Optimal Monetary Policy during a Cost-of-Living Crisis[*]

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Abstract

How should monetary policy react to sectoral shocks in a world where consumption baskets and demand elasticities vary across households? We present a multi-sector New-Keynesian model with generalized, non-homothetic preferences and inequality. The output gap is governed by a *Marginal* Consumer Price Index (MCPI), rather than the regular CPI. Policy trade-offs are shaped by two novel wedges in the New-Keynesian Phillips Curve (NKPC). Analytical results and quantitative simulations show that, following a negative shock to necessity sectors, the NKPC is shifted upward, increasing CPI inflation but decreasing the output gap. We find that the optimal policy response is relatively accommodative.

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1 Introduction

Since 2020, many economies have been confronted with large supply disruptions, resulting from the COVID-19 pandemic, the war in Ukraine, and other shocks. A significant surge in inflation followed, particularly in sectors producing necessities like food and energy. Low-income households have often been disproportionately affected, as they tend to allocate a larger proportion of their expenditures towards such goods.[1] Indeed, the strong squeeze in real incomes, in particular among the poorest households, has led many commentators to declare the situation a "cost-of-living crisis".

To central banks, these events underscored important yet unresolved outstanding questions: How to conduct monetary policy in a world with diverse consumption baskets, and thus heterogeneity in inflation rates across households? Do supply shocks to specific sectors, producing either necessity or luxury goods, call for a specific policy response? How do the distributional implications of such shocks affect monetary policy trade-offs? Is the Consumer Price Index (CPI) still a suitable target for monetary policy?

To answer these questions in a comprehensive way, the standard New Keynesian model –a standard framework for monetary analysis– is arguably not well suited, even when extended with sectoral heterogeneity and inequality in household income and wealth. A key limitation is that preferences are typically assumed to be of a homothetic-CES (Constant Elasticity of Substitution) form, which implies that the composition of consumption baskets is equal across households. As a result, all households share the same price index and are equally affected by sector-specific price increases, unlike in reality.

This paper presents a novel New Keynesian model which incorporates (i) multiple sectors, (ii) permanent income and wealth heterogeneity, and (iii) generalized, non-homothetic preferences, represented through "sufficient statistics" rather than a specific functional form. In this setting, each household has an individual consumption basket, creating heterogeneity in individual inflation rates, real wages and real interest rates. The generalization of preferences may also give rise to heterogeneity in price elasticities of demand across consumers. For example, rich households may not only have high overall levels of expenditures, but may also react less strongly to changes in prices of individual goods. Using the model, we examine both the positive and normative implications of aggregate and sector-level shocks.

Towards this end, we derive an analytical characterization of the model and show that two novel wedges emerge in the New Keynesian Phillips Curve (NKPC): a *non-homotheticity wedge* and an *endogenous markup wedge*. Importantly, the joint movement in these wedges shifts the NKPC in a direction that depends on the sectoral source of the shock. Specifically, a negative

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[1] According to the Office for National Statistics, in October 2022, UK households in the lowest income decile faced on average a nearly 3 percentage points higher rate of inflation than those in the highest income decile, see ONS (2022).

productivity shocks to necessity sectors initially leads to an upward shift of the NKPC, increasing inflation but reducing the output gap. By contrast, shocks to aggregate productivity, or productivity in luxury sectors, tend to move the output gap and inflation in the same direction, as is usually the case in the New Keynesian model. A cost-of-living crisis thus poses a specific challenge to a central bank seeking to stabilize inflation and the output gap, even setting aside any distributional concerns pertaining to such a situation.

In order to draw normative lessons, we study the welfare-optimal monetary policy response to aggregate and sectoral productivity shocks, and compare it to the prescription of a standard interest rate rule targeting CPI inflation. In a simplified version of the model, we show analytically that the optimal policy stance following a negative necessity shock is initially relatively loose, because of the upward shift in the NKPC mentioned above. A swift and strong increase in interest rate could bring down inflation, but only at the expense of a strongly negative output gap, which is not optimal. However, later on the optimal policy tightens, which is qualitatively in line with the delayed tightening by several central banks in response to the recent shocks.

An important implication of non-homothetic preferences is that households devote a relatively large fraction of *marginal* spending to luxuries. Indeed, a household which spends most of its budget on necessities may still allocate a large fraction of any additional spending to luxuries. Accordingly, the real wage which guides marginal saving and labor supply decisions is one which deflates the nominal wage with a *Marginal CPI* (MCPI), weighing sectors by marginal rather than regular budget shares and thus down-weighting necessities compared to the regular CPI. We show that output gap dynamics are associated with the MCPI rather than the regular CPI. Therefore, the MCPI complements the CPI as a natural metric to guide monetary policy.

To better understand the policy trade-offs, we study the two novel NKPC wedges in detail. The first is a *non-homotheticity wedge*. This wedge captures a labor market distortion which arises due to the gap in marginal and regular budget shares on different sectors, in the presence of price rigidities. To understand this wedge intuitively, consider a shock which simultaneously decreases productivity in necessity sectors but increases productivity in luxury sectors. Following this shock, luxury goods become cheaper relative to necessity goods. This increases the real wage in units of households' marginal consumption bundles, since at they margin they spend relatively more on luxuries. In turn, the increase in the marginal real wage induces households to optimally increase labor supply. However, when prices are sticky this increase is diminished, because relative prices react less strongly. As a result, labor supply is distorted downwards and the output gap becomes negative for a given inflation rate or –equivalently– inflation increases for a given output gap. A decrease in the relative productivity of necessity sectors thus shifts up NKPC, while a decrease in the relative productivity of luxury sectors would have the opposite effect.[2]

The second wedge in the NKPC is an *endogenous markup wedge*, which arises from the fact that price elasticities of demand for goods vary across households and over time, once we move beyond homothetic-CES preferences over varieties within sectors. Realistically, poorer households are likely to be more price sensitive and demand elasticities may increase during recessions, as consumption falls. For firms, demand elasticities are in turn a key consideration when setting markups. Fluctuations in the level and distribution of consumption thus create fluctuations in demand elasticities and hence distortions in markups. Specifically, the wedge tends to shift the NKPC *downward* after negative productivity shocks. Compared to the non-homotheticity wedge, the movements in the endogenous markup wedge tend to be smaller but more persistent. Therefore, the combined effect of the two wedges is that, following a negative shock to necessity sectors, the NKPC is initially shifted upward, but downward later on, calling for a specific dynamic policy response which depends on the sectoral origin of the shock.

In addition to the analytical results derived in the simplified model, we conduct a quantitative exploration in a full-blown version of the model, calibrated to the United Kingdom. The model features realistic heterogeneity in income, wealth, expenditure baskets, and marginal propensities to consume, disciplined by data from the Living Costs and Food (LCF) survey. We also allow for heterogeneity in price rigidities across sectors and input-output linkages. Despite its richness, the model is computationally tractable, up to a first-order approximation, as we can characterize the dynamic equilibrium with as a system of sector-level NKPCs and Euler equations, alongside two sector-level equations tracking the relevant aspects of the wealth distribution.

Model simulations reveal that the channels and policy trade-offs highlighted analytically are also important quantitatively. We observe that, under a standard interest rate rule targeting CPI inflation, negative shocks to necessity sectors, such as *Food* or *Electricity & Gas*, lead to an increase in CPI inflation but an initial decline in the output gap, followed by a subsequent upswing. By contrast, after a negative shock to productivity in all sectors, or only in luxury sectors, CPI inflation and the output gap both increase persistently. Regarding the distributional impact of aggregate and sectoral shocks, we also find strong heterogeneity in the consumption responses of individual households, depending not only on their income and wealth but also on their expenditure baskets.[3]

We solve for the welfare-optimal interest rate path in response to sectoral shocks. We do

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[2] While this channel also arises in a representative-agent version of the model with non-homothetic preferences, its strength depends on the degree of long-run inequality. And importantly, empirical discipline on the channel is critically obtained from cross-sectional evidence on the relation between income and expenditures on different goods, which is at odds with a representative-agent assumption.

[3] In addition to non-homothetic preferences, the model includes idiosyncratic preference shifters for goods from different sectors, allowing us to match exactly the heterogeneous consumption baskets observed in micro data.

so analytically in a simplified version of the model, as well as quantitatively in the full-blown model. We find that, compared to a standard interest rate rule, the optimal policy response to a negative necessity shock is initially significantly more accommodative, i.e. the interest rate is held relatively low. For shocks to luxury sectors, we find the opposite. Later on, the optimal policy stance tightens, in particular following necessity shocks. Moreover, we find that potential distributional considerations lead to an overall looser monetary policy reaction to negative productivity shocks, as lower interest rates redistribute wealth towards poorer households, who tend to be more heavily affected by such shocks.[4]

Relation to the literature. A main contribution of this paper is to embed a generalized, non-homothetic preference structure in a multi-sector New Keynesian model, allowing for household inequality. Empirical evidence supporting the relevance of non-homothetic preferences has a long history in the literature. A particularly famous and robust finding is that expenditure shares on food are negatively related to income (Engel, 1857; Houthakker, 1957). It is also understood that these patterns have important implications for the aggregate price indices and the measurement of inequality, see e.g. Hamilton (2001); Kaplan and Schulhofer-Wohl (2017); Jaravel (2019); Argente and Lee (2021). While in this paper we focus on monetary policy and business cycles, others have studied the implications for non-homothetic preferences for growth and structural transformation (e.g. Herrendorf et al. (2014); Boppart (2014); Comin et al. (2021)). Non-homothetic preferences are also recognized to have important implications for tax policy, see Jaravel and Olivi (2021). We further relate to literature which deviates from CES preferences over goods varieties, e.g. Kimball (1995); Amiti et al. (2019); Xhani (2021) and which studies how demand elasticities and markups vary across the income distribution, see e.g. Mongey and Waugh (2023); Nord (2023); Sangani (2023).

The New-Keynesian literature typically sticks to the simplifying assumption of homothetic-CES preferences.[5] Such models therefore abstract from heterogeneity in consumption baskets even when they feature household heterogeneity.[6] Indeed, the mechanisms that we highlight complement the channels highlighted in the literature on monetary policy transmission with heterogeneous agents, see e.g. McKay et al. (2016); Kaplan et al. (2017); Auclert (2019) and many others. This literature often emphasizes the role of heterogeneity in Marginal Propensities to Consume (MPCs), a micro-level non-linearity which makes distributions matter for macroe-

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[4] This is the case even though our assumed social welfare function is such that monetary policy has no motive to affect steady-state inequality.

[5] Some authors in this literature have deviated from CES utility by assuming a Kimball demand function, see e.g. Smets and Wouters (2007). However, such preference preserve homotheticity and do not create endogenous markup fluctuations. Cavallari and Etro (2020) consider a representative-agent model with extended CES preferences which delivers a time-varying price elasticities of demand.

[6] One exception is Blanco and Diz (2021) who study a representative-agent household NK model with two consumption goods, one of which is subject to a subsistence point. Another one is Melcangi and Sterk (2019), who develop a heterogeneous-agents New Keynesian model with an infrequently consumed luxury good.

conomic outcomes. While we connect to this literature, our analysis highlights heterogeneity in consumption behaviour generated by non-homothetic, non-CES preferences. This form of household heterogeneity matters not only for the demand block of the model (characterised by Euler equations and household constraints) but also for the supply block of the model, as characterised by the NKPC. Indeed, we show that household heterogeneity affects both the slope of the NKPC and the time-varying wedges that emerge under generalized preferences.

The normative analysis in this paper connects to the literature on optimal policy in the NK model, see e.g. Galí (2015) and references therein, and on how inequality and redistribution affect optimal monetary policy trade-offs, including redistributive effects, see Challe (2020); Bhandari et al. (2021); Nuno and Thomas (2022); Dávilla and Schaab (2022); Acharya et al. (2023); McKay and Wolf (2023). Our model abstracts from idiosyncratic risk. Instead, we show how non-homothetic, non-CES preferences, combined with permanent inequality, gives rise to novel policy trade-offs. Finally, the multi-sector structure of our model connects our contribution to several recent papers on intersectoral transmission of shocks in neoclassical models and (HA)NK models, including Baqaee and Farhi (2019); Pasten et al. (2020); Rubbo (2023); LaO and Tahbaz-Salehi (2019); Baqaee et al. (2021); Moll et al. (2023); Schaab and Tan (2023); Auclert et al. (2023).

The remainder of this paper is organized as follows. Section 2 lays out the primitive model environment and provide an analytical characterization of the model. In Section 3 we inspect the mechanisms in a relatively simple version of the model, focusing on the role of the two new wedges in the NKPC. Results for the full quantitative model are presented in Section 4. Optimal policy is discussed in Section 5. Section 6 concludes.

2 The model

2.1 Environment

Households. There is a continuum of heterogeneous households, of unit mass and indexed by i. In every period t, a household dies with a probability $\delta \in (0,1)$. Households consume goods from different sectors, indexed by $k = 1, 2.., K$. Within each sector, there is a unit mass continuum of differentiated varieties, indexed by j. The expected utility of household i at time t is given by:

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\beta(1-\delta))^{t+s} \left(u_i(\mathbf{c}_{t+s}(i)) - \chi\left(\frac{n_{t+s}(i)}{\vartheta(i)}\right)\right), \tag{1}$$

where $n_{t+s}(i)$ is effective labor supply, $\vartheta(i)$ is labor productivity, $\beta \in (0,1)$ is the subjective discount factor, and $\mathbb{E}_t$ is the conditional expectations operator. Moreover, the utility from consumption depends on a vector $\mathbf{c}_t(i) = \{\mathbf{c}_{1,t}(i), .., \mathbf{c}_{K,t}(i)\}$, where $\mathbf{c}_{k,t}(i)$ is in turn a vector consisting of the consumption of each variety j in sector k. Specifically, the flow utility from consumption is given by:

$$u_i(\mathbf{c}_t(i)) = U_i(\mathcal{U}(\mathbf{c}_{1,t}(i)), ..., \mathcal{U}(\mathbf{c}_{K,t}(i))),$$

where $U_i(\cdot)$ is an outer utility function, defined over sectoral bundles, which may be household specific. We assume that $U_i(\cdot)$ is differentiable and weakly separable across sectors. The sectoral bundles are in turn given by $\mathcal{U}(\mathbf{c}_{k,t}(i))$. We further assume that the inner utility function $\mathcal{U}(\cdot)$ is a concave, C^3-function which is symmetric over varieties. Moreover, $\chi(\cdot)$ is an increasing, twice differentiable function capturing disutility from labor supply.

Households can save in one-period nominal bonds, denoted by $b_t(i)$ and they are born with different initial levels of nominal wealth. Households also differ in terms of their labor productivity, $\vartheta(i)$, which is constant over time. We thus abstract from idiosyncratic risk, aside from mortality risk. We do allow for the possibility that some households are Hand-to-Mouth (HtM) consumers, which we treat as a permanent characteristic.[7] HtM households cannot adjust their bond holdings, and thus consume their current incomes. Households who are not HtM can choose bond holdings freely, facing only a natural borrowing limit. Households further differ in their ownership of firms. The budget constraint of household i in period t is given by:

$$e_t(i) + \frac{b_{t+1}(i)}{R_t} = b_t(i) + n_t(i)W_t + \sum_k \varsigma_k(i) Div_{k,t}. \tag{2}$$

Here, $e_t(i) = \sum_{k=1}^K e_{k,t}(i) = \sum_{k=1}^K \int_0^1 p_{k,t}(j) c_{k,t}(i,j) dj$ denotes the household's total consumption expenditures, R_t is the gross nominal interest rate on bonds, which is set by a central bank, W_t is the nominal wage per effective unit of labor, $Div_{k,t}$ are total dividends from sector k and $\varsigma_k(i)$ is the equity share of household i in firms in sector k. We assume that equity portfolios are perfectly diversified within and across sectors.

In any period t, household i chooses consumption of each goods variety, $c_{k,t}(i,j)$, bond holdings, $b_t(i)$, and effective labor supply, $n_t(i)$, to maximize utility objective (1), subject to the budget constraint (2) and the laws of motion of equilibrium objects exogenous to households. HtM households in addition face the constraint $b_t(i) = b_{t-1}(i)$.

Some key statistics. In the absence of a parametric form for preferences, let us introduce some key concepts regarding household behavior. As discussed in Appendix A, we can express the demand of household i for a certain goods variety as a function of its price, $p_{k,t}(j)$, a vector of all other prices in the sector, denoted $\mathbf{p}_{k,t}$, and the total expenditures of the household on sector-k goods, $e_{k,t}(i)$. We denote this demand function by $c_{k,t}(i,j) = d_k\left(p_{k,t}(j), \mathbf{p}_{k,t}, e_{k,t}(i)\right)$.

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[7] Even without HtM households, distributional dynamics will generally matter for aggregates, due to the nonlinearities embedded in the generalized, non-homothetic and non-CES preferences.

Table 1. Steady-state statistics

	Individual	Aggregate
Marginal Propensity to Consume:	$MPC(i) = \frac{\partial e_t(i)}{\partial b_t(i)}$	
Budget share:	$s_k(i) = \frac{e_k(i)}{e(i)}$	$\bar{s}_k = \frac{E_k}{E}$
Marginal budget share:	$\partial_e e_k(i) = \frac{\partial e_k(i)}{\partial e(i)}$	$\overline{\partial_e e_k} = \int \frac{e(i)}{E} \partial_e e_k(i) di$
Cross-price elasticity:	$\rho_{k,l}(i) = \frac{\partial c_k(i)}{\partial P_l} \frac{P_l}{c_k(i)}$	$\bar{\rho}_{k,l} = \frac{\partial C_k}{\partial P_l} \frac{P_l}{C_k}$
Demand elasticity:	$\epsilon_k(i) = -\frac{\partial c_k(i,j)}{\partial p_k(j)} \frac{p_k(j)}{c_k(i,j)}$	$\bar{\epsilon}_k = \int \frac{e_k(i)}{E_k} \epsilon_k(i) di$
Super-elasticity:	$\epsilon^s_k(i) = \frac{\partial \epsilon_k(i)}{\partial p_k(j)} \frac{p_k(j)}{\epsilon_k(i)}$	$\bar{\epsilon}^s_k = \frac{\partial \bar{\epsilon}_k}{\partial p_k(j)} \frac{p_k(j)}{\bar{\epsilon}_k}$
Markup sensitivity w.r.t. expenditures:	$\gamma_{e,k}(i) = \frac{\partial \mu_k}{\partial e_k(i)} \frac{E_k}{\mu_k}$	
Markup sensitivity w.r.t. wealth:	$\gamma_{b,k}(i) = \frac{\partial \mu_{k,t}}{\partial b_t(i)} \frac{E}{\mu_k}$	

Note: all statistics are evaluated in the deterministic steady state with zero inflation. $E_k = \int e_k(i)$ are aggregate expenditures on sector k and $E = \sum_k E_k$ are total expenditures across all sectors. Moreover, $C_k = E_k / P_k$ is aggregate sectoral consumption. Finally, $\rho_{k,l}(i)$ is a compensated elasticity.

We can now define a number of household-level statistics, evaluated at the deterministic steady state of the model, which we indicate by omitting the time subscript. We consider a steady state with zero inflation and therefore equal prices within sectors, i.e. $p_k(j) = P_k$ for any variety j in sector k where P_k is the sectoral price level. Note that in such a steady state it holds that $c_k(i,j) = c_k(i)$. Table 1 defines the statistics, which may all vary across households. The table also presents a number of aggregate counterparts that will play a role in the model.

The first statistic is the Marginal Propensity to Consume, often emphasized in the heterogeneous-agents literature. In our setting, we can derive $MPC(i) = \frac{R-1}{R} / \left(1 + \frac{Wn(i)\psi}{e(i)\sigma}\right)$ for non-HtM households and $MPC(i) = 1/\left(1 + \frac{Wn(i)\psi}{e(i)\sigma}\right)$ for HtM households. Within both groups of households, there is MPC heterogeneity resulting from differences in the wealth effect on labor supply, which in turn is due to differences in the composition of financial versus human wealth.

The next three statistics in the table derive from the outer utility function $U_i(\cdot)$ and thus pertain to the allocation of household expenditures across sectors. First, $s_k(i)$, is the regular budget share, i.e. the fraction of expenditures that household i devotes to sector k. Its aggregate counterpart, $\bar{s}_k$, is used to construct the Consumer Price Index, which is defined as $P_{cpi} = \sum_k \bar{s}_k P_k$. Second, $\partial_e e_k(i)$, is the household's *marginal* budget share on sector k. It measures the fraction of each marginal unit of expenditures that the household devotes to goods in sector k. This statistic is not much emphasized in the literature on macroeconomic fluctuations. Indeed, under homothetic preference we obtain $\partial_e e_k(i) = s_k(i)$. However, in our model preference are non-homothetic and the gap between the two statistics will play an important role. The aggregate (expenditure-weighted) counterpart of the marginal budget share is $\overline{\partial_e e_k}$. At the margin, households tend to spend less on necessity goods than they do on average, whereas the opposite is true for luxuries. Accordingly, we label k a necessity sector if $\overline{\partial_e e_k} < \bar{s}_k$, and a luxury sector if $\overline{\partial_e e_k} > \bar{s}_k$.

For later use, we define the *Marginal CPI* (MCPI) as $P_{mcpi} = \sum_k \overline{\partial_e e_k} P_k$. This price index weighs sectors by their marginal rather than their regular budget shares. Relative to the CPI, the MCPI thus overweights luxury sectors and underweights necessity sectors.[8] Note that under homothetic preferences over sectors, marginal and regular budget shares coincide, so that the CPI and MCPI become equal. The final statistic relating to the outer utility function is $\rho_{k,l}(i)$, the compensated elasticity of consumption by household i of sector-k goods with respect to a change in P_l, the price of sector-l goods. Moreover, $\bar{\rho}_{k,l}$ is the aggregate counterpart.

The remaining statistics pertain to the inner utility $\mathcal{U}$, which defines utility over varieties within a sector. These statistics will be key determinants of markups in the model. The first, $\epsilon_k(i)$, is the elasticity of demand for a variety with respect to its price $p_k(j)$. Note that this elasticity varies not only across sectors, but also across households. When setting the markup, firms consider the aggregate demand elasticity for their good, $\bar{\epsilon}_k$, which weighs individual elasticities by expenditure shares. The steady-state markup is given by $\mu_k = \frac{\bar{\epsilon}_k}{\bar{\epsilon}_k - 1}$. While $\epsilon_k(i)$ denotes the demand elasticity at the steady state, the distribution of demand elasticities moves around over time: as households change their levels of expenditures, their demand elasticities change. The response of the individual demand elasticity to a change in the price is given by the price super-elasticity of demand, denoted by $\epsilon_k^s(i)$, as defined in the table.[9] Under CES preferences, demand elasticities are constant and hence $\epsilon_k^s(i) = 0$, but once moving beyond CES this is no longer the case. The super-elasticity of *aggregate* demand for sector-k varieties can be expressed as $\bar{\epsilon}_k^s = (\int \epsilon_k^s(i)\epsilon_k(i)\frac{e_k(i)}{E_k}di - \int (\epsilon_k(i) - \bar{\epsilon}_k)^2 \frac{e_k(i)}{E_k}di)/\bar{\epsilon}_k$. This object takes into account that a change in prices not only affects $\bar{\epsilon}_k$ via changes in individual demand (the first term) elasticities, but also through changes in the composition of demand (the second term).

When moving beyond CES preferences, different households thus contribute differently to markups, depending on their price elasticities of demand, their super-elasticities, and their share in aggregate expenditures. We define two additional statistics which capture the combined effects of this. First, $\gamma_{e,k}(i)$ measures the sensitivity of the markup with respect to individual i's expenditures on sector-k goods: $\gamma_{e,k}(i) = \left(1 - \frac{\epsilon_k(i)}{\bar{\epsilon}_k}\left(1 + \frac{\partial \epsilon_k(i)}{\partial e_k(i)}\frac{e_k(i)}{\epsilon_k(i)}\right)\right)\frac{1}{\bar{\epsilon}_k - 1}$. Intuitively, if there is a relative increase in expenditures among households who have relatively low demand elasticities, the aggregate demand elasticity decreases, pushing up markups. A similar effect takes place if there is a shift in expenditures towards households whose price elasticity of demand is relatively insensitive to the level of expenditures. The second, $\gamma_{b,k}(i)$, captures the markup sensitivity with respect to individual wealth, which we can express as $\gamma_{b,k}(i) = MPC(i)\gamma_{e,k}(i)\partial_e e_l(i)/\bar{s}_k$. Note that under CES preferences we obtain $\gamma_{e,k}(i) = \gamma_{b,k}(i) = 0$.

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[8] One may think of "Core CPI" –a popular index in practice– as an extreme sibling of the MCPI, in the sense that it completely disregards prices in two of the most important necessity sectors: Food and Energy.

[9] Note that, due to symmetry and anticipating that in the steady state firms are identical within sectors, $\epsilon_k(i)$ and $\epsilon_k^s(i)$ do not depend on j, i.e. at the steady state these elasticities are the same for all varieties within a sector.

Finally, we assume that the Elasticity of Intertemporal Substitution (EIS) and the Frisch elasticity of labor supply are homogeneous across households, and denote them by σ and ψ respectively. It is possible to allow for heterogeneity in these objects as well, at the expense of somewhat more complicated algebraic expressions.[10]

Firms. Firms are monopolistically competitive, each producing a single goods variety j in a certain sector k. Within each sector, firms are ex-ante identical but subject to a Calvo-style pricing rigidity: they are able to adjust their price only with a probability $1-\theta_k$ in every period. This probability may vary across sectors. Firms in sector k operate the following technology:

$$y_{k,t}(j) = A_{k,t}F_k(n_{k,t}(j), \tilde{Y}_{1,k,t}(j), \tilde{Y}_{2,k,t}(j), ..., \tilde{Y}_{K,k,t}(j)), \tag{3}$$

where $y_{k,t}(j)$ is output, $F_k(\cdot)$ is a sector-specific production function with constant returns to scale and $A_{k,t}$ is an exogenous, sector-specific productivity variable. In the production function, $n_{k,t}(j)$ are effective units of labor hired by the firm, while $\tilde{Y}_{l,k,t}(j)$ is the quantity of intermediate inputs from sector $l = 1, 2, ..., K$ used in production by firm j in sector k. Intermediate goods are produced by competitive firms who bundle varieties and sell on the these bundles. The technology of these firms is given by $\tilde{Y}_{k,t} = \tilde{F}_k(\tilde{\mathbf{y}}_{k,t})$ where $\tilde{\mathbf{y}}_{k,t}$ is a vector of varieties used in production and where we assume that $\tilde{F}_k$ is twice differentiable, symmetric across varieties and has constant return to scale. We can express the demand of the intermediate goods producers for an individual variety j as $\tilde{y}_{k,t}(j) = \tilde{d}_k\left(p_{k,t}(j), \mathbf{p}_{k,t}\right)\tilde{Y}_{k,t}$.

Firms take as given the aggregate of household demand functions, as well as demand by intermediate goods producers. The total demand for a variety is given by:

$$y_{k,t}(j) = \int_0^1 d_k\left(p_{k,t}(j), \mathbf{p}_{k,t}, e_{k,t}(i)\right) di + \tilde{d}_k\left(p_{k,t}(j), \mathbf{p}_{k,t},\right)\tilde{Y}_{k,t}. \tag{4}$$

where the first term corresponds to household demand and the second to demand from intermediate goods producers. Under CES preferences, household demand for a variety can be expressed as a simple function of its relative price and total demand. In our more general setting, however, the composition of demand matters as well, as demand elasticities and super-elasticities vary across households.

Firms which are allowed to adjust their price do so to maximize the expected present value of profits. The decision problem of those firms is given by:

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[10] It is always possible to renormalize the utility function to obtain a common and arbitrary EIS and Frisch elasticity. Straub (2017) presents a model with EIS heterogeneity.

$$\max_{\substack{p_{k,t}(j),\{n_{k,t+s}(j),\\ y_{k,t+s}(j),\tilde{Y}_{l,k,t+s(j)}\}_{s=0}^{\infty}}} \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta_k^s \left(p_{k,t}(j)y_{k,t+s}(j) - (1-\tau_k)(W_{t+s}n_{k,t+s}(j) + \sum_{l} P_{l,t+s}\tilde{Y}_{l,k,t+s}(j)) - T_{k,t+s} \right), \tag{5}$$

subject to Equations (3) and (4), where $\Lambda_{t,t+s}$ is the firm's stochastic discount factor.[11] In the above equation, τ_k is a time-invariant, sector-specific subsidy which may be used by the government to correct markup distortions in the steady state, and $T_{k,t}$ a lump-sum tax to finance the subsidy, which can be arbitrarily differentiated across sectors, as long as the government budget constraints is satisfied.

Government Policy. We assume that the fiscal authority runs a balanced budget, which implies:

$$\sum_{k=1}^{K} \tau_k \int_0^1 (W_t n_{k,t}(j)) + \sum_{l=1}^{K} P_{l,t}\tilde{Y}_{l,k,t}(j))dj - \sum_k T_{k,t} = 0. \tag{6}$$

The nominal interest rate R_t is set by the monetary authority, taking fiscal policy as given. We will consider two versions of the model. In the first, the central bank follows a simple interest rate rule. In the second version, the interest rate is set optimally.

Demographics and Market Clearing. In any period, a fraction δ of all households dies. We assume that each deceased household is replaced by a new household of the same type. A household's type is pinned down by its labor productivity, $\vartheta(i)$, firm ownership, $\varsigma_k(i)$, initial bond holdings, $b_0(i)$, preferences, U_i, and HtM status. Bond market clearing implies that the average wealth of households is zero, and hence the same is true for deceased and newborn households, due to i.i.d. death probabilities. Therefore, the wealth given to new households can always be financed and the net inheritance from all deceased households is zero. From now on, we will assume that firm ownership is proportional to labor productivity. Clearing in the labor market and the bond market requires, respectively:

$$\begin{aligned} \int_0^1 n_t(i)di &= \sum_k \int_0^1 n_{k,t}(j)dj, \\ \int_0^1 b_t(i)di &= 0. \end{aligned} \tag{7}$$

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[11] We assume that in the steady state $\Lambda_{t,t+s} = (1-\delta)^s\beta^s$. We do not need to make further assumptions on $\Lambda_{t,t+s}$ since we will linearize the model around a steady state with zero inflation.

Goods market clearing requires, for any goods variety:

$$\int_0^1 c_{k,t}(i,j)di + \tilde{y}_{k,t}(j) = y_{k,t}(j). \tag{8}$$

and in every sector:

$$\tilde{Y}_{k,t} = \sum_l \int \tilde{Y}_{l,k,t}(j)dj. \tag{9}$$

An equilibrium is a law of motion for prices and allocations such that households, firms and the government behave as specified above, and markets clear. It is worth noting that in the deterministic steady state of the model, households keep their bond holdings constant over time.[12] The model is thus consistent with any arbitrary steady-state distribution of wealth, which in the calibration we will take from the data.

2.2 Dynamic Equilibrium

In order to study dynamics, we linearize the model around a deterministic steady state. We assume that the central bank targets long-run price stability, so steady-state prices are identical within sectors. We further assume that the government eliminates steady-state markup distortions using the subsidy τ_k.

We now present the system of equations that jointly characterize the dynamic equilibrium of the model, to a first-order approximation. Appendix A provides the underlying derivations, and Appendix B summarizes the equations. To ease the exposition, we present in the main text a simplified model version without HtM households and without Input-Output linkages. In the quantitative applications, we do include these features. Moreover, in Section 3 we will consider a version of the model that is further simplified and derive a number of analytical results which help to sharpen intuition.

New Keynesian Phillips Curve. The central equation in our analysis is the New Keynesian Phillips Curve (NKPC). Let $\hat{P}_{k,t} = \int \hat{p}_{k,t}(j)dj$ be the price of the sector-k goods, where hatted variables denote log deviations from the steady state and where we used that in the steady state prices are identical within sectors. We will denote steady-state variables by omitting the time subscript t. The steady-state interest rate equals $R = \frac{1}{\beta(1-\delta)}$. The net rate of inflation in sector k is given by:

$$\pi_{k,t} = \hat{P}_{k,t} - \hat{P}_{k,t-1}. \tag{10}$$

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[12] It can be shown that, in the absence of idiosyncratic income risk and aggregate shocks, the target level of wealth equals current wealth.

Moreover, individual consumption of sector-k goods is given by $\hat{c}_{k,t}(i) = \hat{e}_{k,t}(i) - \hat{P}_{k,t}$. The NKPC for sector k can be now expressed as:

$$\pi_{k,t} = \kappa_k \tilde{\mathcal{Y}}_t + \lambda_k \left(\mathcal{NH}_t + \mathcal{M}_{k,t} - \mathcal{P}_{k,t} \right) + \beta(1-\delta)\mathbb{E}_t \pi_{k,t+1}, \tag{11}$$

with:

$$\tilde{\mathcal{Y}}_t = \hat{\mathcal{Y}}_t - \hat{\mathcal{Y}}^*_t, \tag{Output gap}$$

$$\mathcal{NH}_t = \sum_{l=1}^{K} (\overline{\partial_e e_l} - \bar{s}_l)(\hat{P}_{l,t} - \hat{P}^*_{l,t}), \tag{Non-homotheticity wedge}$$

$$\mathcal{M}_{k,t} = \int \gamma_{e,k}(i) \frac{c_k(i)}{C_k} \hat{c}_{k,t}(i) di - \Gamma_k \tilde{\mathcal{Y}}_t, \tag{Endogenous markup wedge}$$

$$\mathcal{P}_{k,t} = (\hat{P}_{k,t} - \hat{P}_{cpi,t}) - (\hat{P}^*_{k,t} - \hat{P}^*_{cpi,t}), \tag{Relative price wedge}$$

and the following slope coefficients:

$$\begin{aligned}
\kappa_k &= \lambda_k \left(\frac{1}{\sigma} + \frac{1}{\psi} \right) \left(1 + \frac{\sigma\psi}{\sigma + \psi} \Gamma_k \right), \\
\lambda_k &= \frac{(1-\theta_k)\,(1-\theta_k/R)}{\theta_k} \frac{\bar{\epsilon}_k - 1}{\bar{\epsilon}_k - 1 + \bar{\epsilon}^s_k}, \\
\Gamma_k &= \frac{R}{R-1} \frac{\sigma + \psi}{\sigma} \int \gamma_{b,k}(i) \frac{Wn(i)}{WN} di.
\end{aligned}$$

Before explaining our generalized NKPC in detail, let us note that it is a generalization of the 'standard' NKPC. As usual, the equation relates current sectoral rate of inflation, $\pi_{k,t}$, to the discounted expected rate of inflation, $\beta(1-\delta)\mathbb{E}_t \pi_{k,t+1}$, and an "output gap", $\tilde{\mathcal{Y}}_t$.

In addition, a number of wedges emerge in the NKPC, which affect the joint dynamics of the output gap and inflation. The first of these, $\mathcal{NH}_t$, arises due to non-homothetic preferences over sectors, which makes the composition of consumption baskets vary across households and over time. The second, $\mathcal{M}_{k,t}$, arises due to changes in markups due to fluctuations in the price elasticities of demand faced by firms, which are no longer constant once one deviates from CES preferences over varieties within sectors. We label this wedge the *endogenous markup wedge*. The two new wedges will affect the trade-offs between the output gap and inflation, faced by the central bank. Finally, there is a relative price wedge $\mathcal{P}_{k,t}$ which generally arises in New Keynesian models with sectoral asymmetries.

Slope of the NKPC. Let us now discuss the equation in more detail, starting with κ_k, the slope coefficient with respect to the output gap. The first term within this coefficient, λ_k, captures the micro-level pass-through of marginal costs to prices and in turn consists of two components. The first component within λ_k, i.e. $\frac{(1-\theta_k)(1-\theta_k/R)}{\theta_k}$, is due to sticky prices and is standard in the NK model. The second component, $\frac{\bar{\epsilon}_k - 1}{\bar{\epsilon}_k - 1 + \bar{\epsilon}^s_k}$, arises because of the endogeneity of demand elasticities. Intuitively, a firm realises that if it raises its price, demand will fall and, as a result, consumers may become more price sensitive. This component does not appear under CES preferences ($\bar{\epsilon}^s_k = 0$), but it does appear under for instance Kimball (1995) preferences. In a typical calibration it holds that $\bar{\epsilon}^s_k > 0$, which implies that the pass-through from marginal costs to prices is less than one-for-one, even when prices are fully flexible.

The second term in the definition of κ_k, i.e. $\left(\frac{1}{\sigma} + \frac{1}{\psi}\right)$, is standard in the NK literature. The third term, $\left(1 + \frac{\sigma\psi}{\sigma+\psi}\Gamma_k\right)$, is again due to non-CES preferences. However, this time it captures an aggregate spending effect: when household change their consumption levels, demand elasticities react, which induces firms to change markups. When markups tend to be increasing in wealth ($\gamma_{b,k}(i) > 0$) then an increase in aggregate income makes consumers less price sensitive, therefore pushing up markups. Again, the term vanishes under CES preferences.[13]

Note further that in the general setting, κ_k depends on the entire steady-state distribution of expenditures, through Γ_k and $\bar{\epsilon}^s_k$. Thus, long-run changes in inequality affect the slope of the NKPC. As such, our environment differs from typical HANK settings, in the sense that inequality affects not only the demand block of the model, as represented by consumption Euler equations and budget constraints, but also the supply block, as formed by the NKPCs.

Output gap. The first term on the right hand side of the NKPC is the well-known 'output gap'. Here, $\hat{\mathcal{Y}}_t$ is an aggregate demand index, and $\hat{\mathcal{Y}}^*_t$ is its 'natural' counterpart, indicated by a star and defined as its level in a parallel economy without markup distortions. As in the standard NK model, the output gap captures distortions in the labor market due to time-varying markups. To see this concretely, one can express the output gap alternatively as a (household) wage gap: $\tilde{\mathcal{Y}}_t = \frac{\psi}{1+\frac{\psi}{\sigma}}\left(\hat{w}_{h,t} - \hat{w}^*_{h,t}\right)$, where $\hat{w}_{h,t} = \hat{W}_t - \sum_{l=1}^{K} \overline{\partial_e e_l}\hat{P}_{l,t}$ is the real wage, computed using the Marginal CPI (MCPI) as the deflator, which is the relevant wage for marginal labor supply decisions. Moreover, $\hat{w}^*_{h,t} = \sum_{l=1}^{K} \overline{\partial_e e_l}\hat{A}_{l,t}$ is the natural counterpart of the real wage. This expression for the output gap also obtains in the standard NK model, in which the CPI and MCPI coincide.

In an economy with heterogeneous agents and multiple sectors there are in principle many ways in which one could measure aggregate labor market distortions. However, in Section 6 and Appendix E.3 we show that above formulation of the output gap gives precisely the distortion that enters into a planner's social welfare objective (see Result 7 for a simple case).

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[13] It also vanishes under Kimball (1995) preferences, since such preferences are homothetic, in the sense that they are scaled to be invariant to total demand.

Dynamically, the output gap index evolves according to the following Euler equation:

$$\tilde{\mathcal{Y}}_t = \mathbb{E}_t \tilde{\mathcal{Y}}_{t+1} - \sigma \mathbb{E}_t \left(\hat{R}_t - \pi_{mcpi,t+1} - \hat{r}^*_t \right). \tag{12}$$

This Euler equation has the standard form, except that the real interest rate is computed as $\pi_{mcpi,t} = \sum_{l=1}^{K} \overline{\partial_e e_l} \pi_{l,t}$, i.e. MCPI rate of inflation, rather than the regular CPI. Intuitively, when households decide on consumption today versus consumption tomorrow, they consider on which sectors they spend at the margin. In the Euler equation, $\hat{r}^*_t$ is the natural real interest rate associated with the demand index, i.e. the real interest rate that satisfies the Euler Equation for the natural level of aggregate demand. We can express this rate as:

$$\hat{r}^*_t = \frac{1}{\sigma + \psi} \sum_{l=1}^{K} \left(\psi \overline{\partial_e e_l} + \bar{s}_l \right) (\hat{A}_{l,t+1} - \hat{A}_{l,t}), \tag{13}$$

Moreover, we can express as the natural level of demand and the natural sectoral price as $\hat{\mathcal{Y}}^*_t = \sum_{l=1}^{K} \frac{\psi \overline{\partial_e e_l} + \bar{s}_l}{1 + \psi/\sigma} \hat{A}_{l,t}$ and $\hat{P}^*_{k,t} = -\hat{A}_{k,t}$, respectively.

Note that in the equation for the natural rate, both regular budget shares ($\bar{s}_l$) and the marginal budget shares ($\overline{\partial_e e_l}$) enter. Indeed, in this economy, both the regular CPI and the MCPI matter for aggregate demand. To clarify this point further, let us express the natural level of demand as $\hat{\mathcal{Y}}^*_t = -\frac{1}{1+\psi/\sigma} \hat{P}^*_{cpi,t} - \frac{\psi}{1+\psi/\sigma} \hat{P}^*_{mcpi,t}$, i.e. as a weighted sum of the natural CPI and MCPI. Intuitively, sectoral productivity shocks directly affect aggregate income by shifting the productive capacity of the economy. For this effect, the regular budget shares (i.e. CPI shares) are the relevant sectoral weights. Secondly, sectoral shocks have an indirect equilibrium effect on households' marginal saving and labor supply decisions. For these decisions, the marginal budget shares are the relevant sectoral weights.

Non-homotheticity wedge. We now discuss the two novel NKPC wedges. The first of these, $\mathcal{NH}_t = \sum_{l=1}^{K} (\overline{\partial_e e_l} - \bar{s}_l)(\hat{P}_{l,t} - \hat{P}^*_{l,t})$, is a wedge which arises due to non-homothetic preferences combined with distortions in relative sectoral prices. This wedge increases when prices are distorted downward ($\hat{P}_{l,t} < \hat{P}^*_{l,t}$) in necessity sectors ($\overline{\partial_e e_l} < \bar{s}_l$), but falls when prices are distorted downward in luxury sectors. Indeed, the movements in this wedge will depend critically on the sectoral nature of shocks. Note that $\mathcal{NH}_t$ is the same for all sectors, since it derives from a distortion in the aggregate labor market. Note further that under homothetic preferences, marginal and regular budget shares coincide and hence $\mathcal{NH}_t = 0$. Under non-homothetic preferences, the wedge moves over time. The direction and magnitude of its movement depends on the gap $\overline{\partial_e e_l} - \bar{s}_l$, which in turn depends on the extent of steady-state inequality.[14]

To understand the wedge, it is useful to consider an alternative formulation, given by

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[14] Under non-homothetic preferences, budget shares are non-linear functions total expenditures, hence a long-run change in inequality will generally change the gap between marginal and regular budget shares.

$\mathcal{NH}_t = (\hat{w}_{f,t} - \hat{w}^*_{f,t}) - (\hat{w}_{h,t} - \hat{w}^*_{h,t})$. Here, $\hat{w}_{f,t} = \hat{W}_t - \hat{P}_{cpi,t}$ is the real wage according to the CPI, which is relevant to the marginal cost of the firm (weighted by sales), and $\hat{w}^*_{f,t} = \sum_{l=1}^{K} \bar{s}_l \hat{A}_{l,t}$ is its natural counterpart. Recall that $\hat{w}_{h,t} = \hat{W}_t - \sum_{l=1}^{K} \overline{\partial_e e_l} \hat{P}_{l,t}$ is the real wage according according to the MCPI deflator, which is relevant to households' marginal labor supply decisions, and $\hat{w}^*_{h,t} = \sum_{l=1}^{K} \overline{\partial_e e_l} \hat{A}_{l,t}$ is its natural counterpart. We now observe that $\mathcal{NH}_t$ can be interpreted as a term capturing the extent to which real wage distortions differ between households and firms. As such, $\mathcal{NH}_t$ can be interpreted as a labor wedge, akin to a labor income tax distortion.

To obtain further intuition, note that in an economy with non-homothetic preferences, labor supply optimally responds to changes in relative sectoral productivities, even if aggregate productivity (i.e. weighted sectoral productivity) does not change. Intuitively, when the relative productivity of luxury sectors increases, and relative prices in these sectors fall, households optimally increase labor supply since at the margin they spend relatively more on luxuries. To see this concretely, note that when CPI weighted aggregate productivity does not move, then $\hat{\mathcal{Y}}^*_t = -\frac{\psi}{1+\psi/\sigma} \hat{P}^*_{mcpi,t}$. Given this, any increase in the relative productivity of luxury sectors means that the natural MCPI declines, which leads to an increase in labor supply, increasing the natural level of output. However, when prices are sticky, the relative price movements are muted, and as a result $\mathcal{Y}_t$ increases by less than its natural counterpart, i.e. the output gap becomes negative.

Endogenous markup wedge. The second novel wedge, $\mathcal{M}_{k,t}$, captures the evolution of the distribution of price elasticities of demand for individual goods varieties, which affects the markups set by firms. The distribution of demand elasticities in turn fluctuates with the distribution of expenditures. The distributional origins of the wedge become clear by observing the first term in its definition, $\int \gamma_{e,k}(i) \frac{c_k(i)}{C_k} \hat{c}_{k,t}(i) di$, which integrates over individual households. Here $\hat{c}_{k,t}(i)$ is the consumption change of household i, $\frac{c_k(i)}{C_k}$ is the household's share in total sectoral consumption, and $\gamma_{e,k}(i)$ the sensitivity of markups with respect to individual expenditures. The second term, $-\Gamma_k \tilde{\mathcal{Y}}_t$, subtracts the endogenous markup response due to fluctuations in the output gap, as this effect has been subsumed in κ_k.

The endogenous markup wedge arises due to non-CES utility over varieties within sectors.[15] To see this, note that under CES preference we obtain $\gamma_{b,k}(i) = \Gamma_k = 0$, as demand elasticities are constant, which in turn implies that $\mathcal{M}_{k,t} = 0$. Moving beyond CES, the wedge takes the same form as exogenous markup shocks often considered in New Keynesian models. However, in our setting it is a rich endogenous object, which is shaped by the distribution of expenditures across households, and it therefore moves over time, along with the distribution of wealth. Nonetheless, it turns out that the evolution of the endogenous markup wedge can

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[15] Note that preferences may be homothetic but non-CES and vice versa.

be represented in a tractable way. Specifically, it can be decomposed as:

$$\mathcal{M}_{k,t} = \Gamma_k \hat{\mathcal{Y}}_t^* + \mathcal{M}_{k,t}^P + \mathcal{M}_{k,t}^D. \tag{14}$$

The first component, $\Gamma_k \hat{\mathcal{Y}}_t^*$, arises due to changes in demand elasticities in response to changes in the natural level of aggregate demand. Intuitively, during an economic downturn households cut expenditures and become less price-sensitive, which induces firms to reduce price markups.

The second component captures how substitutions in response to changes in prices in other sectors affect demand elasticities:

$$\mathcal{M}_{k,t}^P = \sum_{l=1}^{K} \mathcal{S}_{k,l} \cdot \left(\hat{P}_{l,t} - \hat{P}_{k,t}\right), \tag{15}$$

where $\mathcal{S}_{k,l} = \int_i \frac{e_k(i)}{E_k} \gamma_{e,k}(i) \rho_{k,l}(i) di$ captures the effect of cross-price substitution on demand elasticities, and hence markups.

The third component, $\mathcal{M}_{k,t}^D$, summarizes the effects of changes in the distribution of household-level real expenditures on markups. For instance, a redistribution from poor to rich agents may give rise to an increase in markups, if rich people are less price sensitive. The evolution of $\mathcal{M}_{k,t}^D$ can be characterized by the following equation:

$$\mathcal{M}_{k,t}^D = \mathbb{E}_t \mathcal{M}_{k,t+1}^D - \sum_{l=1}^{K} \sigma_{k,l}^{\mathcal{M}} (\hat{R}_t - \mathbb{E}_t \pi_{l,t+1}) - \frac{\delta}{1-\delta} \mathbb{E}_t \mathcal{M}_{k,t+1}^0, \tag{16}$$

for any sector k, where $\sigma_{k,l}^{\mathcal{M}} = \sigma \int \gamma_{e,k}(i) \frac{e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \sigma \overline{\partial_e e_l} \Gamma_k$. In Equation (16), $\mathcal{M}_{k,t+1}^0$ captures the dynamics of the wealth distribution, insofar relevant for the markup wedge. It is pinned down by the following equation:

$$\begin{aligned} \mathcal{M}_{k,t}^0 = {} & \frac{1}{(1-\delta) R} \mathbb{E}_t \mathcal{M}_{k,t+1}^0 + \int \gamma_{b,k}(i) \frac{b(i)}{RE} di \left(\hat{R}_t - \mathbb{E}_t \pi_{cpi,t+1}\right) \\ & - \sum_{l=1}^{K} \int \gamma_{b,k}(i) \left(\frac{e(i)}{E} \left(s_l(i) - \bar{s}_l\right) + \frac{\psi W n(i)}{WN} \left(\partial_e e_l(i) - \overline{\partial_e e_l}\right) \right) di \hat{P}_{l,t} - \frac{R-1}{R} \mathcal{M}_{k,t}^D. \end{aligned} \tag{17}$$

Here, the second term on the right hand side captures markup effects due to changes in real interest rates. Intuitively, an increase in interest rates may redistribute wealth towards richer households, who have lower price elasticities of demand. This shifts the composition of demand towards households who are less price sensitive, which induces firms to increase markups. Similarly, the third term captures the markup effects of redistributions due to changes in relative prices.

Relative price wedge. The final wedge in the NKPC, $\mathcal{P}_{k,t} = (\hat{P}_{k,t} - \hat{P}_{cpi,t}) - (\hat{P}^*_{k,t} - \hat{P}^*_{cpi,t})$ arises due to distortions in relative sectoral prices. Specifically, $\hat{P}_{k,t} - \hat{P}_{cpi,t}$ is the sectoral price, relative to the CPI and $\hat{P}^*_{k,t} - \hat{P}^*_{cpi,t}$ is its natural equivalent. The wedge $\mathcal{P}_{k,t}$ is generally present in multi-sector extensions of the standard NK model, if sectors are asymmetric in some way, e.g. if they differ in the degree of price rigidity or if there are sectoral shocks.

Monetary policy. In the positive part of our analysis, we will consider a simple interest rate rule of the following form:

$$\hat{R}_t = \sum_k \phi_k \pi_{k,t}, \tag{18}$$

where setting $\phi_k = \phi \bar{s}_k$ delivers a rule which responds to the CPI inflation rate. In Section 5, we will move beyond the simple rule and instead consider the fully optimal Ramsey policy.

Dynamic Equilibrium. Equations (10)-(18) constitute a system of $5K + 3$ equations in $5K + 3$ endogenous variables, given by $\{\hat{P}_t, \pi_{k,t}, \mathcal{M}^D_{k,t}, \mathcal{M}^P_{k,t}, \mathcal{M}^0_{k,t}\}_{k=1}^K, \tilde{\mathcal{Y}}_t, \hat{R}_t, r^*_t$. We can thus characterize the model with a core block of equations, despite the fact that fluctuations in the distribution of income and wealth matter for the aggregate equilibrium outcomes. The equations for $\mathcal{M}^D_{k,t}$ and $\mathcal{M}^0_{k,t}$ keep track of the relevant distributional moments in a tractable way.

Distributional dynamics. While we do not need to keep track of the full distributional dynamics in order to solve for the aggregate equilibrium, it is straightforward to solve for such dynamics. Here, we focus on the distribution of consumption. Let us define the response of real consumption expenditures of household i as $\hat{c}_t(i) = \hat{e}_t(i) - \sum_{l=1}^K s_l(i)\widehat{P}_{l,t}$. Moreover, let $\boldsymbol{\omega}$ be a vector defining a weight $\omega(i)$ on each household i, with $\int \omega(i)di = 1$. We can thus use $\boldsymbol{\omega}$ to select and weight any arbitrary subset of households.

Now consider some moment of the consumption distribution, $\hat{C}_t(\boldsymbol{\omega}) = \int \omega(i)\hat{c}_t(i)di$. For instance, if we set $\omega(i) = e(i)/E$, then this moment corresponds to the aggregate response of real expenditures. We could also set $\omega(i) = 1$ for only one specific household i and zero for all others. In that case, $\hat{C}_t(\boldsymbol{\omega})$ corresponds to the individual consumption response of a particular household. Alternatively, one can choose $\boldsymbol{\omega}$ to compute the average response among households with certain characteristics. We can characterize $\hat{C}_t(\boldsymbol{\omega})$ with the following Euler equation:

$$\mathbb{E}_t\hat{C}_{t+1}(\boldsymbol{\omega}) - \hat{C}_t(\boldsymbol{\omega}) = \sigma\left(\int \omega(i)di\hat{R}_t - \textstyle\sum_k \int \omega(i)\partial_e e_k(i)di\, \mathbb{E}_t\pi_{k,t+1}\right) + \frac{\delta}{1-\delta}\hat{C}^0_t(\boldsymbol{\omega}), \tag{19}$$

where wealth dynamics are captured by:

$$\hat{C}^0_t(\boldsymbol{\omega}) - \frac{1}{(1-\delta)\,R}\mathbb{E}_t\hat{C}^0_{t+1}(\boldsymbol{\omega}) = \int \omega^0(i)\frac{b(i)}{RE}di\left(\hat{R}_t - \mathbb{E}_t\Sigma_k\bar{s}_k\pi_{k,t+1}\right) + \left(1+\frac{\psi}{\sigma}\right)\int \omega^0(i)\frac{Wn(i)}{WN}di\hat{\mathcal{Y}}_t$$
$$- \Sigma_k \int \omega^0(i)\left(\frac{e(i)}{E}\left(s_k(i)-\bar{s}_k\right) + \frac{Wn(i)}{WN}\psi\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)di\hat{P}_{k,t} - \frac{R-1}{R}\hat{C}_t(\boldsymbol{\omega}), \tag{20}$$

where we defined $\omega^0(i) = \frac{R-1}{R}\frac{\omega(i)}{e(i)/E+Wn(i)/WN\frac{\psi}{\sigma}}$.

3 Understanding the NKPC Wedges

Before studying the model quantitatively and deriving the optimal policy, we present a number of analytical results which help understand how the wedges respond to aggregate and sectoral shocks, how they affect aggregate dynamics, and to what extent it is possible for policy to neutralize the distortions they create. In order to derive these results, we consider two simplifying assumptions which we impose throughout this section, and which we dispose of in the quantitative analysis:

Assumptions:
(**A.1**) The slope of the NKPC with respect to the output gap is homogeneous across sectors, i.e. $\kappa_k = \kappa > 0$ for any sector k.
(**A.2**) There is no steady-state wealth heterogeneity, i.e. $b(i) = 0$ for any household i.[16]

We can now derive a number of results. The proofs of these are provided in Appendix C.

Result 1 (policy invariance of the sectoral wedges): *Under (**A.1**)-(**A.2**), $\mathcal{NH}_t$, $\mathcal{M}_{k,t}$ and $\mathcal{P}_{k,t}$ evolve independently of monetary policy.*

The key insight behind our first analytical result is that all three wedges can be expressed as functions of relative sectoral prices and relative nominal wealth positions only. When the slope of the NKPC with respect to the output gap is homogeneous across sectors and there is no initial nominal wealth heterogeneity, the central bank has no levers to move these relative outcomes, and hence the wedges become invariant to monetary policy. The wedges then become similar to exogenous markup shocks often introduced to NK models, but with potentially richer dynamics depending on movements in the wealth distribution.

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[16] As above, we also abstract from Input-Output linkages and Hand-to-Mouth agents. Note that we do allow for income heterogeneity and for endogenous wealth heterogeneity in response to shocks. Moreover, there is still wealth inequality out of steady state. Finally, all the results go trough under a generalized assumption $\int \gamma_{b,k}(i)b(i)di = 0$, i.e. what matters is that wealth positions are orthogonal to markup contributions.

In the full model, assumptions (**A.1**)-(**A.2**) do not apply. It then becomes possible for policy to affect the wedges, but only via two specific channels: relative sectoral prices and nominal redistributions. Thus, even if a central bank's mandate refers only to aggregate inflation and the output gap, wealth heterogeneity and movements in sectoral prices become intermediate targets for policy.

3.1 The role of the $\mathcal{NH}$ wedge

Let us now explore the wedges in more detail, starting with the non-homotheticity wedge, $\mathcal{NH}$. In order to focus exclusively on this wedge, let us assume, in addition to (**A.1**)-(**A.2**), that $\mathcal{M}_t = 0$, i.e. preferences are are homothetic over sectors. We do preserve the other wedges, i.e. $\mathcal{NH}_t \neq 0$ and $\mathcal{P}_{k,t} \neq 0$. We can now derive our second analytical result, highlighting the relevance of the MCPI index:

Result 2 (Divine coincidence without endogenous markup wedge): *If (**A.1**)-(**A.2**) hold and $\mathcal{M}_t = 0$, then fluctuations in the output gap can be eliminated by stabilising the Marginal CPI index, defined as $\pi_{mcpi,t} \equiv \sum_k \overline{\partial_e e_k} \pi_{k,t}$.*[17]

We thus recover a version of the "Divine Coincidence" often emphasized in the NK literature. But rather than stabilising the CPI index, policy should stabilise the *Marginal* CPI index in order to eliminate fluctuations in the output gap. This result follows from the NKPC for Marginal CPI inflation, which under the assumptions reduces to:

$$\pi_{mcpi,t} = \kappa \tilde{\mathcal{Y}}_t + \beta(1-\delta)\mathbb{E}_t \pi_{mcpi,t+1}. \tag{21}$$

Note that all remaining wedges drop out of this equation. It follows immediately that when $\pi_{mcpi,t} = 0$ at all times, then $\tilde{\mathcal{Y}}_t = 0$.[18]

The MCPI index thus emerges as a natural candidate to be a target for policy. In fact, in this simplified setting the model becomes isomorphic to the standard 3-equation NK model if policy targets the MCPI rather than CPI inflation. To see this, suppose that policy follows a simple interest rate rule targeting the MCPI:

$$\hat{R}_t = \phi \pi_{mcpi,t}.$$

Together with Equation (12), the above two equations form a 3-equation system which take the exact same form as the standard NK model, but with specifically the MCPI index for inflation.

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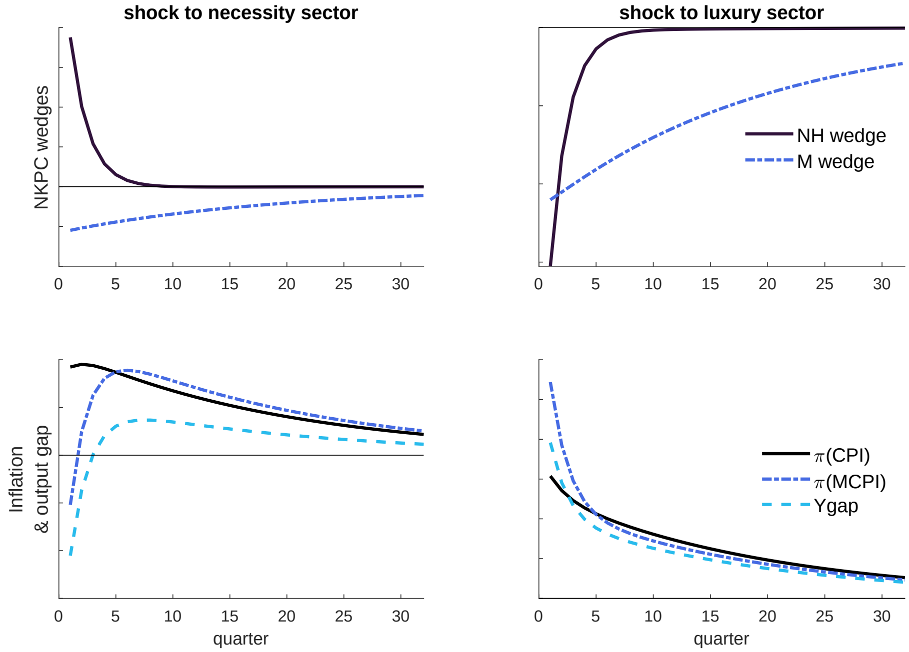
[17] When we relax assumption (**A.1**) and (**A.2**), the divine coincidence index is $\pi_{d,t} \equiv \sum_k \frac{\overline{\partial_e e_k}/\lambda_k}{\sum_l \overline{\partial_e e_l}/\lambda_l} \pi_{k,t}$, the divine coincidence index with I-O linkages is derived in appendix F.

[18] Recall that in this section we abstract from I-O linkages. Rubbo (2023) derives a divine-coincidence index for an NK model with such links.

Illustration: responses to negative sectoral productivity shocks.



Notes: responses to a negative productivity shock to a necessity sector (left panels) and to a luxury sector (right panels). Simplified version satisfying assumptions (**A.1**)-(**A.2**).

Yet, even when the output gap and MCPI inflation are fully stabilized, there are still fluctuations in the CPI. To see this clearly, consider the NKPC for CPI inflation:

$$\pi_{cpi,t} = \kappa \tilde{\mathcal{Y}}_t + \lambda \mathcal{NH}_t + \beta(1-\delta)\mathbb{E}_t \pi_{cpi,t+1}. \tag{22}$$

Thus, due to fluctuations in the $\mathcal{NH}$ wedge, there is a policy trade-off between the output gap and the regular CPI inflation index. Put differently, if monetary policy wishes to neutralise labor market distortions, it must accept fluctuations in CPI inflation. The trade-off between CPI inflation and the output gap depends critically on the sectoral nature of the shock, since the wedge moves in different directions in response to different sectoral shocks:

Result 3 (response of $\mathcal{NH}$ to a sectoral shocks): *If (**A.1**)-(**A.2**) hold and $\mathcal{M}_t = 0$ then, following a negative productivity shock to a necessity (luxury) sector, $\mathcal{NH}_t$ rises (falls) on impact.*

To understand Result 3, consider a negative productivity shock to a necessity sector l. In response, prices rise in that sector due to an increase in marginal costs. However, price stickiness prevents prices from rising as much as in the undistorted case, and therefore $\hat{P}_{l,t} < \hat{P}^*_{l,t}$. That is, prices in the necessity sector are distorted *downward*. Since households consume less necessities at the margin than on average, i.e. $\overline{\partial_e e_l} < \bar{s}_l$, this creates an increase in $\mathcal{NH}_t = \sum_l (\overline{\partial_e e_l} - \bar{s}_l)(\hat{P}_{l,t} - \hat{P}^*_{l,t})$. Intuitively, following a negative productivity shocks to necessities, the *relative* price of luxuries falls. As explained above, this induces households to optimally increase labor supply, since households spend relatively more on luxuries at the margin. However, price rigidities dampen the increase in the relative price of luxuries. Therefore, labor supply is pushed up by less than is optimal, i.e. the output gap falls. Thus, following a negative shock to a necessity (luxury) sector, the non-homotheticity wedge shifts the NKPC upwards (downwards).

To understand the specific policy trade-offs created by sectoral shocks, it is instructive to first consider an extreme policy which strictly targets the CPI, i.e. $\pi_{cpi,t} = \pi_{cpi,t+1} = 0$. It then follows immediately from Result 3 and Equation (22) that a negative productivity shock to a necessity sector results in a negative output gap. Intuitively, the downward distortion in the MCPI-deflated wage depresses workers' labor supply inefficiently. Policy could neutralize this effect by stabilising instead MCPI inflation and the output gap, i.e. by targeting $\pi_{mcpi,t} = \pi_{mcpi,t+1} = \tilde{\mathcal{Y}}_t = 0$, but this would would come at the cost an increase in CPI inflation. Analogously, a negative shock to a luxury sector would *reduce* CPI inflation under this policy.

The response of the output gap thus depends critically (i) the sectoral nature of the shock (luxury vs necessity), and (ii) the inflation index targeted by the central bank. This remains the case once we consider less extreme policies. To show this, let us first consider an MCPI-based rule $\hat{R}_t = \phi \pi_{mcpi,t}$. In this case, the output gap responds to a change in the natural real interest rate exactly as in the standard 3-equation NK model, as the model is isomorphic. Indeed, following a negative productivity shock, the output gap will increase since the natural rate r_t^* increases, regardless of the sectoral nature of the shock.

However, under a CPI-based rule of the form $\hat{R}_t = \phi \pi_{cpi,t}$, the output gap may actually *decline* following a negative necessity shock. To see this, let us rewrite this rule as $\hat{R}_t = \phi \pi_{mcpi,t} + u_t^R$, where $u_t^R = \phi(\pi_{cpi,t} - \pi_{mcpi,t})$. Together with Equations (12) and (21), we obtain a system that is isomorphic to the standard 3-equation NK model but with an additional, endogenous monetary policy shock, u_t^R. Following a negative shock to a necessity sector, CPI inflation increases by more than MCPI inflation, i.e. u_t^R increases, creating an effect akin to a a monetary contraction, pushing down the output gap. If this additional effect is strong enough, the output gap becomes negative. Intuitively, the CPI overweights necessity sectors relative to the MCPI. Therefore, if the central bank targets the CPI, it increases the interest rate by 'too much' when a negative shock to necessities increases prices in that sector. Following a negative shock to a luxury sector, the opposite effect occurs, i.e. there is an additional expansionary effect.[19]

The figure above illustrates the insights so far, by showing impulse response functions for a simplified version of the model in which (**A.1**)-(**A.2**) apply and the central bank follows a CPI-

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[19] The endogenous monetary policy shock is closely related to the non-homotheticity wedge. When λ_k is also homogeneous across sectors, we can express it as $u_t^R = \lambda \sum_{s \geq 0} \frac{1}{R^s} \mathcal{NH}_{t+s}$.

based rule. Following a negative necessity shock, the $\mathcal{NH}$ wedge rises and the output gap falls, whereas CPI inflation rises. With tighter monetary policy, CPI inflation could be reduced, but this would be at the expense of a more negative output gap, i.e. a trade-off arises. By contrast, following a negative shock to luxuries, both the output gap and CPI inflation increase. In this case, a tightening of policy could bring down both. The figure also shows that MCPI inflation can move rather differently from CPI inflation, and that the former tends to co-move more closely with the output gap. In Appendix C, we provide analytical solutions of the model under simple interest rate rules and the simplifying assumptions.

3.2 The role of the $\mathcal{M}$ wedge

Let us now consider movements in the endogenous markup wedge, $\mathcal{M}$. We can show that the Divine Coincidence breaks down once this wedge is active:

Result 4 (Breakdown divine coincidence): *When $\mathcal{M}_t \neq 0$, there generally does not exist an inflation index which can be fully stabilised along with the output gap regardless of the shocks.*

Intuitively, movements the endogenous markup wedge derive from real sources (fluctuations in demand elasticities), which cannot be neutralized with a nominal instrument.

How does the endogenous markup wedge move in respond to shocks? Let us start with an aggregate shock:

Result 5 (dynamics of the endogenous markup wedge): *If (**A.1**)-(**A.2**) hold and $\lambda_k = \lambda \; \forall k$ then $\mathcal{M}_t$ declines following a negative aggregate productivity shock.*

To understand this, it is useful to recall Equation (14) which decomposes the wedge as $\mathcal{M}_{k,t} = \Gamma_k \hat{\mathcal{Y}}^*_t + \mathcal{M}^P_{k,t} + \mathcal{M}^D_{k,t}$. Under the simplifying assumptions, only the first component, $\Gamma_k \hat{\mathcal{Y}}^*_t$ moves in response to aggregate productivity shock, and thus the decline in $\mathcal{M}_t$ is entirely driven by a fall in efficient output. Intuitively, a fall in income creates a decline in aggregate demand, which makes households become more price sensitive, and therefore reduces markups.

Following *sectoral* shocks, the sign of $\mathcal{M}_{k,t}$ is ambiguous, since such shocks bring about relative price changes and redistributions, so that $\mathcal{M}^P_{k,t}$ and $\mathcal{M}^D_{k,t}$ move as well. In other words, the movements in the endogenous markup wedge generally depend on the sectoral source of the shock. To illustrate this point, let us make a further simplifying assumption:

Assumption:
(**A.3**) Outer preferences are of the Stone-Geary form, the superelasticity of the sectoral markup $\gamma_{e,k}(i)\partial_e e_k(i)/E_k$ is equal across sectors, and $\gamma_{e,k}$ is positive and increasing in $e_k(i)$.

We can now derive our final result about the CPI aggregates $\mathcal{M}^P_{cpi,t} = \sum_l \bar{s}_k \mathcal{M}^P_{k,t}$ and $\mathcal{M}^D_{cpi,t} = \sum_l \bar{s}_k \mathcal{M}^D_{k,t}$:

Result 6 (dynamics of the endogenous markup wedge): *If (**A.1**)-(**A.3**) hold, then $\mathcal{M}^P_{cpi,t}$ decreases (increases) and $\mathcal{M}^D_{cpi,t}$ increases (decreases) following a negative productivity shock to a necessity (luxury) sector.*

Under Stone-Geary preferences, the cross price elasticity of demand is given by $\rho_{k,l}(i) = \overline{\partial_e e_l}\,(1 - \underline{c}_k / c_k(i))$. Thus, expenditure switching in response to necessity price changes is relatively low, since the marginal budget share $\overline{\partial_e e_l}$ is low for necessities. Following a negative necessity shock, the substitution towards luxury goods is therefore relatively weak. As a result, expenditures and markups decline in the necessity sector, but this is not fully compensated by an increase in markups in the luxury sector. Therefore, $\mathcal{M}^P_{cpi,t}$ decreases. Moreover, a negative necessity shock disproportionately reduces the spending power of the poor, i.e. there is a relative redistribution towards the rich. Because the rich are are less price sensitive than the poor, the redistribution puts upward pressure on markups, i.e. $\mathcal{M}^D_{cpi,t}$ increases.[21]

The dynamics of the endogenous markup wedge are illustrated in the figure above. Note that the $\mathcal{M}$ wedge declines following both shocks, as the aggregate demand component $\Gamma_k \hat{\mathcal{Y}}^*_t$ dominates in this illustration. Note further that the decline is relatively modest but very persistent, which drives of the upswing in the output gap several quarters after the necessity shock hits, as well as the persistent increase in the output gap following a negative shock to the luxury sector.

4 Quantitative Analysis

The analytical results presented in the previous section show how shifts in the NKPC, and hence policy trade-offs, can depend critically on the sectoral source of the shock. Our next goal is to study quantitatively the effects of productivity shocks to different sectors. To this end, we revert back to the full model, in which the slope of the NKPC may vary across sectors, there is steady-state heterogeneity in nominal wealth, some households are Hand-to-Mouth and there are Input-Output linkages across sectors. We consider the model with an interest rate rule, targeting CPI inflation. In the next section, we consider optimal monetary policy.

4.1 Parameterization

We calibrate the model to the United Kingdom. The model period is set to one quarter. Parameter values are displayed in Tables 3 and 4, and are discussed below in detail. We include

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[20] Here, $\underline{c}_k$ is the subsistence of sector-k consumption. Under (**A.3**), we have $\mathcal{M}^P_{cpi,t} = \sum_k \int \frac{e_k(i) - p_k \underline{c}_k}{E} \gamma_{e,k}(i) di \sum_l \left(\overline{\partial_e e_l} - \bar{s}_l \right) \hat{P}_{l,t}$

[21] Note that we assumed that $\gamma_{e,k}$ is increasing in expenditures. It may be decreasing when demand elasticities of rich households are relatively insensitive to changes in expenditures, compared to the poor. In that case, a redistribution towards the rich may increase the aggregate demand elasticity, as the compositional effect is overturned.

eight COICOP sectors in the model: Food, Clothing, Electricity and Gas, Furniture, Transport, Recreation, Restaurants and Hotels, and Miscellaneous.

Income and wealth distribution. An advantage of the model is that its steady state can be disciplined directly by feeding in observed distributions. To this end, we rely on the Living Costs and Food (LCF) survey, which collects detailed survey data for more than 5600 households in the UK.[22] We think of each household in the survey as a type and we use population weights from the LCF for aggregation.[23]

We construct nominal wealth, $b(i)$, as nominal savings minus mortgage and credit card debt.[24] Total expenditures, $e(i)$, and budget shares by sector, $\bar{s}_k$, are directly observed in the LCF survey. To ensure consistency with the model, we back out labor income $Wn(i)$ as a residual from the budget constraint.[25] Note that we do not explicitly recover individual labour productivities $\vartheta(i)$, however they are not needed since the sufficient statistics are provided by the labour income share $\frac{Wn(i)}{WN}$ and the Frisch elasticity ψ.

Preferences. We further set $\delta = 0.0083$, targeting an adult life expectancy of 60 years. We set $\beta = 0.995$ which implies $R = \frac{1}{(1-\delta)\beta} = 1.0134$ on a quarterly basis. We further set $\psi = \sigma = 1$, in line with conventions in the macroeconomics literature.

Outer utility. In the LCF survey, we directly observe households expenditures on different goods from which we construct the household budget shares for each sector denoted by $s_k(i)$. To recover the marginal budget shares $\partial_e e_k(i)$ and the substitution matrix $\rho_{k,l}(i)$, which are not directly observed, we impose a functional form on the outer utility function and estimate it from the LCF data. Specifically, we parametrize $U_i(\cdot)$ following Comin et al. (2021), who propose a class of non-homothetic CES preferences defined implicitly by:

$$\sum_{k=1}^{K} \mathcal{V}_k(i) \left(\frac{c_k(i)}{g(U(i))^{\zeta_k}} \right)^{\frac{\eta-1}{\eta}} = 1,$$

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[22] The UK consumer price index produced by the ONS is based on expenditure baskets observed in the LCF survey.

[23] We use 2019 data to calibrate the model have 5695 household observations. We think of each of these households as a representative for a particular type. In this sense, our model has about 5695 types of households, with demographic turnover within each type, as households are replaced by steady-state versions of their type at a rate δ.

[24] In the LCF we observe interest income. We convert this into the stock of saving by assuming an interest rate of 1 percent annually. Moreover, to be consistent zero bond holdings on average, we subtract average wealth for each household.

[25] Note that in the model's steady state, household savings, $b(i)$, are constant at the household level and dividends are zero, so total expenditure equals labor income plus interest income for each household j. In a few cases, implied labor income is negative. We then set labor income to zero and expenditures to asset income.

where η is the elasticity of substitution across sectors, ζ_k captures non-homotheticicities in consumption and $\mathcal{V}_k(i)$ are household-specific preference shifters.

As shown by Comin et al. (2021), the non-homothetic CES form implies the following expression for household i's budget share in sector k (relative to some baseline sector $\bar{k}$ whose non-homotheticity parameter $\zeta_{\bar{k}}$ has been normalized to 1):

$$\ln\left(s_k(i)\right) = (1-\eta)\ln\left(\frac{p_k}{p_{\bar{k}}}\right) + (1-\eta)\left(\zeta_k - 1\right)\ln\left(\frac{e(i)}{p_{\bar{k}}}\right) + \zeta_k \ln\left(s_0(i)\right) + \eta \ln\left(\frac{\mathcal{V}_k(i)}{\mathcal{V}_{\bar{k}}(i)^{\zeta_k}}\right).$$

This class of preferences thus allows the sectoral composition of the consumption basket to vary with total expenditures. In particular, sectors that are more of a luxury than the base sector $\bar{k}$ will have a non-homotheticity parameter ζ_k that is larger than one (as long as $\eta < 1$) and the opposite is true for necessity sectors. In the limit, where the ζ's are all equal across sectors we are back to the homothetic CES case.

We model the household-level preference shifters as $\ln \mathcal{V}_k(i) = \beta_k x(i) + v_k(i)$, where $x(i)$ is a vector of demographic characteristics, such as age or couple status, and $v_k(i)$ captures remaining idiosyncratic preference variation. The latter allows to match the model in steady state precisely to the actual distribution of budget shares observed in the LCF data.

We set the elasticity of substitution between sectors as $\eta = 0.1$ and estimate the ζ_k parameters using a GMM procedure, following Comin et al. (2021) but using household-level data. In Appendix D we show that this specification gives a good fit of the empirical relation between expenditures and budget shares, a key object in our model.[26] Nonetheless, even for the same demographic group and expenditure level, there is still considerable variation in budget shares that is driven by the permanent idiosyncratic shifters $v_k(i)$. In Appendix D, we provide details on the estimation. With the estimated equations at hand, we can compute for each household the implied marginal budget shares $\partial_e e_k(i)$, for each sector k, see Appendix B for the formula.

Figure 2 plots histograms of the distribution of the budget shares and marginal budget shares. In necessity sectors such as *Food* and *Electricity & Gas*, budget shares are decreasing in total expenditures and exceed marginal budget shares. In luxury sectors, such as *Recreation* and *Restaurants & Hotels*, the opposite is true. Table 4 shows the marginal budget shares, averaged across households, $\overline{\partial_e e_k}$ along with the average budget share $\bar{s}_k$, as well as the difference $\overline{\partial_e e_k} - \bar{s}_k$, which matters directly for the $\mathcal{NH}$ wedge.

Inner utility. The distributions of demand elasticities within sectors are not directly observed in the data. However, they do have implications, which we can exploit to impose empirical discipline. Specifically, we assume a HARA form for the inner utility function, $\mathcal{U}_k(\cdot)$, which

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[26] The value of η is based on the 10-sector estimation in Comin et al. (2021), Table XII. Our has a relatively short time dimension and, related to this, η does not to appear to be very sharply identified. That said, specifications with low values for η tend to fit the data relatively well.

implies that the elasticity of substitution between goods in sector k, for household i, is then given by:

$$\epsilon_k(i) = a_k + \frac{b_k}{e_k(i)},$$

where $a_k > 0$ and b_k are sector-level constants.[27] When $b_k > 0$, households become less price sensitive as they spend more and it then holds that $\gamma_{e,k}(i) > 0$. It can be shown that the sector-level demand elasticity and super-elasticity are given by, respectively, $\bar{\epsilon}_k = a_k + \frac{b_k}{E_k}$ and $\bar{\epsilon}^s_k = \frac{b_k}{E_k}$. We further assume that intermediate input demand is governed by the same elasticity and superelasticity. Given these objects we can compute the steady-markup at the sector level $\frac{\bar{\epsilon}_k}{\bar{\epsilon}_k - 1}$ and the long-run pass-through of marginal costs to prices as $\frac{\bar{\epsilon}_k - 1}{\bar{\epsilon}_k - 1 + \bar{\epsilon}^s_k}$. We calibrate a_k and b_k by targeting sector-level markup estimates produced by the Office for National Statistics, following the method of De Loecker and Warzynski (2012). Moreover, we target 70 percent pass-through (in all sectors), based on empirical evidence by Amiti et al. (2019). Table 4 presents the implied sector-level coefficients. Given a_k and b_k and the empirical distribution of expenditures at the sector level, $e_k(i)$, we can compute the distributions of individual demand elasticities, $\epsilon_k(i)$, and super-elasticities, $\epsilon^s_k(i)$, which also gives us $\gamma_{e,k}(i)$ and $\gamma_{b,k}(i)$. Expressions for all relevant objects are provided in Appendix B.

Hand-to-Mouth households. The model is flexible regarding $\varphi(i)$, the fraction of hand-to-mouth households within each household of type i. Our calibration strategy targets empirical evidence for the UK on MPCs for different demographic groups, from Albuquerque and Green (2022). Specifically, we assume that $\varphi(i) = \frac{1}{1+exp(-\Upsilon'\mathbf{X}(i))}$, where $\mathbf{X}(i)$ is a vector consisting of a constant and a number of household characteristics observed in the LCF: age (<40 years, 41-58 years, >58 years), and home ownership status (mortgagor, outright owner, renter). We then use a non-linear least squares procedure to find Υ, targeting the estimated difference in MPC of the young and middle age, relative to the old, and of mortgagors and outright owners relative to outright owners. Here, we limit ourselves to characteristics that are found to have significant effects, according to Albuquerque and Green (2022), see Table 4 column 6. We also target their estimated average quarterly MPC which is 0.11. Since Υ contains four coefficients and we have four targets, the fit is nearly perfect. Figure 1 plots the implied distribution of quarterly MPCs across household types, showing substantial heterogeneity.

Price rigidity. To calibrate the price rigidity parameter in each sector, θ_k, we follow empirical evidence on price adjustment frequencies in the United Kingdom, as documented by Dixon and Tian (2017). We convert these into quarterly Calvo probabilities, see Table 4 for the implied

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[27] Note that we are implicitly normalizing the price level in each sector to be 1, which is an innocuous assumption that does not affect our calibration of the HARA utility. The details of why that is so are provided in the Appendix.

values. For *Electricity & Gas*, no direct statistics on price rigidity are available. For this sector, we assume price adjustment probability of 1/6=0.167, corresponding to an energy contract duration of 1.5 years, which is typical in the UK.

Technology. Regarding Input-Output (I-O) linkages, we calibrate the model to the UK data using the matrix of industries' intermediate consumption provided by the ONS. One complication is that the categories on which the I-O tables are supplied are based on the CPA (classification of products by activity) method while our sectors are defined from the COICOP classification. We bridge these differences by constructing a mapping between the two, starting from the 10-digit goods classification and using the correspondence tables provided by the UN's Statistics Division. We also check that adjusting for the intermediate flows to the four COICOP sectors excluded from the model does not significantly change the I-O matrix used in the calibration.

We further assume an AR(1) process in logs for the shock in the model. For both sectoral and aggregate productivity shocks, we assume an autoregressive coefficient $\rho_A = 0.95$. For the monetary policy shock we assume a coefficient $\rho_R = 0.25$. The monetary policy shock is scaled to correspond to an increase in the annualized nominal interest rate of 100 basis points. The aggregate productivity shock correspond to a decline in productivity of one percent. The sectoral productivity shocks are also negative, and for comparability we scale the magnitude of these shocks such that they all have the same impact on the natural demand index $\mathcal{Y}_t^*$ as the aggregate shock. This is achieved by weighting sectoral shock k by a factor $\frac{\sum_{k=1}^{K} \tilde{\Omega}_{k,l}(\psi \overline{\partial_e e_k} + \bar{s}_l)}{\tilde{\Omega}_{k,l}(\psi \overline{\partial_e e_k} + \bar{s}_k)}$, where $\tilde{\Omega}$ is an adjustment for I-O linkages, see Appendix B.

4.2 The full model: results

With the full model at hand, we study to what extent the analytical results of the previous section hold up quantitatively. We also explore the distributional implications of shocks.

Aggregate Responses. Figure 4 plots the responses of the aggregate output gap, the CPI inflation rate, and the MCPI inflation rate, to various shocks. The responses to contractionary monetary policy shocks and negative aggregate productivity shocks, shown in the two top left panels, are typical of the New Keynesian model. Following a monetary contraction, both CPI inflation and the output gap fall, whereas following a negative productivity shock both variables increase.[28] We also observe that, for these two aggregate shocks, the CPI and MCPI inflation indices are closely aligned, although not perfectly, which is due to heterogeneity in the slopes of sectoral NKPCs.

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[28] Following a negative aggregate productivity shock, $\mathcal{Y}_t^*$ and $\mathcal{Y}_t$ both decline. But due to price rigidities, the latter falls by less than the former, and hence the output gap, $\tilde{\mathcal{Y}}_t^* = \mathcal{Y}_t - \mathcal{Y}_t^*$ increases.

The responses to negative sectoral productivity shocks are shown in the remaining panels of Figure 4. To the right side of each panel, we display an index which is negative for necessity sectors and positive for luxury sectors. Let us first consider negative productivity shocks in the two necessity sectors: *Food* and *Electricity & Gas*. In line with the analytical results –and in contrast to the aggregate productivity shock– we observe that the aggregate output gap initially *declines* following such shocks. As explained in the previous section, the non-homothetic wedge in the NKPC, $\mathcal{NH}$ rises, which captures a downward distortion in labor supply, and which pushes down the output gap. Note further that, on impact, the CPI increases by substantially more than the MCPI, underscoring the quantitatively important effects of non-homotheticities.

After about a year the output gap turns positive, which is largely driven by decline in the endogenous markup wedge $\mathcal{M}$. As households reduce consumption, they become more price sensitive, which induces firms to reduce markups. This in turn increases aggregate demand and hence the output gap. This effect propagates with the distribution of wealth and is relatively persistent. Indeed, it tends to dominate in the medium run.

To the central bank, the shifts of the NKPC create specific trade-offs. Initially, a marginally stronger tightening of policy would help contain inflation, but at the expense of a more negative output gap. Later on, however, this would bring down both the output gap and inflation simultaneously. This suggests that in response to negative supply shocks in necessity sectors, a delayed tightening of policy may be optimal. We will explore this more in the next section.

Let us now turn to productivity shocks in three clear luxury sectors: *Furniture*, *Recreation*, *Restaurants & Hotels*.[29] As expected, the output gap initially *increases* strongly, although quantitatively less so for a *Furniture* shock. Much of the initial spike in the output gap diminishes quickly, as the effect of the $\mathcal{NH}$ wedge is relatively transitory. Nonetheless, the output gap remains persistently elevated due to the $\mathcal{M}$ wedge. Note further that the MCPI increases more on impact than the CPI, again illustrating the importance of non-homotheticities. From a policy perspective, a stronger monetary contraction would both close the output gap and reduce inflation, initially as well as later on.

Finally, we consider shocks to sectors which are neither clear luxuries nor necessities (*Clothing*, *Transport*, *Miscellaneous*) the response of the output gap is mixed. This clarifies that quantitative features of the model other than non-homotheticities play a role. In particular, heterogeneity in price rigidity across sectors and I-O linkages matter. In Appendix D.1 we show responses for a version of the model in which we shut down those two features. In that case, the output gap still declines in response to negative productivity shocks in the two necessity sectors (*Food* and *Electricity & Gas*), but not in response to such shocks in any of the other sectors.

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[29] To gauge the extent to which a sector is a necessity or a luxury, we show to the right of each panel a luxury index defined as $100(\overline{\partial_e e_l} - \bar{s}_k)$. This index lies between -1 and 1 and is negative (positive) for necessity (luxury) sectors.

Distributional responses. Let us now consider the response of the full distribution of consumption expenditures to different shocks, see Figure 5. Each dot represents a household in the model (and thus in the LCF survey). The horizontal axis denotes the total steady-state total income (expenditure) of the household, wheres the vertical axes denotes the real consumption expenditure response of the household to various shocks, averaged over the first four quarters following the shock. The red line represents linear regression line fitted through these model-generated data.

Following a monetary contraction, consumption falls, and on average more so for low-income households. Strikingly, for any given income level there is a substantial degree of heterogeneity in the consumption response. For instance, some lower-income households experience consumption gains. This heterogeneity is due to heterogeneity in the composition of labour versus asset income, as well as heterogeneity in steady-state consumption basket, due to taste heterogeneity (we feed the observed consumption shares into the model). Considering the responses to an aggregate productivity shock, we observe a similar pattern, with low-income households being hit slightly more on average, which is for an important part driven by a larger response of wage income. But again, even conditional on total income there is a large amount of heterogeneity, with some households increasing their consumption, for instance because they benefit from the increase in interest rates following the shock.

When we consider productivity shocks to specific sectors, we again observe that on average consumption of the poor responds most negatively, and that there is a large amount of heterogeneity, even conditional on income. Moreover, note that the extent to which poorer households are hit varies strongly across shocks, as indicated by the slope of the red line. Indeed, the slope tends to be relatively flat for luxury sectors (*Recreation* and *Restaurants & Hotels*), but relatively steep for necessity sectors (*Food* and *Electricity & Gas*). This is a natural consequence of the fact that price increases in luxury sectors affect the rich relatively more, whereas the poor are more affected by price increases in necessity sectors.

Overall, these results suggest that, if the central bank considers distributional effects, a cost-of-living crisis may present a particularly challenging situation: in addition to the aggregate trade-off described above, an additional tightening of monetary policy may weigh most heavily on the poor, who are strongly affected by the shock to begin with.

5 Optimal Policy

Having explored the dynamics of the model under an interest rate rule, let us now analyze the normative implications for monetary policy. Specifically we study the optimal interest rate policy under commitment.

5.1 The optimal policy problem

We consider social planner who maximizes, at some initial date 0, a welfare function of the following form:

$$\mathcal{W} = (1-\delta)\int G(V^0(i), i)di + \delta \mathbb{E}_0 \sum_{t_0=0}^{\infty} \beta^{t_0} \int G(V^{t_0}(i), i)di, \tag{23}$$

where the first term on the right-hand side stems from pre-existing households, and the second term from current and future newborns, where the superscript t_0 denotes the period of birth. Moreover, G is a function which captures the social planner's aggregation of welfare levels of different households. The lifetime welfare of household i born at t_0 is given by:

$$V^{t_0}(i) = \mathbb{E}_{t_0} \sum_{s=0}^{\infty} (\beta(1-\delta))^s \left(U_i(\mathbf{c}_{t+s}(i)) - \chi \left(\frac{n_{t+s}(i)}{\vartheta(i)} \right) \right),$$

where setting $t_0 = 0$ gives the value of the pre-existing households. To solve the optimal policy problem, the planner sets the nominal interest rate R_t to maximize the Welfare criterion (23), subject to Equations (10)-(17) holding currently and at any future date.

Our setup allows the planner to have an arbitrary social preference function G. But in order to derive concrete policy prescriptions, we need to make further assumptions on this function. We proceed following the literature on inverse optimal taxation. First, we rule out any motive for the central bank to redistribute wealth in the absence of aggregate shocks. That is, the steady-state distribution is treated as efficient. The underlying idea is that long-run wealth redistribution is considered the domain of fiscal rather than monetary policy. We implement this assumption by imposing that:

$$G'\left(V^{t_0}(i), i\right) \partial_e v\left(e(i)\right) = 1.$$

where $v\left(e(i)\right) = \max_{\mathbf{c}(i)} U_i(\mathbf{c}(i))$ s.t. $\sum_k \int_0^1 p_k(j) c_k(i,j) dj \leq e(i)$ is the indirect utility function. Second, we set $G''\left(V^{t_0}(i), i\right) = 0$, which implies that households' percentage fluctuations in marginal utility are weighed equally by the planner. In Appendix E.1 and E.3 we show that the weight of household i in the central bank's welfare loss function (approximated to a second order) can be expressed as $g(i) = \frac{E}{\psi W n(i) + \sigma e(i)}$. Note that poor households are assigned a higher weight, as those agents are at a point in the utility function with more curvature, i.e. fluctuations in consumption are more costly for them. The equations characterising the optimal policy are derived and presented in Appendix E.

5.2 Analytical results under optimal policy

Before studying the optimal policy quantitatively, we present a number of analytical results in a simplified setting (again without I-O linkages and HtM agents). Proofs are provided in Appendix E.2.

Our first optimal policy result clarifies how heterogeneity and generalized preferences affect the optimal policy problem, relative to a basic NK model. To simplify the problem as much as possible we assume, in addition to (**A.1**)-(**A.2**), that there is sectoral heterogeneity in neither price stickiness nor in demand elasticities and super-elasticities (we relax this in the Appendix).[30] We obtain:

Result 7 (Simplified optimal policy problem): *If (**A.1**)-(**A.2**) hold and* θ_k *,* $\bar{\epsilon}_k$ *and* $\bar{\epsilon}^s_k$ *are equal across sectors, then the optimal policy problem can be expressed as:*

$$\min_{\{\tilde{\mathcal{Y}}_t, \pi_{cpi,t}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{\sigma+\psi}{\sigma\psi}\tilde{\mathcal{Y}}_t^2 + \tilde{\vartheta}\pi_{cpi,t}^2\right)$$

$$s.t.\ \pi_{cpi,t} = \kappa\tilde{\mathcal{Y}}_t + \beta(1-\delta)\mathbb{E}_t\pi_{cpi,t+1} + \lambda(\mathcal{M}_t + \mathcal{NH}_t),$$

where $\tilde{\vartheta} = \frac{\bar{\epsilon}\theta}{(1-\theta)(1-\beta\theta)}$*, and where the wedges* $\mathcal{M}_t \equiv \sum_{k=1}^K \bar{s}_k \mathcal{M}_{k,t}$ *and* $\mathcal{NH}_t$ *evolve independently of monetary policy (Result 1).*

Thus, the optimal policy problem closely resembles the one in the basic NK model, see Galí (2015), Chapter 5. The central bank minimizes a weighted present value of the output gap and CPI inflation subject to an aggregate NKPC. However, in our case the NKPC is shifted by the $\mathcal{M}$ and $\mathcal{NH}$ wedges which, as explained previously, are the result of non-CES and non-homothetic preferences, respectively.

Note that even in this simplified setting, household heterogeneity matters for optimal policy, since it shapes the two wedges. This point highlights the interaction between heterogeneity and generalized preferences. Under homothetic CES preferences, the two wedges would vanish and heterogeneity would become irrelevant for optimal policy, as in McKay and Wolf (2023). But once we move beyond such preferences, heterogeneity affects the NKPC and it affects optimal policy even when monetary policy cannot affect distributions (assumption **A.2**) and/or does not consider inequality part of its policy objective.

Let us now study how the optimal policy is shaped by non-homotheticities. In particular, we are interested in the extent to which optimal policy reacts differently to productivity shocks arising in necessity and luxury sectors. We assume that shocks follow AR(1) processes. In

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[30] When θ_k , $\bar{\epsilon}_k$ and $\bar{\epsilon}^s_k$ vary across sectors, the inflation index becomes $\sum_k \bar{s}_k\bar{\epsilon}_k \frac{\theta_k/\tilde{\vartheta}}{(1-\theta_k)(1-\beta\theta_k)}\pi_{k,t}$, with $\tilde{\vartheta} = \sum_k \bar{s}_k\bar{\epsilon}_k \frac{\theta_k}{(1-\theta_k)(1-\beta\theta_k)}$, and the NKPC slope becomes $\sum_k \bar{s}_k\bar{\epsilon}_k \frac{\theta_k/\tilde{\vartheta}}{(1-\theta_k)(1-\beta\theta_k)}\lambda_k$.

Appendix E.2 we derive analytically the responses under optimal policy to sectoral shocks and show that the sign of the responses switch at some date t^* (which may vary across variables). Result 8 summarizes these findings:

Result 8 (signs of responses under optimal policy): *If (**A.1**)-(**A.2**) hold and $\mathcal{M}_t = 0$, then the responses of the output gap and inflation to necessity and luxury shocks have the opposite sign under optimal policy, in the short, in the medium run, and in present-value terms. The signs of the responses are presented in Table 2.*

Result 8 implies that the sectoral nature of the shock is highly important for the optimal policy response. Table 2 shows that, in response to a negative productivity shock to a necessity sector, the $\mathcal{NH}$ wedge rises, as shown previously. Upon impact, the output gap and MCPI inflation fall, whereas the CPI index increases. Thus, optimal policy does not fully stamp out CPI inflation. Rather, it steers the economy to a point where the corridor between MCPI and CPI inflation includes zero (the former lies below the latter as it down-weights necessity sectors). At the same, optimal policy lets the output gap turn negative. Intuitively, optimal policy strikes a balance between the cost of CPI inflation versus the cost of a negative output gap. After some time, the signs of the responses all switch.[31] However, in present-value terms the short-term effects dominates.

Following a negative productivity shock to luxuries, the precise opposite optimal responses obtain, as shown in the lower half of Table 2. Thus, the optimal policy response critically hinges on the sectoral nature of the shock. To derive Result 8, we have shut down the $\mathcal{M}$ wedge, focusing on the $\mathcal{NH}$ wedge. In Appendix E.2 we derive analytical results on the role of the $\mathcal{M}$ wedge instead.

How does the optimal policy compare to a policy of strict targeting the CPI, i.e. $\pi_{cpi,t} = 0$ at all times? Would it be looser or tighter? Let us define a loose policy as one which targets a higher output gap and higher inflation. We can show the following:

Result 9 (optimal policy versus inflation targeting): *Compared to a strict CPI targeting policy, the optimal policy is initially and in net present value terms relatively loose (tight) following a negative necessity (luxury) shock.*

Intuitively, under a strict CPI targeting policy, the output gap declines initially following a negative necessity shock, as $\mathcal{NH}$ increases. By loosening policy, the decline in the output gap is dampened at the expense of some positive CPI inflation. This improves welfare, since welfare losses are –to a second-order approximation– quadratic in the output gap and CPI inflation.

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[31] Intuitively, the prices of goods in the necessity sector which experiences the fall in productivity are initially distorted downward, due to price stickiness. At some point in time, however, the shock has mostly died out while the price level is still elevated, creating an upward distortion in the sectoral price level.

Table 2. **Sign or responses under optimal policy (Result 8)**

	Y gap	CPI	MCPI	$\mathcal{NH}$
negative necessity shock				
short run	-	+	-	+
medium run	+	-	+	-
present value	-	+	-	+
negative luxury shock				
short run	+	-	+	-
medium run	-	+	-	+
present value	+	-	+	-

Note: sign of the responses results assuming $\mathcal{M} = 0$ and (**A.1**)-(**A.2**). All negative productivity shocks. Short run refers to $t < t^*$ and medium run to $t \geq t^*$. Present value discounts the responses with a factor R^{-t}. See Appendix E.2 for the derivations.

Fully stabilising either the output gap or CPI inflation is therefore never optimal.

5.3 Quantitative dynamics under optimal policy

Result 9 suggests that the optimal policy response to cost-of-living crisis (i.e. a negative shock to necessities) can indeed be rather specific. Following the initial shocks around 2021, central banks were seen to be relatively slow in tightening policy. Interestingly, this appears in line with the optimal policy in the model, at least qualitatively.

We now explore quantitatively the optimal policy responses to various sectoral shocks, and study to what extent the optimal policy response to shocks in sectors like *Food* or *Electricity & Gas* is indeed relatively loose, compared to a typical policy rule $\hat{R}_t = \phi\pi_{cpi,t}$ (with $\phi = 1.5$) and compared to the optimal response to other shocks. In order to make this comparison quantitatively, we exploit that one can implement the optimal interest rate path $\{\hat{R}_t\}_{t=0}^{\infty}$ as a rule $\hat{R}_t = \phi\pi_{cpi,t} + u_t^R$ where $\{u_t^R\}_{t=0}^{\infty}$ is a specific time path for the deviation from the rule ("optimal guidance"), announced when the productivity shock initially hits. We simulate the model both under such an interest rate rule and under optimal policy, and then numerically solve for the guidance path that implements the optimal policy. This path then quantifies how tight or loose the optimal policy is relative to the simple CPI-based rule. In Appendix D.2 we provide the details of this procedure.

The left panel in Figure 6 plots the optimal guidance for the aggregate and sectoral productivity shocks in the full model. In line with the analytical results, optimal policy is initially significantly looser than the rule following a negative necessity shock. Following negative shocks to luxuries, the optimal is also initially looser than the rule, but less so than following necessity shocks. We thus find that the sectoral source of the shock indeed has significant quantitative consequences for optimal policy, in line with the analytical results.[32]

How important are redistributive motives in driving the optimal policy? In the right panel of Figure 6 we shut down the redistributive motives of monetary policy.[33] Qualitatively, the results are unchanged, in the sense that the optimal policy response to negative necessity shocks is significantly looser than the response to necessity shocks. Quantitatively however, the redistributive motives push towards a front-loaded accommodative (i.e. looser) policy for all shocks, as this helps to redistribute towards poorer households, who tend to be more heavily affected in utility terms.

Figure 7 shows the response of the output gap and CPI inflation under optimal policy, for an aggregate productivity shock and productivity shocks to *Food* (lowest luxury index) and *Recreation* (highest luxury index). The quantitative responses are again consistent with the analytical findings. Without redistribution motive, following a negative *Food* shock, the output gap initially is negative, while CPI inflation increases. For a *Recreation* shock, we observe the precise opposite. Once we include a redistribution motive, the responses of the output gap and inflation are both pushed upwards, at least initially.

6 Conclusion

In this paper we addressed the question how monetary policy should respond to sector-specific supply shocks. To this end, we developed a multi-sector New-Keynesian model with household inequality and generalized non-homothetic preferences. An advantage of the framework is that it is relatively tractable, simplifying computations and allowing for analytical results to be derived. Moreover, it can be disciplined directly with data on heterogeneity in income, wealth, MPCs and expenditure baskets.

We showed how, due to non-homothetic and non-CES preferences, two new wedges emerge in the New Keynesian Phillips Curve (NKPC), which directly affect policy trade-offs and which are quantitatively important. In particular, after a negative supply shock to necessity sectors, the NKPC tends to shift upward, creating a policy trade-off between bringing down inflation and avoiding a negative output gap. After studying the optimal policy, we found that –because of this shift in the NKPC– the optimal policy to a negative necessity shock is relatively loose, while later on it tightens.

We have not touched upon on fiscal interventions, which have been widely used in recent years. We explore these policies in ongoing research.

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[32] Consistent with the quantitative results in the previous section, we find that negative shocks to *Transport* (neither a necessity nor a luxury) call for a relatively loose optimal policy, which is again due to the low degree of price rigidity in this sector and the position of this sector in the I-O matrix. Appendix D.1 repeats the exercise shutting down I-O linkages and heterogeneity in price rigidity. In this case, the optimal policy response to a transport shock is similar to the response to an aggregate productivity shock, i.e. much less loose.

[33] In Appendix E.3 we derive conditions shutting down such motives based on the welfare loss function.

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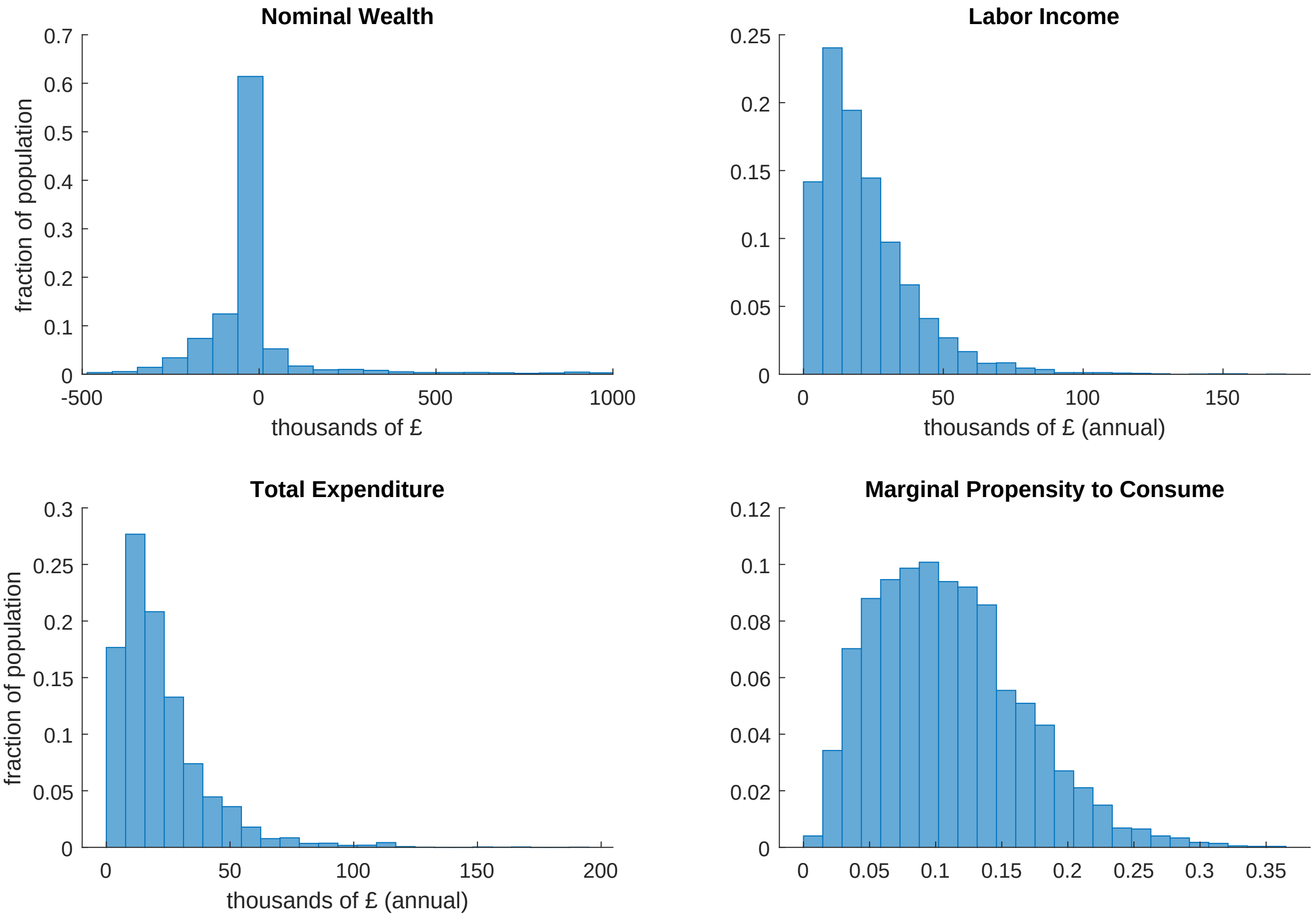
Figures

Figure 1. Steady-state distributions.


Nominal Wealth

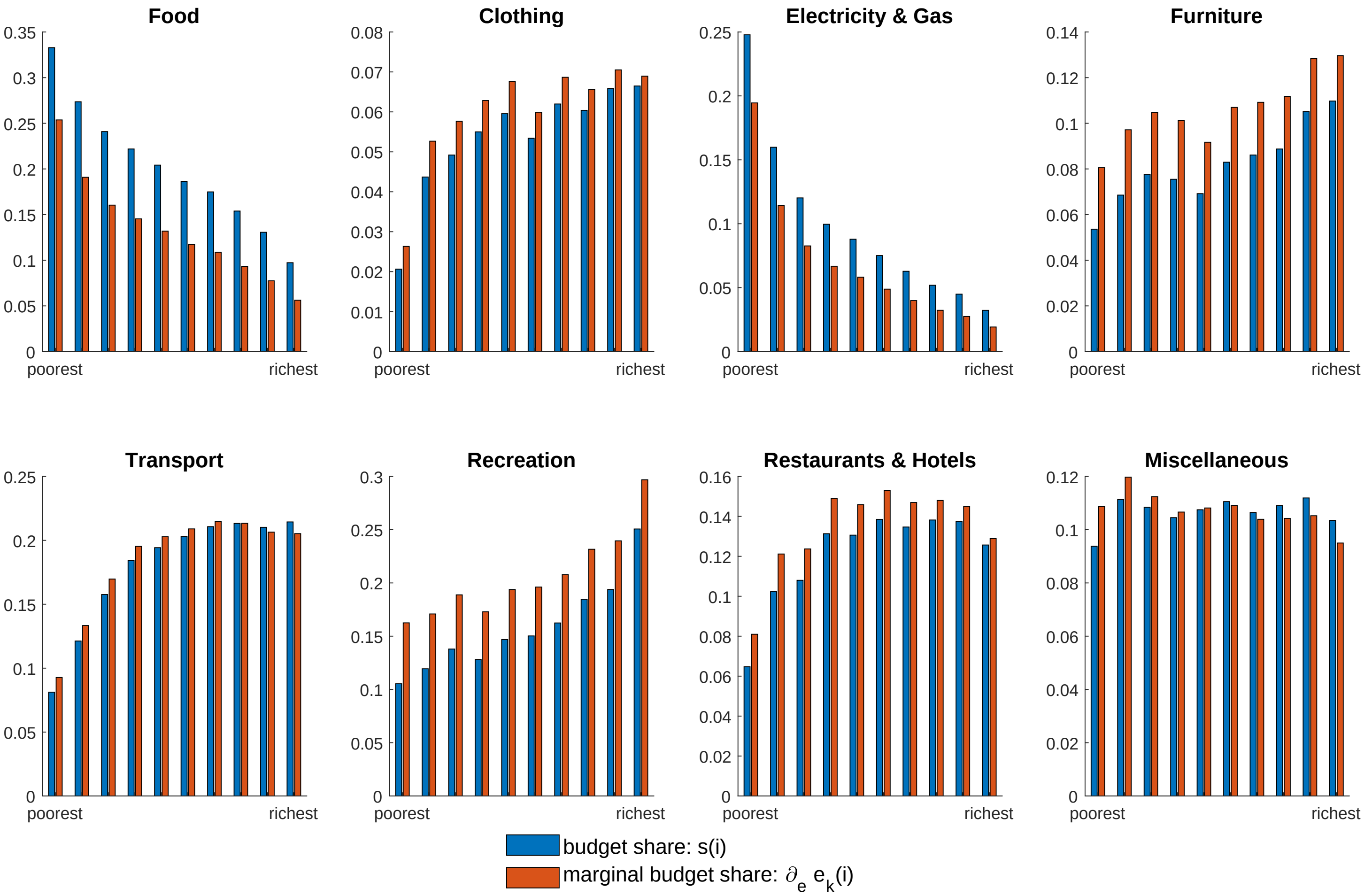
Labor Income

Total Expenditure

Marginal Propensity to Consume

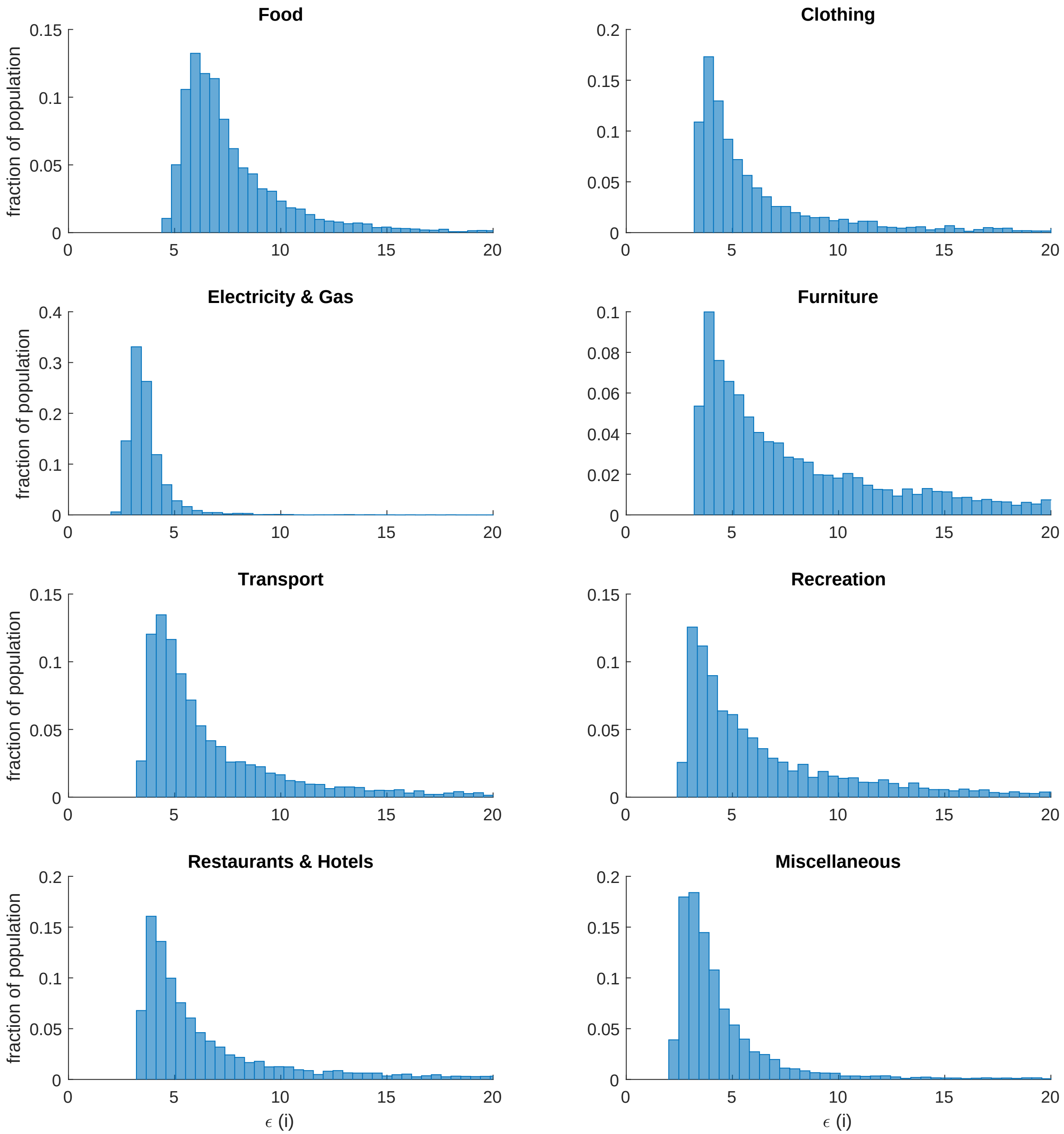
Notes: Data from Living Costs and Food Survey 2019 and authors' calculations, see main text.

Figure 2. Household budget shares by total expenditure decile.



Notes: Budget shares averaged within deciles of total expenditure, ordered from poorest (lowest decile) to richest (highest decile). Source: Living Costs and Food Survey 2019.

Figure 3. Distribution of demand elasticities by sector.



Food

Clothing

Electricity & Gas

Furniture

Transport

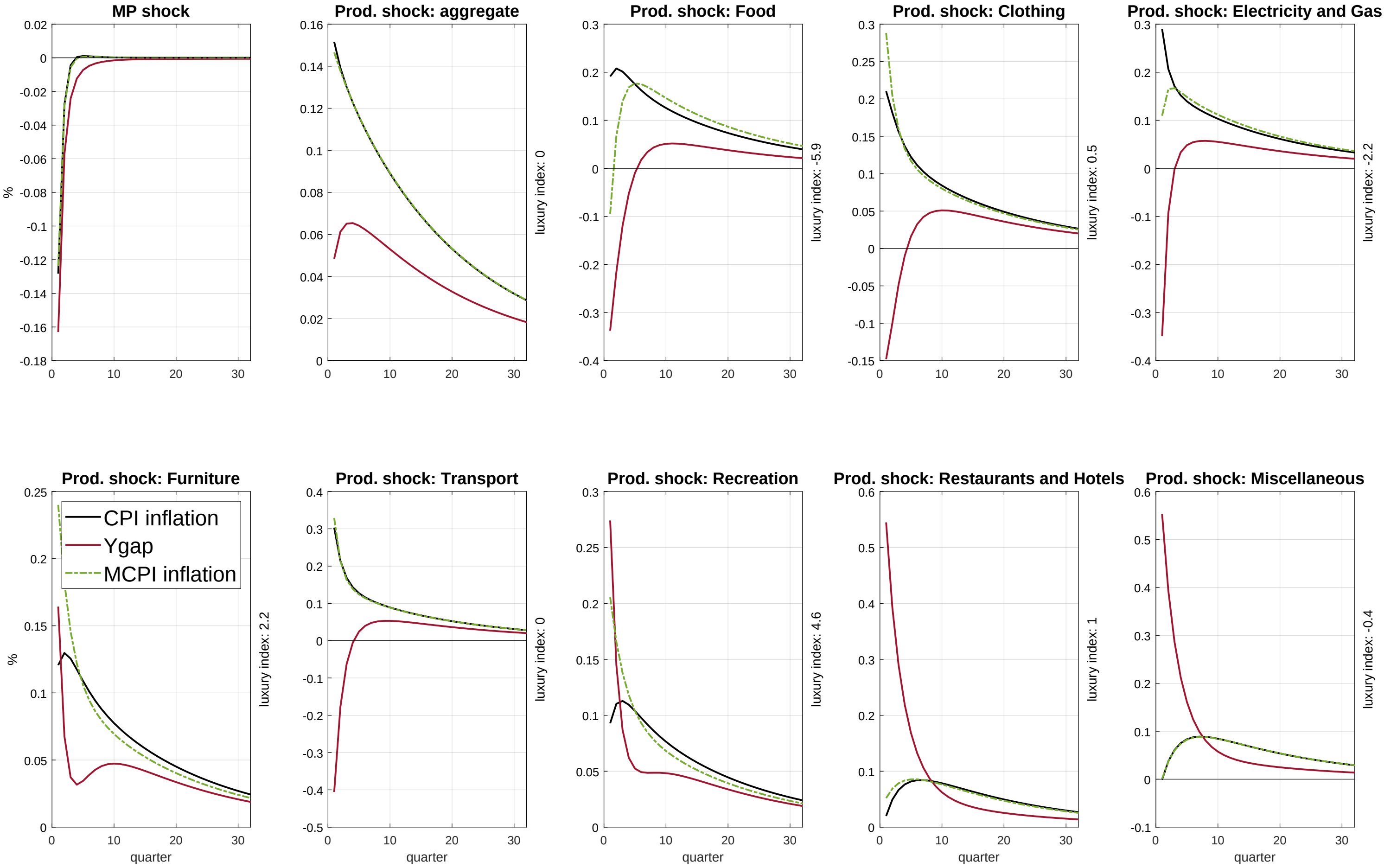
Recreation

Restaurants & Hotels

Miscellaneous

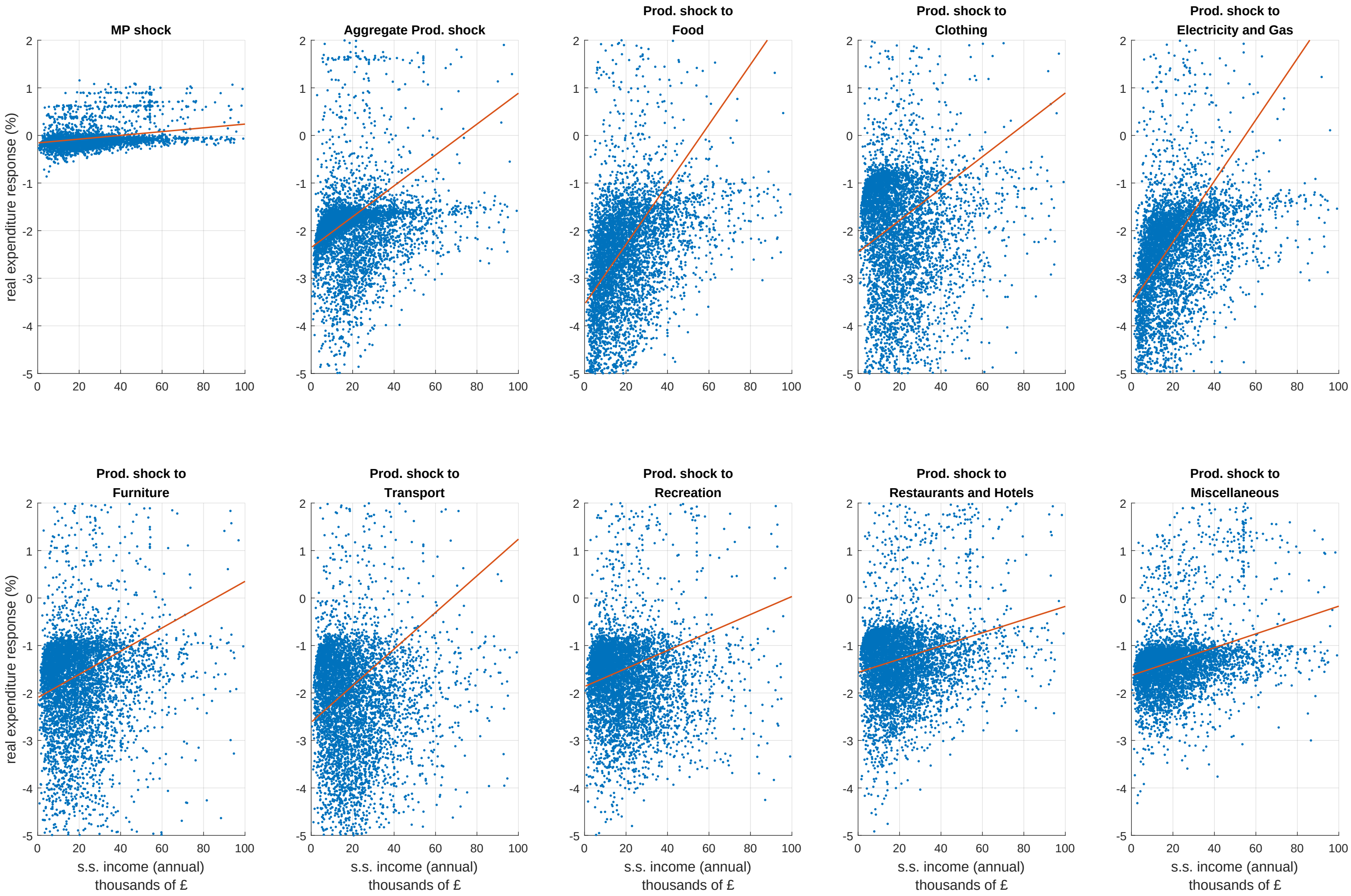
Notes: Histogram of $\epsilon_k(j)$, the demand elasticities across households (by sector).

Figure 4. Responses in the baseline model: all shocks.



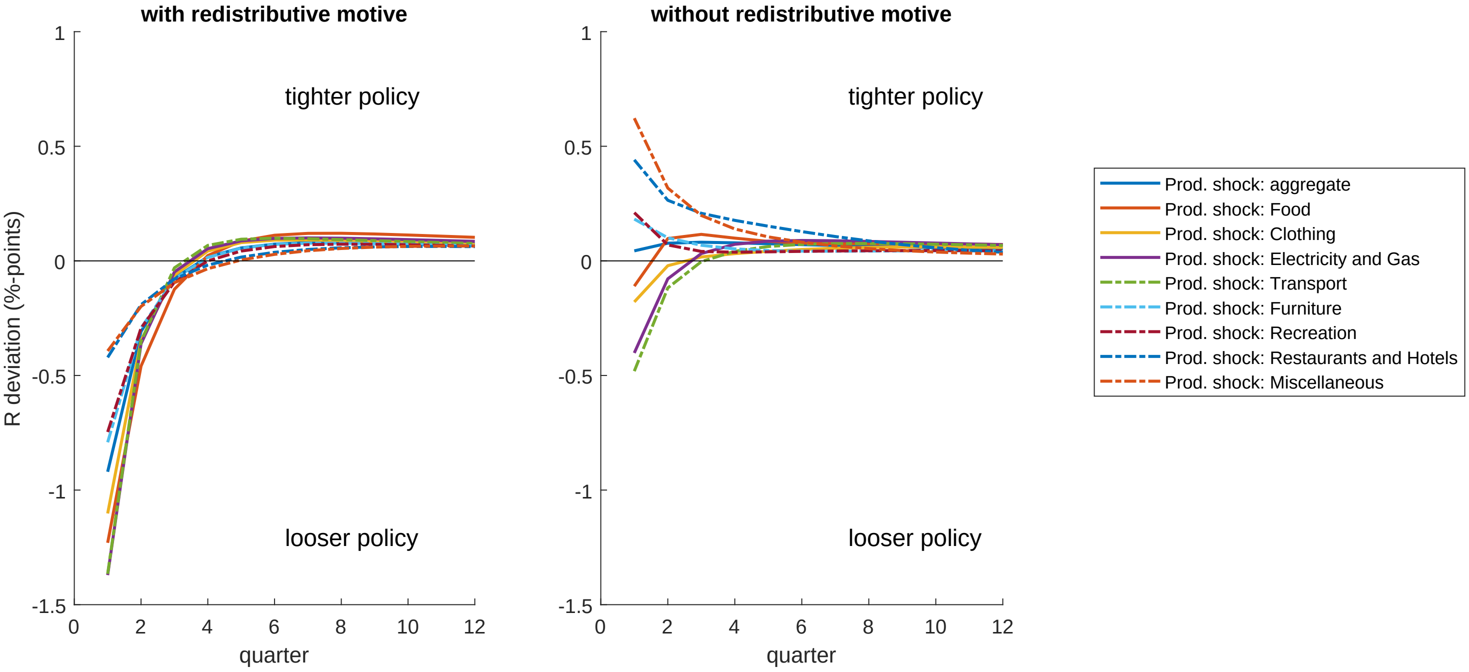
Notes: Impulse Response Functions are generated from the baseline the model, including heterogeneous Calvo probabilities across sectors across sectors, Input-Output linkages, and Hand-to-Mouth households. Responses for productivity shocks are for a 1 percent decline in productivity where scaled for comparability (see main text). On the right axis, the luxury index is defined as $100(\overline{\partial_e e_l} - \bar{s}_k)$.

Figure 5. Heterogeneous consumption responses to aggregate and sectoral shocks.



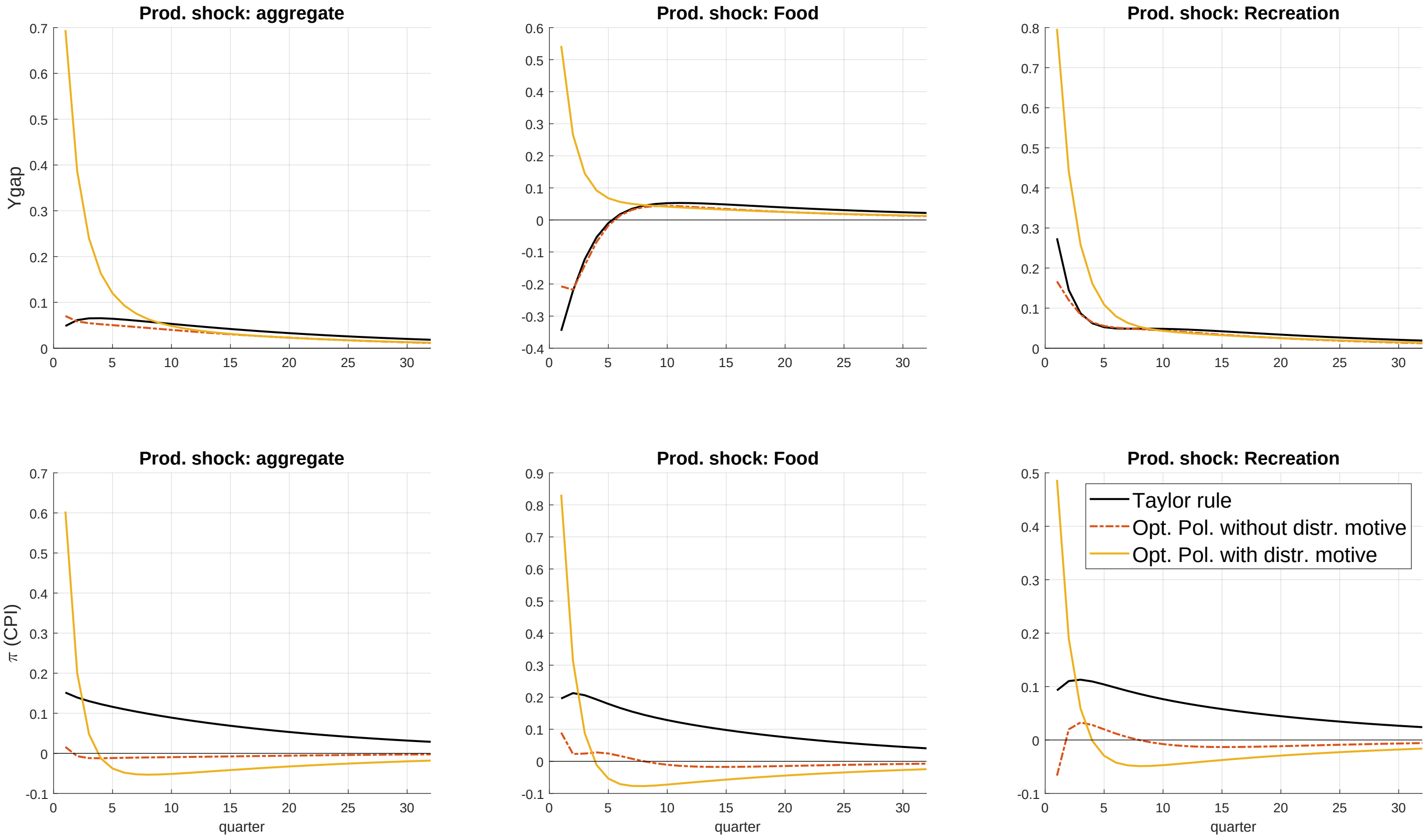
Notes: Response of real expenditures by steady-state income, generated from the baseline model and averaged over first four quarters following the shock. Dots denote individual households. Red lines are fitted 10th order polynomials. All productivity shocks are negative.

Figure 6. Optimal policy relative to Taylor rule.



Notes: Deviations from the Taylor rule $\hat{R}_t = 1.5\pi_{cpi,t}$ which implement Optimal Policy ("optimal guidance"). Higher values mean that optimal monetary policy is tight relative to this rule. See the main text for details. All productivity shocks are negative.

Figure 7. Optimal policy relative to Taylor rule.



Prod. shock: aggregate

Prod. shock: Food

Prod. shock: Recreation

Prod. shock: aggregate

Prod. shock: Food

Prod. shock: Recreation

Notes: Responses of the output gap and CPI inflation. All productivity shocks are negative.

Tables

Table 3. Aggregate parameter values.

Parameter	description	value
β	subjective discount factor	0.99
ψ	Frisch elasticity	1
σ	elasticity of intertemporal substitution	1
δ	death probability	0.0083
ϕ	Taylor rule coefficient	1.5
η	cross-sector elasticity of substitution	0.1
ρ_R	persistence monetary policy shock	0.25
ρ_A	persistence productivity shocks	0.95

Table 4. Sector-level parameter values.

Sector	$\bar{\epsilon}_k$	$\bar{\epsilon}^s_k$	$\bar{s}_k$	$\overline{\partial_e e_l}$	θ_k	κ_k	λ_k	Γ_k
Food	6.5775	2.3903	0.1574	0.0988	0.4100	0.7608	0.5998	0.0386
Clothing	4.8259	1.6397	0.0580	0.0631	0.3900	1.1810	0.6735	0.0929
Electricity & Gas	3.2525	0.9654	0.0630	0.0412	0.1667	2.6410	2.9244	0.0807
Furniture	4.9651	1.6993	0.0910	0.1133	0.4600	0.5731	0.4488	0.1018
Transport	5.0243	1.7247	0.2015	0.2018	0.2600	1.5120	1.4812	0.0806
Recreation	3.8950	1.2407	0.1858	0.2318	0.5100	0.3667	0.3341	0.1248
Restaurants & Hotels	4.7313	1.5991	0.1338	0.1440	0.7200	0.1317	0.0788	0.0883
Miscellaneous	3.1534	0.9229	0.1096	0.1061	0.6700	0.1313	0.1168	0.1210

Notes: $\bar{\epsilon}_k$: demand elasticity (household aggregate), $\bar{\epsilon}^s_k$: superelasticity (household aggregate), $\bar{s}_k$: budget share (household aggregate), $\overline{\partial_e e_l}$: marginal budget share (household aggregate), θ_k: Calvo probability, κ_k: slope NKPC w.r.t. output gap, λ_k slope NKPC w.r.t wedges, Γ_k: slope endogenous markup wedge w.r.t efficient demand index. See the main text and Appendix B for the definitions.

Appendix

A Model derivations

Households

In this section, we derive the optimal response of households' consumption and labor supply decisions to changes in prices (subvariety prices, wage and interest rate) near a steady state where subvariety prices are equal within sectors and the real interest rate satisfies $R_t = (1-\delta)\,\beta$. Preferences are weakly separable for subvarieties across sectors, additively separable in consumption and leisure and additively separable across time. This allows us to characterize households' decisions in three steps. We first study the inner intratemporal consumption problem which determines individual demand for subvarieties conditional on subvarity prices and sectoral expenditure. Second, we determine individual expenditure across sectors and labor supply conditional on subvariety prices, wage and total (intratemporal) consumption expenditure (outer intratemporal problem). These first two problems are the same for both unconstrained and Hand-to-Mouth households. Finally, we determine individual expenditure across time by solving the intertemporal problem of unconstrained households and the decision rule of Hand-to-Mouth households.

Inner intratemporal consumption problem (valid for unconstrained and HtM households)

We start with the allocation of a household's expenditures on varieties within a sector. Note that this is an intratemporal problem. For any such problem, we omit time subscripts in this appendix, unless stated otherwise.

For any sector k, let $\nu_k(\boldsymbol{p}_k, e_k)$ be the indirect subutility function for a given vector of prices $\boldsymbol{p}_k$ and total expenditure e_k, defined as:

$$\nu_k(\boldsymbol{p}_k, e_k) = \max_{\{\boldsymbol{c}_k\}} \mathcal{U}_k\left(\boldsymbol{c}_k\right) \quad s.t. \quad \int p_k(j)c_k(j)dj \leq e_k.$$

Let $d_k(p_k(j^*), \boldsymbol{p}_k, e_k)$ be the household's demand for variety j^* and note that this function is C^2 and symmetric in $\boldsymbol{p}_k$.[34] As noted in the main text, we consider a steady state with identical prices within sectors, i.e. $p_k(j) = P_k$ for all j. Let $\partial_p d_k$ denote the own-price derivative and $\partial_j d_k$ be the Gateaux derivative of d_k with respect to the price of variety j. By symmetry of the subutility function $\mathcal{U}_k$, and the fact that prices are the same in equilibrium, it holds in the steady state that$d_k(p_k(j^*), \boldsymbol{p}_k, e_k) = e_k/P_k$ for any e_k and $\partial_j d_k = \partial_{j'} d_k$ for any two subvarieties. Using the fact that the demand function is homogeneous of degree zero we can apply Euler's theorem to obtain:

$$\left(\partial_p d_k\right) p_k\left(j^*\right) + \int \left(\partial_j d_k\right) p_k(j)dj + \left(\partial_{e_k} d_k\right) e_k = 0.$$

Applying the symmetry property noted above then gives:

$$\left(\partial_p d_k\right) P_k + P_k \left(\partial_j d_k\right) + \left(\partial_{e_k} d_k\right) e_k = 0.$$

After rearranging, we obtain the following expression for the derivative of d_k with respect to the price of variety j:

$$\partial_j d_k = -\partial_p d_k - \frac{1}{P_k^2} e_k.$$

Note that this equation is simply a decomposition of demand for j^* to a change in the price of j into substitution and income effects. This result allows us to derive the first-order change in consumption as:[35]

$$\begin{aligned}
dc_k(j^*) &= \left(\partial_p d_k\right) dp_k(j^*) + \int \left(\partial_j d_k\right) dp_k(j)dj + \partial_{e_k} d_k de_k, \\
&= \left(\partial_p d_k\right) dp_k(j^*) - \left(\left(\partial_p d_k\right) + \frac{1}{P_k}\partial_e d_k e_k\right) \int dp_k(j)dj + \partial_{e_k} d_k de_k, \\
&= \left(\partial_p d_k\right) \left(dp_k(j^*) - dP_k\right) + \frac{1}{P_k}\left(de_k - \frac{e_k}{P_k} dP_k\right).
\end{aligned}$$

This equation relates changes in subvariety consumption with respect to its own relative price $(dp_k(j^*) - dP_k)$ to the inner elasticity of substitution $\epsilon_k = -P_k \partial_p d_k / d_k$which is the standard statistic of the firm pricing problem in steady state. Furthermore, exploiting the fact that $\partial_p d_k$is homogeneous of degree -1, symmetric in $\boldsymbol{p}_k$ one can again apply Euler's theorem to obtain:

$$\left(\partial_{pp} d_k\right) p\left(j^*\right) + \int \left(\partial_{pj} d_k\right) p_k(j)dj + \left(\partial_{pe_k} d_k\right) e_k = -\partial_p d_k,$$

$\Leftrightarrow$

$$P_k \left(\partial_{pp} d_k + \partial_{pj} d_k\right) + \left(\partial_{pe_k} d_k\right) e_k = -\partial_p d_k,$$

$\Leftrightarrow$

$$\partial_{pj} d_k = -\frac{\partial_p d_k}{P_k} - c_k \left(\partial_{pe_k} d_k\right) - \partial_{pp} d_k.$$

---

[34] Note that $\boldsymbol{c}_k$ lives in L^1, since $\mathcal{U}$ is (strictly) concave the problem has a unique solution which satisfies the set of first order conditions. Applying the implicit function theorem – for Banach spaces – shows that $\boldsymbol{c}_k$ is a C^2 function of $\{\boldsymbol{p}_k, e_k\}$.

[35] Recall that, by definition, $c_k(j^*) = d_k(p_k(j^*), \boldsymbol{p}_k, e_k)$.

Using this result: we can derive the following expression for the first-order change in the own-price derivative of sector-k demand:

$$\begin{aligned} d\partial_p d_k &= \left(\partial_{pp} d_k\right) dp_k\left(j^*\right) + \int \left(\partial_{pj} d_k\right) dp_k(j)dj + \left(\partial_{pe_k} d_k\right) de_k, \\ &= \left(\partial_{pp} d_k\right) dp_k\left(j^*\right) + \int \left(-\frac{\partial_p d_k}{P_k} - c_k \partial_{pe} d_k - \partial_{pp} d_k\right) dp_k(j)dj + \left(\partial_{pe_k} d_k\right) de_k, \\ &= \left(\partial_{pp} d_k\right)\left(dp_k\left(j^*\right) - dP_k\right) - \partial_p d_k \frac{dP_k}{P_k} + \left(\partial_{pe_k} d_k\right)\left(de_k - c_k dP_k\right). \end{aligned}$$

This expression will allow us to characterize the changes in elasticities of substitution away from steady state and their impact on firms' pricing decisions – through changes in endogenous markups.

Outer intratemporal consumption problem (valid for unconstrained and HtM households)

We now turn to the allocation of expenditures over different sectors. Let $\boldsymbol{P} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots \boldsymbol{p}_K)$ be the full vector of prices and let $v_i(\boldsymbol{P}, e)$ the indirect utility function of the outer problem which can be household-specific, hence we momentarily re-introduce the subscript i. The problem is to choose expenditure levels across different sectors, conditional on optimally choosing the bundle of varieties $\boldsymbol{c}_k$, which we solved for in the previous section. Recall that we assume that U_i is increasing, strictly concave and C^3. The problem can be expressed as:

$$v_i\left(\mathbf{P}, e\right) = \max_{\{e_1, e_2, \ldots e_K\}} U_i\left(\nu_1(\boldsymbol{p}_1, e_1), \nu_2(\boldsymbol{p}_2, e_2), \ldots, \nu_K(\boldsymbol{p}_K, e_K)\right), \quad s.t. \quad \sum_{k=1}^{K} e_k = e.$$

The associated first-order optimality condition is given by $U^{'}_{i,k}\partial_{e_k}\nu_k = \iota$, where $\iota = \partial_e v_i$ is the Lagrange multiplier. The problem defines a spending function $e_{k,i}\left(e, \boldsymbol{P}\right)$ which is C^2. Note that, by symmetry and since subvariety prices are equal within sectors, it holds in steady state that $\partial_{p_k(j)}\nu_k = \partial_{p_k(j')}\nu_k$ for any j, j' and e_k, so we have $\partial_{p_k(j)} e_k\left(e, \boldsymbol{P}\right) = \partial_{p_k(j')} e_k\left(e, \boldsymbol{P}\right) \equiv \partial_{P_k} e_k\left(e, \boldsymbol{P}\right)$. The derivative of the indirect utility function with respect to the price of a variety j in sector k is given by:

$$\partial_{p_k(j)} v_i = -\partial_e v_i c_k(j),$$

which follows by Roy's identity, where $c_k(j)$ is a shorthand for $d_k(p_k(j), \boldsymbol{p}_k, e_{k,i}\left(e, \boldsymbol{P}\right))$.The expression for the mixed derivative (which we will employ later on) is given by:

$$\begin{aligned} P_k \partial_{ep_k(j)} v_i &= -P_k\left(\partial_{ee} v_i c_k(j) + \partial_e v_i \partial_e e_{k,i} \partial_{e_k} c_k(j)\right), \\ &= -\left(\partial_{ee} v_i e_{k,i} + \partial_e v_i \partial_e e_{k,i}\right). \end{aligned}$$

Given $\partial_{p_k(j)} e_{k,i}\left(e, \boldsymbol{P}\right) = \partial_{P_k} e_{k,i}\left(e, \boldsymbol{P}\right)$ we can now write the change in sector-k expenditures in terms of the change in the sectoral prices, $\frac{dP_k}{p_k} = \hat{P}_k = \int \hat{p}_k(j)dj$:

$$\begin{aligned} de_{k,i} - e_{k,i}\hat{P}_k &= \sum_{l=1}^{K} P_l \partial_{P_l} e_{k,i} \hat{P}_l - e_{k,i}\hat{P}_k + \partial_e e_{k,i} de, \\ &= \left(P_k \partial_{P_k} e_{k,i} + \partial_e e_{k,i} e_{k,i} - e_{k,i}\right)\hat{P}_k + \sum_{l \neq k}\left(P_l \partial_{P_l} e_{k,i} + \partial_e e_{k,i} e_l\right)\hat{P}_l - dP_l + \partial_e e_{k,i}\left(de - \sum_l e_{l,i}\hat{P}_l\right), \\ &\equiv \partial_e e_{k,i}\left(de - \sum_l e_{l,i}\hat{P}_l\right) + e_{k,i}\sum_l \rho_{k,l}\left(i\right)\hat{P}_l. \end{aligned}$$

Note that we have $\sum_l \rho_{k,l} = 0$, as $e_k\left(e, \boldsymbol{P}\right)$ is homogeneous of degree one. In addition, consider the spending responses to a compensated change in the price of sector k:$\hat{P}_k = 1$, $de = e_k$. Inspecting the budget constraint gives $\sum_{l=0}^{K}\left(P_k \partial_{P_k} e_l + \partial_e e_l e_k\right) = e_k$ so we have $\sum_l e_l \rho_{l,k} = 0$.

Labor Supply (valid for unconstrained and HtM households) We start by solving for the labor supply response for an agent of type i in period t, which we derive from the first-order optimality condition for labor supply, which is given by $\chi'\left(\frac{n(i)}{\vartheta(i)}\right)\frac{1}{\vartheta(i)} = \partial_e v_i W$. Taking a first order approximation of this condition, we obtain:

$$\chi''\left(\frac{n(i)}{\vartheta(i)}\right)\frac{1}{\vartheta(i)}\frac{dn(i)}{\vartheta(i)} = \left(\partial_{ee} v_i de(i) + \sum_k \int \left(\partial_{ep_k(j)} v\right) dp_k(j)dj\right) W + \partial_e v_i dW,$$

$\Leftrightarrow$

$$\frac{\chi''\left(n(i)/\vartheta(i)\right)}{\chi'\left(n(i)/\vartheta(i)\right)}\frac{dn(i)}{\vartheta(i)} = \left(\frac{\partial_{ee} v_i}{\partial_e v_i} de(i) - \sum_k \left(\frac{\partial_{ee} v_i}{\partial_e v_i} e_k(i) + \partial_e e_k(i)\right)\hat{P}_k\right) + \frac{dW}{W},$$

$\Leftrightarrow$

$$\hat{n}(i) = \psi\left\{\hat{W} - \sum_k \partial_e e_k(i)\hat{P}_k\right\} - \frac{\psi}{\sigma}\left(\hat{e}\left(i\right) - \sum_l s_l\left(i\right)\hat{P}_l\right).$$

Intertemporal Decision (valid for non-HtM households only)

A household of type i born in t_0 has initial bond holdings $b_{t_0}(i) = b(i)\left(1+\sum_l \bar{s}_l \frac{P_{l,t_0}-P_l^*}{P_l^*}\right)$ with P_l^* the steady state price of l and $P_{l,t_0} = \int p_{l,t_0}(j)\, dj$. Using the definition of the indirect utility function $v_i(\boldsymbol{P}, e)$, one can write the Lagrangian of the non-HtM households intertemporal problem as:

$$\begin{aligned} V(i) = \max_{\{e_{t+s}, n_{t+s}, b_{t+s+1}\}_{s=0}^{\infty}} & \mathbb{E}_t \sum_{s=0}^{\infty} (\beta(1-\delta))^{t+s} \left(v_i\left(\mathbf{P}_{t+s}, e_{t+s}(i)\right) - \chi\left(\frac{n_{t+s}(i)}{\vartheta(i)}\right)\right) \\ & + \theta_{t+s}(i)\left\{ b_{t+s}(i) + n_{t+s}(i)W_{t+s} + \sum_k \varsigma_k(i) Div_{k,t+s} - e_{t+s}(i) - \frac{b_{t+s+1}(i)}{R_{t+s}} \right\}, \end{aligned}$$

with the first-order conditions given by

$$\begin{aligned} \frac{\partial V(i)}{\partial e_{t+s}(i)} &= \mathbb{E}_t\left[(\beta(1-\delta))^{t+s} \partial_e v_i\left(\mathbf{P}_{t+s}, e_{t+s}(i)\right) - \theta_{t+s}(i) \right] = 0, \\ \frac{\partial V(i)}{\partial n_{t+s}(i)} &= \mathbb{E}_t\left[-\chi'\left(\frac{n_{t+s}(i)}{\vartheta(i)}\right)\frac{1}{\vartheta(i)} + \theta_{t+s}(i) W_{t+s} \right] = 0, \\ \frac{\partial V(i)}{\partial b_{t+s+1}(i)} &= \mathbb{E}_t\left[-\frac{\theta_{t+s}(i)}{R_{t+s}} + \theta_{t+s+1}(i) \right] = 0. \end{aligned}$$

We now linearize the consumption Euler Equation, $\partial_e v_{t,i} = \beta(1-\delta)R_t \mathbb{E}_t\left[\partial_e v_{t+1,i}\right]$, around a stationary steady state with no uncertainty:

$$\begin{aligned} \partial_{ee} v_i de_t(i) + \sum_k \int \left(\partial_{ep_k(j)} v_i\right) dp_{k,t}(j) dj &= \beta(1-\delta) dR_t \partial_e v_i \\ & + \beta\left(1-\delta\right) R \left(\partial_{ee} v_i de_{t+1}(i) + \sum_k \int \left(\partial_{ep_k(j)} v_i\right) dp_{k,t+1}(j) dj \right), \\ & \Leftrightarrow \\ \frac{\partial_{ee} v_i}{\partial_e v_i} de_t(i) + \sum_k \int \left(\frac{\partial_{ep_k(j)} v_i}{\partial_e v_i}\right) dp_{k,t}(j) dj &= \frac{dR_t}{R} + \frac{\partial_{ee} v_i}{\partial_e v_i} de_{t+1}(i) + \sum_k \int \left(\frac{\partial_{ep_k(j)} v_i}{\partial_e v_i}\right) dp_{k,t+1}(j) dj, \\ & \Leftrightarrow \\ \frac{\partial_{ee} v_i}{\partial_e v_i}\left(de_t(i) - \sum_k e_k \hat{P}_{k,t} \right) - \sum_k \partial_e e_k(i) \hat{P}_{k,t} &= \hat{R}_t + \frac{\partial_{ee} v_i}{\partial_e v_i}\left(de_{t+1}(i) - \sum_k e_k \hat{P}_{k,t+1} \right) - \sum_k \partial_e e_k(i) \hat{P}_{k,t+1}, \\ & \Leftrightarrow \\ \frac{e\partial_{ee} v_i}{\partial_e v_i}\left(\hat{e}_t - \sum_k s_k \hat{P}_{k,t} \right) - \sum_k \partial_e e_k(i) \hat{P}_{k,t} &= \hat{R}_t + \frac{e\partial_{ee} v_i}{\partial_e v_i}\left(\hat{e}_{t+1} - \sum_k s_k \hat{P}_{k,t+1} \right) - \sum_k \partial_e e_k(i) \hat{P}_{k,t+1}, \\ & \Leftrightarrow \\ \left(\hat{e}_t - \sum_k s_k \hat{P}_{k,t} \right) &= \left(\hat{e}_{t+1} - \sum_k s_k \hat{P}_{k,t+1} \right) - \sigma\left(\hat{R}_t - \sum_k \partial_e e_k(i) \pi_{k,t+1} \right), \end{aligned}$$

where $s_l = e_l(i)/e(i)$ and the third line uses the fact that $P_k \frac{\partial_{ep_k(j)} v_i}{\partial_e v_i} = -\left(\partial_{ee} v_i e_k + \partial_e v_i \partial_e e_k\right)$. We define $\sigma \equiv -\partial_e v_i / e\partial_{ee} v_i$ as the elasticity of intertemporal substitution.

Note: In the formula above and in the labor supply decision problem, we assumed that the EIS $\sigma = -\partial_e v_i / e\partial_{ee} v_i$ is equal across households. It is always possible to renormalize the intratemporal indirect utility of consumption v_i to obtain an arbitrary EIS without affecting the allocation of expenditure (at given $e_t(i)$) across markets and subvarieties. Indeed, if the utility of the households is renormalized to $\Upsilon_i\left(U_i\left(\mathcal{U}_1\left(\boldsymbol{c}_1\right), \ldots, \mathcal{U}_K\left(\boldsymbol{c}_K\right)\right)\right)$, demand for subvarieties $d_k(p_k(j^*), \boldsymbol{p}_k, e_k)$ and the sectoral expenditure functions $e_{k,i}\left(e, \boldsymbol{P}\right)$remains the same while indirect utility of consumption becomes $\Upsilon_i\left(v_i(e, \boldsymbol{P})\right)$. Defining $\Upsilon_i\left(\cdot\right) = \left(v_i^{-1}(\cdot, \boldsymbol{P})\right)^{1-\frac{1}{\sigma}} / \left(1-\frac{1}{\sigma}\right)$ with $\boldsymbol{P}$ fixed at its steady state value allows us to parametrize the EIS to any value σ.

Expenditure of Hand-to-Mouth households.

HtM households consume all their current income, i.e. they never adjust their bond holdings. This allows one to directly solve for the real consumption change in period t from the budget constraint in period t only. In addition, a HtM household of type i born in t_0 has initial bond holdings $b_{t_0}(i) = b(i)\left(1+\sum_l \bar{s}_l \frac{P_{l,t_0}-P_l^*}{P_l^*}\right)$. Differentiating

$$b_{t+1}(i) = R_t\left(b_t(i) + n_t(i)W_t + \sum_k \varsigma_k(i) Div_{k,t} - e_t(i) \right)$$

gives:

$$dR_t\left(b(i)+n(i)W-e(i)\right) \;+\; R\left(dn_t(i)W+n(i)dW_t+\sum_k \varsigma_k(i)dDiv_{k,t}(i)-de_t(i)+b(i)\sum_l \bar{s}_l\hat{P}_{l,t_0}\right) \qquad = \qquad b(i)\sum_l \bar{s}_l\hat{P}_{l,t_0},$$

$\Leftrightarrow$

$$R\left(Wn(i)\left(\left(1+\psi\right)\hat{W}_t-\frac{\psi}{\sigma}\left(\hat{e}_t\left(i\right)-\sum_l s_l\left(i\right)\hat{P}_{l,t}\right)-\sum_k \psi\partial_e e_k(i)\hat{P}_{k,t}\right)+\sum_k \varsigma_k(i)dDiv_{k,t}(i)-e\left(i\right)\left(\hat{e}_t\left(i\right)-\sum_l s_l\left(i\right)\hat{P}_{l,t}+\sum_l s_l(i)\hat{P}_{l,t}\right)\right)$$
$$=\left(1-R\right)b(i)\sum_l \bar{s}_l\hat{P}_{l,t_0}-\hat{R}_t b\left(i\right),$$

$\Leftrightarrow$

$$\hat{R}_t b\left(i\right)+R\left(\psi Wn(i)\hat{W}_t+Wn(i)\sum_k\left(\bar{s}_k-\psi\partial_e e_k(i)\right)\hat{P}_{k,t}-\sum_k e_k(i)\hat{P}_{k,t}+Wn(i)\sum\bar{s}_k\tilde{A}_{k,t}\right)+\left(R-1\right)b(i)\sum_l \bar{s}_l\hat{P}_{l,t_0}$$
$$=R\left(e\left(i\right)+\frac{\psi}{\sigma}Wn(i)\right)\left(\hat{e}_t\left(i\right)-\sum_l s_l\left(i\right)\hat{P}_{l,t}\right),$$

$\Leftrightarrow$

$$\left(e(i)+\frac{\psi}{\sigma}Wn(i)\right)^{-1}\left(\hat{R}_t\frac{b}{R}+Wn(i)\left(\psi\hat{W}_t-\psi\sum_k\overline{\partial_e e}_k\hat{P}_{k,t}+\sum\bar{s}_k\tilde{A}_{k,t}\right)-\sum_k\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\left(\partial_e e_k(i)-\overline{\partial_e e}_k\right)\right)\hat{P}_{k,t}\right)$$
$$+\left(e(i)+\frac{\psi}{\sigma}Wn(i)\right)^{-1}\left(1-\frac{1}{R}\right)b(i)\sum_l \bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t}\right)=\hat{e}_t\left(i\right)-\sum_l s_l\left(i\right)\hat{P}_{l,t}.$$

Using the definition of $\hat{\mathcal{Y}}_t \equiv \frac{\sigma}{\sigma+\psi}\left(\psi\hat{W}_t-\psi\sum_k\overline{\partial_e e}_k\hat{P}_{k,t}+\sum_k\bar{s}_k\tilde{A}_{k,t}\right)$, we obtain:

$$\hat{e}_t\left(i\right)-\sum_l s_l\left(i\right)\hat{P}_{l,t}=\left(e(i)+\frac{\psi}{\sigma}Wn(i)\right)^{-1}\left(\hat{R}_t\frac{b(i)}{R}+\left(1+\frac{\psi}{\sigma}\right)Wn(i)\hat{\mathcal{Y}}_t-\sum_k\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\left(\partial_e e_k(i)-\overline{\partial_e e}_k\right)\right)\hat{P}_{k,t}\right)$$
$$+\left(e(i)+\frac{\psi}{\sigma}Wn(i)\right)^{-1}\left(\left(1-\frac{1}{R}\right)b(i)\sum_l \bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t}\right)\right)$$

where we have used the fact that the equity share of agent i in sector k is the same as the income share and the change in aggregate profits is $d\Pi_t=\sum_k P_k Y_k\left(\hat{P}_{k,t}+\hat{A}_{k,t}-\Omega_{N,k}\hat{W}_t-\sum_l\Omega_{k,l}\hat{P}_{l,t}\right)=\sum_k E_k\left(\hat{P}_{k,t}+\tilde{A}_{k,t}-\hat{W}_t\right)$, with $\tilde{\mathbf{A}}_t=\left(Id-\Omega\right)^{-1}\hat{\mathbf{A}}_t$ so that $dDiv_t(i)=\frac{Wn(i)}{WN}\sum_k E_k\left(\hat{P}_{k,t}+\tilde{A}_{k,t}-\hat{W}_t\right)$ (See subsection on Firm's Input choice for a definition of Ω).[36]

Firms

In this section, we derive the sectoral New Keynesian Phillips Curves. In each sector, identical firms with constant return to scale technology produce subvarieties of good k using labor and a bundle of sector l goods, aggregated by a representative intermediary as inputs. We first derive the firm's pricing equation away from steady state as a function of the change in unit marginal cost. We then study the firm's intratemporal problem to derive changes in demand for intermediate inputs and labor. Finally, using market clearing conditions for goods and labor, we derive the sectoral NKPCs in terms of sectoral prices, the output gap and changes in endogenous markups.

Intermediate inputs producers

We start with competitive intermediaries producing intermediate inputs. They aggregate differentiated varieties into $\tilde{Y}_k$ using a symmetric and CRS technology, and sell them to firms at a price P_k:

$$P_k=\inf_{y_k[j]}\int p_k(j)y_k(j)di$$
$$s.t. 1=\mathcal{F^I}_k(\boldsymbol{y_k})$$

where $\mathcal{F^I}_k$ is symmetric, increasing, strictly concave, C^3 and with $\mathcal{F^I}_k(\boldsymbol{y_k})=1$ if $y_k\left(j\right)=1$ for all j.[37] The intermediary problem defines a unit demand function for subvarieties (indexed by j):

$$D_k^{\mathcal{I}}(p_k[j],\boldsymbol{p_k}).$$

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[36] Note that the real consumption change for HtM agents is given by their MPC times the real income change in a given period that comes from three channels: interest rate changes, output gap and relative prices.

[37] The assumption $\mathcal{F^I}_k(\boldsymbol{y_k})=1 if y_k\left(j\right)=1$ is simply a normalization ensuring that when all prices are equal with $p_k(j)=p_k\,\forall j$, $P_k=p_k$.

Goods varieties firms: price setting

We now turn to the firms producing individual goods varieties. We can re-write the present value of firm profits given in Equation (5) in terms of the reset price and using the fact that production of firms in k has constant returns to scale:[38]

$$\max_{p_{k,t}(j^*)} \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta_k^s \left(p_{k,t}(j^*) D_k \left(p_{k,t}(j^*), \boldsymbol{p}_{k,t+s}, \boldsymbol{e}_{k,t+s}, \tilde{Y}_{k,t+s} \right) - (1-\tau_k) MC_{k,t+s}(j^*) D_k \left(p_{k,t}(j^*), \boldsymbol{p}_{k,t+s}, \boldsymbol{e}_{k,t+s}, \tilde{Y}_{k,t+s} \right) - T_{k,t} \right)$$

with $D_k \left(p_{k,t}(j^*), \boldsymbol{p}_{k,t+s}, \boldsymbol{e}_{k,t+s}, \tilde{Y}_{k,t+s} \right) = \int d_k(p_k(j^*), \boldsymbol{p}_k, e_k(i)) di + D_k^{\mathcal{I}}(p_k[j^*], \{p_k\}) \tilde{Y}_{k,t+s}$ and where MC_k is the marginal cost, to be specified below. The first-order optimality condition is given by:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta_k^s \left(D_{k,t+s}(j^*) + (p_{k,t}(j^*) - (1-\tau_k) MC_{k,t+s}(j^*)) \, \partial_p D_{k,t+s}(j^*) \right) = 0.$$

Using the derivations in the "Households" sub-section and aggregating over the distribution of agents, we can express the change in demand, to a first-order approximation, as:

$$\begin{aligned} dD_{k,t+s}(j^*) &= \int \partial_p d_k(i,j^*) \left(dp_{k,t}(j^*) - dP_{k,t+s} \right) + \partial_{e_k} d_k(i,j^*) \left(de_{k,t+s}(i) - e_k(i) \hat{P}_{k,t+s} \right) di \\ &\quad + \partial_p D_k^{\mathcal{I}} \left(dp_{k,t}(j^*) - dP_{k,t+s} \right) \tilde{Y}_{k,t+s} + D_k^{\mathcal{I}}(p_k[j^*], \{p_k\}) d\tilde{Y}_{k,t+s}, \\ &= \left(\frac{P_k \partial_p D_k^C}{D_k^C} C_k + \frac{P_k \partial_p D_k^{\mathcal{I}}}{D_k^{\mathcal{I}}} \tilde{Y}_{k,t+s} \right) \left(\hat{p}_{k,t}(j^*) - \hat{P}_{k,t+s} \right) \\ &\quad + \frac{1}{P_k} \int \left(de_{k,t+s}(i) - c_k(i) dP_{k,t+s} \right) di + D_k^{\mathcal{I}}(p_k[j^*], \{p_k\}) d\tilde{Y}_{k,t+s}, \end{aligned}$$

where $\partial_p D_k^C = \int \partial_p d_k(i,j^*) \, di$, $D_k^C = \int d_k(i,j^*) \, di$ and we have used that $D_k^{\mathcal{I}}(p_k[j^*], \boldsymbol{p_k}\}) = 1$ in the steady state. Similarly, for the second term:

$$\begin{aligned} d\left(\partial_p D_{k,t+s} \right) &= \int \left(\partial_{pp} d_k(i,j^*) \right) \left(dp_{k,t}(j^*) - dP_{k,t+s} \right) - \partial_p d_k(i,j^*) \hat{P}_{k,t+s} \\ &\quad + \partial_{pe} d_k(i,j^*) \left(de_{k,t+s}(i) - e_k(i) \hat{P}_{k,t+s} \right) di + d \left(\partial_p D_k^{\mathcal{I}} \tilde{Y}_{k,t+s} \right) \\ &= \left(P_k \partial_{pp} D_k^C + P_k \partial_{pp} D_k^{\mathcal{I}} \tilde{Y}_{k,t+s} \right) \left(\hat{p}_{k,t}(j^*) - \hat{P}_{k,t+s} \right) - \left(\frac{P_k \partial_p D_k^C}{D_k^C} C_k + \frac{P_k \partial_p D_k^{\mathcal{I}}}{D_k^{\mathcal{I}}} \tilde{Y}_{k,t+s} \right) \hat{P}_{k,t+s} \\ &\quad + \int \partial_{pe_k} d_k(i,j^*) \left(de_{k,t+s}(i) - c_k(i) dP_{k,t+s} \right) di + \partial_p D_k^{\mathcal{I}} d\tilde{Y}_{k,t+s} \end{aligned}$$

Taking a first-order approximation of the first-order optimality condition and using the expressions above, we obtain:

$$\begin{aligned} 0 &= \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta_k^s \left\{ \left(\hat{p}_{k,t}(j^*) - \hat{P}_{k,t+s} \right) P_k \partial_p D_{k,t+s} + \frac{1}{P_k} \int \left(de_{k,t+s}(i) - e_k(i) \hat{P}_{k,t+s} \right) di + D_k^{\mathcal{I}}(p_k[j^*], \{p_k\}) d\tilde{Y}_{k,t+s} \right\} \\ &\quad + \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta_k^s \left(p_k(j^*) - (1-\tau_k) MC_{k,t+s}(j^*) \right) \left\{ \left(P_k \partial_{pp} D_k^C + P_k \partial_{pp} D_k^{\mathcal{I}} \tilde{Y}_{k,t+s} \right) \left(\hat{p}_{k,t}(j^*) - \hat{P}_{k,t+s} \right) - P_k \partial_p D_{k,t+s} \hat{P}_{k,t+s} \right\} \\ &\quad + \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta_k^s \left(p_k(j^*) - (1-\tau_k) MC_{k,t+s}(j^*) \right) \left\{ \int \partial_{pe_k} d_k(i,j^*) \left(de_{k,t+s}(i) - e_k(i) \hat{P}_{k,t+s} \right) di + \partial_p D_k^{\mathcal{I}} d\tilde{Y}_{k,t+s} \right\} \\ &\quad + \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta_k^s \left(dp_{k,t}(j^*) - (1-\tau_k) dMC_{k,t+s}(j^*) \right) \partial_p D_{k,t+s} \end{aligned}$$

Grouping the terms together and using the fact that in steady state $p_k(j^*) - (1-\tau_k) MC(j^*) = \frac{P_k}{\bar{\epsilon}_k}$, $D_k + (P_k - (1+\tau_k) MC_k) \, \partial_p D_k = 0$, $\frac{P_k \partial_p D_k^C}{D_k^C} = \frac{P_k \partial_p D_k^{\mathcal{I}}}{D_k^{\mathcal{I}}} = -\bar{\epsilon}_k$ and, in the steady state, $\Lambda_{t,t+s} = \tilde{\beta}^s$, where we assume that $\tilde{\beta} = (1-\delta)\beta = 1/R$, we obtain:

$$\begin{aligned} 0 &= \left(1 - \tilde{\beta}\theta_k \right)^{-1} \left(2 P_k \partial_p D_{k,t+s} + \frac{P_k}{\bar{\epsilon}_k} \left(P_k \partial_{pp} D_k^C + P_k \partial_{pp} D_k^{\mathcal{I}} \tilde{Y}_{k,t+s} \right) \right) \hat{p}_{k,t}(j^*) \\ &\quad - \mathbb{E}_t \sum_{s=0}^{\infty} \left(\tilde{\beta}\theta_k \right)^s \left(2 P_k \partial_p D_{k,t+s} + \frac{P_k}{\bar{\epsilon}_k} \left(P_k \partial_{pp} D_k^C + P_k \partial_{pp} D_k^{\mathcal{I}} \tilde{Y}_{k,t+s} \right) \right) \hat{P}_{k,t+s} \\ &\quad + \mathbb{E}_t \sum_{s=0}^{\infty} \left(\tilde{\beta}\theta_k \right)^s \int \left(\frac{1}{P_k} + \frac{P_k}{\bar{\epsilon}_k} \partial_{pe_k} d_k(i,j^*) \right) \left(de_{k,t+s}(i) - e_k(i) \hat{P}_{k,t+s} \right) di + \mathbb{E}_t \sum_{s=0}^{\infty} \left(\tilde{\beta}\theta_k \right)^s \left(\bar{\epsilon}_k - 1 \right) D_k \hat{mc}_{k,t+s} \end{aligned}$$

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[38] This implies that total costs TC can be written as $TC_{k,t}(j) = MC \left(W_t, \boldsymbol{P}_t^I \right) D_{k,t}(j)$.

where $\hat{mc}_{k,t+s} \equiv \hat{MC}_{k,t+s} - \hat{P}_{k,t+s}$ is common across firms. Rewriting this expression recursively gives:

$$\begin{aligned}
&\hat{p}_{k,t}(j^*) = \left(1-\tilde{\beta}\theta_k\right)\hat{P}_{k,t} \\
&- \frac{\left(1-\tilde{\beta}\theta_k\right)\left(\bar{\epsilon}_k - 1\right)}{2P_k\partial_p D_{k,t+s} + \frac{P_k}{\bar{\epsilon}_k}\left(P_k\partial_{pp}D_k^C + P_k\partial_{pp}D_k^{\mathcal{I}}\tilde{Y}_{k,t+s}\right)}\left\{\int\left(\frac{1}{P_k\left(\bar{\epsilon}_k-1\right)} + \frac{P_k}{\bar{\epsilon}_k\left(\bar{\epsilon}_k-1\right)}\partial_{pe_k}d_k\left(i,j^*\right)\right)\left(de_{k,t}\left(i\right) - e_k(i)\hat{P}_{k,t}\right)di + D_k\hat{mc}_{k,t}\right\} \\
&\qquad + \tilde{\beta}\theta_k\mathbb{E}_t\hat{p}_{k,t+1}(j^*)
\end{aligned}$$

Next recall the following definitions:

$$\begin{aligned}
\bar{\epsilon}_k^s &\equiv P_k\partial_p\ln(\bar{\epsilon}_k^s) = \left(-\int\left(\epsilon_k(j)-\bar{\epsilon}_k\right)^2\frac{e_k(j)}{E_k}dj + \int P_k\partial_p\epsilon_k(j)\frac{e_k(j)}{E_k}dj\right)/\bar{\epsilon}_k, \\
\bar{\epsilon}_k^{s,I} &\equiv P_k\partial_p\ln(\epsilon_k^{\mathcal{I}}), \\
\gamma_{e,k}(i) &\equiv \left(1-\frac{\epsilon_k(i)}{\bar{\epsilon}_k}\left(1+\frac{\partial ln(\epsilon_k(i))}{\partial ln(e_k(i))}\right)\right)/\left(\bar{\epsilon}_k-1\right).
\end{aligned}$$

Plugging these definition into the optimal price equation, we obtain:

$$\hat{p}_{k,t}(j^*) = \left(1-\tilde{\beta}\theta_k\right)\hat{P}_{k,t} + \frac{\left(1-\tilde{\beta}\theta_k\right)\left(\bar{\epsilon}_k-1\right)}{\bar{\epsilon}_k - 1 + s_k^C\bar{\epsilon}_k^s + \left(1-s_k^C\right)\bar{\epsilon}_k^{s,I}}\left\{s_k^C\int\gamma_{e,k}(i)\frac{de_{k,t}\left(i\right)-e_k(i)\hat{P}_{k,t}}{E_k}di + \hat{mc}_{k,t}\right\} + \tilde{\beta}\theta_k\mathbb{E}_t\hat{p}_{k,t+1}(j^*).$$

Note that all firms that can reset their prices choose the same $\hat{p}_{k,t}^*$ and $\hat{P}_{k,t} = \left(1-\theta_k\right)\hat{P}_{k,t}^* + \theta_k\hat{P}_{k,t-1}$. It follows that:

$$\pi_{k,t} = \frac{\left(1-\tilde{\beta}\theta_k\right)\left(1-\theta_k\right)}{\theta_k}\frac{\left(\bar{\epsilon}_k-1\right)}{\bar{\epsilon}_k-1+s_k^C\bar{\epsilon}_k^s+\left(1-s_k^C\right)\bar{\epsilon}_k^{s,I}}\left\{s_k^C\int\gamma_{e,k}(i)\frac{de_{k,t}\left(i\right)-e_k(i)\hat{P}_{k,t}}{E_k}di + \hat{mc}_{k,t}\right\} + \tilde{\beta}\mathbb{E}_t\pi_{k,t+1}.$$

Defining

$$\lambda_k \equiv \frac{\left(1-\tilde{\beta}\theta_k\right)\left(1-\theta_k\right)}{\theta_k}\frac{\left(\bar{\epsilon}_k-1\right)}{\bar{\epsilon}_k-1+s_k^C\bar{\epsilon}_k^s+\left(1-s_k^C\right)\bar{\epsilon}_k^{s,I}},$$

we can write the sectoral NKPC as:

$$\pi_{k,t} = \lambda_k\left\{s_k^C\int\gamma_{e,k}(i)\frac{de_{k,t}\left(i\right)-e_k(i)\hat{P}_{k,t}}{E_k}di + \hat{mc}_{k,t}\right\} + \tilde{\beta}\mathbb{E}_t\pi_{k,t+1}.$$

Goods varieties firms: intermediate input choice

The cost-minimization problem of the firm is given by: $\min WL_k(j) + \sum_l P_l\tilde{Y}_{l,k}(j) \quad s.t. \quad A_kF_k(n_k(j), \tilde{Y}_{1,k}(j), \tilde{Y}_{2,k}(j), \ldots, \tilde{Y}_{K,k}(j)) \geq y_k(j)$. Since F_k has constant return to scale we can express the change in the marginal cost as:

$$\begin{aligned}
dMC_k &= \frac{Wn_k}{Y_k}\hat{W} + \sum_l\frac{P_l\tilde{Y}_{l,k}}{Y_k}\hat{P}_l - MC_k\hat{A}_k, \\
&\Leftrightarrow \\
\hat{MC}_k &= \left(1+\mu_k\right)\left(1-\tau_k\right)\left(\Omega_{N,k}\hat{W} + \sum_l\Omega_{k,l}\hat{P}_l\right) - \hat{A}_k, \\
&= \left(\Omega_{N,k}\hat{W} + \sum_l\Omega_{k,l}\hat{P}_l\right) - \hat{A}_k.
\end{aligned}$$

The subsidy is chosen to eliminate markup distortions in the steady state, i.e. $\left(1+\mu_k\right)\left(1-\tau_k\right) = 1$. Ω is the matrix of intermediate input shares ($\Omega_{k,l} = \frac{P_l\tilde{Y}_{l,k}}{P_kY_k}$), Ω_N a column vector of length K of labor shares ($\Omega_{N,k} = 1 - \sum_{l=1}^K\Omega_{k,l}$). Since F_k has CRS, we can write demand for input l has $\tilde{Y}_{l,k}(j) = \mathcal{Y}_{l,k}(\boldsymbol{P}, W)\frac{y_k(j)}{A_k}$ (where $\mathcal{Y}_{l,k}(\boldsymbol{P}, W)$ the unit demand for input l by firms in k is common to all firms in k) and derive change in aggregate demand for input bundle l as:

$$\frac{d\tilde{Y}_l}{\tilde{Y}_l} = \sum_k\mathcal{Q}_{l,k}\left(\hat{Y}_k - \hat{A}_k\right) + \tilde{\mathcal{T}}_{l,W}\hat{W} + \sum_k\tilde{\mathcal{T}}_{l,k}\hat{P}_k.$$

Let $\tilde{\mathcal{T}}$ be the matrix of aggregate input price elasticities such that $\tilde{\mathcal{T}}_{l,k} = \sum_m\frac{\tilde{Y}_{l,m}}{\tilde{Y}_l}\frac{\partial\mathcal{Y}_{l,m}}{\partial P_k}\frac{P_k}{\mathcal{Y}_{l,m}}$, $\tilde{\mathcal{T}}_{l,W} = \sum_m\frac{\tilde{Y}_{l,m}}{\tilde{Y}_l}\frac{\partial\mathcal{Y}_{l,m}}{\partial W}\frac{W}{\mathcal{Y}_{l,m}}$ be the column vector of wage elasticities and $\mathcal{Q}_{l,k} = \mathcal{Y}_{l,k}$ be the matrix of intermediate shares. Since intermediary input producers have a CRS technology we can write the (aggregated) market clearing equation for subvariety k as $\hat{Y}_k = s_k^C\hat{C}_k + (1-s_k^C)\tilde{Y}_k$.We have, denoting $\mathcal{D}\left[s^c\right]$ and $\mathcal{D}\left[PY\right]$ as the diagonal matrices with share of consumption demand and sectoral revenue on the diagonal ($s_k^C = \frac{E_k}{P_kY_k}$and P_kY_k), $\hat{\boldsymbol{Y}}$,$\hat{\boldsymbol{C}}$, $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{P}}$ the column vectors of sectoral output, consumption, TFP shocks and prices:

$$\hat{\boldsymbol{Y}} = \mathcal{D}\left[s^c\right]\hat{\boldsymbol{C}} + \left(Id - \mathcal{D}\left[s^c\right]\right)\left(\mathcal{Q}\left(\hat{\boldsymbol{Y}} - \hat{\boldsymbol{A}}\right) + \tilde{\mathcal{T}}_W\hat{W} + \tilde{\mathcal{T}}\hat{\boldsymbol{P}}\right),$$

$$\Leftrightarrow$$

$$\left(Id - \mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)\hat{\boldsymbol{Y}} = \mathcal{D}\left[s^c\right]\hat{\boldsymbol{C}} - \mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\hat{\boldsymbol{A}} + \mathcal{T}_W\hat{W} + \mathcal{T}\hat{\boldsymbol{P}}.$$

where we use the fact that $\left[\left(Id - \mathcal{D}\left[s^c\right]\right)\mathcal{Q}\right]_{k,l} = \frac{P_k\tilde{Y}_k}{P_kY_k}\frac{\tilde{Y}_{k,l}}{\tilde{Y}_k} = \frac{P_kY_k}{P_lY_l}\Omega_{l,k}$. Note that $\mathcal{T}_W = \left(Id - \mathcal{D}\left[s^c\right]\right)\tilde{\mathcal{T}}_W$ and similarly $\mathcal{T} = \left(Id - \mathcal{D}\left[s^c\right]\right)\tilde{\mathcal{T}}$.

Labor Demand Response

We can similarly write demand for labor for a firm j in sector k as $n_k(j) = \mathcal{N}_k(\boldsymbol{P}, W)\frac{y_k(j)}{A_k}$. Differentiating and aggregating this function, we can express the percentage change in aggregate labor demand as:

$$\hat{N} = \boldsymbol{s^N}\left(\left(\hat{Y} - \hat{A}\right) + \tilde{\mathcal{T}}_W^N\hat{W} + \tilde{\mathcal{T}}^N\hat{P}\right),$$

where $\boldsymbol{s^N} = \left[\frac{W\int n_1(j)dj}{WN}, ..., \frac{W\int n_K(j)dj}{WN}\right]$, $\tilde{\mathcal{T}}_W^N = \left[\partial_{\ln(W)}\ln\left(\mathcal{N}_1\right), ..., \partial_{\ln(W)}\ln\left(\mathcal{N}_K\right)\right]$, $\tilde{\mathcal{T}}_{k,l}^N = \partial_{\ln(P_l)}\ln\left(\mathcal{N}_k\right)$. One can show that the change in labor demand will only depend on the change in consumption and productivities as follows:

$$\begin{aligned}\hat{N} &= \boldsymbol{s^N}\left(\left(Id - \mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)^{-1}\left(\mathcal{D}\left[s^c\right]\hat{C} - \mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\hat{A} + \mathcal{T}_W\hat{W} + \mathcal{T}\hat{P}\right) - \hat{A} + \tilde{\mathcal{T}}_W^N\hat{W} + \tilde{\mathcal{T}}^N\hat{P}\right),\\ &= \boldsymbol{s^N}\left(\left(Id - \mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)^{-1}\left(\mathcal{D}\left[s^c\right] - \hat{A}\right)\right) + \boldsymbol{s^N}\left(\mathcal{T}_W^N\hat{W} + \mathcal{T}^N\hat{P} + \left(Id - \mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)^{-1}\left(\mathcal{T}_W\hat{W} + \mathcal{T}\hat{P}\right)\right).\end{aligned}$$

Note that, as $\left(1 + \mu_k\right)\left(1 - \tau_k\right) = 1$, we have

$$\begin{aligned}\left[WN_1, ..., WN_K\right]\left(Id - \mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)^{-1} &= \left[PY_1, ..., PY_K\right],\\ \partial_{ln(W)}\mathcal{N}_k\hat{W} + \sum_l \partial_{ln(P_l)}\mathcal{Y}_{l,k}\hat{W} &= 0,\\ \partial_{ln(P_l)}\mathcal{N}_k\hat{P}_l + \sum_m \partial_{ln(P_l)}\mathcal{Y}_{m,k}\hat{P}_l &= 0,\end{aligned}$$

where Id denotes the diagonal matrix. We thus obtain:

$$\begin{aligned}\hat{N} &= \boldsymbol{s^N}\left(Id - \mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)^{-1}\left(\mathcal{D}\left[s^c\right]\hat{C} - \hat{A}\right)\\ &= \sum_k \bar{s}_k\left(\hat{E}_k - \hat{P}_k\right) - \sum_k \frac{P_kY_k}{E}\hat{A}_k.\end{aligned}$$

Aggregate Consumption Response

We can derive aggregate spending in sector k by simply aggregating individual decisions:

$$\begin{aligned}\hat{E}_k - \hat{P}_k &= \frac{1}{E_k}\int de_k(i) - e_k(i)\hat{P}_k di,\\ &= \int \frac{e(i)}{E_k}\partial_e e_k(i)\left(\hat{e} - \sum_l s_l\hat{P}_l\right)di + \sum_l S_{k,l}\hat{P}_l,\end{aligned}$$

where $S_{k,l} = \int \rho_{k,l}(i)\frac{e_k(i)}{E_k}di$ is the aggregate compensated price elasticity of sector k with respect to P_l.

Labor Market Clearing

Let us re-introduce time subscripts. Recall that:

$$\hat{n}_t(i) = \psi\left\{\hat{W}_t - \sum_k \partial_e e_k(i)\hat{P}_{k,t}\right\} - \frac{\psi}{\sigma}\left(\hat{e}_t\left(i\right) - \sum_l s_l\left(i\right)\hat{P}_{l,t}\right).$$

Aggregating over all households we obtain:

$$\hat{N}_t = \psi\left(\hat{W}_t - \sum_k \int \frac{Wn(i)}{WN}\partial_e e_k(i)di\hat{P}_{k,t}\right) - \frac{\psi}{\sigma}\int \frac{Wn(i)}{WN}\left(\hat{e}_t\left(i\right) - \sum_l s_l\left(i\right)\hat{P}_{l,t}\right)di.$$

So labor market clearing becomes:

$$\psi\left(\hat{W}_t-\sum_k\int\frac{Wn(i)}{WN}\partial_e e_k(i)di\hat{P}_{k,t}\right)-\frac{\psi}{\sigma}\int\frac{Wn(i)}{WN}\left(\hat{e}_t\left(i\right)-\sum_l s_l\left(i\right)\hat{P}_{l,t}\right)di=\sum_k\left(\bar{s}_k\left(\hat{E}_{k,t}-\hat{P}_{k,t}\right)-\frac{P_kY_k}{E}\hat{A}_{k,t}\right)$$

$$=\sum_k\bar{s}_k\left(\int\frac{e(i)}{E_k}\partial_e e_k(i)\left(\hat{e}_t(i)-\sum_l s_l(i)\hat{P}_{l,t}\right)di+\sum_l S_{k,l}\hat{P}_{l,t}\right)-\sum_k\frac{P_kY_k}{E}\hat{A}_{k,t},$$

$\Leftrightarrow$

$$\psi\hat{W}_t-\psi\sum_k\int\frac{Wn(i)}{WN}\partial_e e_k(i)di\hat{P}_{k,t}+\sum_k\frac{P_kY_k}{E}\hat{A}_{k,t}=\left(\int\frac{e(i)}{E}\left(\sum_k\partial_e e_k(i)+\frac{\psi Wn(i)}{\sigma e(i)}\right)\left(\hat{e}_t(i)-\sum_l s_l(i)\hat{P}_{l,t}\right)di+\sum_l\left(\sum_k\bar{s}_k\mathcal{S}_{k,l}\right)\hat{P}_{l,t}\right),$$

$$=\int\frac{e(i)}{E}\left(1+\frac{\psi Wn(i)}{e(i)\sigma}\right)\left(\hat{e}_t(i)-\sum_l s_l(i)\hat{P}_{l,t}\right)di,$$

where we have used the fact that $\sum_k e_k\rho_{k,l}(i)=0$ for all l,i so $\sum_k\bar{s}_k\mathcal{S}_{k,l}=0$. Finally, recall the definitions:

$$\tilde{\mathbf{A}}_t\equiv\left(Id-\Omega\right)^{-1}\hat{\mathbf{A}}_t,$$

$$\hat{\mathcal{Y}}_t\equiv\frac{\sigma}{\sigma+\psi}\left(\psi\hat{W}_t-\psi\sum_k\overline{\partial_e e}_k\hat{P}_{k,t}+\sum_k\bar{s}_k\tilde{A}_{k,t}\right),$$

where the last line uses the fact that $\left[E_1,...,E_k\right]\left(Id-\Omega\right)^{-1}=\left[P_1Y_1,...,P_kY_k\right]$.So we have:

$$\hat{\mathcal{Y}}_t=\frac{\sigma}{\sigma+\psi}\int\frac{e(i)}{E}\left(1+\frac{\psi Wn(i)}{e(i)\sigma}\right)\left(\hat{e}_t(i)-\sum_l s_l(i)\hat{P}_{l,t}\right)di+\frac{\sigma\psi}{\sigma+\psi}\sum_k\int\frac{wn(i)-e(i)}{E}\partial_e e_k(i)di\hat{P}_{k,t}.$$

Defining the natural level of aggregate demand as the level that prevails in the absence of markups distortions we obtain our formula for the output gap:

$$\hat{\mathcal{Y}^*}_t\equiv\frac{\sigma}{\sigma+\psi}\left(\left(\sum_k\psi\overline{\partial_e e}_k+\bar{s}_k\right)\tilde{A}_{k,t}\right),$$

$$\tilde{\mathcal{Y}}_t=\frac{\sigma\psi}{\sigma+\psi}\left(\hat{W}_t-\sum_k\overline{\partial_e e}_k\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right).$$

see the optimal policy section for a justification of the efficiency of $\hat{\mathcal{Y}^*}_t$. Note that in the absence of markup distortions it holds that $\hat{P}_{k,t}=\hat{W}_t-\tilde{A}_{k,t}$. We will show later, that the output gap shows up in the social welfare function.

Production Efficiency (Detour)

In this section we briefly show that our set of steady state subsidies ($\left(1+\mu_k\right)\left(1-\tau_k\right)=1$) renders production efficient in the steady state. Production is efficient if the steady state consumption bundle $\left\{C_1,...,C_K\right\}$is produced at minimum labor cost.

$$\hat{L}=\sum_k s_k^N\left(\hat{Y}_k+\partial_{ln(W)}ln\left(\mathcal{N}_k\right)\hat{W}+\sum_l\partial_{ln(P_l)}ln\left(\mathcal{N}_k\right)\hat{P}_l\right),$$

$$\left(Id-\mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)\hat{Y}=\left\{\sum_k\frac{W\partial\mathcal{Y}^h_{j,k}}{Y_j\partial W}\right\}_j\hat{W}+\left\{\sum_k\frac{P_l\partial\mathcal{Y}^h_{j,k}}{Y_j\partial P_l}\right\}_{j,l}\hat{P}_l.$$

Therefore:

$$\sum_k s_k^N\partial_{ln(W)}ln\left(\mathcal{N}_k\right)+\boldsymbol{s}^{\mathbf{N}}\left(Id-\mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)^{-1}\left\{\sum_k\frac{W\partial\mathcal{Y}^h_{j,k}}{Y_j\partial W}\right\}_j=0,$$

$$\sum_k s_k^N\partial_{ln(P_l)}ln\left(\mathcal{N}_k\right)+\boldsymbol{s}^{\mathbf{N}}\left(Id-\mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)^{-1}\left\{\sum_k\frac{P_l\partial\mathcal{Y}^h_{j,k}}{Y_j\partial P_l}\right\}_j=0,$$

So $\boldsymbol{s}^{\mathbf{N}}\left(Id-\mathcal{D}\left[PY\right]^{-1}\Omega^T\mathcal{D}\left[PY\right]\right)^{-1}=\left[P_jY_j/WN\right]$, which gives $WN_k=P_kY_k-\sum P_l\tilde{Y}_{l,k}$, or $\left(1+\mu_k\right)\left(1-\tau_k\right)=1$ for all k.

Sectoral NKPC

Recall that

$$\pi_{k,t} = \lambda_k \left\{ s_k^C \int \gamma_{e,k}(i) \frac{de_{k,t}(i) - e_k(i)\hat{P}_{k,t}}{E_k} di + \hat{mc}_{k,t} \right\} + \tilde{\beta}\mathbb{E}_t \pi_{k,t+1},$$

$$\hat{mc}_{k,t} = \left(\Omega_{N,k} \hat{W}_t + \sum_l \Omega_{k,l} \hat{P}_{l,t} \right) - \hat{A}_{k,t} - \hat{P}_{k,t},$$

$$\tilde{\mathcal{Y}}_t = \frac{\sigma\psi}{\sigma + \psi} \left(\hat{W}_t - \sum_k \overline{\partial_e e_k} \left(\hat{P}_{k,t} + \tilde{A}_{k,t} \right) \right),$$

$$de_{k,t}(i) - e_k(i)\hat{P}_{k,t} = \partial_e e_k(i) \left(de_t(i) - \sum_l e_l(i)\hat{P}_{l,t} \right) + e_k(i) \sum_l \rho_{k,l}(i) \hat{P}_{l,t}.$$

Combining these equations, we obtain:

$$\begin{aligned}\pi_{k,t} &= \lambda_k \left\{ s_k^C \tilde{\mathcal{M}}_{k,t} + \Omega_{N,k} \left(\frac{1}{\sigma} + \frac{1}{\psi} \right) \tilde{\mathcal{Y}}_t + \Omega_{N,k} \sum_l \overline{\partial_e e_l} \left(\hat{P}_{l,t} + \tilde{A}_{l,t} \right) + \sum_l \Omega_{k,l} \hat{P}_{l,t} - \hat{A}_{k,t} - \hat{P}_{k,t} \right\} + \tilde{\beta}\mathbb{E}_t \pi_{k,t+1}, \\ &= \lambda_k \left\{ s_k^C \tilde{\mathcal{M}}_{k,t} + \Omega_{N,k} \left(\frac{1}{\sigma} + \frac{1}{\psi} \right) \tilde{\mathcal{Y}}_t + \Omega_{N,k} \sum_l \overline{\partial_e e_l} \left(\hat{P}_{l,t} + \tilde{A}_{l,t} - (\hat{P}_{k,t} + \tilde{A}_{k,t}) \right) + \sum_l \Omega_{k,l} \left(\hat{P}_{l,t} + \tilde{A}_{l,t} - \left(\hat{P}_{k,t} + \tilde{A}_{k,t} \right) \right) \right\} + \tilde{\beta}\mathbb{E}_t \pi_{k,t+1},\end{aligned}$$

with

$$\begin{aligned}\tilde{\mathcal{M}}_{k,t} &= \mathcal{M}_{k,t}^P + \mathcal{M}_{k,t}^E, \\ \mathcal{M}_{k,t}^P &= \sum_l \int \gamma_{e,k}(i) \frac{e_k(i)}{E_k} \rho_{k,l}(i) di \hat{P}_{l,t}, \\ \mathcal{M}_{k,t}^E &= \int \gamma_{e,k}(i) \partial_e e_k \frac{e(i)}{E_k} \left(\hat{e}(i) - \sum_l s_l(i) \hat{P}_{l,t} \right) di\end{aligned}$$

Finally, defining

$$\begin{aligned}\kappa_k &\equiv \lambda_k \left(\frac{1}{\sigma} + \frac{1}{\psi} \right) \left(1 + \frac{\sigma\psi}{\sigma + \psi} \Gamma_k \right), \\ \Gamma_k &\equiv \frac{R}{R-1} \frac{\sigma + \psi}{\sigma} \int \gamma_{b,k}(i) \frac{Wn(i)}{WN} di, \\ \mathcal{M}_{k,t}^D &\equiv \mathcal{M}_{k,t}^E - \frac{1 + \frac{\bar{\psi}}{\bar{\sigma}}}{1 - \frac{1}{R}} \int \gamma_{b,k}(i) \frac{Wn(i)}{WN} di \hat{\mathcal{Y}}_t, \\ \mathcal{M}_{k,t} &\equiv \mathcal{M}_{k,t}^P + \mathcal{M}_{k,t}^D + \frac{1 + \frac{\bar{\psi}}{\bar{\sigma}}}{1 - \frac{1}{R}} \int \gamma_{b,k}(i) \frac{Wn(i)}{WN} di \hat{\mathcal{Y}}_t^*, \\ \mathcal{NH}_t &\equiv \sum_{l=1}^K (\overline{\partial_e e_l} - \bar{s}_l)(\hat{P}_{l,t} + \tilde{A}_{l,t}), \\ \mathcal{P}_{k,t} &\equiv (\hat{P}_{k,t} - \sum_{l=1}^K \bar{s}_l \hat{P}_{l,t}) + (\tilde{A}_{k,t} - \sum_{l=1}^K \tilde{A}_{l,t}), \\ \mathcal{I}_{k,t} &\equiv \sum_{l=1}^K \Omega_{k,l} \left(\mathcal{P}_{l,t} - \mathcal{P}_{k,t} \right),\end{aligned}$$

and noting that $\Omega_{N,k} \sum_{l=1}^K \overline{\partial_e e_l} + \sum_{l=1}^K \Omega_{k,l} = 1$, gives the formula in the model equation appendix. To obtain the equations of the main text without the Input-Output structure, we simply set $\Omega_{N,k} = 1$, $s_k^C = 1$ and $\mathcal{I}_{k,t} = 0$, and obtain:

$$\pi_{k,t} = \kappa_k \tilde{\mathcal{Y}}_t + \lambda_k \left(\mathcal{NH}_t + \mathcal{M}_{k,t} - \mathcal{P}_{k,t} \right) + \tilde{\beta}\mathbb{E}_t \pi_{k,t+1}.$$

Evolution of arbitrary demand indices

In this section, we derive the dynamic equations characterizing the evolution of averages of individual households expenditures for arbitrary weights, taking into account the death/birth process. These equations can be used to compute the full distribution of consumption expenditures. In the next two subsection, we also use these equations to derive the dynamic equation for the output gap and for the endogenous markup wedge.

Denote by $C_t(\omega) = \int \omega(i) \left(\hat{e}_t(i) - \sum_l s_l(i) \hat{P}_{l,t} \right) di$ an arbitrary demand index with weight ω. Moreover, denote by $C_{t+1}^u(\omega) = \int (1 - \varphi(i)) \omega(i) \left(\hat{e}_t(i) - \sum_l s_l(i) \hat{P}_{l,t} \right) di$ the contribution of unconstrained (=non-HtM) households to the demand index. We have:

$$\mathbb{E}_t C^u_{t+1}(\boldsymbol{\omega}) = (1-\delta)\, C^u_t(\boldsymbol{\omega}) + (1-\delta)\,\sigma \int (1-\varphi(i))\,\omega(i) \left(\hat{R}_t - \sum_k \partial_e e_k(i)\pi_{k,t+1}\right) di + \delta \tilde{C}^{u,0}_{t+1}(\boldsymbol{\omega})$$

Here, we use the individual Euler equation, as derived above:

$$\left(\hat{e}_t - \sum_l s_l \hat{P}_{l,t}\right) = \mathbb{E}_t \left(\hat{e}_{t+1} - \sum_l s_l \hat{P}_{l,t+1}\right) - \sigma \mathbb{E}_t \left(\hat{R}_t - \sum_k \partial_e e_k(i)\pi_{k,t+1}\right)$$

for households "born" before $t+1$. $\tilde{C}^{u,0}_{t+1}(\boldsymbol{\omega})$ is the consumption of the households born at $t+1$. Note that the lifetime budget constraint of the households born at t with wealth $b(i)\left(1+\sum_l \bar{s}_l \hat{P}_{l,t}\right)$ is

$$-b(i)\sum_l \bar{s}_l \hat{P}_{l,t} = \mathbb{E}_t \sum_{s=0}^{\infty} \frac{1}{R^s}\left(\frac{b(i)}{R}\hat{R}_{t+s} + Wn(i)\left(\hat{W}_{t+s} - \hat{n}_{t+s}\right) + dDiv_{t+s}(i) - e(i)\sum_k s_k(i)\hat{P}_{k,t+s}\right) - e(i)\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{e}_{t+s} - \sum_k s_k(i)\hat{P}_{k,t+s}\right)$$

Using labor supply decisions and $dDiv_t(i) = \frac{Wn(i)}{WN}\sum_k E_k\left(\hat{P}_k + \tilde{A}_{k,t} - \hat{W}\right)$ we obtain:

$$-b(i)\sum_l \bar{s}_l \hat{P}_{l,t} + \left(e(i) + \frac{\psi}{\sigma}Wn(i)\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{e}_{t+s} - \sum_k s_k(i)\hat{P}_{k,t+s}\right) =$$

$$\mathbb{E}_t \sum_{s=0}^{\infty}\frac{1}{R^s}\left(\frac{b(i)}{R}\hat{R}_{t+s} + \left(1+\frac{\psi}{\sigma}\right)Wn(i)\hat{\mathcal{Y}}_{t+s} - \sum_k \left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\hat{P}_{k,t+s} - \left(1-\frac{1}{R}\right)b(i)\sum_l \bar{s}_l \hat{P}_{l,t+s}\right)$$

$\Leftrightarrow$

$$\left(e(i) + \frac{\psi}{\sigma}Wn(i)\right)\sum \frac{1}{R^s}\left(\hat{e}_{t+s} - \sum_k s_k(i)\hat{P}_{k,t+s}\right) =$$

$$\mathbb{E}_t \sum_{s=0}^{\infty}\frac{1}{R^s}\left(\frac{b(i)}{R}\left(\hat{R}_{t+s} - \pi_{cpi,t+s+1}\right) + \left(1+\frac{\psi}{\sigma}\right)Wn(i)\hat{\mathcal{Y}}_{t+s} - \sum_k \left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\hat{P}_{k,t+s}\right).$$

Using the Euler equation, $\hat{e}_{t+u} - \sum_k s_k(i)\hat{P}_{k,t+u} = \hat{e}_t - \sum_k s_k(i)\hat{P}_{k,t} + \sigma\mathbb{E}_t\sum_{s=0}^{u-1}\left(\hat{R}_{t+s} - \sum_k \partial_e e_k(i)\pi_{k,t+s+1}\right)$ so we obtain:

$$\hat{e}_t - \sum_k s_k(i)\hat{P}_{k,t} =$$

$$\frac{1-\frac{1}{R}}{e(i)+\frac{\psi}{\sigma}Wn(i)}\mathbb{E}_t \sum_{s=0}^{\infty}\frac{1}{R^s}\left(\frac{b(i)}{R}\left(\hat{R}_{t+s} - \pi_{cpi,t+s+1}\right) + \left(1+\frac{\psi}{\sigma}\right)Wn(i)\hat{\mathcal{Y}}_{t+s} - \sum_k \left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\hat{P}_{k,t+s}\right)$$

$$-\sigma\mathbb{E}_t\sum_{s=0}^{\infty}\frac{1}{R^{s+1}}\left(\hat{R}_{t+s} - \sum_k \partial_e e_k(i)\pi_{k,t+s+1}\right).$$

Averaging across households with the arbitrary weights $\boldsymbol{\omega}$, we have:

$$\tilde{C}^{u,0}_t(\boldsymbol{\omega}) - \frac{1}{R}\mathbb{E}_t N\tilde{C}^{u,0}_{t+1}(\boldsymbol{\omega}) = -\sigma\frac{1}{R}\mathbb{E}_t\int(1-\varphi(i))\,\omega(i)\left(\hat{R}_t - \sum_k \partial_e e_k(i)\pi_{k,t+1}\right)di +$$

$$\int \frac{(1-\varphi(i))\,\omega(i)\left(1-\frac{1}{R}\right)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left(\frac{b(i)}{R}\left(\hat{R}_t - \pi_{cpi,t+1}\right) + \left(1+\frac{\psi}{\sigma}\right)Wn(i)\hat{\mathcal{Y}}_{t+s} - \sum_k \left(e(i)\left(s_k(i)-\bar{s}_k\right) + \psi Wn(i)\left(\partial_e e_k(i) - \overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right)di.$$

Defining $C^{u,0}_t(\boldsymbol{\omega}) \equiv \tilde{C}^{u,0}_t(\boldsymbol{\omega}) - C^u_t(\boldsymbol{\omega})$, we have:

$$\mathbb{E}_t C^u_{t+1}(\boldsymbol{\omega}) = C^u_t(\boldsymbol{\omega}) + \sigma\mathbb{E}_t\int(1-\varphi(i))\,\omega(i)\left(\hat{R}_t - \sum_k \partial_e e_k(i)\pi_{k,t+1}\right)di + \frac{\delta}{1-\delta}C^{u,0}_{t+1}(\boldsymbol{\omega}),$$

$$C_t^{u,0}\left(\boldsymbol{\omega}\right)-\frac{1}{R}\mathbb{E}_tC_{t+1}^{u,0}\left(\boldsymbol{\omega}\right)+C_t\left(\boldsymbol{\omega}\right)-\frac{1}{R}\mathbb{E}_tC_{t+1}^{u}\left(\boldsymbol{\omega}\right)=$$

$$\mathbb{E}_t\int\frac{\left(1-\varphi\left(i\right)\right)\omega(i)\left(1-\frac{1}{R}\right)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left(\frac{b(i)}{R}\left(\hat{R}_t-\pi_{cpi,t+1}\right)+\left(1+\frac{\psi}{\sigma}\right)Wn(i)\hat{\mathcal{Y}}_t-\sum_k\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\left(\partial_e e_k(i)-\overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right)di$$

$$-\sigma\frac{1}{R}\mathbb{E}_t\int\left(1-\varphi\left(i\right)\right)\omega(i)\left(\hat{R}_t-\sum_k\partial_e e_k(i)\pi_{k,t+1}\right)di,$$

$$C_t^{u,0}\left(\boldsymbol{\omega}\right)-\frac{1}{R\left(1-\delta\right)}\mathbb{E}_tC_{t+1}^{u,0}\left(\boldsymbol{\omega}\right)+\left(1-\frac{1}{R}\right)C_t^{u}\left(\boldsymbol{\omega}\right)=$$

$$\mathbb{E}_t\int\frac{\left(1-\varphi\left(i\right)\right)\omega(i)\left(1-\frac{1}{R}\right)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left(\frac{b(i)}{R}\left(\hat{R}_t-\pi_{cpi,t+1}\right)+\left(1+\frac{\psi}{\sigma}\right)Wn(i)\hat{\mathcal{Y}}_t-\sum_k\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\left(\partial_e e_k(i)-\overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right)di.$$

Now we consider the contribution of the HtMs, we have:

$$\mathbb{E}_tC_{t+1}^{HtM}\left(\boldsymbol{\omega}\right)=\left(1-\delta\right)C_t^{HtM}\left(\boldsymbol{\omega}\right)+\delta\tilde{C}_{t+1}^{HtM,0}\left(\boldsymbol{\omega}\right)$$

$$+\left(1-\delta\right)\mathbb{E}_t\int\frac{\varphi\left(i\right)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left\{\Delta\hat{R}_{t+1}\frac{b}{R}+\left(1+\frac{\psi}{\sigma}\right)Wn(i)\Delta\hat{\mathcal{Y}}_{t+1}-\sum_k\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\left(\partial_e e_k(i)-\overline{\partial_e e_k}\right)\right)\pi_{k,t+1}\right\}di$$

$$-\left(1-\delta\right)\mathbb{E}_t\int\frac{\varphi\left(i\right)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left\{\left(1-\frac{1}{R}\right)b(i)\pi_{cpi,t+1}\right\}di,$$

$$\tilde{C}_t^{HtM,0}\left(\boldsymbol{\omega}\right)-\frac{1}{R}\mathbb{E}_t\tilde{C}_{t+1}^{HtM,0}\left(\boldsymbol{\omega}\right)=$$

$$-\frac{1}{R}\mathbb{E}_t\int\frac{\varphi\left(i\right)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left\{\Delta\hat{R}_{t+1}\frac{b}{R}+(1+\frac{\psi}{\sigma})Wn(i)\Delta\hat{\mathcal{Y}}_{t+1}-\sum_k\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\left(\partial_e e_k(i)-\overline{\partial_e e_k}\right)\right)\pi_{k,t+1}\right\}di$$

$$-\frac{1}{R}\mathbb{E}_t\int\frac{\varphi\left(i\right)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left\{-\left(1-\frac{1}{R}\right)b(i)\pi_{cpi,t+1}\right\}di$$

$$+\left(1-\frac{1}{R}\right)\mathbb{E}_t\int\frac{\varphi\left(i\right)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left\{\hat{R}_t\frac{b}{R}+(1+\frac{\psi}{\sigma})Wn(i)\hat{\mathcal{Y}}_t-\sum_k\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\left(\partial_e e_k(i)-\overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right\}di$$

$$+\left(1-\frac{1}{R}\right)\mathbb{E}_t\int\frac{\varphi\left(i\right)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left\{-\left(1-\frac{1}{R}\right)b(i)\hat{P}_{cpi,t}\right\}d$$

$$+\mathbb{E}_t\int\varphi\left(i\right)\omega(i)\left(e(i)+\frac{\psi}{\sigma}Wn(i)\right)^{-1}\left(1-\frac{1}{R}\right)b(i)di\left(\hat{P}_{cpi,t}-\frac{1}{R}\hat{P}_{cpi,t+1}\right)$$

Defining $C_t^{HtM,0}\left(\boldsymbol{\omega}\right)\equiv\tilde{C}_t^{HtM,0}\left(\boldsymbol{\omega}\right)-C_t^{HtM}\left(\boldsymbol{\omega}\right)$, we have:

$$\mathbb{E}_tC_{t+1}^{HtM}\left(\boldsymbol{\omega}\right)=C_t^{HtM}\left(\boldsymbol{\omega}\right)+\frac{\delta}{1-\delta}\mathbb{E}_t\tilde{C}_{t+1}^{HtM,0}\left(\boldsymbol{\omega}\right)+\mathbb{E}_t\int\frac{\varphi\left(i\right)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left\{-(1-\frac{1}{R})b(i)\pi_{cpi,t+1}\right\}di$$

$$\mathbb{E}_t\int\frac{\varphi\left(i\right)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left\{\Delta\hat{R}_{t+1}\frac{b}{R}+(1+\frac{\psi}{\sigma})Wn(i)\Delta\hat{\mathcal{Y}}_{t+1}-\sum_k\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\left(\partial_e e_k(i)-\overline{\partial_e e_k}\right)\right)\pi_{k,t+1}\right\}di,$$

$$\tilde{C}_t^{HtM,0}\left(\boldsymbol{\omega}\right)-\frac{1}{R\left(1-\delta\right)}\mathbb{E}_t\tilde{C}_{t+1}^{HtM,0}\left(\boldsymbol{\omega}\right)+\left(1-\frac{1}{R}\right)C_t^{HtM}\left(\boldsymbol{\omega}\right)=$$

$$\left(1-\frac{1}{R}\right)\mathbb{E}_t\int\frac{\varphi\left(i\right)\omega(i)}{e(i)+\frac{\psi}{\sigma}Wn(i)}\left\{\frac{b}{R}\left(\hat{R}_t-\pi_{cpi,t+1}\right)+(1+\frac{\psi}{\sigma})Wn(i)\hat{\mathcal{Y}}_t-\sum_k\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\left(\partial_e e_k(i)-\overline{\partial_e e_k}\right)\right)\hat{P}_{k,t}\right\}di.$$

Putting everything together, we obtain:

$$\mathbb{E}_t C_{t+1}(\boldsymbol{\omega}) = C_t(\boldsymbol{\omega}) + \sigma \mathbb{E}_t \int (1-\varphi(i))\,\omega(i) \left(\hat{R}_t - \sum_k \partial_e e_k(i) \pi_{k,t+1} \right) di + \frac{\delta}{1-\delta} \mathbb{E}_t C^0_{t+1}(\boldsymbol{\omega}) +$$
$$\mathbb{E}_t \int \frac{\varphi(i)\,\omega(i)}{e(i) + \frac{\psi}{\sigma} Wn(i)} \left(\Delta \hat{R}_{t+1} \frac{b}{R} + (1+\frac{\sigma}{\psi}) Wn(i) \Delta \hat{\mathcal{Y}}_{t+1} - \sum_k \left(e(i)\left(s_k(i) - \bar{s}_k\right) + \psi Wn(i) \left(\partial_e e_k(i) - \overline{\partial_e e_k} \right) \right) \pi_{k,t+1} \right) di$$
$$- \mathbb{E}_t \int \frac{\varphi(i)\,\omega(i)}{e(i) + \frac{\psi}{\sigma} Wn(i)} \left((1-\frac{1}{R}) b(i) \pi_{cpi,t+1} \right) di,$$

$$C^0_t(\boldsymbol{\omega}) - \frac{1}{R(1-\delta)} \mathbb{E}_t C^0_{t+1}(\boldsymbol{\omega}) + \left(1 - \frac{1}{R}\right) C_t(\boldsymbol{\omega})$$
$$= \int \frac{\omega(i)(1-\frac{1}{R})}{e(i) + \frac{\psi}{\sigma} Wn(i)} \left(\frac{b(i)}{R} \left(\hat{R}_t - \pi_{cpi,t+1} \right) + \left(1 + \frac{\psi}{\sigma}\right) Wn(i) \hat{\mathcal{Y}}_t - \sum_k \left(e(i)\left(s_k(i) - \bar{s}_k\right) + \psi Wn(i) \left(\partial_e e_k(i) - \overline{\partial_e e_k} \right) \right) \hat{P}_{k,t} \right) di$$

with

$$C^0_0(\boldsymbol{\omega}) = (1-\delta) \int \frac{\omega(i)(1-\frac{1}{R})}{\left(e(i) + \frac{\psi}{\sigma} Wn(i)\right)} b(i) di P_{cpi,0.}$$

Euler Equation for the output gap. We now derive the evolution of the output gap. Recall the definition $\tilde{\mathcal{Y}}_t = \hat{\mathcal{Y}}_t - \hat{\mathcal{Y}}^*{}_t$, with $\hat{\mathcal{Y}}_t = \frac{\sigma}{\sigma+\psi} \left(\psi \hat{W}_t - \psi \sum_k \overline{\partial_e e_k} \hat{P}_{k,t} + \sum_k \bar{s}_k \tilde{A}_{k,t} \right)$. Using the labor market condition, $\hat{\mathcal{Y}}_t$ can be expressed in terms of a demand index $C_t(\boldsymbol{\omega})$, with $\omega(i) = \frac{e(i)}{E} + \frac{\psi}{\sigma} \frac{Wn(i)}{WN}$: $\hat{\mathcal{Y}}_t = \frac{\sigma}{\sigma+\psi} C_t \left(\frac{e}{E} + \frac{\psi}{\sigma} \frac{Wn}{WN} \right) - \frac{\sigma\psi}{\sigma+\psi} \sum_k \int \left(\frac{e(i)}{E} - \frac{Wn(i)}{WN} \right) \partial_e e_k(i) di \hat{P}_{k,t}$. Therefore, applying the formulas derived above, we have:

$$\mathbb{E}_t \hat{\mathcal{Y}}_{t+1} - \hat{\mathcal{Y}}_t = \sigma \int (1-\varphi(i)) \frac{\sigma \frac{e(i)}{E} + \psi \frac{Wn(i)}{WN}}{\sigma + \psi} \left(\hat{R}_t - \sum_k \partial_e e_k(i) \pi_{k,t+1} \right) di$$
$$- \frac{\sigma\psi}{\sigma+\psi} \sum_k \int \left(\frac{e(i)}{E} - \frac{Wn(i)}{WN} \right) \partial_e e_k(i) di \pi_{k,t+1} + \frac{\delta}{1-\delta} \hat{\mathcal{Y}}^0_{t+1} +$$
$$\frac{\sigma}{\sigma+\psi} \mathbb{E}_t \int \varphi(i) \left\{ \Delta \hat{R}_{t+1} \frac{b}{RE} + (1 + \frac{\psi}{\sigma}) \frac{Wn(i)}{WN} \Delta \hat{\mathcal{Y}}_{t+1} \right\} di$$
$$- \frac{\sigma}{\sigma+\psi} \mathbb{E}_t \int \varphi(i) \left\{ \sum_k \left(\frac{e(i)}{E} (s_k(i) - \bar{s}_k) + \psi \frac{Wn(i)}{WN} (\partial_e e_k(i) - \overline{\partial_e e_k}) \right) \pi_{k,t+1} + \left(1 - \frac{1}{R}\right) \frac{b(i)}{E} \pi_{cpi,t+1} \right\} di,$$

$$\hat{\mathcal{Y}}^0_t - \frac{1}{R(1-\delta)} \mathbb{E}_t \hat{\mathcal{Y}}^0_{t+1} + \left(1 - \frac{1}{R}\right) \hat{\mathcal{Y}}_t + \frac{\sigma\psi}{\sigma+\psi} \sum_k \int \left(\frac{e(i)}{E} - \frac{Wn(i)}{WN} \right) \partial_e e_k(i) di \hat{P}_{k,t} =$$
$$\frac{\sigma}{\sigma+\psi} \left(1 - \frac{1}{R}\right) \int \left(\frac{b(i)}{RE} \left(\hat{R}_t - \pi_{cpi,t+1} \right) + \left(1 + \frac{\psi}{\sigma}\right) \frac{Wn(i)}{WN} \hat{\mathcal{Y}}_t - \sum_k \left(\frac{e(i)}{E} \left(s_k(i) - \bar{s}_k\right) + \psi \frac{Wn(i)}{WN} \left(\partial_e e_k(i) - \overline{\partial_e e_k} \right) \right) \hat{P}_{k,t} \right) di.$$

Using

$$\int \frac{b(i)}{RE} di = \int \sum_k \frac{e(i)}{E} \left(s_k(i) - \bar{s}_k\right) di = 0$$

we obtain:

$$\hat{\mathcal{Y}}^0_t - \frac{1}{R(1-\delta)} \mathbb{E}_t \hat{\mathcal{Y}}^0_{t+1} + \left(1 - \frac{1}{R}\right) \hat{\mathcal{Y}}_t + \frac{\sigma\psi}{\sigma+\psi} \sum_k \int \left(\frac{e(i)}{E} - \frac{Wn(i)}{WN} \right) \partial_e e_k(i) di \hat{P}_{k,t} =$$
$$\frac{\sigma}{\sigma+\psi} \left(1 - \frac{1}{R}\right) \int \left(\left(1 + \frac{\psi}{\sigma}\right) \frac{Wn(i)}{WN} \hat{\mathcal{Y}}_t - \sum_k \psi \frac{Wn(i)}{WN} \left(\partial_e e_k(i) - \overline{\partial_e e_k} \right) \hat{P}_{k,t} \right) di,$$
$$\hat{\mathcal{Y}}^0_t - \frac{1}{R(1-\delta)} \mathbb{E}_t \hat{\mathcal{Y}}^0_{t+1} = 0.$$

Using $\hat{\mathcal{Y}}_0^0 = 0$ and $\frac{1}{R(1-\delta)} > 1$, we have $\hat{\mathcal{Y}}_t^0 = 0$ for all t. Defining $\varphi^E \equiv \int \varphi(i)\frac{e(i)}{E}di$, $\varphi^N \equiv \int \varphi(i)\frac{Wn(i)}{WN}di$, we obtain:

$$\begin{aligned}\left(1-\varphi^N\right)\left(\mathbb{E}_t\hat{\mathcal{Y}}_{t+1}-\hat{\mathcal{Y}}_t\right) &= \left(1-\varphi^N\right)\sigma\mathbb{E}_t\left(\hat{R}_t-\sum_k\overline{\partial_e e}_k\pi_{k,t+1}\right)\\ &+\mathbb{E}_t\int\frac{\sigma\varphi\left(i\right)}{\sigma+\psi}\left\{\frac{b(i)}{RE}\left(\Delta\hat{R}_{t+1}-\sigma\left(R-1\right)\hat{R}_t\right)-\frac{e(i)}{E}\sum_k\left(\left(s_k(i)-\bar{s}_k\right)-\sigma(\partial_e e_k(i)-\overline{\partial_e e}_k)\right)\pi_{k,t+1}\right\}di\\ &-\mathbb{E}_t\int\frac{\sigma\varphi\left(i\right)}{\sigma+\psi}\left\{-\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\left(\pi_{cpi,t+1}-\sigma\pi_{mcpi,t+1}\right)\right\}di\end{aligned}$$

By definition, we have $r_t^* = \frac{1}{\sigma}\left(\mathbb{E}_t\hat{\mathcal{Y}}^*{}_{t+1}-\hat{\mathcal{Y}}_t^*\right)$ so $r_t^* \equiv \mathbb{E}_t\frac{1}{\sigma+\psi}\left(\left(\sum_k\psi\overline{\partial_e e}_k+\bar{s}_k\right)\left(\tilde{A}_{k,t+1}-\tilde{A}_{k,t}\right)\right)$. The evolution of the output gap $\tilde{\mathcal{Y}}_t = \hat{\mathcal{Y}}_t-\hat{\mathcal{Y}}_t^*$ is given by:

$$\begin{aligned}\left(1-\varphi^N\right)\left(\mathbb{E}_t\tilde{\mathcal{Y}}_{t+1}-\tilde{\mathcal{Y}}_t\right) &= \left(1-\varphi^N\right)\sigma\mathbb{E}_t\left(\hat{R}_t-\sum_k\overline{\partial_e e}_k\pi_{k,t+1}-r_t^*\right)+\\ &\mathbb{E}_t\int\frac{\sigma\varphi\left(i\right)}{\sigma+\psi}\left\{\frac{b(i)}{RE}\left(\Delta\hat{R}_{t+1}-\sigma\left(R-1\right)\hat{R}_t\right)-\frac{e(i)}{E}\sum_k\left(\left(s_k(i)-\bar{s}_k\right)-\sigma(\partial_e e_k(i)-\overline{\partial_e e}_k)\right)\pi_{k,t+1}\right\}di\\ &-\mathbb{E}_t\int\frac{\sigma\varphi\left(i\right)}{\sigma+\psi}\left\{\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\left(\pi_{cpi,t+1}-\sigma\pi_{mcpi,t+1}\right)\right\}di,\end{aligned}$$

which gives the equation of the main text.

Euler Equation for $\mathcal{M}_{k,t}^D$. Using $\omega(i) = \gamma_{e,k}(i)\partial_e e_k(i)\frac{e(i)}{E_k}$, we obtain:

$$\begin{aligned}\mathbb{E}_t\mathcal{M}_{k,t+1}^E-\mathcal{M}_{k,t}^E &= \sum_l\sigma_{k,l}^{\mathcal{M}^E,u}\left(\hat{R}_t-\pi_{l,t+1}\right)+\frac{\delta}{1-\delta}\mathcal{M}_{k,t+1}^0\\ &\int\left(\varphi(j)\frac{\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)}{\left(1+\frac{Wn(i)\psi}{e(i)\sigma}\right)}\frac{b\left(i\right)}{RE_k}\right)di\Delta\hat{R}_{t+1}+\left(1+\frac{\bar{\psi}}{\bar{\sigma}}\right)\int\left(\varphi\left(i\right)\frac{\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)}{\left(1+\frac{Wn(i)\psi}{e(i)\sigma}\right)}\frac{Wn\left(i\right)}{E_k}\right)di\Delta\hat{\mathcal{Y}}_{t+1}\\ &+\sum_l\int\left(\varphi(i)\frac{\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)}{\left(1+\frac{Wn(i)\psi}{e(i)\sigma}\right)}\left(-\frac{\left(R-1\right)b(i)}{RE_k}\bar{s}_l-\frac{e\left(i\right)}{E_k}\left(s_l(i)-\bar{s}_l\right)+\frac{Wn\left(i\right)}{E_k}\psi\left(\overline{\partial_e e}_l-\partial_e e_l(i)\right)\right)\right)di\pi_{l,t+1}\end{aligned}$$

$$\begin{aligned}\mathcal{M}_{k,t}^0-\frac{1}{\left(1-\delta\right)R}\mathbb{E}_t\mathcal{M}_{k,t+1}^0 &= \int\gamma_{b,k}(i)\frac{b(i)}{RE}di\left(\hat{R}_t-\sum_l\bar{s}_l\pi_{l,t+1}\right)\\ &+\sum_l\int\gamma_{b,k}(i)\left(\frac{e(i)}{E}\left(\bar{s}_l-s_l(i)\right)+\frac{wn(i)}{WL}\left(\bar{\psi}\overline{\partial_e e}_l-\psi(i)\partial_e e_l(i)\right)\right)di\hat{P}_{l,t}\\ &+\left(1+\frac{\bar{\psi}}{\bar{\sigma}}\right)\int\gamma_{b,k}(i)\frac{Wn\left(i\right)}{WN}di\hat{\mathcal{Y}}_t-\frac{R-1}{R}\mathcal{M}_{k,t}^E\end{aligned}$$

with

$$\sigma_{k,l}^{\mathcal{M}^E,u} = \int\gamma_{e,k}(j)\frac{\left(1-\varphi(i)\right)e(i)}{E_k}\partial_e e_k(i)\partial_e e_l(i)di$$

Note that $\mathcal{M}_{k,t}^D = \mathcal{M}_{k,t}^E-\frac{1+\frac{\bar{\psi}}{\bar{\sigma}}}{1-\frac{1}{R}}\int\gamma_{b,k}(i)\frac{Wn(i)}{WN}di\hat{\mathcal{Y}}_t$, so using the equation for the output gap, we have:

$$\begin{aligned}\mathbb{E}_t\mathcal{M}_{k,t+1}^D-\mathcal{M}_{k,t}^D &= \sum_l\sigma_{k,l}^{\mathcal{M},u}\left(\hat{R}_t-\pi_{l,t+1}\right)+\frac{\delta}{1-\delta}\mathbb{E}_t\mathcal{M}_{k,t+1}^0\\ &+\frac{R}{R-1}\int\left(\gamma_{b,k}^u(i)\left(\varphi(i)\frac{b(i)}{RE}-\frac{\left(1-\varphi(i)\right)Wn(i)}{\left(1-\varphi^L\right)WN}\int\left(\varphi(i)\frac{b(i)}{RE}\right)di\right)\right)di\mathbb{E}_t\Delta\hat{R}_{t+1}\\ &-\frac{R}{R-1}\sum_l\int\gamma_{b,k}^u(i)\left\{(1-\frac{1}{R})\left(\varphi(i)\frac{b(i)}{E}-\frac{\left(1-\varphi(i)\right)Wn(i)}{\left(1-\varphi^L\right)WN}\int\left(\varphi(i)\frac{b(i)}{E}\right)di\right)\bar{s}_l\right\}di\mathbb{E}_t\pi_{l,t+1}\\ &-\frac{R}{R-1}\sum_l\int\gamma_{b,k}^u(i)\left\{\left(\varphi(i)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)-\frac{\left(1-\varphi(i)\right)Wn(i)}{\left(1-\varphi^L\right)WN}\int\varphi(i)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)di\right)\right\}di\mathbb{E}_t\pi_{l,t+1}\\ &-\frac{R}{R-1}\sum_l\int\gamma_{b,k}^u(i)\left\{\frac{Wn(i)}{WN}\psi\left(\varphi(i)\left(\partial_e e_l(i)-\overline{\partial_e e}_l\right)-\frac{1-\varphi(i)}{1-\varphi^E}\int\varphi(i)\frac{e(i)}{E}\left(\partial_e e_l(i)-\overline{\partial_e e}_l\right)di\right)\right\}di\mathbb{E}_t\pi_{l,t+1},\end{aligned}$$

$$\mathcal{M}^0_{k,t} - \frac{1}{(1-\delta)\,R}\mathbb{E}_t\mathcal{M}^0_{k,t+1} = \int \gamma^u_{b,k}(i)\frac{b(i)}{RE}di\left(\hat{R}_t - \sum_l \bar{s}_l\mathbb{E}_t\pi_{l,t+1}\right) \\ - \sum_l \int \gamma^u_{b,k}(i)\left(\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) + \psi\frac{Wn(i)}{WN}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right)di\hat{P}_{l,t} - \frac{R-1}{R}\mathcal{M}^D_{k,t}$$

with

$$\sigma^{\mathcal{M},u}_{k,l} = \sigma\int \gamma_{e,k}(i)\frac{(1-\varphi(i))\,e(i)}{E_k}\partial_e e_k(i)\partial_e e_l(i)di - \overline{\partial_e e_l}^u\frac{R}{R-1}\int\frac{(1-\varphi(i))\,Wn}{(1-\varphi^L)\,WN}\gamma^u_{b,k}(i)di\left(\sigma\left(1-\varphi^E\right) + \psi\left(1-\varphi^L\right)\right).$$

B Model equations

Below we present the equations of the full linearized model with an interest rate rule. Derivations are provided in the previous appendix.

Coefficients - households Individual coefficients:

$$
\begin{aligned}
s_k(i) &= \frac{e_k(i)}{e(i)} \\
\partial_e e_k(i) &= \frac{\partial e_k(i)}{\partial e(i)} \overset{(NHCES)}{=} s_k(i)\left(\eta + (1-\eta)\frac{\zeta_k}{\bar{\zeta}(i)}\right) \quad \text{where} \quad \bar{\zeta}(i) = \sum_l s_l(i)\zeta_l \\
\rho_{k,l}(i) &= \partial_{P_l} e_k(i)/P_k + e_l(i)/P_l \partial_e e_k(i)/P_k \overset{(NHCES)}{=} \left(s_l(i) - 1 \cdot \mathbb{I}[k=l]\right)\eta \\
\epsilon_k(i) &= -\frac{\partial e_k(i,j)}{\partial p_k(j)}\frac{p_k(j)}{c_k(i,j)} \overset{(HARA)}{=} a_k + \frac{b_k}{e_k(i)} \\
\epsilon_k^s(i) &= \frac{\partial \epsilon_k(i)}{\partial p_k(j)}\frac{p_k(j)}{\epsilon_k(i)} \overset{(HARA)}{=} \frac{b_k}{c_k(i)} \\
\gamma_{e,k}(i) &= \left(1 - \frac{\epsilon_k(i)}{\bar{\epsilon}_k}\left(1 + \frac{\partial \epsilon_k(i)}{\partial e_k(i)}\frac{e_k(i)}{\epsilon_k(i)}\right)\right) / \left(\bar{\epsilon}_k - 1\right) \overset{(HARA)}{=} \left(1 - \frac{a_k}{\bar{\epsilon}_k}\right)\frac{1}{\bar{\epsilon}_k - 1} \\
MPC(i)^u &= \frac{R-1}{R} / \left(1 + \frac{Wn(i)\psi}{e(i)\sigma}\right) \\
MPC(i)^{HtM} &= 1/\left(1 + \frac{Wn(i)\psi}{e(i)\sigma}\right) \\
MPC(i) &= \varphi(i)MPC(i)^{HtM} + (1-\varphi(i))MPC(i)^u \\
\gamma_{b,k}^u(i) &= MPC(i)^u \gamma_{e,k}(i)\partial_e e_k(i)/\bar{s}_k \\
\gamma_{b,k}^{HtM}(i) &= MPC(i)^{HtM}\gamma_{e,k}(i)\partial_e e_k(i)/\bar{s}_k \\
\gamma_{b,k}(i) &= \varphi(i)\gamma_{b,k}^{htm}(i) + (1-\varphi(i))\gamma_{b,k}^u(i)
\end{aligned}
$$

where the second equality sign imposes the assumed preferences in the calibration.
Aggregate coefficients:

$$
\begin{aligned}
\bar{s}_k &= \frac{E_k}{E} = \frac{\int e_k(i)di}{\int e(i)di} \\
\bar{s}_k^u &= \int \frac{(1-\varphi(i))\, e(i)}{(1-\varphi^E)\, E}\frac{e_l(i)}{e(i)}di \\
\overline{\partial_e e_l} &= \int \frac{e(i)}{E}\partial_e e_k(i)di \\
\overline{\partial_e e_l}^u &= \int \frac{(1-\varphi(i))\, e(i)}{(1-\varphi^E)\, E}\partial_e e_l(i)di \\
\bar{\epsilon}_k &= \int \frac{e_k(i)}{E_k}\epsilon_k(i)di \overset{(HARA)}{=} a_k + \frac{b_k}{E_k} \\
\bar{\epsilon}_k^s &= \left(-\int \left(\epsilon_k(i) - \bar{\epsilon}_k\right)^2 \frac{e_k(i)}{E_k}di + \int \epsilon_k(i)\epsilon_k^s(i)\frac{e_k(i)}{E_k}di\right) / \bar{\epsilon}_k \overset{(HARA)}{=} \frac{b_k}{E_k} \\
s_k^C &= \frac{E_k}{P_k Y_k} \\
\mathcal{S}_{k,l} &= \int \frac{e_k(i)}{E_k}\gamma_{e,k}(i)\rho_{k,l}(i)di
\end{aligned}
$$

$$\begin{aligned}
\varphi^E &= \frac{\int e(i)\varphi(i)di}{E} \\
\varphi^N &= \frac{\int Wn(i)\varphi(i)di}{WN} \\
\sigma^{\mathcal{M},u}_{k,l} &= \sigma \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \overline{\partial_e e_l}^u \frac{R}{R-1} \int \frac{(1-\varphi(i))\, Wn(i)}{(1-\varphi^N)\, WN} \gamma^u_{b,k}(i) di \left(\sigma \left(1-\varphi^E\right) + \psi \left(1-\varphi^L\right)\right) \\
R &= \frac{1}{\beta(1-\delta)}
\end{aligned}$$

Coefficients - firms

$$\begin{aligned}
\Omega_{N,k} &= \frac{WN_k}{P_k Y_k} \\
\Omega_{k,l} &= \frac{P_l \mathcal{Y}_{l,k}}{P_k Y_k} \\
\tilde{\Omega} &= (Id-\Omega)^{-1} \\
\overline{\epsilon}^{\mathcal{I}}_k &= \overline{\epsilon}^s_k
\end{aligned}$$

where Id is the identity matrix.

Coefficients - equations NKPC:

$$\begin{aligned}
\lambda_k &= \frac{(1-\theta_k)(1-\beta\theta_k)}{\theta_k} \frac{\overline{\epsilon}_k - 1}{\overline{\epsilon}_k - 1 + s^C_k \overline{\epsilon}^s_k + \left(1-s^C_k\right) \overline{\epsilon}^{\mathcal{I}}_k} \\
\kappa_k &= \lambda_k \left(\frac{1}{\sigma} + \frac{1}{\psi}\right) \left(\Omega_{N,k} + s^C_k \frac{\sigma\psi}{\sigma+\psi} \Gamma_k\right) \\
\Gamma_k &= \frac{R}{R-1} \frac{\sigma+\psi}{\sigma} \int \gamma^u_{b,k}(i) \frac{Wn(i)}{WN} di
\end{aligned}$$

Other:

$$\begin{aligned}
dhtm\_R_k &= \frac{R}{R-1} \int \left(\gamma^u_{b,k}(i) \left(\varphi(i) \frac{b(i)}{RE} - \frac{(1-\varphi(i))\, Wn(i)}{(1-\varphi^N)\, WN} \int \left(\varphi(i) \frac{b(i)}{RE} \right) di \right) \right) di \\
dhtm\_\pi_{k,l} &= -\frac{R}{R-1} \int \gamma^u_{b,k}(i) \left(1 - \frac{1}{R}\right) \left(\varphi(i) \frac{b(i)}{E} - \frac{(1-\varphi(i))\, Wn(i)}{(1-\varphi^N)\, WN} \int \left(\varphi(i) \frac{b(i)}{E} \right) di \right) \bar{s}_l di \\
&\quad -\frac{R}{R-1} \int \gamma^u_{b,k}(i) \left(\varphi(i) \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l\right) - \frac{(1-\varphi(i))\, Wn(i)}{(1-\varphi^N)\, WN} \int \varphi(i) \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l\right) di \right) di \\
&\quad -\frac{R}{R-1} \int \gamma^u_{b,k}(i) \frac{Wn(i)}{WN} \psi \left(\varphi(i) \left(\partial_e e_l(i) - \overline{\partial_e e_l} \right) - \frac{1-\varphi(i)}{1-\varphi^E} \int \varphi(i) \frac{e(i)}{E} \left(\partial_e e_l(i) - \overline{\partial_e e_l} \right) di \right) di \\
m0\_r_k &= \int \gamma^u_{b,k}(i) \frac{b(i)}{RE} di \\
m0\_P_{k,l} &= -\int \gamma^u_{b,k}(i) \left(\frac{e(i)}{E} \left(s_l(i) - \bar{s}_l\right) + \psi \frac{Wn(i)}{WN} \left(\partial_e e_l(i) - \overline{\partial_e e_l} \right) \right) di \\
ygap\_htm\_\pi_l &= -\left(\sigma \left(\overline{\partial_e e_l}^u - \overline{\partial_e e_l} \right) \left(1-\varphi^E\right) - \left(s^u_l - \bar{s}_l\right) \left(1-\varphi^E\right) - \left(\varphi^E - \varphi^N\right) \left(\sigma \overline{\partial_e e_l} - \bar{s}_l \right) \right)
\end{aligned}$$

Sectoral equations. For every sector $k = 1, ..., K$ we have:

$$\begin{aligned}
\pi_{k,t} &= \hat{P}_{k,t} - \hat{P}_{k,t-1} \\
\pi_{k,t} &= \kappa_k \tilde{\mathcal{Y}}_t + \lambda_k \left(\Omega_{N,k} \mathcal{N}\mathcal{H}_t + s^C_k \mathcal{M}_{k,t} - \Omega_{N,k} \mathcal{P}_{k,t} + \mathcal{I}_{k,t} \right) + \beta(1-\delta)\mathbb{E}_t \pi_{k,t+1} \\
\mathcal{M}_{k,t} &= \Gamma_k \hat{\mathcal{Y}}^*_t + \mathcal{M}^P_{k,t} + \mathcal{M}^D_{k,t} \\
\mathcal{M}^P_{k,t} &= \sum_l \mathcal{S}_{k,l} \left(\hat{P}_{l,t} - \hat{P}_{k,t} \right) \\
\mathbb{E}_t \mathcal{M}^D_{k,t+1} - \mathcal{M}^D_{k,t} &= \sigma^{\mathcal{M},u}_k \hat{R}_t - \sum_l \sigma^{\mathcal{M},u}_{k,l} \pi_{l,t+1} + \frac{\delta}{1-\delta} \mathbb{E}_t \mathcal{M}^0_{k,t+1} + dhtm\_R_k \left(\mathbb{E}_t \hat{R}_{t+1} - \hat{R}_t \right) \\
&\quad + \sum_l dhtm\_\pi_{k,l} \mathbb{E}_t \pi_{l,t+1} \\
\mathcal{M}^0_{k,t-1} - \frac{1}{(1-\delta) R} \mathbb{E}_t \mathcal{M}^0_{k,t} &= m0\_r_k \left(\hat{R}_{t-1} - \mathbb{E}_t \pi_{cpi,t} \right) + \sum_l m0\_P_{k,l} \hat{P}_{l,t-1} - \frac{R-1}{R} \mathcal{M}^D_{k,t-1} \\
\mathcal{P}_{k,t} &= \left(\hat{P}_{k,t} - \hat{P}_{cpi,t} \right) - \left(\hat{P}^*_{k,t} - \hat{P}^*_{cpi,t} \right) \\
\mathcal{I}_{k,t} &= \sum_l \Omega_{k,l} \left(\mathcal{P}_{l,t} - \mathcal{P}_{k,t} \right) \\
\hat{P}^*_{k,t} &= -\tilde{A}_{k,t} \\
\hat{A}_{k,t} &= \rho \hat{A}_{k,t-1} + \varepsilon_{k,t} \\
\tilde{A}_{k,t} &= \sum_l \tilde{\Omega}_{k,l} \hat{A}_{l,t}
\end{aligned}$$

Aggregate equations

$$\begin{aligned}
\tilde{\mathcal{Y}}_t &= \mathbb{E}_t \tilde{\mathcal{Y}}_{t+1} - \sigma \mathbb{E}_t \left(\hat{R}_t - \pi_{mcpi,t+1} - \hat{r}^*_t \right) \\
&\quad - \frac{1}{1-\varphi^N} \frac{\sigma}{\sigma+\psi} \left(\frac{\varphi^E - \varphi^N}{R-1} \left(\mathbb{E}_t \hat{R}_{t+1} - (1 + \sigma(R-1)) \hat{R}_t \right) + \sum_l ygap\_htm\_\pi_l \cdot \pi_{l,t+1} \right) \\
\hat{r}^*_t &= \frac{1}{\sigma+\psi} \sum_l \left(\psi \overline{\partial_e e}_l + \bar{s}_l \right) \left(\mathbb{E}_t \tilde{A}_{l,t+1} - \tilde{A}_{l,t} \right) \\
\hat{\mathcal{Y}}^*_t &= \frac{1}{1+\frac{\psi}{\sigma}} \sum_l \left(\psi \overline{\partial_e e}_l + \bar{s}_l \right) \tilde{A}_{l,t} \\
\mathcal{N}\mathcal{H}_t &= \sum_l \left(\overline{\partial_e e}_l - \bar{s}_l \right) \left(\hat{P}_{l,t} - \hat{P}^*_{l,t} \right) \\
P_{cpi,t} &= \sum_l \bar{s}_l \hat{P}_{l,t} \\
P^*_{cpi,t} &= \sum_l \bar{s}_l \hat{P}^*_{l,t} \\
\pi_{cpi,t} &= \sum_l \bar{s}_l \pi_{l,t} \\
\pi_{mcpi,t} &= \sum_l \overline{\partial_e e}_l \pi_{l,t} \\
\hat{R}_t &= \phi \pi_{cpi,t} + u^R_t \\
u^R_t &= \rho^R u^R_{t-1} + \varepsilon^R_t
\end{aligned}$$

Equations for demand indices. Coefficients:

$$
\begin{aligned}
fracu_{\omega} &= \int \left(1-\varphi(i)\right)\omega(i)di \\
msu_{\omega,l} &= \int \left(1-\varphi(i)\right)\omega(i)\partial_e e_l(i)di \\
chtm\_R_{\omega} &= \int \left(\varphi(i)\frac{\omega(i)}{e(i)+Wn(i)\frac{\psi}{\sigma}}\frac{b(i)}{R}\right)di \\
chtm\_Y_{\omega} &= \left(1+\frac{\psi}{\sigma}\right)\int \left(\varphi(i)\frac{\omega(i)}{e(i)+Wn(i)\frac{\psi}{\sigma}}Wn(i)\right)di \\
chtm\_\pi_{\omega,l} &= \int \left(\varphi(i)\frac{\omega(i)}{e(i)+Wn(i)\frac{\psi}{\sigma}}\left(\frac{(R-1)\,b(i)}{R}\bar{s}_l + e(i)\left(s_l(i)-\bar{s}_l\right) + Wn(i)\psi\left(\partial_e e_l(i)-\overline{\partial_e e_l}\right)\right)\right)di \\
c0\_r_{\omega} &= \frac{R-1}{R}\int \frac{\omega(i)}{e(i)+Wn(i)\frac{\psi}{\sigma}}\frac{b(i)}{R}di \\
c0\_P_{\omega,l} &= -\frac{R-1}{R}\int \frac{\omega(i)}{e(i)+Wn(i)\frac{\psi}{\sigma}}\left(e(i)\left(s_l(i)-\bar{s}_l\right)+\psi Wn(i)\left(\partial_e e_l(i)-\overline{\partial_e e_l}\right)\right)di \\
c0\_Y_{\omega} &= \frac{R-1}{R}\left(1+\frac{\psi}{\sigma}\right)\int \frac{\omega(i)}{e(i)+Wn(i)\frac{\psi}{\sigma}}Wn(i)di
\end{aligned}
$$

Equations:

$$
\begin{aligned}
\mathbb{E}_t\hat{C}_{t+1}\left(\boldsymbol{\omega}\right)-\hat{C}_t\left(\boldsymbol{\omega}\right) = \sigma\left(fracu_{\omega}\hat{R}_t-\sum_l msu_{\omega,l}\pi_{l,t+1}\right)+\frac{\delta}{1-\delta}\mathbb{E}_t\hat{C}^0_{t+1}\left(\boldsymbol{\omega}\right) \\
+chtm\_R_{\omega}\left(\mathbb{E}_t\hat{R}_{t+1}-\hat{R}_t\right)+chtm\_Y_{\omega}\left(\mathbb{E}_t\hat{\mathcal{Y}}_{t+1}-\hat{\mathcal{Y}}_t\right)-\sum_l chtm\_\pi_{\omega,l}\mathbb{E}_t\pi_{l,t+1}
\end{aligned}
$$

$$
\hat{C}^0_{t-1}\left(\boldsymbol{\omega}\right)-\frac{1}{\left(1-\delta\right)R}\mathbb{E}_t\hat{C}^0_t\left(\boldsymbol{\omega}\right) = c0\_r_{\omega}\left(\hat{R}_{t-1}-\mathbb{E}_t\pi_{cpi,t}\right)+\sum_l c0\_P_{\omega,l}\hat{P}_{l,t-1}+c0\_Y_{\omega}\hat{\mathcal{Y}}_{t-1}-\frac{R-1}{R}\hat{C}_{t-1}\left(\boldsymbol{\omega}\right)
$$

C Proofs Analytical results Section 3

Result 1

Denote $\tilde{P}_{k,t} = \hat{P}_{k,t} - \sum_l \frac{\lambda}{\lambda_l}\overline{\partial_e e_l}\hat{P}_{l,t}$ and $\tilde{\pi}_{k,t} = \pi_{k,t} - \sum_l \frac{\lambda}{\lambda_l}\overline{\partial_e e_l}\pi_{l,t}$ the sector price and inflation relative to the 'Divine Coincidence index' $\hat{P}_{d,t} = \sum_l \frac{\lambda}{\lambda_l}\overline{\partial_e e_l}\hat{P}_{l,t}$ with $\frac{1}{\lambda} = \sum_l \frac{\overline{\partial_e e_l}}{\lambda_l}$, define similarly $\tilde{P}^*_{k,t}$. Under (**A.1**), we can aggregate the sectoral NKPCs with the divine coincidence weights to obtain:

$$\pi_{d,t} = \kappa\tilde{\mathcal{Y}}_t + \lambda\sum_k \overline{\partial_e e_k}\mathcal{M}_{k,t} + \beta\left(1-\delta\right)\mathbb{E}_t\pi_{d,t+1},$$

$$\begin{aligned}\tilde{\pi}_{k,t} &= \left(\lambda_k\left(\tilde{P}^*_{k,t} - \tilde{P}_{k,t}\right) - \lambda_k\sum_l \overline{\partial_e e_l}\left(\tilde{P}^*_{l,t} - \tilde{P}_{l,t}\right) + \lambda_k\mathcal{M}_{k,t} - \lambda\sum_l \overline{\partial_e e_l}\mathcal{M}_{l,t}\right) + \beta\left(1-\delta\right)\mathbb{E}_t\tilde{\pi}_{k,t+1}.\\ \tilde{P}_{k,t} &= \tilde{\pi}_{k,t} + \tilde{P}_{k,t-1}.\end{aligned}$$

Next, assume $\int \gamma_{b,k}(i)b(i)di = 0$ for all k, which is a weaker version of assumption (**A.**2). Recall that:

$$\begin{aligned}\mathcal{M}^D_{k,t} &= \mathbb{E}_t\mathcal{M}^D_{k,t+1} - \sum_l \sigma^{\mathcal{M}}_{k,l}(\hat{R}_t - \mathbb{E}_t\pi_{l,t+1}) - \frac{\delta}{1-\delta}\mathbb{E}_t\mathcal{M}^0_{k,t+1},\\ \mathcal{M}^0_{k,t} &= \frac{1}{\left(1-\delta\right)R}\mathbb{E}_t\mathcal{M}^0_{k,t+1} + \int \gamma_{b,k}(i)\frac{b(i)}{RE}di\left(\hat{R}_t - \pi_{cpi,t+1}\right)\\ &\quad - \sum_l \int \gamma_{b,k}(i)\left(\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) + \frac{\psi Wn(i)}{WN}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right)di\hat{P}_{l,t} - \frac{R-1}{R}\mathcal{M}^D_{k,t},\\ \sigma^{\mathcal{M}}_{k,l} &= \sigma\int \gamma_{e,k}(i)\frac{e(i)}{E_k}\partial_e e_k(i)\partial_e e_l(i)di - \sigma\overline{\partial_e e_l}\frac{R}{R-1}\frac{\sigma+\psi}{\sigma}\int \gamma_{b,k}(i)\frac{Wn(i)}{WN}di.\end{aligned}$$

Given $\int \gamma_{b,k}(i)b(i)di = 0$, we can write:

$$\begin{aligned}(\sigma+\psi)\int \gamma_{b,k}(i)\frac{Wn(i)}{WN}di &= \int \gamma_{b,k}(i)\left(\psi\frac{Wn(i)}{WN} + \sigma\left(\frac{Wn(i)}{WN} + \left(1-\frac{1}{R}\right)\frac{b(i)}{E}\right)\right)di,\\ &= \int \gamma_{b,k}(i)\left(\psi\frac{Wn(i)}{WN} + \sigma\frac{e(i)}{E}\right)di,\\ &= \int \left(1-\frac{1}{R}\right)\frac{\gamma_{e,k}(i)\partial_e e_k(i)}{1+\frac{Wn(i)\psi}{e(i)\sigma}}\frac{E}{E_k}\left(\psi\frac{Wn(i)}{WN} + \sigma\frac{e(i)}{E}\right)di,\\ &= \sigma\left(1-\frac{1}{R}\right)\int \gamma_{e,k}(i)\partial_e e_k(i)\frac{e(i)}{E_k}di,\end{aligned}$$

and therefore:

$$\begin{aligned}\sigma^{\mathcal{M}}_{k,l} &= \sigma\int \gamma_{e,k}(i)\frac{e(i)}{E_k}\partial_e e_k(i)\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)di,\\ \sum_l \sigma^{\mathcal{M}}_{k,l} &= 0,\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}^D_{k,t} &= \mathbb{E}_t\mathcal{M}^D_{k,t+1} + \sum_l \sigma^{\mathcal{M}}_{k,l}\mathbb{E}_t\tilde{\pi}_{l,t+1} - \frac{\delta}{1-\delta}\mathbb{E}_t\mathcal{M}^0_{k,t+1},\\ \mathcal{M}^0_{k,t} =& \frac{1}{\left(1-\delta\right)R}\mathbb{E}_t\mathcal{M}^0_{k,t+1} - \sum_l \int \gamma_{b,k}(i)\left(\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) + \frac{\psi Wn(i)}{WN}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right)di\tilde{P}_{l,t} - \frac{R-1}{R}\mathcal{M}^D_{k,t}.\end{aligned}$$

Recall that we can decompose the endogenous markup wedge $\mathcal{M}_{k,t} = \Gamma_k\mathcal{Y}^*_t + \mathcal{M}^P_{k,t} + \mathcal{M}^D_{k,t}$, and note that the first component, $\Gamma_k\mathcal{Y}^*_t$, is exogenous and hence independent of monetary policy. To show that the other components are independent of monetary policy too, we proceed as follows. Since $\sum_l \rho_{k,l}(i) = 0$, we can write the sectoral substitution component of the endogenous markup wedge as:

$$\mathcal{M}^P_{k,t} = \sum_{l=1}^K \int \gamma_{e,k}(i)\frac{e_k}{E_k}\rho_{k,l}(i)di\tilde{P}_{l,t}.$$

Therefore, the relative price equations can be rewritten as:

$$\tilde{\pi}_{k,t} - \beta\left(1-\delta\right)\mathbb{E}_t\tilde{\pi}_{k,t+1} = -\left(\lambda_k - \lambda\right)\left(\frac{1}{\psi}+\frac{1}{\sigma}\right)\hat{\mathcal{Y}}^*_t + \lambda_k\left(\tilde{P}^*_{k,t} - \sum_l \overline{\partial_e e_l}\tilde{P}^*_{l,t}\right) + \sum \alpha_{k,l}\tilde{P}_{l,t} + \sum\left(\lambda_k\tilde{\mathcal{M}}_{k,t} - \lambda\sum_l \overline{\partial_e e_l}\tilde{\mathcal{M}}_{l,t}\right),$$

$$\begin{aligned}
\tilde{P}_{k,t} &= \tilde{\pi}_{k,t} + \tilde{P}_{k,t-1},\\
\tilde{\mathcal{M}}_{k,t} &= \mathbb{E}_t\tilde{\mathcal{M}}_{k,t+1} - \frac{\delta}{1-\delta}\mathbb{E}_t\tilde{\mathcal{M}}^0_{k,t+1},\\
\tilde{\mathcal{M}}^0_{k,t} &= \frac{1}{\left(1-\delta\right)R}\mathbb{E}_t\tilde{\mathcal{M}}^0_{k,t+1} - \sum\beta_{k,l}\tilde{P}_{l,t} - \left(1-\frac{1}{R}\right)\tilde{\mathcal{M}}^D_{k,t},
\end{aligned}$$

with

$$\begin{aligned}
\alpha_{k,l} &= -\lambda_k\mathbb{1}_{k=l} + \overline{\partial_e e_l}\lambda_l + \lambda_k\int\gamma_{e,k}(i)\frac{e}{E_k}\rho_{k,l}(i) - \lambda\sum_n\overline{\partial_e e_n}\int\gamma_{e,n}(i)\frac{e}{E_n}\rho_{n,l}(i)di - \lambda_k\sigma^{\mathcal{M}}_{k,l} + \lambda\sum_n\overline{\partial_e e_n}\sigma^{\mathcal{M}}_{n,l},\\
\beta_{k,l} &= \int\gamma_{b,k}(i)\frac{e(i)}{E}\left(\left(s_l(i)-\bar{s}_l\right) + \sigma\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right)di + \left(1-\frac{1}{R}\right)\sigma^{\mathcal{M}}_{k,l}.
\end{aligned}$$

Since $\hat{\mathcal{Y}}^*_t$ and $\tilde{P}^*_{l,t}$ are exogenous, $\tilde{P}_{k,t}$, $\tilde{\pi}_{k,t}$, $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}^0_{k,t}$ are pinned down by a system of $4(K-1)$ equations which does not involve $\hat{R}_t$. These variables are therefore independent of monetary policy. From the above equations we observe that $\mathcal{M}^D_{k,t}$ and $\mathcal{M}^P_{k,t}$ depend only on $\tilde{\pi}_{k,t}$ and $\tilde{P}_{k,t}$. Therefore, these wedges are independent of monetary policy as well. Finally, the non-homotheticity and relative price wedge can be written as:

$$\begin{aligned}
\mathcal{NH}_t &= \sum_{l=1}^{K}(\overline{\partial_e e_l} - \bar{s}_l)(\tilde{P}_{l,t} - \tilde{P}^*_{l,t}),\\
\mathcal{P}_{k,t} &= (\tilde{P}^*_{k,t} - \sum_l \bar{s}_l\tilde{P}^*_{l,t}) - (\hat{P}^*_{k,t} - \hat{P}^*_{cpi,t}).
\end{aligned}$$

It now follows that all the wedges are independent of monetary policy.

Additions to Result 1. In Appendix F we present a number of additions to Result 1. Specifically, we derive an inflation index implementing the Divine Coinvidence. We also extend Result 1 to the case with HtM households and Input-Output linkages.

Result 2

Note that if $\mathcal{M}_t = 0$, then $\kappa_k = \lambda_k\left(\frac{1}{\sigma}+\frac{1}{\psi}\right)$, so (**A.1**) becomes $\lambda_k = \lambda$ for all k. We can now write the NKPC for the MCPI as :

$$\pi_{mcpi,t} = \kappa\tilde{\mathcal{Y}}_t + \beta\left(1-\delta\right)\mathbb{E}_t\pi_{mcpi,t+1}.$$

And the Euler equations remains:

$$\tilde{\mathcal{Y}}_t = \mathbb{E}_t\tilde{\mathcal{Y}}_{t+1} - \sigma\mathbb{E}_t\left(\hat{R}_t - \pi_{mcpi,t+1} - \hat{r}^*_t\right).$$

As in the standard model, implementing

$$\hat{R}_t = \hat{r}^*_t + \phi\pi_{mcpi,t}$$

therefore stabilizes jointly the output gap and MCPI inflation (when $\phi > 1$). Indeed we obtain:

$$\mathbb{E}_t\pi_{mcpi,t+2} - \left(1+R+R\kappa\sigma\right)\mathbb{E}_t\pi_{mcpi,t+1} + \left(R+R\kappa\sigma\phi\right)\pi_{mcpi,t} = 0.$$

For $\phi > 1$, the roots of the polynomial are strictly larger than 1, so the only non explosive solution is $\pi_{mcpi,t} = 0$ which implies $\tilde{\mathcal{Y}}_t = 0$, see e.g. Woodford (2003).

Result 3

Denote the gap between MCPI and CPI inflation by $\pi_{\Delta,t} = \sum\left(\overline{\partial_e e_l} - \bar{s}_l\right)\pi_{l,t}$, and analogously define $\hat{P}_{\Delta,t}$ and $\hat{A}_{\Delta,t}$. Recall that if if $\mathcal{M}_t = 0$ then (**A.1**) becomes $\lambda_k = \lambda$ for all k. We can write the NKPC for $\pi_{\Delta,t}$ as:

$$R\pi_{\Delta,t} = -\lambda R\left(\hat{P}_{\Delta,t} + \hat{A}_{\Delta,t}\right) + \pi_{\Delta,t+1}$$

$$\Leftrightarrow$$

$$\hat{P}_{\Delta,t+1} - \left(1+R+R\lambda\right)\hat{P}_{\Delta,t} + R\hat{P}_{\Delta,t-1} = \lambda R\hat{A}_{\Delta,t}$$

The eigenvalues of the system are:

$$\mu_{\pm} = \frac{R + R\lambda + 1 \pm \sqrt{(R + R\lambda - 1)^2 + 4R\lambda}}{2}$$

With $\mu_+ > R + R\lambda$, $\mu_- < 1$. We obtain:

$$\hat{P}_{\Delta,t} = -\lambda \sum_0^t \mu_-^{t-s+1} \sum \frac{1}{\mu_+^u} \hat{A}_{\Delta,u+s}.$$

Therefore, we have:

$$\mathcal{NH}_t = -\lambda \sum_0^t \mu_-^{t-s+1} \sum \frac{1}{\mu_+^u} \hat{A}_{\Delta,u+s} + \hat{A}_{\Delta,t}.$$

Now suppose that we have a negative shock in a necessity (luxury) sector, in that case $\hat{A}_{\Delta,t} \geq 0$ ($\hat{A}_{\Delta,t} \leq 0$). Assume in addition that $\left|\hat{A}_{\Delta,t}\right| \leq \left|\hat{A}_{\Delta,0}\right|$ (the shock is larger on impact), then we have for a shock in a necessity sector

$$\begin{aligned}
\mathcal{NH}_0 &\geq \left(1 - \lambda\mu_- \sum_{u\geq 0} \frac{1}{\mu_+^u}\right) \hat{A}_{\Delta,0}, \\
&\geq \left(1 - \frac{\lambda\mu_-\mu_+}{\mu_+ - 1}\right) \hat{A}_{\Delta,0}, \\
&\geq \left(1 - \frac{\lambda R}{R + R\lambda - 1}\right) \hat{A}_{\Delta,0} \geq 0.
\end{aligned}$$

Similarly for a shock in a luxury sector, we have:

$$\mathcal{NH}_0 \leq \left(1 - \frac{\lambda R}{R + R\lambda - 1}\right) \hat{A}_{\Delta,0} \leq 0.$$

Result 3A.0 Analytical formulas for AR(1) shocks

In this section, we assume that shocks vanish at a constant rate ρ_a and derive analytical formulas for $\pi_{cpi,t}$, $\pi_{mcpi,t}$ and $\tilde{\mathcal{Y}}_t$. We show the following:

i. There exists a time $t_{\mathcal{NH}}$ ($t_{\mathcal{NH}} = 0$ if $\rho_a = 0$, $t_{\mathcal{NH}} = \infty$if $\rho_a = 1$) such that for a negative shock in a necessity (luxury) sector and $t \leq t_{\mathcal{NH}}$ then $\mathcal{NH}_t \geq 0$ ($\mathcal{NH}_t \leq 0$) and for $t > t_{\mathcal{NH}}$ $\mathcal{NH}_t \leq 0$ ($\mathcal{NH}_t \geq 0$)

ii. The gap $\pi_{cpi,t} - \pi_{mcpi,t}$ evolves independently of the policy rule. There exists t^* ($t^* = 0$ if $\rho_a = 0$, $t^* = \infty$if $\rho_a = 1$) such that for a negative shock in a necessity (luxury) sector and $t \leq t^*$ then $\pi_{cpi,t} \geq \pi_{mcpi,t}$ ($\pi_{cpi,t} \leq \pi_{mcpi,t}$) and for $t > t^*$ $\pi_{cpi,t} \leq \pi_{mcpi,t}$ ($\pi_{cpi,t} \geq \pi_{mcpi,t}$)

iii. Under the MCPI rule $\hat{R}_t = \phi\pi_{mcpi,t} + \hat{r}_t^*$ (with $\phi > 1$), we have $\pi_{mcpi,t} = \tilde{\mathcal{Y}}_t = 0$ so for a negative shock in a necessity (luxury) sector and $t \leq t^*$ then $\pi_{cpi,t} \geq 0$ ($\pi_{cpi,t} \leq 0$) and for $t > t^*$ $\pi_{cpi,t} \leq 0$ ($\pi_{cpi,t} \geq 0$)

iv. Under the CPI rule $\hat{R}_t = \phi\pi_{cpi,t} + \hat{r}_t^*$, There exists a time $t_{\mathcal{Y}}$ ($t_{\mathcal{Y}} = 0$ if $\rho_a = 0$, $t_{\mathcal{Y}} = \infty$ if $\rho_a = 1$) such that for a negative shock in a necessity (luxury) sector and $t \leq t_{\mathcal{Y}}$ then $\tilde{\mathcal{Y}}_t \leq 0$ ($\tilde{\mathcal{Y}}_t \geq 0$) and for $t > t_{\mathcal{Y}}.\tilde{\mathcal{Y}}_t \geq 0$ ($\tilde{\mathcal{Y}}_t \leq 0$).

v. Under the CPI rule $\hat{R}_t = \phi\pi_{cpi,t} + \hat{r}_t^*$, there exists a level of persistence ρ^*such that for $\rho_a \leq \rho^*$, for negative shocks in a necessity (luxury) sector $\pi_{cpi,t} \geq 0$ ($\pi_{cpi,t} \leq 0$) for all t. For $\rho_a > \rho^*$, There exists t_{CPI} ($t_{CPI} = \infty$if $\rho_a = 1$) such that for a negative shock in a necessity (luxury) sector and $t \leq t_{CPI}$ then $\pi_{cpi,t} \leq 0$ ($\pi_{cpi,t} \geq 0$) and for $t > t_{CPI}$ $\pi_{cpi,t} \geq 0$ ($\pi_{cpi,t} \leq 0$).

vi. Under the alternative rule $\hat{R}_t = \phi\pi_{mcpi,t}$ or $\hat{R}_t = \phi\pi_{cpi,t}$, the response of the output gap and both inflation indices at t are simply shifted up proportionally to $\rho_a^t \hat{r}_0^*$. Normalizing shocks such that $\hat{r}_0^* = -1$ (equal impact of sectoral shocks on efficient output), we have that for $t \leq t_{\mathcal{Y}}$ ($t > t_{\mathcal{Y}}$) and CPI targeting the output gap will be higher (lower) following a shock in a luxury sector rather than in a necessity sector. In addition, for high enough persistence the output gap will be negative under CPI targeting following a shock in a necessity sector.

Dynamics of the $\mathcal{NH}$ wedge. Rewriting $\mathcal{NH}_t = -\lambda \sum_0^t \mu_-^{t-s+1} \sum \frac{1}{\mu_+^u} \hat{A}_{\Delta,u+s} + \hat{A}_{\Delta,t}$, with $\hat{A}_{\Delta,t} = \rho_a^t \hat{A}_{\Delta,0}$ we have:

$$\mathcal{NH}_t = \frac{1}{(\mu_+ - \rho_a)(\mu_- - \rho_a)} \left((R - \rho_a)(1 - \rho_a)\rho_a^t - (R - \mu_-)(1 - \mu_-)\mu_-^t\right) \hat{A}_{\Delta,0}$$

Define $t^* = ln\left(\frac{(R-\mu_-)(1-\mu_-)}{(R-\rho_a)(1-\rho_a)}\right)/ln\left(\frac{\rho_a}{\mu_-}\right)$, for $t \leq t^*$, $\mathcal{NH}_t$same sign as $A_{\Delta,0}$ and for $t > t^*$, $\mathcal{NH}_t$same sign as $-A_{\Delta,0}$. For transitory shock $t^* = 0$, for a permanent shock $t^* = \infty$.
We now derive the evolution of inflation (CPI and MCPI) and the output gap under some particular interest rules.

Case $\hat{R}_t = \phi\pi_{mcpi,t} + \hat{r}^*_t$. The system of equations becomes

$$R\pi_{mcpi,t} = R\kappa\tilde{\mathcal{Y}}_t + \mathbb{E}_t\pi_{mcpi,t+1}$$
$$\tilde{\mathcal{Y}}_{t+1} - \tilde{\mathcal{Y}}_t = \sigma\left(\phi\pi_{mcpi,t} - \pi_{mcpi,t+1}\right)$$

The eigenvalues of the system are

$$\lambda_\pm = \frac{R + R\kappa\sigma + 1 \pm \sqrt{\left(R + R\kappa\sigma - 1\right)^2 - 4R\kappa\sigma\left(\phi - 1\right)}}{2}$$

For $\phi > 1$, the eigenvalues are larger than 1in modulus, we therefore have $\pi_{mcpi,t} = \tilde{\mathcal{Y}}_t = 0$ for all t. The evolution of CPI is then

$$R\pi_{cpi,t} = R\lambda\mathcal{NH}_t + \mathbb{E}_t\pi_{cpi,t+1}$$

$$\pi_{cpi,t} = \frac{R\lambda}{\left(\mu_+ - \rho_a\right)\left(\mu_- - \rho_a\right)}\left(\left(1-\rho_a\right)\rho_a^t - \left(1-\mu_-\right)\mu_-^t\right)\hat{A}_{\Delta,0}$$

We have that $\pi_{cpi,0}$ has the same sign as $A_{\Delta,0}$ (positive for a shock in a necessity sector, negative for a shock in a luxury sector). In addition, define $t^* = ln\left(\frac{(1-\mu_-)}{(1-\rho_a)}\right)/ln\left(\frac{\rho_a}{\mu_-}\right)$, for $t \leq t^*$, $\pi_{cpi,t}$ has same sign as $\hat{A}_{\Delta,0}$ and for $t > t^*$, $\pi_{cpi,t}$ has the same sign as $-\hat{A}_{\Delta,0}$. For transitory shock $t^* = 0$, for a permanent shock $t^* = \infty$.

Case $\hat{R}_t = \phi\pi_{cpi,t} + \hat{r}^*_t$.

$$R\pi_{cpi,t} = R\kappa\tilde{\mathcal{Y}}_t + R\lambda\mathcal{NH}_t + \mathbb{E}_t\pi_{cpi,t+1}$$
$$\tilde{\mathcal{Y}}_{t+1} - \tilde{\mathcal{Y}}_t = \sigma\left(\phi\pi_{cpi,t} - \pi_{mcpi,t+1}\right)$$

In that case, we have

$$\pi_{cpi,t} = \frac{R\lambda}{\left(\mu_+ - \rho_a\right)\left(\mu_- - \rho_a\right)}\left\{\left(1 - \frac{R\kappa\sigma\phi}{\left(\lambda_+ - \rho_a\right)\left(\lambda_- - \rho_a\right)}\right)\left(1-\rho_a\right)\rho_a^t - \left(1 - \frac{R\kappa\sigma\phi}{\left(\lambda_+ - \mu_-\right)\left(\lambda_- - \mu_-\right)}\right)\left(1-\mu_-\right)\mu_-^t\right\}\hat{A}_{\Delta,0}$$

$$\tilde{\mathcal{Y}}_t = -\frac{R\lambda\sigma\phi}{\left(\mu_+ - \rho_a\right)\left(\mu_- - \rho_a\right)}\left\{\frac{\left(1-\rho_a\right)\left(R-\rho_a\right)}{\left(\lambda_+ - \rho_a\right)\left(\lambda_- - \rho_a\right)}\rho_a^t - \frac{\left(1-\mu_-\right)\left(R-\mu_-\right)}{\left(\lambda_+ - \mu_-\right)\left(\lambda_- - \mu_-\right)}\mu_-^t\right\}\hat{A}_{\Delta,0}$$

$$\pi_{cpi,t} - \pi_{mcpi,t} = \frac{R\lambda}{\left(\mu_+ - \rho_a\right)\left(\mu_- - \rho_a\right)}\left(\left(1-\rho_a\right)\rho_a^t - \left(1-\mu_-\right)\mu_-^t\right)\hat{A}_{\Delta,0}$$

Note that the fraction $\frac{(1-x)(R-x)}{(\mu_+-x)(\mu_--x)}$ is decreasing in x. From this we deduce that $\tilde{\mathcal{Y}}_t$ initially has the same sign as $-\hat{A}_{\Delta,0}$ (for $t \leq t^* = t^* = ln\left(\frac{(R-\mu_-)(1-\mu_-)}{(R-\rho_a)(1-\rho_a)}\frac{(\lambda_+-\rho_a)(\lambda_--\rho_a)}{(\lambda_+-\mu_-)(\lambda_--\mu_-)}\right)/ln\left(\frac{\rho_a}{\mu_-}\right)$) then for $t > t^*$ has the same sign as $A_{\Delta,0}$ (for a transitory shock$\tilde{\mathcal{Y}}_t$ has the same sign as $\hat{A}_{\Delta,0}$ for $t > 0$, for a permanent shock, $\tilde{\mathcal{Y}}_t$ has the same sign of $-A_{\Delta,0}$ for all t). This implies that the output gap is always negative on impact in response to a negative shock in the necessity sector, positive for a shock in a luxury sector.

The response of CPI is more ambiguous and depends on the persistence of the shock. There exist a persistence $0 < \nu = \frac{R+R\kappa\sigma+1-\sqrt{(R+R\kappa\sigma-1)^2+4R\kappa\sigma}}{2} < \rho^* < \frac{R+R\lambda+1-\sqrt{(R+R\lambda-1)^2+4R\lambda}}{2} = \mu_-$ such that for $\rho_a \leq \rho^*$, $\pi_{cpi,t}$ always has the same sign as $\hat{A}_{\Delta,0}$. In that case, $\pi_{cpi,t}$ and the output gap initially move in opposite direction. If $\rho_a > \rho^*$, initially cpi inflation has the same sign as $-A_{\Delta,0}$ and then switches sign (keeping the sign of $-\hat{A}_{\Delta,0}$ if $\rho_a = 1$). In that case, $\pi_{cpi,t}$ and the output gap initially co-move. To see this consider the polynomial $P(x) = \left(\left(1-x\right)\left(R-x\right) - R\kappa\sigma x\right)\left(1-x\right) - \left(\lambda_+ - x\right)\left(\lambda_- - x\right)\left(1 - \frac{R\kappa\sigma\phi}{(\lambda_+-\mu_-)(\lambda_--\mu_-)}\right)\left(1-\mu_-\right)$. It is a third order polynomial with a negative dominant term. It is direct to check that $P(x) \geq 0$ for $x \leq \nu$, $P(\mu_-) = 0$, $P(1) = 0$ and $P'(\mu_-) = 0$. This implies $P(x) \geq 0$ for $x \in [0, \rho^*] \cup [\mu_-, 1]$, $P(x) \leq 0$ for $x \in [\rho^*, \mu_-]$ with $\nu < \rho^* < \mu_-$. Inspecting the formula for $\pi_{cpi,t}$ then gives the result. In the extreme case where $\phi \to \infty$, we have $\pi_{cpi,t} = 0$, $\tilde{\mathcal{Y}}_t = -\frac{\sigma\psi}{\sigma+\psi}\mathcal{NH}_t$: stabilizing CPI inflation comes at the cost of distorting the output gap. Finally, since by Result 1 $\pi_{cpi,t} - \pi_{mcpi,t}$ is independent of monetary policy, we have as in the previous case that for a negative shock in a necessity sector, $\pi_{cpi,t}$ is initially higher than $\pi_{mcpi,t}$ and then lower and the opposite is true for a negative shock in a luxury sector.

Case $\hat{R}_t = \phi \pi_{mcpi,t}$. The system of equations becomes

$$R\pi_{mcpi,t} = R\kappa\tilde{\mathcal{Y}}_t + \mathbb{E}_t\pi_{mcpi,t+1}$$
$$\tilde{\mathcal{Y}}_{t+1} - \tilde{\mathcal{Y}}_t = \sigma\left(\phi\pi_{mcpi,t} - \pi_{mcpi,t+1} - \hat{r}^*_t\right)$$

In that case

$$\pi_{mcpi,t} = \frac{R\kappa\sigma}{\left(\mu_+ - \rho_a\right)\left(\mu_- - \rho_a\right)}\rho_a^t\hat{r}^*_0$$
$$\tilde{\mathcal{Y}}_t = \frac{\sigma\left(R - \rho_a\right)}{\left(\mu_+ - \rho_a\right)\left(\mu_- - \rho_a\right)}\rho_a^t\hat{r}^*_0$$

The response is as in the standard model with $\pi_{mcpi,t}$ and $\tilde{\mathcal{Y}}_t$ both increasing in response to a negative shock (sectoral or aggregate) and increase is smaller the stronger the Taylor rule. In addition

$$\pi_{cpi,t} = \frac{R\lambda}{\left(\mu_+ - \rho_a\right)\left(\mu_- - \rho_a\right)}\left(\left(1-\rho_a\right)\rho_a^t - \left(1-\mu_-\right)\mu_-^t\right)\hat{A}_{\Delta,0} + \frac{R\kappa\sigma}{\left(\mu_+ - \rho_a\right)\left(\mu_- - \rho_a\right)}\rho_a^t\hat{r}^*_0$$

$\pi_{cpi,t}$ increases relatively more than $\pi_{mcpi,t}$ for $t \leq t^* = ln\left(\frac{(1-\mu_-)}{(1-\rho_a)}\right)/ln\left(\frac{\rho_a}{\mu_-}\right)$, (less for $t > t^*$) for a negative shock in a necessity sector, relatively less for a negative shock in a luxury sector.

Case $\hat{R}_t = \phi \pi_{cpi,t}$. The system of equations becomes

$$R\pi_{cpi,t} = R\kappa\tilde{\mathcal{Y}}_t + R\lambda\mathcal{NH}_t + \mathbb{E}_t\pi_{cpi,t+1}$$
$$\tilde{\mathcal{Y}}_{t+1} - \tilde{\mathcal{Y}}_t = \sigma\left(\phi\pi_{cpi,t} - \pi_{mcpi,t+1} - \hat{r}^*_t\right)$$

We have:

$$\pi_{cpi,t} = \frac{R\lambda}{\left(\mu_+ - \rho_a\right)\left(\mu_- - \rho_a\right)}\left\{\left(1 - \frac{R\kappa\sigma\phi}{\left(\lambda_+ - \rho_a\right)\left(\lambda_- - \rho_a\right)}\right)\left(1-\rho_a\right)\rho_a^t - \left(1 - \frac{R\kappa\sigma\phi}{\left(\lambda_+ - \mu_-\right)\left(\lambda_- - \mu_-\right)}\right)\left(1-\mu_-\right)\mu_-^t\right\}\hat{A}_{\Delta,0} + \frac{R\kappa\sigma}{\left(\lambda_+ - \rho_a\right)\left(\lambda_- - \rho_a\right)}\rho_a^t\hat{r}^*_0,$$

$$\tilde{\mathcal{Y}}_t = -\frac{R\lambda\bar{\sigma}\phi}{\left(\mu_+ - \rho_a\right)\left(\mu_- - \rho_a\right)}\left\{\frac{\left(1-\rho_a\right)\left(R-\rho_a\right)}{\left(\lambda_+ - \rho_a\right)\left(\lambda_- - \rho_a\right)}\rho_a^t - \frac{\left(1-\mu_-\right)\left(R-\mu_-\right)}{\left(\lambda_+ - \mu_-\right)\left(\lambda_- - \mu_-\right)}\mu_-^t\right\}\hat{A}_{\Delta,0} + \frac{\sigma\left(R-\rho_a\right)}{\left(\lambda_+ - \rho_a\right)\left(\lambda_- - \rho_a\right)}\rho_a^t\hat{r}^*_0.$$

Using the results of the previous cases, we can directly see that following a negative shock in a necessity sector, the output gap is lower under targeting than under MCPI targeting. In addition, if we compare the response of a negative shock in a luxury sector and a necessity sector which have the same impact on efficient output ($\mathcal{Y}^*_t = \frac{1}{1+\frac{\psi}{\sigma}}\sum_l\left(\psi\overline{\partial_e e_l} + \bar{s}_l\right)\hat{A}_{l,t}$), the output gap is relatively lower in response to the shock in the necessity sector. If the shock is sufficiently persistent the output gap is negative in response to a shock in a necessity sector (as $\hat{r}^*_0 \to 0$ when $\rho_a \to 1$).

Additions to Result 3. In Appendix F we extend provide a number of additional analytical results for the case with Hand-to-Mouth households.

Result 4

We first give an example of a shock[39] that is such that there is no inflation index ($\pi_t = \sum_k \tilde{\omega}_k\pi_{k,t}$ with $\sum_k\tilde{\omega}_k \neq 0$) that can be stabilized alongside the output gap under A.1 and A.2. We then argue that for any persistence of the shock ρ_a, the set of shocks for which an inflation index can be jointly stabilized with the output gap is of measure 0. Finally we extend the argument without A.1 and A.2. We denote $\tilde{P}_{k,t}\left(\left\{\hat{\mathbf{A}}_t\right\}_{t\geq 0}\right)$ the solution of the relative price system (described in Result 1) for an arbitrary sequence of shocks $\left\{\hat{\mathbf{A}}_t\right\}_{t\geq 0}$ (and similarly $\tilde{\pi}_{k,t}\left(\left\{\hat{\mathbf{A}}_t\right\}_{t\geq 0}\right)$, $\tilde{\mathcal{M}}_{k,t}\left(\left\{\hat{\mathbf{A}}_t\right\}_{t\geq 0}\right)$ the implied relative price inflation and endogenous markups). Consider a shock $\hat{\mathbf{A}}_t = \left\{\hat{A}_{1,t}, ..., \hat{A}_{k,t}\right\}_{t\geq 0}$ such that for $k = 1, ..., K-1$ and all t:

$$-\left(\frac{1}{\psi} + \frac{1}{\sigma}\right)\left(\lambda_k - \lambda\right)\hat{\mathcal{Y}}^*_t + \lambda_k\tilde{P}^*_{k,t} - \lambda_k\sum_l\overline{\partial_e e_l}\tilde{P}^*_{l,t} = 0$$

Re-expressed in terms of $\hat{\mathbf{A}}_t$ this becomes:

---

[39] There are, of course, other examples as well.

$$-\left(\lambda_k-\lambda\right)\sum_l\left(\overline{\partial_e e}_l+\frac{\bar{s}_l}{\psi}\right)\hat{A}_{l,t}-\lambda_k\hat{A}_{k,t}+\lambda_k\sum_l\overline{\partial_e e}_l\hat{A}_{l,t}=0$$

Note that this is a system of $K-1$ equations in K unknowns, so it admits a non trivial solution $\hat{\mathbf{A}}^*\neq 0$. We necessarily have:

$$\sum_l\left(\overline{\partial_e e}_l+\frac{\bar{s}_l}{\psi}\right)\hat{A}_l^*\neq 0.$$

We reason by contradiction: if $\sum_l\left(\overline{\partial_e e}_l+\frac{\bar{s}_l}{\psi}\right)\hat{A}_l^*=0$, then $\lambda_k\left(\hat{A}_k^*-\sum_n\overline{\partial_e e}_n\hat{A}_n^*\right)=\lambda_l\left(\hat{A}_l^*-\sum_n\overline{\partial_e e}_n\hat{A}_n^*\right)$ for all l,k (note that the K^{th} sector equation is a linear combination of the other $K-1$ equations). Under (**A.1**), we have that $\lambda_k>0$ for all k, which implies $\hat{A}_k^*=0$ for all k. Indeed, noting $\underline{A}^*=min\left(\hat{A}_l^*\right),\overline{A}^*=max\left(\hat{A}_l^*\right)$, we have $0\leq\overline{\lambda}\left(\overline{A}^*-\sum_n\overline{\partial_e e}_n\hat{A}_n^*\right)=\underline{\lambda}\left(\underline{A}^*-\sum_n\overline{\partial_e e}_n\hat{A}_n^*\right)\leq 0$, so $\hat{A}_k^*$ is constant across sectors which implies $\hat{A}_k^*=0$ for all k. This contradicts the fact that $\hat{\mathbf{A}}^*$ is a non trivial solution of the system.

Next, define the shock $\hat{\mathbf{A}}^{*,\rho_a}$ such that $\hat{\mathbf{A}}_t^{*,\rho_a}=\rho_a^t\hat{\mathbf{A}}^*$ for $0\leq\rho_a\leq 1$, in that case, the system for relative prices is given by:

$$\begin{aligned}\tilde{\pi}_{k,t}-\beta\left(1-\delta\right)\mathbb{E}_t\tilde{\pi}_{k,t+1}&=\sum\alpha_{k,l}\tilde{P}_{l,t}+\sum\left(\lambda_k\tilde{\mathcal{M}}_{k,t}-\lambda\sum_l\overline{\partial_e e}_l\tilde{\mathcal{M}}_{l,t}\right)\\ \tilde{P}_{k,t}&=\tilde{\pi}_{k,t}+\tilde{P}_{k,t-1}\\ \tilde{\mathcal{M}}_{k,t}&=\mathbb{E}_t\tilde{\mathcal{M}}_{k,t+1}-\frac{\delta}{1-\delta}\mathbb{E}_t\tilde{\mathcal{M}}_{k,t+1}^0.\\ \tilde{\mathcal{M}}_{k,t}^0&=\frac{1}{\left(1-\delta\right)R}\mathbb{E}_t\tilde{\mathcal{M}}_{k,t+1}^0-\sum\beta_{k,l}\tilde{P}_{l,t}-\left(1-\frac{1}{R}\right)\tilde{\mathcal{M}}_{k,t}^D\end{aligned}$$

We therefore that have $\tilde{P}_{k,t}\left(\hat{\mathbf{A}}^{*,\rho_a}\right)=0$ for all k, t is a solution of the system. This implies that the NKPC for the index $\pi_{d,t}$ is:

$$\pi_{d,t}=\kappa\tilde{\mathcal{Y}}_t+\lambda\sum_k\overline{\partial_e e}_k\Gamma_k\sum_l\frac{\sigma\left(\psi\overline{\partial_e e}_l+\bar{s}_l\right)}{\sigma+\psi}\rho_a^t\hat{A}_l^*++\beta\mathbb{E}_t\pi_{d,t+1}$$

Since $\sum_l\frac{\sigma\left(\psi\overline{\partial_e e}_l+\bar{s}_l\right)}{\sigma+\psi}\hat{A}_l^*\neq 0$, the index $\pi_{d,t}$ cannot be stabilized jointly with the output gap for any $\hat{\mathbf{A}}^{*,\rho_a}$.

Now Take an arbitrary inflation index π_t, decomposing it in the basis of $\pi_{d,t}$ and relative prices, we have:

$$\pi_t=\omega_d\pi_{d,t}+\sum_{k=1}^{K-1}\omega_k\tilde{\pi}_{k,t}$$

Suppose $\omega_d\neq 0$. We have for an arbitrary shock persistence ρ_a, $\hat{\mathbf{A}}^{\rho_a}$ such that $\hat{\mathbf{A}}_t^{\rho_a}=\rho_a^t\hat{\mathbf{A}}$ that the NKPC for the index π_t is:

$$\pi_t-\beta\mathbb{E}_t\pi_{t+1}=\omega_d\kappa\tilde{\mathcal{Y}}_t+\mathcal{W}_t\left(\hat{\mathbf{A}}^{\rho_a}\right)$$

Where the wedge $\mathcal{W}_t\left(\hat{\mathbf{A}}^{\rho_a}\right)$ is given by

$$\begin{aligned}\mathcal{W}_t\left(\hat{\mathbf{A}}^{\rho_a}\right)=&\sum_l\left(\omega_d\lambda\overline{\partial_e e}_l+\sum_{m=1}^{K-1}\omega_m\left(\lambda_m-\lambda\overline{\partial_e e}_m\right)\right)\left\{\Gamma_l\sum_m\frac{\sigma\left(\psi\overline{\partial_e e}_m+\bar{s}_m\right)}{\sigma+\psi}\rho_a^t\hat{A}_m+\left(\mathcal{M}_{l,t}^D\left(\hat{\mathbf{A}}^{\text{æ}_{\mathbf{a}}}\right)+\mathcal{M}_{l,t}^P\left(\hat{\mathbf{A}}^{\rho_a}\right)\right)\right\}\\&-\sum_{l=1}^{k-1}\omega_l\left(\lambda_k\left(\rho_a^t\hat{A}_m+\tilde{P}_{l,t}\left(\hat{\mathbf{A}}^{\rho_a}\right)\right)-\lambda_k\sum_m\overline{\partial_e e}_m\left(\rho_a^t\hat{A}_m+\tilde{P}_{m,t}\left(\hat{\mathbf{A}}^{\rho_a}\right)\right)\right)\end{aligned}$$

Since the system of relative prices (described in Result 1) is linear and that shock enters linearly, we have that the mapping $\hat{\mathbf{A}}\mapsto\tilde{P}_{k,t}\left(\hat{\mathbf{A}}^{\rho_a}\right)$ is linear. Therefore we directly have that the mappings $\hat{\mathbf{A}}\mapsto\mathcal{M}_{k,t}^D\left(\hat{\mathbf{A}}^{\rho_a}\right)$, $\hat{\mathbf{A}}\mapsto\mathcal{M}_{k,t}^P\left(\hat{\mathbf{A}}^{\rho_a}\right)$ are linear (as $\mathcal{M}_{k,t}^D$ and $\mathcal{M}_{k,t}^P$ are linear functions of relative prices). This implies that the mapping $\hat{\mathbf{A}}\mapsto\mathcal{W}_t\left(\hat{\mathbf{A}}^{\rho_a}\right)$ is also linear. Note that since $\mathcal{W}_0\left(\hat{\mathbf{A}}^{*,\rho_a}\right)=\omega_d\lambda\left(\sum\overline{\partial_e e}_k\Gamma_k\right)\sum_l\frac{\sigma\left(\psi\overline{\partial_e e}_l+\bar{s}_l\right)}{\sigma+\psi}\hat{A}_l^*\neq 0$, we have that the kernel of $\hat{\mathbf{A}}\mapsto\mathcal{W}_0\left(\hat{\mathbf{A}}^{\rho_a}\right)$ is at most of dimension $K-1$. Since a subspace of $\mathbb{R}^K$ of dimension $K-1$ has Lebesgue measure 0, that implies that for a any ρ_a, $\mathcal{W}_0\left(\hat{\mathbf{A}}^{\rho_a}\right)\neq 0$ on a subset of measure 1. This implies that no index with $\omega_d\neq 0$ can be stabilized jointly with the output gap. Therefore only relative prices can be stabilized jointly with the output gap. However, as shown in Result 1, relative prices are independent from monetary policy. So the only inflation index that could be stabilized jointly with inflation would be a trivial index which does not respond to any shock.

Note that previous argument remains valid if we relax A.1 and A.2. for permanent shocks. Consider a policy that stabilizes the output gap. We have $\tilde{\mathcal{Y}}_t = 0$, $\hat{R}_t = r^*_t + \pi_{mcpi,t+1}$. We first show that no inflation index $\pi_t = \sum_k \tilde{\omega}_k \pi_{k,t}$ with $\sum_k \tilde{\omega}_k \neq 0$ can be stabilized jointly with the output gap.

$$\begin{aligned}
\tilde{\pi}_{k,t} &= \sum_m \left(\lambda_k \Gamma_k - \lambda \Gamma_m\right) \overline{\partial_e e}_m \sum_l \frac{\sigma\left(\psi \overline{\partial_e e}_l + \bar{s}_l\right)}{\sigma+\psi} \hat{A}_{l,t} - \left(\lambda_k \hat{A}_{k,t} - \lambda_k \sum_l \overline{\partial_e e}_l \hat{A}_{l,t}\right) \\
&\quad - \left(\lambda_k \tilde{P}_{k,t} - \lambda_k \sum_l \overline{\partial_e e}_l \tilde{P}_{l,t}\right) + \lambda_k \left({\mathcal{M}^{\mathcal{D}}}_{k,t} + {\mathcal{M}^{\mathcal{P}}}_{k,t}\right) - \lambda \sum_l \overline{\partial_e e}_l \left({\mathcal{M}^{\mathcal{D}}}_{l,t} + {\mathcal{M}^{\mathcal{P}}}_{l,t}\right) + \beta\left(1-\delta\right) \mathbb{E}_t \tilde{\pi}_{k,t+1}. \\
\tilde{P}_{k,t} &= \tilde{\pi}_{k,t} + \tilde{P}_{k,t-1}.
\end{aligned}$$

Where the endogenous markup now solves:

$$\begin{aligned}
\mathcal{M}^P_{k,t} &= \sum_{l=1}^{K} \int \gamma_{e,k}(i) \frac{e_k}{E_k} \rho_{k,l}(i) di \tilde{P}_{l,t}, \\
\mathcal{M}^D_{k,t} &= \mathbb{E}_t \mathcal{M}^D_{k,t+1} - \sum_l \sigma^{\mathcal{M}}_{k,l} \left(r^*_t + \tilde{\pi}_{mcpi,t+1} - \mathbb{E}_t \widetilde{\pi}_{l,t+1}\right) - \frac{\delta}{1-\delta} \mathbb{E}_t \mathcal{M}^0_{k,t+1}, \\
\mathcal{M}^0_{k,t} &= \frac{1}{\left(1-\delta\right) R} \mathbb{E}_t \mathcal{M}^0_{k,t+1} + \int \gamma_{b,k}(i) \frac{b(i)}{RE} di \left(r^*_t + \tilde{\pi}_{mcpi,t+1} - \tilde{\pi}_{cpi,t+1}\right) \\
&\quad - \sum_l \int \gamma_{b,k}(i) \left(\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) + \frac{\psi W n(i)}{WN}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right) di \widetilde{P}_{l,t} - \frac{R-1}{R} \mathcal{M}^D_{k,t}, \\
\sigma^{\mathcal{M}}_{k,l} &= \sigma \int \gamma_{e,k}(i) \frac{e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \sigma \overline{\partial_e e_l} \frac{R}{R-1} \frac{\sigma+\psi}{\sigma} \int \gamma_{b,k}(i) \frac{Wn(i)}{WN} di.
\end{aligned}$$

Note that under a policy that stabilizes the output gap, the evolution of relative prices is independent from $\pi_{d,t}$: relative prices only depend on themselves. Consider a permanent shock ($\hat{A}_{l,t} = \hat{A}_{l,0}$ for all l, t) such that $\sum_m \left(\lambda_k \Gamma_k - \lambda \Gamma_m\right) \overline{\partial_e e}_m \sum_l \frac{\sigma\left(\psi \overline{\partial_e e}_l + \bar{s}_l\right)}{\sigma+\psi} \hat{A}_{l,t} - \left(\lambda_k \hat{A}_{k,t} - \lambda_k \sum_l \overline{\partial_e e}_l \hat{A}_{l,t}\right) = 0$ for all k and denote it $\hat{\mathbf{A}}^*$. This implies $r^*_t = 0$ for all t and therefore $\tilde{P}_{k,t}\left(\hat{\mathbf{A}}^*\right) = 0$ for all k, t. As before, we necessarily have $\sum_l \left(\overline{\partial_e e}_l + \frac{\bar{s}_l}{\psi}\right) \hat{A}^*_l \neq 0$. If we consider an inflation index $\pi_t = \sum_k \tilde{\omega}_k \pi_{k,t}$ with $\sum_k \tilde{\omega}_k \neq 0$ it can be rewritten $\pi_t = \omega_d \pi_{d,t} + \sum_{k=1}^{K-1} \omega_k \tilde{\pi}_{k,t}$ with $\omega_d \neq 0$. Consider the set of permanent shocks $\hat{\mathbf{A}}^1$, the NKPC for π_t is then $\pi_t - \beta \mathbb{E}_t \pi_{t+1} = \mathcal{W}_t\left(\hat{\mathbf{A}}^1\right)$ and note that $\mathcal{W}_t\left(\hat{\mathbf{A}}^*\right) = \omega_d \sum_m \lambda \Gamma_m \overline{\partial_e e}_m \sum_l \frac{\sigma\left(\psi \overline{\partial_e e}_l + \bar{s}_l\right)}{\sigma+\psi} \hat{A}^*_l \neq 0$. Since $\hat{\mathbf{A}} \mapsto \mathcal{W}_t\left(\hat{\mathbf{A}}^1\right)$ is again a linear map, this implies that the set of permanent shocks such that $\mathcal{W}_0\left(\hat{\mathbf{A}}^1\right) = 0$ as dimension at most $K-1$ and therefore has a Lebesgue measure of 0. We can extend the argument to the set of shocks $\hat{\mathbf{A}}\left(\boldsymbol{\alpha}\right)$such that $\hat{A}_{k,t}\left(\boldsymbol{\alpha}\right) = \sum_{i=0}^{I} \alpha_{i,k} \rho_i^t$ where $0 = \rho_0 < ... < \rho_I = 1$ and $\{\alpha_{i,k}\}_{0 \leq i \leq I, 1 \leq k \leq K}$ are arbitrarily scalars ($\hat{\mathbf{A}}\left(\boldsymbol{\alpha}\right)$ is an arbitrary combination of I shocks with persistence ρ_0,...,ρ_I). Indeed consider $\hat{\mathbf{A}}\left(\boldsymbol{\alpha}^*\right)$ such that $\alpha_{I,k} = \hat{A}^*_k$ and $\alpha_{i,k} = 0$ for $i \neq I$, we have $\mathcal{W}_t\left(\hat{\mathbf{A}}\left(\boldsymbol{\alpha}^*\right)\right) = \omega_d \sum_m \lambda \Gamma_m \overline{\partial_e e}_m \sum_l \frac{\sigma\left(\psi \overline{\partial_e e}_l + \bar{s}_l\right)}{\sigma+\psi} \hat{A}^*_l \neq 0$ so using the same logic, for a given t, the subset of shocks such that $\mathcal{W}_t\left(\hat{\mathbf{A}}\left(\boldsymbol{\alpha}\right)\right) = 0$ is of measure 0 .Therefore no inflation index with $\sum_k \tilde{\omega}_k \neq 0$ can be stabilized jointly with the output gap on any set of combination of AR(1) shocks.

Result 5

Under the assumption that $\lambda = \lambda_k$ for all k, the equations for relative prices (defined with respect to MCPI) can be rewritten as:

$$\begin{aligned}
\tilde{\pi}_{k,t} - \beta\left(1-\delta\right) \mathbb{E}_t \tilde{\pi}_{k,t+1} &= \lambda \left(\tilde{P}^*_{k,t} + \sum \alpha_{k,l} \tilde{P}_{l,t} + \sum \left(\tilde{\mathcal{M}}_{k,t} - \sum_l \overline{\partial_e e_l} \tilde{\mathcal{M}}_{l,t}\right)\right), \\
\tilde{P}_{k,t} &= \tilde{\pi}_{k,t} + \tilde{P}_{k,t-1}, \\
\tilde{\mathcal{M}}_{k,t} &= \mathbb{E}_t \tilde{\mathcal{M}}_{k,t+1} - \frac{\delta}{1-\delta} \mathbb{E}_t \tilde{\mathcal{M}}^0_{k,t+1}, \\
\tilde{\mathcal{M}}^0_{k,t} &= \frac{1}{\left(1-\delta\right) R} \mathbb{E}_t \tilde{\mathcal{M}}^0_{k,t+1} - \sum \beta_{k,l} \tilde{P}_{l,t} - \left(1 - \frac{1}{R}\right) \tilde{\mathcal{M}}^D_{k,t}
\end{aligned}$$

with

$$
\begin{aligned}
\alpha_{k,l} &= -\mathbb{1}_{k=l} + \overline{\partial_e e_l} + \int \gamma_{e,k}(i)\frac{e}{E_k}\rho_{k,l}(i) - \lambda \sum_n \overline{\partial_e e_n} \int \gamma_{e,n}(i)\frac{e}{E_n}\rho_{n,l}(i)di - \sigma^{\mathcal{M}}_{k,l} + \sum_n \overline{\partial_e e_n}\sigma^{\mathcal{M}}_{n,l} \\
\beta_{k,l} &= \int \gamma_{b,k}(i)\frac{e(i)}{E}\left(\left(s_l(i) - \bar{s}_l\right) + \sigma\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right)di + \left(1 - \frac{1}{R}\right)\sigma^{\mathcal{M}}_{k,l}
\end{aligned}
$$

For an aggregate shock, we have $\tilde{P}^*_{k,t} = 0$ for all k so $\tilde{P}_{k,t} = 0$ for all k, t. Since we have

$$
\begin{aligned}
\mathcal{M}^D_{k,t} &= \mathbb{E}_t\mathcal{M}^D_{k,t+1} + \sum_l \sigma^{\mathcal{M}}_{k,l}\mathbb{E}_t\tilde{\pi}_{l,t+1} - \frac{\delta}{1-\delta}\mathbb{E}_t\mathcal{M}^0_{k,t+1} \\
\mathcal{M}^0_{k,t} &= \frac{1}{(1-\delta)R}\mathbb{E}_t\mathcal{M}^0_{k,t+1} - \sum_l \int \gamma_{b,k}(i)\left(\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) + \frac{\psi W n(i)}{WN}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right)di\tilde{P}_{l,t} - \frac{R-1}{R}\mathcal{M}^D_{k,t}.
\end{aligned}
$$

and

$$
\mathcal{M}^P_{k,t} = \sum_l \int \gamma_{e,k}(i)\frac{e_k}{E_k}\rho_{k,l}(i)di\tilde{P}_{l,t}
$$

This implies $\mathcal{M}^D_{k,t} = \mathcal{M}^P_{k,t} = 0$ for all k, t. Therefore,

$$
\mathcal{M}_{k,t} = \Gamma_k\hat{\mathcal{Y}}^*_t < 0
$$

For all k, t if $\Gamma_k > 0$.

Result 6

Assume that the households' utility function associated with intratemporal sectoral consumption takes the form

$$
u(c_k, ..., c_K) = \frac{1}{1-\frac{1}{\sigma}}\left(\prod_{k=1}^{K}\left(c_k - \underline{c_k}\right)^{\alpha_k}\right)^{1-\frac{1}{\sigma}}.
$$

With $c_k = e_k/P_k$ (recall that subvariety prices are equal in steady state) and $\sum \alpha_k = 1$. We have:

$$
\alpha_k\left(e - \sum P_{k=l}\underline{c_l}\right) = P_k\left(c_k - \underline{c_k}\right).
$$

Therefore

$$
\partial_e e_k = \alpha_k
$$

$$
\begin{aligned}
\partial_{P_l}c_k + \frac{\partial_e e_k}{P_k}c_l &= -\frac{\alpha_k}{P_k}\underline{c_l} + \frac{\alpha_k}{P_k}\left(\frac{\alpha_l}{P_l}\left(e - \sum P_k\underline{c_k}\right) + \underline{c_l}\right) - \mathbb{1}_{k=l}\frac{\alpha_k}{P_k^2}\left(e - \sum P_k\underline{c_k}\right), \\
P_l\partial_{P_l}c_k + P_l\frac{\partial_e e_k}{P_k}c_l &= \frac{\alpha_k}{P_k}\left(\alpha_l - \mathbb{1}_{k=l}\right)\left(e - \sum P_k\underline{c_k}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{s}_k &= \int \frac{1}{E}\left(\alpha_k\left(e\left(i\right) - \sum P_l\underline{c_l}\right) + P_k\underline{c_k}\right)di, \\
&= \overline{\partial_e e_k} + \frac{P_k\underline{c_k} - \overline{\partial_e e_k}\sum P_l\underline{c_l}}{E},
\end{aligned}
$$

$$
\begin{aligned}
\frac{e(i)}{E}\left(s_k(i) - \bar{s}_k\right) &= \frac{1}{E}\left(\overline{\partial_e e_k}\left(e(i) - \sum P_l\underline{c_l}\right) + P_k\underline{c_k} - e(i)\left(\overline{\partial_e e_k} + \frac{P_k\underline{c_k} - \overline{\partial_e e_k}\sum P_l\underline{c_l}}{E}\right)\right), \\
&= \frac{1}{E}\left(1 - \frac{e(i)}{E}\right)\left(P_k\underline{c_k} - \overline{\partial_e e_k}\sum P_l\underline{c_l}\right) = \left(1 - \frac{e(i)}{E}\right)\left(\bar{s}_k - \overline{\partial_e e_k}\right).
\end{aligned}
$$

Defining $\tilde{P}_{k,t} = P_{k,t} - \sum_l \overline{\partial_e e_l}P_{l,t}$ and $\tilde{\pi}_{k,t} = \pi_{k,t} - \sum_l \overline{\partial_e e_l}\pi_{l,t}$, we therefore have

$$
\begin{aligned}
\mathcal{M}^P_{k,t} &= \sum_l \int \gamma_{e,k}(i)\frac{\overline{\partial_e e_k}}{E_k}\left(e - \sum P_k\underline{c_k}\right)di\overline{\partial_e e_l}\tilde{P}_{l,t} - \int \gamma_{e,k}(i)\frac{\overline{\partial_e e_k}}{E_k}\left(e - \sum P_k\underline{c_k}\right)di\tilde{P}_{k,t} \\
&= -\int \gamma_{e,k}(i)\frac{\overline{\partial_e e_k}}{E_k}\left(e - \sum P_k\underline{c_k}\right)di\tilde{P}_{k,t}
\end{aligned}
$$

$$\begin{aligned}\mathcal{M}_{k,t}^{D} &= \mathbb{E}_t \mathcal{M}_{k,t+1}^{D} - \frac{\delta}{1-\delta}\mathbb{E}_t \mathcal{M}_{k,t+1}^{0} \\ \mathcal{M}_{k,t}^{0} &= \frac{1}{(1-\delta)\,R}\mathbb{E}_t \mathcal{M}_{k,t+1}^{0} - \sum_l \int \gamma_{b,k}(i)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right) di\tilde{P}_{l,t} - \frac{R-1}{R}\mathcal{M}_{k,t}^{D}\end{aligned}$$

Note that under A3 we have $\gamma_{e,k}(i)\frac{\overline{\partial_e e_k}}{E_k} = \gamma_e\left(i\right)\frac{1}{E}$ and $\gamma_{b,k}(i) = \gamma_{b,l}(i)$ for all k so $\mathcal{M}_{k,t}^{D} = \mathcal{M}_{t}^{D}$,$\mathcal{M}_{k,t}^{0} = \mathcal{M}_{t}^{0}$,

$$\mathcal{M}_{k,t}^{P} = -\int \gamma_e(i)\frac{e(i)-\sum P_k \underline{c_k}}{E} di\tilde{P}_{k,t}$$

$$\begin{aligned}\mathcal{M}_{t}^{D} &= \mathbb{E}_t \mathcal{M}_{t+1}^{D} - \frac{\delta}{1-\delta}\mathbb{E}_t \mathcal{M}_{t+1}^{0} \\ \mathcal{M}_{t}^{0} &= \frac{1}{(1-\delta)\,R}\mathbb{E}_t \mathcal{M}_{t+1}^{0} - \sum_l \int \gamma_{b}(i)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right) di\tilde{P}_{l,t} - \frac{R-1}{R}\mathcal{M}_{t}^{D}\end{aligned}$$

Next under (**A.1**) and (**A.3**), we necessarily have $\Gamma_k = \Gamma$, $\lambda_k = \lambda$ for all k so the NKPC for $\tilde{\pi}_{k,t}$ is

$$\tilde{\pi}_{k,t} = \lambda\left(\left(\tilde{P}_{k,t}^{*} - \tilde{P}_{k,t}\right) - \int \gamma_e(i)\frac{\left(e-\sum P_k \underline{c_k}\right)}{E} di\tilde{P}_{k,t}\right) + \beta\mathbb{E}_t \tilde{\pi}_{k,t+1}$$

The evolution of relative price k only depends on itself. Denoting $\tilde{\lambda} = \lambda\left(1+\int \gamma_e(i)\frac{\left(e-\sum P_k \underline{c_k}\right)}{E}\right)$ the eigenvalues of the system are:

$$\nu_{\pm} = \frac{R+R\tilde{\lambda}+1 \pm \sqrt{\left(R+R\tilde{\lambda}-1\right)^2 - 4R\tilde{\lambda}}}{2}$$

(Note that for $\int \gamma_e(i)\frac{\left(e-\sum P_k \underline{c_k}\right)}{E} > -1, 0 < \nu_- < 1, R+R\tilde{\lambda} < \nu_+$) and the evolution of $\tilde{P}_{k,t}$ is given by:

$$\tilde{P}_{k,t} = \lambda \sum_0^t \nu_-^{t-s+1} \sum \frac{1}{\nu_+^u}\tilde{P}_{k,s+u}^{*}$$

For a negative sequence of shocks in k $\{\hat{P}_{k,t}^{*}\}_{t\geq 0} > 0$, we therefore have $\tilde{P}_{k,t} > 0$ for all t and $\overline{\partial_e e_k}\tilde{P}_{k,t} = -\left(1-\overline{\partial_e e_k}\right)\tilde{P}_{l,t}$ for all $l \neq k$ so we have

$$\begin{aligned}\mathcal{M}_{cpi,t}^{P} &= \sum_l \bar{s}_k \mathcal{M}_{k,t}^{P} = -\int \gamma_e(i)\frac{e(i)-\sum P_k \underline{c_k}}{E} di \sum \bar{s}_l \tilde{P}_{l,t} \\ &= -\int \gamma_e(i)\frac{e(i)-\sum P_k \underline{c_k}}{E} di\left(\bar{s}_k - \overline{\partial_e e_k}\right)\lambda \sum_0^t \nu_-^{t-s+1}\sum \frac{1}{\nu_+^u}\tilde{P}_{k,s+u}^{*}\end{aligned}$$

So $\mathcal{M}_{cpi,t}^{P} < 0$ following a shock in a necessity sector, $\mathcal{M}_{cpi,t}^{P} > 0$ following a shock in a luxury sector. In addition we have for a shock in sector k

$$\begin{aligned}\mathcal{M}_{t}^{D} = &-\int \gamma_b(i)\frac{1}{E}\left(1-\frac{e(i)}{E}\right) di\,(1-\delta)^{t+1}\sum_{u=0}^{\infty}\frac{1}{R^u}\sum_l \left(P_l \underline{c_l} - \overline{\partial_e e_l}\sum P_n \underline{c_n}\right)\tilde{P}_{l,u} \\ &-\delta \int \gamma_b(i)\frac{1}{E}\left(1-\frac{e(i)}{E}\right) di \sum_{s=0}^{t}(1-\delta)^{t-s}\sum_{u=0}^{\infty}\frac{1}{R^u}\sum_l \left(P_l \underline{c_l} - \overline{\partial_e e_l}\sum P_n \underline{c_n}\right)\tilde{P}_{l,s+u} \\ = &-\int \gamma_b(i)\left(1-\frac{e(i)}{E}\right) di\,(1-\delta)^{t+1}\sum_{u=0}^{\infty}\frac{1}{R^u}\left(\bar{s}_k - \overline{\partial_e e_k}\right)\frac{\tilde{P}_{k,u}}{\left(1-\overline{\partial_e e_k}\right)} \\ &-\delta \int \gamma_b(i)\left(1-\frac{e(i)}{E}\right) di \sum_{s=0}^{t}(1-\delta)^{t-s}\sum_{u=0}^{\infty}\frac{1}{R^u}\left(\bar{s}_k - \overline{\partial_e e_k}\right)\frac{\tilde{P}_{k,s+u}}{\left(1-\overline{\partial_e e_k}\right)}\end{aligned}$$

So if $Cov\left(\gamma_b(i), \frac{e(i)}{E}\right) > 0$, $\mathcal{M}_{t}^{D} = \mathcal{M}_{cpi,t}^{D} > 0$ following a shock in a necessity sector. Note that under the stronger assumption that $b\left(i\right) = 0$ for all i, we have $\gamma_b(i) = \gamma_e(i)\sigma/(\sigma+\psi)$ so if $\gamma_e(i)$ is increasing in $e(i)$, $Cov\left(\gamma_b(i), \frac{e(i)}{E}\right) > 0$.

D Calibration procedure and numerical details

Outer Preferences

To calibrate the non-homothetic CES preferences we use the LCF survey, which is the most comprehensive survey on household spending in the UK. Each member of the household keeps a detailed spending diary for a period of two weeks, while expenditure information on bigger items (like cars, vacations, housing etc.) are collected during interviews with the household head. We map these highly disaggregated consumption data into the standard 3-digit COICOP categories using a mapping table provided by the ONS. Aggregating these to the COICOP division level, forms the basis of our definition of sectors for the UK economy as well as providing the data for estimating the household-specific marginal propensities to consume across different sectors.

We exclude housing costs from household expenditures by redefining the relevant consumption category (COICOP4) to only include expenditure on Electricity, Gas and Other Fuels. [40] Furthermore, we exclude the following four sectors from our model: Alcohol & Tobacco, Health, Communication and Education. Health and Education are largely publicly provided in the UK and hence only a very small fraction of households report any private spending in these sectors. The other two sectors account for a small budget share so overall we still capture the vast majority of private expenditure, with the notable exception of housing. [41]

We construct household-specific price indices using the observed consumption shares in the 3-digit subcategories of each COICOP group so that $\ln P_{k,t}(i) = \sum_{m \in M_k} s_{m,k,t}(i) \ln P_{m,k,t}$. Whenever indices of 3-digit COICOP categories are not available (only occurring before 2015 and for a small subset of categories), we use the 2-digit price index of the corresponding group. To guard against any potential endogeneity of prices (similarly to what is done in Comin et al. (2021)) we construct Hausman-type price instruments by using the shares of all other households in the same region and for any given sector. To instrument for total expenditure we use log disposable income as well as the expenditure quintile of the household.

We impose that the individual parameter shifters take the following form:

$$\ln \mathcal{V}_{i,k} = x_i \beta_k + v_i^k,$$

where x_i are household demographic characteristics and v_i^k is an idiosyncratic and time invariant preference shifter that satisfies $\mathbb{E}\left[v_i^k | x_i\right] = 0$. The specific demographic controls include the size of the household $(1, 2+\text{adults})$, number of children $(0, 1+)$ and the age of the household head $(18-37, 38-50, 51-64, 65+)$. Note that since the households are surveyed at different points during the year, we also include quarter dummies to allow for potential seasonal effects in the consumption of different goods. We conduct different robustness checks to show that our results do not qualitatively change with the specific assumptions made in the baseline specification. Table **??** shows the results across a different set of specifications, with the first column showing our baseline version. The other columns show the estimated coefficients for the winsorised sample, adding regional controls (there are 12 regions in the UK) and expanding the sample to include all years available. For the winsorized sample we mark the households that are in the bottom or top 2% of expenditure shares in each of the eight COICOP categories and then drop them from the estimation. The GMM results are pretty robust to outliers so the exact cut-off does not matter much. Note also that in specification 4 we add year dummies on top of the quarter dummies that are present in all specifications. We have also run other robustness checks where we use different instruments or weight the observations by household expenditure and qualitatively the results are unchanged.

	(1)	(2)	(3)	(4)
Food	0.50	0.46	0.49	0.37
	(0.02)	(0.02)	(0.02)	(0.00)
Electricity & Gas	0.52	0.47	0.51	0.30
	(0.02)	(0.02)	(0.02)	(0.00)
Furniture	1.21	1.12	1.19	1.20
	(0.05)	(0.05)	(0.05)	(0.01)
Transport	0.90	0.89	0.88	1.10
	(0.04)	(0.04)	(0.04)	(0.01)
Recreation	1.23	1.15	1.20	1.11
	(0.05)	(0.04)	(0.04)	(0.01)
Restaurants & Hotels	0.98	0.97	0.96	0.99
	(0.04)	(0.04)	(0.03)	(0.01)
Miscellaneous	0.86	0.82	0.84	0.90
	(0.03)	(0.03)	(0.03)	(0.01)
N	3,164	2,815	3,164	56,538

These estimates allows us in turn to construct the marginal budget share $\partial_e e_k(i) = \eta + (1-\eta)\frac{\zeta_k}{\bar{\zeta}(i)}$, where $\bar{\zeta}(i)$ is the household specific 'average' non-homotheticity measure given by $\bar{\zeta}(i) = \sum_k s_k(i)\zeta_k$. This implies that richer households that spend more on luxury goods will have a higher $\bar{\zeta}(i)$. These preferences also imply that the compensated price elasticities take the following form:

$$\rho_{k,l}(i) = \begin{cases} \eta s_l(i) & \text{if } k \neq l, \\ -\eta\left(1 - s_l(i)\right) & \text{if } k = l. \end{cases}$$

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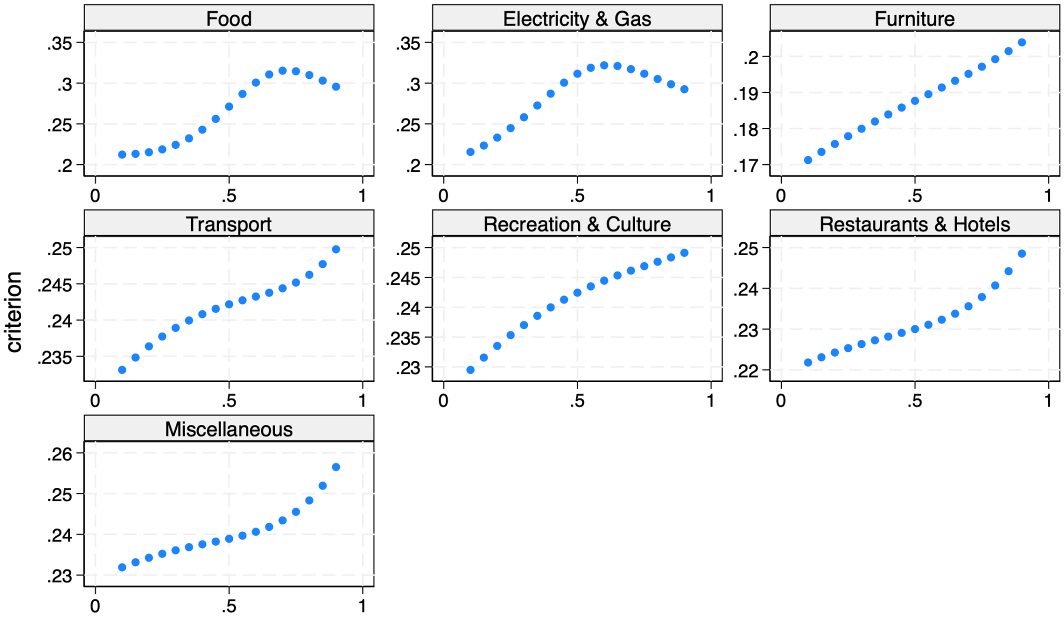
[40] Note that these are not the only direct expenditure on energy as HHs who own vehicles will also spend on diesel and petrol, included in the Transport category.

[41] The correlation between the three different measures of total expenditure (i.e. the original variable, excluding housing and excluding housing plus the four sectors) is always greater than 0.966.

Elasticity of Substitution Parameter. We set the elasticity of substitution parameter equal to 0.1, following the Comin et al. (2021) estimation for their 10-sector model. Here we show that increasing the value of η worsens the fit of the model, as measured by the criterion function of the GMM procedure. Figure 8 plots the criterion value as we vary the value of the elasticity parameter between 0.05 and 0.9. Regardless which of the sectors we choose as the base, the fit of the model worsens with higher values of η. [42]

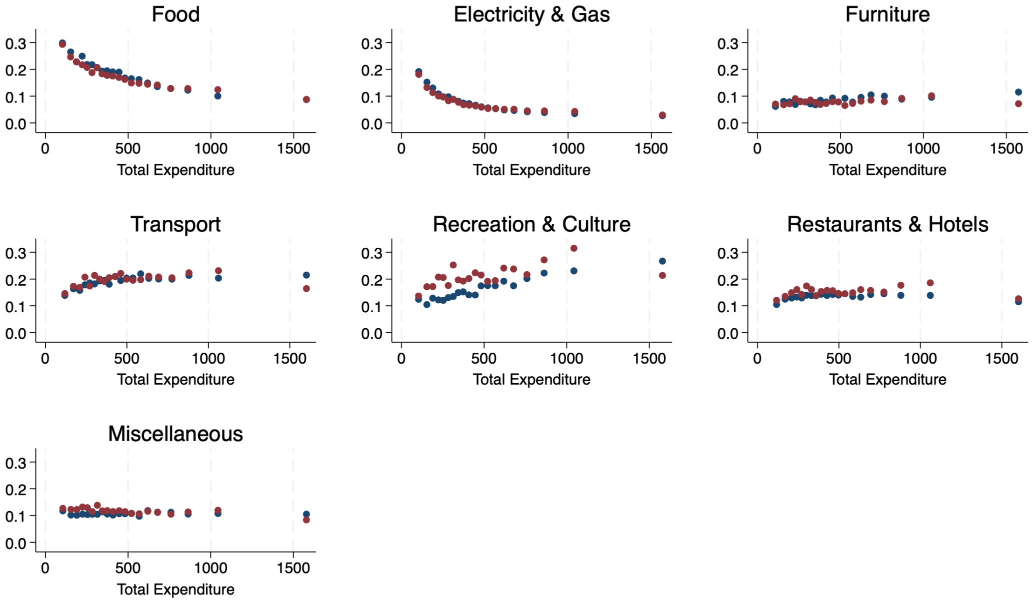
Figure 8. Criterion Value for different values of the elasticity parameter.



Notes: Each panel plots the minimised criterion function for the same GMM procedure for a given base sector.

We also check that the choice of the base sector does not qualitatively change the results of our estimation. Figure **??** plots the variation in the estimated ζ's as we change the sector used as the base one, while the elasticity of substitution is fixed to 0.1 and the set of instrument variables remains unchanged. Note that each estimation proceeds by setting $\zeta_{\bar{k}}$ to one, however in this figure we have rescaled the non-homotheticity parameters by setting the ζ of the food sector to one.

Figure 9. Actual vs. Predicted budget shares by household total expenditure.



Notes: Each point represents the average expenditure share on a given sector by total expenditure bin. The data has been binned into 20 equally sized groups.

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[42] Note that we do not estimate the parameter η jointly with the ζ's because as the figure shows the estimation would demand an η that goes to zero and so the procedure is not well behaved.

Inner Preferences

Our quantitative exercise assumes an inner aggregator that takes the HARA form and is sector specific. The sectoral bundle for household i in sector k is given by

$$\mathcal{U}_k\left(\boldsymbol{c}_k(i)\right) = \frac{1}{a_k - 1} \int \left(b_k + a_k c_k(i,j)\right)^{\frac{a_k - 1}{a_k}} dj,$$

where $\{a_k, b_k\}$ are the two parameters that govern the HARA function. The optimal bundle of varieties given a total sectoral expenditure $e_k(i)$ is the solution to the following problem

$$\max_{\boldsymbol{c}_k(i)} \mathcal{U}_k\left(\boldsymbol{c}_k(i)\right) + \lambda_k(i)\left(e_k(i) - \int p_k(j) c_k(i,j) dj\right),$$

where $\lambda_k(i)$ is the Lagrange multiplier and is household-specific due to the fact that households have different expenditure levels. Taking the FOC of this problem and re-writing allows us to derive the HARA demand function as

$$c_k(i,j) = \frac{1}{a_k}\left(\left(\lambda_k(i) p_k(j)\right)^{-a_k} - b_k\right).$$

We can then use the definition of price elasticity $\epsilon_k(i) \equiv \frac{\partial \ln c_k(i,j)}{\partial \ln p_k(j)}$ and take the derivative of the previous expression to derive that the elasticity is equal to $a_k + \frac{b_k}{c_k(i)}$, as given in the main text. Since subvariety prices are all equal in equilibrium, the household will have the same elasticity of demand for all subvarieties and therefore we suppress the j in the notation. Nonetheless, if $b_k < 0$ households that spend more money on a given sector and therefore consume higher amounts will be less price elastic.

A few more lines of algebra allow us to derive the superelasticity for household i in sector k starting from its definition

$$\begin{aligned} \epsilon_k^s(i) &\equiv \frac{\partial \ln \epsilon_k(i)}{\partial \ln p_k(j)}, \\ &= -\frac{b_k}{c_k^2(i,j)} \frac{\partial c_k(i,j)}{\partial p_k(j)} \frac{p_k(j)}{\epsilon_k(i)}, \\ &= \frac{b_k}{c_k(i,j)}\left(-\frac{\partial c_k(i,j)}{\partial p_k(j)} \frac{p_k(j)}{c_k(i,j)}\right)\frac{1}{\epsilon_k(i)}, \\ &= \frac{b_k}{c_k(i,j)}. \end{aligned}$$

Given the household level elasticity and super-elasticity, we can derive the aggregate counterpart of these objects which will in turn determine the sectoral markup and price passthrough. To recover the aggregate elasticity we take the the average household elasticity, weighted by the expenditure shares to get that $\bar{\epsilon}_k = a_k + \frac{b_k}{C_k}$. Finally, to get the expression for the aggregate super-elasticity, we plug in the expressions for $\epsilon_k(i)$ and $\epsilon_k^s(i)$ in the formula[43] $\bar{\epsilon}_k^s = \left(-\int \left(\epsilon_k(i) - \bar{\epsilon}_k\right)^2 \frac{e_k(i)}{E_k} di + \int \frac{e_k(i)}{E_k} \epsilon_k^s(i) \epsilon_k(i) di\right) / \bar{\epsilon}_k$ and we get that $\bar{\epsilon}_k^s = \frac{b_k}{C_k}$. Note that these formulas are slightly different than the ones given in the main text where we use expenditure rather than actual consumption levels. Normalising the price to one is innocuous since the elasticity and super-elasticity values that are recovered for each household are independent of the assumed price level. The reason for this is that while we can recover a_k for the other coefficient we can only identify $\frac{b_k}{C_k} = \frac{\frac{1}{\text{passthrough}} - 1}{\text{markup} - 1}$. This is sufficient to get the household objects since with a slight re-writing we have that $\epsilon_k(i) = a_k + \left(\frac{e_k(i)}{E_K}\right)^{-1} \frac{b_k}{C_k}$ and $\epsilon_k^s(i) = \left(\frac{e_k(i)}{E_K}\right)^{-1} \frac{b_k}{C_k}$. The same is true for the markup sensitivity parameter which an application of the formula shows to be equal to $\gamma_{e,k}(i) = \left(1 - \frac{a_k}{\bar{\epsilon}_k}\right) \frac{1}{\bar{\epsilon}_k - 1}$.

Input-Output. To calibrate the parameters relating to the IO part of the model, we use the tables of intermediate input consumption provided by the ONS. These tables of input flows are constructed based on the CPA classification that defines 105 industries/products and which are different from the COICOP classification that we use in our model. To bridge this gap, we construct a mapping between the CPA classification and the COICOP one starting from the most disaggregated list of product classification (CPC10) of which there are more than 2000 products, although only 832 are for final consumption. The mapping consists in two steps. The first is to use the CPC10 to COICOP tables and assign weights to each product using the CPI weights available from ONS data. For example, if there are four CPC10 goods for a given COICOP category (we use the most disaggregated one for which we observe consumption weights) that has a weight of 1, each good will receive a weight of 0.25. Also note that the vast majority of CPC10 goods (more than 80%) map to a single COICOP category. Another 12% maps to two categories and only less than 5% maps to 3-5 COICOP categories.

Similarly in the other direction, we map the COICOP10 consumption goods to the CPA industry definitions using the con-

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[43] Note that this formula is valid for any demand system and can be derived directly from the definition of $\bar{\epsilon}_k^s$ as the *elasticity* of the aggregate elasticity with respect to its own price. Taking the derivative wrt price gives $\left(\frac{p_k(j)}{\bar{\epsilon}_k}\right)\left(\int\left(\partial_{p_k}\epsilon_k(i)\frac{e_k(i)}{E_k} + \epsilon_k(i)\left(\frac{\partial_{p_k} e_k(i)}{E_k} - \frac{e_k(i)\int \partial_{p_k} e_k(i) di}{E_k^2}\right)\right) di\right)$. Use the fact that $\partial_{p_k} e_k(i) = c_k(i)\left(1 - \epsilon_k(i)\right)$ and re-arrange to get the expression in the text.

cordance tables available from the UN's Statistics Division.[44] Unsurprisingly, the mapping of consumption goods to industries contains fewer one-to-one cases than with COICOP. Nonetheless, about 60% of goods only map to one or two CPA industries and another 30% map to 3 or 4.

Closed economy adjustment. The intermediate consumption tables provided by the ONS do not specify the share of inputs produced domestically vs what is imported. In our closed-economy world it must be the case that final demand (private consumption) plus intermediate consumption equals to total domestic output $[PY]$. To make this identity hold when we calibrate the model to the real-world data we adjust the vector of domestic total outputs with weights $\{\alpha_1, \alpha_2, \ldots, \alpha_K\}$ such that the following holds

$$[PC]_k + T\,[\alpha]_k = \mathcal{D}\,[\alpha]\,[PY]_k\,,$$

where the matrix T gives the flow of intermediate inputs and specifically $T_{i,j}$ is the amount of product i used in industry j.[45] This correction imposes that all production is done domestically (while not distorting the input mix used by different industries as given by T) and hence sectors in which the UK imports (exports) a lot will have a higher (lower) adjustment factor α.

Table **??** shows the IO matrix Ω for the eights sectors in our model. As is standard, we observe that sectors mostly tend to use goods produced by their own sector and so the diagonal entries dominate.

0.200	0.009	0.023	0.019	0.031	0.049	0.006	0.043
0.003	0.024	0.016	0.024	0.023	0.028	0.001	0.040
0.006	0.011	0.322	0.055	0.036	0.036	0.001	0.094
0.005	0.019	0.060	0.108	0.047	0.064	0.001	0.086
0.008	0.011	0.057	0.039	0.239	0.066	0.003	0.089
0.019	0.011	0.051	0.042	0.068	0.180	0.008	0.109
0.090	0.002	0.043	0.007	0.008	0.014	0.014	0.029
0.005	0.010	0.055	0.029	0.042	0.073	0.007	0.239

D.1 Model without heterogeneity in price stickiness and markups, and without I-O linkages

The baseline model includes various features other than non-homotheticities. In this appendix we study their quantitative importance. Specifically, we shut down sectoral heterogeneity in prices stickiness and steady-state markups, as well as Input-Output linkages. Concretely, we achieve this by targeting in the calibration the (unweighted) average markup across sectors, setting all Calvo parameters equal to the average across sectors, and by setting intermediate input shares to zero.

Figure 10 shows impulse responses under a Taylor rule. As shown by the figure, we preserve the key result that the output gap declines in the two necessity sectors: Food and Electricity & Gas. In sector Transport, the output gap now increases. The increase observed in the baseline model is thus driven by the features that we shut down in this appendix. This is consistent with the fact that Transport is neither a luxury nor a necessity sector (the luxury index equals zero for this sector).

Figure 11 shows the Guidance experiment under the simplified model calibration. The figure shows that the key result, that monetary policy is relatively loose in response to shocks in necessity sectors (Food and Electricity & Gas) is preserved.

Overall these results underscore the importance of non-homotheticities and show that our main results in the baseline model are not driven by sectoral heterogeneity in price setting, markups and I-O linkages.

D.2 Implementing optimal policy with a Taylor rule plus guidance

In this appendix, we show how we back out the "policy guidance" in the exercise of Section 5.3. Guidance is defined as a series of interest rate rule residuals, $\{u^R_{t+s}\}_{s=0}^{\infty}$, where $u^R_{t+s} = \hat{R_{t+s}} - \phi\pi_{t+s}$. These residuals are announced at the moment a certain shock hits (this could be e.g. a sectoral or aggregate productivity shock). The guidance may varies across shocks.

Our goal is to solve for the guidance which, for a certain shock, implements the optimal policy. Let IRF_{OP} be a column vector containing the Impulse Response Function (IRF) of some variable under optimal monetary policy, IRF_{TR} the IRF under a Taylor rule, and $IRF_{MP(s)}$ be the IRF to a purely transitory, unit news shock to the Taylor rule, hitting at date s and announced at date 0. We want to solve for $\{u^R_{t+s}\}_{s=0}^{S-1}$ such that

$$IRF_{OP} = IRF_{TR} + u^R_{t+s}\sum_{s=0}^{S} IRF_{MP(s)} = IRF_{TR} + \boldsymbol{IRF}_{MP}\boldsymbol{u}$$

where S is a truncation date, $\boldsymbol{IRF}_{MP}$ is an $S \times S$ matrix containing the IRFs to the monetary policy shocks on its columns, and $\boldsymbol{u^R}$ is a column vector containing the guidance. We solve for the guidance vector as:

$$\boldsymbol{u^R} = \boldsymbol{IRF}_{MP}^{-1}\left(IRF_{OP} - IRF_{TR}\right).$$

In our implementation, we use the IRF of CPI inflation to aggregate and sector-level shocks. We set the truncation horizon to 75 quarters. We verify ex post that the IRFs of variables are close to identical under optimal policy and the interest rate rule plus guidance.

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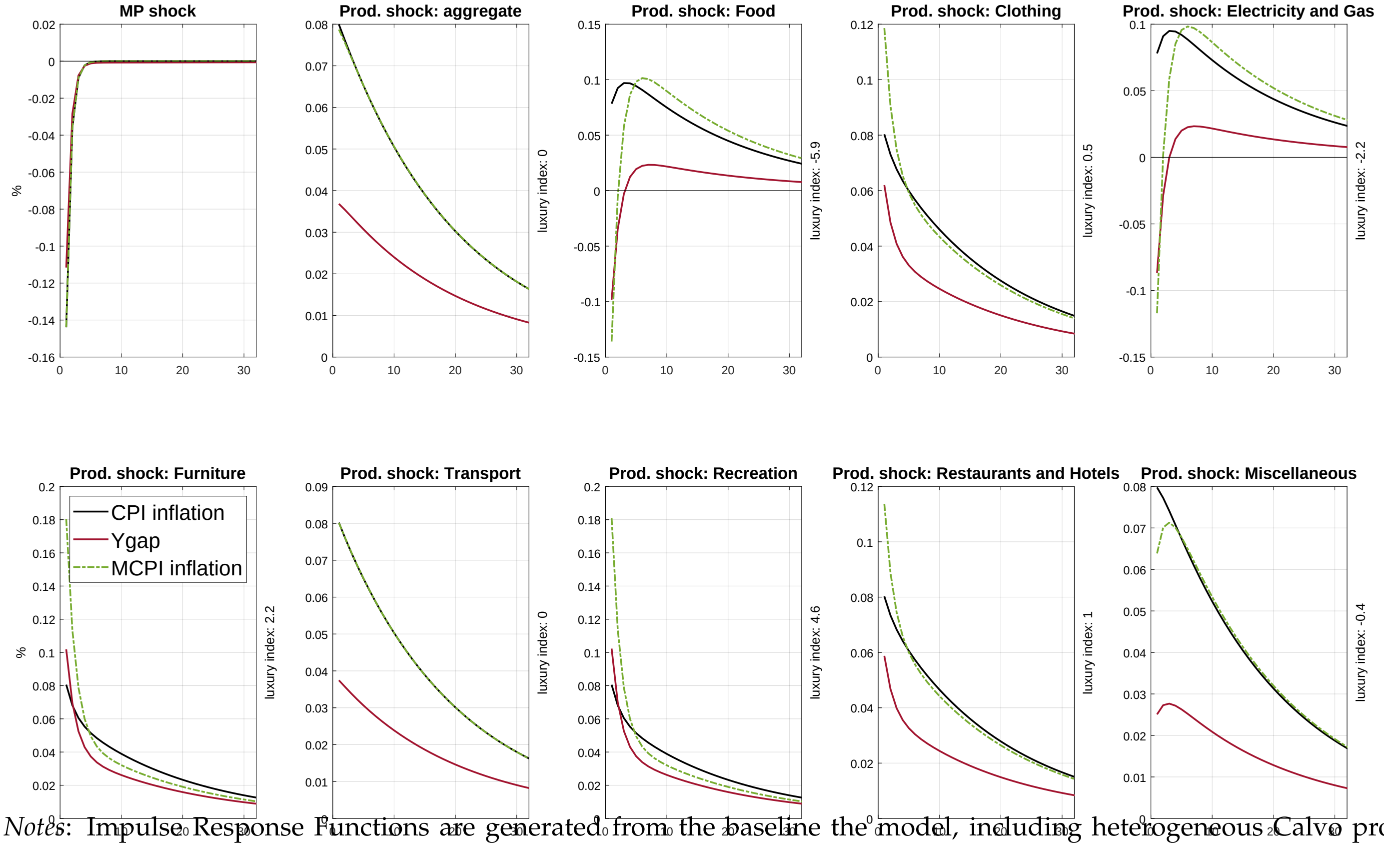
[44] Note that this has to be done in a few steps that consists of the following chain of mapping CPC10 → ISIC3 → ISIC3.1 → ISIC4 → NACE2. That final classification contains 626 categories that can be aggregated to the 105 sectors used in the UK's IO tables.

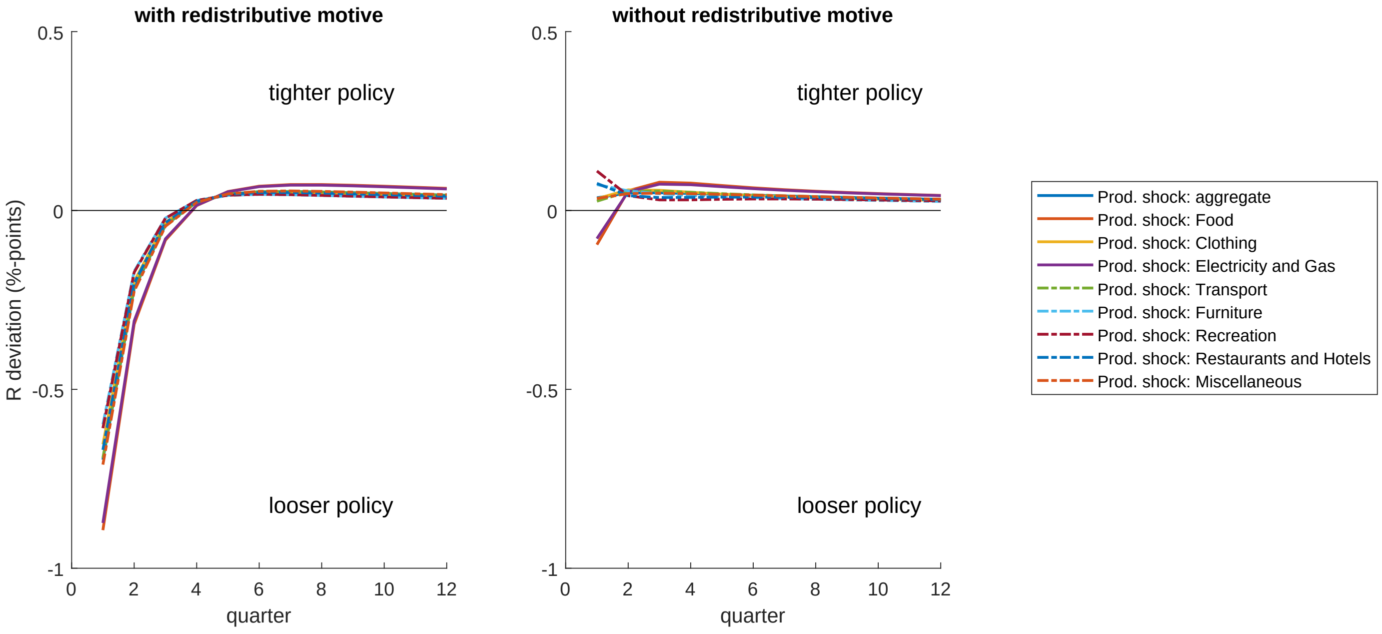
[45] Note that in terms of the Ω matrix one can write the flow matrix as $T = \left(\mathcal{D}\,[PY]\,\Omega\right)^T$.

Figure 10. Responses in the baseline model without heterogeneity in prices stickiness and steady-state markups across sectors, and without Input-Output linkages.



Notes: Impulse Response Functions are generated from the baseline the model, including heterogeneous Calvo probabilities across sectors across sectors, Input-Output linkages, and Hand-to-Mouth households. Responses for productivity shocks are for a 1 percent decline in productivity where scaled for comparability (see main text). On the right axis, the luxury index is defined as $100(\overline{\partial_e e_l} - \bar{s}_k)$.

Figure 11. Optimal policy relative to Taylor rule in the model without heterogeneity in prices stickiness and steady-state markups across sectors, and without Input-Output linkages.



Notes: Deviations from the Taylor rule $\hat{R}_t = 1.5\pi_{cpi,t}$ which implement Optimal Policy ("optimal guidance"). Higher values mean that optimal monetary policy is tight relative to this rule. See the main text for details. All productivity shocks are negative.

E Optimal Policy

E.1 Optimal policy: derivations

As noted in the main text, the Central Bank (CB) values the utility of households according to the social welfare function $\mathcal{W}$ defined has:

$$\mathcal{W} = (1-\delta) \int G\left(V^{-}(i), i\right) di + \delta \mathbb{E}_0 \sum_{t_0=0}^{\infty} \beta^{t_0} \int G\left(V^{t_0}(i), i\right) di$$

Here, a superscript t_0 denotes the birth date of a cohort (within a household type i) and a superscript $_-$ denotes cohorts born before $t = 0$.[46] The value of a cohort t_0 in type i is given by:

$$V^{t_0}(i) = \mathbb{E}_{t_0} \sum_{s=0}^{\infty} \left((1-\delta)\beta\right)^s \left\{ (1-\varphi(i)) \left[U_i\left(\mathcal{U}_1\left(\boldsymbol{c}^{t_0,u}_{1,t_0+s}(i)\right), \ldots \mathcal{U}_K\left(\boldsymbol{c}^{t_0,u}_{K,t_0+s}(i)\right)\right) - \chi\left(\frac{n^{t_0,u}_{t_0+s}(i)}{\vartheta(i)}\right)\right] + \varphi(i) \left[U_i\left(\mathcal{U}_1\left(\boldsymbol{c}^{t_0,HtM}_{1,t_0+s}(i)\right), \ldots \mathcal{U}_K\left(\boldsymbol{c}^{t_0,HtM}_{K,t_0+s}(i)\right)\right) - \chi\left(\frac{n^{t_0,HtM}_{t_0+s}(i)}{\vartheta(i)}\right)\right]\right\}.$$

and note that within each cohort/type a fraction $\varphi(i)$ is HtM, and recall that non-HtM households are denoted by a superscript u. The value of pre-existing cohorts, $V_-(i)$, is defined analogously. The CB maximizes $\mathcal{W}$ under the following set of constraints (for any i, j, k, t, t_0):

- Optimality of intratemporal consumption decisions

  $$c^{t_0,h}_{k,t}(i,j) = d_k\left(p_{k,t}(j), \boldsymbol{p}_{k,t}, e^*_k\left(e^{t_0,h}_t(i), \boldsymbol{P}_t\right)\right)$$

  $$v_i\left(e^{t_0,h}_t(i), \boldsymbol{P}\right) = U_i\left(\mathcal{U}_1\left(\boldsymbol{d}_1\left(p_{1,t}(j), \boldsymbol{p}_{1,t}, e^*_1\left(e^{t_0,h}_t(i), \boldsymbol{P}_t\right)\right)\right), \ldots \mathcal{U}_K\left(\boldsymbol{d}_K\left(p_{K,t}(j), \boldsymbol{p}_{K,t}, e^*_K\left(e^{t_0,h}_t(i), \boldsymbol{P}_t\right)\right)\right)\right)$$

  for $h \in \{u, HtM\}$. Here, d_k and e^*_k are the solutions of the inner and outer consumption problem defined in the previous sections.

- Optimality of labor supply decisions, for $h \in \{u, HtM\}$:

  $$\chi'\left(\frac{n^{t_0,h}_t(i)}{\vartheta(i)}\right)\frac{1}{\vartheta(i)} = W_t \partial_e v_{i,t}\left(e^{t_0,h}_t(i), \boldsymbol{P}\right).$$

- Optimality of intertemporal expenditure decisions for non-HtM households (Euler equation and budget constraint):

  $$\partial_e v_{i,t}\left(e^{t_0,u}_t(i), \boldsymbol{P}_t\right) = \beta(1-\delta)R_t \mathbb{E}_t\left[\partial_e v_{i,t+1}\left(e^{t_0,u}_{t+1}(i), \boldsymbol{P}_t\right)\right],$$

  $$\frac{b^{t_0,u}_{t+1}(i)}{R_t} = b^{t_0,u}_t(i) + n^{t_0,u}_t(i) W_t + \sum_k \varsigma_k(i) Div_{k,t} - e^{t_0,u}_t(i),$$

  with $b^{t_0,u}_{t_0}(i) = b^{t_0,HtM}_{t_0}(i) = \left(1 + \sum_l \bar{s}_l \left(\frac{P_{l,t_0} - P_{l,-}}{P_{l,-}}\right)\right) b^-_0(i)$.

- HtM consumption:

  $$\left(\frac{1}{R_t} - 1\right) b^{t_0,HtM}_{t_0}(i) = n^{t_0,HtM}_t(i) W_t + \sum_k \varsigma_k(i) Div_{k,t} - e^{t_0,HtM}_t(i).$$

- Optimal Price resetting:

  $$\mathbb{E}_t \sum_{s=0}^{\infty} \tilde{\beta}^s \theta^s_k \left(D_{k,t+s}\left(p^*_{k,t}(j^*), \boldsymbol{p}_{k,t+s}, \boldsymbol{e}_{k,t+s}, \tilde{Y}_{k,t+s}\right) + \left(p^*_{k,t}(j^*) - (1-\tau_k) MC_{k,t+s}(j^*)\right) \partial_p D_{k,t+s}\left(p^*_{k,t}(j^*), \boldsymbol{p}_{k,t+s}, \boldsymbol{e}_{k,t+s}, \tilde{Y}_{k,t+s}\right)\right) = 0$$

  where the aggregate demand for subvarieties is defined in the previous section.

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[46] Note that it would be equivalent – to a first order approximation – to differentiate households born before t_0 according to their date of birth, that is consider the social welfare function $\mathcal{W} = \delta \mathbb{E}_0 \sum_{t_0=-\infty}^{\infty} \beta^{t_0} \int G\left(V^{t_0}(i), i\right) di$.

- Labor market clearing:

$$(1-\delta)^{t+1}\int(1-\varphi(i))\,n_t^{-,u}(i)+\varphi(i)\,n_t^{-,HtM}(i)di+\delta\sum_{t_0}(1-\delta)^{t-t_0}\int(1-\varphi(i))\,n_t^{t_0,u}(i)+\varphi(i)\,n_t^{t_0,HtM}(i)di=\sum_{k=1}^{K}\mathcal{N}_k(\boldsymbol{P}_t,W_t)\int\frac{D_{k,t}\left(p_{k,t}(j),\boldsymbol{p}_{k,t},\boldsymbol{e}_{k,t},\tilde{Y}_{k,t}\right)}{A_{k,t}}dj$$

The firm optimal choice of input, the market clearing conditions for intermediate goods and consumption goods and the government budget constraint will be used implicitly.

We denote by $\mathbb{E}_{\delta,t}\left(X_t^{t_0}\right)\equiv(1-\delta)^{t+1}X_t^{-}+\delta\sum_{t_0=0}^{t}(1-\delta)^{t-t_0}X_t^{t_0}$ the inter-generational average of variable $X_t^{t_0}$ at t. We further denote by $\check{\Xi}_t$ and $\tilde{\mu}_{k,t}$ the Lagrange multipliers on the labor market clearing constraint and optimal price setting constraints, and by $\check{\lambda}_t^{t_0}(i)$, $\check{\zeta}_t^{t_0,u}(i)$,$\check{\zeta}_t^{t_0,HtM}(i)$, $\check{\alpha}_t^{t_0}(i)$ and $\check{\aleph}_t^{t_0}(i)$, the Lagrange multipliers on the Euler equation of unconstrained households ($\check{\lambda}_t^{t_0}(i)$), on the optimality of labor supply decisions ($\check{\zeta}_t^{t_0,u}(i)$, $\check{\zeta}_t^{t_0,HtM}(i)$) and on the budget constraints of households ($\check{\alpha}_t^{t_0}(i)$ for unconstrained households and $\check{\aleph}_t^{t_0}(i)$ for HtM households), The Lagrangian of the optimal policy problem is:

$$(1-\delta)\int\frac{1}{E}G\left(V^{-}(i)(i),i\right)di+\delta\mathbb{E}_0\sum\beta^{t_0}\int G\left(V^{t_0}(i),i\right)di$$

$$+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\,\partial_e v_{t,i}\left(e_t^{t_0,u}(i),\boldsymbol{P}\right)\left(\check{\lambda}_t^{t_0}(i)-R_{t-1}\check{\lambda}_{t-1}^{t_0}(i)\right)di$$

$$+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\,\check{\zeta}_t^{t_0,u}(i)\left(W_t\partial_e v_{t,i}\left(e_t^{t_0,u}(i),\boldsymbol{P}\right)-\chi'\left(\frac{n_t^{t_0,u}(i)}{\vartheta(i)}\right)\frac{1}{\vartheta(i)}\right)di+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\varphi(i)\,\check{\zeta}_t^{t_0,HtM}(i)\left(W_t\partial_e v_{t,i}\left(e_t^{t_0,HtM}(i),\boldsymbol{P}\right)-\chi'\left(\frac{n_t^{t_0,HtM}(i)}{\vartheta(i)}\right)\frac{1}{\vartheta(i)}\right)di$$

$$+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\beta^t\int(1-\varphi(i))\,\check{\alpha}_t^{t_0}(i)\left(\frac{b_{t+1}^{t_0,u}(i)}{R_t}-\left(b_t^{t_0,u}(i)+n_t^{t_0,u}(i)W_t+\sum_k\varsigma_k(i)Div_{k,t}-e_t^{t_0,u}(i)\right)\right)di$$

$$+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\check{\aleph}_t^{t_0}(i)\,\varphi(i)\left(\left(\frac{1}{R_t}-1\right)b_{t_0}^{t_0,HtM}(i)-\left(n_t^{t_0,HtM}(i)W_t+\sum_k\varsigma_k(i)Div_{k,t}-e_t^{t_0,HtM}(i)\right)\right)$$

$$+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\check{\Xi}_t W_t\left((1-\delta)^{t+1}\int(1-\varphi(i))\,n_t^{-,u}(i)+\varphi(i)\,n_t^{-,HtM}(i)di+\delta\sum_{t_0}(1-\delta)^{t-t_0}\int(1-\varphi(i))\,n_t^{t_0,u}(i)+\varphi(i)\,n_t^{t_0,HtM}(i)di-\sum_{k=1}^{K}\mathcal{N}_k(\boldsymbol{P}_t,W_t)\int\frac{D_{k,t}\left(p_{k,t}(j),\boldsymbol{p}_{k,t},\boldsymbol{e}_{k,t},\tilde{Y}_{k,t}\right)}{A_{k,t}}dj\right)$$

$$+\sum_{k=1}^{K}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\tilde{\mu}_{k,t}\left(\sum_{s=0}^{\infty}\tilde{\beta}^s\theta_k^s\left(D_{k,t+s}\left(p_{k,t}^*(j^*),\boldsymbol{p}_{k,t+s},\boldsymbol{e}_{k,t+s},\tilde{Y}_{k,t+s}\right)+\left(p_{k,t}^*(j^*)-(1-\tau_k)MC_{k,t+s}(j^*)\right)\partial_p D_{k,t+s}\left(p_{k,t}^*(j^*),\boldsymbol{p}_{k,t+s},\boldsymbol{e}_{k,t+s},\tilde{Y}_{k,t+s}\right)\right)\right)$$

First-order conditions

Let us consider a steady state in which the CB targets zero inflation (and all goods prices and wages are constant), setting $R_t=1/\tilde{\beta}$. Recall also that we normalized $A_{k,t}=1$ and that we assumed that elasticities of substitution across varieties are equal for households and intermediate input producers, i.e. $\frac{P_k\partial_p D_k^C}{D_k^C}=\frac{P_k\partial_p D_k^{\mathcal{I}}}{D_k^{\mathcal{I}}}=-\bar{\epsilon}_k$. In such a steady state, wealth, expenditure and labor supply of households is constant across time and identical for unconstrained and HtM households of the same type i. We first show that, given the presence of a subsidy undoing markups, $(1-\tau_k)\frac{\bar{\epsilon}_k}{\bar{\epsilon}_k-1}=1$, and the first assumption on the social welfare function, $G'\left(V_{t_0}(i),i\right)\partial_e v(i)=1$, this steady state is efficient. We do so by first showing that the first-order conditions to the optimal policy problem hold at the steady state.[47] After doing so, we perturb the first-order conditions around the steady state, in order to solve for the optimal dynamics.

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[47] When we derive the loss function, we also show that the second-order conditions are satisfied.

- First-order conditions for $b_t^{t_0,u}(i)$:

$$\tilde{\beta}\check{\alpha}_t^{t_0}\left(i\right) = \frac{1}{R_{t-1}}\check{\alpha}_{t-1}^{t_0}\left(i\right).$$
$$\Rightarrow \check{\alpha}_t^{t_0}\left(i\right) = \check{\alpha}^{t_0}\left(i\right)$$

where the second line gives the necessary optimality condition in a steady state with constant prices and $R_t = 1/\tilde{\beta}$.

- First-order conditions for the interest rate, R_t :

$$-\beta\mathbb{E}_{\delta,t+1}\left(\int\left(1-\varphi\left(i\right)\right)\check{\lambda}_t^{t_0}\left(i\right)\partial_e v_{t+1,i}\left(e_{t+1}^{t_0,u}\left(i\right),\boldsymbol{P}\right)di\right) - \mathbb{E}_{\delta,t}\left(\int\left(1-\varphi\left(i\right)\right)\check{\alpha}_t^{t_0}\left(i\right)\frac{1}{R_t^2}b_{t+1}^{t_0,u}\left(i\right)di\right) - \mathbb{E}_{\delta,t}\left(\int\varphi\left(i\right)\check{\aleph}_t^{t_0}\left(i\right)\frac{1}{R_t^2}b_{t_0}^{t_0,HtM}\left(i\right)di\right) = 0$$
$$\Rightarrow -\mathbb{E}_{\delta,t}\left(\int\left(1-\varphi\left(i\right)\right)\check{\lambda}_t^{t_0}\left(i\right)\partial_e v_i\left(e\left(i\right),\boldsymbol{P}\right)di\right) - \mathbb{E}_{\delta,t}\left(\int\left(1-\varphi\left(i\right)\right)\check{\alpha}^{t_0}\left(i\right)\frac{b\left(i\right)}{R}di\right) - \mathbb{E}_{\delta,t}\left(\int\varphi\left(i\right)\check{\aleph}_t^{t_0}\left(i\right)\frac{1}{R}b\left(i\right)di\right) = 0$$

where the second line gives the necessary optimality condition in a steady state with constant prices and $R_t = 1/\tilde{\beta}$ (so wealth is constant across time and generations)

- First-order conditions for W_t. Denoting as before $\mathcal{Q}_{l,k} = \mathcal{Y}_{l,k}\frac{Y_k}{Y_l A_k}$ the matrix of intermediate shares, we have:

$$\begin{aligned}
0 = {} & \mathbb{E}_{\delta,t}\int\left(1-\varphi\left(i\right)\right)\check{\zeta}_t^{t_0,u}\left(i\right)\partial_e v_{t,i}\left(e_t^{t_0,u}\left(i\right),\boldsymbol{P}_t\right)di + \mathbb{E}_{\delta,t}\int\varphi\left(i\right)\check{\zeta}_t^{t_0,HtM}\left(i\right)\partial_e v_{t,i}\left(e_t^{t_0,HtM}\left(i\right),\boldsymbol{P}_t\right)di \\
& - \sum_{k=1}^{K}(1-\tau_k)\sum_{s=0}^{t}\left(\left(1-\delta\right)\theta_k\right)^{t-s}\tilde{\mu}_{k,s}\mathcal{N}_k(\boldsymbol{P}_t,W_t)\partial_p D_{k,t}\left(p_{k,s}^*(j^*),\boldsymbol{p}_{k,t},\boldsymbol{e}_{k,t},\tilde{Y}_{k,t}\right) \\
& + \sum_{k=1}^{K}\sum_{s=0}^{t}\left(\left(1-\delta\right)\theta_k\right)^{t-s}\tilde{\mu}_{k,s}\left[d_k^{\mathcal{I}}Y_k + \left(p_{k,s}^*(j^*) - (1-\tau_k)MC_{k,t}(j^*)\right)\partial_p d_k^{\mathcal{I}}\left(p_{k,s}^*(j^*),\boldsymbol{p}_{k,t}\right)Y_k\right]\left(Id-Q\right)^{-1}\left[\sum_k\frac{\partial\mathcal{Y}_{1,k}}{\partial W}\frac{Y_k}{A_{k,t}Y_1},\ldots,\sum_k\frac{\partial\mathcal{Y}_{K,k}}{\partial W}\frac{Y_k}{A_{k,t}Y_K}\right]^T \\
& - \mathbb{E}_{\delta,t}\beta^t\int\left(1-\varphi\left(i\right)\right)\check{\alpha}_t^{t_0}\left(i\right)\left(n_t^{t_0,u}(i) - \varsigma(i)\sum_k\frac{\mathcal{N}_k(\boldsymbol{P}_t,W_t)Y_{k,t}}{A_{k,t}}\right)di - \mathbb{E}_{\delta,t}\int\check{\aleph}_t^{t_0}\left(i\right)\varphi\left(i\right)\left(\left(n_t^{t_0,HtM}(i) - \sum_k\varsigma(i)\frac{\mathcal{N}_k(\boldsymbol{P}_t,W_t)Y_{k,t}}{A_{k,t}}\right)\right) \\
& - \mathbb{E}_{\delta,t}\int\left(\left(1-\varphi\left(i'\right)\right)\check{\alpha}_t^{t_0}\left(i'\right) + \varphi\left(i'\right)\check{\aleph}_t^{t_0}\left(i'\right)\right)\varsigma(i')\sum_k\int d_k^I\partial_W\tilde{Y}_{k,t}\left(p_{k,t}(j) - MC_{k,t}\right)djdi' \\
& - \check{\Xi}_t\sum_{k=1}^{K}\partial_W\mathcal{N}_k(\boldsymbol{P}_t,W_t)\frac{Y_k}{A_{e_k,t}} - \check{\Xi}_t\left[\frac{\mathcal{N}_1(\boldsymbol{P}_t,W_t)Y_1}{A_{1,t}},\ldots,\frac{\mathcal{N}_K(\boldsymbol{P}_t,W_t)Y_K}{A_{K,t}}\right]\left(Id-Q\right)^{-1}\left[\sum_k\frac{\partial\mathcal{Y}_{1,k}}{\partial W}\frac{Y_k}{A_{k,t}Y_1},\ldots,\sum_k\frac{\partial\mathcal{Y}_{K,k}}{\partial W}\frac{Y_k}{A_{k,t}Y_K}\right]^T,
\end{aligned}$$

where the change in demand for intermediary in response to a change in wage solves:

$$\partial_W\tilde{Y}_{l,t} = \sum_k\partial_W\mathcal{Y}_{l,k}(\boldsymbol{P_t},W_t)Y_{k,t} + \sum_k\mathcal{Y}_{l,k}(\boldsymbol{P_t},W_t)\int d_{k,t}^I\left(j\right)dj\partial_W\tilde{Y}_{k,t}.$$

We use the market clearing condition for intermediary and the optimal input demand from firms to obtain the expression on the last line. Using the fact that subvariety prices are constant and equal, that $\frac{P_k\partial_p D_k^C}{D_k^C} = \frac{P_k\partial_p D_k^{\mathcal{I}}}{D_k^{\mathcal{I}}} = -\bar{\epsilon}_k$ and $\left(1-\tau_k\right)\frac{\bar{\epsilon}_k}{\bar{\epsilon}_k-1} = 1$, we can use:

$$d_k^{\mathcal{I}}Y_k + \left(p_{k,s}^*(j^*) - (1-\tau_k)MC_{k,t+s}(j^*)\right)\partial_p d_k^{\mathcal{I}}\left(p_{k,s}^*(j^*),\boldsymbol{p}_{k,t}\right)Y_k = 0 \quad \forall k$$
$$\left[W\mathcal{N}_1(\boldsymbol{P},W)Y_1,\ldots,W\mathcal{N}_K(\boldsymbol{P},W)Y_K\right]\left(Id-Q\right)^{-1} = \left[P_1Y_1,\ldots,P_KY_K\right]$$
$$W_t\partial_W\mathcal{N}_k(\boldsymbol{P},W) + \sum_l P_l\frac{\partial\mathcal{Y}_{l,k}}{\partial W} = 0 \quad \forall k.$$

In addition, we define $\check{\mu}_{k,T}$ which constrains the growth rate of sectoral inflation: :

$$\check{\mu}_{k,T} \equiv \frac{\theta_k}{1-\theta_k} \sum_{t=0}^{T} \left((1-\delta)\,\theta_k\right)^{T-t} \tilde{\mu}_{k,t} \sum_{s=0}^{\infty} \tilde{\beta}^s \theta_k^s \left(2p_{k,t}(j^*)\partial_p D_{k,t+s} + \left(p_{k,t}(j^*) - (1-\tau_k)MC_{k,t+s}(j^*)\right) p_{k,t}(j^*)\partial_{pp} D_{k,t+s}\right)$$

and note that around our steady steady state:

$$\begin{aligned}
\check{\mu}_{k,T} &= \frac{\theta_k}{(1-\theta_k)\left(1-\tilde{\beta}\theta_k\right)} \left(2P_k\partial_p D_{k,} + \left(P_k - (1-\tau_k)MC_k\right) P_k \partial_{pp} D_k\right) \sum_{t=0}^{T} \left((1-\delta)\,\theta_k\right)^{T-t} \tilde{\mu}_{k,t} \\
&= P_k \partial_p D_k \frac{\bar{\epsilon}_k - 1}{\bar{\epsilon}_k} \frac{1}{\lambda_k} \sum_{t=0}^{T} \left((1-\delta)\,\theta_k\right)^{T-t} \tilde{\mu}_{k,t}.
\end{aligned}$$

Using this, we rewrite the first order condition as:

$$\begin{aligned}
0 = \mathbb{E}_{\delta,t} \int (1-\varphi(i))\, \check{\zeta}_t^{t_0,u}(i)\, \partial_e v_{t,i}^{u,t_0} di + \mathbb{E}_{\delta,t} \int \varphi(i)\, \check{\zeta}_t^{t_0,HtM}(i)\, \partial_e v_{t,i}^{HtM,t_0} di - \sum_{k=1}^{K} \lambda_k \check{\mu}_{k,s} \mathcal{N}_k(\boldsymbol{P}_t, W_t) \\
- \mathbb{E}_{\delta,t} \beta^t \int (1-\varphi(i))\, \check{\alpha}_t^{t_0}(i) \left(n(i) - \varsigma(i) \sum_k \mathcal{N}_k(\boldsymbol{P}_t, W_t) Y_{k,t}\right) di - \mathbb{E}_{\delta,t} \int \check{\aleph}_t^{t_0}(i)\, \varphi(i) \left(\left(n(i) - \sum_k \varsigma_k(i) \mathcal{N}_k(\boldsymbol{P}_t, W_t) Y_{k,t}\right)\right) di.
\end{aligned}$$

Using the fact that $\varsigma(i) = n(i)/N$, the steady state equation is:

$$0 = \mathbb{E}_{\delta,t} \int (1-\varphi(i))\, \check{\zeta}_t^{t_0,u}(i)\, \partial_e v di + \mathbb{E}_{\delta,t} \int \varphi(i)\, \check{\zeta}_t^{t_0,HtM}(i)\, \partial_e v di - \sum_{k=1}^{K} \lambda_k \check{\mu}_{k,s} \mathcal{N}_k(\boldsymbol{P}_t, W_t).$$

- First order conditions with respect to labor supply, $n_t^{t_0,u}(i)$ and , $n_t^{t_0,HtM}(i)$:

$$\begin{aligned}
&\check{\zeta}_t^{t_0,u}(i)\, \partial_e v_{t,i}^{u,t_0} = \psi n_t^{t_0,u}(i) \left\{\check{\Xi}_t - \check{\alpha}_t^{t_0}(i) - G'(V_{t_0}(i)) \partial_e v_{t,i}^{u,t_0}\right\}. \\
&\Rightarrow \check{\zeta}_t^{t_0,u}(i)\, \partial_e v = \psi n(i) \left\{\check{\Xi}_t - \check{\alpha}^{t_0}(i) - 1\right\} \\
&\check{\zeta}_t^{t_0,HtM}(i)\, \partial_e v_{t,i}^{HtM,t_0} = \psi n_t^{t_0,HtM}(i) \left\{\check{\Xi}_t - \check{\aleph}_t^{t_0}(i) - G'(V_{t_0}(i)) \partial_e v_{t,i}^{HtM,t_0}\right\} \\
&\Rightarrow \check{\zeta}_t^{t_0,HtM}(i)\, \partial_e v = \psi n(i) \left\{\check{\Xi}_t - \check{\aleph}_t^{t_0}(i) - 1\right\}
\end{aligned}$$

where we used the definition $\psi = \chi'\left(\frac{n_t^{t_0,u}(i)}{\vartheta(i)}\right) / \left(\frac{n_t^{t_0,u}(i)}{\vartheta(i)} \chi''\left(\frac{n_t^{t_0,u}(i)}{\vartheta(i)}\right)\right)$ and the optimality of labor supply decisions, and the second and fourth line are the steady state equations.

- First order condition with respect to expenditure of the non-HtM, $e_t^u(i)$:

$$\begin{aligned}
&G'\left(V_{t_0}(i), i\right)\partial_e v_{t,i}^{u,t_0} + \left(\check{\lambda}_t^{t_0}(i) - R\check{\lambda}_{t-1}^{t_0}(i)\right)\partial_{ee} v_{t,i}^{u,t_0} + \check{\zeta}_t^{t_0}(i)\, W_t \partial_{ee} v_{t,i}^{u,t_0} \\
&\quad + \check{\alpha}_t^{t_0}(i) - \mathbb{E}_{\delta,t}\int \left(\left(1-\varphi(i')\right)\check{\alpha}_t^{t_0}(i') + \varphi(i')\,\check{\aleph}_t^{t_0}(i')\right)\varsigma(i')\sum_k \int \left(\partial_e e_k \partial_e d_k + d_k^I \partial_e \tilde{Y}_k\right)\left(p_{k,t}(j) - MC_{k,t}\right) dj di' \\
&\quad - \check{\Xi}_t \sum_{k=1}^{K} \frac{\mathcal{N}_k(\boldsymbol{P}_t, W_t)}{A_k}\int \left(\partial_e e_k \partial_e d_k + d_k^I \partial_e \tilde{Y}_k\right) dj \\
&\quad + \sum_{k=1}^{K}\sum_{s=0}^{t}\left((1-\delta)\theta_k\right)^{t-s}\tilde{\mu}_{k,s}\left(\partial_e d_{k,t}\left(p_{k,s}^*(j^*), \boldsymbol{p}_{k,t}, e_{k,t}\right) + \left(p_{k,s}^*(j^*) - (1-\tau_k)MC_{k,t}(j^*)\right)\partial_{pe} d_{k,t}\left(p_{k,s}^*(j^*), \boldsymbol{p}_{k,t}, e_{k,t}\right)\right) \\
&\quad + \sum_{k=1}^{K}\sum_{s=0}^{t}\left((1-\delta)\theta_k\right)^{t-s}\tilde{\mu}_{k,s}\left[d_k^{\mathcal{I}} + \left(p_{k,s}^*(j^*) - (1-\tau_k)MC_{k,t+s}(j^*)\right)\partial_p d_k^{\mathcal{I}}\left(p_{k,s}^*(j^*), \boldsymbol{p}_{k,t}\right)\right]\partial_e \tilde{Y}_k = 0
\end{aligned}$$

With some abuse of notation, $\partial_e \tilde{Y}_k$ is the Gateaux derivative (keeping prices fixed) of demand for intermediary output with respect to a change in $e_t^{t_0,u}(i)$. Note that we use $(1-\tau_k)\frac{\bar{\epsilon}_k}{\bar{\epsilon}_k - 1} = 1$ and the adjustment of the lump sum tax to express the total change in dividends. We have, denoting $\tilde{Q}_{l,k} = \mathcal{Y}_{l,k}/A_k$:

$$\left[\partial_e \tilde{Y}_1, ..., \partial_e \tilde{Y}_K\right]^T = (Id - \tilde{Q})^{-1}\left[\sum_k \mathcal{Y}_{1,k}(\boldsymbol{P}, W)\int \partial_e e_k \partial_e d_k(i,j) dj, ..., \sum_k \mathcal{Y}_{K,k}(\boldsymbol{P}, W)\int \partial_e e_k \partial_e d_k(i,j) dj\right]^T = (Id - \tilde{Q})^{-1}\tilde{Q}\left[\int \left(\partial_e e_1 \partial_e d_1\right) dj, ..., \int \left(\partial_e e_K \partial_e d_K\right) dj\right]^T,$$

$$\sum_{k=1}^{K}\frac{\mathcal{N}_k(\boldsymbol{P}_t, W_t)}{A_k}\int \left(\partial_e e_k \partial_e d_k + d_k^I \partial_e \tilde{Y}_k\right) dj = \sum_{k=1}^{K}\frac{\mathcal{N}_k(\boldsymbol{P}_t, W_t) + \sum_l \mathcal{Y}_{l,k}(\boldsymbol{P}_t, W_t)}{A_k}\int \left(\partial_e e_k \partial_e d_k\right) dj = 1.$$

Simplifying we have, in steady state:

$$1 + \left(\check{\lambda}_t^{t_0}(i) - R\lambda_{t-1}^{t_0}(i)\right)\partial_{ee} v + \check{\zeta}_t^{t_0,u}(i)\, W \partial_{ee} v + \check{\alpha}^{t_0}(i) - \check{\Xi}_t - \sum_{k=1}^{K}\frac{1}{P_k Y_k}\lambda_k \check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i) = 0.$$

Following the same steps, the first order conditions for the expenditure of HtM households, $e_t^{HtM}(i)$, is

$$\begin{aligned}
&G'\left(V_{t_0}(i), i\right)\partial_e v_{t,i}^{HtM,t_0} + \check{\zeta}_t^{t_0,u}(i)\, W_t \partial_{ee} v_{t,i}^{HtM,t_0} + \check{\aleph}_t^{t_0}(i) - \mathbb{E}_{\delta,t}\int \left(\left(1-\varphi(i')\right)\check{\alpha}_t^{t_0}(i') + \varphi(i')\,\check{\aleph}_t^{t_0}(i')\right)\varsigma(i')\sum_k \int \left(\partial_e e_k \partial_e d_k + d_k^I \partial_e \tilde{Y}_k\right)\left(p_{k,t}(j) - MC_{k,t}\right) dj di' \\
&- \check{\Xi}_t \sum_{k=1}^{K}\frac{\mathcal{N}_k(\boldsymbol{P}_t, W_t)}{A_{e_k,t}}\int \left(\partial_e e_k \partial_e d_k + d_k^I \partial_e \tilde{Y}_k\right) dj + \sum_{k=1}^{K}\sum_{s=0}^{t}\left((1-\delta)\theta_k\right)^{t-s}\tilde{\mu}_{k,s}\left(\partial_e d_{k,t}\left(p_{k,s}^*(j^*), \boldsymbol{p}_{k,t}, e_{k,t}\right) + \left(p_{k,s}^*(j^*) - (1-\tau_k)MC_{k,t}(j^*)\right)\partial_{pe} d_{k,t}\left(p_{k,s}^*(j^*), \boldsymbol{p}_{k,t}, e_{k,t}\right)\right) \\
&\quad + \sum_{k=1}^{K}\sum_{s=0}^{t}\left((1-\delta)\theta_k\right)^{t-s}\tilde{\mu}_{k,s}\left[d_k^{\mathcal{I}} Y_k + \left(p_{k,s}^*(j^*) - (1-\tau_k)MC_{k,t+s}(j^*)\right)\partial_p d_k^{\mathcal{I}}\left(p_{k,s}^*(j^*), \boldsymbol{p}_{k,t}\right) Y_k\right]\partial_e \tilde{Y}_k = 0,
\end{aligned}$$

and in steady state simplifies to:

$$1 + \check{\zeta}_t^{t_0}(i)\, W \partial_{ee} v + \check{\aleph}_t^{t_0}(i) - \check{\Xi}_t - \sum_{k=1}^{K}\frac{1}{P_k Y_k}\lambda_k \check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i) = 0.$$

- Finally consider the first order conditions for a compensated change in resetted prices $p_{k,t}^*(j^*)$ (That is, each household receives a transfer in period $t+s$, $s \geq 0$ which cancels the income effect of the price change. For a household consuming a bundle $\boldsymbol{d}_{k,t+s}(i,j)$of the varieties in sector k at $t+s$, the transfer would be $(1-\theta_k)\,\theta_k^s \int d_{k,t+s}(i,j)\, dj$. Note that we can alternatively consider an uncompensated change in prices, but the terms corresponding to the income effects can then be simplified using the first-order condition corresponding to the optimality of expenditure of unconstrained and HtM households.):

$$
\begin{aligned}
0 = {} & \tilde{\mu}_{k,t} \Bigg(\sum_{s=0}^{\infty} \tilde{\beta}^{s} \theta_k^{s} \Big(2\partial_p D_{k,t+s} \big(p_{k,t}^{*}(j^{*}), \boldsymbol{p}_{k,t+s}, \boldsymbol{e}_{k,t+s}, \tilde{Y}_{k,t+s} \big) + \big(p_{k,t}^{*}(j^{*}) - (1-\tau_k) MC_{k,t+s}(j^{*}) \big) \partial_{pp} D_{k,t+s} \big(p_{k,t}^{*}(j^{*}), \boldsymbol{p}_{k,t+s}, \boldsymbol{e}_{k,t+s}, \tilde{Y}_{k,t+s} \big) \Big) \Bigg) \\
& + (1-\theta_k)\, \mathbb{E}_0 \sum_{T=t} (\beta\theta_k)^{T-t} \sum_{s=0}^{T} \left((1-\delta)\,\theta_l\right)^{T-s} \tilde{\mu}_{k,s} \Big(\partial_P D_{k,T} + \int \partial_e d_{l,T} \big(p_{l,s}^{*}(j^{*}), \boldsymbol{p}_{l,T}, e_{l,T} \big) \frac{e_{k,T}}{p_{k,T}\,(j)} di \Big) \\
& + (1-\theta_k)\, \mathbb{E}_0 \sum_{T=t} (\beta\theta_k)^{T-t} \sum_{s=0}^{T} \left((1-\delta)\,\theta_l\right)^{T-s} \tilde{\mu}_{k,s} \Big((p_{k,s}(j^{*}) - (1-\tau_k) MC_{k,T}(j^{*})) \Big(\partial_{pP} D_{k,T} + \int \partial_{pe} d_{l,T} \big(p_{l,s}^{*}(j^{*}), \boldsymbol{p}_{l,T}, e_{l,T} \big) \frac{e_{k,T}}{p_{k,T}\,(j)} di \Big) \Big) \\
& + (1-\theta_k) \sum_{T=t}^{\infty} (\beta\theta_k)^{T-t} \sum_{l=1}^{K} \sum_{s=0}^{T} \left((1-\delta)\,\theta_l\right)^{T-s} \tilde{\mu}_{l,s} \\
& \cdot \int \Big(\partial_e d_{l,T} \big(p_{l,s}^{*}(j^{*}), \boldsymbol{p}_{l,T}, e_{l,T} \big) + \big(p_{l,s}^{*}(j^{*}) - (1-\tau_k) MC_{l,T}(j^{*}) \big) \partial_{pe} d_{l,T} \big(p_{l,s}^{*}(j^{*}), \boldsymbol{p}_{l,T}, e_{l,T} \big) \Big(\partial_{p_k(j^{*})} e_{l,T} - \mathbb{1}_{l=k} \frac{e_{k,T}}{p_{k,T}\,(j)} + \partial_e e_{l,T} d_{k,T}(i,j^{*}) \Big) \Big) di \\
& + (1-\theta_k) \sum_{T=t}^{\infty} (\beta\theta_k)^{T-t} \sum_{l=1}^{K} \sum_{s=0}^{T} \left((1-\delta)\,\theta_l\right)^{T-s} \tilde{\mu}_{l,s} \Big[d_l^{\mathcal{I}} \tilde{Y}_l + \big(p_{l,s}^{*}(j^{*}) - (1-\tau_l) MC_{l,T}(j^{*}) \big) \partial_p d_k^{\mathcal{I}} \big(p_{l,T}^{*}(j^{*}), \boldsymbol{p}_{l,t} \big) \tilde{Y}_l \Big] (Id-Q)^{-1} \cdot \left[\sum_l \frac{\partial \mathcal{Y}_{1,l}}{\partial p(j^{*})} \frac{Y_l}{A_{l,t} Y_1}, \dots, \sum_l \frac{\partial \mathcal{Y}_{K,l}}{\partial p(j^{*})} \frac{Y_l}{A_{l,t} Y_K} \right]^{T} \\
& + (1-\theta_k) \sum_{T=t}^{\infty} (\beta\theta_k)^{T-t} \sum_{l=1}^{K} \sum_{s=0}^{T} \left((1-\delta)\,\theta_l\right)^{T-s} \tilde{\mu}_{l,s} \Big[d_l^{\mathcal{I}} \big(p_{l,s}^{*}(j^{*}) - (1-\tau_l) MC_{l,T}(j^{*}) \big) \partial_p d_l^{\mathcal{I}} \big(p_{l,s}^{*}(j^{*}), \boldsymbol{p}_{l,T} \big) \Big] \partial_{p_k(j^{*})} \tilde{Y}_{l,T} \\
& - (1-\theta_k) \sum_{T=t}^{\infty} (\beta\theta_k)^{T-t} \sum_{l=1}^{K} \sum_{s=0}^{T} \left((1-\delta)\,\theta_l\right)^{T-s} \tilde{\mu}_{l,s} (1-\tau_k) \frac{\mathcal{Y}_{k,l}}{A_{l,T}} \partial_p D_{l,T} - (1-\theta_k) \sum_s (\beta\theta_k)^{s}\, \check{\Xi}_{t+s} \Big(\sum_{l=1}^{K} \partial_{p_k(j^{*})} \mathcal{N}_l(\boldsymbol{P}_t, W_t) \frac{Y_l}{A_{l,t}} \Big) \\
& - (1-\theta_k) \sum_s (\beta\theta_k)^{s}\, \check{\Xi}_{t+s} \Bigg(\left[\frac{\mathcal{N}_1(\boldsymbol{P}_t, W_t) Y_1}{A_{1,t}}, \dots, \frac{\mathcal{N}_K(\boldsymbol{P}_t, W_t) Y_K}{A_{K,t}} \right] (Id-Q)^{-1} \left[\sum_l \frac{\partial \mathcal{Y}_{1,l}}{\partial p(j^{*})} \frac{Y_l}{A_{l,t} Y_1}, \dots, \sum_k \frac{\partial \mathcal{Y}_{K,l}}{\partial p(j^{*})} \frac{Y_l}{A_{l,t} Y_K} \right]^{T} \Bigg) \\
& - (1-\theta_k) \sum_s (\beta\theta_k)^{s}\, \check{\Xi}_{t+s} \sum_{l=1}^{K} \frac{\mathcal{N}_l(\boldsymbol{P}_{t+s}, W_{t+s})}{A_{l,t}} \mathbb{E}_{\delta,t+s} \int\!\!\int \Big(\big(\partial_{p_k(j^{*})} e_{l,t+s} + \partial_e e_{l,t+s} d_{k,t+s}(i,j^{*}) \big) \partial_e d_{l,t+s} + d_{l,t+s}^{I} \partial_{p_k(j^{*})} \tilde{Y}_{l,t+s} \Big) dj di \\
& - (1-\theta_k) \sum_s (\beta\theta_k)^{s}\, \check{\Xi}_{t+s} \frac{\mathcal{N}_k(\boldsymbol{P}_{t+s}, W_{t+s})}{A_{e_k,t}} \mathbb{E}_{\delta,t+s} \int \partial_p d_{k,t+s} + \int \partial_P d_{k,t+s} dj di + \Big(\partial_{p_k(j^{*})} d_{k,t+s}^{I} + \int \partial_P d_{k,t+s}^{I} dj \Big) \tilde{Y}_{k,t+s} \\
& - (1-\theta_k) \sum_s (\beta\theta_k)^{s}\, \mathbb{E}_{\delta,t+s} \Big(\int \Big[(1-\varphi(i)) \big(\check{\lambda}_{t+s}^{t_0}(i) - R \check{\lambda}_{t+s-1}^{t_0}(i) \big) + \check{\zeta}_{t+s}^{t_0}(i)\, W \Big] \partial_e v_{t+s,i}^{t_0,u} \partial_e d_{k_{t+s,i}}^{t_0,u} \partial_e e_{k_{t+s,i}}^{t_0,u} \Big) \\
& - (1-\theta_k) \sum_s (\beta\theta_k)^{s}\, \mathbb{E}_{\delta,t+s} \Big(\int \varphi(i)\, \check{\zeta}_{t+s}^{t_0,HtM}(i)\, W \partial_e v_{t+s,i}^{t_0,HtM} \partial_e d_{k_{t+s,i}}^{t_0,HtM} \partial_e e_{k_{t+s,i}}^{t_0,HtM} di \Big) \\
& + (1-\theta_k) \sum_s (\beta\theta_k)^{s}\, \mathbb{E}_{\delta,t+s} \Big(\int (1-\varphi(i))\, \check{\alpha}_t^{t_0}(i) \Big[d_{k,t+s}(i,j^{*}) - \varsigma(i)\, \partial_{p(j^{*})} Div_t \Big] di \Big) - \delta \int (1-\varphi(i))\, \check{\alpha}_{t+s}^{t+s}(i)\, b_{t+s}^{t+s,u}(i)\, di \bar{s}_k \frac{1}{P_k} \\
& + (1-\theta_k) \sum_{s=0} (\beta\theta_k)^{s} \Bigg(-\mathbb{E}_{\delta t+s} \Big(\int \varphi(i)\, \check{\aleph}_{t+s}^{t_0} \Big[d_{k,t+s}(i,j^{*}) - \varsigma(i)\, \partial_{p(j^{*})} Div_t \Big] \Big) di - \delta \sum_{u=0} \left((1-\delta)\,\beta\right)^{u} \int \varphi(i)\, \check{\aleph}_{t+u+s}^{t+s}(i)\, b_{t+s}^{t+s,HtM}(i)\, di \bar{s}_k \frac{R_{t+u}-1}{R_{t+u}} \frac{1}{P_k} \Bigg).
\end{aligned}
$$

with

$$\left[\partial_{p_k(j^*)}\tilde{Y}_{1,t+s},...,\partial_{p_k(j^*)}\tilde{Y}_{K,t+s}\right]^T = (Id-\tilde{Q})^{-1}\tilde{Q}\left[\int\int\left(\partial_{p_k(j^*)}e_{1,t+s}+\partial_e e_{1,t+s}d_{k,t+s}(i,j^*)\right)\partial_e d_{1,t}djdi,...,\int\int\left(\partial_{p_k(j^*)}e_{K,t+s}+\partial_e e_{K,t}d_{k,t+s}(i,j^*)\right)\partial_e d_{K,t}djdi\right]^T$$

$$+(Id-\tilde{Q})^{-1}\tilde{Q}\left[0,...,\mathbb{E}_{\delta,t+s}\int\partial_p d_{k,t+s}+\int\partial_P d_{k,t+s}djdi+\left(\partial_{p_k(j^*)}d^I_{k,t+s}+\int\partial_P d^I_{k,t+s}dj\right)\tilde{Y}_{k,t+s},...,0\right]$$

$$\partial_{p(j^*)}Div_{t+s}=\sum_l\int\int\left(\left(\partial_{p_k(j^*)}e_{l,t+s}+\partial_e e_{l,s}d_{k,t+s}(i,j^*)\right)\partial_e d_{l,t+s}+d^I_{l,t+s}\partial_{p_k(j^*)}\tilde{Y}_{l,t+s}\right)\left(p_{l,t+s}(j)-MC_{l,t+s}\right)didj$$

$$+\int\left(\partial_p d_{k,t+s}+\partial_p d^I_{k,t+s}\right)\left(p_{l,t+s}(j)-MC_{l,t+s}\right)di+\int\int\left(\partial_P d_{k,t+s}+\partial_P d^I_{k,t+s}\right)\left(p_{l,t+s}(j)-MC_{l,t+s}\right)didj+y_{k,t+s}(j)-\sum_l\mathcal{Y}_{k,l,t+s}Y_{l,t+s}$$

We define $\check{\mu}_{k,T}$, which constrains the growth rate of sectoral inflation:

$$\check{\mu}_{k,T}\equiv\frac{\theta_k}{1-\theta_k}\sum_{t=0}^{T}\left((1-\delta)\,\theta_k\right)^{T-t}\tilde{\mu}_{k,t}\sum_{s=0}^{\infty}\tilde{\beta}^s\theta_k^s\left(2p_{k,t}(j^*)\partial_p D_{k,t+s}+\left(p_{k,t}(j^*)-(1-\tau_k)MC_{k,t+s}(j^*)\right)p_{k,t}(j^*)\partial_{pp}D_{k,t+s}\right)$$

Using the properties of the steady state, we get:

$$0=\frac{(1-\theta_k)}{\theta_k}\frac{1}{P_k}\left(\check{\mu}_{k,t}-\left((1-\delta)\,\theta_l\right)\check{\mu}_{k,t-1}\right)$$

$$-(1-\theta_k)\sum_{T=t}(\beta\theta_k)^{T-t}\left(\frac{(1-\theta_k)\left(1-\tilde{\beta}\theta_k\right)}{\theta_k}\check{\mu}_{k,T}\frac{1}{P_k}+\frac{1}{P_k}\lambda_k\check{\mu}_{k,T}\right)$$

$$-(1-\theta_k)\sum_{T=t}^{\infty}(\beta\theta_k)^{T-t}\sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,T}}{P_lY_l}\int\left(\gamma_{e,l}(i)\frac{e_l(i)\,\rho_{l,k}(i)}{P_k}\right)di-(1-\theta_k)\sum_{T=t}^{\infty}(\beta\theta_k)^{T-t}\sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,T}}{P_lY_l}\frac{\mathcal{Y}_{k,l}Y_l}{A_{l,T}}$$

$$-(1-\theta_k)\sum_s(\beta\theta_k)^s\,\mathbb{E}_{\delta,t+s}\left(\int\left[(1-\varphi(i))\left(\check{\lambda}^{t_0}_{t+s}(i)-R\check{\lambda}^{t_0}_{t+s-1}(i)\right)+\check{\zeta}^{t_0,u}(i)\,W\right]\partial_e v\frac{1}{P_k}\partial_e e_k+\varphi(i)\,\check{\zeta}^{t_0,HtM}(i)\,W\partial_e v_i\frac{1}{P_k}\partial_e e_k di\right)$$

$$+(1-\theta_k)\sum_s(\beta\theta_k)^s\,\mathbb{E}_{\delta,t+s}\left(\int(1-\varphi(i))\,\check{\alpha}^{t_0}(i)\frac{1}{P_k}\left[e_k(i)-\varsigma(i)\,E_k\right]di\right)-\delta\int(1-\varphi(i))\,\check{\alpha}^{t+s}(i)\,b^{t+s}(i)\,di\bar{s}_k\frac{1}{P_k}$$

$$+(1-\theta_k)\sum_s(\beta\theta_k)^s\left(-\mathbb{E}_{\delta,t+s}\left(\int\varphi(i)\,\check{\aleph}^{t_0}_{t+s}(i)\frac{1}{P_k}\left[e_k(i)-\varsigma(i)\,E_k\right]\right)di-\delta\frac{R-1}{R}\sum_{u=0}\frac{1}{R^u}\int\varphi(i)\,\check{\aleph}^{t+s}_{t+u+s}(i)\,b(i)\,di\bar{s}_k\frac{1}{P_k}\right)$$

Taking the difference between the equation at $t+1$ times $\beta\theta_k$ and the equation at t we obtain

$$\left(\beta\check{\mu}_{k,t+1}-\left(1+\left((1-\delta)\,\beta\right)\right)\check{\mu}_{k,t}+(1-\delta)\,\check{\mu}_{k,t-1}\right)=\lambda_k\check{\mu}_{k,t}-\sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,t}}{Y_lP_l}\int\left(\gamma_{e,l}(i)\,e_l(i)\,\rho_{k,l}(i)\right)di-\sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,t}}{P_lY_l}\frac{P_k\mathcal{Y}_{k,l}Y_l}{A_l}$$

$$-\mathbb{E}_{\delta,t}\left(\int\left[(1-\varphi(i))\left((i)-R\lambda^{t_0}_{t-1}(i)\right)+\check{\zeta}^{t_0,u}_t(i)\right]\partial_e v\partial_e e_k+\varphi(i)\,\check{\zeta}^{t_0,HtM}_t(i)\,\partial_e v_i\partial_e e_k di\right)$$

$$+\mathbb{E}_{\delta,t}\left(\int\left((1-\varphi(i))\,\check{\alpha}^{t_0}(i)+\varphi(i)\,\check{\aleph}_t^{t_0}\right)\left[e_k(i)-\varsigma(i)\,E_k\right]di\right)-\delta\int\left((1-\varphi(i))\,\check{\alpha}^t(i)+\varphi(i)\frac{R-1}{R}\sum_{u=0}\frac{1}{R^u}\check{\aleph}^t_{t+u}(i)\right)b(i)\,di\bar{s}_k$$

We can now verify that a steady state with $R=1/\tilde{\beta}$, constant wages and prices (chosen such that the good markets and labor market clear, recall that this implies that wealth, expenditure and labor supply of households is constant across time and identical for unconstrained and HtM households) and $\check{\zeta}^{t_0,u}_t=\check{\zeta}^{t_0,HtM}_t=\check{\Xi}_t=\check{\mu}_{k,t}=\check{\lambda}^{t_0}_t=0$,

$\check{\alpha}^{t}\left(i\right)=\check{\aleph}_{t}^{t}\left(i\right)=\check{\aleph}^{t}\left(i\right)=-1$ solves the set of first-order conditions[48]

$$-\mathbb{E}_{\delta,t}\left(\left(1-\varphi\left(i\right)\right)\check{\lambda}_{t}^{t_0}\left(i\right)\partial_{e}v\left(e\left(i\right),\boldsymbol{P}\right)\right)-\mathbb{E}_{\delta,t}\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}^{t_0}\left(i\right)\frac{b\left(i\right)}{R}\right)-\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\check{\aleph}^{t_0}\left(i\right)\frac{1}{R}b\left(i\right)\right)=0\Rightarrow\mathbb{E}_{\delta,t}\left(\frac{b\left(i\right)}{R}\right)=0$$

$$\check{\zeta}_{t}^{t_0,u}\left(i\right)\partial_{e}v=\psi n(i)\left\{\check{\Xi}_{t}-\check{\alpha}^{t_0}\left(i\right)-1\right\}\quad\check{\zeta}_{t}^{t_0,HtM}\left(i\right)\partial_{e}v=\psi n(i)\left\{\check{\Xi}_{t}-\check{\alpha}^{t_0}\left(i\right)-1\right\}\Rightarrow0=\psi n(i)\left\{1-1\right\}\quad0=\psi n(i)\left\{1-1\right\}$$

$$0=\mathbb{E}_{\delta,t}\int\left(1-\varphi\left(i\right)\right)\check{\zeta}_{t}^{t_0,u}\left(i\right)\partial_{e}v\,di+\mathbb{E}_{\delta,t}\int\varphi\left(i\right)\check{\zeta}_{t}^{t_0,HtM}\left(i\right)\partial_{e}di-\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,s}\mathcal{N}_{k}(\boldsymbol{P}_{t},W_{t})\Rightarrow0=0$$

$$1+\left(\check{\lambda}_{t}^{t_0}\left(i\right)-R\check{\lambda}_{t-1}^{t_0}\left(i\right)\right)\partial_{ee}v+\check{\zeta}_{t}^{t_0,u}\left(i\right)W\partial_{ee}v+\check{\alpha}^{t_0}\left(i\right)-\check{\Xi}_{t}-\sum_{k=1}^{K}\frac{1}{P_{k}Y_{k}}\lambda_{k}\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_{e}e_{k}\left(i\right)=0\Rightarrow1-1=0$$

$$1+\check{\zeta}_{t}^{t_0}\left(i\right)W\partial_{ee}v+\check{\aleph}^{t_0}\left(i\right)-\check{\Xi}_{t}-\frac{1}{P_{k}Y_{k}}\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_{e}e_{k}\left(i\right)=0\Rightarrow1-1=0$$

$$\left(\beta\check{\mu}_{k,t+1}-\left(1+\left(\left(1-\delta\right)\beta\right)\right)\check{\mu}_{k,t}+\left(1-\delta\right)\check{\mu}_{k,t-1}\right)=\lambda_{k}\check{\mu}_{k,t}-\sum_{l=1}^{K}\lambda_{l}\frac{\check{\mu}_{l,t}}{Y_{l}P_{l}}\int\left(\gamma_{e,l}\left(i\right)e_{l}\left(i\right)\rho_{k,l}\left(i\right)\right)di$$

$$\begin{aligned}-\mathbb{E}_{\delta,t}\left(\int\left[\left(1-\varphi\left(i\right)\right)\left(\check{\lambda}_{t}^{t_0}\left(i\right)-R\check{\lambda}_{t-1}^{t_0}\left(i\right)\right)+\check{\zeta}_{t}^{t_0,u}\left(i\right)W\right]\partial_{e}v\partial_{e}e_{k}\left(i\right)+\varphi\left(i\right)\check{\zeta}_{t}^{t_0,HtM}\left(i\right)W\partial_{e}v_{i}\partial_{e}e_{k}\left(i\right)di\right)+\mathbb{E}_{\delta,t}\left(\int\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}^{t_0}\left(i\right)+\varphi\left(i\right)\check{\aleph}^{t_0}\right)\left[e_{k}(i)-\varsigma\left(i\right)E_{k}\right]di\right)\\-\delta\int\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}^{t}\left(i\right)+\varphi\left(i\right)\check{\aleph}^{t}\left(i\right)\right)b\left(i\right)di\bar{s}_{k}P_{k}\end{aligned}$$

$$\Rightarrow0=-\mathbb{E}_{\delta,t}\left(\int\left[e_{k}(i)-E_{k}\right]di\right)+\delta\int b\left(i\right)di\bar{s}_{k}P_{k}$$

Differentiating the first-order conditions

We now differentiate the first-order conditions around the steady state constructed in the previous section. Prices are in log-deviation while Lagrange multipliers are in absolute deviations.

- First order conditions with respect to $b_{t}^{t_0,u}(i)$:

$$\begin{aligned}\check{\alpha}_{t}^{t_0}\left(i\right)&=\check{\alpha}_{t-1}^{t_0}\left(i\right)+\hat{R}_{t-1}\\&=\check{\alpha}^{t_0}\left(i\right)+\sum_{s=0}^{t-t_0-1}\hat{R}_{t_0+s}\end{aligned}$$

- First Order conditions for the interest rate

$$\begin{aligned}-\mathbb{E}_{\delta,t}\left(\int\left(1-\varphi\left(i\right)\right)\check{\lambda}_{t}^{t_0}\left(i\right)\partial_{e}v\left(e\left(i\right),\boldsymbol{P}\right)di\right)&=\mathbb{E}_{\delta,t}\left(\int\left(1-\varphi\left(i\right)\right)\check{\alpha}^{t_0}\left(i\right)\frac{b\left(i\right)}{R}di\right)+\mathbb{E}_{\delta,t}\left(\int\varphi\left(i\right)\check{\aleph}_{t}^{t_0}\left(i\right)\frac{1}{R}b\left(i\right)di\right)\\&+\mathbb{E}_{\delta,t}\left(\int\left(1-\varphi\left(i\right)\right)\sum_{s=0}^{t-t_0-1}\hat{R}_{t_0+s}\frac{b\left(i\right)}{R}di\right)\end{aligned}$$

---

[48] Note that to solve the steady state system we only need $\check{\Xi}-\check{\alpha}^{t_0}=-1$ and $\check{\Xi}-\check{\aleph}^{t_0}=-1$. It's direct to verify that choosing any values for $\check{\Xi}$, $\check{\alpha}^{t_0}$, and $\check{\aleph}^{t_0}$ that satisfy this would give the same system of differentiated first order conditions.

- First Order conditions for W_t:

$$0 = \mathbb{E}_{\delta,t} \int \left(\left(1 - \varphi\left(i\right)\right) \check{\zeta}_t^{t_0,u}\left(i\right) + \varphi\left(i\right) \check{\zeta}_t^{t_0,HtM}\left(i\right) \right) \partial_e v di - \sum_{k=1}^{K} \lambda_k \check{\mu}_{k,s} \mathcal{N}_k(\boldsymbol{P}_t, W_t) + \sum_{k,l} \frac{P_l \partial_W \mathcal{Y}_{l,k}}{A_k} Y_k \left(\hat{P}_{l,t} + \tilde{A}_{l,t,}\right) + \sum_k \frac{W \partial_W \mathcal{N}_k}{A_k} Y_k \hat{W}_t$$

- First-order conditions with respect to labor supply:

$$\begin{aligned}
\check{\zeta}_t^{t_0,u}\left(i\right) \partial_e v &= \psi n(i) \left\{ \check{\Xi}_t - \sum_{s=0}^{t-t_0-1} \hat{R}_{t_0+s} - \check{\alpha}^{t_0}\left(i\right) - \frac{V\left(i\right) G''\left(V\left(i\right), i\right)}{G'\left(V\left(i\right), i\right)} \hat{V}_{t_0}\left(i\right) + \frac{1}{\sigma}\left(\hat{e}_t^{t_0,u}\left(i\right) - \sum_l s_l\left(i\right) \hat{P}_{l,t} \right) + \sum_l \partial_e e_l\left(i\right) \hat{P}_{l,t} \right\} \\
&= \psi n(i) \left\{ \check{\Xi}_t - \check{\alpha}^{t_0}\left(i\right) - \frac{V\left(i\right) G''\left(V\left(i\right), i\right)}{G'\left(V\left(i\right), i\right)} \hat{V}_{t_0}\left(i\right) + \frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}\left(i\right) - \sum_l s_l\left(i\right) \hat{P}_{l,t_0} \right) + \sum_l \partial_e e_l\left(i\right) \hat{P}_{l,t_0} \right\} \\
\check{\zeta}_t^{t_0,HtM}\left(i\right) \partial_e v &= \psi n(i) \left\{ \check{\Xi}_t - \check{\aleph}_t^{t_0}\left(i\right) - \frac{V\left(i\right) G''\left(V\left(i\right), i\right)}{G'\left(V\left(i\right), i\right)} \hat{V}_{t_0}\left(i\right) + \frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}\left(i\right) - \sum_l s_l\left(i\right) \hat{P}_{l,t} \right) + \sum_l \partial_e e_l\left(i\right) \hat{P}_{l,t} \right\}
\end{aligned}$$

- First-order condition with respect to expenditure. We only need to rexpress the impact of individual consumption on profits.
- Log linearizing $\sum_k \int \left(\partial_e e_k \partial_e d_k + d_k^I \partial_e \tilde{Y}_k\right) \left(p_{k,t}(j) - MC_{k,t}\right) dj$ we have:

$$\begin{aligned}
\frac{d \sum_k \int \left(\partial_e e_k \partial_e d_k + d_k^I \partial_e \tilde{Y}_k\right) \left(p_{k,t}(j) - MC_{k,t}\right) dj}{\sum_k \int \left(\partial_e e_k \partial_e d_k + d_k^I \partial_e \tilde{Y}_k\right) \left(p_{k,t}(j) - MC_{k,t}\right) dj} &= \left(\hat{P}_{k,t} + \hat{A}_{k,t} - \Omega_{N,k} \hat{W}_t + \sum_l \Omega_{k,l} \hat{P}_{l,t} \right) \mathcal{D}\left(P\right) \left(Id - \tilde{Q}\right)^{-1} \mathcal{D}^{-1}\left(P\right) \left[\partial e_1, ..., \partial_e e_K\right]^T \\
&= \left[\partial_e e_1, ..., \partial_e e_K\right]^T \left(Id - \Omega\right)^{-1} \left[\left(\hat{P}_{k,t} + \hat{A}_{k,t} - \Omega_{N,k} \hat{W}_t - \sum_l \Omega_{k,l} \hat{P}_{l,t} \right) \right] \\
&= -\hat{W}_t + \sum_l \partial_e e_l \left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right)
\end{aligned}$$

So we have:

$$\begin{aligned}
&\frac{V\left(i\right) G''\left(V\left(i\right), i\right)}{G'\left(V\left(i\right), i\right)} \hat{V}_{t_0}\left(i\right) - \frac{1}{\sigma}\left(\hat{e}_t^{t_0,u}\left(i\right) - \sum_l s_l\left(i\right) \hat{P}_{l,t} \right) - \sum_l \partial_e e_l\left(i\right) \hat{P}_{l,t} + \left(\lambda_t^{t_0}\left(i\right) - R \lambda_{t-1}^{t_0}\left(i\right) \right) \partial_{ee} v + \check{\zeta}_t^{t_0,u}\left(i\right) W \partial_{ee} v \\
&\qquad + \check{\alpha}^{t_0}\left(i\right) + \sum_{s=0}^{t-t_0-1} \hat{R}_{t_0+s} - \check{\Xi}_t - \left(\hat{W}_t - \sum_l \partial_e e_l \left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right) \right) - \frac{1}{P_k Y_k} \sum_{k=1}^{K} \lambda_k \check{\mu}_{k,t} \gamma_{e,k}\left(i\right) \partial_e e_k\left(i\right) = 0 \\
&\frac{V\left(i\right) G''\left(V\left(i\right), i\right)}{G'\left(V\left(i\right), i\right)} \hat{V}_{t_0}\left(i\right) - \frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}\left(i\right) - \sum_l s_l\left(i\right) \hat{P}_{l,t_0} \right) - \sum_l \partial_e e_l\left(i\right) \hat{P}_{l,t} + \left(\lambda_t^{t_0}\left(i\right) - R \lambda_{t-1}^{t_0}\left(i\right) \right) \partial_{ee} v + \check{\zeta}_t^{t_0,u}\left(i\right) W \partial_{ee} v \\
&\qquad + \check{\alpha}^{t_0}\left(i\right) - \check{\Xi}_t - \left(\hat{W}_t - \sum_l \partial_e e_l \left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right) \right) - \frac{1}{P_k Y_k} \sum_{k=1}^{K} \lambda_k \check{\mu}_{k,t} \gamma_{e,k}\left(i\right) \partial_e e_k\left(i\right) = 0
\end{aligned}$$

For the expenditure of HtM households:

$$\begin{aligned}
&\frac{V\left(i\right) G''\left(V\left(i\right), i\right)}{G'\left(V\left(i\right), i\right)} \hat{V}_{t_0}\left(i\right) - \frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}\left(i\right) - \sum_l s_l\left(i\right) \hat{P}_{l,t} \right) - \sum_l \partial_e e_l\left(i\right) \hat{P}_{l,t} + \check{\zeta}_t^{t_0,HtM}\left(i\right) W \partial_{ee} v \\
&\qquad + \check{\aleph}_t^{t_0}\left(i\right) - \check{\Xi}_t - \left(\hat{W}_t - \sum_l \partial_e e_l \left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right) \right) - \frac{1}{P_k Y_k} \sum_{k=1}^{K} \lambda_k \check{\mu}_{k,t} \gamma_{e,k}\left(i\right) \partial_e e_k\left(i\right) = 0
\end{aligned}$$

- Finally for resetted prices, note that we have:

$$d\left(\sum_s (\beta\theta_k)^s \partial_{p(j^*)} Div_{t+s} - \sum_s (\beta\theta_k)^{s+1} \partial_{p(j^*)} Div_{t+1+s}\right) = \sum_l \left(\int e_l(i)\,\rho_{l,k}(i)\,\frac{1}{P_l P_k} di + \partial_{p_k(j^*)} \tilde{Y}_l\right) P_l \left(\hat{P}_{l,t} + \hat{A}_{l,t} - \hat{M}C_{l,t+s}\right)$$
$$+ \frac{Y_k}{P_k}\bar{\epsilon}_k \frac{\theta_k}{(1-\beta\theta_k)(1-\theta_k)} \left(\beta\pi_{k,t+1} - \pi_{k,t}\right) + dE_{k,t}$$

Defining $\vartheta_k = \bar{\epsilon}_k \frac{\theta_k}{(1-\beta\theta_k)(1-\theta_k)}$, we obtain:

$$\left(\beta\check{\mu}_{k,t+1} - (1 + ((1-\delta)\beta))\,\check{\mu}_{k,t} + (1-\delta)\,\check{\mu}_{k,t-1}\right) = \lambda_k\check{\mu}_{k,t} - \sum_{l=1}^{K} \lambda_l \frac{\check{\mu}_{l,t}}{Y_l P_l} \int \left(\gamma_{e,l}(i)\, e_l(i)\,\rho_{l,k}(i)\right) di - \sum_{l=1}^{K} \lambda_l \frac{\check{\mu}_{l,t}}{P_l Y_l} \frac{P_k \mathcal{Y}_{k,l} Y_l}{A_l}$$
$$- \mathbb{E}_{\delta,t}\left(\int \left[(1-\varphi(i))\left(\check{\lambda}_t^{t_0}(i) - R\check{\lambda}_{t-1}^{t_0}(i)\right) + \check{\zeta}_t^{t_0,u}(i)\, W\right] \partial_e v \partial_e e_k + \varphi(i)\,\check{\zeta}_t^{t_0,HtM}(i)\, W \partial_e v_i \partial_e e_k di\right)$$
$$+ \mathbb{E}_{\delta,t}\left(\int \left((1-\varphi(i))\,\check{\alpha}^{t_0}(i) + \varphi(i)\,\check{\aleph}_t^{t_0}\right)\left[e_k(i) - \varsigma(i)\, E_k\right] di\right) + \mathbb{E}_{\delta,t}\left(\int (1-\varphi(i))\left[e_k(i) - \varsigma(i)\, E_k\right] di \sum_{s=0}^{t-t_0-1} \hat{R}_{t_0+s}\right) - \delta \int (1-\varphi(i))\,\check{\alpha}^t(i)\, b(i)\, di \bar{s}_k$$
$$- \delta \sum_{u=0} ((1-\delta)\beta)^u \int \varphi(i)\,\check{\aleph}_{t+u}^t(i)\, b(i)\, di \bar{s}_k \frac{R-1}{R} + \delta \sum_{u=0} \frac{1}{R^u} \int \varphi(i)\, b(i)\, di \bar{s}_k \frac{\hat{R}_{t+u}}{R}$$
$$+ \sum_{l,m} \frac{P_l P_k \partial_{P_k} \mathcal{Y}_{l,m}}{A_m} Y_m \left(\hat{P}_{l,t} + \tilde{A}_{l,t,}\right) + \sum_l \frac{W P_k \partial_{P_k} \mathcal{N}_l}{A_l} Y_l \hat{W}_t + \sum_l E_l \bar{\rho}_{l,k}\left(\hat{P}_{l,t} + \tilde{A}_{l,t}\right) - P_k Y_k \vartheta_k \left(\pi_{k,t} - \beta\pi_{k,t+1}\right)$$

Solving the Labor market equation

The next step is to re-write the (infinite number of) linearized conditions, into a system of a limited number of equations and variables. Our first main equation is the optimality of the wage

$$0 = \mathbb{E}_{\delta,t} \int \left((1-\varphi(i))\,\check{\zeta}_t^{t_0,u}(i) + \varphi(i)\,\check{\zeta}_t^{t_0,HtM}(i)\right) W \partial_e v di - \sum_{k=1}^{K} \lambda_k \check{\mu}_{k,s} W \mathcal{N}_k + \sum_{k,l} \frac{P_l W \partial_W \mathcal{Y}_{l,k}}{A_k} Y_k \left(\hat{P}_{l,t} + \tilde{A}_{l,t,}\right) + \sum_k \frac{W^2 \partial_W \mathcal{N}_k}{A_k} Y_k \hat{W}_t$$

Let us define the first component as $\tilde{Z}_t \equiv \mathbb{E}_{\delta,t} \int \left((1-\varphi(i))\,\check{\zeta}_t^{t_0,u}(i) + \varphi(i)\,\check{\zeta}_t^{t_0,HtM}(i)\right) W \partial_e v di$. Substituting out the Lagrange multipliers gives:

$$\tilde{Z}_t = \psi W N \check{\Xi}_t - \mathbb{E}_{\delta,t} \int \psi n(i) \frac{V(i)\, G''(V(i), i)}{G'(V(i), i)} \hat{V}_{t_0}(i)\, di + \mathbb{E}_{\delta,t} \int (1-\varphi(i))\, \psi W n(i) \left\{\frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}(i) - \sum_l s_l(i)\, \hat{P}_{l,t_0}\right) + \sum_l \partial_e e_l(i)\, \hat{P}_{l,t_0} - \check{\alpha}^{t_0}(i)\right\}$$
$$+ \varphi(i)\, \psi W n(i) \left\{\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}(i) - \sum_l s_l(i)\, \hat{P}_{l,t}\right) + \sum_l \partial_e e_l(i)\, \hat{P}_{l,t} - \check{\aleph}_t^{t_0}(i)\right\} di$$

Our first goal is to solve for $\check{\Xi}_t$, $\check{\alpha}^{t_0}(i)$ and $\check{\aleph}_t^{t_0}(i)$. Using the optimality of household's expenditure and substituting the $\check{\zeta}_t^{t_0,u}(i)$ term

$$\begin{aligned}\Big(\check{\lambda}_t^{t_0}(i)-R\check{\lambda}_{t-1}^{t_0}(i)\Big)\partial_e v &= \sigma e(i)\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i)-\sigma e(i)\Bigg(\frac{1}{\sigma}\Bigg(\hat{e}_{t_0}^{t_0,u}(i)-\sum_l s_l(i)\,\hat{P}_{l,t_0}\Bigg)+\sum_l \partial_e e_l(i)\,\hat{P}_{l,t_0}\Bigg)-\check{\zeta}_t^{t_0,u}(i)\,W\partial_e v+\sigma e(i)\,\check{\alpha}^{t_0}(i)\\
&\quad -\sigma e(i)\,\check{\Xi}_t-\sigma e(i)\Bigg(\Bigg(\hat{W}_t-\sum_l \partial_e e_l\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\Bigg)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\Bigg)\\
\Big(\check{\lambda}_t^{t_0}(i)-R\check{\lambda}_{t-1}^{t_0}(i)\Big)\partial_e v &= (\sigma e(i)+\psi n(i))\Bigg(\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i)-\Bigg(\frac{1}{\sigma}\Bigg(\hat{e}_{t_0}^{t_0,u}(i)-\sum_l s_l(i)\,\hat{P}_{l,t_0}\Bigg)+\sum_l \partial_e e_l(i)\,\hat{P}_{l,t_0}\Bigg)\Bigg)\\
&\quad +(\sigma e(i)+\psi n(i))\left(\check{\alpha}^{t_0}(i)-\check{\Xi}_t\right)-\sigma e(i)\Bigg(\Bigg(\hat{W}_t-\sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\Bigg)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\Bigg)\end{aligned}$$

So using, $\check{\lambda}_{t_0-1}^{t_0}(i)=0\,,\frac{1}{R^t}\check{\lambda}_t^{t_0}(i)\to 0$, we have:

$$\begin{aligned}0 &= (\sigma e(i)+\psi Wn(i))\Bigg(\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i)-e(i)\Bigg(\frac{1}{\sigma}\Bigg(\hat{e}_{t_0}^{t_0,u}(i)-\sum_l s_l(i)\,\hat{P}_{l,t_0}\Bigg)+\sum_l \partial_e e_l(i)\,\hat{P}_{l,t_0}\Bigg)\Bigg)+(\sigma e(i)+\psi Wn(i))\,\check{\alpha}^{t_0}(i)\\
&\quad -\left(1-\frac{1}{R}\right)(\sigma e(i)+\psi Wn(i))\sum_{s=0}^{\infty}\frac{1}{R^s}\check{\Xi}_{t_0+s}-\sigma e(i)\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\Bigg(\hat{W}_{t_0+s}-\sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t_0+s}+\hat{P}_{l,t_0+s}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t_0+s}\gamma_{e,k}(i)\,\partial_e e_k(i)\Bigg)\\
\check{\alpha}^{t_0}(i) &= -\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i)+\Bigg(\frac{1}{\sigma}\Bigg(\hat{e}_{t_0}^{t_0,u}(i)-\sum_l s_l(i)\,\hat{P}_{l,t_0}\Bigg)+\sum_l \partial_e e_l(i)\,\hat{P}_{l,t_0}\Bigg)\\
&\quad +\sum_{s=0}^{\infty}\frac{1}{R^{s+1}}\Delta\Omega_{t_0+1+s}+\Omega_{t_0}+\frac{\sigma e(i)}{\sigma e(i)+\psi n(i)}\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\Bigg(\hat{W}_{t_0+s}-\sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t_0+s}+\hat{P}_{l,t_0+s}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t_0+s}\gamma_{e,k}(i)\,\partial_e e_k(i)\Bigg)\end{aligned}$$

Similarly for the budget constraint multiplier of HtM agents,

$$\begin{aligned}0 &= (\sigma e(i)+\psi Wn(i))\Bigg(\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i)-\Bigg(\frac{1}{\sigma}\Bigg(\hat{e}_{t_0}^{t_0,HtM}(i)-\sum_l s_l(i)\,\hat{P}_{l,t_0}\Bigg)+\sum_l \partial_e e_l(i)\,\hat{P}_{l,t_0}\Bigg)\Bigg)\\
&\quad +(\sigma e(i)+\psi Wn(i))\Big(\check{\aleph}_t^{t_0}(i)-\check{\Xi}_t\Big)-\sigma e(i)\Bigg(\hat{W}_t-\sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\Bigg)\\
\check{\aleph}_t^{t_0}(i) &= -\Bigg(\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i)-\Bigg(\frac{1}{\sigma}\Bigg(\hat{e}_t^{t_0,HtM}(i)-\sum_l s_l(i)\,\hat{P}_{l,t}\Bigg)+\sum_l \partial_e e_l(i)\,\hat{P}_{l,t}\Bigg)\Bigg)\\
&\quad +\check{\Xi}_t+\frac{\sigma e(i)}{\sigma e(i)+\psi Wn(i)}\Bigg(\hat{W}_t-\sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\Bigg)\end{aligned}$$

Next, define

$$\tilde{\Lambda}_t\equiv\mathbb{E}_{\delta,t}\Big((1-\varphi(i))\Big(\check{\lambda}_t^{t_0}(i)-R\check{\lambda}_{t-1}^{t_0}(i)\Big)\partial_e v\Big)$$

Using the first-order condition for the optimality of expenditure of unconstrained households to substitute $\Big(\check{\lambda}_t^{t_0}(i)-R\check{\lambda}_{t-1}^{t_0}(i)\Big)\partial_e v$ and the overlapping generation sturcture, the evolution of $\tilde{\Lambda}_t$ is given by:

$$\tilde{\Lambda}_{t+1}-(1-\delta)\,\tilde{\Lambda}_t=-(1-\delta)\left\{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di\Delta\check{\Xi}_{t+1}+\int(1-\varphi(i))\,\sigma e(i)\left(\Delta\hat{W}_{t+1}-\sum_l\partial_e e_l(i)\,\Delta\left(\tilde{A}_{l,t+1}+\hat{P}_{l,t+1}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\Delta\check{\mu}_{k,t+1}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)\right\}di$$
$$+\delta\int(1-\varphi(i))\,\check{\lambda}_{t+1}^{t+1}(i)\,\partial_e v di.$$

Next to derive the evolution of $\int(1-\varphi(i))\,\check{\lambda}_{t+1}^{t+1}(i)\,\partial_e v di$ we define

$$\tilde{\Lambda}_t^0\equiv\int(1-\varphi(i))\,\lambda_t^t(i)\,\partial_e v di-\tilde{\Lambda}_t.$$

The evolution of $\tilde{\Lambda}_t$ in terms of $\tilde{\Lambda}_t^0$ is simply:

$$\tilde{\Lambda}_{t+1}-\tilde{\Lambda}_t=-\left\{\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di\Delta\check{\Xi}_{t+1}+\int(1-\varphi(i))\,\sigma e(i)\left(\Delta\hat{W}_{t+1}-\sum_l\partial_e e_l(i)\,\Delta\left(\tilde{A}_{l,t+1}+\hat{P}_{l,t+1}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\Delta\check{\mu}_{k,t+1}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)\right\}di+\frac{\delta}{1-\delta}\tilde{\Lambda}_{t+1}^0.$$

While using the definition of $\lambda_t^t(i)$from the FOC for expenditure, the evolution of $\tilde{\Lambda}_t^0$ is given by:

$$\begin{aligned}\int(1-\varphi(i))\,\lambda_t^t\partial_e v di&=(\sigma e(i)+\psi Wn(i))\left(\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_t(i)-\left(\frac{1}{\sigma}\left(\hat{e}_t^{t,u}(i)-\sum_l s_l(i)\,\hat{P}_{l,t}\right)+\sum_l\partial_e e_l(i)\,\hat{P}_{l,t}\right)\right)\\&\quad+(\sigma e(i)+\psi Wn(i))\left(\check{\alpha}^t(i)-\check{\Xi}_t\right)-\sigma e(i)\left(\hat{W}_t-\sum_l\partial_e e_l(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)\\&=\int(1-\varphi(i))\,(\sigma e(i)+\psi Wn(i))\,di\sum_{s=0}^{\infty}\frac{1}{R^{s+1}}\Delta\Omega_{t+1+s}\\&\quad+\int(1-\varphi(i))\,\sigma e(i)\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\Delta\hat{W}_{t+1+s}-\sum_l\partial_e e_l(i)\,\Delta\left(\tilde{A}_{l,t+1+s}+\hat{P}_{l,t+1+s}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\Delta\check{\mu}_{k,t+1+s}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)di\\\Rightarrow\tilde{\Lambda}_t^0-\frac{1}{(1-\delta)\,R}\tilde{\Lambda}_{t+1}^0&=-\left(1-\frac{1}{R}\right)\tilde{\Lambda}_t\end{aligned}$$

Coming back to $\tilde{Z}_t$, and defining a new variable $\tilde{\tilde{Z}}_t$ which captures the contribution of the unconstrained households to the variable $\tilde{Z}_t$, we can write

$$\begin{aligned}\tilde{Z}_t&=\psi WN\check{\Xi}_t-\mathbb{E}_{\delta,t}\int\psi Wn(i)\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i)\,di\\&\quad+\mathbb{E}_{\delta,t}\int(1-\varphi(i))\,\psi Wn(i)\left\{\frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}(i)-\sum_l s_l(i)\,\hat{P}_{l,t_0}\right)+\sum_l\partial_e e_l(i)\,\hat{P}_{l,t_0}-\check{\alpha}^{t_0}(i)\right\}+\varphi(i)\,\psi Wn(i)\left\{\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}(i)-\sum_l s_l(i)\,\hat{P}_{l,t}\right)+\sum_l\partial_e e_l(i)\,\hat{P}_{l,t}-\check{\aleph}_t^{t_0}(i)\right\}di\\\tilde{\tilde{Z}}_t&\equiv\tilde{Z}_t+\int\varphi(i)\,\psi Wn(i)\left(\frac{\sigma e(i)}{\sigma e(i)+\psi n(i)}\left(\hat{W}_t-\sum_l\partial_e e_l(i)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)\right)di\\&=\int(1-\varphi(i))\,\psi Wn(i)di\check{\Xi}_t+\mathbb{E}_{\delta,t}\int(1-\varphi(i))\,\psi Wn(i)\left\{\frac{1}{\sigma}\left(\hat{e}_{t_0}^{t_0,u}(i)-\sum_l s_l(i)\,\hat{P}_{l,t_0}\right)+\sum_l\partial_e e_l(i)\,\hat{P}_{l,t_0}-\frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t_0}(i)-\check{\alpha}^{t_0}(i)\right\}.\end{aligned}$$

The evolution of $\tilde{\tilde{Z}}_t$ is given by (using the fact that the second term in the definition of $\tilde{\tilde{Z}}_t$ is independent of t):

$$
\begin{aligned}
\tilde{\tilde{Z}}_{t+1} - (1-\delta)\,\tilde{\tilde{Z}}_t &= (1-\delta)\int (1-\varphi(i))\, Wn(i)\,\psi di \Delta\check{\Xi}_{t+1} \\
&\quad + \delta\left(\int (1-\varphi(i))\,\psi Wn(i) di \check{\Xi}_{t+1} + \int (1-\varphi(i))\,\psi Wn(i)\left\{\frac{1}{\sigma}\left(\hat{e}^{t+1,u}_{t+1}(i) - \sum_l s_l(i)\,\hat{P}_{l,t+1}\right) + \sum_l \partial_e e_l(i)\,\hat{P}_{l,t+1} - \frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t+1}(i) - \check{\alpha}^{t+1}(i)\right\}\right).
\end{aligned}
$$

The second line correspond to the contribution of unconstrained households born at $t+1$ to $\tilde{\tilde{Z}}_{t+1}$. To characterize the dynamics of this term, we define:

$$
\tilde{\tilde{Z}}^0_{t+1} \equiv \int (1-\varphi(i))\,\psi Wn(i) di \check{\Xi}_{t+1} + \int (1-\varphi(i))\,\psi Wn(i)\left\{\frac{1}{\sigma}\left(\hat{e}^{t+1,u}_{t+1}(i) - \sum_l s_l(i)\,\hat{P}_{l,t+1}\right) + \sum_l \partial_e e_l(i)\,\hat{P}_{l,t+1} - \frac{V(i)\,G''(V(i))}{G'(V(i))}\hat{V}_{t+1}(i) - \check{\alpha}^{t+1}(i)\right\} - \tilde{\tilde{Z}}_{t+1}
$$

Using the definition of $\tilde{\tilde{Z}}^0_{t+1}$, the joint evolution of $\tilde{\tilde{Z}}_t$ and $\tilde{\tilde{Z}}^0_t$ is given by:

$$
\begin{aligned}
\tilde{\tilde{Z}}_{t+1} - \tilde{\tilde{Z}}_t &= \int (1-\varphi(i))\, Wn(i)\,\psi di \Delta\check{\Xi}_{t+1} + \frac{\delta}{1-\delta}\tilde{\tilde{Z}}^0_{t+1} \\
\tilde{\tilde{Z}}^0_t - \frac{1}{(1-\delta)R}\tilde{\tilde{Z}}^0_{t+1} &= -\left(1-\frac{1}{R}\right)\int (1-\varphi(i))\,\psi Wn(i)\frac{\sigma e(i)}{\sigma e(i) + \psi Wn(i)}\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right) + \frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k \check{\mu}_{k,t+s}\gamma_{e,k}(i)\,\partial_e e_k(i)\right) di - \left(1-\frac{1}{R}\right)\tilde{\tilde{Z}}_t
\end{aligned}
$$

Finally, define

$$
\begin{aligned}
Z_t &\equiv \tilde{\tilde{Z}}_t + \frac{\int (1-\varphi(i))\, Wn(i)\,\psi di}{\int (1-\varphi(i))\,(\sigma e(i) + \psi Wn(i))\, di}\tilde{\Lambda}_t \\
&\quad + \frac{\int (1-\varphi(i))\, Wn(i)\,\psi di}{\int (1-\varphi(i))\,(\sigma e(i) + \psi Wn(i))\, di}\int (1-\varphi(i))\,\sigma e(i)\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right) + \frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k \check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\right) di \\
Z^0_t &\equiv \tilde{\tilde{Z}}^0_t + \tilde{\Lambda}^0_t
\end{aligned}
$$

The dynamics of Z_t are characterized by the following two equations:

$$
Z_{t+1} - Z_t = \frac{\delta}{1-\delta}Z^0_{t+1}
$$

$$
\begin{aligned}
&Z^0_t - \frac{1}{(1-\delta)R}Z^0_{t+1} + \left(1-\frac{1}{R}\right)Z_t \\
&\quad = \left(1-\frac{1}{R}\right)\int (1-\varphi(i))\,\sigma e(i)\left(\frac{\int (1-\varphi(i))\, Wn(i)\,\psi di}{\int (1-\varphi(i))\,(\sigma e(i) + \psi Wn(i))\, di} - \frac{\psi Wn(i)}{\sigma e(i) + \psi Wn(i)}\right)\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right) + \frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k \check{\mu}_{k,t+s}\gamma_{e,k}(i)\,\partial_e e_k(i)\right) di
\end{aligned}
$$

And we can rewrite our original equation characterizing the optimality of the nominal wage as:

$$
\begin{aligned}
\sum_{k=1}^{K}\lambda_k \check{\mu}_{k,s} W\mathcal{N}_k(\boldsymbol{P}_t, W_t) &= Z_t - \frac{\int (1-\varphi(i))\, Wn(i)\,\psi di}{\int (1-\varphi(i))\,(\sigma e(i) + \psi Wn(i))\, di}\tilde{\Lambda}_t \\
&\quad - \frac{\int (1-\varphi(i))\, Wn(i)\,\psi di}{\int (1-\varphi(i))\,(\sigma e(i) + \psi Wn(i))\, di}\int (1-\varphi(i))\,\sigma e(i)\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right) + \frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k \check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\right) di \\
&\quad - \int \varphi(i)\,\psi Wn(i)\left(\frac{\sigma e(i)}{\sigma e(i) + \psi Wn(i)}\left(\hat{W}_t - \sum_l \partial_e e_l(i)\left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right) + \frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k \check{\mu}_{k,t}\gamma_{e,k}(i)\,\partial_e e_k(i)\right)\right) di + \sum_{k,l}\frac{P_l W \partial_W \mathcal{Y}_{l,k}}{A_k}Y_k\left(\hat{P}_{l,t} + \tilde{A}_{l,t,}\right) + \sum_k \frac{W^2 \partial_W \mathcal{N}_k}{A_k}Y_k \hat{W}_t
\end{aligned}
$$

Next, we derive the evolution of $\tilde{\Lambda}_t$. The optimality of $\hat{R}_t$ allows us to express $\tilde{\Lambda}_t$ in terms of the Lagrange multipliers of the budget constraints of unconstrained and HtM households:

$$
\begin{aligned}
-\mathbb{E}_{\delta,t}\left(\left(1-\varphi\left(i\right)\right)\breve{\lambda}_t^{t_0}\left(i\right)\left(i\right)\partial_e v\left(e\left(i\right),\boldsymbol{P}\right)\right) &= \mathbb{E}_{\delta,t}\left(\left(1-\varphi\left(i\right)\right)\breve{\alpha}_t^{t_0}\left(i\right)\frac{b\left(i\right)}{R}\right)+\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\breve{\aleph}_t^{t_0}\left(i\right)\frac{1}{R}b\left(i\right)\right)\\
-\tilde{\Lambda}_t = &\mathbb{E}_{\delta,t}\left(\left(1-\varphi\left(i\right)\right)\breve{\alpha}_t^{t_0}\left(i\right)\frac{b\left(i\right)}{R}\right)+\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\breve{\aleph}_t^{t_0}\left(i\right)\frac{1}{R}b\left(i\right)\right)\\
&-\left(1-\delta\right)R\mathbb{E}_{\delta,t-1}\left(\left(1-\varphi\left(i\right)\right)\breve{\alpha}_{t-1}^{t_0}\left(i\right)\frac{b\left(i\right)}{R}\right)+\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\breve{\aleph}_{t-1}^{t_0}\left(i\right)\frac{1}{R}b\left(i\right)\right)
\end{aligned}
$$

Define

$$
\tilde{A}_{b,t}\equiv\mathbb{E}_{\delta,t}\left(\left(1-\varphi\left(i\right)\right)\breve{\alpha}_t^{t_0}\left(i\right)\frac{b\left(i\right)}{R}\right)+\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\breve{\aleph}_t^{t_0}\left(i\right)\frac{1}{R}b\left(i\right)\right)
$$

Using our formulas for $\breve{\alpha}_t^{t_0}$ and $\breve{\aleph}_t^{t_0}$ derived above, the evolution of $\tilde{A}_{b,t}$ is given by:

$$
\begin{aligned}
\tilde{A}_{b,t+1}-\left(1-\delta\right)\tilde{A}_{b,t} &= \left(1-\delta\right)\hat{R}_t\int\left(1-\varphi\left(i\right)\right)\frac{b\left(i\right)}{R}di+\left(1-\delta\right)\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)di\Delta\breve{\Xi}_{t+1}\\
&+\left(1-\delta\right)\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(\Delta\hat{W}_{t+1}-\sum_l\partial_e e_l\left(i\right)\left(\Delta\tilde{A}_{l,t+1}+\Delta\hat{P}_{l,t+1}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\Delta\breve{\mu}_{k,t+1}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di\\
&+\left(1-\delta\right)\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\frac{1}{R}b\left(i\right)\Delta\frac{1}{\sigma}\left(\hat{e}_{t+1}^{t_0,HtM}\left(i\right)-\sum_l s_l\left(i\right)\hat{P}_{l,t+1}\right)+\Delta\sum_l\partial_e e_l\left(i\right)\hat{P}_{l,t+1}\right)\\
&+\delta\int\left(\left(1-\varphi\left(i\right)\right)\breve{\alpha}_{t+1}^{t+1}\left(i\right)+\varphi\left(i\right)\breve{\aleph}_{t+1}^{t+1}\left(i\right)\right)\frac{1}{R}b\left(i\right)di
\end{aligned}
$$

The last line gives the contribution of the newborn households, to characterize its evolution, we define:

$$
\tilde{A}_{b,t+1}^{0}=\int\left(\left(1-\varphi\left(i\right)\right)\breve{\alpha}_{t+1}^{t+1}\left(i\right)+\varphi\left(i\right)\breve{\aleph}_{t+1}^{t+1}\left(i\right)\right)\frac{1}{R}b\left(i\right)di-\tilde{A}_{b,t+1}
$$

The decisions of households born at t in terms of expenditure at t and their change in welfare at t (usingRoy's identity) are given by:

$$
\hat{e}_t^{t_0,HtM}\left(i\right)-\sum_l s_l\left(i\right)\hat{P}_{l,t}=\frac{\sigma}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left\{\hat{R}_t\frac{b\left(i\right)}{R}+Wn(i)\left(\psi\hat{W}_t+\sum\bar{s}_k\tilde{A}_{k,t}\right)-\sum_k\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\partial_e e_k(i)\right)\hat{P}_{k,t}+\left(1-\frac{1}{R}\right)b(i)\sum_l\bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t}\right)\right\}
$$

$$
\begin{aligned}
dV_{t_0}\left(i\right) &= \partial_e v\sum_{s=0}\frac{1}{R_s}\left\{\frac{b\left(i\right)}{R}\hat{R}_{t+s}+\hat{W}_{t+s}Wn\left(i\right)-\sum_k e_k\left(i\right)\hat{P}_{k,t}+\varsigma\left(i\right)\sum_k E_k\left(\hat{P}_{k,t+s}+\tilde{A}_{k,t+s}-\hat{W}_{t+s}\right)+\frac{R-1}{R}b\left(i\right)\sum_k\bar{s}_k\hat{P}_{k,t_0}\right\}\\
&= \partial_e v\sum_{s=0}\frac{1}{R_s}\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+1+s}\right)-e\left(i\right)\sum_k\left(s_k\left(i\right)-\bar{s}_k\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_k\bar{s}_k\tilde{A}_{k,t+s}\right\}\\
dV_{-}\left(i\right) &= \partial_e v\left(\sum_{s=0}\frac{1}{R_s}\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_s-\pi_{cpi,1+s}\right)-e\left(i\right)\sum_k\left(s_k\left(i\right)-\bar{s}_k\right)\hat{P}_{k,s}+Wn\left(i\right)\sum_k\bar{s}_k\tilde{A}_{k,s}\right\}-b\left(i\right)\sum_k\bar{s}_k\hat{P}_{k,0}\right)
\end{aligned}
$$

$$
\begin{aligned}
\hat{e}_t^{t,u}-\sum_k s_k(i)\hat{P}_{k,t} &= -\sigma\sum\frac{1}{R^{s+1}}\left(\hat{R}_{t+s}-\sum_k\partial_e e_k(i)\pi_{k,t+s+1}\right)\\
&+\frac{\left(1-\frac{1}{R}\right)\sigma}{\left(\sigma e(i)+\psi Wn(i)\right)}\sum\frac{1}{R^s}\left(\frac{b(i)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+s+1}\right)+\psi Wn(i)\hat{W}_{t+s}-\sum_k\left(e(i)\left(s_k(i)-\bar{s}_k\right)+\psi Wn(i)\partial_e e_k(i)\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_k\bar{s}_k\tilde{A}_{k,t+s}\right)
\end{aligned}
$$

.

$$\begin{aligned}\hat{e}_0^{-,u} - \sum_k s_k(i)\hat{P}_{k,t} &= -\sigma \sum \frac{1}{R^{s+1}} \left(\hat{R}_{t+s} - \sum_k \partial_e e_k(i) \pi_{k,t+s+1} \right) \\ &\quad + \frac{\left(1-\frac{1}{R}\right)\sigma}{(\sigma e(i) + \psi Wn(i))} \Bigg\{ \sum \frac{1}{R^s} \Bigg(\frac{b(i)}{R} \left(\hat{R}_{t+s} - \pi_{cpi,t+s+1}\right) + \psi Wn(i)\hat{W}_{t+s} - \sum_k \left(e(i)\left(s_k(i) - \bar{s}_k\right) + \psi Wn(i)\partial_e e_k(i)\right) \hat{P}_{k,t+s} + Wn\left(i\right) \sum_k \bar{s}_k \tilde{A}_{k,t+s} \Bigg) - b\left(i\right) \sum_k \bar{s}_k \hat{P}_{k,0} \Bigg\}\end{aligned}$$

Using these expressions, $\check{\alpha}_t^t$ can be rewritten as:

$$\begin{aligned}\check{\alpha}_t^t\left(i\right) = & \left(-\frac{\partial_e v G''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)} + \frac{\left(1-\frac{1}{R}\right)}{(\sigma e(i) + \psi Wn(i))} \right) \sum_{s=0} \frac{1}{R_s} \left\{ \frac{b\left(i\right)}{R} \left(\hat{R}_{t+s} - \pi_{cpi,t+1+s}\right) - e\left(i\right) \sum_k \left(s_k\left(i\right) - \bar{s}_k\right) \hat{P}_{k,t+s} + Wn\left(i\right) \sum_k \bar{s}_k \tilde{A}_{k,t+s} \right\} \\ & + \left(1-\frac{1}{R}\right) \sum \frac{1}{R^s} \hat{W}_{t+s} - \sum \frac{1}{R^{s+1}} \hat{R}_{t+s} + \sum_{s=0}^{\infty} \frac{1}{R^{s+1}} \Delta\Omega_{t+1+s} + \check{\Xi}_t \\ & + \frac{\sigma e\left(i\right)}{\sigma e\left(i\right) + \psi Wn\left(i\right)} \left(1-\frac{1}{R}\right) \sum_{s=0}^{\infty} \frac{1}{R^s} \left(-\sum_l \partial_e e_l\left(i\right) \tilde{A}_{l,t+s} + \frac{1}{P_k Y_k} \sum_{k=1}^{K} \lambda_k \check{\mu}_{k,t+s} \gamma_{e,k}\left(i\right) \partial_e e_k\left(i\right) \right)\end{aligned}$$

The evolution of $\tilde{A}_{b,t}$ and $\tilde{A}_{b,t}^0$ is therefore characterized by:

$$\begin{aligned}\tilde{A}_{b,t+1} - \tilde{A}_{b,t} = & \hat{R}_t \int \left(1-\varphi\left(i\right)\right) \frac{b\left(i\right)}{R} di + \int \varphi\left(i\right) \frac{1}{R} b\left(i\right) di \Delta\check{\Xi}_{t+1} + \int \varphi\left(i\right) \frac{b\left(i\right)}{R} \Delta\hat{W}_{t+1} di \\ & + \int \varphi\left(i\right) \frac{b\left(i\right)}{R} \frac{\sigma e\left(i\right)}{\sigma e\left(i\right) + \psi Wn\left(i\right)} \left(-\sum_l \partial_e e_l\left(i\right) \left(\Delta\tilde{A}_{l,t+1}\right) + \frac{1}{P_k Y_k} \sum_{k=1}^{K} \lambda_k \Delta\check{\mu}_{k,t+1} \gamma_{e,k}\left(i\right) \partial_e e_k\left(i\right) \right) di \\ & + \mathbb{E}_{\delta,t} \left(\varphi\left(i\right) \frac{b\left(i\right)}{R} \frac{1}{\sigma e\left(i\right) + \psi Wn\left(i\right)} \left\{ \frac{b\left(i\right)}{R} \left(\Delta\hat{R}_{t+1} - \left(R-1\right)\pi_{cpi,t+1}\right) + Wn(i) \sum \bar{s}_k \Delta\tilde{A}_{k,t+1} - \sum_k e(i)\left(s_k(i) - \bar{s}_k\right)\pi_{k,t+1} \right\} \right) + \frac{\delta}{1-\delta} \tilde{A}_{b,t+1}^0\end{aligned}$$

$$\begin{aligned}\tilde{A}_{b,t}^0 - \frac{1}{\left(1-\delta\right)R} \tilde{A}_{b,t+1}^0 + \left(1-\frac{1}{R}\right) \tilde{A}_{b,t} = & \int \frac{b\left(i\right)}{R} \left(-\frac{\partial_e v G''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)} + \frac{\left(1-\frac{1}{R}\right)}{\sigma e\left(i\right) + \psi Wn\left(i\right)} \right) \left\{ \frac{b\left(i\right)}{R} \left(\hat{R}_{t+s} - \pi_{cpi,t+1+s}\right) - e\left(i\right) \sum_k \left(s_k\left(i\right) - \bar{s}_k\right) \hat{P}_{k,t+s} + Wn\left(i\right) \sum_k \bar{s}_k \tilde{A}_{k,t+s} \right\} di \\ & + \left(1-\frac{1}{R}\right) \int \frac{b\left(i\right)}{R} \frac{\sigma e\left(i\right)}{\sigma e\left(i\right) + \psi Wn\left(i\right)} \left(-\sum_l \partial_e e_l\left(i\right) \tilde{A}_{l,t} + \frac{1}{P_k Y_k} \sum_{k=1}^{K} \lambda_k \check{\mu}_{k,t} \gamma_{e,k}\left(i\right) \partial_e e_k\left(i\right) \right) di\end{aligned}$$

Using the relationship between $\tilde{\Lambda}_t$ and $\check{\Xi}_t$, we obtain:

$$
\begin{aligned}
&\left(1-\frac{\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)A_{b,t+1}-\left(1-\frac{\left(1-\delta+R\right)\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)A_{b,t}-\frac{(1-\delta)R\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}A_{b,t-1}-\frac{\delta}{1-\delta}A^{0}_{b,t+1}=\\
&\qquad-\sum\left(\int\varphi\left(i\right)\frac{b\left(i\right)}{R}\frac{\sigma e\left(i\right)\partial_{e}e_{l}\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\frac{b(i)}{R}di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_{e}e_{l}\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\Delta\tilde{A}_{l,t+1}\\
&\qquad+\sum\left(\int\varphi\left(i\right)\frac{b\left(i\right)}{R}\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\gamma_{e,k}\left(i\right)\partial_{e}e_{k}\left(i\right)di+\frac{\int\left(1-\varphi\left(i\right)\right)\frac{b(i)}{R}di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_{e}e_{k}\left(i\right)\gamma_{e,k}\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\frac{\lambda_{k}}{P_{k}Y_{k}}\lambda_{k}\Delta\check{\mu}_{k,t+1}\\
&\qquad+\left(\int\varphi\left(i\right)\frac{\left(\frac{b(i)}{R}\right)^{2}}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\left(\int\left(1-\varphi\left(i\right)\right)\frac{b(i)}{R}di\right)^{2}}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right)\\
&\qquad+\left(\int\varphi\left(i\right)\frac{\frac{b(i)}{R}Wn(i)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\frac{b(i)}{R}di\int\left(1-\varphi\left(i\right)\right)Wn(i)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\sum\bar{s}_{k}\Delta\tilde{A}_{k,t+1}\\
&\qquad-\sum\left(\int\varphi\left(i\right)\frac{\frac{b(i)}{R}e(i)\left(s_{k}(i)-\bar{s}_{k}\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\frac{b(i)}{R}di\int\left(1-\varphi\left(i\right)\right)\frac{e(i)}{E}\left(s_{k}(i)-\bar{s}_{k}\right)}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\pi_{k,t+1}
\end{aligned}
$$

$$
\begin{aligned}
&A^{0}_{b,t}-\frac{1}{\left(1-\delta\right)R}A^{0}_{b,t+1}+\left(1-\frac{1}{R}\right)A_{b,t}=\int\frac{b\left(i\right)}{R}\left(-\frac{\partial_{e}vG''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}+\frac{\left(1-\frac{1}{R}\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+1+s}\right)-e\left(i\right)\sum_{k}\left(s_{k}\left(i\right)-\bar{s}_{k}\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_{k}\bar{s}_{k}\tilde{A}_{k,t+s}\right\}di\\
&\qquad+\left(1-\frac{1}{R}\right)\int\frac{b\left(i\right)}{R}\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(-\sum_{l}\partial_{e}e_{l}\left(i\right)\tilde{A}_{l,t}+\frac{1}{P_{k}Y_{k}}\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_{e}e_{k}\left(i\right)\right)di
\end{aligned}
$$

And the original wage equation in terms of Z_t and $A_{b,t}$ is

$$
\begin{aligned}
&\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,s}W\mathcal{N}_{k}(\boldsymbol{P}_{t},W_{t})=Z_{t}+\frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi n\left(i\right)\right)di}\left(A_{b,t}-(1-\delta)RA_{b,t-1}\right)\\
&\qquad-\frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi n\left(i\right)\right)di}\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_{k}Y_{k}}\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_{e}e_{k}\left(i\right)\right)di\\
&\qquad-\int\varphi\left(i\right)\psi Wn(i)\left(\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_{k}Y_{k}}\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_{e}e_{k}\left(i\right)\right)\right)di\\
&\qquad+\sum_{k,l}\frac{P_{l}W\partial_{W}\mathcal{Y}_{l,k}}{A_{k}}Y_{k}\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_{k}\frac{W^{2}\partial_{W}\mathcal{N}_{k}}{A_{k}}Y_{k}\hat{W}_{t}
\end{aligned}
$$

Solving the Price Setting equation

The second set of main equations are given by the optimality of price setting:

$$
\begin{aligned}
\left(\beta\check{\mu}_{k,t+1}-\left(1+\left(\left(1-\delta\right)\beta\right)\right)\check{\mu}_{k,t}+\left(1-\delta\right)\check{\mu}_{k,t-1}\right) &= \lambda_{k}\check{\mu}_{k,t}-\sum_{l=1}^{K}\lambda_{l}\frac{\check{\mu}_{l,t}}{Y_{l}P_{l}}\int\left(\gamma_{e,l}\left(i\right)e_{l}\left(i\right)\rho_{l,k}\left(i\right)\right)di-\sum_{l=1}^{K}\lambda_{l}\frac{\check{\mu}_{l,t}}{P_{l}Y_{l}}\frac{P_{k}\mathcal{Y}_{k,l}Y_{l}}{A_{l}} \\
&\quad -\mathbb{E}_{\delta,t}\left(\int\left[\left(1-\varphi\left(i\right)\right)\left(\check{\lambda}_{t}^{t_{0}}\left(i\right)-R\check{\lambda}_{t-1}^{t_{0}}\left(i\right)\right)+\check{\zeta}_{t}^{t_{0},u}\left(i\right)W\right]\partial_{e}v\partial_{e}e_{k}\left(i\right)+\varphi\left(i\right)\check{\zeta}_{t}^{t_{0},HtM}\left(i\right)W\partial_{e}v\partial_{e}e_{k}\left(i\right)di\right) \\
&\quad +\mathbb{E}_{\delta,t}\left(\int\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}_{t}^{t_{0}}\left(i\right)+\varphi\left(i\right)\check{\aleph}_{t}^{t_{0}}\left(i\right)\right)\left[e_{k}(i)-\varsigma\left(i\right)E_{k}\right]di\right)-\delta\int\left(1-\varphi\left(i\right)\right)\check{\alpha}^{t}\left(i\right)b\left(i\right)di\bar{s}_{k} \\
&\quad -\delta\sum_{u=0}\left(\left(1-\delta\right)\beta\right)^{u}\int\varphi\left(i\right)\check{\aleph}_{t+u}^{t}\left(i\right)b\left(i\right)di\bar{s}_{k}\frac{R-1}{R}+\delta\sum_{u=0}\frac{1}{R^{u}}\int\varphi\left(i\right)b\left(i\right)di\bar{s}_{k}\frac{\hat{R}_{t+u}}{R} \\
&\quad +\sum_{l,m}\frac{P_{l}P_{k}\partial_{P_{k}}\mathcal{Y}_{l,m}}{A_{m}}Y_{m}\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_{l}\frac{WP_{k}\partial_{P_{k}}\mathcal{N}_{l}}{A_{l}}Y_{l}\hat{W}_{t}+\sum_{l}E_{l}\bar{\rho}_{l,k}\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)-Y_{k}\vartheta_{k}\left(\pi_{k,t}-\beta\pi_{k,t+1}\right)
\end{aligned}
$$

As in the previous subsection the goal here is to derive the dynamics of the components of these equation using only a finite number of variables.
First note that we have

$$
\begin{aligned}
\mathbb{E}_{\delta,t}\left(\int\varphi\left(i\right)\check{\zeta}_{t}^{t_{0},HtM}\left(i\right)W\partial_{e}v\partial_{e}e_{k}\left(i\right)di\right) &= \mathbb{E}_{\delta,t}\left(\int\varphi\left(i\right)\partial_{e}e_{k}\left(i\right)\psi Wn\left(i\right)\left\{\check{\Xi}_{t}-\check{\aleph}_{t}^{t_{0}}\left(i\right)-\frac{G''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}dV_{t_{0}}\left(i\right)+\frac{1}{\sigma}e_{t}^{t_{0},HtM}\left(i\right)+\partial_{e}e_{l}\left(i\right)\cdot\hat{P}_{l,t}\right\}di\right) \\
&= -\mathbb{E}_{\delta,t}\left(\int\varphi\left(i\right)\partial_{e}e_{k}\left(i\right)\psi Wn\left(i\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_{k}Y_{k}}\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_{e}e_{k}\left(i\right)\right)di\right) \\
&= -\int\varphi\left(i\right)\partial_{e}e_{k}\left(i\right)\psi Wn\left(i\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_{k}Y_{k}}\sum_{k=1}^{K}\lambda_{k}\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_{e}e_{k}\left(i\right)\right)di
\end{aligned}
$$

Next, we solve for $\tilde{L}_{k,t}\equiv\mathbb{E}_{\delta,t}\left(\int\left[\left(1-\varphi\left(i\right)\right)\left(\check{\lambda}_{t}^{t_{0}}\left(i\right)-R\check{\lambda}_{t-1}^{t_{0}}\left(i\right)\right)+\check{\zeta}_{t}^{t_{0},u}\left(i\right)\right]\partial_{e}v\partial_{e}e_{k}\left(i\right)\right)$, that we can re-express as:

$$
\tilde{L}_{k,t}=\mathbb{E}_{\delta,t}\left(\int\left(1-\varphi\left(i\right)\right)\partial_{e}e_{k}\left(i\right)\sigma e\left(i\right)\left\{\frac{G''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}-\frac{1}{\sigma\left(i\right)}\hat{e}_{t_{0}}^{t_{0},u}\left(i\right)-\partial_{e}e_{l}\left(i\right)\cdot\hat{P}_{l,t_{0}}+\check{\alpha}_{t_{0}}\left(i\right)-\left(\hat{W}_{t}-\partial_{e}e_{l}\left(i\right)\cdot\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)-\check{\Xi}_{t}-\frac{1}{P_{l}Y_{l}}\sum_{l=1}^{K}\lambda_{l}\check{\mu}_{l,t}\gamma_{e,l}\left(i\right)\partial_{e}e_{l}\left(i\right)\right\}di\right)
$$

Define

$$
\begin{aligned}
\tilde{\tilde{L}}_{k,t} &= \mathbb{E}_{\delta,t}\left(\left(1-\varphi\left(i\right)\right)\partial_{e}e_{k}\left(i\right)\sigma e\left(i\right)\left\{\frac{G''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}-\frac{1}{\sigma\left(i\right)}\hat{e}_{t_{0}}^{t_{0},u}\left(i\right)-\partial_{e}e_{l}\left(i\right)\cdot\hat{P}_{l,t_{0}}+\check{\alpha}_{t_{0}}\left(i\right)-\check{\Xi}_{t}\right\}\right) \\
&= \tilde{L}_{k,t}+\int\left(1-\varphi\left(i\right)\right)\partial_{e}e_{k}\left(i\right)\sigma e\left(i\right)\left\{\left(\hat{W}_{t}-\partial_{e}e_{l}\left(i\right)\cdot\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)+\frac{1}{P_{l}Y_{l}}\sum_{l=1}^{K}\lambda_{l}\check{\mu}_{l,t}\gamma_{e,l}\left(i\right)\partial_{e}e_{l}\left(i\right)\right\}di
\end{aligned}
$$

We have

$$
\Delta\tilde{\tilde{L}}_{k,t+1}=-\int\left(1-\varphi\left(i\right)\right)\partial_{e}e_{k}\left(i\right)\sigma e\left(i\right)di\Delta\check{\Xi}_{t+1}-\frac{\lambda_{l}}{P_{l}Y_{l}}\sum_{l=1}^{K}\int\left(1-\varphi\left(i\right)\right)\partial_{e}e_{k}\left(i\right)\sigma e\left(i\right)\partial_{e}e_{l}\left(i\right)di\Delta\check{\mu}_{l,t+1}+\frac{\delta}{1-\delta}\tilde{\tilde{L}}_{k,t+1}^{0}
$$

With

$$
\tilde{\tilde{L}}_{k,t}^{0}-\frac{1}{R\left(1-\delta\right)}\tilde{\tilde{L}}_{k,t+1}^{0}+\left(1-\frac{1}{R}\right)\tilde{\tilde{L}}_{k,t}=\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\frac{\sigma e\left(i\right)}{\psi Wn\left(i\right)+\sigma e\left(i\right)}\sigma e\left(i\right)\partial_{e}e_{k}\left(i\right)\left(\hat{W}_{t}-\sum_{l}\partial_{e}e_{l}\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_{l}Y_{l}}\sum_{l=1}^{K}\lambda_{l}\check{\mu}_{l,t}\gamma_{e,l}\left(i\right)\partial_{e}e_{l}\left(i\right)\right)
$$

Define

$$
\begin{aligned}
L_{k,t} &\equiv \tilde{\tilde{L}}_{k,t} - \frac{\int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi W n\left(i\right)\right)di}\tilde{\Lambda}_t \\
&- \frac{\int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi W n\left(i\right)\right)di}\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\left(\hat{W}_t - \sum_l \partial_e e_l\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di \\
L^0_{k,t} &\equiv \tilde{\tilde{L}}^0_{k,t} - \frac{\int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi W n\left(i\right)\right)di}\tilde{\Lambda}^0_t
\end{aligned}
$$

We have

$$
\Delta L_{k,t+1} = \frac{\delta}{1-\delta}L^0_{k,t+1}
$$

$$
\begin{aligned}
L^0_{k,t} - \frac{1}{R\left(1-\delta\right)}L^0_{k,t+1} + \left(1-\frac{1}{R}\right)L_{k,t} &= \left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\frac{\sigma e\left(i\right)}{\psi w n\left(i\right)+\sigma e\left(i\right)}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t - \sum_l \partial_e e_l\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_l Y_l}\sum_{l=1}^{K}\lambda_l\check{\mu}_{l,t}\gamma_{e,l}\left(i\right)\partial_e e_l\left(i\right)\right) \\
&- \frac{\int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi W n\left(i\right)\right)di}\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\left(\hat{W}_t - \sum_l \partial_e e_l\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di
\end{aligned}
$$

The resetting equation becomes

$$
\begin{aligned}
\left(\beta\check{\mu}_{k,t+1} - \left(1+\left(\left(1-\delta\right)\beta\right)\right)\check{\mu}_{k,t} + \left(1-\delta\right)\check{\mu}_{k,t-1}\right) &= \lambda_k\check{\mu}_{k,t} - \sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,t}}{Y_l P_l}\int\left(\gamma_{e,l}\left(i\right)e_l\left(i\right)\rho_{l,k}\left(i\right)\right)di - \sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,t}}{P_l Y_l}\frac{P_k\mathcal{Y}_{k,l}Y_l}{A_l} \\
&- L_{k,t} + \int\varphi\left(i\right)\partial_e e_k\psi n\left(i\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi W n\left(i\right)}\left(\hat{W}_t - \sum_l \partial_e e_l\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di \\
&+ \int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\sigma e\left(i\right)\left\{\left(\hat{W}_t - \partial_e e_l\left(i\right)\cdot\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)+\frac{1}{P_l Y_l}\sum_{l=1}^{K}\lambda_l\check{\mu}_{l,t}\gamma_{e,l}\left(i\right)\partial_e e_l\left(i\right)\right\}di \\
&- \frac{\int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi W n\left(i\right)\right)di}\tilde{\Lambda}_t \\
&- \frac{\int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi W n\left(i\right)\right)di}\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\left(\hat{W}_t - \sum_l \partial_e e_l\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_k Y_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di \\
&+ \mathbb{E}_{\delta,t}\left(\int\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}^{t_0}_t\left(i\right)+\varphi\left(i\right)\check{\aleph}^{t_0}_t\left(i\right)\right)\left[e_k(i)-\varsigma\left(i\right)E_k\right]di\right) - \delta\int\left(1-\varphi\left(i\right)\right)\check{\alpha}^t\left(i\right)b\left(i\right)di\bar{s}_k \\
&- \delta\sum_{u=0}\left(\left(1-\delta\right)\beta\right)^u\int\varphi\left(i\right)\check{\aleph}^t_{t+u}\left(i\right)b\left(i\right)di\bar{s}_k\frac{R-1}{R} + \delta\sum_{u=0}\frac{1}{R^u}\int\varphi\left(i\right)b\left(i\right)di\bar{s}_k\frac{\hat{R}_{t+u}}{R} \\
&+ \sum_{l,m}\frac{P_l P_k\partial_{P_k}\mathcal{Y}_{l,m}}{A_m}Y_m\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right) + \sum_l\frac{W P_k\partial_{P_k}\mathcal{N}_l}{A_l}Y_l\hat{W}_t + \sum_l E_l\bar{\rho}_{l,k}\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right) - Y_k\vartheta_k\left(\pi_{k,t}-\beta\pi_{k,t+1}\right)
\end{aligned}
$$

Next note that we have

$$
\begin{aligned}
\mathbb{E}_{\delta,t}\left(\int\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}^{t_0}_t\left(i\right)+\varphi\left(i\right)\check{\aleph}^{t_0}_t\left(i\right)\right)\left[e_k(i)-\varsigma\left(i\right)E_k\right]di\right) &= \mathbb{E}_{\delta,t}\left(\int\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}^{t_0}_t\left(i\right)+\varphi\left(i\right)\check{\aleph}^{t_0}_t\left(i\right)\right)\left[e_k(i)-e(i)\bar{s}_k\right]di\right) \\
&+ \left(1-\frac{1}{R}\right)\bar{s}_k\mathbb{E}_{\delta,t}\left(\int\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}^{t_0}_t\left(i\right)+\varphi\left(i\right)\check{\aleph}^{t_0}_t\left(i\right)\right)b(i)di\right)
\end{aligned}
$$

Define

$$\tilde{A}_{e_k,t} \equiv \mathbb{E}_{\delta,t}\left(\int \left(\left(1-\varphi\left(i\right)\right)\check{\alpha}_t^{t_0}\left(i\right)+\varphi\left(i\right)\check{\aleph}_t^{t_0}\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\right)$$

We have

$$\begin{aligned}
\tilde{A}_{e_k,t+1}-\left(1-\delta\right)\tilde{A}_{e_k,t} &= \left(1-\delta\right)\hat{R}_t\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di+\left(1-\delta\right)\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\Delta\check{\Xi}_{t+1} \\
&\quad+\left(1-\delta\right)\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(\Delta\hat{W}_{t+1}-\sum_l\partial_e e_l\left(i\right)\left(\Delta\tilde{A}_{l,t+1}+\Delta\hat{P}_{l,t+1}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\Delta\check{\mu}_{k,t+1}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di \\
&\quad+\left(1-\delta\right)\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\Delta\frac{1}{\sigma}\left(\hat{e}_{t+1}^{t_0,HtM}\left(i\right)-\sum_l s_l\left(i\right)\hat{P}_{l,t+1}\right)+\Delta\sum_l\partial_e e_l\left(i\right)\hat{P}_{l,t+1}\right) \\
&\quad+\delta\int\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}_{t+1}^{t+1}+\varphi\left(i\right)\check{\aleph}_{t+1}^{t+1}\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di
\end{aligned}$$

Define

$$\tilde{A}^0_{e_k,t+1}=\int\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}_{t+1}^{t+1}\left(i\right)+\varphi\left(i\right)\check{\aleph}_{t+1}^{t+1}\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di-\tilde{A}_{e_k,t+1}$$

We have

$$\begin{aligned}
\tilde{A}_{e_k,t+1}-\tilde{A}_{e_k,t} &= \hat{R}_t\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di+\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\Delta\check{\Xi}_{t+1}+\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\Delta\hat{W}_{t+1}di \\
&\quad+\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(-\sum_l\partial_e e_l\left(i\right)\left(\Delta\tilde{A}_{l,t+1}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\Delta\check{\mu}_{k,t+1}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di \\
&\quad+\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{1}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left\{\frac{b\left(i\right)}{R}\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right)+Wn(i)\sum\bar{s}_k\Delta\tilde{A}_{k,t+1}-\sum_k e(i)\left(s_k(i)-\bar{s}_k\right)\pi_{k,t+1}\right\}\right)+\frac{\delta}{1-\delta}\tilde{A}^0_{e_k,t+1}
\end{aligned}$$

$$\begin{aligned}
\tilde{A}^0_{e_k,t}-\frac{1}{\left(1-\delta\right)R}\tilde{A}^0_{e_k,t+1}+\left(1-\frac{1}{R}\right)\tilde{A}_{e_k,t} &= +\left(1-\frac{1}{R}\right)\int\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(-\sum_l\partial_e e_l\left(i\right)\tilde{A}_{l,t}+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di \\
&\quad+\int\left(e_k(i)-e(i)\bar{s}_k\right)\left(-\frac{\partial_e vG''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}+\frac{\left(1-\frac{1}{R}\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+1+s}\right)-e\left(i\right)\sum_k\left(s_k\left(i\right)-\bar{s}_k\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_k\bar{s}_k\tilde{A}_{k,t+s}\right\}di
\end{aligned}$$

Define

$$\begin{aligned}
A_{e_k,t} &\equiv \tilde{A}_{e_k,t}+\frac{\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}di\tilde{\Lambda}_t \\
A^0_{e_k,t} &\equiv \tilde{A}^0_{e_k,t}+\frac{\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}di\tilde{\Lambda}^0_t
\end{aligned}$$

We have

$$\begin{aligned}
A_{e_k,t+1}-A_{e_k,t} &= \hat{R}_t\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di \\
&\quad -\frac{\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\left\{\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\left(\Delta\hat{W}_{t+1}-\sum_l\partial_e e_l\left(i\right)\Delta\left(\tilde{A}_{l,t+1}+\hat{P}_{l,t+1}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\Delta\check{\mu}_{k,t+1}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)\right\} \\
&\quad +\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\Delta\hat{W}_{t+1}di \\
&\quad +\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(-\sum_l\partial_e e_l\left(i\right)\left(\Delta\tilde{A}_{l,t+1}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\Delta\check{\mu}_{k,t+1}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di \\
&\quad +\mathbb{E}_{\delta,t}\left(\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{1}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left\{\frac{b\left(i\right)}{R}\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right)+Wn(i)\sum\bar{s}_k\Delta\tilde{A}_{k,t+1}-\sum_k e(i)\left(s_k(i)-\bar{s}_k\right)\pi_{k,t+1}\right\}\right)+\frac{\delta}{1-\delta}A^0_{e_k,t+1}
\end{aligned}$$

$$\begin{aligned}
A^0_{e_k,t}-\frac{1}{\left(1-\delta\right)R}A^0_{e_k,t+1}+\left(1-\frac{1}{R}\right)A_{e_k,t} &= \left(1-\frac{1}{R}\right)\int\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(-\sum_l\partial_e e_l\left(i\right)\tilde{A}_{l,t}+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di \\
&\quad +\int\left(e_k(i)-e(i)\bar{s}_k\right)\left(-\frac{\partial_e v G''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}+\frac{\left(1-\frac{1}{R}\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+1+s}\right)-e\left(i\right)\sum_k\left(s_k\left(i\right)-\bar{s}_k\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_k\bar{s}_k\tilde{A}_{k,t+s}\right\}di
\end{aligned}$$

Using the evolution of the wage (derived from the output gap Euler)

$$\begin{aligned}
\left(1-\varphi^N\right)\psi\Delta\hat{W}_{t+1} &= \left(\left(1-\varphi^N\right)\left(\sigma+\psi\right)-\sigma\left(1-\frac{1}{R}\right)\int\varphi\left(i\right)\frac{b(i)}{RE}\right)\hat{R}_t-\left(1-\varphi^N\right)\bar{s}_k\cdot\Delta\tilde{A}_{l,t+1} \\
&\quad +\int\varphi\left(i\right)\left\{\frac{b(i)}{RE}\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right)-\frac{e(i)}{E}\sum_k\left(\left(s_k(i)-\bar{s}_k\right)\right)\pi_{k,t+1}\right\}di \\
&\quad +\sigma\left(\sum_k-\int\left(1-\varphi\left(i\right)\right)\frac{e(i)}{E}\partial_e e_k(i)\pi_{k,t+1}\right)
\end{aligned}$$

we have

$$\begin{aligned}
A_{e_k,t+1}-A_{e_k,t}-\frac{\delta}{1-\delta}A^0_{e_k,t+1} &= -\sum\left(\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)\partial_e e_l\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_e e_l\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\Delta\tilde{A}_{l,t+1} \\
&\quad +\sum\left(\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\gamma_{e,l}\left(i\right)\partial_e e_l\left(i\right)di+\frac{\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_e e_l\gamma_{e,l}\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\frac{\lambda_l}{P_lY_l}\lambda_l\Delta\check{\mu}_{l,t+1} \\
&\quad +\left(\int\varphi\left(i\right)\frac{\left(\frac{b(i)}{R}\right)\left(e_k(i)-e(i)\bar{s}_k\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\left(\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\right)^2}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right) \\
&\quad +\left(\int\varphi\left(i\right)\frac{\left(e_k(i)-e(i)\bar{s}_k\right)Wn(i)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\int\left(1-\varphi\left(i\right)\right)Wn(i)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\sum\bar{s}_l\Delta\tilde{A}_{l,t+1} \\
&\quad -\sum\left(\int\varphi\left(i\right)\frac{\left(e_k(i)-e(i)\bar{s}_k\right)e(i)\left(s_l(i)-\bar{s}_l\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\left(e_k(i)-e(i)\bar{s}_k\right)di\int\left(1-\varphi\left(i\right)\right)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\pi_{k,t+1}
\end{aligned}$$

$$\begin{aligned}A^{0}_{e_k,t}-\frac{1}{(1-\delta)R}A^{0}_{e_k,t+1}+\left(1-\frac{1}{R}\right)A_{e_k,t}=\int\left(e_k(i)-e(i)\bar{s}_k\right)\left(-\frac{\partial_e vG''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}+\frac{\left(1-\frac{1}{R}\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+1+s}\right)-e\left(i\right)\sum_k\left(s_k\left(i\right)-\bar{s}_k\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_k\bar{s}_k\tilde{A}_{k,t+s}\right\}\\+\left(1-\frac{1}{R}\right)\int\left(e_k(i)-e(i)\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(-\sum_l\partial_e e_l\left(i\right)\tilde{A}_{l,t}+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di\end{aligned}$$

The resetting equation becomes

$$\begin{aligned}\left(\beta\check{\mu}_{k,t+1}-\left(1+\left(\left(1-\delta\right)\beta\right)\right)\check{\mu}_{k,t}+\left(1-\delta\right)\check{\mu}_{k,t-1}\right)=\lambda_k\check{\mu}_{k,t}-\sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,t}}{Y_lP_l}\int\left(\gamma_{e,l}\left(i\right)e_l\left(i\right)\rho_{l,k}\left(i\right)\right)di-\sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,t}}{P_lY_l}\frac{P_k\mathcal{Y}_{k,l}Y_l}{A_l}\\-L_{k,t}+\int\varphi\left(i\right)\partial_e e_k\psi n\left(i\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(\hat{W}_t-\sum_l\partial_e e_l\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di\\+\int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\sigma e\left(i\right)\left\{\left(\hat{W}_t-\partial_e e_l\left(i\right)\cdot\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)\right)+\frac{1}{P_lY_l}\sum_{l=1}^{K}\lambda_l\check{\mu}_{l,t}\gamma_{e,l}\left(i\right)\partial_e e_l\left(i\right)\right\}di\\-\frac{\int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\tilde{\Lambda}_t\\-\frac{\int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\left(\hat{W}_t-\sum_l\partial_e e_l\left(i\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right)+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di\\+A_{e_k,t}-\frac{\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}di\tilde{\Lambda}_t+\left(1-\frac{1}{R}\right)\bar{s}_k\mathbb{E}_{\delta,t}\left(\int\left(\left(1-\varphi\left(i\right)\right)\check{\alpha}^{t_0}_t\left(i\right)+\varphi\left(i\right)\check{\aleph}^{t_0}_t\left(i\right)\right)b(i)di\right)\\-\delta\int\left(1-\varphi\left(i\right)\right)\check{\alpha}^t\left(i\right)b\left(i\right)di\bar{s}_k-\delta\sum_{u=0}\left(\left(1-\delta\right)\beta\right)^u\int\varphi\left(i\right)\check{\aleph}^t_{t+u}\left(i\right)b\left(i\right)di\bar{s}_k\frac{R-1}{R}+\delta\sum_{u=0}\frac{1}{R^u}\int\varphi\left(i\right)b\left(i\right)di\bar{s}_k\frac{\hat{R}_{t+u}}{R}\\+\sum_{l,m}\frac{P_lP_k\partial_{P_k}\mathcal{Y}_{l,m}}{A_m}Y_m\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_l\frac{WP_k\partial_{P_k}\mathcal{N}_l}{A_l}Y_l\hat{W}_t+\sum_l E_l\bar{\rho}_{l,k}\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)-Y_k\vartheta_k\left(\pi_{k,t}-\beta\pi_{k,t+1}\right)\end{aligned}$$

Finally, define

$$\mathcal{A}_{b,t}=\int\left(1-\varphi\left(i\right)\right)\check{\alpha}^t\left(i\right)b\left(i\right)di+\sum_{u=0}\left(\left(1-\delta\right)\beta\right)^u\int\varphi\left(i\right)\check{\aleph}^t_{t+u}\left(i\right)b\left(i\right)di\frac{R-1}{R}-\sum_{u=0}\frac{1}{R^u}\int\varphi\left(i\right)b\left(i\right)di\bar{s}_k\frac{\hat{R}_{t+u}}{R}$$

We have

$$\begin{aligned}\mathcal{A}_{b,t}-\frac{1}{R}\mathcal{A}_{b,t+1}=\left(1-\frac{1}{R}\right)\int b\left(i\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\left(-\sum_l\partial_e e_l\left(i\right)\tilde{A}_{l,t}+\frac{1}{P_kY_k}\sum_{k=1}^{K}\lambda_k\check{\mu}_{k,t}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)\right)di\\+\int b\left(i\right)\left(-\frac{\partial_e vG''\left(V\left(i\right)\right)}{G'\left(V\left(i\right)\right)}+\frac{\left(1-\frac{1}{R}\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)\left\{\frac{b\left(i\right)}{R}\left(\hat{R}_{t+s}-\pi_{cpi,t+1+s}\right)-e\left(i\right)\sum_k\left(s_k\left(i\right)-\bar{s}_k\right)\hat{P}_{k,t+s}+Wn\left(i\right)\sum_k\bar{s}_k\tilde{A}_{k,t+s}\right\}di\end{aligned}$$

Optimal Policy Equations: summary

We now collect and slightly simplify the optimal policy equations derived above.[49] We obtain a system of $6 + K * 6$ equations in the following variables: $\hat{W}_t, Z_t, Z_t^0$, $A_{b,t}, A_{b,t}^0$ and $\mathcal{A}_{b,t}$ and $M_{k,t}, \breve{\mu}_{k,t}, L_{k,t}, L_{k,t}^0$ and $A_{e_k,t}, A_{e_k,t}^0$. These replace the interest rate rule. Note that the evolution of $\hat{W}_t$ is given by

$$\begin{aligned}
\left(1-\varphi^N\right)\psi\Delta\hat{W}_{t+1} = & \left(\left(1-\varphi^E\right)\sigma + \left(1-\varphi^N\right)\psi\right)\hat{R}_t - \left(1-\varphi^N\right)\bar{s}_k \cdot \Delta\tilde{A}_{l,t+1} \\
& + \int \varphi\left(i\right)\left\{\frac{b(i)}{RE}\left(\Delta\hat{R}_{t+1} - \left(R-1\right)\pi_{cpi,t+1}\right) - \frac{e(i)}{E}\sum_k\left(\left(s_k(i) - \bar{s}_k\right)\right)\pi_{k,t+1}\right\}di \\
& + \sigma\left(\sum_k - \int\left(1-\varphi\left(i\right)\right)\frac{e(i)}{E}\partial_e e_k(i)\pi_{k,t+1}\right)
\end{aligned}$$

We renormalize $\check{\mu}_{k,t} \equiv E\breve{\mu}_{k,t}$, and define $g(i) = \left(-\frac{\partial_e v G''(V(i))\frac{R}{R-1}}{G'(V(i))} + \frac{1}{\sigma e(i) + \psi W n(i)}\right)E$. Our Labor Market equation equation becomes

$$\begin{aligned}
\sum_{k=1}^{K}\Omega_{N,k}\lambda_k\breve{\mu}_{k,t} = & Z_t + \frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\frac{1}{R}\left(A_{b,t} - (1-\delta)RA_{b,t-1}\right) \\
& - \left(\frac{\int\left(1-\varphi\left(i\right)\right)\frac{Wn(i)}{WN}\psi di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di} + \int\varphi\left(i\right)\psi\frac{Wn(i)}{WN}\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi n\left(i\right)}di\right)\hat{W}_t \\
& + \sum_k\left(\frac{\int\left(1-\varphi\left(i\right)\right)\frac{Wn(i)}{WN}\psi di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_e e_k\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di} + \int\varphi\left(i\right)\psi\frac{Wn(i)}{WN}\frac{\sigma e\left(i\right)\partial_e e_k\left(i\right)}{\sigma e\left(i\right)+\psi n\left(i\right)}di\right)\left(\tilde{A}_{k,t}+\hat{P}_{k,t}\right) \\
& - \sum_k\lambda_k\frac{E_k}{P_kY_k}\left(\frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e(i)}{E_k}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di} + \int\varphi\left(i\right)\frac{\sigma\psi Wn(i)}{\sigma e\left(i\right)+\psi n\left(i\right)}\frac{e\left(i\right)}{E_k}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)di\right)\check{\mu}_{k,t} \\
& + \sum_{k,l}\frac{P_lW\partial_W\mathcal{Y}_{l,k}Y_k}{A_kE}\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right) + \sum_k\frac{W\partial_W\mathcal{N}_k}{A_kN}Y_k\hat{W}_t
\end{aligned}$$

With

$$Z_{t+1} - Z_t = \frac{\delta}{1-\delta}Z_{t+1}^0$$

$$\begin{aligned}
Z_{t-1}^0 - \frac{1}{\left(1-\delta\right)R}Z_t^0 + \left(1-\frac{1}{R}\right)Z_{t-1} = & \left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left(\frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di} - \frac{\psi Wn(i)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)di\hat{W}_{t-1} \\
& - \sum_k\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\partial_e e_k\left(i\right)\left(\frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di} - \frac{\psi Wn(i)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)di\left(\tilde{A}_{k,t-1}+\hat{P}_{k,t-1}\right) \\
& + \sum_k\frac{\lambda_kE_k}{P_kY_k}\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E_k}\partial_e e_k\left(i\right)\gamma_{e,k}\left(i\right)\left(\frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di} - \frac{\psi Wn(i)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)di\breve{\mu}_{k,t-1}
\end{aligned}$$

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[49] We also derived these equations in a different way, starting from the welfare loss function derived in the next appendix.

And

$$
\begin{aligned}
&\left(1-\frac{\int \varphi\left(i\right)\frac{1}{R}b\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)A_{b,t+1}-\left(1-\frac{\left(1-\delta+R\right)\int \varphi\left(i\right)\frac{1}{R}b\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)A_{b,t}-\frac{(1-\delta)R\int \varphi\left(i\right)\frac{1}{R}b\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}A_{b,t-1}-\frac{\delta}{1-\delta}A^{0}_{b,t+1}=\\
&\qquad -\sum_{l}\left(\int\varphi\left(i\right)\frac{b\left(i\right)}{E}\frac{\sigma e\left(i\right)\partial_e e_l\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\frac{b(i)}{E}di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_e e_l\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\Delta\tilde{A}_{l,t+1}\\
&\qquad +\sum_{k}\left(\int\varphi\left(i\right)b\left(i\right)\frac{\sigma}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\frac{e\left(i\right)}{E_k}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)di+\frac{\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e(i)}{E_k}\partial_e e_k\left(i\right)\gamma_{e,k}\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\frac{\lambda_k E_k}{P_k Y_k}\Delta\check{\mu}_{k,t+1}\\
&\qquad +\left(\int\varphi\left(i\right)\frac{\left(b\left(i\right)\right)^2}{ER}\frac{1}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{1}{ER}\frac{\left(\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\right)^2}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right)\\
&\qquad +\left(\int\varphi\left(i\right)\frac{\frac{b(i)}{E}Wn(i)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\frac{b(i)}{E}di\int\left(1-\varphi\left(i\right)\right)Wn(i)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\sum\bar{s}_k\Delta\tilde{A}_{k,t+1}\\
&\qquad -\sum\left(\int\varphi\left(i\right)\frac{\frac{b(i)}{E}e(i)\left(s_k(i)-\bar{s}_k\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\frac{b(i)}{E}di\int\left(1-\varphi\left(i\right)\right)e(i)\left(s_k(i)-\bar{s}_k\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\pi_{k,t+1}
\end{aligned}
$$

$$
\begin{aligned}
A^{0}_{b,t-1}-\frac{1}{\left(1-\delta\right)R}A^{0}_{b,t}+\left(1-\frac{1}{R}\right)A_{b,t-1}=&\left(1-\frac{1}{R}\right)\int\frac{b\left(i\right)}{E}g(i)\frac{b\left(i\right)}{RE}di\left(\hat{R}_{t-1}-\pi_{cpi,t}\right)\\
&-\left(1-\frac{1}{R}\right)\sum_k\int\frac{b\left(i\right)}{E}g(i)\frac{e\left(i\right)}{E}\left(s_k\left(i\right)-\bar{s}_k\right)di\hat{P}_{k,t-1}\\
&+\left(1-\frac{1}{R}\right)\int\frac{b\left(i\right)}{E}g(i)\frac{Wn\left(i\right)}{WN}di\sum_k\bar{s}_k\tilde{A}_{k,t-1}\\
&-\left(1-\frac{1}{R}\right)\sum_k\int\frac{b\left(i\right)}{E}\frac{\sigma e\left(i\right)\partial_e e_k\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di\tilde{A}_{k,t-1}\\
&+\left(1-\frac{1}{R}\right)\sum_k\frac{\lambda_k E_k}{P_k Y_k}\int\frac{b\left(i\right)}{E_k}\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)di\check{\mu}_{k,t-1}
\end{aligned}
$$

The Price resetting equation becomes

$$M_{k,t}=\check{\mu}_{k,t}-\left(1-\delta\right)\check{\mu}_{k,t-1}$$

$$
\begin{aligned}
\beta M_{k,t+1}-M_{k,t}=\lambda_k\check{\mu}_{k,t}-\sum_{l=1}^{K}\frac{\lambda_l E_l}{P_l Y_l}\int\gamma_{e,l}\left(i\right)\frac{e_l\left(i\right)}{E_l}\rho_{l,k}\left(i\right)di\check{\mu}_{l,t}-\sum_{l=1}^{K}\lambda_l\frac{\check{\mu}_{l,t}}{P_l Y_l}\frac{P_k\mathcal{Y}_{k,l}Y_l}{A_l}+\sum_{l,m}\frac{P_l P_k\partial_{P_k}\mathcal{Y}_{l,m}}{A_m E}Y_m\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_l\frac{P_k\partial_{P_k}\mathcal{N}_l}{A_l N}Y_l\hat{W}_t \\
+\sum_l\bar{s}_l\bar{\rho}_{l,k}\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)-\frac{P_k Y_k}{E}\vartheta_k\left(\pi_{k,t}-\beta\pi_{k,t+1}\right)-L_{k,t}+\left(\frac{\int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\sigma e\left(i\right)di\int\left(1-\varphi\left(i\right)\right)\psi\frac{Wn(i)}{WN}}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}+\int\varphi\left(i\right)\partial_e e_k\left(i\right)\frac{\psi Wn\left(i\right)}{WN}\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di\right)\hat{W}_t \\
-\sum_l\left(\int\varphi\left(i\right)\partial_e e_l\left(i\right)\partial_e e_k\left(i\right)\psi\frac{Wn\left(i\right)}{WN}\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}-\frac{\int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\sigma e\left(i\right)di\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e(i)}{E}\partial_e e_l\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}+\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\partial_e e_k\left(i\right)\partial_e e_l\left(i\right)di\right)\left(\tilde{A}_{l,t}+\hat{P}_{l,t}\right) \\
+\sum_{l=1}^{K}\frac{\lambda_l E_l}{P_l Y_l}\left(\int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\frac{\sigma e\left(i\right)}{E_l}\gamma_{e,l}\left(i\right)\partial_e e_l\left(i\right)di+\int\varphi\left(i\right)\partial_e e_k\psi Wn\left(i\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\frac{1}{E_l}\gamma_{e,l}\left(i\right)\partial_e e_l\left(i\right)di\right)\check{\mu}_{l,t} \\
-\sum_{l=1}^{K}\frac{\lambda_l E_l}{P_l Y_l}\left(\frac{\int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\int\left(1-\varphi\left(i\right)\right)\partial_e e_l\left(i\right)\frac{\sigma e\left(i\right)}{E_l}\gamma_{e,l}\left(i\right)di\right)\check{\mu}_{l,t} \\
+\left(\frac{\int\left(1-\varphi\left(i\right)\right)\partial_e e_k\left(i\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}+\frac{\int\varphi\left(i\right)\left(e_k(i)-e(i)\bar{s}_k\right)}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\frac{1}{R}\left(A_{b,t}-R\left(1-\delta\right)A_{b,t-1}\right)+\left(1-\frac{1}{R}\right)\bar{s}_k A_{b,t}-\delta\bar{s}_k\mathcal{A}_{b,t}+A_{e_k,t}
\end{aligned}
$$

With $\vartheta_k=\bar{\epsilon}_k\frac{\theta_k}{(1-\beta\theta_k)(1-\theta_k)}$ and:

$$
\Delta L_{k,t+1}=\frac{\delta}{1-\delta}L^0_{k,t+1}
$$

$$
\begin{aligned}
L^0_{k,t-1}-\frac{1}{R\left(1-\delta\right)}L^0_{k,t}+\left(1-\frac{1}{R}\right)L_{k,t-1}=\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left(\frac{\sigma e\left(i\right)\partial_e e_k\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}-\frac{\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_e e_k\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)di\hat{W}_{t-1} \\
-\sum_l\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)\left(\frac{\sigma e\left(i\right)\partial_e e_k\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}-\frac{\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_e e_k\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)di\left(\tilde{A}_{l,t-1}+\hat{P}_{l,t-1}\right) \\
+\sum_l\frac{\lambda_l E_l}{P_l Y_l}\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E_l}\partial_e e_l\left(i\right)\gamma_{e,l}\left(i\right)\left(\frac{\sigma e\left(i\right)\partial_e e_k\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}-\frac{\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_e e_k\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)di\check{\mu}_{l,t-1}
\end{aligned}
$$

$$
\begin{aligned}
A_{e_k,t+1}-A_{e_k,t}-\frac{\delta}{1-\delta}A^0_{e_k,t+1}=-\sum\left(\int\varphi\left(i\right)\frac{e(i)}{E}\left(s_k(i)-\bar{s}_k\right)\frac{\sigma e\left(i\right)\partial_e e_l\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\frac{e(i)}{E}\left(s_k(i)-\bar{s}_k\right)di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_e e_l\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\Delta\tilde{A}_{l,t+1} \\
+\sum\left(\int\varphi\left(i\right)e(i)\left(s_k(i)-\bar{s}_k\right)\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\frac{1}{E_l}\gamma_{e,l}\left(i\right)\partial_e e_l\left(i\right)di+\frac{\int\left(1-\varphi\left(i\right)\right)e(i)\left(s_k(i)-\bar{s}_k\right)di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_e e_l\left(i\right)\frac{1}{E_l}\gamma_{e,l}\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\frac{\lambda_l E_l}{P_l Y_l}\lambda_l\Delta\check{\mu}_{l,t+1} \\
+\left(\int\varphi\left(i\right)\frac{\left(\frac{b(i)}{R}\right)\frac{e(i)}{E}\left(s_k(i)-\bar{s}_k\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\int\left(1-\varphi\left(i\right)\right)\frac{e(i)}{E}\left(s_k(i)-\bar{s}_k\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\left(\Delta\hat{R}_{t+1}-\left(R-1\right)\pi_{cpi,t+1}\right) \\
+\left(\int\varphi\left(i\right)\frac{\frac{e(i)}{E}\left(s_k(i)-\bar{s}_k\right)Wn(i)}{\sigma X_t\equiv e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\frac{e(i)}{E}\left(s_k(i)-\bar{s}_k\right)di\int\left(1-\varphi\left(i\right)\right)Wn(i)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\sum\bar{s}_l\Delta\tilde{A}_{l,t+1} \\
-\sum\left(\int\varphi\left(i\right)\frac{\frac{e(i)}{E}\left(s_k(i)-\bar{s}_k\right)e(i)\left(s_l(i)-\bar{s}_l\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}di+\frac{\int\left(1-\varphi\left(i\right)\right)\frac{e(i)}{E}\left(s_k(i)-\bar{s}_k\right)di\int\left(1-\varphi\left(i\right)\right)e(i)\left(s_l(i)-\bar{s}_l\right)}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\pi_{l,t+1}
\end{aligned}
$$

$$\begin{aligned}
A^{0}_{e_k,t-1} - \frac{1}{(1-\delta)\,R} A^{0}_{e_k,t} + \left(1-\frac{1}{R}\right) A_{e_k,t-1} = {} & \left(1-\frac{1}{R}\right) \int \frac{e(i)}{E} \left(s_k(i) - \bar{s}_k\right) g(i) \frac{b\,(i)}{RE} di \left(\hat{R}_{t-1} - \pi_{cpi,t}\right) \\
& - \left(1-\frac{1}{R}\right) \sum_l \int \frac{e(i)}{E} \left(s_k(i) - \bar{s}_k\right) g(i) \frac{e\,(i)}{E} \left(s_l\,(i) - \bar{s}_l\right) di \hat{P}_{l,t-1} \\
& + \left(1-\frac{1}{R}\right) \int \frac{e(i)}{E} \left(s_k(i) - \bar{s}_k\right) g(i) \frac{Wn\,(i)}{WN} di \sum_l \bar{s}_l \tilde{A}_{l,t-1} \\
& - \left(1-\frac{1}{R}\right) \sum_l \int \frac{e(i)\left(s_k(i) - \bar{s}_k\right)}{E} \frac{\sigma e\,(i)\,\partial_e e_l\,(i)}{\sigma e\,(i) + \psi Wn\,(i)} di \tilde{A}_{l,t-1} \\
& + \sum_k \lambda_l \frac{E_l}{P_l Y_l} \left(1-\frac{1}{R}\right) \int e(i) \left(s_k(i) - \bar{s}_k\right) \frac{\sigma e\,(i)\,/E_l}{\sigma e\,(i) + \psi Wn\,(i)} \gamma_{e,l}\,(i)\,\partial_e e_l\,(i)\, di \check{\mu}_{l,t-1}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_{b,t} - \frac{1}{R} \mathcal{A}_{b,t+1} = {} & \left(1-\frac{1}{R}\right) \int \frac{b\,(i)}{E} g(i) \frac{b\,(i)}{RE} di \left(\hat{R}_t - \pi_{cpi,t+1}\right) \\
& - \left(1-\frac{1}{R}\right) \sum_k \int \frac{b\,(i)}{E} g(i) \frac{e\,(i)}{E} \left(s_k\,(i) - \bar{s}_k\right) di \hat{P}_{k,t} \\
& + \left(1-\frac{1}{R}\right) \int \frac{b\,(i)}{E} g(i) \frac{e\,(i)}{E} \frac{Wn\,(i)}{WN} di \sum_k \bar{s}_k \tilde{A}_{k,t} \\
& - \left(1-\frac{1}{R}\right) \sum_k \int \frac{b\,(i)}{E} \frac{\sigma e\,(i)\,\partial_e e_k\,(i)}{\sigma e\,(i) + \psi Wn\,(i)} di \tilde{A}_{k,t} \\
& + \left(1-\frac{1}{R}\right) \sum_k \lambda_k \frac{E_k}{P_k Y_k} \int \frac{b\,(i)}{E_k} \frac{\sigma e\,(i)}{\sigma e\,(i) + \psi Wn\,(i)} \gamma_{e,k}\,(i)\,\partial_e e_k\,(i)\, di \check{\mu}_{k,t}
\end{aligned}$$

E.2 Optimal policy: proofs analytical results Section 5

Result 7

We first show that under (A.1) and (A.2), optimal policy attempts to jointly stabilize the output gap $\tilde{\mathcal{Y}}_t$ and an inflation index $\pi_t^\theta \equiv \sum_{k=1}^K \frac{\bar{s}_k \vartheta_k}{\vartheta} \pi_{k,t}$ (with $\vartheta = \sum_{k=1}^K \bar{s}_k \vartheta_k$), in the sense that optimal policy can equivalently be derived by solving:

$$\inf_{\{\tilde{\mathcal{Y}}_t, \pi_t^\theta\}_{t\geq 0}} \quad \mathbb{E}_0 \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left(\frac{\sigma + \psi}{\sigma \psi} \tilde{\mathcal{Y}}_t^2 + \vartheta \left(\pi_t^\theta \right)^2 \right)$$

$$s.t. \quad \mathbb{E}_t \pi_{t+1}^\theta - R \pi_t^\theta = -R \kappa \tilde{\mathcal{Y}}_t - R \lambda^\theta \hat{\mathcal{W}}_t^\theta$$

We consider the general case in which $\bar{\epsilon}_k, \bar{\epsilon}_k^s$ and θ_k may vary across sectors. Note that under inner CES preferences the inflation index can be rewritten $\pi_t^\theta = \frac{1}{\sum_{k=1}^K \frac{\bar{s}_k \bar{\epsilon}_k}{\lambda_k}} \sum_{k=1}^K \frac{\bar{s}_k \bar{\epsilon}_k}{\lambda_k} \pi_{k,t}$, π_t^θ overweight larger sectors (higher $\bar{s}_k$) more rigid sectors (lower λ_k) and more elastic sector (higher $\bar{\epsilon}_k$). If we have that θ_k and $\bar{\epsilon}_k$ are equal across sector then π_t^θ is simply the CPI index. The NKPC associated with π_t^θ is given by

$$\mathbb{E}_t \pi_{t+1}^\theta - R \pi_t^\theta = -R \kappa \tilde{\mathcal{Y}}_t - R \lambda^\theta \hat{\mathcal{W}}_t^\theta$$

Where $\hat{\mathcal{W}}_t^\theta$ is a wedge that is independent from monetary policy (Result 1 of the positive section). Under the optimal policy, we have $\tilde{\mathcal{Y}}_0 = -\frac{\sigma\psi}{\sigma+\psi} \kappa \vartheta \pi_0^\theta$, and $\tilde{\mathcal{Y}}_t$ partially absorbs the wedge: if $\hat{\mathcal{W}}_t^\theta \geq 0$ at all t then $\tilde{\mathcal{Y}}_t \leq 0$ at all t. In addition, when ϑ goes to infinity keeping all other parameters fixed, we have $\tilde{\mathcal{Y}}_t = -\frac{\lambda^\theta}{\kappa} \hat{\mathcal{W}}_t^\theta$ and $\pi_t^\theta = 0$: the output gap fully absorbs the wedge. Inversely, when ϑ goes to 0, $\tilde{\mathcal{Y}}_t = 0$: the inflation index fully absorbs the wedge.

Note that under **(A.2)** we have $A_{b,t} = A_{b,t}^0 = 0$ for all t and since $e(i) = Wn(i)$, $Z_t = Z_t^0 = 0$ for all t. Defining $\check{\mu}_t \equiv \sum_{k=1}^K \check{\mu}_{k,t}$ We can rewrite the Labor Market equation as:

$$\frac{\sigma\psi}{\sigma + \psi} \kappa \check{\mu}_t = -\tilde{\mathcal{Y}}_t.$$

The system of price resetting equations becomes

$$M_{k,t} = \check{\mu}_{k,t} - (1 - \delta) \check{\mu}_{k,t-1},$$

$$\begin{aligned}
\beta \mathbb{E}_t M_{k,t+1} - M_{k,t} = {} & -\bar{s}_k \vartheta_k \left(\pi_{k,t} - \beta \mathbb{E}_t \pi_{k,t+1} \right) \\
& + \overline{\partial_e e_k} \frac{\sigma\psi}{\sigma + \psi} \kappa \sum_{l=1}^K \check{\mu}_{l,t} + \overline{\partial_e e_k} \tilde{\mathcal{Y}}_t - L_{k,t} + A_{e_k,t} \\
& - \sum_{l=1}^K \lambda_l \int \gamma_{e,l}(i) \frac{e_l(i)}{E_l} \rho_{l,k}(i)\, di \check{\mu}_{l,t} + \sum_l \bar{s}_l \bar{\rho}_{l,k} \left(\hat{P}_{l,t} + \tilde{A}_{l,t} \right) - \sigma \sum_l \left(\int \frac{e}{E} \partial_e e_k(i) \partial_e e_l(i)\, di - \overline{\partial_e e_k}\, \overline{\partial_e e_l} \right) \left(\tilde{A}_{l,t} + \hat{P}_{l,t} \right) \\
& \lambda_k \check{\mu}_{k,t} - \overline{\partial_e e_k} \sum_{l=1}^K \lambda_l \check{\mu}_{l,t} + \sum_{l=1}^K \lambda_l \left(\int \partial_e e_k(i) \frac{\sigma e(i)}{E_l} \gamma_{e,l}(i) \partial_e e_l(i)\, di - \overline{\partial_e e_k} \int \partial_e e_l(i) \frac{\sigma e(i)}{E_l} \gamma_{e,l}(i)\, di \right) \check{\mu}_{l,t}.
\end{aligned}$$

Note that we have:

$$\sum_{k=1}^K L_{k,t} = Z_t,$$

$$\sum_{k=1}^K e_l(i) \rho_{l,k}(i) = \sum_{k=1}^K \bar{s}_l \bar{\rho}_{l,k} = 0,$$

$$\sum_{k=1}^K s_k(i) = \sum_{k=1}^K \bar{s}_k = \sum_{k=1}^K \overline{\partial_e e_k} = \sum_{k=1}^K \partial_e e_k(i) = 1.$$

We therefore have $\sum_{k=1}^{K} L_{k,t} = \sum_{k=1}^{K} A_{e_k,t} = \sum_{k=1}^{K} A^0_{e_k,t} = 0$. Defining $M_t \equiv \sum_{k=1}^{K} M_{k,t}$,we have

$$
\begin{aligned}
M_t &= \check{\mu}_t - (1-\delta)\,\check{\mu}_{t-1}, \\
\beta \mathbb{E}_t M_{k,t+1} - M_{k,t} &= -\sum_{k=1}^{K} \bar{s}_k \vartheta_k \left(\pi_{k,t} - \beta \mathbb{E}_t \pi_{k,t+1}\right),
\end{aligned}
$$

Defining;

$$
\begin{aligned}
\vartheta &\equiv \sum_{k=1}^{K} \bar{s}_k \vartheta_k, \\
\pi_t^{\theta} &\equiv \sum_{k=1}^{K} \frac{\bar{s}_k \vartheta_k}{\vartheta} \pi_{k,t}, \\
\lambda^{\theta} &\equiv \sum_{k=1}^{K} \frac{\bar{s}_k \vartheta_k}{\vartheta} \lambda_k, \\
\hat{\mathcal{W}}_t^{\theta} &\equiv \sum_{k=1}^{K} \frac{\bar{s}_k \vartheta_k \lambda_k}{\vartheta \lambda^{\theta}} \mathcal{M}_{k,t} + \sum_{k=1}^{K} \left(\overline{\partial_e e}_k - \frac{\bar{s}_k \vartheta_k \lambda_k}{\vartheta \lambda^{\theta}} \right) \tilde{P}_{k,t},
\end{aligned}
$$

the evolution of the output gap under optimal policy is determined by:

$$
\begin{aligned}
\tilde{\mathcal{Y}}_t &= -\frac{\sigma\psi}{\sigma+\psi} \kappa \check{\mu}_t, \\
\check{\mu}_t - (1-\delta)\,\check{\mu}_{t-1} &= \vartheta \pi_t^{\theta}, \\
\mathbb{E}_t \pi_{t+1}^{\theta} - R\pi_t^{\theta} &= -R\kappa \tilde{\mathcal{Y}}_t - R\lambda^{\theta} \hat{\mathcal{W}}_t^{\theta}.
\end{aligned}
$$

Note that we would obtain the same system of equation if the central bank were instead to solve:

$$
\begin{aligned}
\inf_{\left\{\tilde{\mathcal{Y}}_t, \pi_t^{\theta}\right\}_{t\geq 0}} \quad & \mathbb{E}_0 \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left(\frac{\sigma+\psi}{\sigma\psi} \tilde{\mathcal{Y}}_t^2 + \vartheta \left(\pi_t^{\theta}\right)^2 \right) \\
s.t. \quad & \mathbb{E}_t \pi_{t+1}^{\theta} - R\pi_t^{\theta} = -R\kappa \tilde{\mathcal{Y}}_t - R\lambda^{\theta} \hat{\mathcal{W}}_t^{\theta}.
\end{aligned}
$$

In the special case in which$\bar{\epsilon}_k, \bar{\epsilon}_k^s$ and θ_k are common across sectors we obtain the problem stated in result 6. Denoting by $\beta^t \check{\check{\mu}}_t$ the Lagrange multiplier on the NKPC, the first-order conditions are:

$$
\begin{aligned}
\tilde{\mathcal{Y}}_t &= -\frac{\sigma\psi}{\sigma+\psi} R\kappa \check{\check{\mu}}_t \\
\vartheta \pi_t^{\theta} &= R\check{\check{\mu}}_t - \beta^{-1} R\check{\check{\mu}}_{t-1}
\end{aligned}
$$

Redefining $\check{\mu}_t = R\check{\check{\mu}}_t$we obtain:

$$
\begin{aligned}
\tilde{\mathcal{Y}}_t &= -\frac{\sigma\psi}{\sigma+\psi} \kappa \check{\mu}_t, \\
\check{\mu}_t - (1-\delta)\,\check{\mu}_{t-1} &= \vartheta \pi_t^{\theta}, \\
\mathbb{E}_t \pi_{t+1}^{\theta} - R\pi_t^{\theta} &= -R\kappa \tilde{\mathcal{Y}}_t - R\lambda^{\theta} \hat{\mathcal{W}}_t^{\theta},
\end{aligned}
$$

which is the same system.

Note that under (**A.1**) and (**A.2**), the wedge $\hat{\mathcal{W}}_t^{\theta}$ evolves independently of monetary policy. The OP system can be rewritten as

$$
\mathbb{E}_t \tilde{\mathcal{Y}}_{t+1} - \left((1-\delta) + R\left(1 + \frac{\sigma\psi}{\sigma+\psi} \vartheta \kappa^2\right) \right) \tilde{\mathcal{Y}}_t - (1-\delta)\, R\tilde{\mathcal{Y}}_{t-1} = R\lambda^{\theta} \frac{\sigma\psi}{\sigma+\psi} \vartheta \kappa \hat{\mathcal{W}}_t^{\theta}.
$$

Defining

$$\mu_{\pm} \equiv \frac{(1-\delta) + R\left(1 + \frac{\sigma\psi}{\sigma+\psi}\vartheta\kappa^2\right) \pm \sqrt{\left((1-\delta) + R\left(1 + \frac{\sigma\psi}{\sigma+\psi}\vartheta\kappa^2\right)\right)^2 - 4(1-\delta)R}}{2}$$

and noting that we have $0 < \mu_- < 1-\delta < R < \mu_+$, we have

$$\tilde{\mathcal{Y}}_t = -\mathbb{E}_t \frac{\lambda^\theta \frac{\sigma\psi}{\sigma+\psi}\vartheta\kappa}{(1-\delta)} \sum_{s=0}^{t} \mu_-^{t+1-s} \sum_{u=0}^{+\infty} \mu_+^{-u} \hat{\mathcal{W}}_{s+u}^\theta$$

We directly obtain that if $\hat{\mathcal{W}}_t^\theta \gtreqless 0$ for all t then $\tilde{\mathcal{Y}}_t \lesseqgtr 0$ for all t. In addition we have $\lim_{\vartheta\to\infty} \mu_+^{-1} = \lim_{\vartheta\to\infty} \mu_- = 0$ and $\mu_- = (1-\delta) / \left(\frac{\sigma\psi}{\sigma+\psi}\vartheta\kappa\right) + o(1/\vartheta)$, so as ϑ goes to infinity keeping all other parameters fixed, we have

$$\tilde{\mathcal{Y}}_t = -\frac{\lambda^\theta}{\kappa}\hat{\mathcal{W}}_t^\theta$$

Inversely when ϑ goes to 0, the output gap goes to 0 and π_t^θ fully absorbs the wedge $\hat{\mathcal{W}}_t^\theta$.

Result 8

In addition to (**A.1**) and (**A.2**), we now assume that there are no endogenous markups ($\gamma_{e,k}(i) = 0$ for all i, k) and that sectoral shocks in k follow vanish geometrically $\hat{A}_{k,t} = \rho_a^t \hat{A}_{k,0}$. We derive analytical formulas for the evolution of $\tilde{\mathcal{Y}}_t, \pi_{mcpi,t}$ and π_t^θ and characterize their sign. First note that for aggregate shocks, we have $\tilde{\mathcal{Y}}_t = \pi_{mcpi,t} = \pi_t^\theta = 0$. If $\overline{\partial_e e}_k < \frac{\bar{s}_k\vartheta_k}{\vartheta}$ (note that if ϑ_k are equal across sector the condition simply characterize necessity), following a negative shock in sector k, $\tilde{\mathcal{Y}}_t$ is negative on impact and there t^* such that for $t \geq t^*$, $\tilde{\mathcal{Y}}_t$ is positive. $\pi_{mcpi,t}$ is negative on impact and there t^* such that for $t \geq t^*$, $\pi_{mcpi,t}$ is positive. π_t^θ is positive on impact and if δ is small enough there t^* such that for $t \geq t^*$, π_t^θ is positive. In net present value term, we have $\sum_{t\geq 0} \frac{1}{R^t}\tilde{\mathcal{Y}}_t, \sum_{t\geq 0} \frac{1}{R^t}\pi_{mcpi,t} > 0$ and$\sum_{t\geq 0} \frac{1}{R^t}\pi_t^\theta < 0$ following a negative shock in k with $\overline{\partial_e e}_k < \frac{\bar{s}_k\vartheta_k}{\vartheta}$.

Under $(A.2)$ and $\gamma_{e,k}(i) = 0$, we have $\lambda_k = \frac{\sigma\psi}{\sigma+\psi}\kappa \equiv \lambda$ and $\mathcal{M}_{k,t} = 0$ for all k, so we have that the exogenous wedge is given by:

$$\hat{\mathcal{W}}_t^\theta = \tilde{P}_t^\Delta,$$

with $P_t^\Delta = \sum_{k=1}^K \left(\overline{\partial_e e}_k - \frac{\bar{s}_k\vartheta_k}{\vartheta}\right)\hat{P}_{k,t}$, $A_t^\Delta = \sum_{k=1}^K \left(\overline{\partial_e e}_k - \frac{\bar{s}_k\vartheta_k}{\vartheta}\right)\hat{A}_{k,t}$, $\tilde{P}_t^\Delta = P_t^\Delta + A_t^\Delta$. The relative price $\tilde{P}_t^\Delta$ satisfies

$$\mathbb{E}_t\hat{P}_{t+1}^\Delta - (1 + R(1+\lambda))\hat{P}_t^\Delta + R\hat{P}_t^\Delta = R\lambda\rho_a^t\hat{A}_0^\Delta$$

Denoting the roots of the equation polynomial as $\nu_\pm$, we have

$$\nu_\pm = \frac{1 + R(1+\lambda) \pm \sqrt{(1 + R(1+\lambda))^2 - 4R}}{2},$$

with $0 < \nu_- < 1 < R < \nu_+$. And $\hat{P}_t^\Delta$ is given by:

$$P_t^\Delta = -\frac{R\lambda}{(\nu_- - \rho_a)(\nu_+ - \rho_a)}\left(\nu_-^{t+1} - \rho_a^{t+1}\right)\hat{A}_0^\Delta$$

P_t^Δ is independent of policy and always has the same sign as $-\hat{A}_{k,0}$. The wedge is then given by:

$$\tilde{P}_t^\Delta = -\frac{1}{(\nu_- - \rho_a)(\nu_+ - \rho_a)}\left((R-\nu_-)(1-\nu_-)\nu_-^t - (R-\rho_a)(1-\rho_a)\rho_a^t\right)\hat{A}_0^\Delta.$$

Noting that $(R-x)(1-x)$ is positive and decreasing on $[0,1]$, we conclude that the wedge (independently of policy) initially has the same sign as $\hat{A}_0^\Delta$ for $t < t^*$ (with t^* the smallest t such that $(R-\rho_a)(1-\rho_a)\rho_a^t > (R-\nu_-)(1-\nu_-)\nu_-^t$ if $\rho_a > \nu_-$,such that $(R-\rho_a)(1-\rho_a)\rho_a^t < (R-\nu_-)(1-\nu_-)\nu_-^t$ if $\rho_a < \nu_-$) and thas the same sign as $-\hat{A}_{k,0}$ for $t \geq t^*$. Note that $t^* = 1$ for transitory shocks, $t^* = \infty$ for permanent shocks.

Plugging this formula in our general expression for the output gap and using the NKPC for the indices π_t^θ and $\pi_{mcpi,t}$, and the definition of the nominal interest rate, we obtain:

$$\begin{aligned}
\tilde{\mathfrak{Y}}_t &= \frac{R\lambda^2\vartheta}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \left\{ \frac{(R-\rho_a)(1-\rho_a)}{(\rho_a - \mu_+)(\rho_a - \mu_-)} \left\{ \rho_a^{t+1} - \mu_-^{t+1} \right\} - \frac{(R-\nu_-)(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)} \left\{ \nu_-^{t+1} - \mu_-^{t+1} \right\} \right\} \hat{A}_0^{\Delta} \\
\pi_{mcpi,t} &= \frac{(R\lambda)^2\vartheta\kappa}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \left\{ \frac{(1-\rho_a)(R-\rho_a)}{(\rho_a - \mu_+)(\rho_a - \mu_-)} \left\{ \frac{\rho_a}{R-\rho_a}\rho_a^t - \frac{\mu_-}{R-\mu_-}\mu_-^t \right\} - \frac{(R-\nu_-)(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)} \left\{ \frac{\nu_-}{R-\nu_-}\nu_-^t - \frac{\mu_-}{R-\mu_-}\mu_-^t \right\} \right\} \hat{A}_0^{\Delta} \\
\pi_t^{\theta} &= -\frac{R\lambda}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \left\{ \frac{(R-\rho_a)(1-\rho_a)}{(\rho_a - \mu_+)(\rho_a - \mu_-)} \left\{ (\rho_a - (1-\delta))\rho_a^t - (\mu_- - (1-\delta))\mu_-^t \right\} - \frac{(R-\nu_-)(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)} \left\{ (\nu_- - (1-\delta))\nu_-^t - (\mu_- - (1-\delta))\mu_-^t \right\} \right\} \hat{A}_0^{\Delta} \\
\hat{R}_t &= -\frac{1+\psi}{\sigma+\psi}(1-\rho_a)\rho_a^t \sum_k \bar{s}_k A_{k,0} - \frac{\psi}{\sigma+\psi}(1-\rho_a)\rho_a^t \hat{A}_0^{\Delta} \\
&\quad - \frac{1}{\sigma}\frac{R\lambda^2\vartheta}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \left\{ \frac{(R-\rho_a)(1-\rho_a)}{(\rho_a - \mu_+)(\rho_a - \mu_-)} \left\{ (1-\rho_a)\rho_a^{t+1} - (1-\mu_-)\mu_-^{t+1} \right\} - \frac{(R-\nu_-)(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)} \left\{ (1-\nu_-)\nu_-^{t+1} - (1-\mu_-)\mu_-^{t+1} \right\} \right\} \hat{A}_0^{\Delta} \\
&\quad + \frac{R\lambda^2\vartheta}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \left\{ \frac{(1-\rho_a)(R-\rho_a)}{(\rho_a - \mu_+)(\rho_a - \mu_-)} \left\{ \frac{R\kappa\rho_a}{R-\rho_a}\rho_a^{t+1} - \frac{R\kappa\mu_-}{R-\mu_-}\mu_-^{t+1} \right\} - \frac{(R-\nu_-)(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)} \left\{ \frac{R\kappa\nu_-}{R-\nu_-}\nu_-^{t+1} - \frac{R\kappa\mu_-}{R-\mu_-}\mu_-^{t+1} \right\} \right\} \hat{A}_0^{\Delta}
\end{aligned}$$

For aggregate shocks we have $\hat{A}_0^{\Delta} = 0$ so $\tilde{\mathfrak{Y}}_t = \pi_{mcpi,t} = \pi_t^{\theta} = 0$.
On impact, after some algebra, we obtain

$$\begin{aligned}
\tilde{\mathfrak{Y}}_0 &= -\frac{R\lambda^2\vartheta}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \left(-R + \rho_a\nu_- - \mu_+(\rho_a + \nu_-) + \mu_+(R+1) \right) \hat{A}_0^{\Delta} \\
\pi_{mcpi,0} &= -\frac{(R\lambda)^2\vartheta\kappa}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \frac{R}{R-\mu_-} (\mu_+ - 1) \hat{A}_0^{\Delta} \\
\pi_0^{\theta} &= -\frac{R\lambda}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \left(-R + \rho_a\nu_- - \mu_+(\rho_a + \nu_-) + \mu_+(R+1) \right) \hat{A}_0^{\Delta}
\end{aligned}$$

Note that since $\rho_a, \nu_- < 1$, $\mu_+ > R$, we have:

$$\begin{aligned}
-R + \rho_a\nu_- - \mu_+(\rho_a + \nu_-) + \mu_+(R+1) &= \mu_+(R + 1 - (\rho_a + \nu_-)) - R + \rho_a\nu_- \\
&\geq R(R + 1 - (\rho_a + \nu_-)) - R + \rho_a\nu_- \\
&= (R - \rho_a)(R - \nu_-) \geq 0
\end{aligned}$$

$\tilde{\mathfrak{Y}}_0, \pi_{mcpi,0} \gtreqless 0$ and $\pi_0^{\theta} \lesseqgtr 0$ if $\hat{A}_0^{\Delta} \lesseqgtr 0$. In addition, $\tilde{\mathfrak{Y}}_0 = -\lambda\vartheta\pi_0^{\theta}$. Note in particular that if $\bar{\epsilon}_k^s = 0$ (CES inner utility) and $\bar{\epsilon}_k = \bar{\epsilon}$ across sector, we have $\pi_t^{\theta} = \pi_{cpi,t}$ and $\hat{A}_t^{\Delta} = \sum_{k=1}^{K} \left(\overline{\partial_e e_k} - \bar{s}_k \right) \hat{A}_{k,t}$: $\hat{A}_t^{\Delta}$ is negative (positive) for negative shocks in luxury (necessity) sectors.

In the medium run the behavior of $\tilde{\mathfrak{Y}}_t, \pi_t^{\theta}, \pi_{mcpi,t}$ a priori depends on which of the parameters ρ_a, μ_- or ν_- dominates. If $\rho_a > \mu_-, \nu_-$, we have:

$$\begin{aligned}
\tilde{\mathfrak{Y}}_t &= \frac{R\lambda^2\vartheta}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \frac{(R-\rho_a)(1-\rho_a)}{(\rho_a - \mu_+)(\rho_a - \mu_-)} \rho_a^{t+1} \hat{A}_0^{\Delta} + o\left(\rho_a^t\right) \\
\pi_{mcpi,t} &= \frac{(R\lambda)^2\vartheta\kappa}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \frac{(1-\rho_a)}{(\rho_a - \mu_+)(\rho_a - \mu_-)} \rho_a^{t+1} \hat{A}_0^{\Delta} + o\left(\rho_a^t\right) \\
\pi_t^{\theta} &= -\frac{R\lambda}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \frac{(R-\rho_a)(1-\rho_a)}{(\rho_a - \mu_+)(\rho_a - \mu_-)} (\rho_a - (1-\delta)) \rho_a^t \hat{A}_0^{\Delta} + o\left(\rho_a^t\right)
\end{aligned}$$

for t large enough we have $\tilde{\mathfrak{Y}}_t, \pi_{mcpi,t} \gtreqless 0$ if $\hat{A}_0^{\Delta} \gtreqless 0$. $\pi_t^{\theta} \gtreqless 0$ ($\pi_t^{\theta} \lesseqgtr 0$) if $\hat{A}_0^{\Delta} \gtreqless 0$ and $\rho_a < (1-\delta)$ ($\rho_a > (1-\delta)$). Similarly, if $\nu_- > \mu_-, \rho_a$, we have:

$$\tilde{\mathcal{Y}}_t = -\frac{R\lambda^2\vartheta}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \frac{(R-\nu_-)(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)} \nu_-^{t+1} \hat{A}_0^\Delta + o(\nu_-^t)$$

$$\pi_{mcpi,t} = -\frac{(R\lambda)^2\vartheta\kappa}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \frac{(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)} \nu_-^{t+1} \hat{A}_0^\Delta + o(\nu_-^t)$$

$$\pi_t^\theta = \frac{R\lambda}{(\nu_- - \rho_a)(\nu_+ - \rho_a)} \frac{(R-\nu_-)(1-\nu_-)}{(\nu_- - \mu_+)(\nu_- - \mu_-)} (\nu_- - (1-\delta))\, \nu_-^t \hat{A}_0^\Delta + o(\nu_-^t)$$

for t large enough we have $\tilde{\mathcal{Y}}_t, \pi_{mcpi,t} \gtrless 0$ if $\hat{A}_0^\Delta \gtrless 0$. $\pi_t^\theta \gtrless 0$ ($\pi_t^\theta \lesseqgtr 0$) if $\hat{A}_0^\Delta \gtrless 0$ and $\nu_- < (1-\delta)$ ($\nu_- > (1-\delta)$). Finally, if $\mu_- > \nu_-, \rho_a$:

$$\tilde{\mathcal{Y}}_t = \frac{R\lambda^2\vartheta}{(\nu_+ - \rho_a)} \frac{\lambda^2\vartheta R(R+\rho_a\nu_-) + \delta(R-\rho_a)(R-\nu_-)}{(\rho_a - \mu_+)(\rho_a - \mu_-)(\nu_- - \mu_+)(\nu_- - \mu_-)} \mu_-^{t+1} \hat{A}_0^\Delta + o(\mu_-^t)$$

$$\pi_{mcpi,t} = \frac{(R\lambda)^2\vartheta\kappa}{(\nu_+ - \rho_a)} \frac{\lambda^2\vartheta R(R+\rho_a\nu_-) + \delta(R-\rho_a)(R-\nu_-)}{(\rho_a - \mu_+)(\rho_a - \mu_-)(\nu_- - \mu_+)(\nu_- - \mu_-)} \frac{1}{R-\mu_-} \mu_-^{t+1} \hat{A}_0^\Delta + o(\mu_-^t)$$

$$\pi_t^\theta = -\frac{R\lambda}{(\nu_+ - \rho_a)} \frac{\lambda^2\vartheta R(R+\rho_a\nu_-) + \delta(R-\rho_a)(R-\nu_-)}{(\rho_a - \mu_+)(\rho_a - \mu_-)(\nu_- - \mu_+)(\nu_- - \mu_-)} (\mu_- - (1-\delta))\, \mu_-^t \hat{A}_0^\Delta + o(\mu_-^t)$$

Recall that $\mu_- < 1-\delta$ so we have $\tilde{\mathcal{Y}}_t, \pi_{mcpi,t}, \pi_t^\theta \gtrless 0$ if $\hat{A}_0^\Delta \gtrless 0$.

Finally we derive the net present value of $\tilde{\mathcal{Y}}_t, \pi_{mcpi,t}, \pi_t^\theta$ under optimal policy. We have:

$$\mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \tilde{\mathcal{Y}}_t = -\frac{(R\lambda)^2\vartheta}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \frac{R(\mu_+ - 1)}{R-\mu_-} \hat{A}_0^\Delta$$

$$\mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \pi_{mcpi,t} = -\frac{(R\lambda)^2\vartheta\kappa}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \frac{R}{(R-\mu_-)^2(R-\rho_a)(R-\nu_-)} \left\{ R^2(\mu_+(R-\delta) - R) + \delta R^2(\rho_a + \nu_-) + \left(\mu_-(R-1) + \delta R^2\right)\rho_a\nu_- \right\}$$

$$\mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \pi_t^\theta = \frac{R^2\lambda(R-(1-\delta))}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \frac{\mu_+ - 1}{R-\mu_-} \hat{A}_0^\Delta$$

Note that $\mu_+ > R$, $0 \leq \rho_a, \nu_- \leq 1$ and as $\beta(1-\delta)R = 1$, $R - \delta > 1$ so $R^2(\mu_+(R-\delta) - R) + \delta R^2(\rho_a + \nu_-) + \left(\mu_-(R-1) + \delta R^2\right)\rho_a\nu_- > 0$. We therefore have $\sum_{t\geq 0} \frac{1}{R^t} \tilde{\mathcal{Y}}_t, \sum_{t\geq 0} \frac{1}{R^t} \pi_{mcpi,t} \gtrless 0$ and$\sum_{t\geq 0} \frac{1}{R^t} \pi_t^\theta \lesseqgtr 0$ if $\hat{A}_0^\Delta \lesseqgtr 0$

Result 9

Under the assumption θ_k and $\bar{\epsilon}_k$ are equal across sector then $\pi_t^\theta = \pi_{cpi,t}$, using the result of the previous subsection, we have:

$$\tilde{\mathcal{Y}}_0 = -\frac{R\lambda\lambda\vartheta}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \left(-R + \rho_a\nu_- - \mu_+(\rho_a + \nu_-) + \mu_+(R+1)\right) \hat{A}_0^\Delta$$

$$\mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \tilde{\mathcal{Y}}_t = -\frac{R^2\lambda\lambda\vartheta}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \frac{R(\mu_+ - 1)}{R-\mu_-} \hat{A}_0^\Delta$$

$$\pi_{cpi,0} = -\frac{R\lambda}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \left(-R + \rho_a\nu_- - \mu_+(\rho_a + \nu_-) + \mu_+(R+1)\right) \hat{A}_0^\Delta$$

$$\mathbb{E}_0 \sum_{t\geq 0} \frac{1}{R^t} \pi_{cpi,t} = \frac{R^2\lambda(R-(1-\delta))}{(\nu_+ - \rho_a)(\mu_+ - \rho_a)(\mu_+ - \nu_-)} \frac{\mu_+ - 1}{R-\mu_-} \hat{A}_0^\Delta$$

Note that using $(\mu_+ - 1)(\mu_+ - R) - R\lambda\vartheta\kappa\mu_+ = 0$ and $-R + \rho_a\nu_- - \mu_+(\rho_a + \nu_-) + \mu_+(R+1) \geq 0$ we have:

$$
\begin{aligned}
\tilde{\mathcal{Y}}_0 &= -\frac{\lambda}{\kappa}\frac{1}{(\nu_+-\rho_a)}\frac{(\mu_+-1)}{(\mu_+-\rho_a)}\frac{(\mu_+-R)\left(-R+\rho_a\nu_- - \mu_+(\rho_a+\nu_-)+\mu_+(R+1)\right)}{(\mu_+-\nu_-)\,\mu_+}\hat{A}_0^{\Delta} \\
\left|\tilde{\mathcal{Y}}_0\right| &= \frac{\lambda}{\kappa}\frac{1}{(\nu_+-\rho_a)}\frac{(\mu_+-1)}{(\mu_+-\rho_a)}\frac{(\mu_+-R)\left(-R+\rho_a\nu_- - \mu_+(\rho_a+\nu_-)+\mu_+(R+1)\right)}{(\mu_+-\nu_-)\,\mu_+}\left|\hat{A}_0^{\Delta}\right| \\
&< \frac{\lambda}{\kappa}\frac{1}{(\nu_+-\rho_a)}\frac{(\mu_+-1)}{(\mu_+-\rho_a)}\frac{(\mu_+-R)\left(-(\rho_a+\nu_-)+(R+1)\right)\mu_+}{(\mu_+-\nu_-)\,\mu_+}\left|\hat{A}_0^{\Delta}\right| \\
&= \frac{\lambda}{\kappa}\frac{R+1-(\rho_a+\nu_-)}{(\nu_+-\rho_a)}\frac{(\mu_+-1)}{(\mu_+-\rho_a)}\frac{(\mu_+-R)}{(\mu_+-\nu_-)}\left|\hat{A}_0^{\Delta}\right| \\
&\leq \frac{\lambda}{\kappa}\frac{R+1-(\rho_a+\nu_-)}{(\nu_+-\rho_a)}\left|\hat{A}_0^{\Delta}\right|
\end{aligned}
$$

where the last line uses the fact that $\rho_a, \nu_- \leq 1 < R$. Similarly, using $(\mu_- - 1)(\mu_- - R) - R\lambda\vartheta\kappa\mu_- = 0$

$$
\begin{aligned}
\mathbb{E}_0\sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t &= -\frac{\lambda}{\kappa}\frac{R}{(\nu_+-\rho_a)}\frac{(\mu_+-1)}{(\mu_+-\rho_a)}\frac{(1-\mu_-)}{(\mu_+-\nu_-)\,\mu_-}\hat{A}_0^{\Delta} \\
&= -\frac{\lambda}{\kappa}\frac{R}{(\nu_+-\rho_a)}\frac{(\mu_+-1)}{(\mu_+-\rho_a)}\frac{(1-\mu_-)}{\left(\frac{1}{\beta}-\nu_-\mu_-\right)}\hat{A}_0^{\Delta}
\end{aligned}
$$

$$
\mathbb{E}_0\left|\sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t\right| < \frac{\lambda}{\kappa}\frac{R}{(\nu_+-\rho_a)}\left|\hat{A}_0^{\Delta}\right|
$$

where the second line uses $\mu_+\mu_- = R(1-\delta) = 1/\beta$ and the last line uses $\frac{1}{\beta} - \nu_-\mu_- \geq 1 - \nu_-\mu_- \geq 1 - \mu_-$ and $\rho_a \leq 1$.

Under strict CPI targeting we have $\pi_{cpi,t} = 0$ at all dates and

$$
\tilde{\mathcal{Y}}_t = -\frac{\lambda}{\kappa}\mathcal{NH}_t = -\frac{1}{(\nu_- - \rho_a)(\nu_+ - \rho_a)}\left((R-\rho_a)(1-\rho_a)\rho_a^t - (R-\nu_-)(1-\nu_-)\nu_-^t\right)\hat{A}_0^{\Delta}.
$$

So under strict CPI targeting:

$$
\begin{aligned}
\tilde{\mathcal{Y}}_0 &= - -\frac{\lambda}{\kappa}\frac{1}{(\nu_+-\rho_a)}\left(R+1-(\rho_a+\nu_-)\right)\hat{A}_0^{\Delta} \\
\mathbb{E}_0\sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t &= -\frac{\lambda}{\kappa}\frac{R}{(\nu_+-\rho_a)}\hat{A}_0^{\Delta}.
\end{aligned}
$$

Denoting with a superscript CPI the variables under CPI targeting, OP the variables the variables under optimal policy we therefore have after a negative shock in a necessity sector:

$$
\begin{gathered}
\tilde{\mathcal{Y}}_0^{CPI} < \tilde{\mathcal{Y}}_t^{OP} < 0 \\
\mathbb{E}_0\sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t^{CPI} < \mathbb{E}_0\sum_{t\geq 0}\frac{1}{R^t}\tilde{\mathcal{Y}}_t^{OP} < 0 \\
\pi_{cpi,0}^{OP} > \pi_{cpi,0}^{CPI} = 0 \\
\mathbb{E}_0\sum_{t\geq 0}\frac{1}{R^t}\pi_{cpi,t}^{OP} > \mathbb{E}_0\sum_{t\geq 0}\frac{1}{R^t}\pi_{cpi,t}^{CPI} = 0
\end{gathered}
$$

monetary policy is more accomodative after a negative shock in a necessity sector than strict targeting. After a shock in a luxury sector, we have:

$$\tilde{\mathcal{Y}}_0^{CPI} > \tilde{\mathcal{Y}}_t^{OP} > 0$$
$$\mathbb{E}_0 \sum_{t \geq 0} \frac{1}{R^t} \tilde{\mathcal{Y}}_t^{CPI} > \mathbb{E}_0 \sum_{t \geq 0} \frac{1}{R^t} \tilde{\mathcal{Y}}_t^{OP} > 0$$
$$\pi_{cpi,0}^{OP} < \pi_{cpi,0}^{CPI} = 0$$
$$\mathbb{E}_0 \sum_{t \geq 0} \frac{1}{R^t} \pi_{cpi,t}^{OP} < \mathbb{E}_0 \sum_{t \geq 0} \frac{1}{R^t} \pi_{cpi,t}^{CPI} = 0$$

i.e. monetary policy is more strict.

E.3 Welfare loss function

Welfare Loss Function

In this appendix, we derive the second order approximation of our social welfare function:

$$\mathcal{W} = (1-\delta)\int G\left(V_{-}\left(i\right),i\right)di + \delta\mathbb{E}_0\sum_{t_0=0}^{\infty}\beta^{t_0}\int G\left(V_{t_0}\left(i\right),i\right)di.$$

Recall that the value function of household i born at $t_0$is given by:

$$V_{t_0}\left(i\right) = \mathbb{E}_{t_0}\sum_{s=0}^{\infty}\left(\left(1-\delta\right)\beta\right)^s\left\{\left(1-\varphi\left(i\right)\right)\left[U_i\left(\mathcal{U}_1\left(\boldsymbol{c}^u_{1,t_0+s}\left(i\right)\right),\ldots\mathcal{U}_K\left(\boldsymbol{c}^u_{K,t_0+s}\left(i\right)\right)\right)-\chi\left(\frac{n^u_{t_0+s}(i)}{\vartheta(i)}\right)\right]+\varphi\left(i\right)\left[U_i\left(\mathcal{U}_1\left(\boldsymbol{c}^{HtM}_{1,t_0+s}\left(i\right)\right),\ldots\mathcal{U}_K\left(\boldsymbol{c}^{HtM}_{K,t_0+s}\left(i\right)\right)\right)-\chi\left(\frac{n^{HtM}_{t_0+s}(i)}{\vartheta(i)}\right)\right]\right\}.$$

Where the quantities $\left\{\boldsymbol{c}^u_{k,t_0+s}\left(i\right),\boldsymbol{c}^{HtM}_{1,t_0+s}\left(i\right),n^u_{t_0+s}(i),n^{Htm}_{t_0+s}(i)\right\}_{s\geq 0}$ are chosen optimally, as described in the derivation appendix. Applying Roy's identity (and using the fact that in steady state consumption and labor supply is constant across constrained and unconstrained households of the same type i) , the derivative of the value function is given by:

$$\begin{aligned} dV_{t_0}\left(i\right) = {} & \mathbb{E}_{t_0}\sum_{s=0}^{\infty}\left(\left(1-\delta\right)\beta\right)^s\partial_e v_{t_0+s}\left\{\frac{b_{t_0+s+1}\left(i\right)}{R_{t_0+s}}\hat{R}_{t_0+s}+\hat{W}_{t_0+s}W_{t_0+s}n_{t_0+s}\left(i\right)-\sum_{l=1}^{K}\int d^{t_0}_{l,t_0+s}(i,j)dp_{l,t_0+s}\left(j\right)dj+\varsigma\left(i\right)dDiv_{t_0+s}\right\} \\ & +\left(1-\varphi\left(i\right)\right)\partial_e v_{t_0}b_{t_0}\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\varphi\left(i\right)\mathbb{E}_{t_0}\sum_{s=0}^{\infty}\left(\left(1-\delta\right)\beta\right)^s\partial_e v_{t_0+s}\left(1-\frac{1}{R_{t_0+s}}\right)b_{t_0}\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}, \end{aligned}$$

with the first order change in dividends given by:

$$\begin{aligned} dDiv_t = {} & \sum_{k=1}^{K}\int y_k\left(j\right)\left(dp_{k,t}\left(j\right)-\frac{1}{A_{k,t}}\left(\mathcal{N}_k(\boldsymbol{P}_t,W_t)dW_t+\sum_{l=1}^{K}\mathcal{Y}_{l,k}(\boldsymbol{P}_t,W_t)dP_{l,t}-\left(\mathcal{N}_k(\boldsymbol{P}_t,W_t)W_t+\sum_{l=1}^{K}\mathcal{Y}_{l,k}(\boldsymbol{P}_t,W_t)P_{l,t}\right)\hat{A}_{k,t}\right)\right) \\ & +\sum_{k=1}^{K}\int dy_k\left(j\right)\left(p_{k,t}\left(j\right)-\frac{1}{A_{k,t}}\left(\mathcal{N}_k(\boldsymbol{P}_t,W_t)W_t+\sum_{l=1}^{K}\mathcal{Y}_{l,k}(\boldsymbol{P}_t,W_t)P_{l,t}\right)\right), \end{aligned}$$

where the change in demand for variety j is given by:

$$\begin{aligned} dy_{k,t}\left(j\right) = {} & \mathbb{E}_{\delta,t}\left(\int\partial_p d_{k,t}(i,j)dp_{k,t}\left(j\right)+\int\partial_P d_{k,t}(i,j)\left[j^*\right]dp_{k,t}\left(j^*\right)dj^*+\partial_e d_{k,t}(i,j)\left(\left(1-\varphi\left(i\right)\right)de^{t_0,u}_{k,t}\left(i\right)+\varphi\left(i\right)de^{t_0,HtM}_{k,t}\right)di\right) \\ & +\left(\partial_p d^I_{k,t}\left(j\right)dp_{k,t}\left(j\right)+\int\partial_P d^I_{k,t}\left(j\right)\left[j^*\right]dp_{k,t}\left(j^*\right)dj^*\right)\tilde{Y}_{k,t}+d\tilde{Y}_{k,t}, \end{aligned}$$

and demand for intermediary $d\tilde{Y}_{k,t}$ solves the system

$$\begin{aligned} d\tilde{Y}_{k,t} & = \sum_{l}\left(\partial_W\mathcal{Y}_{k,l}(\boldsymbol{P}_t,W_t)dW_t+\sum_{m=1}^{K}\partial_{P_m}\mathcal{Y}_{k,l}(\boldsymbol{P}_t,W_t)dP_{m,t}\right)Y_{k,t}+\sum_{l}\mathcal{Y}_{k,l}(\boldsymbol{P}_t,W_t)dY_{k,t}, \\ dY_{k,t} & = \int dy_{k,t}\left(j\right)dj. \end{aligned}$$

Around a steady state where prices, consumption, wealth and labor supply are constant (and equal across generation, constrained and unconstrained households of the same type i) and with $p_k(j) = \frac{1}{A}\left(\mathcal{N}_k(\boldsymbol{P}, W)W + \sum_{l=1}^K \mathcal{Y}_{l,k}(\boldsymbol{P}, W)P_l\right)$, this simplifies to

$$
\begin{aligned}
dV_{t_0}(i) = {} & \partial_e v(i)\,\mathbb{E}_{t_0}\sum_{s=0}^{\infty}((1-\delta)\beta)^s\left\{\frac{b(i)}{R}\hat{R}_{t_0+s} + \hat{W}_{t_0+s}Wn(i) - \sum_{l=1}^K e_l(i)\,\hat{P}_{l,t_0+s} + \varsigma(i)\sum_{k=1}^K P_kY_k\left(\hat{P}_{k,t_0+s} - \frac{1}{A_kP_k}\left(W\mathcal{N}_k(\boldsymbol{P}_t, W_t)\hat{W}_{t_0+s} + \sum_{l=1}^K P_l\mathcal{Y}_{l,k}(\boldsymbol{P}_t, W_t)\hat{P}_{l,t}\right) + \hat{A}_{k,t_0+s}\right)\right\} \\
& + \partial_e v(i)\,b(i)\sum_{l=1}^K \bar{s}_l\hat{P}_{l,t_0} \\
= {} & \partial_e v(i)\,\mathbb{E}_{t_0}\sum_{s=0}^{\infty}((1-\delta)\beta)^s\left\{\frac{b(i)}{R}\left(\hat{R}_{t_0+s} - \sum_{l=1}^K \bar{s}_l\pi_{l,t_0+s+1}\right) - \sum_{l=1}^K e(i)\,(s_l(i) - \bar{s}_l)\,\hat{P}_{l,t_0+s} + Wn(i)\sum_{l=1}^K \bar{s}_l\tilde{A}_{l,t_0+s}\right\}
\end{aligned}
$$

Next we derive the second order approximation of the social welfare function. We use the fact that in steady state, prices, consumption, wealth and labor supply are constant (and equal across generation, constrained and unconstrained households of the same type i) , $G'(V_{t_0}(i), i)\,\partial_e v_{t_0} = 1$ and $p_k(j) = \frac{1}{A}\left(\mathcal{N}_k(\boldsymbol{P}, W)W + \sum_{l=1}^K \mathcal{Y}_{l,k}(\boldsymbol{P}, W)P_l\right)$. Using clearing of the bond market, we have:

$$
\begin{aligned}
d^2\mathcal{W} = {} & (1-\delta)\int G''(i)\,(dV_-(i))^2\,di + \delta\mathbb{E}_0\sum_{t_0=0}^{\infty}\beta^{t_0}\int G''(i)\,(dV_{t_0}(i))^2\,di \\
& - \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int(1-\varphi(i))\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,u} - \sum_k s_k(i)\,\hat{P}_{k,t}\right) + \sum_k \partial_e e_k(i)\,\hat{P}_{k,t}\right)\left\{\frac{b(i)}{R}\hat{R}_t - \sum_{l=1}^K(e(i)\,s_l(i) - Wn(i)\,\bar{s}_l)\,\hat{P}_{l,t} + Wn(i)\sum_{l=1}^K \bar{s}_l\tilde{A}_{l,t} + \mathbb{1}_{t=t_0}b(i)\sum_{l=1}^K \bar{s}_l\hat{P}_{l,t_0}\right\}di \\
& - \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\varphi(i)\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM} - \sum_k s_k(i)\,\hat{P}_{k,t}\right) + \sum_k \partial_e e_k(i)\,\hat{P}_{k,t}\right)\left\{\frac{b(i)}{R}\hat{R}_t - \sum_{l=1}^K(e(i)\,s_l(i) - Wn(i)\,\bar{s}_l)\,\hat{P}_{l,t} + Wn(i)\sum_{l=1}^K \bar{s}_l\tilde{A}_{l,t} + \left(1 - \frac{1}{R}\right)b(i)\sum_{l=1}^K \bar{s}_l\hat{P}_{l,t_0}\right\}di \\
& + \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left\{dW_tdN_t - \sum_{l=1}^K\int d\left\{d_{l,t}^{t_0}(i,j)\right\}dp_{l,t_0+s}(j)\,dj + d^2Div_t\right\} + \mathbb{E}_0\sum\beta^t\mathbb{E}_{\delta,t}\left(\frac{dR_t}{R^2}b^{HtM,t_0}(i)\sum_{l=1}^K \bar{s}_l\hat{P}_{l,t_0}\right),
\end{aligned}
$$

with

$$
\int d\left\{d_{l,t_0+s}^{t_0}(i,j)\right\}dp_{l,t_0+s}(j)\,dj = \int \partial_p d_{l,t_0+s}^{t_0}(i,j)dp_{l,t_0+s}(j)\left(dp_{l,t_0+s}(j) - dP_{l,t_0+s}\right) + \partial_e d_{l,t_0+s}^{t_0}(i,j)\left(de_{l,t_0+s}(i) - e_l(i)\,\hat{P}_{l,t_0+s}\right)dp_{l,t_0+s}(j)\,dj,
$$

and

$$
\begin{aligned}
d^2Div_t = {} & -dW_tdN_t^d - \sum_{k=1}^K d\tilde{Y}_{k,t}^d dP_{k,t} - \sum_{k=1}^K \tilde{Y}_{k,t}^d d^2P_{k,t} + \sum_{k=1}^K\int dy_k(j)\left(dp_{k,t}(j) + P_k\hat{A}_{k,t}\right)dj \\
& + \sum_{k=1}^K \frac{Y_k}{A_{k,t}}\left(\mathcal{N}_k(\boldsymbol{P}_t, W_t)dW_t + \sum_{l=1}^K \mathcal{Y}_{l,k}(\boldsymbol{P}_t, W_t)dP_{l,t} - \left(\mathcal{N}_k(\boldsymbol{P}_t, W_t)W_t + \sum_{l=1}^K \mathcal{Y}_{l,k}(\boldsymbol{P}_t, W_t)P_{l,t}\right)\hat{A}_{k,t}\right)\hat{A}_{k,t} \\
& + \sum_{k=1}^K\int dy_k(j)\left(dp_{k,t}(j) - \frac{1}{A_{k,t}}\left(\mathcal{N}_k(\boldsymbol{P}_t, W_t)dW_t + \sum_{l=1}^K \mathcal{Y}_{l,k}(\boldsymbol{P}_t, W_t)dP_{l,t}\right) + p_{k,t}(j)\,\hat{A}_{k,t}\right)dj,
\end{aligned}
$$

Simplifying – using market clearing conditions for labor and markets and properties of the steady state – and removing the terms independent of monetary policy, we have:

$$
\begin{aligned}
dW_tdN_t - \sum_{l=1}^K\int d\left\{d_{l,t}^{t_0}(i,j)\right\}dp_{l,t_0+s}(j)\,dj + d^2Div_t = {} & -\sum_k P_kdC_{k,t}\left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{\mathbf{A}}_{k,t}\right) \\
& - \left(\sum_{k,l}\frac{1}{A_l}P_kY_l\partial_W\mathcal{Y}_{k,l}\hat{W}_t\left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{A}_{k,t}\right) + \sum_{k,l,m}\frac{1}{A_l}P_kY_l\hat{P}_{m,t}\partial_{P_m}\mathcal{Y}_{k,l}\left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{\mathbf{A}}_{k,t}\right)\right) \\
& + \hat{W}_t\sum_k E_k\tilde{\mathbf{A}}_{k,t} - \sum_{k=1}^K P_kY_k\left(\left(\frac{E_k}{P_kY_k}\bar{\epsilon}_k + \left(1 - \frac{E_k}{P_kY_k}\right)\bar{\epsilon}_k^I\right)\int\left(\hat{p}_{k,t}(j) - \hat{P}_{k,t}\right)\hat{p}_{k,t}(j)\,dj\right)
\end{aligned}
$$

Next we have, noting $\hat{p}^*_{k,t}$ the reset price at t and $\int \hat{p}^2_{k,t}(j)dj = (1-\theta_k)\sum_{m=0}^{t}\theta_k^m\left(\hat{p}^*_{k,t-m}\right)^2$:

$$\begin{aligned}
\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\int\left(\hat{p}_{k,t}\left(j\right)-\hat{P}_{k,t}\right)\hat{p}_{k,t}\left(j\right)dj &= \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left(\frac{1-\theta_k}{1-\beta\theta_k}\left(\hat{p}^*_{k,t}\right)^2-\left(\hat{P}_{k,t}\right)^2\right)\\
&= \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left\{\frac{1}{\left(1-\beta\theta_k\right)\left(1-\theta_k\right)}\left(\pi_{k,t}\right)^2+2\frac{1}{\left(1-\beta\theta_k\right)}\pi_{k,t}\hat{P}_{k,t-1}+\frac{\left(1-\theta_k\right)}{1-\beta\theta_k}\left(\hat{P}_{k,t-1}\right)^2-\left(\hat{P}_{k,t}\right)^2\right\}\\
&= \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left\{\frac{\theta_k}{\left(1-\beta\theta_k\right)\left(1-\theta_k\right)}\left(\pi_{k,t}\right)^2-\frac{\theta_k}{1-\beta\theta_k}\left(\hat{P}_{k,t-1}\right)^2+\frac{\beta\theta_k}{\left(1-\beta\theta_k\right)}\left(\hat{P}_{k,t}\right)^2\right\}\\
&= \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{\theta_k}{\left(1-\beta\theta_k\right)\left(1-\theta_k\right)}\left(\pi_{k,t}\right)^2.
\end{aligned}$$

We therefore have

$$\begin{aligned}
\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left\{dW_tdN_t-\sum_{l=1}^{K}\int d\left\{d^{t_0}_{l,t}(i,j)\right\}dp_{l,t_0+s}\left(j\right)dj+d^2Div_t\right\} &= -\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_k P_k dC_{k,t}\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{\mathbf{A}}_{k,t}\right)\\
&\quad-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left(\sum_{k,l}\frac{1}{A_l}P_kY_lW\partial_W\mathcal{Y}_{k,l}\hat{W}_t\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{\mathbf{A}}_{k,t}\right)+\sum_{k,l,m}\frac{1}{A_l}P_mP_kY_l\hat{P}_{m,t}\partial_{P_m}\mathcal{Y}_{k,l}\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{\mathbf{A}}_{k,t}\right)\right)\\
&\quad+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_k E_k\hat{W}_t\tilde{\mathbf{A}}_{k,t}-\frac{\theta_k\bar{\epsilon}_k}{\left(1-\beta\theta_k\right)\left(1-\theta_k\right)}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left(\pi_{k,t}\right)^2
\end{aligned}$$

Let us first slightly rewrite the terms coming from substitution in production. We have:

$$\begin{aligned}
&-\left(\sum_{k,l}\frac{1}{A_l}P_kY_lW\partial_W\mathcal{Y}_{k,l}\hat{W}_t\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{\mathbf{A}}_{k,t}\right)+\sum_{k,l,m}\frac{1}{A_l}P_kP_mY_l\hat{P}_{m,t}\partial_{P_m}\mathcal{Y}_{k,l}\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{\mathbf{A}}_{k,t}\right)\right)\\
&\qquad-\sum\frac{1}{A_l}Y_lW^2\partial_W\mathcal{N}_l\hat{W}_t^{\,2}-\sum_{k,l}\frac{1}{A_l}P_kY_lW\partial_W\mathcal{Y}_{k,l}\hat{W}_t\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)-\sum_{k,l}\frac{1}{A_l}WY_l\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)P_k\partial_{P_k}\mathcal{N}_l\hat{W}_t-\sum_{k,l,m}\frac{1}{A_l}P_mP_kY_l\left(\hat{P}_{m,t}+\tilde{\mathbf{A}}_{m,t}\right)\partial_{P_m}\mathcal{Y}_{k,l}\left(\hat{P}_{k,t}+\tilde{\mathbf{A}}_{k,t}\right)
\end{aligned}$$

where we used

$$\begin{gathered}
\sum_k P_k\partial_W\mathcal{Y}_{k,l}=-W\partial_W\mathcal{N}_l\\
\sum_k P_k\partial_{P_m}\mathcal{Y}_{k,l}=-W\partial_{P_m}\mathcal{N}_l\\
\partial_W\mathcal{Y}_{k,l}=\partial_{P_k}\mathcal{N}_l\\
\partial_{P_m}\mathcal{Y}_{k,l}=\partial_{P_k}\mathcal{Y}_{m,l}
\end{gathered}$$

We define

$$\mathcal{N}\left(W,\boldsymbol{P}\right)\equiv\sum_l\frac{1}{A_l}Y_l\mathcal{N}_l\left(W,\boldsymbol{P}\right)$$

$$\mathcal{Y}_k\left(W,\boldsymbol{P}\right)=\sum_l\frac{1}{A_l}Y_l\mathcal{Y}_{k,l}\left(W,\boldsymbol{P}\right)$$

We have

$$
\begin{aligned}
&-\sum \frac{1}{A_l} Y_l W \partial_W \mathcal{N}_l \hat{W}_t{}^2 - \sum_{k,l} \frac{1}{A_l} P_k Y_l W \partial_W \mathcal{Y}_{k,l} \hat{W}_t \left(\hat{P}_{k,t} + \tilde{\mathbf{A}}_{k,t}\right) - \sum_{k,l} \frac{1}{A_l} W Y_l \left(\hat{P}_{k,t} + \tilde{\mathbf{A}}_{k,t}\right) P_k \partial_{P_k} \mathcal{N}_l \hat{W}_t - \sum_{k,l,m} \frac{1}{A_l} P_m P_k Y_l \left(\hat{P}_{m,t} + \tilde{\mathbf{A}}_{m,t}\right) \partial_{P_m} \mathcal{Y}_{k,l} \left(\hat{P}_{k,t} + \tilde{\mathbf{A}}_{k,t}\right) \\
&\qquad = -W^2 \partial_W \mathcal{N} \hat{W}_t{}^2 - \sum_k P_k W \partial_W \mathcal{Y}_k \hat{W}_t \left(\hat{P}_{k,t} + \tilde{\mathbf{A}}_{k,t}\right) - \sum_k W P_k \partial_{P_k} \mathcal{N} \left(\hat{P}_{k,t} + \tilde{\mathbf{A}}_{k,t}\right) \hat{W}_t - \sum_{k,l} P_l P_k \partial_{P_l} \mathcal{Y}_k \left(\hat{P}_{k,t} + \tilde{\mathbf{A}}_{k,t}\right) \left(\hat{P}_{l,t} + \tilde{\mathbf{A}}_{l,t}\right)
\end{aligned}
$$

Note that we have

$$
P_k dC_{k,t} = \mathbb{E}_{\delta,t} \left(\int \left((1-\varphi(i)) P_k dc_{k,t}^{t_0,u}(i) + \varphi(i) P_k dc_{k,t}^{t_0,HtM}(i) \right) \right)
$$

with

$$
P_k dc_{k,t}^{t_0}(i) = e(i) \partial_e e_k(i) \left(\hat{e}_t^{t_0}(i) - \sum_k s_k(i) \hat{P}_{k,t} \right) + e_k(i) \sum_l \rho_{k,l}(i) \hat{P}_{l,t}
$$

So rearranging, we obtain:

$$
\begin{aligned}
P_k dC_{k,t} \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{\mathbf{A}}_{k,t}\right) = \mathbb{E}_{\delta,t} \int \frac{1}{\sigma} \left(\hat{e}_t^{t_0}(i) - \sum_k s_k(i) \hat{P}_{k,t} \right) \sigma e(i) \partial_e e_k(i) \, di \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{A}_{k,t}\right) \\
- \sum_k E_k \sum_l \bar{\rho}_{k,l} \left(\hat{P}_{l,t} + \tilde{A}_{l,t}\right) \left(\hat{P}_{k,t} + \tilde{A}_{k,t}\right)
\end{aligned}
$$

So we can rewrite the welfare loss as

$$
\begin{aligned}
d^2\mathcal{W} = {} & (1-\delta) \int G''(i) \left(dV_-(i)\right)^2 di + \delta \mathbb{E}_0 \sum_{t_0=0}^{\infty} \beta^{t_0} \int G''(i) \left(dV_{t_0}(i)\right)^2 di \\
& - \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbb{E}_{\delta,t} \int (1-\varphi(i)) \left(\frac{1}{\sigma} \left(\hat{e}_t^{t_0,u} - \sum_k s_k(i) \hat{P}_{k,t} \right) + \sum_k \partial_e e_k(i) \hat{P}_{k,t} \right) \left\{ \frac{b(i)}{R} \hat{R}_t - \sum_{l=1}^{K} \left(e(i) s_l(i) - W n(i) \bar{s}_l\right) \hat{P}_{l,t} + W n(i) \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t} \right\} di \\
& - \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbb{E}_{\delta,t} \int (1-\varphi(i)) \left(\frac{1}{\sigma} \left(\hat{e}_t^{t_0,u} - \sum_k s_k(i) \hat{P}_{k,t} \right) + \sum_k \partial_e e_k(i) \hat{P}_{k,t} \right) \left\{ \mathbb{1}_{t=t_0} b(i) \sum_{l=1}^{K} \bar{s}_l \hat{P}_{l,t_0} + \sum_{l=1}^{K} \sigma e(i) \partial_e e_k(i) \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{A}_{k,t}\right) \right\} di \\
& - \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbb{E}_{\delta,t} \int \varphi(i) \left(\frac{1}{\sigma} \left(\hat{e}_t^{t_0,HtM} - \sum_k s_k(i) \hat{P}_{k,t} \right) + \sum_k \partial_e e_k(i) \hat{P}_{k,t} \right) \left\{ \frac{b(i)}{R} \hat{R}_t - \sum_{l=1}^{K} \left(e(i) s_l(i) - W n(i) \bar{s}_l\right) \hat{P}_{l,t} + W n(i) \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t} \right\} di \\
& - \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbb{E}_{\delta,t} \int \varphi(i) \left(\frac{1}{\sigma} \left(\hat{e}_t^{t_0,HtM} - \sum_k s_k(i) \hat{P}_{k,t} \right) + \sum_k \partial_e e_k(i) \hat{P}_{k,t} \right) \left\{ \left(1 - \frac{1}{R}\right) b(i) \sum_{l=1}^{K} \bar{s}_l \hat{P}_{l,t_0} + \sum_{l=1}^{K} \sigma e(i) \partial_e e_k(i) \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{A}_{k,t}\right) \right\} di \\
& + \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{k,l}^{K} \int \sigma e(i) \partial_e e_k(i) \partial_e e_l(i) \, di \hat{P}_{l,t} \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{A}_{k,t}\right) \\
& - \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(-W^2 \partial_W \mathcal{N} \hat{W}_t{}^2 - \sum_k P_k W \partial_W \mathcal{Y}_k \hat{W}_t \left(\hat{P}_{k,t} + \tilde{\mathbf{A}}_{k,t}\right) - \sum_k W P_k \partial_{P_k} \mathcal{N} \left(\hat{P}_{k,t} + \tilde{\mathbf{A}}_{k,t}\right) \hat{W}_t - \sum_{k,l} P_l P_k \partial_{P_l} \mathcal{Y}_k \left(\hat{P}_{k,t} + \tilde{\mathbf{A}}_{k,t}\right) \left(\hat{P}_{l,t} + \tilde{\mathbf{A}}_{l,t}\right) \right) \\
& + \sum_k E_k \sum_l \bar{\rho}_{k,l} \left(\hat{P}_{l,t} + \tilde{A}_{l,t}\right) \left(\hat{P}_{k,t} + \tilde{A}_{k,t}\right) + \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_k E_k \hat{W}_t \tilde{\mathbf{A}}_{k,t} - \frac{\theta_k \bar{\epsilon}_k}{(1-\beta\theta_k)(1-\theta_k)} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\pi_{k,t}\right)^2 \\
& + \mathbb{E}_0 \sum \beta^t \mathbb{E}_{\delta,t} \left(\frac{dR_t}{R^2} b(i) \sum_{l=1}^{K} \bar{s}_l \hat{P}_{l,t_0} \right)
\end{aligned}
$$

Simplification of the terms corresponding to the expenditure of unconstrained households

Here we simplify the second line of the expression. To do so, define

$$\begin{aligned}X_t \equiv & \frac{1}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma\frac{e(i)}{E}+\frac{Wn(i)}{WN}\psi\right)di}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_l \bar{s}_l\tilde{A}_{l,t}+\frac{Wn\left(i\right)}{WN}\bar{\psi}\hat{W}_t+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t}\right\}di\\ & -\frac{1}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma\frac{e(i)}{E}+\frac{Wn(i)}{WN}\psi\right)di}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_t-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t}\right\}\end{aligned}$$

X_t is an aggregate variable growing at rate $\hat{R}_t$. Indeed, recall that we have, from the labor market clearing condition:

$$\begin{aligned}\sum_l\bar{s}_l\tilde{A}_{l,t}+\psi\left(\hat{W}_t-\sum_l\int\frac{Wn\left(i\right)}{WN}\partial_e e_l\left(i\right)\hat{P}_{l,t}\right) & =\mathbb{E}_{\delta,t}\left(\frac{e\left(i\right)}{E}\left(1+\frac{Wn\left(i\right)\psi}{e\left(i\right)\sigma}\right)\left(\left(1-\varphi\left(i\right)\right)\left(\hat{e}^{t_0,u}_{t_0+s}-\sum_k s_k\left(i\right)\hat{P}_{k,t_0+s}\right)+\varphi\left(i\right)\left(\hat{e}^{t_0,HtM}_{t_0+s}-\sum_k s_k\left(i\right)\hat{P}_{k,t_0}\right)\right)\right)\\ & \equiv\mathcal{E}_t\end{aligned}$$

And it is direct to show that:

$$\begin{aligned}\mathbb{E}_t\mathcal{E}_{t+1}-\mathcal{E}_t= & \int\left(1-\varphi\left(i\right)\right)\left(\sigma\frac{e\left(i\right)}{E}+\frac{Wn\left(i\right)}{WN}\psi\right)\left(\hat{R}_t-\sum_l\partial_e e_l\left(i\right)\pi_{l,t+1}\right)di\\ & +\int\varphi\left(i\right)\left\{\frac{b_0}{ER}\Delta\hat{R}_{t+1}+\frac{Wn\left(i\right)}{WN}\psi\Delta\hat{W}_{t+1}-\sum_l\frac{Wn\left(i\right)}{WN}\psi\partial_e e_l\left(i\right)\pi_{l,t+1}-\frac{e\left(i\right)}{E}s_k\left(i\right)\cdot\hat{\pi}_{l,t+1}+\frac{Wn\left(i\right)}{WN}\sum\left\{\bar{s}_k\left(\hat{\pi}_{l,t+1}+\Delta\hat{A}_{l,t+1}\right)\right\}di\right\}\end{aligned}$$

So we have:

$$\mathbb{E}_tX_{t+1}-X_t=\hat{R}_t.$$

Since:

$$\mathbb{E}_t\left(\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_{t+1}-\sum_k s_k\left(i\right)\hat{P}_{k,t+1}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t+1}\right)-\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_t-\sum_k s_k\left(i\right)\hat{P}_{k,t}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t}=\hat{R}_t,$$

we have

$$\begin{aligned} & -\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\left(1-\varphi\left(i\right)\right)\left(\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_t-\sum_k s_k\left(i\right)\hat{P}_{k,t}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t}\right)\left\{\frac{b\left(i\right)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di\\ & -\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\left(1-\varphi\left(i\right)\right)\left(\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_t-\sum_k s_k\left(i\right)\hat{P}_{k,t}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t}\right)\left\{\mathbb{1}_{t=t_0}b\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di\\ & =-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\left(1-\varphi\left(i\right)\right)\\ & \left(\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_t-\sum_k s_k\left(i\right)\hat{P}_{k,t}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t}-X_t\right)\left\{\frac{b\left(i\right)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\mathbb{1}_{t=t_0}b\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di\\ & -\mathbb{E}_0\sum_{t=0}^{\infty}\beta^tX_t\mathbb{E}_{\delta,t}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\mathbb{1}_{t=t_0}b\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di\end{aligned}$$

Next we have:

$$\mathbb{E}_{t_0}\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_{t_0+s}-\sum_k s_k\left(i\right)\hat{P}_{k,t_0+s}\right)+\sum_k \partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}-X_{t_0+s}\right)$$

$$=\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\frac{E}{\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)}\left\{\frac{b\left(i\right)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\frac{e\left(i\right)}{E}\sigma\sum_l\partial_e e_l\left(i\right)\hat{P}_{l,t_0+s}\right\}-X_{t_0+s}\right)$$

so

$$-\mathbb{E}_0\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left(\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_{t_0+s}-\sum_k s_k\left(i\right)\hat{P}_{k,t_0+s}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}-X_{t_0+s}\right)$$

$$\cdot\left\{\frac{b\left(i\right)}{R}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\mathbb{1}_{t=t_0}b\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}$$

$$=-\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\Bigg(\frac{E}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\Bigg\{\frac{b\left(i\right)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\frac{e\left(i\right)}{E}\sigma\sum_l\partial_e e_l\left(i\right)\hat{P}_{l,t_0+}$$

$$\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$=-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\frac{E^2}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left\{\frac{b\left(i\right)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}-\frac{e\left(i\right)}{E}\sigma\sum_l\partial_e e_l\left(i\right)\tilde{A}_{k,t_0+s}\right\}\right\}^2di$$

$$-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{E^2}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\frac{e\left(i\right)}{E}\sigma\sum_l\partial_e e_l\left(i\right)\hat{P}_{l,t_0+s}\right\}\sum_{s=0}^{\infty}\frac{1}{R^s}\left\{\sum_{l=1}^{K}\sigma\frac{e\left(i\right)}{E}\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sum_{s=0}^{\infty}\frac{1}{R^s}E\left\{\frac{b\left(i\right)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}$$

$$+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

First note that we have, up to terms independent from monetary policy:

$$-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{E^2}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\frac{e\left(i\right)}{E}\sigma\sum_l\partial_e e_l\left(i\right)\hat{P}_{l,t_0+s}\right\}\sum_{s=0}^{\infty}\frac{1}{R^s}\left\{\sum_{l=1}^{K}\sigma\frac{e\left(i\right)}{E}\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di$$

$$=-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)\psi\sigma e\left(i\right)}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}-\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)^2di$$

$$+\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\sum_l\partial_e e_l\left(i\right)\left(\hat{P}_{l,t_0+s}+\tilde{A}_{l,t_0+s}\right)\right\}^2di$$

$$-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^s}\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}di$$

We therefore obtain:

$$\begin{aligned}
& -\mathbb{E}_0 \sum_{s=0}^{\infty} \frac{1}{R^s} \int \left(1-\varphi\left(i\right)\right) \left(\frac{1}{\sigma} \left(\hat{e}_{t_0+s}^{t_0,u} - \sum_k s_k\left(i\right) \hat{P}_{k,t_0+s} \right) + \sum_k \partial_e e_k\left(i\right) \hat{P}_{k,t_0+s} - X_{t_0+s} \right) \\
& \qquad \cdot \left\{ \frac{b\left(i\right)}{R} \hat{R}_{t_0+s} - \sum_{l=1}^{K} \left(e\left(i\right) s_l\left(i\right) - Wn\left(i\right) \bar{s}_l\right) \hat{P}_{l,t_0+s} + Wn\left(i\right) \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t_0+s} + \mathbb{1}_{t=t_0} b\left(i\right) \sum_{l=1}^{K} \bar{s}_l \hat{P}_{l,t_0} + \sum_{l=1}^{K} \sigma e\left(i\right) \partial_e e_k\left(i\right) \left(\hat{W}_{t_0+s} - \hat{P}_{k,t_0+s} - 2\tilde{A}_{k,t_0+s}\right) \right\} di \\
& = -\left(1-\frac{1}{R}\right) \int \left(1-\varphi\left(i\right)\right) \frac{E^2}{\sigma e\left(i\right) + Wn\left(i\right)\psi} \left\{ \sum_{s=0}^{\infty} \frac{1}{R^s} \left\{ \frac{b\left(i\right)}{RE} \left(R_{t_0+s} - \sum_{l=1}^{K} \bar{s}_l \pi_{l,t_0+s+1} \right) - \sum_{l=1}^{K} \frac{e\left(i\right)}{E} \left(s_l\left(i\right) - \bar{s}_l\right) \hat{P}_{l,t_0+s} + \frac{Wn\left(i\right)}{WN} \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t_0+s} - \frac{e\left(i\right)}{E} \sigma \sum_l \partial_e e_l\left(i\right) \tilde{A}_{k,t_0+s} \right\} \right\}^2 di \\
& \qquad - \left(1-\frac{1}{R}\right) \int \left(1-\varphi\left(i\right)\right) \frac{Wn\left(i\right)\psi\sigma e\left(i\right)}{\sigma e\left(i\right) + Wn\left(i\right)\psi} \left(\sum_{s=0}^{\infty} \frac{1}{R^s} \hat{W}_{t_0+s} - \sum_k \partial_e e_k\left(i\right) \left(\hat{P}_{k,t_0+s} + \tilde{A}_{k,t_0+s}\right) \right)^2 di \\
& \qquad + \left(1-\frac{1}{R}\right) \int \left(1-\varphi\left(i\right)\right) \sigma e\left(i\right) \left\{ \sum_{s=0}^{\infty} \frac{1}{R^s} \sum_l \partial_e e_l\left(i\right) \left(\hat{P}_{l,t_0+s} + \tilde{A}_{l,t_0+s}\right) \right\}^2 di \\
& \qquad - \left(1-\frac{1}{R}\right) \int \left(1-\varphi\left(i\right)\right) \sigma e\left(i\right) \sum_{s=0}^{\infty} \frac{1}{R^s} \hat{W}_{t_0+s} \sum \frac{1}{R^s} \sum_k \partial_e e_k\left(i\right) \hat{P}_{k,t_0+s} di \\
& \qquad - \left(1-\frac{1}{R}\right) \int \left(1-\varphi\left(i\right)\right) \sum_{s=0}^{\infty} \frac{1}{R^s} E \left\{ \frac{b\left(i\right)}{RE} \left(R_{t_0+s} - \sum_{l=1}^{K} \bar{s}_l \pi_{l,t_0+s+1} \right) - \sum_l \frac{e\left(i\right)}{E} \left(s_l\left(i\right) - \bar{s}_l\right) \hat{P}_{l,t_0+s} + Wn\left(i\right) \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t} \right\} di \sum_{s=0}^{\infty} \frac{1}{R^s} \hat{W}_{t_0+s} \\
& + \left(1-\frac{1}{R}\right) \sum_{s=0}^{\infty} \frac{1}{R^s} X_{t_0+s} \sum_{s=0}^{\infty} \frac{1}{R^s} \int \left(1-\varphi\left(i\right)\right) \left\{ \frac{b\left(i\right)}{R} \left(R_{t_0+s} - \sum_{l=1}^{K} \bar{s}_l \pi_{l,t_0+s+1} \right) - \sum_l \frac{e\left(i\right)}{E} \left(s_l\left(i\right) - \bar{s}_l\right) \hat{P}_{l,t_0+s} + Wn\left(i\right) \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t} + \sum_{l=1}^{K} \sigma e\left(i\right) \partial_e e_k\left(i\right) \left(\hat{W}_{t_0+s} - \hat{P}_{k,t_0+s} - 2\tilde{A}_{k,t_0+s}\right) \right\} di
\end{aligned}$$

Next we have:

$$\begin{aligned}
&\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^{s}}X_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^{s}}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di\\
&=\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^{E}\right)\sigma+\left(1-\varphi^{N}\right)\psi\right)^{-1}}{R^{s}}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_{l}\bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\sum_{l}\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t_0+s}\right\}di\\
&\cdot\sum_{s=0}^{\infty}\frac{1}{R^{s}}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_{l}\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di\\
&-\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^{E}\right)\sigma+\left(1-\varphi^{N}\right)\psi\right)^{-1}}{R^{s}}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_0+s}-\left(R-1\right)\sum_{l}\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di\\
&\cdot\sum_{s=0}^{\infty}\frac{1}{R^{s}}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_{l}\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}\\
&+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^{s}}X_{t_0+s}\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\sum_{l}\bar{s}_l\hat{P}_{l,t_0}\\
&=\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^{E}\right)\sigma+\left(1-\varphi^{N}\right)\psi\right)^{-1}}{R^{s}}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_{l}\bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\bar{\psi}\hat{W}_{t_0+s}+\sum_{l}\sigma\frac{e\left(i\right)}{E}\partial_e e_l di\hat{P}_{l,t_0+s}\right\}di\\
&\cdot\sum_{s=0}^{\infty}\frac{1}{R^{s}}\int\left(1-\varphi\left(i\right)\right)\left\{\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di\\
&+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^{E}\right)\sigma+\left(1-\varphi^{N}\right)\psi\right)^{-1}}{R^{s}}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_{l}\bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\sum_{l}\sigma\frac{e\left(i\right)}{E}\partial_e e_l di\hat{P}_{l,t_0+s}\right\}di\\
&\cdot\sum_{s=0}^{\infty}\frac{1}{R^{s}}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_{l}\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{k,t_0+s}\right\}di\\
&-\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^{E}\right)\sigma+\left(1-\varphi^{N}\right)\psi\right)^{-1}}{R^{s}}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_0+s}-\left(R-1\right)\sum_{l}\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di\\
&\cdot\sum_{s=0}^{\infty}\frac{1}{R^{s}}\int\left(1-\varphi\left(i\right)\right)\left\{\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di\\
&-\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{\left(\left(1-\varphi^{E}\right)\sigma+\left(1-\varphi^{N}\right)\psi\right)^{-1}}{R^{s}}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_0+s}-\left(R-1\right)\sum_{l}\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}\right\}di\\
&\cdot\sum_{s=0}^{\infty}\frac{1}{R^{s}}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_{l}\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di\\
&+\left(1-\frac{1}{R}\right)\sum\frac{1}{R^{s}}X_{t_0+s}\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\sum_{l}\bar{s}_l\hat{P}_{l,t_0}
\end{aligned}$$

where $\varphi^E \equiv \int \varphi(i)\frac{e(i)}{E}di$,$\varphi^N \equiv \int \varphi(i)\frac{Wn(i)}{WN}di$. Let us first rewrite the first term of this expression. We have:

$$
\begin{aligned}
&\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_l \bar{s}_l\tilde{A}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t_0+s}\right\}di \\
&\qquad\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di \\
&=\left(1-\frac{1}{R}\right)\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{W}_{t_0+s}-\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)\right\}^2 di \\
&\quad-\left(1-\frac{1}{R}\right)\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\sum\frac{1}{R^s}\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right\}^2 di \\
&\quad-\left(1-\frac{1}{R}\right)\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)di\right)^2 \\
&+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}di+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\frac{1}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma\frac{e(i)}{E}+\frac{Wn(i)}{WN}\psi\right)di}\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t_0+s}di \\
&\qquad\cdot\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di
\end{aligned}
$$

Next, the second and third terms can be rewritten:

$$\begin{aligned}
&\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^{s}}\frac{1}{\left(1-\varphi^{E}\right)\sigma+\left(1-\varphi^{N}\right)\psi}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_{l}\bar{s}_{l}\tilde{A}_{l,t_{0}+s}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_{0}+s}+\sum_{l}\sigma\frac{e\left(i\right)}{E}\partial_{e}e_{l}\left(i\right)di\hat{P}_{l,t_{0}+s}\right\}di\\
&\qquad\cdot\sum_{s=0}^{\infty}\frac{1}{R^{s}}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_{0}+s}-\left(R-1\right)\sum_{l}\bar{s}_{l}\hat{P}_{l,t_{0}+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_{l}\left(i\right)-\bar{s}_{l}\right)\hat{P}_{l,t_{0}+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_{l}\tilde{A}_{l,t_{0}+s}\right\}di\\
&\qquad-\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^{s}}\frac{1}{\left(1-\varphi^{E}\right)\sigma+\left(1-\varphi^{N}\right)\psi}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_{0}+s}-\left(R-1\right)\sum_{l}\bar{s}_{l}\hat{P}_{l,t_{0}+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_{l}-\bar{s}_{l}\right)\hat{P}_{l,t_{0}+s}\right\}di\\
&\qquad\cdot\sum_{s=0}^{\infty}\frac{1}{R^{s}}\int\left(1-\varphi\left(i\right)\right)\left\{\sum_{l=1}^{K}\sigma e\left(i\right)\partial_{e}e_{k}\left(i\right)\left(\hat{W}_{t_{0}+s}-\hat{P}_{k,t_{0}+s}-2\tilde{A}_{k,t_{0}+s}\right)\right\}di\\
&=\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^{s}}\hat{W}_{t_{0}+s}\sum\frac{1}{R^{s}}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_{0}+s}-\left(R-1\right)\sum_{l}\bar{s}_{l}\hat{P}_{l,t_{0}+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_{l}\left(i\right)-\bar{s}_{l}\right)\hat{P}_{l,t_{0}+s}\right\}di\\
&\quad+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^{s}}\frac{1}{\left(1-\varphi^{E}\right)\sigma+\left(1-\varphi^{N}\right)\psi}\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_{0}+s}di\sum_{s=0}^{\infty}\frac{1}{R^{s}}\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_{l}\tilde{A}_{l,t_{0}+s}di\\
&\quad+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^{s}}\frac{1}{\left(1-\varphi^{E}\right)\sigma+\left(1-\varphi^{N}\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sum_{l}\sigma\frac{e\left(i\right)}{E}\partial_{e}e_{l}di\hat{P}_{l,t_{0}+s}di\sum\frac{1}{R^{s}}\int\left(1-\varphi\left(i\right)\right)\left\{Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_{l}\tilde{A}_{l,t_{0}+s}\right\}di\\
&\quad+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^{s}}\frac{1}{\left(1-\varphi^{E}\right)\sigma+\left(1-\varphi^{N}\right)\psi}\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}\sum_{l}\bar{s}_{l}\tilde{A}_{l,t_{0}+s}di\\
&\qquad\cdot\sum_{s=0}^{\infty}\frac{1}{R^{s}}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_{0}+s}-\left(R-1\right)\sum_{l}\bar{s}_{l}\hat{P}_{l,t_{0}+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_{l}\left(i\right)-\bar{s}_{l}\right)\hat{P}_{l,t_{0}+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_{l}\tilde{A}_{l,t_{0}+s}\right\}di\\
&+2\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^{s}}\frac{1}{\left(1-\varphi^{E}\right)\sigma+\left(1-\varphi^{N}\right)\psi}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_{0}+s}-\left(R-1\right)\sum_{l}\bar{s}_{l}\hat{P}_{l,t_{0}+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_{l}\left(i\right)-\bar{s}_{l}\right)\hat{P}_{l,t_{0}+s}\right\}di\sum\frac{1}{R^{s}}\int\left(1-\varphi\left(i\right)\right)\sum_{l=1}^{K}\sigma e\left(i\right)\partial_{e}e_{k}\left(i\right)\tilde{A}_{k,t_{0}+s}di
\end{aligned}$$

Using:

$$\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_{0}+s}-\left(R-1\right)\sum_{l}\bar{s}_{l}\hat{P}_{l,t_{0}+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_{l}\left(i\right)-\bar{s}_{l}\right)\hat{P}_{l,t_{0}+s}\right\}di=-\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_{t_{0}+s}-\left(R-1\right)\sum_{l}\bar{s}_{l}\hat{P}_{l,t_{0}+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_{l}\left(i\right)-\bar{s}_{l}\right)\hat{P}_{l,t_{0}+s}\right\}di$$

Since in steady state:

$$\int b\left(i\right)di=\int\frac{e\left(i\right)}{E}\left(s_{l}\left(i\right)-\bar{s}_{l}\right)di=0$$

we have:

$$\begin{aligned}
&\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}\\
&=\left(1-\frac{1}{R}\right)\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{W}_{t_0+s}-\sum_{k}\partial_e e_k\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)\right\}^2 di\\
&-\left(1-\frac{1}{R}\right)\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\sum\frac{1}{R^s}\sum_{k}\partial_e e_k\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right\}^2 di\\
&-\left(1-\frac{1}{R}\right)\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\sum_{k}\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)di\right)^2\\
&+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}\sum\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\sum_{k}\partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}di\\
&+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_{l}\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l}\bar{s}_l\tilde{A}_{l,t_0+s}\right\}di\\
&+\left(1-\frac{1}{R}\right)\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_{l}\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_{l}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}-\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\tilde{A}_{k,t_0+s}\right\}di\right)^2\\
&+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\sum_{l}\bar{s}_l\hat{P}_{l,t_0}
\end{aligned}$$

Putting everything together, we have:

$$\begin{aligned}
&-\mathbb{E}_0\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left(\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_{t_0+s}\left(i\right)-\sum_k s_k\left(i\right)\hat{P}_{k,t_0+s}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}-X_{t_0+s}\right)\\
&\qquad\cdot\left\{\frac{b\left(i\right)}{R}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\mathbb{1}_{t=t_0}b\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di\\
&=-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\frac{E^2}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left\{\frac{b\left(i\right)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}-\frac{e\left(i\right)}{E}\sigma\sum_l\partial_e e_l\left(i\right)\tilde{A}_{k,t_0+s}\right\}\right\}^2di\\
&\quad-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)\psi\sigma e\left(i\right)}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}-\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)^2di\\
&\quad+\left(1-\frac{1}{R}\right)\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{W}_{t_0+s}-\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)\right\}^2di\\
&\quad-\left(1-\frac{1}{R}\right)\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)di\right)^2\\
&\quad+\left(1-\frac{1}{R}\right)\frac{\left(1-\varphi^E\right)\sigma}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right\}^2di\\
&\quad-\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\sum_l\bar{s}_l\hat{P}_{l,t_0}di\sum\frac{1}{R^s}\hat{W}_{t_0+s}+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\sum_l\bar{s}_l\hat{P}_{l,t_0}\\
&+\frac{1-\frac{1}{R}}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\left(R_{t_0+s}-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}-\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\tilde{A}_{k,t_0+s}\right\}di\right)^2
\end{aligned}$$

Finally, we rewrite some coefficients, we have:

$$\begin{aligned}
&\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)\psi\sigma e\left(i\right)}{\sigma e\left(i\right)+Wn\left(i\right)\psi}di-\frac{\int\left(1-\varphi\left(i\right)\right)\psi Wn\left(i\right)di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}\\
&=\frac{\sigma\psi}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}\left\{\int\left(1-\varphi\left(i\right)\right)e\left(i\right)\frac{Wn\left(i\right)\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}{\sigma e\left(i\right)+Wn\left(i\right)\psi}di-\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)di\int\left(1-\varphi\left(i\right)\right)e\left(i\right)\frac{\sigma e\left(i\right)+Wn\left(i\right)\psi}{\sigma e\left(i\right)+Wn\left(i\right)\psi}di\right\}\\
&=\frac{\sigma\psi}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}\left\{\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\frac{Wn\left(i\right)\int\left(1-\varphi\left(i\right)\right)e\left(i\right)di-e\left(i\right)\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)di}{\sigma e\left(i\right)+Wn\left(i\right)\psi}di\right\}\\
&=\left(1-\frac{1}{R}\right)\frac{\sigma\int\left(1-\varphi\left(i\right)\right)\psi Wn\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}\left\{\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\frac{\frac{Wn\left(i\right)}{\int\left(1-\varphi\left(i\right)\right)\psi Wn\left(i\right)di}\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di-b\left(i\right)}{\sigma e\left(i\right)+Wn\left(i\right)\psi}di\right\}\\
&=-\frac{\int\left(1-\varphi\left(i\right)\right)\psi\frac{Wn\left(i\right)}{WN}di}{\left(1+\frac{\int\left(1-\varphi\left(i\right)\right)\left(\frac{Wn\left(i\right)}{WN}\psi\right)di}{\int\left(1-\varphi\left(i\right)\right)\frac{\sigma e\left(i\right)}{E}di}\right)}\int\left(1-\frac{1}{R}\right)\frac{\frac{\left(1-\varphi\left(i\right)\right)b\left(i\right)}{\int\left(1-\varphi\left(i\right)\right)e\left(i\right)di}-\frac{\left(1-\varphi\left(i\right)\right)Wn\left(i\right)}{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)di}\frac{\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)e\left(i\right)di}}{1+\frac{Wn\left(i\right)\psi}{\sigma e\left(i\right)}}
\end{aligned}$$

and we define the variance covariance matrix of marginal propensities to spend:

$$\mathcal{E}_{k,l}=\int\frac{(1-\varphi(i))\,e(i)}{\int(1-\varphi(i))\,e(i)\,di}\partial_e e_l(i)\,\partial_e e_k(i)\,di-\frac{\int(1-\varphi(i))\,e(i)\,\partial_e e_l(i)\,di\int(1-\varphi(i))\,e(i)\,\partial_e e_l(i)\,di}{\left(\int(1-\varphi(i))\,e(i)\,di\right)^2}$$

So we have:

$$\begin{aligned}
&-\mathbb{E}_0\sum_{s=0}^{\infty}\frac{1}{R^s}\int(1-\varphi(i))\left(\frac{1}{\sigma}\left(\hat{e}^{t_0,u}_{t_0+s}(i)-\sum_k s_k(i)\,\hat{P}_{k,t_0+s}\right)+\sum_k\partial e_k(i)\,\hat{P}_{k,t_0+s}-X_{t_0+s}\right)\\
&\qquad\cdot\left\{\frac{b(i)}{R}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\left(e(i)\,s_l(i)-Wn(i)\,\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\mathbb{1}_{t=t_0}b(i)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e(i)\,\partial_e e_k(i)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}\\
&=-\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\frac{E^2}{\sigma e(i)+Wn(i)\,\psi}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left\{\frac{b(i)}{RE}\left(R_{t_0+s}-\sum_{l=1}^{K}\bar{s}_l\pi_{l,t_0+s+1}\right)-\sum_l\frac{e(i)}{E}\left(s_l-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn(i)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}-\frac{e(i)}{E}\sigma\sum_l\partial_e e_l(i)\,\tilde{A}_{k,t_0+s}\right\}\right\}^2 di\\
&\quad+\left(1-\frac{1}{R}\right)E\frac{\int(1-\varphi(i))\,\psi\frac{Wn(i)}{WN}di}{\left(1+\frac{\int(1-\varphi(i))\left(\frac{Wn(i)}{WN}\psi\right)di}{\int(1-\varphi(i))\frac{\sigma e(i)}{E}di}\right)}\int\left(1-\frac{1}{R}\right)\frac{\frac{(1-\varphi(i))b(i)}{\int(1-\varphi(i))e(i)di}-\frac{(1-\varphi(i))Wn(i)}{\int(1-\varphi(i))Wn(i)di}\frac{\int(1-\varphi(i))b(i)di}{\int(1-\varphi(i))e(i)di}}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\left\{\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{W}_{t_0+s}-\sum_k\partial_e e_k(i)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)\right\}^2 di\\
&\quad+\left(1-\frac{1}{R}\right)\frac{\left(\left(1-\varphi^E\right)\sigma\right)^2}{\int(1-\varphi(i))\left(\sigma\frac{e(i)}{E}+\frac{Wn(i)}{WN}\psi\right)di}\sum_{k,l}\mathcal{E}_{k,l}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right)\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\left(\hat{P}_{l,t_0+s}+\tilde{A}_{l,t_0+s}\right)\right)\\
&\quad-\left(1-\frac{1}{R}\right)\int(1-\varphi(i))\,b(i)\,di\sum_l\bar{s}_l\hat{P}_{l,t_0}di\sum_{s=0}^{\infty}\frac{1}{R^s}\hat{W}_{t_0+s}+\left(1-\frac{1}{R}\right)\sum_{s=0}^{\infty}\frac{1}{R^s}X_{t_0+s}\int(1-\varphi(i))\,b(i)\,di\sum_l\bar{s}_l\hat{P}_{l,t_0}\\
&\quad+\frac{1-\frac{1}{R}}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\sum_{s=0}^{\infty}\frac{1}{R^s}\int(1-\varphi(i))\left\{\frac{b(i)}{R}\left(R_{t_0+s}-(R-1)\sum_l\bar{s}_l\hat{P}_{l,t_0+s}\right)-\sum_l\frac{e(i)}{E}\left(s_l-\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn(i)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}-\sum_{l=1}^{K}\sigma e(i)\,\partial_e e_k(i)\,\tilde{A}_{k,t_0+s}\right\}di\right)^2
\end{aligned}$$

Simplification of the terms corresponding to the expenditure of unconstrained households

Here we simplify the second line of the of the social welfare function. Recall that we have:

$$\begin{aligned}
&\frac{1}{\sigma}\left(\hat{e}^{t_0,HtM}_t-\sum_k s_k(i)\,\hat{P}_{k,t}\right)+\sum_k\partial_e e_k(i)\,\hat{P}_{k,t}\\
&=\frac{E}{(\sigma e(i)+Wn(i)\,\psi)}\left\{\frac{b(i)}{RE}R_{t_0+s}+\frac{Wn(i)}{WN}\psi\sum_{l=1}^{K}\left(\hat{W}_t-\partial_e e_k(i)\,\hat{P}_{l,t}\right)-\sum_l\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn(i)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)b(i)\sum_l\bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t}\right)\right\}+\sum_k\partial_e e_k(i)\,\hat{P}_{k,t}\\
&=\frac{E}{(\sigma e(i)+Wn(i)\,\psi)}\left\{\frac{b(i)}{RE}R_{t_0+s}+\frac{Wn(i)}{WN}\psi\hat{W}_t+\frac{\sigma e(i)}{E}\sum_{l=1}^{K}\partial_e e_k(i)\,\hat{P}_{l,t}-\sum_l\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn(i)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_l\bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t_0+s}\right)\right\}
\end{aligned}$$

We then get

$$
\begin{aligned}
&-\sum_{s=0}^{\infty}\frac{1}{R^s}\int\varphi\left(i\right)\left(\frac{1}{\sigma}\left(\hat{e}_{t_0+s}^{t_0,HtM}-\sum_k s_k\left(i\right)\hat{P}_{k,t_0+s}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}\right)\\
&\qquad\cdot\left\{\frac{b\left(i\right)}{R}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t_0+s}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)b\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di\\
&=-\sum_{s=0}^{\infty}\frac{1}{R^s}\int\varphi\left(i\right)\frac{E^2}{\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)}\left\{\frac{b\left(i\right)}{RE}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_l\bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t_0+s}\right)-\frac{\sigma e\left(i\right)}{E}\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\tilde{A}_{k,t_0+s}\right\}^2\\
&\qquad-\sum_{s=0}^{\infty}\frac{1}{R^s}\int\varphi\left(i\right)\frac{E^2}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\frac{e\left(i\right)}{E}\sigma\sum_l\partial_e e_l\left(i\right)\hat{P}_{l,t_0+s}\right\}\left\{\sum_{l=1}^{K}\sigma\frac{e\left(i\right)}{E}\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di\\
&\qquad-\sum_{s=0}^{\infty}\hat{W}_{t_0+s}\frac{1}{R^s}\int\varphi\left(i\right)E\left\{\frac{b\left(i\right)}{RE}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_l\bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t_0+s}\right)\right\}
\end{aligned}
$$

First note that we have, up to terms independent from monetary policy

$$
\begin{aligned}
&-\mathbb{E}_0\sum_{s\geq 0}\frac{1}{R^s}\int\varphi\left(i\right)\frac{E^2}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\frac{Wn\left(i\right)}{WN}\psi\hat{W}_{t_0+s}+\frac{e\left(i\right)}{E}\sigma\sum_l\partial_e e_l\left(i\right)\hat{P}_{l,t_0+s}\right\}\left\{\sum_{l=1}^{K}\sigma\frac{e\left(i\right)}{E}\partial_e e_k\left(i\right)\left(\hat{W}_{t_0+s}-\hat{P}_{k,t_0+s}-2\tilde{A}_{k,t_0+s}\right)\right\}di\\
&\qquad=-\mathbb{E}_0\sum_{s\geq 0}\frac{1}{R^s}\int\varphi\left(i\right)\frac{\sigma\psi e\left(i\right)Wn\left(i\right)}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\hat{W}_{t_0+s}-\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right\}^2di\\
&\qquad\quad+\mathbb{E}_0\sum_{s\geq 0}\frac{1}{R^s}\int\varphi\left(i\right)\sigma e\left(i\right)\left\{\sum_{l,k}\partial_e e_l\left(i\right)\left(\hat{P}_{k,t_0+s}+\tilde{A}_{k,t_0+s}\right)\right\}^2di-\mathbb{E}_0\sum_{s\geq 0}\frac{1}{R^s}\int\varphi\left(i\right)\sigma e\left(i\right)\hat{W}_{t_0+s}\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t_0+s}di
\end{aligned}
$$

So we have:

$$
\begin{aligned}
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\varphi\left(i\right)\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}-\sum_k s_k\left(i\right)\hat{P}_{k,t}\right)+\sum_k\partial e_k\left(i\right)\hat{P}_{k,t}\right)\\
&\qquad\cdot\left\{\frac{b\left(i\right)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\left(1-\frac{1}{R}\right)b\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di\\
&=-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\frac{E^2}{\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)}\left\{\frac{b\left(i\right)}{RE}\hat{R}_{t_0+s}-\sum_{l=1}^{K}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t_0+s}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_l\bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t_0+s}\right)-\frac{\sigma e\left(i\right)}{E}\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\tilde{A}_{k,t_0+s}\right\}^2\\
&\quad-\mathbb{E}_0\sum\beta^t\int\varphi\left(i\right)\frac{\sigma\psi e\left(i\right)Wn\left(i\right)}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\hat{W}_t-\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right\}^2di+\sum\beta^t\int\varphi\left(i\right)\sigma e\left(i\right)\left\{\sum_{l,k}\partial_e e_l\left(i\right)\left(\hat{P}_{l,t}+\tilde{A}_{l,t}\right)\right\}^2di-\sum\beta^t\int\varphi\left(i\right)\sigma e\left(i\right)\hat{W}_t\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t}di\\
&\qquad-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\hat{W}_t\int\varphi\left(i\right)E\left\{\frac{b\left(i\right)}{RE}\hat{R}_t-\sum_{l=1}^{K}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t}+\frac{Wn\left(i\right)}{WN}\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t_0+s}+\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_l\bar{s}_l\left(\hat{P}_{l,t_0}-\hat{P}_{l,t}\right)\right\}
\end{aligned}
$$

Next, we gather all the terms of the form $\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t x_t z_t$, that is:

$$
\begin{aligned}
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\int\varphi\left(i\right)\left(\frac{1}{\sigma}\left(\hat{e}_t^{t_0,HtM}-\sum_k s_k\left(i\right)\hat{P}_{k,t}\right)+\sum_k\partial_e e_k\left(i\right)\hat{P}_{k,t}\right)\\
&\qquad\cdot\left\{\frac{b\left(i\right)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\left(1-\frac{1}{R}\right)b\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di\\
&\qquad+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_{k,l}^{K}\int\sigma e\left(i\right)\partial_e e_k\left(i\right)\partial_e e_l\left(i\right)di\hat{P}_{l,t}\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_k E_k\hat{W}_t\tilde{\mathbf{A}}_{k,t}\\
&\qquad-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t X_t\mathbb{E}_{\delta,t}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\mathbb{1}_{t=t_0}b\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di
\end{aligned}
$$

Following the same step as above, we rewrite the third line as:

$$
\begin{aligned}
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t X_t\mathbb{E}_{\delta,t}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}+\mathbb{1}_{t=t_0}b\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t_0}+\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di\\
&=-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left\{\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t}+\frac{Wn\left(i\right)}{WN}\bar{\psi}\hat{W}_t+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t}di\right\}\int\left(1-\varphi\left(i\right)\right)\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)di\\
&\qquad-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left\{\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t}+\frac{Wn\left(i\right)}{WN}\bar{\psi}\hat{W}_t+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t}di\right\}\\
&\qquad\cdot\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di\\
&\quad+\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_t-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t}di\right\}\int\left(1-\varphi\left(i\right)\right)\left\{\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di\\
&\qquad+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_t-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t}di\right\}\\
&\qquad\cdot\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di-\delta\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t X_t\int\left(1-\varphi\left(i\right)\right)b\left(i\right)\sum_{l=1}^{K}\bar{s}_l\hat{P}_{l,t}di
\end{aligned}
$$

First, we have

$$\begin{aligned}
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left\{\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_t+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t}di\right\}\int\left(1-\varphi\left(i\right)\right)\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)di\\
&=-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t E\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\hat{W}_t-\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right\}^2 di\\
&+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t E\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left\{\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)\right\}^2 di\\
&+\sum_{t=0}^{\infty}\beta^t E\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left(\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\sum_k\partial_e e_k\left(i\right)\left(\hat{P}_{k,t}+\tilde{A}_{k,t}\right)di\right)^2\\
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t E\sum_k\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\partial_e e_k\left(i\right)di\hat{W}_t\hat{P}_{k,t}\\
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}di\sum_l\bar{s}_l\tilde{A}_{l,t}di\int\left(1-\varphi\left(i\right)\right)\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)di
\end{aligned}$$

Next,

$$\begin{aligned}
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\mathbb{E}_{\delta,t}\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\left\{\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t}+\frac{Wn\left(i\right)}{WN}\psi\hat{W}_t+\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t}\right\}di\\
&\cdot\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{R}\hat{R}_t-\sum_{l=1}^{K}\left(e\left(i\right)s_l\left(i\right)-Wn\left(i\right)\bar{s}_l\right)\hat{P}_{l,t}+Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}\right\}di\\
&+\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_t-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t}\right\}di\int\left(1-\varphi\left(i\right)\right)\left\{\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\left(\hat{W}_t-\hat{P}_{k,t}-2\tilde{A}_{k,t}\right)\right\}di\\
&=\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\hat{W}_t\int\varphi\left(i\right)E\left\{\frac{b\left(i\right)}{RE}\hat{R}_t-\sum_{l=1}^{K}\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t}-\left(1-\frac{1}{R}\right)\frac{b(i)}{E}\sum_l\bar{s}_l\hat{P}_{l,t}\right\}di\\
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{\left(1-\varphi^N\right)\psi}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\hat{W}_t\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}di\\
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\left\{\int\left(1-\varphi\left(i\right)\right)\sum_l\sigma\frac{e\left(i\right)}{E}\partial_e e_l\left(i\right)di\hat{P}_{l,t}di\right\}\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\sum_{l=1}^{K}\bar{s}_l\tilde{A}_{l,t}di\\
&-\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\left(1-\varphi\left(i\right)\right)\frac{Wn\left(i\right)}{WN}\sum_l\bar{s}_l\tilde{A}_{l,t}di\int\left(1-\varphi\left(i\right)\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_t-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t}\right\}di\\
&-2\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\frac{1}{\left(1-\varphi^E\right)\sigma+\left(1-\varphi^N\right)\psi}\int\varphi\left(i\right)\left\{\frac{b\left(i\right)}{ER}\left(\hat{R}_t-\left(R-1\right)\sum_l\bar{s}_l\hat{P}_{l,t}\right)-\sum_l\frac{e\left(i\right)}{E}\left(s_l\left(i\right)-\bar{s}_l\right)\hat{P}_{l,t}\right\}di\int\left(1-\varphi\left(i\right)\right)\sum_{l=1}^{K}\sigma e\left(i\right)\partial_e e_k\left(i\right)\tilde{A}_{k,t}di
\end{aligned}$$

Gathering the terms, we obtain:

$$
\begin{aligned}
& -\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbb{E}_{\delta,t} \int \varphi\left(i\right) \left(\frac{1}{\sigma} \left(\hat{e}_t^{t_0,HtM} - \sum_k s_k\left(i\right) \hat{P}_{k,t} \right) + \sum_k \partial_e e_k\left(i\right) \hat{P}_{k,t} \right) \\
& \qquad \cdot \left\{ \frac{b\left(i\right)}{R} \hat{R}_t - \sum_{l=1}^{K} \left(e\left(i\right) s_l\left(i\right) - Wn\left(i\right) \bar{s}_l\right) \hat{P}_{l,t} + Wn\left(i\right) \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t} + \left(1 - \frac{1}{R}\right) b\left(i\right) \sum_{l=1}^{K} \bar{s}_l \hat{P}_{l,t_0} + \sum_{l=1}^{K} \sigma e\left(i\right) \partial_e e_k\left(i\right) \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{A}_{k,t}\right) \right\} di \\
& \qquad + \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{k,l}^{K} \int \sigma e\left(i\right) \partial_e e_k\left(i\right) \partial e_l\left(i\right) di \hat{P}_{l,t} \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{A}_{k,t}\right) + \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_k E_k \hat{W}_t \tilde{\mathbf{A}}_{k,t} \\
& \qquad - \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t X_t \mathbb{E}_{\delta,t} \int \left(1 - \varphi\left(i\right)\right) \left\{ \frac{b\left(i\right)}{R} \hat{R}_t - \sum_{l=1}^{K} \left(e\left(i\right) s_l\left(i\right) - Wn\left(i\right) \bar{s}_l\right) \hat{P}_{l,t} + Wn\left(i\right) \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t} + \mathbb{1}_{t=t_0} b\left(i\right) \sum_{l=1}^{K} \bar{s}_l \hat{P}_{l,t_0} + \sum_{l=1}^{K} \sigma e\left(i\right) \partial_e e_k\left(i\right) \left(\hat{W}_t - \hat{P}_{k,t} - 2\tilde{A}_{k,t}\right) \right\} di \\
= & -\sum_{s=0}^{\infty} \frac{1}{R^s} \int \varphi\left(i\right) \frac{E^2}{\left(\sigma e\left(i\right) + Wn\left(i\right) \psi\right)} \left\{ \frac{b\left(i\right)}{RE} \hat{R}_{t_0+s} - \sum_{l=1}^{K} \frac{e\left(i\right)}{E} \left(s_l\left(i\right) - \bar{s}_l\right) \hat{P}_{l,t_0+s} + \frac{Wn\left(i\right)}{WN} \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t_0+s} + \left(1 - \frac{1}{R}\right) \frac{b(i)}{E} \sum_l \bar{s}_l \left(\hat{P}_{l,t_0} - \hat{P}_{l,t_0+s}\right) - \frac{\sigma e\left(i\right)}{E} \sum_{l=1}^{K} \sigma e\left(i\right) \partial_e e_k\left(i\right) \tilde{A}_{k,t_0+s} \right\}^2 \\
& \qquad - \mathbb{E}_0 \sum \beta^t \int \varphi\left(i\right) \frac{\sigma \psi e\left(i\right) Wn\left(i\right)}{\sigma e\left(i\right) + Wn\left(i\right) \psi} \left\{ \hat{W}_t - \sum_k \partial_e e_k\left(i\right) \left(\hat{P}_{k,t} + \tilde{A}_{k,t}\right) \right\}^2 di \\
& \qquad - \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t E \frac{\left(1 - \varphi^N\right) \psi}{\left(1 - \varphi^E\right) \sigma + \left(1 - \varphi^N\right) \psi} \int \left(1 - \varphi\left(i\right)\right) \sigma \frac{e\left(i\right)}{E} \left\{ \hat{W}_t - \sum_k \partial_e e_k\left(i\right) \left(\hat{P}_{k,t} + \tilde{A}_{k,t}\right) \right\}^2 di \\
& \qquad - \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t E \frac{\left(\left(1 - \varphi^E\right) \sigma\right)^2}{\left(1 - \varphi^E\right) \sigma + \left(1 - \varphi^N\right) \psi} \sum_{k,l} \mathcal{E}_{k,l} \left(\hat{P}_{k,t} + \tilde{A}_{k,t}\right) \left(\hat{P}_{l,t} + \tilde{A}_{l,t}\right) - \delta \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(1 - \frac{1}{R}\right) \int \left(1 - \varphi\left(i\right)\right) b\left(i\right) di \sum_l \bar{s}_l \hat{P}_{l,t} di \sum_{s \geq 0} \frac{1}{R^s} \hat{W}_{t+s} \\
& \qquad - \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{1}{\left(1 - \varphi^E\right) \sigma + \left(1 - \varphi^N\right) \psi} \left(\int \left(1 - \varphi\left(i\right)\right) \left\{ \frac{b\left(i\right)}{ER} \left(\hat{R}_t - \left(R - 1\right) \sum_l \bar{s}_l \hat{P}_{l,t} \right) - \sum_l \frac{e\left(i\right)}{E} \left(s_l\left(i\right) - \bar{s}_l\right) \hat{P}_{l,t} + \frac{Wn\left(i\right)}{WN} \sum_{l=1}^{K} \bar{s}_l \tilde{A}_{l,t} - \sigma \frac{e\left(i\right)}{E} \sum_{l=1}^{K} \partial_e e_k\left(i\right) \tilde{A}_{k,t} di \right\} \right)^2 \\
& \qquad - \delta \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t X_t \int \left(1 - \varphi\left(i\right)\right) b\left(i\right) \sum_{l=1}^{K} \bar{s}_l \hat{P}_{l,t} di
\end{aligned}
$$

Social Welfare Function

Note that we have, as $\mathbb{E}_t X_{t+1} - X_t = \hat{R}_t$

$$\mathbb{E}_0 X_t \int \left(1 - \varphi\left(i\right)\right) b\left(i\right) \sum_{l=1}^{K} \bar{s}_l \hat{P}_{l,t} di + \mathbb{E}_0 \left(1 - \frac{1}{R}\right) \sum \frac{1}{R^s} X_{t+s} \int \left(1 - \varphi\left(i\right)\right) b\left(i\right) di \sum_l \bar{s}_l \hat{P}_{l,t} = \sum \frac{\hat{R}_s}{R^{s+1}} b(i) \sum_{l=1}^{K} \bar{s}_l \hat{P}_{l,t_0}$$

In addition, for any variable x_t, z_t, we have

$$\sum_{t=0}^{\infty} \frac{1}{R^t} \left(x_{t_0+t} + \left(1 - \frac{1}{R}\right) z_{t_0} \right)^2 - \left(1 - \frac{1}{R}\right) \left(\sum_{t=0}^{\infty} \frac{1}{R^t} x_{t_0+t} + z_{t_0} \right)^2 = \mathbb{E}_0 \sum_{t=0}^{\infty} \frac{1}{R^t} \left(x_{t_0+t}\right)^2 - \mathbb{E}_0 \left(1 - \frac{1}{R}\right) \left(\sum_{t=0}^{\infty} \frac{1}{R^t} x_{t_0+t} \right)^2$$

Finally, we denote for any variable Z_{t_0},

$$\mathbb{E}_\beta\left(Z_{t_0}\right) \equiv \mathbb{E}_0 \left(1 - \beta\right) \left(\frac{1}{1 - \frac{1}{R}} \left(1 - \delta\right) Z_{-} + \delta \frac{1}{1 - \frac{1}{R}} \sum_{t_0} \beta^{t_0} Z_{t_0} \right)$$

the social average over the population of the variable Z_{t_0}(Note that if $Z_{t_0}$is constant, we have $\mathbb{E}_\beta\left(Z_{t_0}\right) = Z$). Using this facts and the previous derivations, we can write – after normalization – the social welfare function as:

$$
\begin{aligned}
\mathcal{L} = {} & (1-\beta)\sum \beta^{s}\frac{\left(1-\varphi^{N}\right)\psi}{\left(1+\frac{\left(1-\varphi^{N}\right)\psi}{\left(1-\varphi^{E}\right)\sigma}\right)}\left\{\underbrace{\int \frac{(1-\varphi(i))\,e(i)}{\int(1-\varphi(i))\,e(i)\,di}\left(\hat{W}_{s}-\sum_{l}\partial_{e}e_{l}(i)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)^{2}di}_{\text{Labor Distortions with average wealth effect}}\right\} \\
& -(1-\beta)\sum \beta^{s}\frac{\left(1-\varphi^{N}\right)\psi}{\left(1+\frac{\left(1-\varphi^{N}\right)\psi}{\left(1-\varphi^{E}\right)\sigma}\right)}\underbrace{\mathbb{E}_{\beta}\left(\left\{\int\left(1-\frac{1}{R}\right)\frac{\frac{(1-\varphi(i))b(i)}{\int(1-\varphi(i))e(i)di}-\frac{(1-\varphi(i))Wn(i)}{\int(1-\varphi(i))Wn(i)di}\frac{\int(1-\varphi(i))b(i)di}{\int(1-\varphi(i))e(i)di}}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\left(\left(1-\tilde{\beta}\right)\sum_{u\geq t_{0}}\tilde{\beta}^{u-t_{0}}\left(\hat{W}_{u}-\sum_{l}\partial_{e}e_{l}(i)\left(\hat{P}_{l,u}+\tilde{A}_{l,u}\right)\right)\right)^{2}\right\}\right)}_{\text{Correctionfor the dispersion in wealth effects}} \\
& +(1-\beta)\sum \beta^{s}\underbrace{\int \varphi(i)\,\frac{Wn(i)}{WN}\frac{\psi}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\left(\hat{W}_{s}-\sum_{l}\partial_{e}e_{l}(i)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)^{2}di}_{\text{Labor distortion HtM}} \\
& +(1-\beta)\sum \beta^{s}\frac{\left(1-\varphi^{E}\right)\sigma}{\left(1+\frac{\left(1-\varphi^{N}\right)\psi}{\left(1-\varphi^{E}\right)\sigma}\right)}\underbrace{\sum_{k,l}\mathcal{E}_{k,l}\left\{\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)-\mathbb{E}_{\beta}\left(\left(1-\tilde{\beta}\right)\left(\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\right)\left(\left(1-\tilde{\beta}\right)\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)\right)\right\}}_{\text{Intertemporal misallocation of expenditures}} \\
& -(1-\beta)\sum \beta^{s}\underbrace{\sum_{k}\bar{s}_{k}\sum_{l}\bar{\sqsubset}_{k,l}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)}_{\text{Intratemporal misallocation of expenditures}}+(1-\beta)\;\sum \beta^{s}\underbrace{\sum_{k}\bar{s}_{k}\vartheta_{k}\left(\pi_{k,s}\right)^{2}}_{\text{Within Sector misallocation}}\;+(1-\beta)\sum \beta^{s}\underbrace{\left(\hat{W}_{s},\hat{P}_{s}+\tilde{\mathbf{A}}_{s}\right)^{T}\mathcal{O}\left(\hat{W}_{s},\hat{P}_{s}+\tilde{\mathbf{A}}_{s}\right)}_{\text{Input misallocation}} \\
& +\underbrace{\mathbb{E}_{\beta}\left(\int g(i)\left(\nu_{t_{0}}(i)+\left(1-\tilde{\beta}\right)\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\sum_{l}\frac{Wn(i)}{WN}\bar{s}_{l}\tilde{A}_{l,s}\right)^{2}di\right)}_{\text{Redistribution}}-\underbrace{2\mathbb{E}_{\beta}\left(\left(1-\tilde{\beta}\right)\sum_{s\geq t_{0}}\int\left(\nu_{t_{0}}(i)\tilde{\beta}^{s-t_{0}}\sum_{l}\frac{(1-\varphi(i))\,\partial_{e}e_{l}(i)}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\tilde{A}_{l,s}\right)di\right)}_{\text{"Bang for Buck non HtM''}} \\
& +\underbrace{\mathbb{E}_{\beta}\left(\left(1-\tilde{\beta}\right)\sum \tilde{\beta}^{s-t_{0}}\int \varphi(i)\,\frac{E}{\sigma e(i)+Wn(i)\,\psi}\left\{\aleph_{s}^{t_{0}}(i)+\sum_{l}\frac{Wn(i)}{WN}\bar{s}_{l}\tilde{A}_{l,s}\right\}^{2}-\int \varphi(i)\,\frac{E}{\sigma e(i)+Wn(i)\,\psi}\left\{\nu_{t_{0}}(i)+\left(1-\tilde{\beta}\right)\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\frac{Wn(i)}{WN}\bar{s}_{l}\tilde{A}_{l,s}\right\}^{2}\right)}_{\text{Non Smoothing compensation for HtM}} \\
& -\underbrace{2\mathbb{E}_{\beta}\left(\left(1-\tilde{\beta}\right)\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\int \aleph_{s}^{t_{0}}(i)\sum_{l}\frac{\varphi(i)\,\partial_{e}e_{l}(i)}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\tilde{A}_{l,s}di\right)}_{\text{"Bang for Buck HtM''}} \\
& +\frac{E}{\int(1-\varphi(i))\,(\sigma e(i)+Wn(i)\,\psi)\,di}\mathbb{E}_{\beta}\underbrace{\left\{\left(1-\tilde{\beta}\right)\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\int(1-\varphi(i))\left\{\aleph_{s}^{t_{0}}(i)+\sum_{l}\frac{Wn(i)}{WN}\bar{s}_{l}\tilde{A}_{l,s}-\sum_{l}\frac{\sigma e(i)}{E}\partial_{e}e_{l}(i)\,\tilde{A}_{l,s}\right\}\right)^{2}\right\}}_{\text{Smoothing Premium for non HtM}} \\
& -\frac{E}{\int(1-\varphi(i))\,(\sigma e(i)+Wn(i)\,\psi)\,di}\mathbb{E}_{\beta}\underbrace{\left\{\left(\int(1-\varphi(i))\left\{\nu_{t_{0}}(i)+\left(1-\tilde{\beta}\right)\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\sum_{l}\left(\frac{Wn(i)}{WN}\bar{s}_{l}\tilde{A}_{l,s}-\frac{\sigma e(i)}{E}\partial_{e}e_{l}(i)\,\tilde{A}_{l,s}\right)\right\}di\right)^{2}\right\}}_{\text{Smoothing Premium for non HtM}}
\end{aligned}
$$

where $\mathcal{O}$ denote the aggregate input substitution matrix and is given by $\mathcal{O} = \begin{bmatrix} \frac{W\partial_W \mathcal{N}}{N} & \left(\frac{P_k \partial_{P_k} \mathcal{N}}{N}\right)^T \\ \frac{P_k \partial_{P_k} \mathcal{N}}{N} & \frac{P_k \mathcal{Y}_k}{E} \frac{P_l \partial_{P_l} \mathcal{Y}_k}{\mathcal{Y}_k} \end{bmatrix}$.

The last five lines constitute the redistributive motive. the first term is standard it is 0 when households are compensated for their loss of income. the second term is an adjustment taking into account that it is relatively more efficient to compensate households who consume more in relatively more productive sectors. the last twho terms are adjustments taking into account that HtM households cannot smooth their consumption.
With

$$\mathcal{E}_{k,l} = \int \frac{(1-\varphi(i))\, e(i)}{\int (1-\varphi(i))\, e(i)\, di} \partial_e e_l(i)\, \partial_e e_k(i)\, di - \frac{\int (1-\varphi(i))\, e(i)\, \partial_e e_k(i)\, di \int (1-\varphi(i))\, e(i)\, \partial_e e_l(i)\, di}{\left(\int (1-\varphi(i))\, e(i)\, di\right)^2}$$

$$\nu_{t_0}(i) = \left(1-\tilde{\beta}\right) \sum_{s \geq t_0} \tilde{\beta}^{s-t_0} \left(\frac{b(i)}{ER} \left(\hat{R}_{t_0+s} - \sum_{l=1}^{K} \bar{s}_l \pi_{l,t_0+s+1} \right) - \sum_{l=1}^{K} \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l\right) \hat{P}_{l,t_0+s} \right)$$

$$\nu_{-}(i) = \left(1-\tilde{\beta}\right) \sum_{s \geq 0} \tilde{\beta}^{s} \left(\frac{b(i)}{ER} \left(\hat{R}_s - (R-1) \sum_l \bar{s}_l \hat{P}_{l,s} \right) - \sum_l \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l\right) \hat{P}_{l,s} \right)$$

$$\aleph_s^{t_0}(i) = \left(\frac{b(i)}{ER} \left(\hat{R}_s - (R-1) \sum_l \bar{s}_l \hat{P}_{l,s} \right) - \sum_l \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l\right) \hat{P}_{l,s} \right) + \sum_l \frac{R-1}{R} \frac{b(i)}{E} \bar{s}_l \hat{P}_{l,t_0}$$

$$\aleph_{-}(i) = \left(\frac{b(i)}{ER} \left(\hat{R}_s - (R-1) \sum_l \bar{s}_l \hat{P}_{l,s} \right) - \sum_l \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l\right) \hat{P}_{l,s} \right)$$

$$g(i) = -\frac{G''(V(i))\, v'(e(i))\, E}{\left(1-\tilde{\beta}\right) G'(V(i))} + \frac{E}{\sigma e(i) + Wn(i)\, \psi}$$

Without HtM households and IO, this simplifies to:

$$\mathcal{L} = (1-\beta) \sum \beta^s \frac{\psi}{1+\frac{\psi}{\sigma}} \left\{ \underbrace{\int \frac{e(i)}{E} \left(\hat{W}_s - \sum_l \partial_e e_l(i) \left(\hat{P}_{l,s} + \tilde{A}_{l,s} \right) \right)^2 di}_{\text{Labor Distortions with average wealth effect}} - \underbrace{\mathbb{E}_\beta \left(\left\{ \int \left(1 - \frac{1}{R}\right) \frac{\frac{b(i)}{E}}{1+\frac{Wn(i)\psi}{\sigma e(i)}} \left(\left(1-\tilde{\beta}\right) \sum_{u \geq t_0} \tilde{\beta}^{u-t_0} \hat{W}_u - \sum_l \partial_e e_l(i) \left(\hat{P}_{l,u} + \tilde{A}_{l,u} \right) \right)^2 \right\} \right)}_{\text{Correctionfor the dispersion in wealth effects}} \right\}$$

$$(1-\beta) \sum \beta^s \frac{\sigma}{1+\frac{\psi}{\sigma}} \underbrace{\sum_{k,l} \mathcal{E}_{k,l} \left\{ \left(\hat{P}_{k,s} + \tilde{A}_{k,s} \right) \left(\hat{P}_{l,s} + \tilde{A}_{l,s} \right) - \mathbb{E}_\beta \left(\left(1-\tilde{\beta}\right) \left(\sum_{s \geq t_0} \tilde{\beta}^{s-t_0} \left(\hat{P}_{k,s} + \tilde{A}_{k,s} \right) \right) \left(\left(1-\tilde{\beta}\right) \sum_{s \geq t_0} \tilde{\beta}^{s-t_0} \left(\hat{P}_{l,s} + \tilde{A}_{l,s} \right) \right) \right) \right\}}_{\text{Intertemporal misallocation of expenditures}}$$

$$-(1-\beta) \sum \beta^s \underbrace{\sum_k \bar{s}_k \sum_l \bar{\complement}_{k,l} \left(\hat{P}_{k,s} + \tilde{A}_{k,s} \right) \left(\hat{P}_{l,s} + \tilde{A}_{l,s} \right)}_{\text{Intratemporal misallocation of expenditures}} + (1-\beta) \sum \beta^s \underbrace{\sum_k \bar{s}_k \vartheta_k \left(\pi_{k,s}\right)^2}_{\text{Within Sector misallocation}}$$

$$+ \underbrace{\mathbb{E}_\beta \left(\int g(i) \left(\nu_{t_0}(i) + \left(1-\tilde{\beta}\right) \sum_{s \geq t_0} \tilde{\beta}^{s-t_0} \sum_l \frac{Wn(i)}{WN} \bar{s}_l \tilde{A}_{l,s} \right)^2 di \right)}_{\text{Redistribution}} - \underbrace{2\mathbb{E}_\beta \left(\left(1-\tilde{\beta}\right) \sum_{s \geq t_0} \tilde{\beta}^{s-t_0} \int \left(\nu_{t_0}(i) \sum_l \frac{\partial_e e_l(i)}{1+\frac{Wn(i)\psi}{\sigma e(i)}} \tilde{A}_{l,s} \right) di \right)}_{\text{"Bang for Buck ''}}$$

Defining $\tilde{P}_{l,s} = \hat{P}_{l,s} + \tilde{A}_{l,s}$, the markup in k, and using $\tilde{\mathcal{Y}}_t = \frac{\sigma\psi}{\sigma+\psi}\left(\hat{W}_t - \sum_l \overline{\partial_e e_l}\left(\hat{P}_{l,t} + \tilde{A}_{l,t}\right)\right)$ we can re-express the loss function directly in terms of the output gap :

$$
\begin{aligned}
\mathcal{L} = {} & (1-\beta)\sum \beta^s \frac{\psi}{1+\frac{\psi}{\sigma}} \left\{ \left(\frac{1}{\sigma}+\frac{1}{\psi}\right)^2 \underbrace{\tilde{\mathcal{Y}}_s^2} + \mathcal{E}_{k,l}\tilde{P}_{k,s}\tilde{P}_{l,s} - \mathbb{E}_\beta \left(\left\{ \int \left(1-\frac{1}{R}\right) \frac{\frac{b(i)}{E}}{1+\frac{Wn(i)\psi}{\sigma e(i)}} \left(\left(1-\tilde{\beta}\right) \sum_{u\geq t_0} \tilde{\beta}^{u-t_0} \left(\frac{1}{\sigma}+\frac{1}{\psi}\right) \tilde{\mathcal{Y}}_u - \sum_l \left(\partial_e e_l\left(i\right) - \overline{\partial_e e_l}\right) \tilde{P}_{l,s} \right)^2 \right\} \right) \right\} \\
& (1-\beta)\sum \beta^s \frac{\sigma}{1+\frac{\psi}{\sigma}} \sum_{k,l} \mathcal{E}_{k,l} \left\{ \tilde{P}_{k,s}\tilde{P}_{l,s} - \mathbb{E}_\beta \left(\left(\left(1-\tilde{\beta}\right) \sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \tilde{P}_{k,s} \right) \left(\left(1-\tilde{\beta}\right) \sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \tilde{P}_{l,s} \right) \right) \right\} \\
& - (1-\beta)\sum \beta^s \sum_k \bar{s}_k \sum_l \bar{\complement}_{k,l} \tilde{P}_{k,s}\tilde{P}_{l,s} + (1-\beta)\sum \beta^s \sum \bar{s}_k \vartheta_k \left(\pi_{k,s}\right)^2 \\
& + \mathbb{E}_\beta \left(\int g\left(i\right) \left(\nu_{t_0}(i) + \left(1-\tilde{\beta}\right) \sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \sum_l \frac{Wn\left(i\right)}{WN} \bar{s}_l \tilde{A}_{l,s} \right)^2 di \right) - 2\mathbb{E}_\beta \left(\left(1-\tilde{\beta}\right) \sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \int \left(\nu_{t_0}(i) \sum_l \frac{\partial_e e_l\left(i\right)}{1+\frac{Wn(i)\psi}{\sigma e(i)}} \tilde{A}_{l,s} \right) di \right)
\end{aligned}
$$

System of Equations without redistributive motive:

We now derive the system of optimal equation when the central bank ignores redistributive motives (last five lines of the loss function for the full model). Defining:

$$
\begin{aligned}
\mathcal{L}^{nd} \equiv {} & (1-\beta)\sum \beta^s \frac{\left(1-\varphi^N\right)\psi}{\left(1+\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma}\right)} \left\{ \underbrace{\int \frac{\left(1-\varphi\left(i\right)\right) e\left(i\right)}{\int \left(1-\varphi\left(i\right)\right) e\left(i\right) di} \left(\hat{W}_s - \sum_l \partial_e e_l\left(i\right) \left(\hat{P}_{l,s} + \tilde{A}_{l,s}\right) \right)^2 di}_{\text{Labor Distortions with average wealth effect}} \right\} \\
& - (1-\beta)\sum \beta^s \frac{\left(1-\varphi^N\right)\psi}{\left(1+\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma}\right)} \underbrace{\mathbb{E}_\beta \left(\left\{ \int \left(1-\frac{1}{R}\right) \frac{\frac{(1-\varphi(i))b(i)}{\int(1-\varphi(i))e(i)di} - \frac{(1-\varphi(i))Wn(i)}{\int(1-\varphi(i))Wn(i)di} \frac{\int(1-\varphi(i))b(i)di}{\int(1-\varphi(i))e(i)di}}{1+\frac{Wn(i)\psi}{\sigma e(i)}} \left(\left(1-\tilde{\beta}\right) \sum_{u\geq t_0} \tilde{\beta}^{u-t_0} \left(\hat{W}_u - \sum_l \partial_e e_l\left(i\right) \left(\hat{P}_{l,u} + \tilde{A}_{l,u}\right) \right) \right)^2 \right\} \right)}_{\text{Correctionfor the dispersion in wealth effects}} \\
& + (1-\beta)\sum \beta^s \underbrace{\int \varphi\left(i\right) \frac{Wn\left(i\right)}{WN} \frac{\psi}{1+\frac{Wn(i)\psi}{\sigma e(i)}} \left(\hat{W}_s - \sum_l \partial_e e_l\left(i\right) \left(\hat{P}_{l,s} + \tilde{A}_{l,s}\right) \right)^2 di}_{\text{Labor distortion HtM}} \\
& (1-\beta)\sum \beta^s \frac{\left(1-\varphi^E\right)\sigma}{\left(1+\frac{(1-\varphi^N)\psi}{(1-\varphi^E)\sigma}\right)} \underbrace{\sum_{k,l} \mathcal{E}_{k,l} \left\{ \left(\hat{P}_{k,s} + \tilde{A}_{k,s}\right)\left(\hat{P}_{l,s} + \tilde{A}_{l,s}\right) - \mathbb{E}_\beta \left(\left(1-\tilde{\beta}\right) \left(\sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \left(\hat{P}_{k,s} + \tilde{A}_{k,s}\right) \right) \left(\left(1-\tilde{\beta}\right) \sum_{s\geq t_0} \tilde{\beta}^{s-t_0} \left(\hat{P}_{l,s} + \tilde{A}_{l,s}\right) \right) \right) \right\}}_{\text{Intertemporal misallocation of expenditures}} \\
& - (1-\beta)\sum \beta^s \underbrace{\sum_k \bar{s}_k \sum_l \bar{\complement}_{k,l} \left(\hat{P}_{k,s} + \tilde{A}_{k,s}\right)\left(\hat{P}_{l,s} + \tilde{A}_{l,s}\right)}_{\text{Intratemporal misallocation of expenditures}} + (1-\beta) \sum \beta^s \underbrace{\sum_k \bar{s}_k \vartheta_k \left(\pi_{k,s}\right)^2}_{\text{Within Sector misallocation}} + (1-\beta)\sum \beta^s \underbrace{\left(\hat{W}_s, \hat{P}_s + \tilde{\mathbf{A}}_s\right)^T \mathcal{O}\left(\hat{W}_s, \hat{P}_s + \tilde{\mathbf{A}}_s\right)}_{\text{Input misallocation}}
\end{aligned}
$$

The central bank solves

$$\inf_{\left\{\hat{W}_t,\hat{\boldsymbol{P}}_t\right\}_{t\geq 0}} \mathcal{L}^{nd}$$

under the constraints

$$\begin{aligned}
\pi_{k,t} &= \kappa_k \tilde{\mathcal{Y}}_t + \lambda_k \left(\Omega_{N,k} \sum_l \overline{\partial_e e_l} \left(\hat{P}_{l,t} + \tilde{A}_{l,t} - \left(\hat{P}_{k,t} + \tilde{A}_{k,t} \right) \right) + \sum_l \Omega_{k,l} \left(\hat{P}_{l,t} + \tilde{A}_{l,t} - \left(\hat{P}_{k,t} + \tilde{A}_{k,t} \right) \right) + s_k^C \mathcal{M}_{k,t} \right) + \beta \pi_{k,t+1}, \\
\hat{P}_{k,t} &= \hat{P}_{k,t-1} + \pi_{k,t}, \\
\mathcal{M}_{k,t} &= \sum_l \int \gamma_{e,k}(i) \frac{e_k(i)}{E_k} \rho_{k,l}(i) di \hat{P}_{l,t} + \frac{1+\frac{\bar{\psi}}{\bar{\sigma}}}{1-\frac{1}{R}} \int \gamma_{b,k}(i) \frac{Wn\left(i\right)}{WN} di \hat{\mathcal{Y}}_t^* + \mathcal{M}_{k,t}^D, \\
\mathbb{E}_t \mathcal{M}_{k,t+1}^D - \mathcal{M}_{k,t}^D &= \sum_l \sigma_{k,l}^{\mathcal{M},u} \left(\hat{R}_t - \pi_{l,t+1} \right) + \frac{\delta}{1-\delta} \mathbb{E}_t \mathcal{M}_{k,t+1}^0 \\
&+ \frac{R}{R-1} \int \left(\gamma_{b,k}^u(i) \left(\varphi(i) \frac{b(i)}{RE} - \frac{\left(1-\varphi(i)\right) Wn(i)}{\left(1-\varphi^N\right) WN} \int \left(\varphi(i) \frac{b(i)}{RE} \right) di \right) \right) di \mathbb{E}_t \Delta \hat{R}_{t+1} \\
&- \frac{R}{R-1} \sum_l \int \gamma_{b,k}^u(i) \left\{ (1-\frac{1}{R}) \left(\varphi(i) \frac{b(i)}{E} - \frac{\left(1-\varphi(i)\right) Wn(i)}{\left(1-\varphi^N\right) WN} \int \left(\varphi(i) \frac{b(i)}{E} \right) di \right) \bar{s}_l \right\} di \mathbb{E}_t \pi_{l,t+1} \\
&- \frac{R}{R-1} \sum_l \int \gamma_{b,k}^u(i) \left\{ \left(\varphi(i) \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l \right) - \frac{\left(1-\varphi(i)\right) Wn(i)}{\left(1-\varphi^N\right) WN} \int \varphi(i) \frac{e(i)}{E} \left(s_l(i) - \bar{s}_l \right) di \right) \right\} di \mathbb{E}_t \pi_{l,t+1} \\
&- \frac{R}{R-1} \sum_l \int \gamma_{b,k}^u(i) \left\{ \frac{Wn(i)}{WN} \psi \left(\varphi(i) \left(\partial_e e_l(i) - \overline{\partial_e e_l} \right) - \frac{1-\varphi(i)}{1-\varphi^E} \int \varphi(i) \frac{e(i)}{E} \left(\partial_e e_l(i) - \overline{\partial_e e_l} \right) di \right) \right\} di \mathbb{E}_t \pi_{l,t+1}, \\
\mathcal{M}_{k,t}^0 - \frac{1}{\left(1-\delta\right) R} \mathbb{E}_t \mathcal{M}_{k,t+1}^0 &= \int \gamma_{b,k}^u(i) \frac{b(i)}{RE} di \left(\hat{R}_t - \sum_l \bar{s}_l \mathbb{E}_t \pi_{l,t+1} \right) \\
&- \sum_l \int \gamma_{b,k}^u(i) \left(\frac{e(i)}{E} \left(s_l(i) - \bar{s}_l \right) + \psi \frac{Wn(i)}{WN} \left(\partial_e e_l(i) - \overline{\partial_e e_l} \right) \right) di \hat{P}_{l,t} - \frac{R-1}{R} \mathcal{M}_{k,t}^D, \\
\left(1-\varphi^N\right) \left(\mathbb{E}_t \tilde{\mathcal{Y}}_{t+1} - \tilde{\mathcal{Y}}_t \right) &= \left(1-\varphi^N\right) \sigma \mathbb{E}_t \left(\hat{R}_t - \sum_k \overline{\partial_e e_k} \pi_{k,t+1} - r_t^* \right) + \mathbb{E}_t \int \frac{\sigma \varphi\left(i\right)}{\sigma+\psi} \left\{ \frac{b(i)}{RE} \left(\Delta \hat{R}_{t+1} - \sigma \left(R-1\right) \hat{R}_t \right) - \frac{e(i)}{E} \sum_k \left(\left(s_k(i) - \bar{s}_k \right) - \sigma (\partial_e e_k(i) - \overline{\partial_e e_k}) \right) \pi_{k,t+1} \right\} di \\
&- \mathbb{E}_t \int \frac{\sigma \varphi\left(i\right)}{\sigma+\psi} \left\{ \left(1-\frac{1}{R}\right) \frac{b(i)}{E} \left(\pi_{cpi,t+1} - \sigma \pi_{mcpi,t+1} \right) \right\} di, \\
\tilde{\mathcal{Y}}_t &= \frac{\sigma \psi}{\sigma+\psi} \left(\hat{W}_t - \sum_k \overline{\partial_e e_k} \left(\hat{P}_{k,t} + \tilde{A}_{k,t} \right) \right).
\end{aligned}$$

Denoting $\check{\mu}_{k,t}$ the lagrange multiplier on the NKPC of sector k and $M_{k,t}$ the lagrange multiplier on the price evolution equation and taking derivatives with respect to the wage at t directly gives:

$$\begin{aligned}\sum_{k=1}^{K}\Omega_{N,k}\lambda_k\breve{\mu}_{k,t} = Z_t &+ \frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\frac{1}{R}\left(A_{b,t}-(1-\delta)RA_{b,t-1}\right)\\ &- \left(\frac{\int\left(1-\varphi\left(i\right)\right)\frac{Wn(i)}{WN}\psi di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}+\int\varphi\left(i\right)\psi\frac{Wn(i)}{WN}\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi n\left(i\right)}di\right)\hat{W}_t\\ &+ \sum_k\left(\frac{\int\left(1-\varphi\left(i\right)\right)\frac{Wn(i)}{WN}\psi di\int\left(1-\varphi\left(i\right)\right)\sigma e\left(i\right)\partial_e e_k\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}+\int\varphi\left(i\right)\psi\frac{Wn(i)}{WN}\frac{\sigma e\left(i\right)\partial_e e_k\left(i\right)}{\sigma e\left(i\right)+\psi n\left(i\right)}di\right)\left(\tilde{A}_{k,t}+\hat{P}_{k,t}\right)\\ &- \sum_k\lambda_k\frac{E_k}{P_kY_k}\left(\frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e(i)}{E_k}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}+\int\varphi\left(i\right)\frac{\sigma\psi Wn(i)}{\sigma e\left(i\right)+\psi n\left(i\right)}\frac{e\left(i\right)}{E_k}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)di\right)\breve{\mu}_{k,t}\\ &+ \sum_{k,l}\frac{P_lW\partial_W\mathcal{Y}_{l,k}Y_k}{A_kE}\left(\hat{P}_{l,t}+\tilde{A}_{l,t,}\right)+\sum_k\frac{W\partial_W\mathcal{N}_k}{A_kN}Y_k\hat{W}_t\end{aligned}$$

With

$$Z_{t+1}-Z_t=\frac{\delta}{1-\delta}Z_{t+1}^0$$

$$\begin{aligned}Z_{t-1}^0-\frac{1}{\left(1-\delta\right)R}Z_t^0+\left(1-\frac{1}{R}\right)Z_{t-1} &= \left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\left(\frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}-\frac{\psi Wn(i)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)di\hat{W}_{t-1}\\ &- \sum_k\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E}\partial_e e_k\left(i\right)\left(\frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}-\frac{\psi Wn(i)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)di\left(\tilde{A}_{k,t-1}+\hat{P}_{k,t-1}\right)\\ &+ \sum_k\frac{\lambda_kE_k}{P_kY_k}\left(1-\frac{1}{R}\right)\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e\left(i\right)}{E_k}\partial_e e_k\left(i\right)\gamma_{e,k}\left(i\right)\left(\frac{\int\left(1-\varphi\left(i\right)\right)Wn\left(i\right)\psi di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}-\frac{\psi Wn(i)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\right)di\breve{\mu}_{k,t-1}\end{aligned}$$

And

$$\begin{aligned}&\left(1-\frac{\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)A_{b,t+1}-\left(1-\frac{\left(1-\delta+R\right)\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)A_{b,t}-\frac{(1-\delta)R\int\varphi\left(i\right)\frac{1}{R}b\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}A_{b,t-1}-\frac{\delta}{1-\delta}A_{b,t+1}^0=\\ &\qquad+\sum_k\left(\int\varphi\left(i\right)b\left(i\right)\frac{\sigma}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\frac{e\left(i\right)}{E_k}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)di+\frac{\int\left(1-\varphi\left(i\right)\right)b\left(i\right)di\int\left(1-\varphi\left(i\right)\right)\sigma\frac{e(i)}{E_k}\partial_e e_k\left(i\right)\gamma_{e,k}\left(i\right)di}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+\psi Wn\left(i\right)\right)di}\right)\frac{\lambda_kE_k}{P_kY_k}\Delta\breve{\mu}_{k,t+1}\end{aligned}$$

$$A_{b,t-1}^0\quad-\quad\frac{1}{\left(1-\delta\right)R}A_{b,t}^0\quad+\quad\left(1-\frac{1}{R}\right)A_{b,t-1}\quad=\quad\left(1-\frac{1}{R}\right)\sum_k\frac{\lambda_kE_k}{P_kY_k}\int\frac{b\left(i\right)}{E_k}\frac{\sigma e\left(i\right)}{\sigma e\left(i\right)+\psi Wn\left(i\right)}\gamma_{e,k}\left(i\right)\partial_e e_k\left(i\right)di\breve{\mu}_{k,t-1}$$

Taking derivatives with respect to the price of k at t gives:

$$M_{k,t}=\breve{\mu}_{k,t}-\left(1-\delta\right)\breve{\mu}_{k,t-1}$$

$$
\begin{aligned}
\beta M_{k,t+1} - M_{k,t} = {} & \lambda_k \check{\mu}_{k,t} - \sum_{l=1}^{K} \frac{\lambda_l E_l}{P_l Y_l} \int \gamma_{e,l}(i) \frac{e_l(i)}{E_l} \rho_{l,k}(i)\, di \check{\mu}_{l,t} - \sum_{l=1}^{K} \lambda_l \frac{\check{\mu}_{l,t}}{P_l Y_l} \frac{P_k \mathcal{Y}_{k,l} Y_l}{A_l} + \sum_{l,m} \frac{P_l P_k \partial_{P_k} \mathcal{Y}_{l,m}}{A_m E} Y_m \left(\hat{P}_{l,t} + \tilde{A}_{l,t,}\right) + \sum_l \frac{P_k \partial_{P_k} \mathcal{N}_l}{A_l N} Y_l \hat{W}_t \\
& + \sum_l \bar{s}_l \bar{\rho}_{l,k} \left(\hat{P}_{l,t} + \tilde{A}_{l,t}\right) - \frac{P_k Y_k}{E} \vartheta_k \left(\pi_{k,t} - \beta \pi_{k,t+1}\right) - L_{k,t} \\
& + \left(\frac{\int (1-\varphi(i))\, \partial_e e_k(i)\, \sigma e(i)\, di \int (1-\varphi(i))\, \psi \frac{Wn(i)}{WN}}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} + \int \varphi(i)\, \partial_e e_k(i)\, \frac{\psi Wn(i)}{WN} \frac{\sigma e(i)}{\sigma e(i) + \psi Wn(i)} di \right) \hat{W}_t \\
& - \sum_l \left(\int \varphi(i)\, \partial_e e_l(i)\, \partial_e e_k(i)\, \psi \frac{Wn(i)}{WN} \frac{\sigma e(i)}{\sigma e(i) + \psi Wn(i)} - \frac{\int (1-\varphi(i))\, \partial_e e_k(i)\, \sigma e(i)\, di \int (1-\varphi(i))\, \sigma \frac{e(i)}{E} \partial_e e_l di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} + \int (1-\varphi(i))\, \sigma \frac{e(i)}{E} \partial_e e_k(i)\, \partial_e e_l(i)\, di \right) \left(\tilde{A}_{l,t} + \hat{P}_{l,t}\right) \\
& + \sum_{l=1}^{K} \frac{\lambda_l E_l}{P_l Y_l} \left(\int (1-\varphi(i))\, \partial_e e_k(i)\, \frac{\sigma e(i)}{E_l} \gamma_{e,l}(i)\, \partial_e e_l(i)\, di + \int \varphi(i)\, \partial_e e_k(i)\, \psi Wn(i)\, \frac{\sigma e(i)}{\sigma e(i) + \psi Wn(i)} \frac{1}{E_l} \gamma_{e,l}(i)\, \partial_e e_l(i)\, di \right) \check{\mu}_{l,t} \\
& - \sum_{l=1}^{K} \frac{\lambda_l E_l}{P_l Y_l} \frac{\int (1-\varphi(i))\, \partial_e e_k(i)\, \sigma e(i)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \int (1-\varphi(i))\, \partial_e e_l(i)\, \frac{\sigma e(i)}{E_l} \gamma_{e,l}(i)\, di \check{\mu}_{l,t} \\
& + \left(\frac{\int (1-\varphi(i))\, \partial_e e_k \sigma e di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} + \frac{\int \varphi(i) \left(e_k(i) - e(i) \bar{s}_k\right)}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) \frac{1}{R} \left(A_{b,t} - R(1-\delta) A_{b,t-1}\right) + \left(1 - \frac{1}{R}\right) \bar{s}_k A_{b,t} - \delta \bar{s}_k \mathcal{A}_{b,t} + A_{e_k,t}
\end{aligned}
$$

With

$$
\Delta L_{k,t+1} = \frac{\delta}{1-\delta} L^0_{k,t+1}
$$

$$
\begin{aligned}
L^0_{k,t-1} - \frac{1}{R(1-\delta)} L^0_{k,t} + \left(1 - \frac{1}{R}\right) L_{k,t-1} = {} & \left(1 - \frac{1}{R}\right) \int (1-\varphi(i))\, \sigma \frac{e(i)}{E} \left(\frac{\sigma e(i)\, \partial_e e_k(i)}{\sigma e(i) + \psi Wn(i)} - \frac{\int (1-\varphi(i))\, \sigma e(i)\, \partial_e e_k(i)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) di \hat{W}_{t-1} \\
& - \sum_l \left(1 - \frac{1}{R}\right) \int (1-\varphi(i))\, \sigma \frac{e(i)}{E} \partial_e e_l(i) \left(\frac{\sigma e(i)\, \partial_e e_k(i)}{\sigma e(i) + \psi Wn(i)} - \frac{\int (1-\varphi(i))\, \sigma e(i)\, \partial_e e_k(i)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) di \left(\tilde{A}_{l,t-1} + \hat{P}_{l,t-1}\right) \\
& + \sum_l \frac{\lambda_l E_l}{P_l Y_l} \left(1 - \frac{1}{R}\right) \int (1-\varphi(i))\, \sigma \frac{e(i)}{E_l} \partial_e e_l(i)\, \gamma_{e,l}(i) \left(\frac{\sigma e(i)\, \partial_e e_k(i)}{\sigma e(i) + \psi Wn(i)} - \frac{\int (1-\varphi(i))\, \sigma e(i)\, \partial_e e_k(i)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) di \check{\mu}_{l,t-1}
\end{aligned}
$$

$$
\begin{aligned}
A_{e_k,t+1} - A_{e_k,t} - \frac{\delta}{1-\delta} A^0_{e_k,t+1} = {} & \sum_l \left(\int \varphi(i)\, e(i) \left(s_k(i) - \bar{s}_k\right) \frac{\sigma e(i)}{\sigma e(i) + \psi Wn(i)} \frac{1}{E_l} \gamma_{e,l}(i)\, \partial_e e_l(i)\, di \right) \frac{\lambda_l E_l}{P_l Y_l} \lambda_l \Delta \check{\mu}_{l,t+1} \\
& + \sum_l \left(\frac{\int (1-\varphi(i))\, e(i) \left(s_k(i) - \bar{s}_k\right) di \int (1-\varphi(i))\, \sigma e(i)\, \partial_e e_l(i)\, \frac{1}{E_l} \gamma_{e,l}(i)\, di}{\int (1-\varphi(i)) (\sigma e(i) + \psi Wn(i))\, di} \right) \frac{\lambda_l E_l}{P_l Y_l} \lambda_l \Delta \check{\mu}_{l,t+1}
\end{aligned}
$$

$$
A^0_{e_k,t-1} - \frac{1}{(1-\delta) R} A^0_{e_k,t} + \left(1 - \frac{1}{R}\right) A_{e_k,t-1} = \sum_k \lambda_l \frac{E_l}{P_l Y_l} \left(1 - \frac{1}{R}\right) \int e(i) \left(s_k(i) - \bar{s}_k\right) \frac{\sigma e(i) / E_l}{\sigma e(i) + \psi Wn(i)} \gamma_{e,l}(i)\, \partial_e e_l(i)\, di \check{\mu}_{l,t-1}
$$

$$
\mathcal{A}_{b,t} - \frac{1}{R} \mathcal{A}_{b,t+1} = \left(1 - \frac{1}{R}\right) \sum_k \lambda_k \frac{E_k}{P_k Y_k} \int \frac{b(i)}{E_k} \frac{\sigma e(i)}{\sigma e(i) + \psi Wn(i)} \gamma_{e,k}(i)\, \partial_e e_k(i)\, di \check{\mu}_{k,t}
$$

Efficiency of the Steady State

For first order change in prices, the change in welfare is

$$
d\mathcal{W} = \mathbb{E}_\beta \int G'(V(i))\, \partial_e v(i)\, \mathbb{E}_{t_0} \sum_{s=0}^{\infty} ((1-\delta)\beta)^s \left\{ \left(\frac{b(i)}{ER} \left(\hat{R}_s - (R-1) \sum_l \bar{s}_l \hat{P}_{l,s} \right) - \sum_l \frac{e(i)}{E} \left(s_l - \bar{s}_l\right) \hat{P}_{l,s} \right) + \sum_l \frac{R-1}{R} \frac{b(i)}{E} \bar{s}_l \hat{P}_{l,t_0} \right\} di
$$

Using the fact that $G'\left(V(i)\right)\partial_e v\left(i\right) = 1$ and that the market for bonds clear, we have that

$$d\mathcal{W} = 0$$

For any change in prices (subvariety prices, wage and interest rate). Therefore there is no change in monetary policy that can improve upon the steady state. It is also direct to verify that there are no taxes at any date t, financed by a lump sum at t than can improve the steady state. To see this consider, for example a wage subsidy in sector k at t subsidized by an arbitrary lump sum. The total impact on welfare is given by

$$\beta^t \mathbb{E}_0 \int G'\left(V(i)\right)\partial_e v\left(i\right)\left\{\varsigma\left(i\right)\frac{Y_k}{A_k}\mathcal{N}_k(\boldsymbol{P}_t, W_t) dW_{k,t} - d\tau_t\left(i\right)\right\} di$$

With

$$\int \frac{Y_k}{A_{k,t}}\mathcal{N}_k(\boldsymbol{P}_t, W_t) dW_{k,t} di = \int d\tau_t\left(i\right) di$$

Using again $G'\left(V(i)\right)\partial_e v\left(i\right) = 1$ and $\int \varsigma\left(i\right) di = 1$, this subsidy has no impact on Welfare. Similarly, wage subsidy to workers, input taxes and commodity taxes have no impact on welfare. The economy is therefore constrained efficient: the only inefficiency is the uninsured death risk which could be corrected through an annuity market but neither through monetary policy or through taxes (if the government cannot have debt).

We now verify that the second order approximation of the Welfare Function is always negative. This will imply that a steady state with 0 inflation and output gap is a local maximum. First note that a direct application of Cauchy-Schwartz gives that for any variable x_t and any weight $0 < \alpha < 1$,

$$\mathbb{E}_0\left(\sum_{s\geq 0}\alpha^s x_{t+s}\right)^2 \leq \mathbb{E}_0 \frac{1}{(1-\alpha)}\sum_{s\geq 0}\alpha^s\left(x_{t+s}\right)^2$$

Given the definition of $\mathbb{E}_\beta$we have that $\sum_{s=0}^{\infty}\beta^s X_s = \mathbb{E}_\beta\left[\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}X_s\right]$ so we can rewrite the average labor distortion term (which we denote by L_1) as

$$(1-\beta)\sum\beta^s\left\{\int\frac{\left(1-\varphi\left(i\right)\right)e\left(i\right)}{\int\left(1-\varphi\left(i\right)\right)e\left(i\right)di}\left(\hat{W}_s - \sum_l \partial_e e_l\left(i\right)\left(\hat{P}_{l,s} + \tilde{A}_{l,s}\right)\right)^2 di\right\} = \left(1-\tilde{\beta}\right)\mathbb{E}_\beta\left(\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\int\frac{\left(1-\varphi\left(i\right)\right)e\left(i\right)}{\int\left(1-\varphi\left(i\right)\right)e\left(i\right)di}\left(\hat{W}_s - \sum_l \partial_e e_l\left(i\right)\left(\hat{P}_{l,s} + \tilde{A}_{l,s}\right)\right)^2 di\right)$$

We can re-write the term that appears in the dispersion of wealth effect as

$$\left(1-\frac{1}{R}\right)\frac{\frac{(1-\varphi(i))b(i)}{\int(1-\varphi(i))e(i)di} - \frac{(1-\varphi(i))Wn(i)}{\int(1-\varphi(i))Wn(i)di}\frac{\int(1-\varphi(i))b(i)di}{\int(1-\varphi(i))e(i)di}}{1+\frac{Wn(i)\psi}{\sigma e(i)}} = \frac{\left(1-\varphi\left(i\right)\right)e\left(i\right)}{\int\left(1-\varphi\left(i\right)\right)e\left(i\right)di} - \left(1-\varphi\left(i\right)\right)\frac{e\left(i\right)Wn\left(i\right)/E}{\sigma e\left(i\right) + Wn\left(i\right)\psi}\left(\frac{\sigma}{\left(1-\varphi^N\right)} + \frac{\psi}{\left(1-\varphi^E\right)}\right),$$

where the second term is always weakly negative and hence

$$L_2 \equiv \mathbb{E}_\beta\left(\left\{\int\left(1-\frac{1}{R}\right)\frac{\frac{(1-\varphi(i))b(i)}{\int(1-\varphi(i))e(i)di} - \frac{(1-\varphi(i))Wn(i)}{\int(1-\varphi(i))Wn(i)di}\frac{\int(1-\varphi(i))b(i)di}{\int(1-\varphi(i))e(i)di}}{1+\frac{Wn(i)\psi}{\sigma e(i)}}\left(\left(1-\tilde{\beta}\right)\sum_{u\geq t_0}\tilde{\beta}^{u-t_0}\left(\hat{W}_u - \sum_l \partial_e e_l\left(i\right)\left(\hat{P}_{l,u} + \tilde{A}_{l,u}\right)\right)\right)^2 di\right\}\right) \leq$$

$$\mathbb{E}_\beta\left(\left\{\int\frac{\left(1-\varphi\left(i\right)\right)e\left(i\right)}{\int\left(1-\varphi\left(i\right)\right)e\left(i\right)di}\left(\left(1-\tilde{\beta}\right)\sum_{u\geq t_0}\tilde{\beta}^{u-t_0}\left(\hat{W}_u - \sum_l \partial_e e_l\left(i\right)\left(\hat{P}_{l,u} + \tilde{A}_{l,u}\right)\right)\right)^2 di\right\}\right)$$

Putting these results together we have that

$$L_1 + L_2 \geq \left(1-\tilde{\beta}\right)\mathbb{E}_\beta\left(\int\frac{\left(1-\varphi\left(i\right)\right)e\left(i\right)}{\int\left(1-\varphi\left(i\right)\right)e\left(i\right)di}\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\hat{W}_s - \sum_l \partial_e e_l\left(i\right)\left(\hat{P}_{l,s} + \tilde{A}_{l,s}\right)\right)^2 di\right) - \mathbb{E}_\beta\left(\int\frac{\left(1-\varphi\left(i\right)\right)e\left(i\right)}{\int\left(1-\varphi\left(i\right)\right)e\left(i\right)di}\left(\left(1-\tilde{\beta}\right)\sum_{u\geq t_0}\tilde{\beta}^{u-t_0}\left(\hat{W}_u - \sum_l \partial_e e_l\left(i\right)\left(\hat{P}_{l,u} + \tilde{A}_{l,u}\right)\right)\right)\right.$$

$$\geq 0$$

where the last line follows from our preliminary Cauchy-Schwartz result.

Next consider the term

$$(1-\beta)\sum\beta^{s}\left\{\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)-\mathbb{E}_{\beta}\left(\left(1-\tilde{\beta}\right)\left(\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\right)\left(\left(1-\tilde{\beta}\right)\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)\right)\right\}$$

$$=\left(1-\tilde{\beta}\right)\mathbb{E}_{\beta}\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)-\mathbb{E}_{\beta}\left(\left(1-\tilde{\beta}\right)\left(\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\right)\left(\left(1-\tilde{\beta}\right)\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)\right)$$

$$=\left(1-\tilde{\beta}\right)\mathbb{E}_{\beta}\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}-\left(1-\tilde{\beta}\right)\left(\sum_{u\geq t_{0}}\tilde{\beta}^{u-t_{0}}\left(\hat{P}_{k,u}+\tilde{A}_{k,s}\right)\right)\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}-\left(1-\tilde{\beta}\right)\sum_{s\geq t_{0}}\tilde{\beta}^{u-t_{0}}\left(\hat{P}_{l,u}+\tilde{A}_{l,u}\right)\right)$$

Since $\mathcal{E}$ is a variance-covariance matrix, it is positive semi-definite, therefore:

$$(1-\beta)\sum\beta^{s}\frac{\left(1-\varphi^{E}\right)\sigma}{\left(1+\frac{\left(1-\varphi^{N}\right)\psi}{\left(1-\varphi^{E}\right)\sigma}\right)}\sum_{k,l}\mathcal{E}_{k,l}\left\{\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)-\mathbb{E}_{\beta}\left(\left(1-\tilde{\beta}\right)\left(\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\right)\left(\left(1-\tilde{\beta}\right)\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\right)\right)\right\}\geq 0$$

Next, we have:

$$-(1-\beta)\sum\beta^{s}\sum_{k}\bar{s}_{k}\sum_{l}\bar{\mathcal{C}}_{k,l}\left(\hat{P}_{k,s}+\tilde{A}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{A}_{l,s}\right)\geq 0$$

$$-(1-\beta)\sum\beta^{s}\left\{\frac{W\partial_{W}\mathcal{N}}{N}\hat{W}_{s}^{\,2}+\sum_{k}\frac{P_{k}\mathcal{Y}_{k}}{E}\frac{W\partial_{W}\mathcal{Y}_{k}}{\mathcal{Y}_{k}}\hat{W}_{s}\left(\hat{P}_{k,s}+\tilde{\mathbf{A}}_{k,s}\right)+\sum_{k}\frac{P_{k}\partial_{P_{k}}\mathcal{N}}{N}\left(\hat{P}_{k,s}+\tilde{\mathbf{A}}_{k,s}\right)\hat{W}_{s}+\sum_{k,l}\frac{P_{k}\mathcal{Y}_{k}}{E}\frac{P_{l}\partial_{P_{l}}\mathcal{Y}_{k}}{\mathcal{Y}_{k}}\left(\hat{P}_{k,s}+\tilde{\mathbf{A}}_{k,s}\right)\left(\hat{P}_{l,s}+\tilde{\mathbf{A}}_{l,s}\right)\right\}\geq 0$$

Since both the substitution and the transformation matrix are negative semi-definite.
Finally, we can rewrite the redistribution terms as

$$+\mathbb{E}_{\beta}\left(\int-\frac{G''\left(V(i)\right)v'\left(e\left(i\right)\right)E}{\left(1-\tilde{\beta}\right)G'\left(V(i)\right)}\left(\nu_{t_{0}}(i)+\left(1-\tilde{\beta}\right)\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\sum_{l}\frac{e\left(i\right)}{E}s_{l}\left(i\right)\tilde{A}_{l,s}\right)^{2}di\right)$$

$$+\mathbb{E}_{\beta}\left(\left(1-\tilde{\beta}\right)\sum_{s\geq t_{0}}\int\left(1-\varphi\left(i\right)\right)\frac{E}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left(\nu_{t_{0}}(i)+\left(1-\tilde{\beta}\right)\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\sum_{l}\frac{e\left(i\right)}{E}s_{l}\left(i\right)\tilde{A}_{l,s}-\partial_{e}e_{l}\left(i\right)\tilde{A}_{l,s}\right)\right)di\right)$$

$$+\mathbb{E}_{\beta}\left(\left(1-\tilde{\beta}\right)\sum\tilde{\beta}^{s-t_{0}}\int\varphi\left(i\right)\frac{E}{\sigma e\left(i\right)+Wn\left(i\right)\psi}\left\{\aleph_{s}^{t_{0}}(i)+\sum_{l}\frac{e\left(i\right)}{E}s_{l}\left(i\right)\tilde{A}_{l,s}-\partial_{e}e_{l}\left(i\right)\tilde{A}_{l,s}\right\}^{2}\right)$$

$$+\frac{E}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}\mathbb{E}_{\beta}\left\{\left(1-\tilde{\beta}\right)\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\left(\int\left(1-\varphi\left(i\right)\right)\left\{\aleph_{s}^{t_{0}}(i)+\sum_{l}\frac{e\left(i\right)}{E}s_{l}\left(i\right)\tilde{A}_{l,s}-\sum_{l}\frac{\sigma e\left(i\right)}{E}\partial_{e}e_{l}\left(i\right)\tilde{A}_{l,s}\right\}\right)^{2}\right\}$$

$$-\frac{E}{\int\left(1-\varphi\left(i\right)\right)\left(\sigma e\left(i\right)+Wn\left(i\right)\psi\right)di}\mathbb{E}_{\beta}\left\{\left(\int\left(1-\varphi\left(i\right)\right)\left\{\nu_{t_{0}}(i)+\left(1-\tilde{\beta}\right)\sum_{s\geq t_{0}}\tilde{\beta}^{s-t_{0}}\sum_{l}\left(\frac{e\left(i\right)}{E}s_{l}\left(i\right)\tilde{A}_{l,s}-\frac{\sigma e\left(i\right)}{E}\partial_{e}e_{l}\left(i\right)\tilde{A}_{l,s}\right)\right\}di\right)^{2}\right\}$$

Note that

$$\left(1-\tilde{\beta}\right)\sum\tilde{\beta}^{s-t_{0}}\aleph_{s}^{t_{0}}(i)=\nu_{t_{0}}(i)$$

So

$$\mathbb{E}_{\beta}\left\{\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\left(\int\left(1-\varphi\left(i\right)\right)\left\{\aleph_s^{t_0}(i)+\sum_l\frac{e\left(i\right)}{E}s_l\left(i\right)\tilde{A}_{l,s}-\sum_l\frac{\sigma e\left(i\right)}{E}\partial_e e_l\left(i\right)\tilde{A}_{l,s}\right\}\right)^2\right\}$$

$$-\mathbb{E}_{\beta}\left\{\left(\int\left(1-\varphi\left(i\right)\right)\left\{\nu_{t_0}(i)+\left(1-\tilde{\beta}\right)\sum_{s\geq t_0}\tilde{\beta}^{s-t_0}\sum_l\left(\frac{e\left(i\right)}{E}s_l\left(i\right)\tilde{A}_{l,s}-\frac{\sigma e\left(i\right)}{E}\partial_e e_l\left(i\right)\tilde{A}_{l,s}\right)\right\}di\right)^2\right\}\geq 0$$

Therefore, all terms in the social welfare function are positive: there are no second order deviations in prices, wage and interest rates improving on the steady state.

F Additional analytical results

Addition to Result 1: divine Coincidence indices without endogenous markups

We now derive an inflation index which can be fully stabilized alongside the output gap, for the case with the endogenous markup wedge. In this case, the sectoral NKPC can be written as:

$$\pi_{k,t} = \kappa_k \tilde{\mathcal{Y}}_t + \lambda_k \left(\sum_l \left(\Omega_{N,k} \overline{\partial_e e_l} + \Omega_{k,l} \right) \left(\hat{P}_{l,t} - \hat{P}^*_{l,t} \right) - \left(\hat{P}_{k,t} - \hat{P}^*_{k,t} \right) \right) + \beta \mathbb{E}_t \pi_{k,t+1},$$

or in matrix form:

$$\boldsymbol{\pi_t} = \boldsymbol{\kappa} \tilde{\mathcal{Y}}_t + \mathcal{D}\left[\lambda\right] \left(\tilde{\Omega} - Id \right) \left(\boldsymbol{\hat{P}_t} - \boldsymbol{\hat{P}^*_t} \right) + \beta \mathbb{E}_t \boldsymbol{\pi_{t+1}}.$$

where $\boldsymbol{\kappa} = [\kappa_1, ..., \kappa_K]^T$, $\mathcal{D}\left[\lambda\right]$ is a $K \times K$ diagonal matrix with λ_k on the diagonal and $\tilde{\Omega}_{k,l} = \Omega_{N,k} \overline{\partial_e e_l} + \Omega_{k,l}$. Note that

$$\tilde{\Omega}_{k,l} \geq 0,$$
$$\sum_l \tilde{\Omega}_{k,l} = 1.$$

The Perron-Frobenius' theorem for row-stochastic matrices implies that we have an eigenvector $\boldsymbol{\tilde{\omega}} = [\tilde{\omega}_1, ..., \tilde{\omega}_K]$ with $\tilde{\omega}_k \geq 0$ and $\sum_k \tilde{\omega}_k = 1$ (normalization) such that

$$\boldsymbol{\tilde{\omega}} \tilde{\Omega} = \boldsymbol{\tilde{\omega}}.$$

Now, define

$$\boldsymbol{\omega} = \left[\frac{\lambda}{\lambda_1} \tilde{\omega}_1, ..., \frac{\lambda}{\lambda_K} \tilde{\omega}_K \right],$$
$$\frac{1}{\lambda} = \sum_k \frac{1}{\lambda_k} \tilde{\omega}_k.$$

Note that we have

$$\boldsymbol{\omega} \mathcal{D}\left[\lambda\right] \left(\tilde{\Omega} - Id \right) = \lambda \boldsymbol{\tilde{\omega}} \mathcal{D}^{-1}\left[\lambda\right] \mathcal{D}\left[\lambda\right] \left(\tilde{\Omega} - Id \right) = 0.$$

Now define

$$\pi_{d,t} = \sum \omega_k \pi_{k,t},$$

We have

$$\begin{aligned} \pi_{d,t} &= \kappa \tilde{\mathcal{Y}}_t + \boldsymbol{\omega} \mathcal{D}\left[\lambda\right] \left(\tilde{\Omega} - Id \right) \left(\boldsymbol{\hat{P}_t} - \boldsymbol{\hat{P}^*_t} \right) + \beta \mathbb{E}_t \pi_{d,t+1}, \\ &= \kappa \tilde{\mathcal{Y}}_t + \beta \mathbb{E}_t \pi_{d,t+1}. \end{aligned}$$

With $\kappa = \sum \omega_k \kappa_k$. Therefore $\pi_{d,t}$ can be stabilized jointly with the output gap.

Addition to Result 1: HtM and I-O

In this section we show how to extend Result 1 to the case with HtM households and Input-Output links. We slightly amend the assumption (**A.1**) and (**A.2**):

- **Assumption A1:** $\kappa_k = \kappa$ for all k (recall that with IO $\kappa_k = \lambda_k \left(\frac{1}{\sigma} + \frac{1}{\psi} \right) \left(\Omega_{N,k} + s^C_k \frac{\sigma \psi}{\sigma + \psi} \Gamma_k \right)$)
- **Assumption A2:** $\int \gamma^u_{b,k}(i) \left\{ \frac{(1-\varphi(i))b}{E} - \frac{(1-\varphi(i))Wn}{(1-\varphi^L)WN} \int (1-\varphi(i)) \frac{b(i)}{E} di \right\} di = \int \gamma^{HtM}_{b,k}(i) \left\{ \frac{\varphi(i)b}{E} - \frac{(1-\varphi(i))Wn}{(1-\varphi^L)WN} \int \varphi(i) \frac{b(i)}{E} di \right\} di = 0$

Note that (**A.2**) is slightly strengthened with HtM. Without HtM we only need to assume that $\gamma_{b,k}(i)$ is uncorrelated with wealth. With HtM we assume that $\gamma_{b,k}(i)$ is uncorrelated with both the wealth of the HtM and the unconstrained households. We rewrite once again the system of equation of relative prices $\tilde{P}_{k,t} = \hat{P}_{k,t} - \hat{P}_{d,t}$, with $\hat{P}_{d,t}$ defined in the previous section:

$$\tilde{\pi}_{k,t} = \left(\lambda_k s_k^C \mathcal{M}_{k,t} - \sum_l \lambda_l s_l^C \omega_l \mathcal{M}_{l,t}\right) + \lambda_k \left(+\sum \left(\Omega_{N,k}\overline{\partial_e e_l} + \Omega_{k,l}\right)\left(\tilde{P}_{l,t} - \tilde{P}^*_{l,t}\right) - \left(\tilde{P}_{k,t} - \tilde{P}^*_{k,t}\right)\right) + \beta \mathbb{E}_t \tilde{\pi}_{k,t+1}$$

$$\begin{aligned}\mathcal{M}_{k,t} &= \Gamma_k \hat{\mathcal{Y}}_t^* + \mathcal{M}^P_{k,t} + \mathcal{M}^D_{k,t} \\ \mathcal{M}^P_{k,t} &= \sum_l \mathcal{S}_{k,l} \tilde{P}_{l,t}\end{aligned}$$

Therefore to prove that relative prices evolve independently of monetary policy, we only need show that under (**A.2**), $\mathcal{M}^D_{k,t}$ only depends on relative prices. Recall that we have:

$$\begin{aligned}\mathbb{E}_t \mathcal{M}^D_{k,t+1} - \mathcal{M}^D_{k,t} = &\left(\sigma_k^{\mathcal{M},u} \hat{R}_t - \sum_l \sigma_{k,l}^{\mathcal{M},u} \pi_{l,t+1}\right) + \frac{\delta}{1-\delta}\mathbb{E}_t \mathcal{M}^0_{k,t+1} \\ &+ \frac{R}{R-1}\int \left(\gamma^u_{b,k}(i)\left(\varphi(i)\frac{b(i)}{RE} - \frac{(1-\varphi(i))\, Wn(i)}{(1-\varphi^L)\, WN}\int \left(\varphi(i)\frac{b(i)}{RE}\right) di\right)\right) di \mathbb{E}_t \Delta \hat{R}_{t+1} \\ &- \frac{R}{R-1}\sum_l \int \gamma^u_{b,k}(i)\left\{\left(1 - \frac{1}{R}\right)\left(\varphi(i)\frac{b(i)}{E} - \frac{(1-\varphi(i))\, Wn(i)}{(1-\varphi^L)\, WN}\int \left(\varphi(i)\frac{b(i)}{E}\right) di\right) \bar{s}_l\right\} di \mathbb{E}_t \pi_{l,t+1} \\ &- \frac{R}{R-1}\sum_l \int \gamma^u_{b,k}(i)\left\{\left(\varphi(i)\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) - \frac{(1-\varphi(i))\, Wn(i)}{(1-\varphi^L)\, WN}\int \varphi(i)\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) di\right)\right\} di \mathbb{E}_t \pi_{l,t+1} \\ &- \frac{R}{R-1}\sum_l \int \gamma^u_{b,k}(i)\left\{\frac{Wn(i)}{WN}\psi\left(\varphi(i)\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right) - \frac{1-\varphi(i)}{1-\varphi^E}\int \varphi(i)\frac{e(i)}{E}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right) di\right)\right\} di \mathbb{E}_t \pi_{l,t+1}\end{aligned}$$

$$\mathcal{M}^0_{k,t} - \frac{1}{(1-\delta)\, R}\mathbb{E}_t \mathcal{M}^0_{k,t+1} = \int \gamma^u_{b,k}(i)\frac{b(i)}{RE} di \left(\hat{R}_t - \sum_l \bar{s}_l \mathbb{E}_t \pi_{l,t+1}\right) - \sum_l \int \gamma^u_{b,k}(i)\left(\frac{e(i)}{E}\left(s_l(i) - \bar{s}_l\right) + \psi \frac{Wn(i)}{WN}\left(\partial_e e_l(i) - \overline{\partial_e e_l}\right)\right) di \hat{P}_{l,t} - \frac{R-1}{R}\mathcal{M}^D_{k,t}$$

Under (**A.2**) we can rewrite $\sigma_{k,l}^{\mathcal{M},u}$ as:

$$
\begin{aligned}
\sigma_{k,l}^{\mathcal{M},u} &= \sigma \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \overline{\partial_e e_l}^u \frac{R}{R-1} \int \frac{(1-\varphi(i))\, Wn}{(1-\varphi^L)\, WN} \gamma_{b,k}^u(i) di \left(\sigma \left(1-\varphi^E \right) + \psi \left(1-\varphi^L \right) \right) \\
&= \sigma \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \sigma \overline{\partial_e e_l}^u \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{(1-\varphi^L)\, E_k} \partial_e e_k(i) di \frac{\left(\sigma \left(1-\varphi^E\right) + \psi \left(1-\varphi^L\right)\right)}{\sigma+\psi} \\
&+ \sigma \overline{\partial_e e_l}^u \int \frac{(1-\varphi(i))\, b}{(1-\varphi^L)\, WN} \gamma_{b,k}^u(i) di \frac{\left(\sigma \left(1-\varphi^E\right) + \psi \left(1-\varphi^L\right)\right)}{\sigma+\psi} \\
&= \sigma \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \sigma \overline{\partial_e e_l}^u \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{(1-\varphi^L)\, E_k} \partial_e e_k(i) di \\
&- \sigma \overline{\partial_e e_l}^u \int \gamma_{b,k}^u(i) \frac{(1-\varphi(i))\, (\sigma e + \psi Wn)}{(1-\varphi^L)\, WN} di \frac{\left(\left(1-\varphi^E\right) - \left(1-\varphi^L\right)\right)}{\sigma+\psi} \frac{R}{R-1} \\
&+ \sigma \overline{\partial_e e_l}^u \int \frac{(1-\varphi(i))\, b}{(1-\varphi^L)\, WN} \gamma_{b,k}^u(i) di \frac{\left(\sigma \left(1-\varphi^E\right) + \psi \left(1-\varphi^L\right)\right)}{\sigma+\psi} \\
&= \sigma \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \sigma \overline{\partial_e e_l}^u \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{(1-\varphi^L)\, E_k} \partial_e e_k(i) di \\
&+ \sigma \overline{\partial_e e_l}^u \int \frac{1}{(1-\varphi^L)\, WN} \gamma_{b,k}^u(i) \left\{ (1-\varphi(i))\, b \int (1-\varphi(i)) \frac{\sigma e + \psi Wn}{(\sigma+\psi)\, E} di - (1-\varphi(i)) \frac{\sigma e + \psi Wn}{\sigma+\psi} \int (1-\varphi(i)) \frac{b(i)}{E} di \right\} di \\
&= \sigma \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \sigma \overline{\partial_e e_l}^u \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{(1-\varphi^L)\, E_k} \partial_e e_k(i) di \\
&+ \sigma \overline{\partial_e e_l}^u \int \frac{1}{(1-\varphi^L)\, WN} \gamma_{b,k}^u(i) \left\{ (1-\varphi(i))\, b \int (1-\varphi(i)) \left(\frac{Wn}{WN} + \frac{\sigma b}{\sigma+\psi} \frac{R-1}{RE} \right) - (1-\varphi(i)) \left(Wn + \frac{\sigma b}{\sigma+\psi} \frac{R-1}{R} \right) \int (1-\varphi(i)) \frac{b(i)}{E} di \right\} di \\
&= \sigma \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \sigma \overline{\partial_e e_l}^u \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{(1-\varphi^L)\, E_k} \partial_e e_k(i) di \\
&+ \sigma \overline{\partial_e e_l}^u \frac{\sigma}{\sigma+\psi} \frac{R-1}{R} \int \gamma_{b,k}^u(i) \left\{ \frac{(1-\varphi(i))\, b}{E} - \frac{(1-\varphi(i))\, Wn}{(1-\varphi^L)\, WN} \int (1-\varphi(i)) \frac{b(i)}{E} di \right\} didi
\end{aligned}
$$

We therefore have under (**A.2**)

$$
\begin{aligned}
\sigma_{k,l}^{\mathcal{M},u} &= \sigma \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{E_k} \partial_e e_k(i) \partial_e e_l(i) di - \sigma \overline{\partial_e e_l}^u \int \gamma_{e,k}(i) \frac{(1-\varphi(i))\, e(i)}{(1-\varphi^L)\, E_k} \partial_e e_k(i) di \\
\sum_l \sigma_{k,l}^{\mathcal{M},u} &= \sigma_k^{\mathcal{M},u} = 0
\end{aligned}
$$

In addition we have

$$\begin{aligned}\int \gamma^u_{b,k}(i)\frac{b(i)}{RE}di &= \int \gamma^u_{b,k}(i)\frac{(1-\varphi(i))\,b}{E}di + \int \gamma^u_{b,k}(i)\frac{\varphi(i)b}{E}di \\ &= \int \gamma^u_{b,k}(i)\frac{(1-\varphi(i))\,b}{E}di + \int \gamma^u_{b,k}(i)\frac{\varphi(i)b}{E}di - \int \gamma^u_{b,k}(i)\frac{(1-\varphi(i))\,Wn}{(1-\varphi^L)\,WN}\left(\int \frac{b(i)}{E}di\right)di \\ &= \int \gamma^u_{b,k}(i)\left\{\frac{(1-\varphi(i))\,b}{E} - \frac{(1-\varphi(i))\,Wn}{(1-\varphi^L)\,WN}\int (1-\varphi(i))\,\frac{b(i)}{E}di\right\}di + \frac{R-1}{R}\int \gamma^{HtM}_{b,k}(i)\left\{\frac{\varphi(i)b}{E} - \frac{(1-\varphi(i))\,Wn}{(1-\varphi^L)\,WN}\int \varphi(i)\frac{b(i)}{E}di\right\}di \\ &= 0\end{aligned}$$

Therefore the equations for $\mathcal{M}^D_{k,t}$ can be rewritten:

$$\begin{aligned}\mathbb{E}_t\mathcal{M}^D_{k,t+1} - \mathcal{M}^D_{k,t} = &-\sum_l \sigma^{\mathcal{M},u}_{k,l}\tilde{\pi}_{l,t+1} + \frac{\delta}{1-\delta}\mathbb{E}_t\mathcal{M}^0_{k,t+1} \\ &- \frac{R}{R-1}\sum_l\int \gamma^u_{b,k}(i)\left\{\left(\varphi(i)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right) - \frac{(1-\varphi(i))\,Wn(i)}{(1-\varphi^L)\,WN}\int \varphi(i)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)di\right)\right\}di\mathbb{E}_t\tilde{\pi}_{l,t+1} \\ &- \frac{R}{R-1}\sum_l\int \gamma^u_{b,k}(i)\left\{\frac{Wn(i)}{WN}\psi\left(\varphi(i)\left(\partial_e e_l(i)-\overline{\partial_e e_l}\right) - \frac{1-\varphi(i)}{1-\varphi^E}\int \varphi(i)\frac{e(i)}{E}\left(\partial_e e_l(i)-\overline{\partial_e e_l}\right)di\right)\right\}di\mathbb{E}_t\tilde{\pi}_{l,t+1}\end{aligned}$$

$$\mathcal{M}^0_{k,t} - \frac{1}{(1-\delta)\,R}\mathbb{E}_t\mathcal{M}^0_{k,t+1} = -\sum_l\int \gamma^u_{b,k}(i)\left(\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right) + \psi\frac{Wn(i)}{WN}\left(\partial_e e_l(i)-\overline{\partial_e e_l}\right)\right)di\tilde{P}_{l,t} - \frac{R-1}{R}\mathcal{M}^D_{k,t}$$

Where we use

$$\sum_l\int \gamma^u_{b,k}(i)\left(\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right) + \psi\frac{Wn(i)}{WN}\left(\partial_e e_l(i)-\overline{\partial_e e_l}\right)\right)di = 0$$

$$\begin{aligned}&\sum_l\int \gamma^u_{b,k}(i)\left\{\left(\varphi(i)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right) - \frac{(1-\varphi(i))\,Wn(i)}{(1-\varphi^L)\,WN}\int \varphi(i)\frac{e(i)}{E}\left(s_l(i)-\bar{s}_l\right)di\right)\right\} \\ &+\sum_l\int \gamma^u_{b,k}(i)\left\{\frac{Wn(i)}{WN}\psi\left(\varphi(i)\left(\partial_e e_l(i)-\overline{\partial_e e_l}\right) - \frac{1-\varphi(i)}{1-\varphi^E}\int \varphi(i)\frac{e(i)}{E}\left(\partial_e e_l(i)-\overline{\partial_e e_l}\right)di\right)\right\} = 0\end{aligned}$$

to replace $\hat{P}_{l,t}$, $\pi_{l,t}$ with $\tilde{P}_{l,t}$, $\tilde{\pi}_{l,t}$. Therefore, relative prices are determined by a system of $4(K-1)$ equations which independent of $\hat{R}_t$ and are therefore independent from monetary policy. We conclude, since $\mathcal{NH}_t, \mathcal{M}_{k,t}, \mathcal{P}_{k,t}, \mathcal{I}_{k,t}$ only depend on relative prices that the wedges are independent from monetary policy.

Additions to Result 3: Analytical Formulas with HtM

We first re-derive the evolution of any relative price $\tilde{P}_{k,t} = \hat{P}_{k,t} - \sum \overline{\partial_e e_l}\hat{P}_{l,t}$

$$R\tilde{\pi}_{k,t} = -\lambda R\left(\tilde{P}_{k,t} - \tilde{P}^*_{k,t}\right) + \tilde{\pi}_{k,t}$$

$$P_{\Delta,t+1} - (1+R+R\lambda)\,P_{\Delta,t} + RP_{\Delta,t-1} = \lambda R\hat{A}_{\Delta,t}$$

The eigenvalues of the system are:

$$\mu_\pm = \frac{R+R\lambda+1 \pm \sqrt{(R+R\lambda-1)^2 + 4R\lambda}}{2}$$

With $\mu_+ > R + R\lambda$, $\mu_- < 1$. We obtain:

$$\tilde{P}_{k,t} = \lambda \sum_{0}^{t} \mu_{-}^{t-s+1} \sum \frac{1}{\mu_{+}^{u}} \tilde{P}_{k,t}^{*}$$

We now assume shock vanishes at a constant rate ρ_a, we have:

$$\tilde{P}_{k,t} = \frac{1}{\left(\mu_{+} - \rho_a\right)\left(\mu_{-} - \rho_a\right)} \left(\mu_{-}^{t+1} - \rho_a^{t+1}\right) \tilde{P}_{k,0}^{*}$$

Next we slightly rewrite the output gap equation

$$\begin{aligned}\tilde{\mathcal{Y}}_{t+1} - \tilde{\mathcal{Y}}_t &= \sigma\left(\left(1-\Phi_b\right)\hat{R}_t - \left(1-\Phi_b\right)\pi_{mcpi,t+1} - \hat{r}_t^{*}\right) \\ &\quad + \Phi_b\left(\frac{1}{R-1}\left(\mathbb{E}_t\hat{R}_{t+1} - \hat{R}_t\right) - \bar{s}_l \pi_{l,t+1}\right) - \frac{\left(1-\varphi^E\right)}{1-\varphi^L}\frac{\sigma}{\sigma+\psi}\sum_l \left(\sigma\left(\overline{\partial_e e_l}^{u} - \overline{\partial_e e_l}\right) - \left(s_l^u - \bar{s}_l\right)\right)\tilde{\pi}_{l,t+1}\end{aligned}$$

$$\Phi_b = \frac{\varphi^E - \varphi^L}{1-\varphi^L}\frac{\sigma}{\sigma+\psi}$$

$$-1 < \Phi_b < 1$$

Response to aggregate shocks, inflation targeting. For aggregate shocks ($A_{k,t} = A_t$), all relative sectoral prices are constant so the response does not depend on which inflation index is targeted. Assume $\hat{R}_t = \phi\pi_t$ (with π_t an arbitrary index), we have

$$R\pi_t = R\kappa\tilde{\mathcal{Y}}_t + \pi_{t+1}$$

$$\pi_{t+2} - \left(1 + R + R\kappa\sigma\left(1-\Phi_b\right) + R\kappa\Phi_b\right)\pi_{t+1} + R\pi_t + R\kappa\sigma\left(1-\Phi_b\right)\hat{R}_t + R\kappa\Phi_b\frac{1}{R-1}\Delta\hat{R}_{t+1} - R\kappa\sigma\hat{r}_t^{*} = 0$$

$$\pi_{t+2} - \left(1 + R + R\kappa\left(\sigma - \Phi_b\left(\sigma + \phi\frac{1}{R-1} - 1\right)\right)\right)\pi_{t+1} + R\left(1 + \kappa\phi\left(\sigma\left(1-\Phi_b\right) - \Phi_b\frac{1}{R-1}\right)\right)\pi_t - R\kappa\sigma\hat{r}_t^{*} = 0$$

The eigenvalues of the system are

$$\lambda_{\pm}^{HtM} = \frac{\left(1 + R + R\kappa\left(\sigma - \Phi_b\left(\sigma + \phi\frac{1}{R-1} - 1\right)\right)\right) \pm \sqrt{\left(\left(R + R\kappa\left(\sigma - \Phi_b\left(\sigma + \phi\frac{1}{R-1} - 1\right)\right)\right) - 1\right)^2 - 4R\kappa\left[\left(\phi - 1\right)\left(\sigma\left(1-\Phi_b\right)\right) - \Phi_b\right]}}{2}$$

Note that when $\Phi_b < 0$, $\phi \geq 1$ implies that both eigenvalues of the system are larger than one[50]

$$\begin{aligned}\pi_t &= \frac{R\kappa}{\rho_a^2 - \left(1 + R + R\kappa\left(\sigma - \Phi_b\left(\sigma + \phi\frac{R}{R-1} - 1\right)\right)\right)\rho_a + R\left(1 + \kappa\phi\left(\sigma\left(1-\Phi_b\right) - \Phi_b\frac{R}{R-1}\right)\right)}\rho_a^t\hat{r}_0^{*} \\ &= \frac{R\kappa\sigma}{\left(\lambda_{+} - \rho_a\right)\left(\lambda_{-} - \rho_a\right) + R\kappa\mathcal{C}(\rho_a)}\rho_a^t\hat{r}_0^{*}\end{aligned}$$

$$\tilde{\mathcal{Y}}_t = \frac{\sigma\left(R - \rho_a\right)}{\left(\lambda_{+} - \rho_a\right)\left(\lambda_{-} - \rho_a\right) + R\kappa\mathcal{C}(\rho_a)}\rho_a^t\hat{r}_0^{*}$$

with

$$\mathcal{C}(\rho_a) = -\Phi_b\left(\sigma\left(\phi - \rho_a\right) + \phi\frac{R}{R-1}\left(1 - \rho_a\right) + \rho_a\right) > 0$$

---

[50] Note that we only need $\phi > \max\{1 + \frac{\Phi_b}{\sigma(1-\Phi_b)}, -\left(R-1\right)\left(\frac{1}{-\Phi_b}\left(\frac{1}{\kappa}\left(1 - \frac{1}{R}\right) + \sigma\right) + \sigma - 1\right)\}$in particular if $\kappa \leq 1$or $\sigma \geq 1$ this simplifies to $\phi > 1 + \frac{\Phi_b}{\sigma(1-\Phi_b)}$.

And $\mathcal{C}(\rho_a)$ is decreasing in ρ_A. For a given policy rule, the presence of HtM households decreases the impact of technology or monetary shocks on inflation and the output gap. Intuitively, as HtM have negative wealth on average they respond to an increase in inflation by cutting consumption, since they respond more strongly than non HtM this makes monetary policy more effective.

Response to sectoral shocks, inflation targeting. Now assume that CB targets CPI: $\hat{R}_t = \phi \pi_{cpi,t}$. The system becomes

$$\pi_{cpi,t+2} - \left(1 + R + R\kappa\left(\sigma - \Phi_b\left(\sigma + \phi\frac{1}{R-1} - 1\right)\right)\right)\pi_{cpi,t+1} + R\left(1 + \kappa\phi\left(\sigma\left(1-\Phi_b\right) - \Phi_b\frac{1}{R-1}\right)\right)\pi_{cpi,t}$$
$$-R\kappa\sigma\hat{r}^*_t + R\lambda\left(\mathcal{NH}_{t+1} - \mathcal{NH}_t\right) - R\kappa\frac{\left(1-\varphi^E\right)}{1-\varphi^L}\frac{\sigma}{\sigma+\psi}\sum_l\left(\sigma\left(\overline{\partial_e e_l}^u - \overline{\partial_e e_l}\right) - \left(s^u_l - \bar{s}_l\right)\right)\tilde{\pi}_{l,t+1} - R\kappa\bar{\sigma}\left(\tilde{\pi}_{mcpi,t+1} - \tilde{\pi}_{cpi,t+1}\right) = 0$$

We have

$$\pi_t = \frac{R\lambda}{\left(\mu_+ - \rho_a\right)\left(\mu_- - \rho_a\right)}\left\{\left(1 - \frac{R\kappa\sigma\phi + R\kappa\mathcal{C}(\rho_a)}{\left(\lambda_+ - \rho_a\right)\left(\lambda_- - \rho_a\right) + R\kappa\mathcal{C}(\rho_a)}\right)\left(1-\rho_a\right)\rho_a^t - \left(1 - \frac{R\kappa\sigma\phi + R\kappa\mathcal{C}(\mu_-)}{\left(\lambda_+ - \mu_-\right)\left(\lambda_- - \mu_-\right) + R\kappa\mathcal{C}(\mu_-)}\right)\left(1-\mu_-\right)\mu_-^t\right\}\hat{A}_{\Delta,0}$$
$$+ \frac{R\kappa\sigma}{\left(\lambda_+ - \rho_a\right)\left(\lambda_- - \rho_a\right) + R\kappa\mathcal{C}(\rho_a)}\rho_a^t\hat{r}^*_0$$
$$- \frac{R\kappa\sigma}{\left(\mu_+ - \rho_a\right)\left(\mu_- - \rho_a\right)}\frac{\left(1-\varphi^E\right)}{1-\varphi^L}\frac{\sigma}{\sigma+\psi}\sum_l\left(\sigma\left(\overline{\partial_e e_l}^u - \overline{\partial_e e_l}\right) - \left(s^u_l - \bar{s}_l\right)\right)\left(\frac{\left(1-\mu_-\right)}{\left(\lambda_+ - \mu_-\right)\left(\lambda_- - \mu_-\right) + \mathcal{C}(\mu_-)}\mu_-^{t+1} - \frac{\left(1-\rho_a\right)}{\left(\lambda_+ - \rho_a\right)\left(\lambda_- - \rho_a\right) + \mathcal{C}(\rho_a)}\rho_a^{t+1}\right)\tilde{P}^*_{l,0}$$

$$\tilde{\mathcal{Y}}_t = \underbrace{-\frac{R\lambda}{\left(\mu_+ - \rho_a\right)\left(\mu_- - \rho_a\right)}\left\{\frac{\left(\sigma\phi + \mathcal{C}(\rho_a)\right)\left(1-\rho_a\right)\left(R-\rho_a\right)}{\left(\lambda_+ - \rho_a\right)\left(\lambda_- - \rho_a\right) + R\kappa\mathcal{C}(\rho_a)}\rho_a^t - \frac{\left(\sigma\phi + \mathcal{C}(\mu_-)\right)\left(1-\mu_-\right)\left(R-\mu_-\right)}{\left(\lambda_+ - \mu_-\right)\left(\lambda_- - \mu_-\right) + R\kappa\mathcal{C}(\mu_-)}\mu_-^t\right\}\hat{A}_{\Delta,0}}_{\text{Impactof NH wedge}}$$
$$+ \frac{\sigma\left(R-\rho_a\right)}{\left(\lambda_+ - \rho_a\right)\left(\lambda_- - \rho_a\right) + R\kappa\mathcal{C}(\rho_a)}\rho_a^t\hat{r}^*_0$$
$$\underbrace{-\frac{1}{\left(\mu_+ - \rho_a\right)\left(\mu_- - \rho_a\right)}\frac{\varphi^E}{1-\varphi^L}\frac{\sigma}{\sigma+\psi}\sum_l\left(\left(s^{HtM}_l - \bar{s}_l\right) - \sigma\left(\overline{\partial_e e_l}^{HtM} - \overline{\partial_e e_l}\right)\right)\left(\frac{\left(1-\mu_-\right)\left(R-\mu_-\right)}{\left(\lambda_+ - \mu_-\right)\left(\lambda_- - \mu_-\right) + R\kappa\mathcal{C}(\mu_-)}\mu_-^{t+1} - \frac{\left(1-\rho_a\right)\left(R-\rho_a\right)}{\left(\lambda_+ - \rho_a\right)\left(\lambda_- - \rho_a\right) + R\kappa\mathcal{C}(\rho_a)}\rho_a^{t+1}\right)\tilde{P}^*_{l,0}}_{\text{Impact of relative price on HtM consumption}}$$

The introduction of HtM has an ambiguous impact on both CPI inflation and the output gap. The response of the output gap is the sum of three terms. The first one is the contribution of the $\mathcal{NH}$ wedge:

$$-\frac{R\lambda}{\left(\mu_+ - \rho_a\right)\left(\mu_- - \rho_a\right)}\left\{\frac{\left(\sigma\phi + \mathcal{C}(\rho_a)\right)\left(1-\rho_a\right)\left(R-\rho_a\right)}{\left(\lambda_+ - \rho_a\right)\left(\lambda_- - \rho_a\right) + R\kappa\mathcal{C}(\rho_a)}\rho_a^t - \frac{\left(\sigma\phi + \mathcal{C}(\mu_-)\right)\left(1-\mu_-\right)\left(R-\mu_-\right)}{\left(\lambda_+ - \mu_-\right)\left(\lambda_- - \mu_-\right) + R\kappa\mathcal{C}(\mu_-)}\mu_-^t\right\}\hat{A}_{\Delta,0}$$

As before, $\frac{(\sigma\phi+\mathcal{C}(\rho_a))(1-\rho_a)(R-\rho_a)}{(\lambda_+-\rho_a)(\lambda_--\rho_a)+R\kappa\mathcal{C}(\rho_a)}$ is decreasing in ρ_a, so the sign of this term is the same with or without HtM. The amplitude is however ambiguous. For transitory shocks in necessity sectors without HtM, as see in the previous subsection CPI inflation is positive and decreasing. This implies that the interest rate implemented is positive and decreasing which increases the growth rate of HtM demand which implies lower out output gap (output gap converges to 0in the long run): the response is amplified. By contrast, the response would be muted for a permanent shock

The second term summarizes the impact of the change in real rate. As explain previously, this response is always muted with HtM.

The third term corresponds to the difference in demand growth rate between HtM and unconstrained households in response to changes in sectoral prices:

$$-\frac{1}{\left(\mu_+ - \rho_a\right)\left(\mu_- - \rho_a\right)}\frac{\varphi^E}{1-\varphi^L}\frac{\sigma}{\sigma+\psi}\sum_l\left(\left(s^{HtM}_l - \bar{s}_l\right) - \sigma\left(\overline{\partial_e e_l}^{HtM} - \overline{\partial_e e_l}\right)\right)\left(\frac{\left(1-\mu_-\right)\left(R-\mu_-\right)}{\left(\lambda_+ - \mu_-\right)\left(\lambda_- - \mu_-\right) + R\kappa\mathcal{C}(\mu_-)}\mu_-^{t+1} - \frac{\left(1-\rho_a\right)\left(R-\rho_a\right)}{\left(\lambda_+ - \rho_a\right)\left(\lambda_- - \rho_a\right) + R\kappa\mathcal{C}(\rho_a)}\rho_a^{t+1}\right)\tilde{P}^*_{l,0}$$

For transitory shocks, after the first period, there is deflation of necessity goods. If the growth rate of HtM necessary good consumption is relatively higher in response to deflation in the necessity sector ($\left(s_l^{HtM} - \bar{s}_l\right) - \sigma \left(\overline{\partial_e e_l}^{HtM} - \overline{\partial_e e_l}\right) > 0$), the output gap is lower at all dates, which further amplifies the response of the output gap to a transitory necessity shock. This is reversed for close to permanent shocks: in that case there is inflation of necessity goods which reduces the growth rate of HtM demand and implies a relatively higher output gap.