Bargaining with Binary Private Information

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This paper studies bargaining between a seller and a buyer with binary private valuation. The tractability of the setting (with respect to the case of general valuation distribution, see Gul et al., 1986) permits explicitly constructing the full equilibrium set using an intuitive induction argument, providing a simple proof of the Coase conjecture, and obtaining new results: The seller extracts all surplus as she becomes more patient and that the equilibrium outcome converges to the full-information outcome as private information vanishes. We also fully characterize the case where there is a deadline. In this case, if the probability of a high valuation is high enough, the seller always charges a high price, there are trade bursts at the outset and the deadline, and trade occurs at a constant rate in between.

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1 Introduction

In a seminal paper, Gul et al. (1986) studied the problem faced by a monopolist selling durable goods over time to a mass of buyers with different valuations. They showed that the Coase (1972)'s conjecture holds when there is a gap between the seller's cost and the support of buyers' valuations: As the frequency with which the monopolist can change prices increases, all equilibrium prices converge to the lowest buyer valuation. The same result applies in a setting where a seller and a single buyer bargain over the price of a durable good, the buyer has private information on his valuation, and the seller makes price offers over time.

An important part of the subsequent bargaining literature has focused on cases where demand is binary. Binary demand has been extensively used in the study of dynamic monopolist markets with arriving buyers (Sobel, 1991), bargaining in decentralized markets (Lauermann and Wolinsky, 2016), bargaining with arriving buyers (Kaya and Kim, 2018), revenue management (Dilmé and Li, 2019), bargaining with news (Daley and Green, 2020), bargaining with divisble goods (Gerardi et al., 2022) or repeated bargaining (Kaya and Roy, 2022, and Dilmé, 2023a). The reasons for focusing on the binary valuations are typically simplicity (it provides a minimal, canonical departure from perfect information), solvability (it is sometimes necessary to make the model tractable), and clarity (it allows for more straightforward arguments and closed-form solutions).

This paper studies the dynamic monopoly setting in Gul et al. (1986) for the case where the demand is binary. Even though Gul et al.'s analysis for a general demand accommodates the case of a binary demand, we believe it is important to study the binary setting in isolation for the following reasons.

First, studying the binary setting permits obtaining an explicit and complete characterization of equilibrium behavior. We can do so without the need to deal with most of the technicalities required to address the general case, which often cloud the exposition. Even though our proof is not short, it is intuitive and fully self-contained. The key step in our construction consists of pinning down the largest range of seller's prior beliefs about the buyer's valuation where, in all equilibria, the seller offers a low price in the first period, and the buyer accepts it immediately. From this first range, one can inductively identify subsequent ranges of priors where, whenever the prior lays within a given range, the seller offers a given price for sure in the first period for sure in all equilibria. The Coase conjecture follows from a simple argument using the closed-form expressions describing the equilibrium outcome.

Second, the tractability of the analysis permits studying the case where the seller's and buyer's discount rates differ and obtaining new comparative-statics results. We show that, in the limit where the seller gets more patient, the Coasian forces weaken, so the seller extracts the full trade surplus from the buyer. Similarly, as the prior degenerates towards a high valuation, the equilibrium outcome coincides with the unique outcome of the perfect information game, so the buyer fails to benefit from reputation effects. We also obtain the seller's optimal pricing under commitment, which is generically deterministic and features price discrimination when the seller is more patient than the buyer.

Third, we use our analysis and results to obtain the equilibrium dynamics for the case horizon is finite, hence we are able to compare the infinite and finite horizon cases within the same framework. Again, we provide a complete characterization of equilibrium behavior, now using a backward induction argument from the deadline. We show that, as the frequency with which offers are made increases, the equilibrium outcome is determined by a threshold in the seller's prior about the buyer's valuation. If the seller's prior belief about the buyer having a high valuation is lower than the threshold, the Coasian outcome – where trade occurs immediately at a price equal to the low valuation – is the unique equilibrium outcome. Alternatively, if the seller assigns a probability to the buyer having a high valuation higher than the threshold, the unique equilibrium outcome is drastically different. Unlike the no-gap case (see Fuchs and Skrzypacz, 2013, and Dilmé, 2023b), there is a burst of trade at the outset at a price equal to the high buyer's valuation. Then, at all times before the deadline, the price equals the high buyer's valuation, and trade occurs at a constant rate. Finally, there is another trade burst at the deadline, also at a price equal to the high buyer's valuation.

Overall, we provide a complete analysis of monopoly pricing with binary demand. Our analysis sheds light on a central result that permeates a large body of literature and permits obtaining new results. We hope our analysis can be used beyond the classical setting, for example, in bargaining with news or endogenous types.

Structure of the paper: Section 2 presents the model. Section 3 and 4 analyze the infinite-horizon and finite-horizon cases, respectively. Section 5 concludes. The Appendix contains the proofs of all results.

2 Model

A seller and a buyer bargain over time over the price of an indivisible durable good.[1] The buyer's private valuation for the good is θ, which is either high (h), with probability $\phi_0 \in (0,1)$, or low (ℓ), with probability $1-\phi_0$, with $h > \ell > 0$.

Time is discrete, $t \in \mathcal{T} := \{0,1,...,T\}$, where either $T = +\infty$ (infinite horizon) or $T \in \mathbb{Z}_+$ (finite horizon). In each period, the seller offers a price p from $\mathcal{P} := [0,h]$.[2] Then, the buyer accepts, in which case the game ends, or rejects, in which case the game continues. The discount factors of the seller and the buyer are $\delta_s \equiv e^{-r_\text{s}\Delta}$ and $\delta_b \equiv e^{-r_\text{b}\Delta}$, respectively, where $r_\text{s}, r_\text{b} > 0$ are interpreted as their discount rates and $\Delta > 0$ is interpreted as the length of the period. If the buyer accepts price p in period t, he gets $\delta_b^t(\theta - p)$, and the seller gets $\delta_s^t p$. They both get 0 if there is the buyer never accepts. If trade occurs in period t, we will sometimes say it occurs at "physical time $t\Delta$."

2.1 Strategies and equilibrium concept

For each $t \in \mathcal{T}$, a *t-history* of the game is a sequence of prices $p^t \equiv (p_0, ..., p_{t-1}) \in \mathcal{P}^t$. We let $H := \cup_{t=0}^{T} \mathcal{P}^t$ be the set of histories. A *strategy of the seller* is a map from each history to a distribution over price offers, $\pi : H \to \Delta(\mathcal{P})$. A *strategy of the θ-buyer* is a map from each history and price to a probability of accepting the offer, $\alpha_\theta : H \times \mathcal{P} \to [0,1]$.

A *belief system* is a map $\phi : H \to [0,1]$, where $\phi(p^t)$ is interpreted as the probability the seller assigns to the buyer's valuation being h at history p^t. An assessment is a pair composed of a strategy profile and a belief system. Given an assessment and a history, the continuation payoff of the seller and the buyer after any history is given, respectively, by

$$C(p^t; \pi, \alpha_\ell, \alpha_h, \phi) := \phi(p^t)\, \mathbb{E}\left[\delta_s^{\tilde{\tau}-t}\, \tilde{p}_{\tilde{\tau}} \middle| p^t; \pi, \alpha_h\right] + (1 - \phi(p^t))\, \mathbb{E}\left[\delta_s^{\tilde{\tau}-t}\, \tilde{p}_{\tilde{\tau}} \middle| p^t; \pi, \alpha_\ell\right]$$

[1] Unlike Gul et al. (1986), and following the convention in most of the bargaining literature, we analyze a bargaining setting with a seller and a privately informed buyer. As pointed out in Ausubel and Deneckere (1989), the model is mathematically equivalent to one where a monopolist sells to an atomless market.

[2] The assumption that the set of prices is bounded is necessary for a well-defined payoff for any strategy profile. Even if $\mathcal{P}$ is larger, only prices in the range $[0,h]$ are relevant for studying equilibrium behavior.

and, for each $\theta \in \{\ell, h\}$,

$$V_\theta(p^t;\pi,\alpha_\theta) := \mathbb{E}\left[\delta_s^{\tilde{\tau}-t}\left(\theta - \tilde{p}_{\tilde{\tau}}\right) \middle| p^t;\pi,\alpha_\theta\right] .$$

We will look for *perfect Bayesian equilibria*, referred to as just equilibria.

Definition 2.1. A *perfect Bayesian equilibrium* is a triple (π,α,ϕ) satisfying that

1. For all $p^t \in H$, π maximizes $C(p^t;\hat{\pi},\alpha_\ell,\alpha_h,\phi)$ among all seller strategies $\hat{\pi}$.
2. For all $p^t \in H$ and $\theta \in \{\ell,h\}$, α_θ maximizes $V_\theta(p^t;\pi,\hat{\alpha}_\theta)$ among all buyer strategies $\hat{\alpha}_\theta$.
3. For all $p^t \in H$ and $p_t \in \mathcal{P}$, ϕ is updated according to Bayes rule whenever possible, that is,

$$\phi(p^t,p_t) = \frac{\phi(p^t)\left(1-\alpha_h(p_t|p^t)\right)}{\phi(p^t)\left(1-\alpha_h(p_t|p^t)\right) + \left(1-\phi(p^t)\right)\left(1-\alpha_\ell(p_t|p^t)\right)}$$

   whenever the denominator is positive, and also satisfies $\phi(\emptyset) = \phi_0$.

3 Infinite-horizon

In this section, we study the case where $T = +\infty$, which is the case studied in Gul et al. (1986). As the introduction explains, while the binary demand is covered by Gul et al.'s analysis, we believe studying the binary case in isolation is worthwhile for different reasons. For binary demand, our characterization is more general than Gul et al.'s in that it allows the discount factors of the seller and the buyer to be different. We first present some preliminary results, then the equilibrium characterization, then some comparative statics results (and, in particular, the Coase conjecture), and finally, we characterize the optimal pricing of a seller with commitment.

3.1 Preliminary results

We now present two preliminary results, which are standard in the literature on bargaining with asymmetric information. We first establish the standard "Diamond paradox," which states that a dynamic monopolist never sets a price below the lowest valuation of the buyer. We use a different (but equivalent) formulation to the standard one.

Lemma 3.1. *A price strictly lower than ℓ is accepted for sure in any equilibrium and history.*

An immediate implication of Lemma 3.1 is that the seller never offers a price $p_t > \ell$ in equilibrium because such a price is accepted for sure and while, for example, offering $(p_t +$

$\ell)/2 > p_t$ (which is also accepted for sure) is a profitable deviation. We next establish the standard "skimming property," which states that if a buyer is willing to accept an (on- or off-path) offer, then a buyer with a higher valuation is strictly willing to accept it. For binary demand, this result follows trivially from the Diamond's paradox. The implication is that, along any on- or off-path history, the seller's posterior decreases unless trade occurs for sure in a given period; that is, for any history p^t such that $p_t > \ell$, we have $\phi(p^t) \leq \phi(p^{t-1})$.

Lemma 3.2. *In any equilibrium, if an on- or off-path price offer is accepted with positive probability by the ℓ-buyer, then it is accepted for sure by the h-buyer.*

3.2 Equilibrium characterization

In this section, we provide a characterization of the equilibrium behavior, which is analogous to Theorem 1 in Gul et al. (1986).

Theorem 3.1. *There exists a strictly increasing sequence $(\hat{\phi}_k, \hat{p}_k)_{k=0}^{\infty}$, with $\hat{\phi}_0 = 0$ and $\hat{p}_0 = \ell$, such that, in any equilibrium, the following holds true:*

1. *If k is such that $\phi_0 \in (\hat{\phi}_k, \hat{\phi}_{k+1})$ then the on-path history is $(\hat{p}_k, \hat{p}_{k-1}, ..., \hat{p}_0)$ and the corresponding beliefs are $(\phi_0, \hat{\phi}_{k-1}, ..., \hat{\phi}_0)$.*
2. *If k is such that $\phi_0 = \hat{\phi}_k$, then the seller mixes between the on-path history $(\hat{p}_k, \hat{p}_{k-1}, ..., \hat{p}_0)$, with corresponding beliefs $(\phi_0, \hat{\phi}_{k-1}, ..., \hat{\phi}_0)$, and the on-path history $(\hat{p}_{k-1}, \hat{p}_{k-2}, ..., \hat{p}_0)$, with corresponding beliefs $(\phi_0, \hat{\phi}_{k-2}, ..., \hat{\phi}_0)$.*

Theorem 3.1 characterizes equilibrium behavior in terms of a sequence $(\hat{\phi}_k, \hat{p}_k)_{k=0}^{\infty}$. This sequence can be easily computed using a simple system of difference equations, presented in Section 3.3. Note that if $\phi_0 \notin \{\hat{\phi}_k\}_{k=1}^{\infty}$, then the equilibrium outcome is unique, while if $\phi_0 \in \{\hat{\phi}_k\}_{k=1}^{\infty}$, then all randomizations between the two described price paths are equilibrium outcomes.

Sketch of the proof

We now provide a sketch of the proof of Theorem 3.1. The reader interested in the Coase conjecture may jump directly to Section 3.3.

The proof proceeds inductively as follows. We let $\hat{\phi}_1 \in [0,1]$ be the largest prior such that whenever $\phi_0 \in [0, \hat{\phi}_1)$, the seller offers ℓ in the first period with probability one in all equilibria.

Note that if $\phi_0 < \hat{\phi}_1$, then the h-buyer accepts for sure any price strictly lower than $\hat{p}_1$, the price that makes him indifferent between accepting in a given period or waiting to accept ℓ the next period (i.e., $\hat{p}_1 > \ell$ satisfies $h - \hat{p}_1 = \delta_b (h - \ell)$). Hence, if $\phi_0 \in [0, \hat{\phi}_1)$, the seller can obtain a payoff arbitrarily close to

$$\phi_0 \, \hat{p}_1 + \delta_s (1 - \phi_0) \, \ell \tag{1}$$

by offering a price slightly below $\hat{p}_1$. Since (1) must be smaller than ℓ for $\phi_0 < \hat{\phi}_1$ (because, by the definition of $\hat{\phi}_1$, there is an equilibrium where ℓ is offered in the first period), we have that $\hat{\phi}_1 \leq \hat{\phi}'_1$, where $\hat{\phi}'_1 := \frac{\ell - \delta_s \ell}{\hat{p}_1 - \delta_s \ell}$ is the prior that makes (1) equal to ℓ. The argument to show that $\hat{\phi}_1 \geq \hat{\phi}'_1$ is more involved. We prove this by showing that, for any ϕ_0 close but higher than $\hat{\phi}_1$ satisfying that an equilibrium where the seller offers a price higher than ℓ exists, the price offered in the first period of such an equilibrium is $\hat{p}_1$ and the seller's payoff is given by expression (1). Given that the seller can obtain arbitrarily close ℓ by offering a price arbitrarily close ℓ, it must be that $\phi_0 \geq \hat{\phi}'_1$, which implies that $\hat{\phi}_1 \geq \hat{\phi}'_1$ as well. The implication is that expression (1) is equal to ℓ when $\phi_0 = \hat{\phi}_1$, that is,

$$\hat{\phi}_1 = \left(1 + \tfrac{1-\delta_b}{1-\delta_s} \tfrac{h-\ell}{\ell}\right)^{-1} . \tag{2}$$

The fact that whenever $\phi_0 < \hat{\phi}_1$, the seller sets a price equal to ℓ in all equilibria serves as an anchor to begin the inductive construction of the equilibrium behavior. In particular, this implies that if the posterior of the seller is strictly below $\hat{\phi}_1$ at some (on- or off-path) history of some equilibrium (i.e., not necessarily in the first period), then the seller offers ℓ for sure at this history.

The next step (and all others) of the inductive argument proceeds similarly. We now let $\hat{\phi}_2 \geq \hat{\phi}_1$ be the maximal posterior satisfying that, whenever $\phi_0 \in (\hat{\phi}_1, \hat{\phi}_2)$, the seller offers $\hat{p}_1$ with probability one in the first period in all equilibria. Let $\hat{p}_2$ be the price that makes the h-buyer indifferent between accepting now or waiting to accept $\hat{p}_1$ next period (i.e., satisfying $h - \hat{p}_2 = \delta_b (h - \hat{p}_1)$). Now, if $\phi_0 \in (\hat{\phi}_1, \hat{\phi}_2)$, and the seller offers a price p_0 slightly lower than $\hat{p}_2$, such price must be accepted a probability such that the posterior of the seller in the second period is $\hat{\phi}_1$. Indeed, if the second period's posterior is strictly above $\hat{\phi}_1$, then the h-buyer strictly benefits from accepting p_0, while if this posterior is strictly below $\hat{\phi}_1$, the h-buyer strictly benefits from rejecting p_0; hence, both cases lead to a contradiction. Therefore, if $\phi_0 \in (\hat{\phi}_1, \hat{\phi}_2)$, the seller can obtain a payoff arbitrarily close to[3]

$$\tfrac{\phi_0-\hat{\phi}_1}{1-\hat{\phi}_1}\,\hat{p}_2+\tfrac{1-\phi_0}{1-\hat{\phi}_1}\,\delta_s\left(\hat{\phi}_1\,\hat{p}_1+\delta_s\,(1-\hat{\phi}_1)\,\ell\right)\,. \tag{3}$$

This is no larger than the payoff the seller obtains from offering a price slightly below $\hat{p}_1$ (hence obtaining (1)) only if $\phi_0 \leq \hat{\phi}_2'$, where $\hat{\phi}_2'$ is the prior that makes (1) and (3) equal, hence $\hat{\phi}_2 \leq \hat{\phi}_2'$. Again, the proof argues that $\hat{\phi}_2 \geq \hat{\phi}_2'$ by showing that, when ϕ_0 is higher but close to $\hat{\phi}_2$, the seller offers $\hat{p}_2$ for sure in the first period in all equilibria; so, $\hat{\phi}_2 = \hat{\phi}_2'$.

Remark 3.1 (Comparison with Gul et al. (1986)'s proof). The reader is advised to read the proof of Theorem 1 in Gul et al. (1986). In our view, their proof is involved and difficult to summarize, as they have to introduce a significant amount of notation, concepts, and intermediate results to address the case of a general demand adequately. Even though some of our steps resemble some of theirs (for example, they begin proving the Diamond's paradox and also that if the residual demand is small enough, then the lowest valuation is offered in the first period in all equilibria), we have not been able to find a clear mapping between the two proofs.

3.3 Comparative statics

In this section, we provide some comparative statics results. The first is the classical Coase conjecture: as $\Delta \to 0$, the physical time it takes for trade to happen shrinks to 0, and the transaction price tends to ℓ. The second result establishes that, as the seller becomes more patient, Coasian forces vanish, and she extracts all trade surplus from the buyer. The third result establishes that Coasian forces also vanish as the seller becomes more convinced that the buyer's valuation is high, and the equilibrium outcome converges to the full-information outcome.

The frequent-offer limit

We now consider the limit as the length of each period vanishes, that is, $\Delta \to 0$. We will prove that the Coase conjecture holds for the binary-demand case, independently of the values of r_{s} or r_{b}.

We first provide an explicit construction of the sequence described in the statement of Theorem 3.1, denoted $(\hat{\phi}_k, \hat{p}_k)_{k=0}^{\infty}$. We use an auxiliary sequence $(\hat{C}_k)_{k=0}^{\infty}$ to do so. We initialize

[3] Note that if the prior is ϕ_0 and the posterior in period 1 is equal to $\hat{\phi}_1$, then the Bayes' rule specifies that $\hat{\phi}_1 = \frac{\phi_0\,\alpha_h(\hat{p}_1)}{\phi_0\,\alpha_h(\hat{p}_1)+1-\phi_0}$, which implies that the probability of trade is $\phi_0\,\alpha_h(\hat{p}_1) = \frac{\phi_0-\hat{\phi}_1}{1-\hat{\phi}_1}$.

$(\hat{\phi}_0, \hat{p}_0, \hat{C}_0) := (0, \ell, \ell)$ and, for each $k \geq 1$, we have

$$\hat{p}_k = (1-\delta_b)\, h + \delta_b\, \hat{p}_{k-1} \ , \ \text{ and} \tag{4}$$

$$\hat{C}_k = \tfrac{\hat{\phi}_k - \hat{\phi}_{k-1}}{1-\hat{\phi}_{k-1}}\, \hat{p}_k + \tfrac{1-\hat{\phi}_k}{1-\hat{\phi}_{k-1}}\, \delta_s\, \hat{C}_{k-1} = \tfrac{\hat{\phi}_k - \hat{\phi}_{k-1}}{1-\hat{\phi}_{k-1}}\, \hat{p}_{k-1} + \tfrac{1-\hat{\phi}_k}{1-\hat{\phi}_{k-1}}\, \hat{C}_{k-1} \ . \tag{5}$$

For each $k \geq 1$ and given $(\hat{\phi}_{k-1}, \hat{p}_{k-1}, \hat{C}_{k-1})$, expressions (4) and (5) uniquely define $(\hat{\phi}_k, \hat{p}_k, \hat{C}_k)$. Condition (4) requires that the h-buyer is indifferent between accepting $\hat{p}_k$ in a given period or accepting $\hat{p}_{k-1}$ the next period (recall the iterative construction of $\hat{p}_k$ described above). Condition (5) requires that, when the prior is $\hat{\phi}_k$, the seller is indifferent between offering a price equal to $\hat{p}_k$ or equal to $\hat{p}_{k-1}$ (recall the statement of Theorem (3.1) and the sketch of the proof). Indeed, the first equality in Condition (5) says that the seller's equilibrium payoff when the posterior is $\hat{\phi}_k$ (which is denoted by $\hat{C}_k$) is obtained if she offers $\hat{p}_k$, which has an acceptance probability by the h-buyer that makes next period's posterior $\hat{\phi}_{k-1}$ and next period's continuation value $\hat{C}_{k-1}$. The second equality in Condition (5) says that the seller can obtain the same payoff by charging $\hat{p}_{k-1}$, in which case the next period's posterior is $\hat{\phi}_{k-2}$.[4]

The following result, which is analogous to Theorem 3 in Gul et al. (1986), shows that the Coase conjecture holds when the demand is binary.

Corollary 3.1 (Coase conjecture). *As $\Delta \to 0$, the price offered in the initial period converges to ℓ, and the physical time it takes for the price to be equal to ℓ shrinks to 0.*

The proof of Corollary 3.1 is simple. Note that, from equations (4) and (5), the following expression can be obtained:

$$1 - \hat{\phi}_k = \frac{h - \hat{C}_k}{h - \frac{\delta_s - \delta_b}{1-\delta_b}\, \hat{C}_{k-1}}\, (1 - \hat{\phi}_{k-1}) = \prod_{k'=1}^{k} \frac{h - \hat{C}_{k'}}{h - \frac{\delta_s - \delta_b}{1-\delta_b}\, \hat{C}_{k'-1}} \ ,$$

where the second equality follows from the iterated use of the first equality. It is clear from equation (4) that, because the cost for the h-buyer of waiting k periods for the price to be ℓ vanishes as $\Delta \to 0$, we have that $\hat{p}_k \to \ell$ as $\Delta \to 0$ for each fixed k. This implies that each $\hat{C}_k$ tends to ℓ as $\Delta \to 0$. We can then use the previous expression to obtain the limit of the belief threshold

[4] By charging $\hat{p}_{k-1}$, the seller obtains $\frac{\hat{\phi}_k - \hat{\phi}_{k-2}}{1-\hat{\phi}_{k-2}}\, \hat{p}_{k-1} + \frac{1-\hat{\phi}_k}{1-\hat{\phi}_{k-2}}\, \delta_s\, \hat{C}_{k-2}$. Using the expression for $\hat{C}_{k-1}$, it is easy to see that this is equal to the expression on the right-hand side of the second equality in equation (5).

for each k as $\Delta \to 0$:

$$\lim_{\Delta \to 0} \hat{\phi}_k = 1 - \left(1 + \frac{r_s}{r_b} \frac{\ell}{h-\ell}\right)^{-k} . \tag{6}$$

Equation (6) illustrates the severity of the commitment problem of the seller: even when Δ is small, she sells with a significantly high probability to the h-buyer just before selling to the ℓ-buyer. In other words, for each ϕ_0, the number of periods before selling to the ℓ-buyer is bounded; hence, the physical time it takes from trade to occur in equilibrium vanishes as $\Delta \to 0$. Then, since we argued that $\hat{p}_k \to \ell$ as $\Delta \to 0$ for all k, the seller sells to the h-buyer at a price very close to ℓ, and so her payoff tends to ℓ.[5]

To obtain further intuition, let t denote the period where trade occurs in equilibrium, so $\phi_{t-1} \in [\hat{\phi}_1, \hat{\phi}_2]$. That is, in period $t-1$, the seller is willing to trade with the h-buyer at price $\hat{p}_1$ and postpone trading with the ℓ-buyer for one period, instead of trading immediately at price ℓ. When Δ is small, $\hat{p}_1 \simeq \ell + (h-\ell)\, r_{\mathrm{b}}\, \Delta$: Because the cost of delaying is small for the buyer, the price he is willing to accept is close to ℓ. Hence, the seller's continuation payoff in period t is approximately equal to

$$(\ell + (h-\ell)\, r_{\mathrm{b}}\, \Delta)\, \phi_{t-1} + (1 - r_{\mathrm{s}}\, \Delta)\, (1 - \phi_{t-1})\, \ell = \ell + \left(r_{\mathrm{b}}\, \phi_{t-1}\, (h-\ell) - r_{\mathrm{s}}\, (1-\phi_{t-1})\, \ell\right) \Delta .$$

We see that both the benefit from selling to the h-buyer in period $t-1$, which is proportional ϕ_{t-1}, and the cost of delaying selling to the ℓ-byter, which is proportional to $1-\phi_{t-1}$, are of order Δ. Hence, price discrimination is beneficial only if ϕ_{t-1} is significantly away from 0 even when Δ is small; formally, we have $\lim_{\Delta \to 0} \hat{\phi}_1 > 0$. As we have argued, this leads to very large acceptance probabilities of the equiliibrium prices by the h-buyer in the periods before the game ends at prices close to ℓ, which leads to the Coase conjecture.

Patient seller

We now study the limit where the seller gets patient while keeping the discount rate of the buyer fixed (note that the limit where the buyer gets patient is trivial: trade occurs immediately at price ℓ). Even when the seller is more patient than the buyer, the seller competes with her future selves, especially if the buyer believes so: If the buyer believes that tomorrow's price is

[5] Formally, for any $\phi_0 \in (0,1)$, there is some $k^*(\phi_0) \in \mathbb{N}$ such that $\phi_0 < \lim_{\Delta \to 0} \hat{\phi}_{k^*(\phi_0)}$. Then, as $\Delta \to 0$, the equilibrium payoff of the seller is no larger $\lim_{\Delta \to 0} \hat{C}_{k^*(\phi_0)} = \ell$.

low, the seller may be induced to offer a low price today. The following result establishes that, as the seller gets patient, such self-fulfilling prophecies do not occur in equilibrium, so Coasian dynamics are averted and the seller obtains the full trade surplus.

Corollary 3.2. *As $r_\mathrm{s} \to 0$, the seller's payoff converges toward $\phi_0 h + (1-\phi_0)\,\ell$.*

To see why Corollary 3.2 holds, note first that if $\phi_0 > \hat{\phi}_k$, the payoff of the seller from offering first a price slightly below $\hat{p}_k$ and then offering ℓ the next period is approximately equal to

$$\tfrac{\phi_0 - \hat{\phi}_{k-1}}{1-\hat{\phi}_{k-1}}\, \hat{p}_k + \tfrac{1-\phi_0}{1-\hat{\phi}_{k-1}}\, \delta_s\, \ell\ . \tag{7}$$

This expression is close to the total trade surplus (which is an upper bound on the seller's equilibrium payoff) when δ_s is close to 1, $\hat{\phi}_{k-1}$ is close to 0, and $\hat{p}_k$ is close to h. Since $\hat{p}_k$ does not depend on r_s, it is readily seen from equation (5) that, for each fixed k, $\hat{\phi}_k - \hat{\phi}_{k-1} \to 0$ as $r_\mathrm{s} \to 0$; that is, the seller finely screens the h-buyer before selling to the ℓ-buyer. Note also that, from equation (2), we have that $\hat{\phi}_1 \to 0$ as $r_\mathrm{s} \to 0$, hence $\lim_{r_\mathrm{s}\to 0} \hat{\phi}_k = 0$ for all k. Hence, for each ϕ_0, the value of k such that $\phi_0 \in [\hat{\phi}_k, \hat{\phi}_{k+1})$ increases towards infinity as r_s shrinks. Similarly, from equation (4), we have that $\hat{p}_k \to h$ as $k \to \infty$. It is then clear that if r_s is small enough, the seller's equilibrium payoff must be close to the total trade surplus.[6][7]

A similar result is obtained in the limit where $\ell \to 0$, that is, in the limit where the "gap" between the seller's cost and the lowest buyer's valuation vanishes. That is, for each $\phi_0 \in (0,1)$ and $\varepsilon > 0$, there is some $\overline{\ell}_\varepsilon > 0$ such that if $\ell < \overline{\ell}_\varepsilon$, then the equilibrium payoff of the seller (in all equilibria) is higher than $\phi_0 h - \varepsilon$. This result is somehow consistent with Ausubel and Deneckere (1989)'s "folk theorem" for the "no gap" case: In our setting, for any $C \in [0, \phi_0 h]$, there is a sequence $(r_\mathrm{s}^n, \ell^n)_n \to (0,0)$ such that the sequence of corresponding seller's equilibrium payoffs converges to C.

[6] Indeed, for all $\varepsilon > 0$, there is some $\overline{r}_\mathrm{s} > 0$ such that, if $r_\mathrm{s} < \overline{r}_\mathrm{s}$, there is some $k(r_\mathrm{s})$ satisfying that $\hat{\phi}_{k(r_\mathrm{s})-1} < \varepsilon$ and $\hat{p}_{k(r_\mathrm{s})-1} > h - \varepsilon$. Then, for all ϕ_0, if r_s is close enough to 0, the seller can obtain a payoff close to the trade surplus by offering $\hat{p}_{k(r_\mathrm{s})-1}$ in the first period and ℓ the second period.

[7] Note that the Coasian logic of competition with one-selves applies before trade occurs: For example, the seller offers price $\hat{p}_1$ (which is smaller than h and independent of r_s) the period before offering ℓ. Still, because $\lim_{r_\mathrm{s}\to 0} \hat{\phi}_1 = 0$, the seller only offers $\hat{p}_1$ in equilibrium when the posterior is very close to 0 if r_s is small, hence Coasian forces do not affect the seller's equilibrium payoff much.

Full-information limit

The next result regards the limit $\phi_0 \to 1$, that is, the limit where the buyer is known to have a high valuation (note that the limit where $\phi_0 \to 0$ is trivial). This limit can be interpreted as the study of a perturbed version of a full-information bargaining model where there is a small-probability behavioral type (who only accepts offers weakly lower than ℓ), in the spirit of reputational bargaining (see Abreu and Gul, 2000). In our setting, as the likelihood of the "tough buyer" vanishes, the seller extracts the full trade surplus from the buyer. In other words, the equilibrium outcome of the asymmetric-information game converges to the outcome of the full-information game (where trade occurs immediately at price h) as the private information vanishes.

Corollary 3.3. *The seller's equilibrium payoff approximates h as $\phi_0 \to 1$.*

Corollary 3.3 follows from the following intuitive argument. First, note that because the buyer discounts the future, $\hat{p}_k \to h$ as $k \to \infty$. Furthermore, for every k and $\phi_0 > \hat{\phi}_k$, the equilibrium payoff of the seller is no smaller than expression (7), which converges to $\hat{p}_k$ as $\phi_0 \to 1$. It then follows that the seller's payoff converges to h as $\phi_0 \to 1$.

3.4 Seller's commitment

In this section, we briefly present the optimal strategy of the seller when she can commit. We do so for completeness and because analyzing the commitment solution helps us understand the severity of the commitment problem in our previous analysis.

To level the field, we focus on a seller who can commit to a pricing strategy instead of a general mechanism. We now consider a game where the seller first commits to a pricing strategy, and then the buyer chooses the best response. We let $\phi^* := \ell/h$.

Theorem 3.2. *Any equilibrium outcome when the seller has commitment power is given by:*

1. *If $\delta_s \leq \delta_b$ then there is only trade in period 0, either at price ℓ (if $\phi_0 < \phi^*$) or at h (if $\phi_0 > \phi^*$), or at a price drawn from $\{\ell, h\}$ (if $\phi_0 = \phi^*$).*
2. *If $\delta_s > \delta_b$ then*
   - *(a) if $\phi_0 < \hat{\phi}_1$ then trade occurs immediately at price ℓ, and*
   - *(b) if $\phi_0 > \hat{\phi}_1$ then some price $p_0 \in (\ell, h)$ is charged in period 0, which is accepted by the h-buyer, and there is some $\bar{t} > 0$ such that ℓ-buyer trades for sure in $\{\bar{t}, \bar{t}+1\}$ at price ℓ.*

(c) *When $\phi_0 = \hat{\phi}_1$ the seller potentially mixes between offering ℓ in the first period or offering $\hat{p}_1$ in the first period and ℓ in the second period.*

The first result is expected: If the seller is more impatient than the buyer, she commits not to intertemporally price discriminate and obtains the static payoff. When $\delta_s = \delta_b$, this follows from the classical result in Stokey (1979). To see why the result holds true also when the seller is more impatient than the buyer, assume for a contradiction that, for some δ_s strictly smaller than δ_b, there is a (weakly) optimal strategy for the seller which involves price discrimination. Then, a more patient seller with a discount factor equal to δ_b would obtain a strictly higher payoff by using the same price-discriminating strategy instead non-price-discriminating strategy prescribed in Theorem 3.2, contradicting Stokey's result. While an impatient seller with commitment fails to capture all trade surplus, she benefits from commitment when $\phi_0 > \phi^*$: In this case, when the seller is weakly more impatient than the buyer, she would prefer to commit not to price discriminate, and the failure to do so implies that she is strictly worse off in the absence of commitment.

When the seller is more patient than the buyer, a seller with commitment power price-discriminates, taking advantage of the higher delay cost of the buyer (see Fudenberg and Tirole, 1983, and Landsberger and Meilijson, 1985). Unlike when $\delta_s \leq \delta_b$, the set of priors where she offers ℓ is independent of the commitment power of the seller (and equal to $[0, \hat{\phi}_1]$). For higher priors, the seller with commitment sells to the h-buyer earlier at a higher price and sells to the ℓ-buyer later in time. From the proof of Theorem 3.2, it is easy to see that, generically in ϕ_0, a time where the seller offers ℓ is deterministic.

4 Finite-horizon

We now consider the finite-horizon case, where the game ends in period $T \in \mathbb{Z}_+$. This setting is a binary version of the model in Fuchs and Skrzypacz (2013), which assumes that the buyer's valuation is distributed according to a power distribution; hence, the distribution is absolutely continuous and exhibits no gap. It is also a generalization of Fudenberg and Tirole (1983), who study a two-period binary setting, to an arbitrarily long horizon.

The following result is analogous to Theorem 3.1 in that it provides a full characterization of equilibrium behavior.

Theorem 4.1. *There exists an increasing sequence $(k_T, \overline{\phi}_T)_{T=0}^{\infty}$, with $\overline{\phi}_T \geq \hat{\phi}_{k_T}$ for all T and $(k_0, \overline{\phi}_0) = (0, \phi^*)$, such that, in any equilibrium, the following holds true:*

1. *If $\phi_0 > \overline{\phi}_T$ then the on-path history is $(h, h, ..., h)$ and the corresponding beliefs are $(\phi_0, \overline{\phi}_{T-1}, ..., \overline{\phi}_0)$.*
2. *If $\phi_0 < \overline{\phi}_T$ then,*
   - *(a) if $\phi_0 \in (\hat{\phi}_{k_T}, \overline{\phi}_T)$ then the on-path history is $(\hat{p}_{k_T}, \hat{p}_{k_T-1}, ..., \ell)$ and the corresponding beliefs are $(\phi_0, \hat{\phi}_{k_T-1}, ..., \hat{\phi}_0)$, and,*
   - *(b) otherwise, the equilibrium is as specified in Theorem 3.1.*
3. *If $\phi_0 = \overline{\phi}_T$ then the seller randomizes between part 1 and part 2.*[8]

Theorem 4.1 establishes that there are two types of equilibrium outcomes. The first is a "Coasian outcome", like the one described in Theorem 3.1: The seller offers a sequence of prices $(\hat{p}_k, \hat{p}_{k-1}, ..., \ell)$, where ℓ may be offered before the deadline is reached. The second type is a "high-price outcome," where the seller maintains a tough bargaining position by offering h at all times. In this case, because the h-buyer does not obtain any surplus from trading, he is indifferent between accepting or rejecting the price h in any of the periods, and he accepts the price with positive probability in all periods before the deadline.

It is not difficult to see that $\lim_{T\to\infty} \overline{\phi}_T = 1$ and $\lim_{T\to\infty} \overline{k}_T = \infty$, hence the equilibrium outcome converges to the outcome of the infinite horizon case as the horizon gets longer. That is, for a given ϕ_0, the equilibrium outcome coincides with that in Theorem 3.1 if T is large enough.

Proof sketch

The proof of Theorem 4.1 is different from that of Theorem 3.1 because finite horizon permits using backward induction from the last period, simplifying each step of the iteration. Still, the nonstationarity of the setting with finite horizon complicates some of the arguments. An additional complication is that, unlike Fuchs and Skrzypacz (2013), depending on the prior and the time horizon, different types of equilibria may arise, some exhibiting Coasian dynamics where trade occurs for sure before the deadline and some where the seller sustains a tough bargaining position until the deadline.

The inductive backward construction of the equilibrium set now begins studying a model with $T = 0$ (i.e., the game has one period). In all equilibria of a one-period game, the seller

[8] That is, if $\phi_0 = \overline{\phi}_T > \hat{\phi}_{k_T}$ then the seller randomizes between the paths 1 and 2(a), and if $\phi_0 = \overline{\phi}_T = \hat{\phi}_{k_T}$, then the seller randomizes between the paths 1 and 2(b).

offers ℓ if $\phi_0 < \phi^*$, h if $\phi_0 > \phi^*$, and mixes between these two price offers if $\phi_0 = \phi^*$ (recall that $\phi^* = \ell/h$). Hence, $\overline{\phi}_0 = \phi^*$ and $k_0 = 0$.

Consider now the second step of the inductive backward construction, where $T = 1$. We first observe that if $\phi_0 > \phi^*$, the seller can ensure a payoff arbitrarily close to

$$\tfrac{\phi_0 - \phi^*}{1-\phi^*}\, h + \delta_s\, \tfrac{1-\phi_0}{1-\phi^*}\, \phi^*\, h \tag{8}$$

by charging a price slightly below h in periods 0 and 1. Indeed, suppose the seller (on- or off-path) offers a price p_0 slightly below h in period 0. In that case, the usual argument applies to show that the posterior in the second period, $\phi_1(p_0)$, is equal to ϕ^*: (i) If the h-buyer rejects p_0 for sure, then $\phi_1(p_0) = \phi_0 > \phi^*$ and so $p_1(p_0) = h$, but this gives the h-buyer a strict incentive to accept $p_0 < h$ in the first period; (ii) if the h-buyer accepts for sure, then $\phi_1(p_0) = 0$ and so $p_1(p_0) = \ell$, but this gives the h-buyer a strict incentive to reject p_0 if p_0 is close enough to h, (iii) so it must be that the buyer is indifferent between accepting p_0 or not, which requires that the seller mixes between ℓ and h in the second period, hence $\phi_1(p_0) = \phi^*$. We then define $\overline{\phi}_1$ as the unique solution of

$$\tfrac{\overline{\phi}_1 - \phi^*}{1-\phi^*}\, h + \delta_s\, \tfrac{1-\overline{\phi}_1}{1-\phi^*}\, \phi^*\, h = \max\left\{\, \ell\,, \overline{\phi}_1\, \hat{p}_1 + \delta_s\,(1-\overline{\phi}_1)\, \ell\,\right\}\ ;$$

that is, if $\phi_0 = \overline{\phi}_1$, then the seller is indifferent between, on one hand, offering a price arbitrarily close to h (and obtaining a payoff arbitrarily close to the left-hand side), and on the other hand, following the Coasian equilibrium by either offering $\hat{p}_1$ (and then ℓ in the second period) or offering ℓ. If $\overline{\phi}_1 < \hat{\phi}_1$, then $\overline{k}_1 = 0$ and, in the first period of all equilibria, either the seller offers ℓ (if $\phi_0 < \overline{\phi}_1$), or offers h (if $\phi_0 > \overline{\phi}_1$), or randomizes between these two prices (if $\phi_0 = \overline{\phi}_1$). If $\overline{\phi}_1 > \hat{\phi}_1$, then $\overline{k}_1 = 1$ and, in the first period of all equilibria, either the seller offers ℓ (if $\phi_0 < \overline{\phi}_1$), or offers $\hat{p}_1$ (if $\hat{\phi}_1 < \phi_0 < \overline{\phi}_1$), or offers h (if $\phi_0 > \overline{\phi}_1$), or uses a corresponding mixing whenever $\phi_0 \in \{\hat{\phi}_1, \overline{\phi}_1\}$. The non-generic case where $\overline{\phi}_1 = \hat{\phi}_1$ is equivalent to the case where $\overline{\phi}_1 < \hat{\phi}_1$ except that $\overline{k}_1 = 1$ and, if $\phi_0 = \overline{\phi}_1$, the seller potentially mixes between ℓ, $\hat{p}_1$, and h in the first period.

The characterization of equilibrium behavior for $T \geq 2$ is analogous.

Frequent offers limit

We now consider the frequent offer limit, as we did in Section 3.3 for the infinite-horizon case. In particular, we are interested in the limit where the length of the period gets smaller (i.e.,

$\Delta \to 0$), but the horizon's physical time converges to some value $\overline{\tau} > 0$: For each Δ, we will consider the model with a finite horizon with number of periods equal to $T_\Delta := \max\{t | t\,\Delta < \overline{\tau}\}$. Hence, while $T_\Delta \to \infty$ as $\Delta \to 0$ (i.e., the number of periods increases to infinity), the physical time until the deadline, $T_\Delta\,\Delta$, converges to $\overline{\tau}$.[9]

Before providing an expression for the limit of $\overline{\phi}_{T_\Delta}$ as $\Delta \to 0$, which we denote by $\overline{\phi}_{\overline{\tau}}$, we make the following observation. Note that, if $\overline{\tau} > 0$ is not too high, it must be that $\overline{\phi}_{\overline{\tau}} \in (\phi^*, 1)$. Indeed, $\overline{\phi}_{\overline{\tau}} = 1$ would imply that, for all $\phi_0 \in (0,1)$, the equilibrium payoff of the seller would be (close to) ℓ for Δ small enough. Nevertheless, the seller has the option of waiting until the deadline and offering h, which gives her a payoff equal to $e^{-r_s \overline{\tau}} \phi_0 h$, which is larger than ℓ if ϕ_0 is high enough and $\overline{\tau}$ is small enough. Similarly, if $\overline{\phi}_{\overline{\tau}} = \phi^*$ (recall that $\overline{\phi}_T \geq \phi^*$ for all T), then there is (almost) no trade between the second period and the deadline if Δ is small enough. This implies that the continuation value of the seller after the first period is close to $e^{-r_s \overline{\tau}} \ell$, but then the seller can profitably deviate by offering ℓ in the second period. The following result establishes that, in fact, $\overline{\phi}_{\overline{\tau}} \in (\phi^*, 1)$ for all $\overline{\tau} \in (0, +\infty)$.

Corollary 4.1. *We have that* $\lim_{\Delta \to 0} \overline{\phi}_{T_\Delta} = \overline{\phi}_{\overline{\tau}}$, *where*

$$\overline{\phi}_{\overline{\tau}} := 1 - (1-\phi^*)\, e^{-r_s \frac{\ell}{h-\ell} \overline{\tau}} \,. \tag{9}$$

Equation (9) is obtained as follows. Note first that if $\phi_0 > \overline{\phi}_{T_\Delta}$ for a small $\Delta > 0$, the continuation payoff after the first period should be close to ℓ. Indeed, note that the posterior in the second period is $\overline{\phi}_{T_\Delta - 1}$, and from the proof of Theorem 4.1 (and from the sketch of the proof above), we can see that the continuation payoff in the second period is equal to

$$\frac{\overline{\phi}_{T_\Delta-1} - \hat{\phi}_{\overline{k}_{T_\Delta-1}-1}}{1-\hat{\phi}_{\overline{k}_{T_\Delta-1}-1}}\, \hat{p}_{\overline{k}_{T_\Delta-1}-1} + \frac{1-\overline{\phi}_{T_\Delta-1}}{1-\hat{\phi}_{\overline{k}_{T_\Delta-1}-1}}\, \delta_s\, \hat{C}_{\overline{k}_{T_\Delta-1}-2}\,,$$

since the seller is indifferent between offering h or offering $\hat{p}_{\overline{k}_{T_\Delta-1}-1}$, in which case the next period's posterior would be $\hat{\phi}_{\overline{k}_{T_\Delta-1}-1}$. A straightforward argument similar to the one used to prove the Coase conjecture (Corollary 3.1) shows that both $\lim_{\Delta \to 0} \hat{p}_{\overline{k}_{T_\Delta-1}-1} = \ell$ and $\lim_{\Delta \to 0} \hat{C}_{\overline{k}_{T_\Delta-1}-2} = \ell$; hence, the seller's continuation payoff in the second period converges to ℓ as $\Delta \to 0$.[10] That

[9]It is not difficult to see that, as $\Delta \to 0$ while keeping T fixed, the buyer purchases at a price increasingly close to the static model (with $T = 0$) in equilibrium.

[10]Note that the claim that $\lim_{\Delta \to 0} \hat{p}_{\overline{k}_{T_\Delta-1}-1} = \ell$ and $\lim_{\Delta \to 0} \hat{C}_{\overline{k}_{T_\Delta-1}-2} = \ell$ does not follow directly from Corollary 3.1 because $\overline{k}_{T_\Delta-1}$ could potentially increase towards infinity as $\Delta \to 0$. Nevertheless, from equation (6), we see

is, if $\phi_0 > \overline{\phi}_{\overline{\tau}}$ and $\Delta > 0$ small enough, the seller sells with a probability approximately equal to $\frac{\phi_0 - \overline{\phi}_{\overline{\tau}}}{1-\overline{\phi}_{\overline{\tau}}}$ in the first period, and her continuation payoff is approximately equal to ℓ. Hence, as $\Delta \to 0$, her equilibrium payoff converges to

$$\frac{\phi_0 - \overline{\phi}_{\overline{\tau}}}{1-\overline{\phi}_{\overline{\tau}}}\, h + \frac{1-\phi_0}{1-\overline{\phi}_{\overline{\tau}}}\, \ell \; .$$

After the initial trade burst, the seller's continuation payoff stays close to ℓ. It must then be that, in equilibrium, the h-buyer accepts h at a slow rate so that the posterior at physical time τ is equal to $\overline{\phi}_{\overline{\tau}-\tau}$ (note that $\overline{\tau} - \tau$ is the physical time remaining until the deadline). Hence, it must be that, for $\varepsilon > 0$ small enough, we have

$$\ell = \frac{\overline{\phi}_{\overline{\tau}} - \overline{\phi}_{\overline{\tau}-\varepsilon}}{1-\overline{\phi}_{\overline{\tau}-\varepsilon}}\, h + e^{-r_s \varepsilon}\, \frac{1-\overline{\phi}_{\overline{\tau}}}{1-\overline{\phi}_{\overline{\tau}-\varepsilon}}\, \ell + o(\varepsilon) \;\; \Rightarrow \;\; r_s\, \ell = \frac{h-\ell}{1-\overline{\phi}_{\overline{\tau}}}\, \frac{\mathrm{d}}{\mathrm{d}\overline{\tau}} \overline{\phi}_{\overline{\tau}} \; . \tag{10}$$

Hence, the probability of trade in a per unit of physical time is $\lim_{\varepsilon \to 0} \frac{\overline{\phi}_{\overline{\tau}} - \overline{\phi}_{\overline{\tau}-\varepsilon}}{1-\overline{\phi}_{\overline{\tau}-\varepsilon}} = r_s\, \ell / h$. The solution to the differential equation on the right side of (10) with boundary condition $\overline{\phi}_0 = \phi^*$ is (9). Therefore, if $\phi_0 > \overline{\phi}_{\overline{\tau}}$, there is an initial trade burst (where the posterior jumps from ϕ_0 to $\overline{\phi}_{\overline{\tau}}$), a constant rate of trade at all times between the beginning and the deadline (where the posterior slowly declines, being equal to $\overline{\phi}_{\overline{\tau}-\tau}$ at physical time τ), and a trade burst at the deadline (where trade occurs with probability ϕ^*).[11]

It is easy to see that, as one could expect, the range of priors where the seller offers h in equilibrium (for a fixed $\overline{\tau}$), which is $[1 - \overline{\phi}_{\overline{\tau}}, 1]$, shrinks when the seller is more impatient, the low valuation increases, or when the high valuation decreases.

We briefly compare Corollary 4.1 with Fuchs and Skrzypacz (2013) and Dilmé (2023b).[12] A first important difference is that we obtain trade bursts both at the outset and the deadline, while they only find it at the deadline. Such a trade burst is the source of most of the surplus the seller obtains from trade (after that, her continuation payoff is ℓ). In fact, in both of their models, the seller obtains in equilibrium the same payoff she would get from waiting until

that that is not the case: for a fixed ϕ_0, trade occurs for sure in a bounded number of periods (with a bound independent of Δ) under the Coase outcome.

[11] Note that the rate of trade (not conditioning on the buyer's valuation) at times between the beginning and the deadline must be indeed constant: From equation (10), the probability of trade between τ and $\tau + \varepsilon$ is $\frac{\ell}{h}\, r_s\, \varepsilon + o(\varepsilon)$.

[12] Recall that, like Fuchs and Skrzypacz (2013), Dilmé (2023b) studies a versions of Gul et al. (1986)'s setting with a deadline. Unlike Fuchs and Skrzypacz's, Dilmé's model is set directly in continuous-time, allows for a general absolutely continuous distribution (within the no-gap case), and also allows for different discount rates.

the deadline and charging the monopolist price, but the seller obtains a higher payoff in our model: It is easy to see that our seller obtains more than $e^{-r_s \overline{\tau}} \phi_0 h$ as she gets to sell to the h-buyer at price h before the deadline with positive probability in equilibrium. In some sense, rather paradoxically, the commitment problem of the seller is worse in their settings, where there is no gap, than in ours, where there is a gap. Note also that, in both our model and Dilmé's, the seller's payoff is independent of r_b (Fuchs and Skrzypacz, 2013, only consider the case of equal discounting).

5 Conclusions

The Coase conjecture permeates the study of dynamic monopolists and bargaining with asymmetric information. Its counter-intuitiveness contrasts with the strength of Coasian forces in numerous settings, which completely determine the equilibrium outcome unless strong countervailing effects are present.[13] Unfortunately, the proof of the Coase conjecture is often clouded with necessary technicalities and complex arguments to cover all possibilities. Hence, it is difficult to comprehend the logic in full.[14]

Our main contribution is deriving a complete characterization of the equilibrium pricing of a monopolist facing a binary demand. Even though binary demand is a particular case of the setting studied in Gul et al. (1986) with a general demand, the simplicity of our arguments and the tractability of our setting help shed light on the logic behind the Coase conjecture. Using this tractability, we are able to provide new results and fully characterize the case of a finite horizon. Our analysis may be helpful in obtaining similar characterizations in similar bargaining settings.

[13] Such countervailing effects may arise from discrete demand (Bagnoli et al., 1989), adverse selection (Deneckere and Liang, 2006), capacity choice (McAfee and Wiseman, 2008), arrival of traders (Fuchs and Skrzypacz, 2010), outside options (Board and Pycia, 2014), or differentiated goods (Nava and Schiraldi, 2019). Note also that Ausubel and Deneckere (1989) shows a "folk theorem" for the no-gap case, where there is no gap between the buyers' lowest valuation and the seller's cost.

[14] Example 1 in Gul et al. (1986) shows that stationary equilibria can be analyzed in the uniform demand case in a tractable way (see Güth and Ritzberger, 1998, for a detailed analysis of this case). Still, because there is no gap between the buyers' lowest valuation and the seller's cost, such an example fails to illustrate why the Coase conjecture holds in all equilibria. To our knowledge, no tractable example under the gap case has been studied.

A Proofs

Proof of Lemmas 3.1 and 3.2

Proof. Even though the proofs of Lemmas 3.1 and 3.2 are standard, we include them here for completeness. To prove Lemma 3.1, let $\underline{p}$ be the infimum of all prices that are optimal for the seller in some equilibrium at some history. For the sake of contradiction, assume that $\underline{p}<\ell$. Consider a history and an equilibrium where it is optimal for the seller to offer $\underline{p}+\varepsilon$, for $\varepsilon\geq 0$ small enough that both types of the buyer strictly prefer accepting $\underline{p}+\varepsilon$ this period than $\underline{p}$ next period, that is, satisfying that $\varepsilon\in[0,(1-\delta_b)\,(\ell-\underline{p}))$ (which exists by the definition of $\underline{p}$ and the assumption that $\underline{p}<\ell$). Then, the seller can profitably deviate by setting a price slightly above $\underline{p}+\varepsilon$ that keeps both types of the buyer strictly willing to accept it, a contradiction. Hence, since no price strictly lower than ℓ is offered in equilibrium, if the seller deviates and offers a price strictly lower than ℓ, such a price is accepted for sure.

Lemma 3.2 follows immediately from Lemma 3.1: If an on- or off-path price offer is accepted with positive probability by the ℓ-buyer, such price offer should be weakly lower than ℓ, but then it is accepted for sure by the h-buyer because it is equal or lower than the infimum price the seller offers under any equilibrium. □

Proof of Theorem 3.1

Proof. **Notation:** We define the correspondence $\mathcal{C}^*:[0,1]\rightrightarrows[\ell,h]$ so that, for each $\phi_0\in[0,1]$, $\mathcal{C}^*(\phi_0)$ denotes the set of seller's equilibrium payoffs for prior ϕ_0. We will show that, in fact, there is a unique seller's equilibrium payoff for each $\phi_0\in[0,1]$. Somehow abusively, for a property "Q" of the set of equilibria, we will use $\mathcal{C}^*(\phi_0|Q)$ to denote the equilibrium payoffs of the among the equilibria satisfying property Q when the prior is ϕ_0.

Induction argument: We will provide a proof by induction over $k=0,1,...$ Our induction hypothesis in the k-th step is the following:

Induction hypothesis for k**:** There is a strictly increasing sequence $(\hat{\phi}_{k'},\hat{p}_{k'})_{k'=0}^{k+1}$, with $(\hat{\phi}_0,\hat{p}_0)=(0,\ell)$, such that the following is true in all equilibria:

1. For all $\phi_0\leq\hat{\phi}_k$, let $k'<k$ be such that $\phi_0\in(\hat{\phi}_{k'},\hat{\phi}_{k'+1}]$. Then:
   (a) If $\phi_0\in(\hat{\phi}_{k'},\hat{\phi}_{k'+1})$ then the on-path history is $(\hat{p}_{k'},\hat{p}_{k'-1},...,\hat{p}_0)$ and the corresponding beliefs are $(\phi_0,\hat{\phi}_{k'-1},...,\hat{\phi}_0)$..

(b) If $\phi_0 = \hat{\phi}_{k'}$ then the seller mixes between the on-path history $(\hat{p}_{k'}, \hat{p}_{k'-1}, ..., \hat{p}_0)$, with corresponding beliefs $(\phi_0, \hat{\phi}_{k'-1}, ..., \hat{\phi}_0)$, and the on-path history $(\hat{p}_{k'-1}, \hat{p}_{k'-2}, ..., \hat{p}_0)$, with corresponding beliefs $(\phi_0, \hat{\phi}_{k'-2}, ..., \hat{\phi}_0)$.

2. For all $\phi_0 > \hat{\phi}_{k+1}$, there is no equilibrium where the seller charges a price strictly below $\hat{p}_{k+1}$ in the first period with positive probability, and the payoff of the h-buyer is weakly below $h - \hat{p}_{k+1}$ in all equilibria.

Part 1: Proof for $k=0$. We first prove that there is a pair $(\hat{\phi}_1, \hat{p}_1)$, satisfying the properties stated in the induction hypothesis. This part illustrates the second part of the proof, which will generalize some arguments to a general k.

We let $\hat{\phi}_1 \in [0,1]$ be the highest prior satisfying that, for all $\phi_0 < \hat{\phi}_1$, trade occurs for sure in the first period at price ℓ in any equilibrium. Fix an equilibrium. Note that, because the seller never offers a price below ℓ at any history of any equilibrium (by the Diamond's paradox), if the seller charges a price strictly below $\hat{p}_1 := (1-\delta_b)\,h + \delta_b\,\ell$ (on- or off-path), the h-buyer accepts such an offer for sure. Therefore, it must be that, for all $\phi_0 < \hat{\phi}_1$,

$$\ell \geq \phi_0\,\hat{p}_1 + \delta_s\,(1-\phi_0)\,\ell \;\;\Rightarrow\;\; \phi_0 \leq \hat{\phi}_1' := \tfrac{(1-\delta_s)\,\ell}{(1-\delta_s)\,\ell + (1-\delta_b)\,(h-\ell)}\;.$$

We then have that $\hat{\phi}_1 \leq \hat{\phi}_1'$.

For each $\varepsilon > 0$, we define

$$\overline{C}_1^{\varepsilon} := \sup\big(\overbrace{\cup_{\phi_0 \in [\hat{\phi}_1, \hat{\phi}_1+\varepsilon]} \mathcal{C}^*(\phi_0 | \Pr(p_0 = \ell) < 1)}^{(*)}\big)\;.$$

Note that, for all $\varepsilon > 0$, $(*)$ is non-empty (by the definition of $\hat{\phi}_1$). We let $\overline{C}_1^0 := \lim_{\varepsilon \searrow 0} \overline{C}_1^{\varepsilon}$, which exists because $\overline{C}_1^{\varepsilon}$ is non-decreasing in ε. We let $(\phi_0^n)_n$ be a sequence decreasing toward $\hat{\phi}_1$ such that there is a corresponding sequence $(\pi^n, \alpha^n, \phi^n)_n$ satisfying that (i) for each n, $(\pi^n, \alpha^n, \phi^n)$ is an equilibrium when the prior is ϕ_0^n satisfying $\pi^n(p_0^n > \ell) > 0$, and (ii) $\lim_{n\to\infty} C_0^n = \overline{C}_1^0$ (where C_0^n is the seller's equilibrium payoff in the n-th equilibrium). Without loss of generality for the argument, we assume that the seller does not randomize in the first period, that is, there is a sequence $(p_0^n)_n$ such that $p_0^n > \ell$ and $\pi^n(p_0^n) = 1$ for all n.[15] There are four possibilities:

1. Assume first, for the sake of contradiction, that there is a strictly increasing sequence

[15]The reason is that if an equilibrium where the seller offers a given p_0 with positive probability in the first period exists, then there is an equilibrium where the seller offers p_0 for sure in the first period.

$(n_m)_m$ such that $\pi^{n_m}(p_1^{n_m} > \ell | p_0^{n_m}) > 0$ for all m (i.e., the seller offers a price above ℓ with positive probability in the second period after offering $p_0^{n_m}$ in the first period). This implies that $\phi_1^{n_m}(p_0^{n_m}) \in [\hat{\phi}_1, \phi_0^{n_m}]$ (i.e., after the seller offers $p_0^{n_m}$ (on-path), the second period's posterior is in $[\hat{\phi}_1, \phi_0^{n_m}]$). The payoff of the seller is then

$$C_0^{n_m} = \frac{\phi_0^{n_m} - \phi_1^{n_m}(p_0^{n_m})}{1-\phi_1^{n_m}(p_0^{n_m})}\, p_0^{n_m} + \frac{1-\phi_0^{n_m}}{1-\phi_1^{n_m}(p_0^{n_m})}\, \delta_s\, C_1^{n_m}(p_0^{n_m}) \ .$$

By assumption, the left-hand side tends to $\overline{C}_1^0$ as $m \to \infty$. The first term on the right-hand side tends to 0 because $\phi_0^{n_m} \to \hat{\phi}_1$ as $m \to \infty$, hence $\phi_1^{n_m}(p_0^{n_m}) \to \hat{\phi}_1$ as $m \to \infty$ as well. Since $C_1^{n_m}(p_0^{n_m}) \le \overline{C}_1^{\varepsilon}$ for $\varepsilon = \phi_1^{n_m}(p_0^{n_m}) - \hat{\phi}_1$, we have that

$$\overline{C}_1^0 = \lim_{m\to\infty} C_0^{n_m} = \delta_s \lim_{m\to\infty} C_1^{n_m}(p_0^{n_m}) \le \delta_s\, \overline{C}_1^0 \ ,$$

which implies that $C_1^0 \le 0$. This is a contradiction because $\overline{C}_1^0 \ge \ell > 0$.

2. Assume now, again for the sake of contradiction, that there is a strictly increasing sequence $(n_m)_m$ such that $p_0^{n_m} > \hat{p}_1$ for all m. Since, from the previous result, we have that $\pi^{n_m}(p_1^{n_m} > \ell | p_0^{n_m}) = 0$ if m is large enough, we have that the h-buyer accepts with probability zero the first price, and so $\phi_1^{n_m}(p_0^{n_m}) = \phi_0^{n_m}$ for m large enough. Now, we have $C_0^{n_m} = \delta_s\, C_1^{n_m}(p_0^{n_m})$, which again implies that $\overline{C}_1^0 \le \delta_s\, \overline{C}_1^0$, a contradiction.

3. Assume, again for the sake of contradiction, that there is a strictly increasing sequence $(n_m)_m$ such that $p_0^{n_m} \in (\ell, \hat{p}_1)$ for all m. We argued that in this case, by the Diamond's paradox, the h-buyer accepts the first offer for sure for all m. Nevertheless, the seller can profitably deviate by offering a price in $(p_0^{n_m}, \hat{p}_1)$ (which is accepted for sure by h-buyer by the Diamond's paradox), a contradiction.

4. The only possibility left is that $p_0^n = \hat{p}_1$ and $\pi^n(p_1^n > \ell | p_0^n) = 0$ for n large enough. The payoff of the seller is then

$$C_0^n = \phi_0^n\, \hat{p}_1 + (1-\phi_0^n)\, \delta_s\, \ell \ \ \Rightarrow \ \ \overline{C}_1^0 = \hat{\phi}_1\, \hat{p}_1 + \delta_s\, (1-\hat{\phi}_1)\, \ell \ .$$

Since it must be that $\overline{C}_1^0 \ge \ell$, we have that $\hat{\phi}_1 \ge \hat{\phi}_1'$ (recall that $\hat{\phi}_1'\, \hat{p}_1 + \delta_s\, (1-\hat{\phi}_1')\, \ell = \ell$).

We then conclude that, since $\hat{\phi}_1 \le \hat{\phi}_1'$ and $\hat{\phi}_1 \ge \hat{\phi}_1'$, we have $\hat{\phi}_1 = \hat{\phi}_1'$. Note finally that if $\phi_0 > \hat{\phi}_1$, then the seller has the option of offering a price slightly below $\hat{p}_1$, which ensures that the h-buyer accepts the price for sure. The arguments above show that if $\phi_0 > \hat{\phi}_1$ then it is strictly suboptimal for the seller to offer a price strictly lower than $\hat{p}_1$. Furthermore, it easily follows from the previous arguments that if $\phi_0 = \hat{\phi}_1$, all equilibria begin with a (possibly degenerated) randomization between offering ℓ and $\hat{p}_1$, and for all mixing probabilities, there is an equilibrium with such a mixing probability in the first period.

Part 2: Proof for $k \geq 1$. We now assume that the statement of Theorem 3.1 hypothesis holds for $k-1$, and we will prove it holds for k. We let $\hat{\phi}_{k+1}$ be the posterior such that, for all $\phi_0 \in (\hat{\phi}_k, \hat{\phi}_{k+1})$, the seller offers $\hat{p}_k$ in the first period for sure and the continuation value for the h-buyer is $h - \hat{p}_k$ in all equilibria (we let $\hat{\phi}_{k+1} := \hat{\phi}_k$ if such a posterior does not exist). We divide this part of the proof into four subparts.

Part 2.1. Assume $\phi_0 \in (\hat{\phi}_k, \hat{\phi}_{k+1})$ and fix an equilibrium (π, α, ϕ) (hence $\pi(p_0 = \hat{p}_k) = 1$). We show that the seller's equilibrium payoff is at most

$$\tfrac{\phi_0 - \hat{\phi}_{k-1}}{1 - \hat{\phi}_{k-1}} \, \hat{p}_k + \tfrac{1 - \phi_0}{1 - \hat{\phi}_{k-1}} \, \delta_s \, \hat{C}_{k-1} \ , \tag{11}$$

where $\hat{C}_{k-1}$ is the seller's payoff in any equilibrium when the prior is $\hat{\phi}_{k-1}$ (which satisfies (5) and $\hat{C}_0 = \ell$). We divide the argument into three cases:

1. We first assume, for the sake of contradiction, that $\phi_1(\hat{p}_k) < \hat{\phi}_{k-1}$. In this case, by the induction hypothesis, the h-buyer's continuation payoff in the second period is at least $h - \hat{p}_{k-2}$, but then it has a strict incentive to reject $\hat{p}_k$, contradicting that $\phi_1(\hat{p}_k) < \hat{\phi}_{k-1} < \phi_0$.
2. Assume now, again for the sake of contradiction, that $\phi_1(\hat{p}_k) \in (\hat{\phi}_k, \hat{\phi}_{k+1})$. In this case, the h-buyer's continuation payoff in the second period is $h - \hat{p}_k$, but then he has a strict incentive to accept $\hat{p}_k$ in the first period, a contradiction.
3. Assume finally that $\phi_1(\hat{p}_k) \in [\hat{\phi}_{k-1}, \hat{\phi}_k]$. In this case, because the h-buyer accepts $\hat{p}_k$ with a non-degenerated probability, his continuation payoff in the second period should be $h - \hat{p}_{k-1}$, which using the induction hypothesis implies that $\pi(p_1 = \hat{p}_{k-1} | \hat{p}_k) = 1$, and hence $\phi_2(\hat{p}_k, \hat{p}_{k-1}) = \hat{\phi}_{k-2}$. The seller's equilibrium payoff is then given by

$$\tfrac{\phi_0 - \phi_1(\hat{p}_k)}{1 - \phi_1(\hat{p}_k)} \, \hat{p}_k + \tfrac{1 - \phi_0}{1 - \phi_1(\hat{p}_k)} \, \delta_s \Big(\underbrace{\tfrac{\phi_1(\hat{p}_k) - \hat{\phi}_{k-2}}{1 - \hat{\phi}_{k-2}} \, \hat{p}_{k-1} + \tfrac{1 - \phi_1(\hat{p}_k)}{1 - \hat{\phi}_{k-2}} \, \delta_s \, \hat{C}_{k-2}}_{(*)} \Big) \ . \tag{12}$$

   The derivative of the previous expression with respect to $\phi_1(\hat{p}_k)$ is

$$-\tfrac{(1 - \phi_0)}{(1 - \phi_1(\hat{p}_k))^2} \, (\hat{p}_k - \delta_s \, \hat{p}_{k-1}) < 0 \ .$$

Hence, the seller's equilibrium payoff is bounded by expression (12) evaluated at $\phi_1(\hat{p}_k) = \hat{\phi}_{k-1}$, which is equal to expression (11) (note that $(*)$ in expression (12) is equal to $\hat{C}_{k-1}$ when $\phi_1(\hat{p}_k) = \hat{\phi}_{k-1}$).

Part 2.2. We now argue that, in any equilibrium and for any $\phi_0 > \hat{\phi}_k$, a payoff arbitrarily close to expression (11) can be achieved by charging a price slightly below $\hat{p}_k$. Indeed, assume that the seller offers $\hat{p}_k - \varepsilon$, for a small $\varepsilon > 0$. If the h-buyer rejects the offer for sure, then $\phi_1(\hat{p}_k - \varepsilon) > \hat{\phi}_k$. Still, the continuation payoff of the h-buyer from rejecting is at most $\delta_b(h - \hat{p}_k)$ (by the induction hypothesis), which is smaller than $h - (\hat{p}_k - \varepsilon)$ (i.e., the payoff of accepting $\hat{p}_k - \varepsilon$ in the first period) if ε is small enough, contradicting the incentive to reject the first offer. Alternatively, the h-buyer cannot be strictly willing to accept $\hat{p}_k - \varepsilon$, since otherwise, the next period's price is ℓ, making rejection a strictly profitable deviation. Hence, the h-buyer must be indifferent between accepting and rejecting $\hat{p}_k - \varepsilon$, and so his continuation payoff in the second period should be $\delta_b^{-1}(h - (\hat{p}_k - \varepsilon))$. From the definition of $\hat{p}_k$ we have that, if ε is small enough, such a continuation payoff is in $(h - \hat{p}_{k-1}, h - \hat{p}_{k-2})$, so it must be that $\phi_1(\hat{p}_k - \varepsilon) = \hat{\phi}_{k-1}$.

An implication is that, since we argued that (11) is an upper bound on the seller's payoff when $\phi_0 \in [\hat{\phi}_k, \hat{\phi}_{k+1})$, and now we obtain that it is also a lower bound, we have that (11) is the unique equilibrium seller's payoff in this range of prior beliefs. It is only left to obtain the value of $\hat{\phi}_{k+1}$.

Part 2.3. Define $\hat{p}_{k+1} := (1 - \delta_b)h + \delta_b \hat{p}_k$. Note that if $\phi_0 \in (\hat{\phi}_k, \hat{\phi}_{k+1})$ and the seller offers a price $p_0 \in (\hat{p}_k, \hat{p}_{k+1})$ then, necessarily, it must be that $\phi_1 = \hat{\phi}_k$. Indeed, if $\phi_1 > \hat{\phi}_k$ then the continuation value of the h-buyer is at most $\delta_b(h - \hat{p}_k) < h - p_0$, contradicting that $\phi_1 > \hat{\phi}_k$; while if $\phi_1 < \hat{\phi}_k$ then the continuation value of the h-buyer is larger than $\delta_b(h - \hat{p}_k) > h - p_0$, contradicting that $\phi_1 < \hat{\phi}_k$. This implies that a lower bound on the seller's equilibrium payoff when $\phi_0 \in (\hat{\phi}_k, \hat{\phi}_{k+1})$ is

$$\tfrac{\phi_0 - \hat{\phi}_k}{1 - \hat{\phi}_k} \hat{p}_{k+1} + \tfrac{1 - \phi_0}{1 - \hat{\phi}_k} \delta_s \hat{C}_k \, . \tag{13}$$

The previous expression increases in ϕ_0 faster than expression (11). Indeed, both expressions are linear in ϕ_0, expression (13) is higher than expression (11) when $\phi_0 = 1$, and expression (13) is lower than expression (11) when $\phi_0 = \hat{\phi}_k$.[16] It is then the case that offering $\hat{p}_k$ in period 1 is optimal only if $\phi_0 \leq \hat{\phi}'_{k+1}$, where $\hat{\phi}'_{k+1} > \hat{\phi}_k$ is the unique ϕ_0 which makes expression (13) equal

[16] Note that, for $\phi_0 = \hat{\phi}_k$, (11) is equal to $\hat{C}_k$ and (13) is equal to $\delta_s \hat{C}_k < \hat{C}_k$.

to (11). Hence, we have $\hat{\phi}_{k+1} \le \hat{\phi}'_{k+1}$.

Part 2.4. As in Part 1 of this proof, define

$$\overline{C}^{\varepsilon}_{k+1} := \sup\left\{ \cup_{\phi_0 \in [\hat{\phi}_{k+1}, \hat{\phi}_{k+1}+\varepsilon]} \mathcal{C}^*(\phi_0 | p_0 > \hat{p}_k) \right\}$$

for all $\varepsilon > 0$, and we let $\overline{C}^0_{k+1} := \lim_{\varepsilon \searrow 0} \overline{C}^{\varepsilon}_{k+1}$. We let $(\phi_0^n)_n$ be a decreasing sequence converging to $\hat{\phi}_{k+1}$ with a corresponding sequence $(\pi^n, \alpha^n, \phi^n)_n$ satisfying that (i) for all n, $(\pi^n, \alpha^n, \phi^n)$ is an equilibrium when the prior is ϕ_0^n satisfying $\pi^n(p_0^n > \hat{p}_k) > 0$, and (ii) $\lim_{n \to \infty} C_0^n = \overline{C}^0_{k+1}$. Note that, without loss of generality for the argument, we can assume that the seller plays a pure strategy (i.e., $\pi^n(p_0^n) = 1$ for some $p_0^n > \hat{p}_k$) for all n (see Footnote 15). We now consider five cases, which are analogous to the four cases in Part 1 with one extra case:

1. Assume, for the sake of contradiction, that there is a strictly increasing sequence $(n_m)_m$ such that $\pi^{n_m}(p_1^{n_m} > \hat{p}_k | p_0^{n_m}) > 0$ for all m. By the induction hypothesis, this implies that $\phi_1^{n_m}(p_0^{n_m}) \in [\hat{\phi}_{k+1}, \phi_0^{n_m}]$. The payoff of the seller is

$$C_0^{n_m} = \frac{\phi_0^{n_m} - \phi_1^{n_m}(p_0^{n_m})}{1 - \phi_1^{n_m}(p_0^{n_m})} p_0^{n_m} + \frac{1 - \phi_0^{n_m}}{1 - \phi_1^{n_m}(p_0^{n_m})} \delta_s C_1^{n_m}(p_0^{n_m}) \; .$$

   By assumption, the left-hand side tends to $\overline{C}^0_{k+1}$ as $m \to \infty$. The first term on the right-hand side tends to 0 because $\phi_0^{n_m} \to \hat{\phi}_{k+1}$ as $m \to \infty$, hence $\phi_1^{n_m}(p_0^{n_m}) \to \hat{\phi}_{k+1}$ as $m \to \infty$ as well. Since $C_1^{n_m}(p_0^{n_m}) \le \overline{C}_1^{\varepsilon}$ for $\varepsilon = \phi_1^{n_m}(p_0^{n_m}) - \hat{\phi}_{k+1}$, we have that

$$\overline{C}^0_{k+1} = \lim_{m \to \infty} C_0^{n_m} = \delta_s \lim_{m \to \infty} C_1^{n_m}(p_0^{n_m}) \le \delta_s \overline{C}^0_{k+1} \; ;$$

   hence, $\overline{C}^0_{k+1} \le 0$. This is a contradiction because $\overline{C}^0_{k+1} \ge \ell > 0$.

2. Assume, again for the sake of contradiction, that there is a strictly increasing sequence $(n_m)_m$ such that $\pi^n(p_1^{n_m} < \hat{p}_k | p_0^{n_m}) > 0$ for all m, which implies that $\phi_1^{n_m}(p_0^{n_m}) \le \hat{\phi}_k$ and that the continuation value of the h-buyer in the second period is higher than $h - \hat{p}_k$. This implies that the h-buyer rejects $p_0^{n_m}$ for sure, hence $\phi_1^{n_m}(p_0^{n_m}) = \phi_0^{n_m} > \hat{\phi}_k$, a contradiction.

3. Assume now, again for the sake of contradiction, that there is a strictly increasing sequence $(n_m)_m$ such that $p_0^{n_m} > \hat{p}_{k+1}$ for all m. Since, from the previous result, we have that $\pi^{n_m}(p_1^{n_m} = \hat{p}_k | p_0^{n_m}) = 1$ if m is large enough, we have that the h-buyer accepts with probability zero the first price. Now, we have $C_0^{n_m} = \delta_s C_1^{n_m}(p_0^{n_m})$, which again implies that $\overline{C}^0_{k+1} \le \delta_s \overline{C}^0_{k+1}$, a contradiction.

4. Assume, again for the sake of contradiction, that there is a strictly increasing sequence $(n_m)_m$ such that $p_0^{n_m} \in (\hat{p}_k, \hat{p}_{k+1})$ for all m. Since, from the previous results, we have $\pi^{n_m}(p_1^{n_m} = \hat{p}_k | p_0^{n_m}) = 1$, the h-buyer accepts such price for sure, implying again that $p_1^{n_m}(p_0^{n_m}) = \ell$, and so that the h-buyer is strictly willing to reject $p_0^{n_m}$, a contradiction.

5. The only possibility left is that, if n is large enough, then $p_0^n = \hat{p}_{k+1}$ and $\pi^n(p_1^n = \hat{p}_k | p_0^n) = 1$. Hence, we have that $\overline{C}_{k+1}$ equal to (13). Since, as we argued in Part 2.2, $\overline{C}_{k+1}$ is weakly higher than (11), we have that $\hat{\phi}_{k+1} \geq \hat{\phi}'_{k+1}$.

Overall, we conclude that $\hat{\phi}_{k+1} = \hat{\phi}'_{k+1}$. Hence, in any equilibrium, if $\phi_0 \in (\hat{\phi}_k, \hat{\phi}_{k+1})$, the seller offers $\hat{p}_k$ for sure in the first period (and the continuation play is according to the statement of the induction hypothesis), while if $\phi_0 = \hat{\phi}_{k+1}$, the seller potentially mixes between $\hat{p}_k$ and $\hat{p}_{k+1}$ (and also the continuation play is according to the statement of the induction hypothesis). Since we argued that offering a price below $\hat{p}_{k+1}$ is dominated by offering a price slightly below $\hat{p}_{k+1}$ if $\phi_0 > \hat{\phi}_{k+1}$, which proves the second point of the induction hypothesis. □

Proofs of Corollaries 3.1-3.3

Proof. The proofs follow from the arguments in the main text. □

Proof of Theorem 3.2

Proof. We now look for seller strategies π and maps from each strategy profile π' to a strategy of the buyer $(\alpha_\ell(\cdot|\pi'), \alpha_h(\cdot|\pi'))$ such that (i) π maximizes the $C(\emptyset|\pi', \alpha_\ell, \alpha_h, \phi_0)$ among all π' and (ii) each $\alpha_\theta(\cdot|\pi')$ maximizes $V_\theta(p^t; \pi', \alpha_\theta)$ for all θ, p^t, and π' (note that, differently from Definition 2.1, we do not require the seller's strategy to be sequentially optimal and we allow the buyers to observe the strategy of the seller). It is not difficult to see that we can assume, without loss of generality, that the buyer purchases when he is indifferent.[17] We then assign, to each seller's strategy π, the payoff $C_0(\pi)$ computed under the assumption that each type of the buyer buys in the first period it is optimal for him to do so. We look for π maximizing $C_0(\pi)$.

We first argue that it is without loss of optimality to focus on equilibria where the seller's price in the first period is non-stochastic. To see that, pose a seller strategy π and let p_0 be a price in the support of the seller's offer distribution in the first period such that, conditionally

[17] The argument is standard: It is easy to see that, for a given π, the seller prefers each type of the buyer to buy at the earliest sequentially optimal time, and that she can alter the payoffs to make such acceptance strictly optimal.

on offering p_0 in the first period, the seller obtains a payoff $\hat{C}_0 \geq C_0(\pi)$. Note that there exists a seller strategy where the seller offers p_0 for sure in the first period and the corresponding continuation price path afterward, which gives a payoff equal to $\hat{C}_0$ to the seller.

From the previous observation, it follows that if there is a seller's strategy where the h-buyer does not purchase at time 0 for sure, there is another seller's strategy where the h-buyer purchases at time 0 for sure, which gives the seller a higher payoff. Hence, we focus without loss of generality on seller strategies the h-buyer purchases at time 0 for sure.

Now, see that if a seller strategy is such that the transaction price is strictly below ℓ with positive probability, there is another seller strategy where no price is strictly below ℓ, which gives the seller a higher payoff. Such price path can be obtained by replacing each instance where a price $p_t < \ell$ is offered by a price equal to $\beta p_t + (1-\beta)\ell$ for some $\beta \in (0,1)$. It is easy to see that the h-buyer still prefers to buy at time 0, while the ℓ-buyer buys at a weakly earlier time at a weakly higher price; hence, the seller is weakly better off.

We now study stochastic price paths that maximize the seller's payoff conditional on a price $p_0 \in \{\ell\} \cup [\hat{p}_1, h]$ being offered in period 0 and accepted for sure by the h-buyer, while the ℓ-seller buys at the first (random) time $\tilde{t}$ where the price is ℓ.[18] It is clear that the seller's optimality requires that the h-buyer is indifferent between buying at p_0 and mimicking the ℓ-buyer. Hence, it must be that

$$h - p_0 = \mathbb{E}[\delta_b^{\tilde{t}}]\,(h-\ell)\ .$$

Assume that $\tilde{t}$ is optimal and assigns positive probability to some time $t_1 > 1$. The seller can replace the event where ℓ is offered at t_1 by a lottery between $t_1 - 1$ (with probability q) and $t_1 + 1$ (with probability $1-q$) so that the h-buyer remains indifferent, that is, such that

$$\delta_b^{t_1} = q\,\delta_b^{t_1-1} + (1-q)\,\delta_b^{t_1+1} \quad \Rightarrow \quad q = \tfrac{\delta_b}{1+\delta_b}\ .$$

The change in the seller's payoff conditional on this even occurring is

$$\left(q\,\delta_s^{t_1-1} + (1-q)\,\delta_s^{t_1+1}\right)\ell - \delta_s^{t_1}\,\ell = \tfrac{1-\delta_s}{1+\delta_b}\,\delta_s^{t_1-1}\,(\delta_b - \delta_s)\,\ell\ .$$

There are two cases:

[18] Note that if $p_0 \in (\ell, \hat{p}_1)$, there is no strategy of the seller that makes the h-buyer indifferent between buying in period 0 or buying at a later date.

1. Assume first $\delta_b \geq \delta_s$.[19] In this case, the seller weakly benefits from the previous change. An analogous argument to that before shows that, without loss, we can assume that the support of $\tilde{t}$ in an optimal seller strategy is either $\{0\}$ or a subset of $\{1, +\infty\}$ in this case. That is, either the seller offers ℓ in period 0, or offers p_0 in period 0 and then price ℓ in period 1 with probability $q_1 \in [0,1]$ satisfying

$$h - p_0 = \delta_b \, q_1 \, (h - \ell) \; .$$

Then, the seller's payoff is

$$\max \left\{ \ell, \max_{q_1 \in [0,1]} \left(\phi_0 \, (h - \delta_b \, q_1 \, (h - \ell)) + (1 - \phi_0) \, \delta_s \, q_1 \, \ell \right) \right\} \; .$$

It is then easy to see that, in the first period and under any optimal seller strategy, the seller either offers ℓ (if $\phi_0 < \phi^*$) or h (if $\phi_0 > \phi^*$) or randomizes between the two (if $\phi_0 = \phi^*$), and that no trade takes place for all $t > 0$.

2. If $\delta_b < \delta_s$, the seller loses from the change described above. The implication now is that the support of $\tilde{t}$ in any seller strategy is either $\{0\}$ or a subset of $\{\bar{t}, \bar{t}+1\}$ for some $\bar{t} > 0$. It is clear that, under an optimal seller strategy, offering ℓ in the first period is optimal for the seller if $\phi_0 < \hat{\phi}_1$, while offering some $p_0 > \ell$ in the first period is optimal if $\phi_0 > \hat{\phi}_1$. Assume the second case, and let $q \in [0,1]$ be such that the seller offers price ℓ with probability q at $\bar{t}$ and with probability $1 - q$ at $\bar{t} + 1$. The seller maximizes

$$\phi_0 \, p_0 + (1 - \phi_0) \, (q \, \delta_s^{\bar{t}} + (1 - q) \, \delta_s^{\bar{t}+1}) \, \ell$$

over $\bar{t}$ and q subject to

$$h - p_0 = (q \, \delta_b^{\bar{t}} + (1 - q) \, \delta_b^{\bar{t}+1}) \, (h - \ell) \; .$$

Standard analysis implies that the optimal $\bar{t}$ is the smallest satisfying

$$(1 - \phi_0) \, (1 - \delta_s) \, \delta_s^{\bar{t}} \, \ell - \phi_0 \, (1 - \delta_b) \, \delta_b^{\bar{t}} \, (h - \ell) \geq 0 \tag{14}$$

[19] The main text describes a shorter argument using the result in Stokey (1979). We provide here a self-contained proof.

while the optimal q is 1 if the previous expression holds with strict inequality for the optimal $\bar{t}$, or any value in $[0,1]$ if the previous expression holds with equality. The statement of the Theorem 3.2 follows from these observations. Note that, since equation (14) holds with equality for some $\bar{t} \in \{1,2,...\}$ non-generically in the parameters of the model,[20] we have that, generically, $q = 1$. □

Proof of Theorem 4.1

Proof. Like the proof of Theorem 3.1, the proof proceeds by induction, this time over the length of the horizon $T = 0,1,2,...$ The following will be our induction hypothesis:

Induction hypothesis for T: There exists an increasing sequence $(k_{T'}, \overline{\phi}_{T'})_{T'=0}^{T}$, with $\overline{\phi}_{T'} \geq \hat{\phi}_{k_{T'}}$ for all T' and $\overline{\phi}_0 = \phi^* := \ell/h$, such that, in any equilibrium, the following holds true:

1. If $\phi_0 > \overline{\phi}_T$ then the on-path history is $(h,h,...,h)$ and the corresponding beliefs are $(\phi_0, \overline{\phi}_{T-1}, ..., \overline{\phi}_0)$.
2. If $\phi_0 < \overline{\phi}_T$ then,
   (a) if $\phi_0 \in (\hat{\phi}_{k_T}, \overline{\phi}_T)$ then the on-path history is $(\hat{p}_{k_T}, \hat{p}_{k_T-1}, ..., \ell)$ and the corresponding beliefs are $(\phi_0, \hat{\phi}_{k_T-1}, ..., \hat{\phi}_0)$, and,
   (b) otherwise, the equilibrium is as specified in Theorem 3.1.
3. If $\phi_0 = \overline{\phi}_T$ then the seller randomizes between part 1 and part 2.[21]

Part 1: Proof for $T = 0$. The result is clear for $T = 0$, where $\overline{\phi}_0 = \phi^*$ and $k_0 = 0$.

Part 2: Proof for $T+1 > 1$. Fix some $T \geq 1$ and assume that the induction hypothesis holds for T. By the same argument as in the proof of Theorem 3.1, in any equilibrium, if the seller offers (on or off-path) a price $p_0 \in (\hat{p}_k, \hat{p}_{k+1})$ for some $k \leq k_T$ such that $\phi_0 \geq \hat{\phi}_k$, then the h-buyer must be indifferent between accepting it or not; so $\phi_1 = \hat{\phi}_k$ and the seller obtains

$$\tfrac{\phi_0 - \phi_1}{1-\phi_1}\, p_0 + \tfrac{1-\phi_0}{1-\phi_1}\, \delta_s\, \hat{C}_k \; . \tag{15}$$

If, instead, the seller offers $\hat{p}_{k+1}$ for some $k \leq k_T$ such that $\phi_0 \geq \hat{\phi}_k$, then $\phi_1 \in [\hat{\phi}_k, \min\{\phi_0, \hat{\phi}_{k+1}, \overline{\phi}_T\}]$, and the payoff of the seller is again given by (15). If $\phi_0 \geq \overline{\phi}_T$ and $p_0 \in (\hat{p}_{k_T+1}, h)$, then $\phi_1 = \overline{\phi}_T$

[20] That is, equation (14) holds with equality only if $\log\left(\frac{\phi_0}{1-\phi_0}\,\frac{1-\delta_b}{1-\delta_s}\,\frac{h-\ell}{\ell}\right) / \log(\delta_s/\delta_b)$ is a natural number.

[21] That is, if $\phi_0 = \overline{\phi}_T > \hat{\phi}_{k_T}$ then the seller randomizes between the paths described in parts 1 and 2(a), and if $\phi_0 = \overline{\phi}_T = \hat{\phi}_{k_T}$ then the seller randomizes between the paths described in parts 1 and 2(b).

and the seller's payoff is

$$\overline{C}_{T+1}(\phi_0,p_0) := \tfrac{\phi_0-\overline{\phi}_T}{1-\overline{\phi}_T}\,p_0 + \tfrac{1-\phi_0}{1-\overline{\phi}_T}\,\delta_s\,\overline{C}_T\ , \tag{16}$$

where $\overline{C}_T$ is the continuation payoff at T when the posterior is $\overline{\phi}_T$. Finally, if $\phi_0 \geq \overline{\phi}_T$ and $p_0 = h$, then $\phi_1 \in [\overline{\phi}_T, \phi_0]$ and the payoff of the seller is

$$\tfrac{\phi_0-\phi_1}{1-\phi_1}\,h + \tfrac{1-\phi_0}{1-\phi_1}\,\delta_s\left(\tfrac{\phi_1-\overline{\phi}_T}{1-\overline{\phi}_T}\,h + \tfrac{1-\phi_1}{1-\overline{\phi}_T}\,\overline{C}_T\right)\ , \tag{17}$$

It is easy to see that both (15) and (16) are increasing in ϕ_1, and that expression (15) is smaller than (16) for p_0 close enough to h.

Because, in the T-period model, the seller is indifferent between offering h and $\hat{p}_{k_T}$ when $\phi_0 = \overline{\phi}_T$, we have

$$\overline{C}_T = \tfrac{\overline{\phi}_T-\hat{\phi}_{k_T-1}}{1-\hat{\phi}_{k_T-1}}\,\hat{p}_k + \tfrac{1-\overline{\phi}_T}{1-\hat{\phi}_{k_T-1}}\,\delta_s\,\hat{C}_{k_T-1}\ .$$

Hence, for ϕ_0 slightly below $\overline{\phi}_T$, the payoff of offering a price slightly below h in the $(T+1)$-period model is approximately $\delta_s\overline{C}_T$. The payoff from offering a price slightly below $\hat{p}_k$ is approximately equal to $\overline{C}_T$. It is then clear that for all $\phi_0 \leq \overline{\phi}_T$, the price offered in the first period in the $(T+1)$-model is that given by the induction hypothesis.

Assume $\phi_0 \geq \overline{\phi}_T$. In this case, by offering a price slightly below h, the seller can obtain a payoff arbitrarily close to $\overline{C}_{T+1}(\phi_0, h)$ defined in equation (16). It is then easy to see that k_{T+1} and $\overline{\phi}_{T+1}$ are determined as follows:

1. If $\overline{C}_{T+1}(\overline{\phi}_T, h) = \hat{C}_{k_T}$ then $k_{T+1} := k_T + 1$ and $\overline{\phi}_{T+1} := \hat{\phi}_{k_T+1}$.
2. If $\overline{C}_{T+1}(\overline{\phi}_T, h) < \hat{C}_{k_T}$ then $k_{T+1} := k_T + 1$ and $\overline{\phi}_{T+1}$ is the unique solution to

$$\overline{C}_{T+1}(\overline{\phi}_{T+1}, h) = \tfrac{\overline{\phi}_{T+1}-\hat{\phi}_{k_T}}{1-\hat{\phi}_{k_T}}\,\hat{p}_{k_T+1} + \tfrac{\overline{\phi}_{T+1}-\hat{\phi}_{k_T}}{1-\hat{\phi}_{k_T}}\,\delta_s\,\hat{C}_{k_T-1}\ .$$

3. If $\overline{C}_{T+1}(\overline{\phi}_T, h) > \hat{C}_{k_T}$ then $k_{T+1} := k_T$ and $\overline{\phi}_{T+1}$ is the unique solution to

$$\overline{C}_{T+1}(\overline{\phi}_{T+1}, h) = \tfrac{\overline{\phi}_{T+1}-\hat{\phi}_{k_T-1}}{1-\hat{\phi}_{k_T-1}}\,\hat{p}_{k_T} + \tfrac{\overline{\phi}_{T+1}-\hat{\phi}_{k_T-1}}{1-\hat{\phi}_{k_T-1}}\,\delta_s\,\hat{C}_{k_T-2}\ .$$

The proof is then concluded. □

Proof of Corollary 4.1

Proof. The proof follows from a formalization of the arguments in the main text. □

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