma 2. *If the equilibrium contract wage is such that,* $w^* \leq \tilde{w}$*, then the equilibrium is a quitting equilibrium, i.e.* $T^* = \overline{T}_q(\mu, w^*)$*. Otherwise, the equilibrium is a tenure equilibrium.*

Proof. For $w^* \leq \tilde{w}$, suppose that $T^* < \overline{T}_q(\mu, w^*)$. At the equilibrium wage, $\partial V(\mu, w^*, T^*)/\partial T > 0$ and $\partial v(\mu, w^*, T^*)/\partial T > 0$. Therefore, a subset of firms could always deviate to $T = \overline{T}_q(\mu, w^*)$. The deviating firms and the workers will be better off with this new contract. Hence, for $w^* \leq \tilde{w}$, $T < \overline{T}_q(\mu, w^*)$ cannot be an equilibrium. We have already ruled out $T > \overline{T}_q(\mu, w^*)$ because the worker quits at $\overline{T}_q(\mu, w^*)$ and the offered contract term is irrelevant. Therefore, if $w^* \leq \tilde{w}$, then $T^* = \overline{T}_q(\mu, w^*)$.

[6] A detailed proof is given in Appendix A.

On the other hand, if $w^* > \tilde{w}$ and $T^* = \overline{T}_q(\mu, w^*) > \overline{T}$, then the firms can deviate to a new contract (w', T') where $w' > w^*$ and $\overline{T} \leq T' < \overline{T}_q(\mu, w^*)$ such that $v(\mu, w^*, \overline{T}_q(\mu, w^*)) = v(\mu, w', T')$. The workers' indifference along with $T' \in [\overline{T}, \overline{T}_q(\mu, w^*))$ implies that $V(\mu, w', T') > V(\mu, w^*, \overline{T}_q(\mu, w^*))$. This follows because with the workers indifferent, reducing the contract duration improves the overall match surplus $V^p(\mu, T)$ $(\partial V^p(\mu, T)/\partial T < 0$ for all $T > \overline{T})$, and increases the firms' expected payoff. The deviating firms are better off and the workers are indifferent. Therefore, for $w^* > \tilde{w}$, $T^* = \overline{T}_q(\mu, w^*)$ cannot be an equilibrium. □

The above lemma shows that if the equilibrium wage is below the threshold wage $\tilde{w}$, then the equilibrium features workers quitting. However, if the equilibrium wage strictly exceeds this threshold then the offered contract term binds and the workers are fired by the firm if they do not succeed before the term expires. But what is the equilibrium contract term offered by the firms and how does it compare with the optimal experimentation duration $\overline{T}$? To answer this question, note that Lemma 2 implies that if $T^* < \overline{T}_q(\mu, w^*)$ it must be that $w^* > \tilde{w}$. From the definition of $\tilde{w}$, this implies that the optimal experimentation duration is shorter than the quitting time, $\overline{T} < \overline{T}_q(\mu, w^*)$. If $T^* > \overline{T}$, then the firms can deviate to a contract offering $\boldsymbol{\omega}' = (w', T')$ where $\tilde{w} < w^* < w'$ and $\overline{T} < T' < T^*$ such that $v(\mu, w', T') = v(\mu, w^*, T^*)$, so that the workers are indifferent. The workers' indifference along with $T' \in (\overline{T}, T^*)$ implies that $V(\mu, w', T') > V(\mu, w^*, T^*)$. This follows because with the workers indifferent, reducing the contract duration improves the overall match surplus $V^p(\mu, T)$ $(\partial V^p(\mu, T)/\partial T < 0$ for all $T > \overline{T})$, and increases the firms' expected payoff. The workers are indifferent, and the deviating firms are better off. Hence, $T^* > \overline{T}$ cannot be an equilibrium contract duration. An exactly similar argument applies for $T^* < \overline{T}$ where the firm deviates to offer $\tilde{w} < w' < w^*$ and $\overline{T} > T' > T^*$ which keeps the workers indifferent and the deviating firms are better off. This implies that $T^* < \overline{T}$ cannot arise in equilibrium either. Therefore, it must be that $T^* = \overline{T}$ for all $w^* > \tilde{w}$, giving us the following theorem.

Theorem 1. *The tenure equilibrium results in optimal experimentation, i.e.* $T^* = \overline{T}$, *and the quitting equilibrium results in sub-optimal experimentation, i.e.* $T^* < \overline{T}$

The reader may refer to Appendix A for a more detailed proof. The tenure equilibrium contract is $\omega^* = (w^*, \overline{T})$ with $w^* > \tilde{w}$, otherwise $\omega^* = (w^*, \overline{T}_q(\mu, w^*))$. From Definition 1 the equilibrium contract, and the queue length are the tangency point between the indifference curve of the worker corresponding to the payoff $\overline{v}^*$, and the iso-profit curve of the firms corresponding to 0 profit (due to free-entry).[7] The following proposition highlights the equilibrium split of the surplus between the worker and the firm,

Proposition 5. *The post-match expected payoff for the worker in equilibrium is,*

$$v(\mu, w^*, T^*) = \overline{v}^* + \frac{1-\varepsilon(q^*)}{1-\varepsilon(q^*)(1-\Theta(\mu, w^*))} \cdot (V^P(\mu, T^*) - \overline{v}^*) \tag{21}$$

where $\varepsilon(q) = -q\alpha_u'(q)/\alpha_u(q)$ is the elasticity of the arrival rate of jobs and

$$1-\Theta(\mu, w^*) = \frac{\frac{\partial V^p(\mu,T^*)}{\partial T} \cdot \frac{dT^*}{dw}}{\frac{dv(\mu,w^*,T^*)}{dw}} \in (0,1]$$

is the marginal gain in total surplus from incremental wage increase relative to the marginal increase in worker's payoff.

Proposition 5 shows that the equilibrium post-match payoff of the worker can be interpreted as a solution to the Nash-Bargaining problem between the firm and the worker, in which $(1-\varepsilon(q^*))/(1-\varepsilon(q^*)(1-\Theta(\mu, w^*)))$ is analogous to the worker's bargaining power and $\overline{v}^*$ is the disagreement payoff of the worker. The firm's bargaining power is $(\varepsilon(q^*)\Theta(\mu, w^*))/(1-\varepsilon(q^*)(1-\Theta(\mu, w^*)))$ and the disagreement payoff is 0. A wage increase leads to an increase in the contract duration and raises the total surplus. If this increase is large relative to the direct increase in the worker's payoff, then the worker's bargaining power is higher, and the worker parts with a larger proportion of the net surplus. Also note that if $\varepsilon(q)$ is weakly increasing in q, then as q declines, the worker's bargaining power is higher.[8]

[7] If not, then at (w^*, q^*), the indifference curve of the worker intersects the 0 profit iso-profit curve either from below or above. If the former, then there is a wage, queue length pair, $(w', q') > (w^*, q^*)$ such that $v^u(\mu, w^*, q^*) = v^u(\mu, w', q')$ but $V^u(\mu, w', q') > 0$. Therefore, (w^*, q^*) cannot be an equilibrium. An exactly similar argument applies for the latter case when the indifference curve intersects the iso-profit from above. only that $(w', q') < (w^*, q^*)$.

[8] Which it is for standard matching functions such as Cobb-Douglas and the urn-ball matching function

5. Efficiency

Moen (1997) shows that without learning and experimentation competitive search with wage posting is efficient even with fixed wages. However, with workers and firms learning about the worker's type through experimentation, it is possible that in equilibrium, the match dissolves too soon or too late and the equilibrium is no longer efficient.

The planner's goal is to maximize the total surplus generated in the economy taking the search frictions (i.e. the matching function and search cost) as given. To achieve this goal, the planner chooses the number of vacancies and the experimentation duration subject to the evolution of unemployed workers in the economy. In other words, the planner solves the following dynamic optimization:

$$\max_{q(t),T} \int_0^\infty u(t) \left(v_0(\rho + \delta) + \alpha_u(q(t))V^p(\mu, T) - \frac{\kappa}{q(t)} \right) e^{-\rho t} dt$$

subject to the law of motion,

$$\dot{u}(t) = \delta - (\delta + \alpha_u(q(t)))u(t).$$

We denote the planner's value function by $\overline{V}(\mu, u)$. The planner's solution satisfies the following HJB equation,

$$\rho \overline{V}(\mu, u) = \max_{q,T} \left\{ u \left(v_0(\rho + \delta) + \alpha_u(q) \cdot V^p(\mu, T) - \frac{\kappa}{q} \right) + \frac{d\overline{V}(\mu, u)}{du} \dot{u} \right\} \tag{22}$$

where $\dot{u} = \delta - (\delta + \alpha_u(q))u$. The first expression on the RHS is the flow output generated in the economy net of search cost. The second part captures the change in continuation payoff, given the law of motion for the measure of unemployed workers.[9]

To solve for the value function of the planner, $\overline{V}(\mu, u)$, the solution to (22), we guess and verify

$$\overline{V}(u) = A + B \cdot u$$

[9]The mapping in (22) is a contraction, therefore, the value function is unique.

We can interpret A as the continuation value if there are no unemployed workers, and B, the flow payoff generated by a single unemployed worker. Note that we have suppressed μ but (A, B) are functions of μ along with the other parameters. We solve for the value function, using the first-order condition with respect to (q, T),

$$\varepsilon(q)\alpha_u(q)B = \varepsilon(q)\alpha_u(q)V^p(\mu, T) - \frac{\kappa}{q}\ ,\ T = \overline{T} \tag{23}$$

and the envelope theorem,

$$(\rho + \delta + \alpha_u(q))B = v_0(\rho + \delta) + \alpha_u(q)V^p(\mu, \overline{T}) - \frac{\kappa}{q} \tag{24}$$

Combining the two conditions, (23) and (24),

$$\frac{(\rho + \delta)(V^p(\mu, \overline{T}) - v_0)}{\kappa} = \frac{(\rho + \delta) + (1 - \varepsilon(q))\alpha_u(q)}{\alpha_v(q)\varepsilon(q)} \tag{25}$$

Equation (25) pins down the queue length chosen by the planner. If $\varepsilon(q)$ is weakly increasing in q, then the right-hand side is strictly decreasing in q. The left-hand side is the flow post-match surplus relative to the search cost and is independent of the queue length. This gives a unique solution, q. Notice that the efficient (q, T) are independent of u, which verifies our initial guess about the planner's value function.

To see how this condition compares to the equilibrium, note that Theorem 1 states that in the tenure equilibrium, $T^* = \overline{T}$, and the tangency condition (21) simplifies to,

$$v(\mu, w^*, \overline{T}) = \overline{v}^* + (1 - \varepsilon(q^*)) \cdot \left(V^p(\mu, \overline{T}) - \overline{v}^*\right)$$

This is because $\Theta(\mu, w^*) = 1$ when $T^* = \overline{T}$. To further simplify the tangency condition to eliminate w^*, we make the following substitutions,

$$(\rho+\delta+\alpha_u(q^*))(v(\mu, w^*, \overline{T})-\overline{v}^*) = (\rho+\delta)(v(\mu, w^*, \overline{T})-v_0) = (\rho+\delta)\left(V^p(\mu, \overline{T}) - v_0 - \frac{\kappa}{\alpha_v(q^*)}\right)$$

The first equality comes from the worker's indifference condition and the second from the free-entry. Rearranging the terms, results in

$$\frac{(\rho+\delta)(V^p(\mu,\overline{T})-v_0)}{\kappa}=\frac{(\rho+\delta)+(1-\varepsilon(q^*))\alpha_u(q^*))}{\alpha_v(q^*)\varepsilon(q^*)} \tag{26}$$

This is identical to Equation (25). A higher post-match value relative to the search cost induces more firms to enter, and the queue length declines. Therefore, we have the following theorem,

Theorem 2. *The competitive search equilibrium is efficient*

(1) under fully flexible wage contracts.

(2) under fixed-wage contracts if the equilibrium wage is high enough, $w^* \geq \tilde{w}$*, or the offered contract duration binds.*

To understand the intuition behind this result, note that under directed search, the firms internalize the workers' marginal rate of substitution of queue lengths for the payoff from posted contracts. With free-entry, firms out-compete each other by posting more lucrative contracts and doing so until the expected payoff from a vacancy is driven to 0. In equilibrium, the firms' and the workers' marginal rates of substitution of queue lengths for the contract payoff are equalized. As long as the experimentation under this tangency contract is $\overline{T}$, the planner cannot make the firms better off without making the workers worse off, and vice versa by changing the queue length. With fully flexible wages, the firms offer contracts that allow for optimal experimentation (Proposition 2), and hence the equilibrium queue length is the same as the efficient benchmark. With fixed-wage contracts, as shown in Theorem 1, under a tenure equilibrium the experimentation duration is optimal. Therefore, a tenure equilibrium with fixed wages is efficient.

Note that if the equilibrium fixed-wage is below the threshold wage $\tilde{w}$, or in other words the wage is so low that workers end up quitting, then the post-match surplus is lower and hence, the benefit from faster job arrival is lower. This implies that the equilibrium queue length is longer than the efficient benchmark as fewer firms post vacancies. The next result (Corollary 1) highlights that such an inefficient outcome arises in equilibrium if the search cost faced by firms is sufficiently high.

Corollary 1. *If κ is greater than $\overline{\kappa}$, then the fixed-wage competitive search equilibrium with contract duration commitment is not efficient. The threshold $\overline{\kappa}$ is such that the q that solves the equation* (25) *with $\kappa = \overline{\kappa}$, satisfies the free-entry condition at $w = \tilde{w}$.*

To better understand this result, note that for efficiency the queue length must solve (25). This queue length when substituted in the free-entry condition,

$$v(\mu, w, \overline{T}) = V^p(\mu, \overline{T}) - \frac{\kappa}{\alpha_v(q)}$$

gives us the wage at the point of tangency, conditional on $T = \overline{T}$ because the RHS is independent of w and the LHS is increasing in w. As long as this wage is higher than the threshold $\tilde{w}$, the experimentation duration $\overline{T}$, the queue length, and the wage are consistent with the equilibrium. Notice that as κ increases, the wage solving the free-entry declines. Therefore, there must exist a $\overline{\kappa}$ such that for all $\kappa > \overline{\kappa}$, the wage is below the threshold $\tilde{w}$ and therefore in equilibrium $T^* < \overline{T}$. Put differently, under a tenure equilibrium as the search cost faced by the firms increases, fewer firms enter, and the equilibrium wage declines. There is a threshold search cost at which the equilibrium wage in the tenure equilibrium coincides with the threshold wage $\tilde{w}$. For all search cost values above this threshold, the equilibrium entails quitting, and sub-optimal experimentation.

To summarize, we find that the competitive search equilibrium is efficient if the firms can choose fully flexible wage contracts, or under fixed wages if the workers do not quit in equilibrium. It is important to point out that the equilibrium duration is optimal because the firms can commit to the contract duration. In the next section, we analyze the equilibrium when firms cannot commit to any arbitrary contract duration.

6. Fixed Wages Without Commitment to Contract Duration

In this section, we analyze fixed-wage contracts when the firms cannot commit to a contract duration to understand how commitment affects the equilibrium. In practice, such a situation may arise when contract duration commitments cannot be enforced legally. For example, in developing countries with poor legal infrastructure or countries where worker termination laws are not strictly enforced. Without commitment, a firm may deviate from the contractual term and fire the worker sooner if it improves their expected payoff. Such deviations should not be possible in equilibrium, and the firing time must be such that it maximizes the firms' post-match expected payoff.

6.1. **Firms' Firing Decision.**

Suppose that the match continues until time T. The expected payoff for the firm after the match is formed is derived following similar steps as in Section 4. The firm's expected payoff from the match is,

$$V(\mu, w, T) = \mu \cdot \frac{\underline{y} - w + \lambda(V_h + \overline{y})}{\rho + \delta + \lambda} \left(1 - e^{-(\rho+\delta+\lambda)T}\right) + (1-\mu)\frac{\underline{y} - w}{\rho + \delta} \left(1 - e^{-(\rho+\delta)T}\right) \tag{27}$$

The firm value given by (27) is analogous to the worker's value in (20). The first term is weighted by the belief that the match is productive and captures the firm's post-match payoff from a productive match. Similarly, the second term captures the post-match payoff conditional on being unproductive. We can solve for the firm's choice of $\overline{T}_f$ by assuming that the worker does not quit employment. We proceed as we did in Section 4. If $w < \underline{y}$ then the expected post-match payoff is increasing in T, and the firm never fires the worker ($\overline{T}_f \to \infty$). Similarly, if $w > \underline{y} + \lambda(V_h + \overline{y})$ then the expected payoff is decreasing in T and the firm chooses to fire the worker immediately ($\overline{T}_f = 0$). We focus on the the wage $w \in \left(\underline{y}, \underline{y} + \lambda(V_h + \overline{y})\right)$.

The firm chooses $T = \overline{T}_f$ to maximize $V(\mu, w, T)$.

$$\overline{T}_f(\mu, w) = \arg\max_T V(\mu, w, T)$$

For $w \in \left(\underline{y}, \underline{y} + \lambda(V_h + \overline{y})\right)$, the first expression in equation (27) is increasing in T and the second decreasing in T. If the worker works incrementally longer, the first part results in a marginal benefit to the firm if the match is productive, and the second part results in a marginal loss if the match is unproductive. The firm chooses $\overline{T}_f$ where the marginal loss $w - \underline{y}$ and the benefit $\lambda\mu(\overline{y} + V_h)$ are equalized.

$$\lambda \underline{\mu}_f(\overline{y} + V_h) = w - \underline{y}$$

Just like the belief threshold $\underline{\mu}(w)$ for the worker, we have a similar belief threshold such that the firm fires the worker immediately if $\mu \leq \underline{\mu}_f(w)$.

The first order condition with respect to T of (27) gives us,

$$\overline{T}_f(\mu, w) = \begin{cases} \infty & w \leq \underline{y} \\ \frac{1}{\lambda}\left[\ln\left(\frac{\mu}{1-\mu}\right) - \ln\left(\frac{\underline{\mu}_f(w)}{1-\underline{\mu}_f(w)}\right)\right] & w \in (\underline{y}, \underline{y} + \lambda\mu(V_h + \overline{y})) \\ 0 & w \geq \underline{y} + \lambda\mu(V_h + \overline{y}) \end{cases} \tag{28}$$

The firing time $\overline{T}_f$ is higher if the log odds ratio of the worker being productive is higher relative to the log odds ratio of the threshold belief, $\underline{\mu}_f(w)/(1 - \underline{\mu}_f(w))$ and this difference is weighted by the inverse of the learning rate λ. For the same difference in the log odds ratio, if the learning rate is faster the worker is fired sooner.

6.2. **Endogenous Experimentation Duration.**

The endogenous experimentation duration is the shorter of the endogenous quitting time and the firms' choice of firing time.

$$\hat{T}(\mu, w) = \min\{\overline{T}_q(\mu, w), \overline{T}_f(\mu, w)\}$$

Notice that $\overline{T}_q(\mu, w)$ is increasing in w and $\overline{T}_f(\mu, w)$ is decreasing in w. Inequalities in (1) imply that $v_0(\rho + \delta) > \underline{y}$ and $v_0(\rho + \delta) - \lambda\mu(v_h - v_0) < \underline{y} + \lambda\mu(V_h + \overline{y})$[10]. For all

[10] The above follows from the second inequality in (1). To see this, substitute the total surplus $V_h + v_h = \dfrac{\lambda\overline{y} + \underline{y}}{\rho + \delta}$ into $v_0(\rho + \delta) + \lambda\mu v_0 - \lambda\mu(v_h + V_h) - (\underline{y} + \lambda\overline{y}) = \left(\dfrac{\rho + \delta + \lambda\mu}{\rho + \delta}\right)\left(v_0(\rho + \delta) - \dfrac{\lambda\overline{y} + \underline{y}}{\rho + \delta}\right) < 0$

$w < \max\{v_0(\rho+\delta) - \lambda\mu(v_h - v_0), \underline{y}\}$, $\overline{T}_q(\mu, w) < \overline{T}_f(\mu, w)$. For low enough wage, the workers are not compensated enough and they prefer quitting before the firm fires. For all $w > \min\{v_0(\rho+\delta), \underline{y}+\lambda\mu(V_h+\overline{y})\}$, $\overline{T}_f(\mu, w) < \overline{T}_q(\mu, w)$. For a high enough wage, the firms do not earn sufficient profit and they prefer firing the worker. Moreover, the continuity of $\overline{T}_q(\mu, w)$ and $\overline{T}_f(\mu, w)$ for $w \in \big(\max\{v_0(\rho+\delta) - \lambda\mu(v_h - v_0), \underline{y}\}, \min\{v_0(\rho+\delta), \underline{y} + \lambda\mu(V_h + \overline{y})\}\big)$ implies that the difference between the quitting time and the firing time, $(\overline{T}_q(\mu, w) - \overline{T}_f(\mu, w))$ is continuously increasing in w and is equal to 0 for some value of w. That is, there is some intermediate wage at which the quitting time for the worker and the firing time for the firm coincide. Interestingly, this wage is the same as $\tilde{w}$.

To see this, note that

$$V(\mu, w, T) = V^p(\mu, T) - v(\mu, w, T)$$

and, at $w = \tilde{w}$, $\overline{T}(\mu) = \overline{T}_q(\mu, w)$

$$\frac{\partial V(\mu, w, T)}{\partial T} = \underbrace{\frac{\partial V^p(\mu, T)}{\partial T}}_{=0} - \underbrace{\frac{\partial v(\mu, w, T)}{\partial T}}_{=0} = 0$$

Therefore, it must be that

$$\overline{T}_f(\mu, \tilde{w}) = \overline{T}_q(\mu, \tilde{w}) = \overline{T}(\mu)$$

Figure (1) illustrates, how $\hat{T}(\mu, w)$ is determined. The endogenous quitting time increases with wage, whereas the firing time $\overline{T}_f$ decreases. They intersect at $\tilde{w}$.

6.3. **Expected Match Payoff.** We can now state the expected post-match payoffs for the worker and the firm given that the match dissolves endogenously at $\hat{T}(\mu, w)$.

$$v(\mu, w) = \begin{cases} v(\mu, w, \overline{T}_q(w)) & w \le \tilde{w} \\ v(\mu, w, \overline{T}_f(w)) & w > \tilde{w} \end{cases}$$

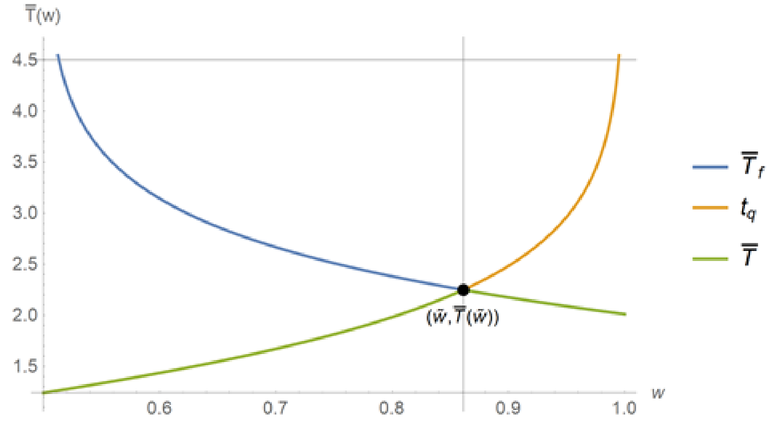


FIGURE 1. Quitting/Firing time vs. offered fixed wage w

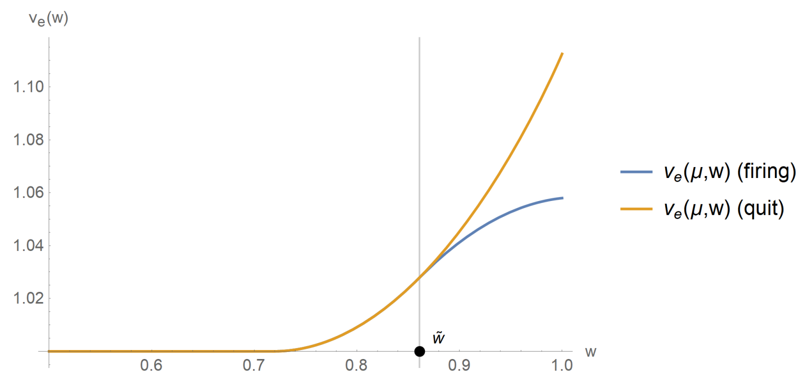


FIGURE 2. $v(\mu, w, \overline{T})$ vs. offered fixed wage w

and

$$V(\mu, w) = \begin{cases} V(\mu, w, \overline{T}_q(w)) & w \leq \tilde{w} \\ V(\mu, w, \overline{T}_f(w)) & w > \tilde{w} \end{cases}$$

Figures (2) and (3) indicate the payoffs, and how it changes when the firms fire the workers for $w > \tilde{w}$. Note that the curves are smooth at $\tilde{w}$, i.e. the workers' indifference curves and the firms' iso-profit curves are also going to be smooth in the w-q space.

6.4. **Competitive Search Equilibrium.** The competitive search equilibrium in the case of fixed wage contracts is as per the Definition 1 with $\boldsymbol{\omega} = w$. The offered wage endogenously pins down the experimentation duration, and therefore the contracts are fully characterized

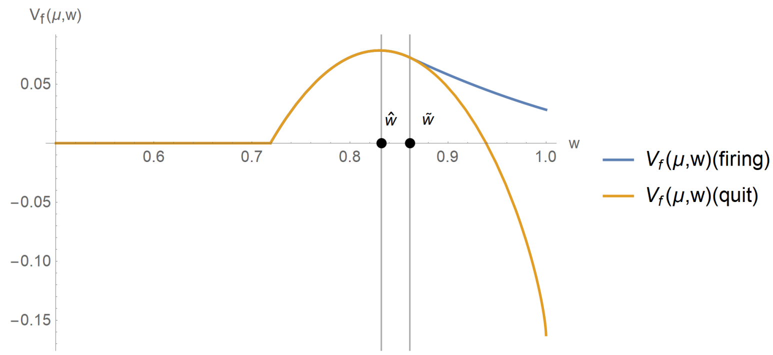


FIGURE 3. $V(\mu, w, \overline{T})$ vs. offered fixed wage w

by only the wage. The firms' expected payoff is

$$\rho V^u(\mu, w, q) = -\kappa + \alpha_v(q)(V(\mu, w) - V^u(\mu, w, q))$$

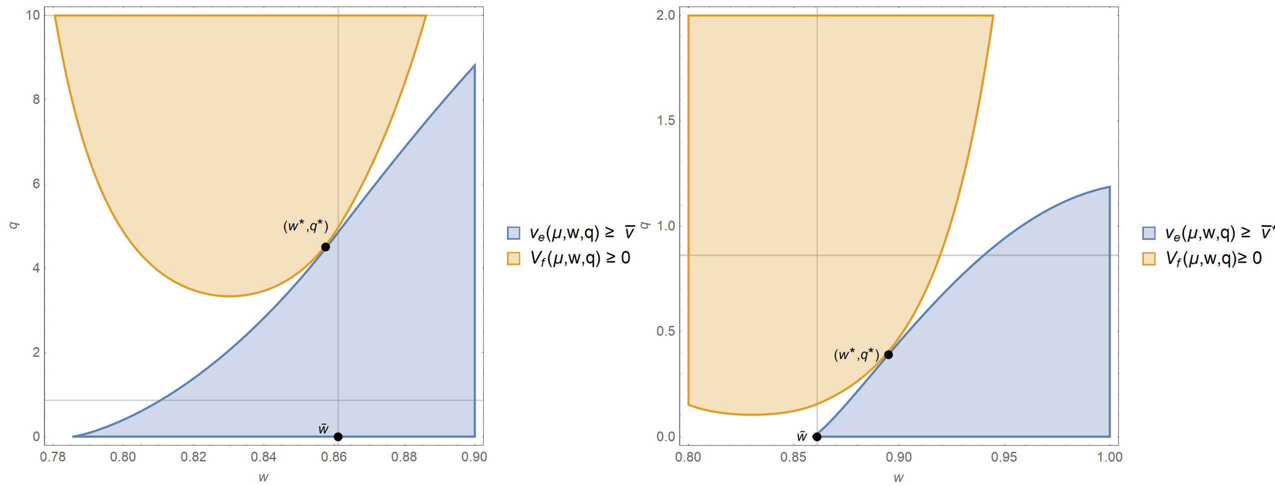
and the workers' expected payoff is

$$(\rho + \delta)v^u(\mu, w, q) = v_0(\rho + \delta) + \alpha_u(q)(v(\mu, w) - v^u(\mu, w, q))$$

Proposition 6. *The competitive search equilibrium* $(w^*, q^*, u^*, \overline{v}^*)$ *exists under a fixed-wage contract. The equilibrium contract entails two possibilities, (a) Quitting: workers quit employment (b) Tenure: workers are fired after a certain duration if they remain unproductive.*

The arguments for existence are similar to Proposition 4. Please refer to Appendix A for the detailed proof.

Figure (4) illustrates the two possibilities for the equilibrium. Which type of equilibrium firms and workers end up with depends on the parameters of the model. For example, as already shown in Corollary (1), if $\kappa > \overline{\kappa}$, then there is quitting in equilibrium, i.e. $w^* < \tilde{w}$. At $\kappa = \overline{\kappa}$, $T^* = \overline{T}_f = \overline{T}_q = \overline{T}$ and $w^* = \tilde{w}$. For any value of $\kappa > \overline{\kappa}$, $w^* < \tilde{w}$, and therefore there is quitting in equilibrium.



(A) Quitting Equilibrium, $w^* \leq \tilde{w}$ ($\kappa = \kappa_0$) (B) Tenure Equilibrium, $w^* > \tilde{w}$ ($\kappa = 0.5\kappa_0$)

FIGURE 4. Competitive Search Equilibrium under Fixed-wage contracts.

7. Discussion

Interestingly, the fixed-wage contract without commitment to contract duration is efficient only for the edge case, when $w^* = \tilde{w}$. For any other value of the equilibrium wage, the experimentation is sub-optimal, and therefore, the equilibrium queue length is smaller than what a planner prefers.

To understand why any wage other than $\tilde{w}$ results in sub-optimal experimentation, we can think of the firm and the worker as being involved in a two-sided two-armed bandit problem. Both sides, the firm, and the worker have the option to dissolve the match (quit or fire, the safe arm), or continue gambling to get a high output realization, the risky arm. Only if both parties choose the risky arm, does the experimentation continue, otherwise the match breaks down. With the workers and firms constrained to fixed wage contracts, the split of the surplus is such that either the firm gains too much compared to the worker, so that the worker quits earlier when $w < \tilde{w}$ or vice versa for $w > \tilde{w}$. Only at $w = \tilde{w}$, the split is such that the time of separation coincides with the optimal experimentation duration, otherwise the match dissolves sub-optimally.

Under fully flexible wage contracts, the firm compensates the worker just sufficiently so that she doesn't quit. Then conditional on the expected compensation to the worker the firm's maximization problem coincides with that of the planner. Therefore, the firm chooses an optimal stopping time that maximizes the total expected post-match surplus. We can interpret the result as if under fixed wages, the firm and the worker are restricted in dividing the expected surplus between them which leads to sub-optimal experimentation. This implies that in instances where they can write flexible contracts based on the future output realizations, the job tenure will be longer, or the turnover will be lower. There is empirical evidence that shows that employee ownership contracts even for rank-and-file employees reduce turnover, and increase job tenure, Blasi et al. (2008). The ability to commit to a contract duration also mitigates this inefficiency. However, even in that case, it must be that the search friction is not too high, otherwise the worker quits in equilibrium.

8. Extensions

8.1. **Minimum wage or Worker Liquidity constraints.** Suppose that the firm faces a constraint that it cannot pay a wage below w_L. The reason for such a constraint could be due to an exogenously imposed minimum wage or the workers may face a liquidity constraint and they cannot accept a very low wage. Consider that the firms can offer fully flexible wages, however, they cannot commit to the contract duration. In this case, the equilibrium experimentation duration may not be optimal depending on the value of the minimum wage relative to the optimal experimentation wage, $\tilde{w}$.

Proposition 7. *If the firm cannot credibly commit to the tenure of the contract, there are two cases,*

1. *If the minimum wage $w_L \leq \tilde{w}$, then the firm offers a variable wage contract, and the match dissolves efficiently at time $\overline{T}$*
2. *If the minimum wage $w_L > \tilde{w}$, the equilibrium contract is such that the match dissolves inefficiently at $T^* < \overline{T}$.*

Proof. **Case 1:** $w_L \leq \tilde{w}$: Let us define $\overline{v}_L$, the minimum value it must pay the worker under the variable wage contract. Note that the minimum value to the worker under this contract must be such that the worker is paid w_L until $w(t) = v_0(\rho+\delta) - \gamma(\mu,t)(v_h - v_0) \leq w_L$ and $w(t)$ thereafter. The continuation payoff to the worker at time $\hat{t}(w_L)$ is v_0 where $\hat{t}(w_L)$ is such that

$$w(\hat{t}) = w_L = v_0(\rho+\delta) - \gamma(\mu,\hat{t})(v_h - v_0)$$

Note that if $w_L \leq \tilde{w}$, then $\hat{t}(w_L) = \overline{T}_q(w_L)$. That is the time at which the firm starts to increase the flow wage of the worker is the same at which the worker would have quit had he been awarded a fixed wage contract w_L. The minimum the firm has to promise the worker is the following wage schedule

$$\boldsymbol{\omega} = w(t) = \begin{cases} w_L & t < \hat{t} = \overline{T}(w_L) \\ v_0(\rho+\delta) - \lambda\gamma(\mu,t)(v_h - v_0) & t \geq \hat{t} \end{cases}$$

The value to the worker under this wage schedule is the same as under a fixed wage w_L. Therefore, the minimum the worker earns under the wage restriction w_L is,

$$\overline{v}_L = v(\mu, w_L, \overline{T}_q(w_L)) = v(\mu, w_L)$$

The expected post-match income for the firm when it dissolves the match at time T,

$$V(\mu, \boldsymbol{\omega}, T) = V^p(\mu, T) - v(\mu, w_L)$$

Under no commitment to a contract term, the firm dissolves the match at $\overline{T}$ which maximizes the firm's payoff. In a competitive search equilibrium, the firm may have to promise the worker a continuation above the minimum v_0 at the start of employment. In this case, the firm offers a bonus payment, ω at the start of employment, and then keeps the worker indifferent thereafter.

$$\boldsymbol{\omega} = w(t) = \begin{cases} w_L & t < \hat{t}(w_L) \\ v_0(\rho+\delta) - \lambda\gamma(\mu,t)(v_h - v_0) & t \geq \hat{t}(w_L) \end{cases}$$

The worker is paid a lump-sum amount at the start $\omega = \hat{v} - v(\mu, \boldsymbol{\omega})$ where $\hat{t}(w_L)$ is as before. The firms do not deviate from this tenure because it maximizes their value. The equilibrium $\hat{v}^* = \omega^* + v(\mu, \boldsymbol{\omega}^*)$ is determined as outlined in Section 3.

Case 2: $w_L > \tilde{w}$: Suppose the firm continues to pay the variable wage contract, and will start increasing the wage so as to avoid the worker from quitting for $t \geq \hat{t}(w_L) = \overline{T}_q(w_L)$. However, note that because $w_L > \tilde{w}$, so that $\overline{T}_q(w_L) > \overline{T} > \overline{T}_f(w_L)$. Under $\boldsymbol{\omega}$, the firm wants to fire the worker before starting to raise the wage. In other words,

$$\underbrace{V(\mu, w_L, \overline{T}_f(w_L))}_{\text{Maximum value under fixed wage } w_L} > \underbrace{V(\mu, w_L, \overline{T})}_{\text{Value to firm when the match dissolved at } \overline{T}} > V(\mu, w_L, \overline{T}_q(w_L))$$

Therefore for $w_L > \tilde{w}$, the firm is better off firing the worker earlier than optimal experimentation duration, i.e. $T_f(w_L) < \overline{T}$. Therefore, the firm can credibly only offer a contract such that $T_f(\mu, w) < \overline{T}$ for all $w_L > \tilde{w}$. □

The proposition highlights how the minimum wage restrictions interact with the equilibrium contract. Even under minimal restrictions on the offered wages, if the firms cannot commit to the contract duration, then a very high minimum wage will result in inefficient experimentation and a lower expected payoff for the workers. This implies that if there are frictions in the economy that prevent the firms from credibly committing to a contract term, such as a poor legal infrastructure, then setting a very high minimum wage could hurt the workers in equilibrium.

8.2. **Relaxing the Common Prior.** So far we have assumed that the firm and the worker share a common prior over the initial match productivity. However, it could be that one of the parties is too optimistic or pessimistic. We highlight the implications for the experimentation duration if the worker and firms do not share a common prior. Let us first consider how the worker and firm decisions are affected because of this mismatch in priors and how it affects their expected income.

The initial beliefs are μ_f for the firms and μ_w for the worker with μ, the true likelihood of a productive match. For simplicity, consider the fully flexible wage schedule. The variable wage offered to the worker is,

$$w(t) = v_0(\rho + \delta) - \lambda\gamma(\mu_w, t)(v_h - v_0)$$

with $\hat{v} - v(\mu_w, \boldsymbol{\omega}, T)$ paid as a bonus at recruitment. For ease of notation define the following function which is proportional to the density of mixture models of exponential distributions of different hazard rates.

$$h(\mu, T) := \frac{d}{dT} H(\mu, T) = \left(\mu + (1-\mu)e^{\lambda T}\right) e^{-(\rho+\lambda+\delta)T}$$

The post-match payoff of the workers is,

$$v(\mu_w, \boldsymbol{\omega}) = \hat{v}$$

The value to the firm is,

$$V(\mu_f, \boldsymbol{\omega}) = \int_0^T \left(\underline{y} - v_0(\rho+\delta) + \frac{\lambda\mu_w(v_h - v_0)}{\mu_w + (1-\mu_w)e^{\lambda\overline{z}}} + \frac{\lambda\mu_f(V_h + \overline{y})}{\mu_f + (1-\mu_f)e^{\lambda\overline{z}}} \right) (\mu_f + (1-\mu_f)e^{\lambda\overline{z}}) e^{-(\rho+\delta+\lambda)\overline{z}} dz$$

The firms should not have a profitable deviation from T_f, therefore it should be chosen in order to maximize the firm's value. Differentiating with respect to T,

$$\begin{aligned}
(HP) \qquad \mu_f(\rho+\delta+\lambda)\left(\frac{\lambda\overline{y}+\underline{y}}{\rho+\delta}-v_0\right) &= (1-\mu_f)(\rho+\delta)\left(v_0-\frac{\underline{y}}{\rho+\delta}\right)e^{\lambda T} \\
&+ \underbrace{\lambda\mu_w(v_h-v_0)\left(\frac{\mu_f}{\mu_w}-\frac{h(\mu_f,T)}{h(\mu_w,\overline{T})}\right)}_{\text{difference due to heterogeneous prior}}
\end{aligned}$$

8.2.1. *Case 1: Optimistic Workers.* This amounts to $\mu_w > \mu_f = \mu$, i.e. the firms have the true prior and the workers overestimate the likelihood of match success. The worker is paid a lower wage at a given time t, as they are more optimistic. In this case, the variable wage is not efficient and the firm retains the worker for longer than the optimal experimentation duration. It is more profitable for the firm to retain the worker longer because the worker is cheaper due to his optimism. Therefore, the firm fires the worker later than the optimal experimentation duration.

8.2.2. *Case 2: Pessimistic Workers.* This amounts to $\mu_w < \mu_f = \mu$. The firm has to pay the worker a higher wage as the worker is pessimistic. Because the firm pays the worker a higher wage, the marginal benefit from increasing the experimentation duration is lower, and the firm fires the worker sooner than the optimal experimentation duration.

8.2.3. *Case 3: Optimistic Firms.* This amounts to $\mu = \mu_w < \mu_f$. Because the firm believes that the worker is more likely to succeed she wants to experiment longer. If the productive workers' bargaining power is high, then the firm reduces the experimentation duration. If the worker parts away with most of the surplus from a productive match, then despite the firm's optimism the matches will dissolve sooner than the optimal experimentation duration.

8.2.4. *Case 4: Pessimistic Firms.* This amounts to $\mu = \mu_w > \mu_f$. The workers are paid the same wages as in the common prior. However, because the firm is pessimistic it prefers to fire the worker sooner. If the bargaining power of productive workers is very high (or their outside option income is low), then the firm pays a very low wage to the worker which allows it to experiment for a duration longer than the optimal experimentation duration.

With misspecified priors and workers having the correct belief, if the worker's post-success payoff is close to their outside option wage ($v_h - v_0 \to 0^+$), then optimistic firms experiment for too long, and the pessimistic firms experiment too little. An increase in the post-success payoff of the worker may move the endogenous experimentation toward the optimal. This observation provides a rationale for policies that enhance the property rights of innovators. Such policies by improving the post-success payoff for the innovators curtail the sub-optimal experimentation when the firms are either too optimistic, as during investment bubbles, or pessimistic as in the aftermath of a crisis.

9. Conclusion

In this paper, we propose a theoretical framework to analyze endogenous experimentation and tenure contracts under uncertainty about match productivity and search frictions. We provide a rationale for the commonly observed tenure contracts awarded to researchers and show that such contracts are efficient. However, if the search frictions are too severe, then the equilibrium wages decline and result in sub-optimal experimentation and inefficiency. We also highlight the importance of firms' ability to commit to contract tenure which may depend on the quality of institutions enforcing employment contracts. If the firms cannot commit to the tenure, then the equilibrium is generally inefficient. Our model applies not just to labor markets in performance-driven professions such as entrepreneurship, and scientific research, but also to firms licensing patents from patent owners, patients experimenting with physicians, etc. Using our model, we can speak to the implications of several policies such as minimum wage for apprentices or entry-level workers negotiated by trade unions. We highlight that a minimum wage may increase the surplus that accrues to the worker, however, if the minimum wage is set too high it may result in sub-optimal experimentation, lower surplus for the workers, and lower social welfare.

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Appendix A. Proofs

Proof of Proposition 4. From Definition 1, the equilibrium contract must satisfy,

$$V^*(\mu;\overline{v}) := \max_{\boldsymbol{\omega},q} V^u(\mu,\boldsymbol{\omega},q)$$

subject to $v(\mu,\boldsymbol{\omega},q)=\overline{v}$. The firms enter the sub-markets until the value is driven to 0, i.e. $V^*(\mu;\overline{v})=0$. First, let us show that the firm's maximization is well-defined. Define the function $\mathcal{W}:\mathbb{R}^+\to\mathbb{R}^+$ such that $V(\mu,\mathcal{W}(T),T)=0$. Since $\overline{T}_q(\mu,w)$ is monotonic in w, let its inverse be $t_q^{-1}(T;\mu)$. The feasible set for the firm to choose $\boldsymbol{\omega}$ is $\boldsymbol{\Omega}=\{(w,T):T\in[0,\overline{T}_q(\mu,w)], w\in[t_q^{-1}(T;\mu),\mathcal{W}(T;\mu)]\}$. Given $q(\mu,\boldsymbol{\omega};\overline{v})$ such that for $\boldsymbol{\omega}\in\boldsymbol{\Omega}$, the value to the worker $v^u(\mu,\boldsymbol{\omega},q(\mu,\boldsymbol{\omega};\overline{v}))=\overline{v}$, the maximization problem

$$\max_{\boldsymbol{\omega}\in\boldsymbol{\Omega}} V^u(\mu,\boldsymbol{\omega},q(\mu,\boldsymbol{\omega};\overline{v}))$$

is well-defined as $V^u(\mu,\boldsymbol{\omega},q(\mu,\boldsymbol{\omega};\overline{v}))$ is bounded in the domain $\boldsymbol{\omega}\in\boldsymbol{\Omega}$. Since $\boldsymbol{\Omega}$ is compact-valued, and $V^u(\mu,\boldsymbol{\omega},q(\mu,\boldsymbol{\omega};\overline{v}))$ is continuous in $\boldsymbol{\omega}$, $V^*(\mu;\overline{v})$ is continuous in $\overline{v}$ by the Theorem of Maximum. By the Envelope Theorem, the $V^*(\mu;\overline{v})$ is decreasing in $\overline{v}$. Clearly for $\overline{v}=V^p(\mu,\overline{T})$, $q(\mu,\boldsymbol{\omega};\overline{v})=0$ is singleton and $V^*(\mu;\overline{v})=-\kappa/\rho<0$. Moreover for $\overline{v}=v_0$, if $v(\mu,\boldsymbol{\omega})>v_0$ then $q(\mu,\boldsymbol{\omega};\overline{v})\to\infty$ and if $v(\mu,\boldsymbol{\omega})=v_0$, then $q(\mu,\boldsymbol{\omega};\overline{v})\in[0,\infty)$. In this case, $V^*(\mu;\overline{v})\to\max_{w,T}(V^p(\mu,T)-v(\mu,w,T))>0$. Therefore, there exists some $\overline{v}^*$ such that $V^*(\mu,\overline{v}^*)=0$. This gives us equilibrium contract, $\omega^*=\arg\max V^u(\mu,\boldsymbol{\omega},q(\mu,\boldsymbol{\omega};\overline{v}^*))$, the queue length $q^*=q(\mu,\boldsymbol{\omega};\overline{v}^*)$ and the equilibrium unemployment $u^*=\delta/(\delta+\alpha_u(q^*))$. □

Proof of Theorem 1. From Definition 1, the equilibrium contract must satisfy,

$$V^*(\mu;\overline{v}) := \max_{\boldsymbol{\omega},q} V^u(\mu,\boldsymbol{\omega},q)$$

subject to $v(\mu,\boldsymbol{\omega},q)\geq\overline{v}$. The firms enter the sub-markets until the value is driven to 0, i.e. $V^*(\mu;\overline{v})=0$. We can re-write this optimization problem in its dual form as below,

$$\max_{\boldsymbol{\omega},q} v^u(\mu,\boldsymbol{\omega},q)$$

subject to $V(\mu, \boldsymbol{\omega}, q) \geq V^*(\mu; \overline{v})$. Combining this with the free-entry condition, the constraint set is modified to,

$$(FE) \qquad V^u(\mu, \boldsymbol{\omega}, q) \geq V^*(\mu; \overline{v}) = 0$$

Note that the above inequality constraint must bind. If not, then the optimizing $(\boldsymbol{\omega}, q)$ are in the interior of the constraint set. In this case, more firms will enter this sub-market (as there is positive value to be made) so that q declines, and the workers are better off, implying that $(\boldsymbol{\omega}, q)$ cannot be the solution to the optimization problem.

We now write the optimization problem where $\boldsymbol{\omega} = (w, T)$.

$$\max_{w,T,q} v^u(\mu, w, T, q) = v_0 + \max_{w,T,q} \frac{\alpha_u(q)(v(\mu, w, T) - v_0)}{\rho + \delta + \alpha_u(q)} \tag{29}$$

subject to

$$V^u(\mu, w, T, q) = \frac{-\kappa + \alpha_v(q)(V^p(\mu, T) - v(\mu, w, T))}{\rho + \alpha_v(q)} = 0 \tag{30}$$

and

$$T \leq t_q(\mu, w) \tag{31}$$

The final constraint follows from the requirement that only the firm has commitment power when it comes to the duration of the contract and the worker can choose to quit anytime she pleases. Solving the optimization,

$$\begin{aligned} \mathcal{L}(w, T, q, \theta_1, \theta_2) = & \frac{\alpha_u(q)(v(\mu, w, T) - v_0)}{\rho + \delta + \alpha_u(q)} \\ & + \theta_1(-\kappa + \alpha_v(q)(V^p(\mu, T) - v(\mu, w, T))) + \theta_2(t_q(\mu, w) - T) \end{aligned}$$

The solution (w, T, q) must satisfy ,

$$\left(\frac{\alpha_u(q)}{\rho + \delta + \alpha_u(q)} - \theta_1 \alpha_v(q)\right) v_w(\mu, w, T) + \theta_2 t_{qw}(\mu, w) = 0 \tag{32}$$

$$\left(\frac{\alpha_u(q)}{\rho + \delta + \alpha_u(q)} - \theta_1 \alpha_v(q)\right) v_T(\mu, w, T) + \theta_1 \alpha_v(q) V^p_T(\mu, T) - \theta_2 = 0 \tag{33}$$

$$(34) \qquad \frac{\varepsilon(q)\rho}{(1-\varepsilon(q))(\alpha_u(q)+\rho+\delta)} = \frac{V^p(\mu,T)-v(\mu,w,T)}{(v(\mu,w,T)-v_0)}$$

the free-entry condition $\kappa = \alpha_v(q)(V^p(\mu,T)-v(\mu,w,T))$ and $\overline{T}_q(\mu,w) \geq T$. In a tenure equilibrium, the constraint $\overline{T}_q(\mu,w) > T$ (no quitting) does not bind. Therefore, $\theta_2 = 0$. Since $v_w(\mu,w,T) > 0$ for $\overline{T}_q(\mu,w) > T$, from Equation (32) it must be that $\left(\frac{\alpha_u(q)}{\rho+\delta+\alpha_u(q)} - \theta_1\alpha_v(q)\right) = 0$. Substituting into Equation (33) and noting that the free-entry condition must bind so that $\theta_1 > 0$, gives us $V^p_T(\mu,T) = 0$. That is, $T^* = \overline{T}$, the optimal tenure that maximizes the total expected value from the match $V^p(\mu,T)$. □

Proof of Proposition 5. In equilibrium, the free-entry condition, along with the tangency implies that we can equate the implicit derivatives of the iso-profit curve corresponding to 0 profit and the indifference curve corresponding to the payoff $\overline{v}^*$ in the q-w space. The implicit derivative of the iso-profit curve corresponding to the free-entry condition,

$$\frac{dq}{dw} = -\frac{\frac{\partial(V^p(\mu,T^*)-v(\mu,w^*,T^*))}{\partial w}}{-\frac{\partial\frac{\kappa}{\alpha_v(q)}}{\partial q}} = -\frac{\frac{dV^p(\mu,T^*)}{dT^*}\frac{dT^*}{dw} - \frac{dv(\mu,w^*,T^*)}{dw}}{\frac{\kappa\alpha_v'(q)}{(\alpha_v(q))^2}}$$

and the implicit derivative of the indifference curve corresponding to $\overline{v}^*$,

$$\frac{dq}{dw} = -\frac{\frac{\partial}{\partial w}\frac{(v_0(\rho+\delta)+\alpha_u(q)v(\mu,w^*,T^*))}{(\rho+\delta+\alpha_u(q))}}{\frac{\partial}{\partial q}\frac{(v_0(\rho+\delta)+\alpha_u(q)v(\mu,w^*,T^*))}{(\rho+\delta+\alpha_u(q))}} = \frac{(\rho+\delta+\alpha_u(q))\alpha_u(q)\frac{dv(\mu,w^*,T^*)}{dw}}{(\rho+\delta)(v(\mu,w^*,T^*)-v_0)\alpha_u'(q)}$$

We first equate the two slopes, and make the following substitution from the worker's indifference condition,

$$\frac{(\rho+\delta)(v(\mu,w^*,T^*)-v_0)}{(\rho+\delta+\alpha_u(q))} = v(\mu,w^*,T^*) - \overline{v}^*$$

and the free-entry condition, $\kappa = \alpha_v(q)(V^p(\mu,T^*) - v(\mu,w^*,T^*))$. Finally, we substitute

$$1 - \Theta(\mu,w^*) = \frac{\frac{\partial V^p(\mu,T^*)}{\partial T}\cdot\frac{dT^*}{dw}}{\frac{dv(\mu,w^*,T^*)}{dw}}$$

which gives us the following condition,

$$\frac{(v(\mu,w^*,T^*)-\overline{v}^*)}{(V^P(\mu,T^*)-v(\mu,w^*,T^*))} = \frac{(1-\varepsilon(q))}{\varepsilon(q)\Theta(\mu,w^*)}$$

Rearranging the terms, and dividing by $(1-\varepsilon(q)(1-\Theta(\mu,w^*)))$

$$v(\mu,w^*,T^*)=\overline{v}^*+\left(\frac{1-\varepsilon(q^*)}{1-\varepsilon(q^*)(1-\Theta(\mu,w^*))}\right)\cdot(V^p(\mu,T^*)-\overline{v}^*)$$

□

Proof of Theorem 2. We denote the planner's value function by $\overline{V}(\mu,u)$. The planner's solution satisfies the following HJB equation,

$$\rho\overline{V}(\mu,u)=\max_{q,T}\left\{u\left(v_0(\rho+\delta)+\alpha_u(q)\cdot(\overline{V}(\mu,T)+v_0)-\frac{\kappa}{q}\right)+\frac{d\overline{V}(\mu,u)}{du}\cdot\frac{du}{dt}\right\} \tag{35}$$

where $\dot{u}=\delta-u(\delta+\alpha_u(q))$.

We can see that the value function $\overline{V}(\mu,u)=A+Bu$, i.e. it is linear in u. Using the FOCs w.r.t. $(q,\overline{T})$ and the Envelope Theorem,

$$\frac{\partial}{\partial T}=0\implies\frac{dV^p(\mu,\overline{T})}{d\overline{T}}=0\implies\overline{T}=\frac{1}{\lambda}\left[\ln\left(\frac{\mu}{1-\mu}\right)+\ln\left(1+\frac{\lambda}{\rho+\delta}\right)+\ln\left(\frac{\lambda\overline{y}}{v_0(\rho+\delta)-\underline{y}}-1\right)\right]$$

$$\frac{\partial}{\partial q}=0\implies B=\frac{\alpha_u'(q)V^p(\mu,\overline{T})+\frac{\kappa}{q^2}}{\alpha_u'(q)}=V^p(\mu,\overline{T})+\frac{\kappa}{q^2\alpha_u'(q)}$$

Using the Envelope Theorem,

$$B=v_0+\frac{\alpha_u(q)(V^p(\mu,\overline{T})-v_0)-\frac{\kappa}{q}}{(\rho+\delta+\alpha_u(q))}$$

The efficient queue length q_p satisfies,

$$\frac{(\rho+\delta)(V^p(\mu,\overline{T})-v_0)}{\kappa}=-\frac{q\alpha_u'(q)+\alpha_u(q)+\rho+\delta}{q^2\alpha_u'(q)}$$

Note that the LHS is independent of q, and therefore, q_p is independent of u and only depends on the parameters and μ.

We now compare the equilibrium (q^*, T^*) with the planner's choice. The equivalent condition for the competitive search equilibrium is,

$$-\frac{(\rho+\delta+\alpha_u(q))}{\alpha_u(q)}\left(\frac{\alpha_u(q)+q\alpha_u'(q)}{q^2\alpha_u'(q)}\right) = \frac{(\overline{v}-v_0)}{\kappa}(\rho+\delta)$$

$$\frac{(\overline{v}-v_0)}{\kappa}(\rho+\delta) = -\frac{\rho+\delta+\alpha_u(q)+q\alpha_u'(q)}{q^2\alpha_u'(q)} - \frac{(\rho+\delta)}{q\alpha_u(q)}$$

Note that $q\alpha_u(q) = \alpha_v(q)$, and from the free entry

$$V(\mu, \boldsymbol{\omega}) = V^p(\mu, T^*) - \overline{v} = \frac{\kappa}{\alpha_v(q)}$$

Substituting and rearranging,

$$\frac{(\rho+\delta)(V^p(\mu, T^*)-v_0)}{\kappa} = -\frac{q\alpha_u'(q)+\alpha_u(q)+\rho+\delta}{q^2\alpha_u'(q)}$$

If we are in the fully-flexible wages benchmark, then $T^* = \overline{T}(\mu) = \overline{T}$, and $q_p = q^*$. This proves that the competitive search equilibrium allocation under the fully-flexible wages benchmark is efficient.

Under the fixed-wage competitive search equilibrium, if $w^* \geq \tilde{w}$, then $T^* = \overline{T}$. Therefore, $q_p = q^*$, i.e. under fixed-wage contracts if the equilibrium is one where the offered contract duration is shorter than the quitting time, then the equilibrium is efficient.

Note that for $w^* < \tilde{w}$, $T^* = \overline{T}_q(\mu, w^*) < \overline{T}$. Note that the RHS of Equation (A) is decreasing in q. Therefore, $q_p < q^*$ as $V^p(\mu, \overline{T}_q(\mu, w^*) < V^p(\mu, \overline{T})$. It follows that the decentralized equilibrium under fixed-wages is not efficient if workers quit in equilibrium, or $w^* < \tilde{w}$. $\square$

Proposition 8. *The threshold wage, $\tilde{w}$ increases when v_0 increases and β (worker's bargaining power) decreases. The effect of the rate of learning on $\tilde{w}$ depends on the bargaining* *power of the worker, with*

$$\frac{d\tilde{w}}{d\lambda} = \begin{cases} \geq 0 & \beta \leq \dfrac{v_p - v_0}{v_p - v_0 + \overline{y}} \\ < 0 & otherwise \end{cases}$$

Proof. The comparative statics of $\tilde{w}$ with respect to v_0 is intuitive. The workers quit faster when their outside option is more generous. As a result, the threshold beyond which the firm fires is higher. Similarly, a higher β translates to a higher v_h and a lower V_h with $(V_h + v_h)$, the total post-high output match surplus remaining constant. Therefore, for the same wage, the quitting time is higher. On the other hand, the firm has less to gain from a productive worker. The firm fires them sooner for a given wage. Hence, the $\tilde{w}$ is lower for a higher β.

It is crucial to note that the effect of λ does not operate through the rate of learning but through the value of a productive match for the firm and the worker. The rate of learning is the same for both the firm and the worker and therefore does not play a role in the determination of $\tilde{w}$. However, a higher λ is associated with a higher surplus. This implies that the worker would prefer to quit later and the firms delay firing the worker for the same wage. However, when β is sufficiently small, then the firms' benefit from firing later dominates.

□

Proof of Proposition 6. Define

$$\tilde{V}(\mu, w; \overline{v}) = V^u(\mu, w, q(w; \overline{v}))$$

which is continuous in w. $\tilde{V}(\mu, w; \overline{v}) = V^u(\mu, w, q(w; \overline{v})) = -\kappa/\rho$ for $w = \underline{w}(\overline{v})$ and $w = \overline{w}(\overline{v}) < \underline{y} + \lambda(\overline{y} + V_h)$ where for $w \in \{\underline{w}(\overline{v}), \overline{w}(\overline{v})\}$, $v(\mu, w) = \overline{v}$ $(q(w; \overline{v}) \to 0)$. $\overline{V}(\mu, w, \overline{v})$ is continuous in this domain, and

$$V^*(\mu, \overline{v}) = \max_{w \in [\underline{w}(\overline{v}), \overline{w}(\overline{v})]} \left\{ \tilde{V}(\mu, w; \overline{v}) \right\}$$

is well-defined. (Weirstrass Theorem).

$V^*(\mu, \overline{v})$ is decreasing in $\overline{v}$ and the free-entry condition pins down $\overline{v}^*$ (Envelope Theorem).

$$w^* = \arg \max_{w \in [\underline{w}(\overline{v}^*), \overline{w}(\overline{v}^*)]} \left\{ \tilde{V}(\mu, w, \overline{v^*}) \right\}$$

and $q^* = q(\mu, w^*; \overline{v}^*)$. The workers' entry condition pins down the $u^* = \delta/(\delta + \alpha_u(q^*))$. $\square$

Appendix B. Continuation Values Derivations

B.1. **Post High-output Bargaining.** Once the worker produces a high output, there is no uncertainty about the worker's type. Assume that the disagreement payoff for the firm is V^u and that for the worker is v_p.

We impose free entry condition on the firm and therefore, $V^u = 0$. The Nash-Bargaining solution is

$$\max_w \left\{ \left(\frac{w}{\rho + \delta} - v_p \right)^\beta \left(\frac{\lambda \overline{y} + \underline{y} - w}{\rho + \delta} - V^u \right)^{1-\beta} = \left(\frac{w}{\rho + \delta} - v_p \right)^\beta \left(\frac{\lambda \overline{y} + \underline{y} - w}{\rho + \delta} \right)^{1-\beta} \right\}$$

The FOC and rearrangement gives us the value to the worker $W(w)$,

$$W(w^*) = \frac{w^*}{\rho + \delta} = \beta \left(\frac{\lambda \overline{y} + \underline{y}}{\rho + \delta} - v_p \right) + v_p$$

and that to the firm,

$$V(w^*) = \frac{\lambda \overline{y} + \underline{y} - w^*}{\rho + \delta} = (1 - \beta) \left(\frac{\lambda \overline{y} + \underline{y}}{\rho + \delta} - v_p \right)$$

In the main paper we abstract away from the mechanics that determine v_p. However, here we provide a microfoundation for the same. Let us assume that when the worker disagrees, he quits and the wage is determined in competitive search for productive workers with free entry. The arrival rate of jobs to the worker is α_u and that of the workers to the firm is α_v.

$$(\rho + \delta) v_p = v_0 (\rho + \delta) + \alpha_u (q_p) \left(\frac{w}{\rho + \delta} - v_p \right) \tag{36}$$

$$\rho V^u = \max_{w,q_p} \left\{ -\kappa + \alpha_v(q_p) \left(\frac{\lambda \overline{y} + \underline{y} - w}{\rho + \delta} - V^u \right) \right\} \tag{37}$$

subject to

$$q_p \in Q(\overline{v}, w) \equiv \{q : v_p(q, w) \geq \overline{v}\}$$

and $V^u = 0$ (free entry). We can write the dual of the above problem after imposing the free entry condition,

$$(\rho + \delta)v_p = \max_{q_p} \left\{ v_0(\rho + \delta) + \alpha_u(q_p) \left(\frac{w}{\rho + \delta} - v_p \right) \right\} \tag{38}$$

subject to,

$$V^u = 0 = -\kappa + \alpha_v(q_p) \frac{\lambda \overline{y} + \underline{y} - w}{\rho + \delta} \tag{39}$$

Substituting $q\alpha_u = \alpha_v$ and using the first order condition with respect to q_p,

$$\begin{aligned} \left(\frac{\alpha_v'}{q_p} - \frac{\alpha_v}{q_p^2} \right) \left(\frac{w}{\rho + \delta} - v_p \right) &= -\frac{\alpha_v' \kappa}{\alpha_v} \\ &= \frac{\alpha_v'(w - \lambda \overline{y} - \underline{y})}{q_p(\rho + \delta)} \end{aligned} \tag{40}$$

Let $\dfrac{q\alpha_v'(q)}{\alpha_v(q)} = \varepsilon(q)$. Rearranging,

$$\frac{w^*}{\rho + \delta} = \varepsilon(q_p) \left(\frac{\lambda \overline{y} + \underline{y}}{\rho + \delta} - v_p \right) + v_p \tag{41}$$

Equations (38), (39), and (41) determine the equilibrium (q_p^*, v_p, w^*). Moreover, note that the arrival rate of workers to the recruitment pool of productive workers simply scales the number of firms v, so that $q_p^* = u/v$ holds. Therefore, the prior wage and arrival rate of workers has no effect on v_p which we take as exogenous in the main section. This allows us to subsume these values into (v_h, V_h) and focus on the wage contracts that will arise in equilibrium.

B.2. **Microfounding Post-High-Output Valuations:** So far we have assumed that q_p^* is determined in equilibrium. However, note that the corresponding u depends on the arrival rate of workers to the recruitment pool for productive workers. With the Nash-Bargaining protocol described earlier, in equilibrium no worker who produces a high output disagrees so that $u = 0$ and therefore, no sub-market for productive workers recruitment exists. In this sub-section we justify the formulation of equations (2) and (3) from the main paper.

Suppose that when the worker produces a high output, the match dissolves with probability π. This can also be rationalized as the workers adopting a mixed strategy and choosing to leave employment to search in the productive pool with probability π. The firm must offer the worker at least her outside option value v_p in order to retain the worker when renegotiating. We assume that the firm gives the worker lowest possible value so that she remains indifferent between quitting and continuing. The value to the worker is,

$$v_h = v_p$$

and the firm,

$$V_h = \pi V^u + (1-\pi)\left(\frac{\lambda\overline{y} + \underline{y}}{\rho + \delta} - v_p\right)$$

As is clear from the derivation in the previous section,

$$v_h = v_p = v_0 + \left(\frac{\alpha_u(q_p^*)}{\alpha_u(q_p^*) + \rho + \delta}\right)\left(\frac{\lambda\overline{y} + \underline{y}}{\rho + \delta} - v_0\right)$$

and

$$V_h = (1-\pi)\left(1 - \frac{\alpha_u(q_p^*)}{\alpha_u(q_p^*) + \rho + \delta}\right)\left(\frac{\lambda\overline{y} + \underline{y}}{\rho + \delta} - v_0\right)$$

Let $\pi = \epsilon$ where $\epsilon \to 0^+$ and $\left(\frac{\alpha_u(q_p^*)}{\alpha_u(q_p^*) + \rho + \delta}\right) = \tilde{\beta}$,

$$v_h = v_0 + \tilde{\beta}\left(\frac{\lambda\overline{y} + \underline{y}}{\rho + \delta} - v_0\right)$$

$$V_h = (1-\tilde{\beta})\left(\frac{\lambda\overline{y}+\underline{y}}{\rho+\delta} - v_0\right)$$

This is akin to the formulation with Nash Bargaining in Equations (2) and (3) except we replace $\beta \to \tilde{\beta}$, $v_p \to v_0$. The share of the surplus that the productive workers can get depends on the conditions in the productive workers' market. For example, a higher queue length q_p^*, reduces their share of surplus.

B.3. **Worker's Quitting Decision.**

Proof of Proposition 3. Worker's productivity is $\theta \in \{\theta^H, \theta^L\}$. Worker's type is $\mu \in (0,1)$ where $\mu = Pr(\theta = \theta^H)$ is the belief that the worker has high productivity. The firm has a prior over the types of the workers, $\mu \sim F[0,1]$.

The state variables can be summarized in time τ, the time for which the worker has been unproductive and the output in the previous instant. If the worker after producing $\underline{y}$ until time t, produces $y_t = \overline{y}$, then the firm and the worker both update their belief $Pr(\theta = \theta^H | y_t = \overline{y}) = 1$. If the worker continues to produce $\underline{y}$ the worker updates his belief about his productivity,

$$\mu(\tau) = Pr(\theta = \theta^H | \underline{y}, \tau) = \frac{\mu e^{-\lambda\tau}}{\mu e^{-\lambda\tau} + 1 - \mu}$$

The worker has an outside option that pays b_0 per unit time. Hence, the worker's lifetime value from taking the outside option is

$$v_0 = \frac{b_0}{\rho+\delta}$$

The worker's choice is the decision whether to quit and take his outside option. At each point of time τ, the worker after observing the state (τ, y_τ) will make a decision whether to continue or quit his employment in order to take the outside option. Let $d_q \in \{0,1\}$ where $d_q = 1$ refers to the decision to quit and $d_q = 0$ to continue with the current employment. The Bellman equation for the worker's optimal decision can be written as follows.

$$v(\mu,\tau) = \max_{d_q \in \{0,1\}} \begin{cases} v(\mu,\tau) & \text{if } d_q = 0 \\ v_0 & \text{if } d_q = 1 \end{cases}$$

$v(\mu,\tau)$ is the continuation value of a worker who has produced a low output and has been unproductive for time τ. Note that at any instant the worker believes that he has high productivity with probability $\gamma(\mu,\tau)$. Let $v(\theta,\tau)$ is the continuation value for a worker of type $\theta \in \{\theta^H, \theta^L\}$, then

$$v(\mu,\tau) = \gamma(\mu,\tau) \cdot v(\theta^H,\tau) + (1-\gamma(\mu,\tau)) \cdot v(\theta^L,\tau)$$

If the worker produces $\overline{y}$ then $\gamma(\mu,\tau) = 1$, i.e beyond this point as the uncertainty over his type is resolved. The worker now renegotiates wage, so that the worker's continuation value after producing a high output is v_h.

$$v_h = v_p + \beta\left(\frac{\underline{y} + \lambda\overline{y}}{\rho+\delta} - v_p\right)$$

Since, the worker updates his belief each instant depending on the output produced, we can write the expression for it as below,

$$\begin{aligned} v(\mu,\tau) = w\Delta t + e^{-(\rho+\delta)\Delta t}\gamma(\mu,\tau) \cdot (Pr(y_{\tau+\Delta t} = \underline{y}|\theta^H) \cdot v(\theta^H, \tau+\Delta t) \\ + (1 - Pr(y_{\tau+\Delta t} = \overline{y}|\theta^H)) \cdot v_h) \\ + e^{-(\rho+\delta)\Delta t} \cdot (1-\gamma(\mu,\tau)) \cdot v(\theta^L, \tau+\Delta t) \end{aligned}$$

The first expression is the flow payoff $w\Delta t$. The remaining expression is the expected continuation value depending on the belief over the worker's type and the output produced. $e^{-(\rho+\delta)\Delta t}$ is the discounted value (rate ρ) given the worker does not die (rate δ) in the period Δt.

The probability that a worker of type θ^H, will produce a low output is,

$$Pr(y_{\tau+\Delta t} = \underline{y}|\theta^H) = e^{-\lambda\Delta t}$$

Therefore the continuation value to the worker,

$$\begin{aligned}
v(\mu,\tau) = w\Delta t + e^{-(\rho+\delta)\Delta t}\gamma(\mu,\tau)\cdot\big(e^{-\lambda\Delta t}\cdot v(\theta^H,\tau+\Delta t) + (1-e^{-\lambda\Delta t})\cdot v_h\big) \\
+ e^{-(\rho+\delta)\Delta t}\cdot(1-\gamma(\mu,\tau))\cdot v(\theta^L,\tau+\Delta t)
\end{aligned} \tag{42}$$

Subtract $v(\mu,\tau+\Delta t) = \gamma(\tau+\Delta t)\cdot v(\theta^H,\tau+\Delta t) + (1-\gamma(\tau+\Delta t))\cdot v(\theta^L,\tau+\Delta t)$ from both sides, add and subtract $\gamma(\mu,\tau)v(\theta^H,\tau+\Delta t) + (1-\gamma(\mu,\tau))v(\theta^L,\tau+\Delta t)$, divide throughout by Δt, and take the limit $\Delta t \to 0$, the left hand side (42) is $\lim_{\Delta t\to 0} -\frac{(v(\mu,\tau+\Delta t)-v(\mu,\tau))}{\Delta t} = -\frac{dv(\mu,\tau)}{d\tau}$.

(43)

$$\begin{aligned}
-\frac{dv(\mu,\tau)}{d\tau} = w - \lim_{\Delta t\to 0}\left(\frac{1-e^{-(\rho+\delta+\lambda)\Delta t}}{\Delta t}\gamma(\mu,\tau) + \frac{\gamma(\tau+\Delta t)-\gamma(\mu,\tau)}{\Delta t}e^{-(\rho+\delta+\lambda)\Delta t}\right)v(\theta^H,\tau \\
+\Delta t) - \lim_{\Delta t\to 0}\left(\frac{1-e^{-(\rho+\delta)\Delta t}}{\Delta t}(1-\gamma(\mu,\tau)) - \frac{\gamma(\tau+\Delta t)-\gamma(\mu,\tau)}{\Delta t}\right)v(\theta^L,\tau+\Delta t) \\
+ \lim_{\Delta t\to 0}\frac{(1-e^{-\lambda\Delta t})}{\Delta t}\gamma(\mu,\tau)e^{-(\rho+\delta)\Delta t}v_h
\end{aligned}$$

Computing the expression at the limit $\Delta t \to 0$,

$$\begin{aligned}
-\frac{dv(\mu,\tau)}{d\tau} = w - \left((\rho+\delta+\lambda)\gamma(\mu,\tau) + \frac{d\gamma(\mu,\tau)}{d\tau}\right)v(\theta^H,\tau) \\
- \left((\rho+\delta)(1-\gamma(\mu,\tau)) + \frac{d(1-\gamma(\mu,\tau))}{d\tau}\right)v(\theta^L,\tau) \\
+ \lambda\gamma(\mu,\tau)v_h
\end{aligned} \tag{44}$$

Note that

$$\frac{d\gamma(\mu,\tau)}{d\tau} = -\lambda\frac{\mu(1-\mu)}{(\mu+(1-\mu)e^{\lambda\tau})^2}e^{\lambda\tau} \tag{45}$$

$$\frac{d\gamma(\mu,\tau)}{d\tau} = -\lambda\gamma(\mu,\tau)(1-\gamma(\mu,\tau)) \tag{46}$$

Substituting (46) in Equation (44),

$$-\frac{dv(\mu,\tau)}{d\tau} = w + \lambda\gamma(\mu,\tau)v_h - (\rho+\delta)(\gamma(\mu,\tau)v(\theta^H,\tau) + (1-\gamma(\mu,\tau))v(\theta^L,\tau)) \\ - \lambda\gamma(\mu,\tau)\left(\gamma(\mu,\tau)v(\theta^H,\tau) + (1-\gamma(\mu,\tau))v(\theta^L,\tau)\right)$$

From the definition of $v(\mu,\tau) = \gamma(\mu,\tau)v(\theta^H,\tau) + (1-\gamma(\mu,\tau))v(\theta^L,\tau)$,

$$-\frac{dv(\mu,\tau)}{d\tau} = w + \lambda\gamma(\mu,\tau)v_h - (\rho+\delta+\lambda\gamma(\mu,\tau))v(\mu,\tau) \tag{47}$$

Equation (47) is a linear differential equation in $v(\mu,\tau)$. Rearranging,

$$\frac{dv(\mu,\tau)}{d\tau} - (\rho+\delta+\lambda\gamma(\mu,\tau))v(\mu,\tau) = -(w + \lambda\gamma(\mu,\tau)v_h)$$

Multiply throughout by $e^{-\int_0^\tau(\rho+\delta+\lambda\gamma(s))ds}$

$$\frac{d}{d\tau}\left(v(\mu,\tau)e^{-\int_0^\tau(\rho+\delta+\lambda\gamma(s))ds}\right) = -(w+\lambda\gamma(\mu,\tau)v_h)e^{-\int_0^\tau(\rho+\delta+\lambda\gamma(s))ds}$$

$$e^{-\int_0^\tau(\rho+\delta+\lambda\gamma(s))ds} = (\mu + (1-\mu)e^{\lambda\tau})\cdot e^{-(\rho+\delta+\lambda)\tau}$$

Suppose, the worker quits at time t_q. The worker then earns b_0 until perpetuity. Therefore,

$$v(\mu,t_q) = v_0 = \frac{b_0}{\rho+\delta}$$

This provides us with the boundary conditions,

$$\int_t^{t_q} d\left(v(\mu,\tau)e^{-\int_0^\tau(\rho+\delta+\lambda\gamma(s))ds}\right) = -\int_t^{t_q}(w+\lambda\gamma(\mu,\tau)v_h)(\mu+(1-\mu)e^{\lambda\tau})\cdot e^{-(\rho+\delta+\lambda)\tau}d\tau$$

The solution to the differential equation is,

$$v(\mu,t,t_q) - v_0 = \gamma(\mu,t)\left(\frac{w+\lambda v_h}{\rho+\delta+\lambda} - v_0\right)\left(1 - e^{-(\rho+\delta+\lambda)(t_q-t)}\right) \\ + (1-\gamma(\mu,t))\left(\frac{w}{\rho+\delta} - v_0\right)\left(1 - e^{-(\rho+\delta)(t_q-t)}\right) \tag{48}$$

Lemma 3. *Any worker of type μ, maximizes value at time $t \in [0, t_q]$ by quitting at time*

$$\overline{T}_q(\mu) = \frac{1}{\lambda} \cdot \left[\ln\left(1 + \frac{\lambda}{\rho+\delta}\right) + \ln\left(\frac{\mu}{1-\mu}\right) + \ln\left(\frac{\lambda}{\rho+\delta+\lambda} \cdot \left(\frac{v_h - \frac{w}{\rho+\delta}}{v_0 - \frac{w}{\rho+\delta}}\right) - 1\right)\right]$$

Proof. Differentiate with respect to t_q and rearrange,

$$\frac{\partial v(\mu,t,t_q)}{\partial t_q} = \frac{(1-\mu)(\rho+\delta)e^{-(\rho+\delta+\lambda)(t_q-t)}}{\mu+(1-\mu)e^{\lambda t}} \left[\left(1+\frac{\lambda}{\rho+\delta}\right)\left(\frac{\mu}{1-\mu}\right)\left(\frac{w+\lambda v_h}{\rho+\delta+\lambda} - v_0\right) - e^{\lambda t_q}\left(v_0 - \frac{w}{\rho+\delta}\right)\right]$$

$$\frac{\partial v(\mu,t,t_q)}{\partial t_q} = \begin{cases} \geq 0 & t_q \leq \overline{T}_q(\mu) \\ < 0 & t_q > \overline{T}_q(\mu) \end{cases}$$

Therefore, the worker will prefer $\overline{T}_q$ over any other $t_q \in [0, \infty)$ to maximize value at any time t. □

However, we need to verify whether $v(\mu, t, \overline{T}_q(\mu)) - v_0 \geq 0$ for all $t \in [0, \overline{T}_q(\mu)]$. We write $v(\mu, t, \overline{T}_q)$ as,

$$v(\mu,t,\overline{T}_q) - v_0 = \frac{1}{\mu+(1-\mu)e^{\lambda t}} \left[\mu\left(\frac{w+\lambda v_h}{\rho+\delta+\lambda} - v_0\right) + (1-\mu)e^{\lambda \overline{T}_q}\left(\frac{w}{\rho+\delta} - v_0\right)e^{-\lambda(\overline{T}_q - t)} - e^{-(\rho+\delta+\lambda)(\overline{T}_q-t)}\left(\mu\left(\frac{w+\lambda v_h}{\rho+\delta+\lambda} - v_0\right) + (1-\mu)e^{\lambda\overline{T}_q}\left(\frac{w}{\rho+\delta} - v_0\right)\right)\right]$$

Substituting, $\overline{T}_q(\mu)$ so that $(1-\mu)e^{\lambda \overline{T}_q}\left(v_0 - \frac{w}{\rho+\delta}\right) = \left(1+\frac{\lambda}{\rho+\delta}\right)\mu\left(\frac{w+\lambda v_h}{\rho+\delta+\lambda} - v_0\right)$,

$$\begin{aligned} v(\mu,t,\overline{T}_q) - v_0 &= \frac{\mu}{\mu+(1-\mu)e^{\lambda t}}\left(\frac{w+\lambda v_h}{\rho+\delta+\lambda} - v_0\right)\left[1 + \frac{\lambda}{\rho+\delta}e^{-(\rho+\delta+\lambda)(\overline{T}_q-t)} - \left(1+\frac{\lambda}{\rho+\delta}\right)e^{-\lambda(\overline{T}_q-t)}\right] \\ &\geq 0 \end{aligned}$$

The inequality follows from the fact that the function,

$$h(x) = \frac{\lambda}{\rho+\delta}e^{-(\rho+\delta+\lambda)x} - \left(1+\frac{\lambda}{\rho+\delta}\right)e^{-\lambda x}$$

is increasing in x for all $x \geq 0$. Hence, for all $t \leq \overline{T}_q$,

$$v(\mu, t, \overline{T}_q) - v_0 = \frac{\mu}{\mu + (1-\mu)e^{\lambda t}} \left(\frac{w + \lambda v_h}{\rho + \delta + \lambda} - v_0 \right) (1 + h(\overline{T}_q - t)) \geq 0$$

The worker earns a value strictly higher than v_0 for all time $t \in [0, \overline{T}_q(\mu))$ and quits at time $\overline{T}_q(\mu)$. $\square$