(Un-)Common Preferences, Ambiguity, and Coordination*

Simone Cerreia-Vioglio[a], Roberto Corrao[b], Giacomo Lanzani[c]

January 2024

Abstract

We study the "common prior" assumption when agents have differential information and preferences beyond subjective expected utility (SEU). We consider interim preferences consistent with respect to the same ex-ante evaluation and characterize them. Notably, agents are mutually dynamic consistent with respect to the same ex-ante evaluation if and only if all the limits of higher-order expectations coincide. Within this framework, we characterize the properties of equilibrium prices in financial beauty contests. Unlike the SEU case, the limit price does not coincide with the common ex-ante expectation. Moreover, high-coordination motives create a divergence between the market price and the fundamental value.

*[a]Università Bocconi, [b]MIT, [c]Harvard University. We wish to thank Ian Ball, Pierpaolo Battigalli, Xiaoyu Cheng, Tommaso Denti, Andrew Ellis, Joel Flynn, Mira Frick, Drew Fudenberg, Ben Golub, Brian Hill, Ryota Iijima, Peter Klibanoff, Asen Kochov, Paolo Leonetti, Fabio Maccheroni, Massimo Marinacci, Stephen Morris, Daisuke Oyama, Giulio Principi, Antonio Penta, Lorenzo Stanca, Tomasz Strzalecki, and Muhamet Yildiz for very useful suggestions. Simone Cerreia-Vioglio gratefully acknowledges the financial support of ERC (grant SDDM-TEA).

1 Introduction

The common prior assumption is one of the most used and debated concepts in economic theory (see, e.g., Morris [36]). When the agents are subjective expected utility (SEU) maximizers, this assumption captures the idea of mutual ex-ante agreement on the preferences over uncertain prospects. However, preferences that do not reduce uncertainty to a single probability are normatively convincing and consistent with experimental findings. Notably, these departures are consistent with the rationality of decision makers who acknowledge their *ambiguity* about an objective probabilistic model and have nonneutral attitudes toward it. Therefore, it is crucial to understand whether the ex-ante mutual agreement can be expressed independently of agents' attitudes toward ambiguity and, in this case, to study the implications for the agents' interim preferences and behavior of this mutual agreement. This paper answers these questions by formalizing increasing degrees of mutual ex-ante agreement among agents with differential information and rational preferences such as maxmin expected utility, Choquet expected utility, and variational preferences.

We first impose restrictions on the agents' interim preferences that guarantee the existence of a *single* ex-ante preference that is *jointly* "consistent" for all the agents. Next, we show that, as for the baseline SEU case, all these restrictions can be fully characterized by properties of the *higher-order interim preferences* of the agents.

We then embed rational preferences in standard coordination games (e.g., beauty contests and price competitions), and we derive a complete characterization of equilibrium behavior in the high-coordination limit in terms of the agents' higher-order preferences without any ex-ante agreement restriction. However, when we impose some ex-ante agreement, we find a striking result: the desire for coordination considerably tames the attitudes toward uncertainty, and the limit equilibrium behavior in some critical cases is indistinguishable from the ones obtained under SEU.

Common ex-ante preferences and beyond First, we generalize the notion of conditional expectation for preferences that are not necessarily SEU but just rational. We start with a pair of *ex-ante* and *interim expectations*, modeling the preferences of the agent before and after the arrival of information, and require them to be "consistent" in the sense that they jointly exhibit a reduction/increase in the uncertainty aversion at the interim stage compared to the ex-ante one. These consistency properties, which are satisfied by several existing updating rules for non-SEU preferences (e.g., full Bayesian updating and proxy updating for maxmin preferences), give rise to the notion of *lower* and *upper conditional expectations* respectively. When both consistency properties are satisfied, we define the notion of *nonlinear conditional expectation.* This last case fully maintains the dynamic consistency of conditional SEU while relaxing linearity.

Armed with this novel taxonomy, we analyze a multi-agent setting with differential information. We extend to rational preferences the notion of (*nonlinear) higher-order expectations,* capturing the idea of preferences over acts formed by the evaluations attached by other agents to an original act. This notion is essential for our analysis and is illustrated through a simple asset-pricing model where agents care about the willingness to pay of the other traders rather than the fundamental value of an asset. Our first result shows that, under a full-support condition and the presence of null public information, the higher-order expectations over sequences of agents converge to a state-independent limit, provided that all the agents appear infinitely often in the sequence. This result greatly generalizes the equivalent result of Samet [42] beyond SEU and lays the foundation of our analysis. Moreover, it is easily illustrated in our asset pricing example by implying the existence of a well-defined and state-independent equilibrium price that does not depend on the order of trades among agents.

Next, we say that agents share a *lower* (resp. *upper*) *common ex-ante expectation* if their conditional preferences are less (resp. more) uncertainty averse with respect to the same ex-ante expectation. The interpretation is that, before observing their private information, the agents share the same perceived ambiguity about the probabilistic model and the same attitude toward it. Then, in the interim stage, the agents' preferences may differ, but only insofar as the nature of their private information was different. Therefore, our consistency properties impose restrictions between periods for each individual as well as restrictions across all individuals. For every profile of interim expectations, there always exist a lower and an upper common ex-ante preferences exhibiting the minimal degrees of changes in the attitudes toward uncertainty between the ex-ante and interim stage.

We characterize these extreme common ex-ante preferences in different ways. First, under the deterministic convergence property highlighted above, these ex-ante preferences are characterized by the extreme limits of higher-order expectations of the agents. Second, we make this concept operational by providing an algorithm to recover them from the agents' interim preferences. Finally, we provide behavioral axioms that identify the testable conditions that an ex-ante preference needs to satisfy to be the extreme lower (upper) common ex-ante expectation for a collection of interim preferences.

When all the agents are dynamically consistent with respect to the same unconditional preference, we say that they share a *nonlinear common ex-ante expectation.* In other words, we weaken the assumption of mutual agreement about an objective probabilistic model to that of *mutual dynamic consistency* with respect to a common ex-ante rational preference. We provide a characterization of the existence of a nonlinear common ex-ante expectation that purely concerns the interim preferences of the agents. There is a nonlinear common ex-ante expectation if and only if all the interim higher-order (nonlinear) expectations of the agents converge to the same limit, which coincides with the nonlinear common ex-ante expectation. On the one hand, this result significantly generalizes the characterization of the common prior assumption in Samet [42]. On the other hand, it points out that it is the *invariance property* of dynamic consistency that allows us to characterize mutual ex-ante agreement in terms of interim higher-order beliefs, as opposed to the probabilistic nature of beliefs. However, dynamic consistency with respect to all the information structures of the agents is very restrictive under ambiguity-averse preferences, as pointed out, for example, by Ellis [11]. Therefore, our result implies that the order of traders in our asset pricing example is generally relevant under ambiguity aversion.

Coordination and ambiguity We next move to the implications for coordination games of the assumptions on an ex-ante agreement under variational preferences, a large subclass of rational preferences. We first consider an application of our results to beauty contests in market networks under incomplete information. Here, we show that the (bid) prices in the unique equilibrium become independent of the state and agent as the coordination motives prevail. Notably, we provide bounds on the equilibrium price dispersion that only depends on the joint connectivity of the network and information structure.

Next, we analyze the unique equilibrium price in the limit for strong coordination motives. In general, this limit is characterized by a worst-case weighted average of the models that are maximally trusted by the agents at the interim stage. With this result, we can already see that a significant part of the ambiguity aversion of the agents disappears in the limit equilibrium, as all the probabilistic models that are not maximally trusted become irrelevant. Moreover, we provide bounds on the limit evaluation of the asset in terms of the ex-ante preferences that we presented, thereby assessing the price effect of interim information.

Our theorem implies that whenever the agents share the same unique ex-ante benchmark probability model, the limit equilibrium price collapses to the expected value of the asset under this unique benchmark. This establishes a strong irrelevance result: as coordination motives prevail, the limit price is unaffected by the uncertainty attitudes of the agents. In turn, this has important implications for our financial beauty contest application. If the common benchmark probability model of the agents is misspecified, then our result implies mispricing with respect to the true fundamental value of the asset, despite agents that are concerned about misspecification. Intuitively, the agents attach a much higher value to coordination than to the fundamental value of the asset; hence, in equilibrium, they have little reason to reduce their willingness to pay due to the concern for misspecification of the *shared* benchmark probability model.

In general, if a nonlinear common ex-ante expectation exists, then the limit price can lie strictly above the ex-ante preference, pointing out a key difference with the limit result under SEU of Golub and Morris [18]. However, this wedge exists only if the agents are ambiguous with respect to each other information structure. Indeed, when agents are unambiguous about the aggregate information, the standard limit equivalence of the SEU case is restored. Notably, in this case, agents might still perceive ambiguity about the fundamental, and their full-coordination limit price decreases in their ambiguity aversion.

The previous results depend only on the best-response structure of the game. In particular, we can derive the same best-response functions from different games with strong coordination motives. An example is a price-competition game where firms produce partially differentiated goods under incomplete information about the demand function.

2 Nonlinear conditional expectations

In this section, we introduce nonlinear conditional expectations. We start by recalling the usual notion of (linear) conditional expectation. Consider a finite state space Ω endowed with the power set $\mathcal{P}(\Omega)$. We let Π be a partition of Ω, and for every $\omega \in \Omega$, we denote as $\Pi(\omega)$ the unique element of Π that contains ω.

2.1 Linear case

Consider a probability $\mu \in \Delta(\Omega)$. If Π is a partition of Ω, then a map $p_\mu : \Omega \times \mathcal{P}(\Omega) \to [0,1]$ is a regular conditional probability of μ given Π if and only if: (i) For each $\omega \in \Omega$, $p_\mu(\omega, \cdot) \in \Delta(\Omega)$; (ii) For each $F \in \mathcal{P}(\Omega)$, the function $p_\mu(\cdot, F) : \Omega \to [0,1]$ is a conditional probability of F given Π.

With this, the function $V_\mu : \Omega \times \mathbb{R}^\Omega \to \mathbb{R}$, defined by

$$V_\mu(\omega, f) = \mathbb{E}_{p_\mu(\omega,\cdot)}[f] \qquad \forall \omega \in \Omega, \forall f \in \mathbb{R}^\Omega,$$

is a regular conditional expectation and has the following properties:

a. For each $\omega \in \Omega$ the function $V_\mu(\omega, \cdot) : \mathbb{R}^\Omega \to \mathbb{R}$ is normalized, monotone, and linear;[1]

b. For each $f \in \mathbb{R}^\Omega$ the function $V_\mu(\cdot, f) : \Omega \to \mathbb{R}$ is Π-measurable and satisfies

$$\mathbb{E}_\mu(f) = \mathbb{E}_\mu(V_\mu(\cdot, f)) \text{ and } V_\mu\left(\omega, f1_{\Pi(\omega)} + h1_{\Pi(\omega)^c}\right) = V_\mu(\omega, f) \quad \forall \omega \in \Omega, \forall h \in \mathbb{R}^\Omega. \tag{1}$$

In words, (1) contains two properties: the law of iterated expectations and that the update of μ assigns probability one to the realized partition cell.

2.2 Nonlinear case

Mimicking what we discussed above, we consider two functions $\bar{V} : \mathbb{R}^\Omega \to \mathbb{R}$ and $V : \Omega \times \mathbb{R}^\Omega \to \mathbb{R}$. In terms of interpretation, $\bar{V}(f)$ is the unconditional expectation of f, while $V(\cdot, f)$ describes its conditional expectation.

Definition 1. *Let $\bar{V} : \mathbb{R}^\Omega \to \mathbb{R}$. We say that $\bar{V}$ is an* ex-ante (generalized) expectation *if and only if $\bar{V}$ is normalized and monotone.*

This definition amounts to saying that the preference $\succsim$ represented by an ex-ante expectation $\bar{V}$ is *rational* (Cerreia-Vioglio et al. [9]). This is a large class of preferences that includes maxmin expected utility (Gilboa and Schmeidler [16]), Choquet expected

[1] A functional $T : \mathbb{R}^\Omega \to \mathbb{R}$ is normalized if and only if $T(k1_\Omega) = k$ for all $k \in \mathbb{R}$.

utility (Schmeidler [43]), variational preferences (Maccheroni et al. [32]), quantile maximization (Rostek [41]), and uncertainty averse preferences (Cerreia-Vioglio et al. [8]). Monotonicity is a conceptual (although mild) requirement implying that the agents prefer larger monetary outcomes, whereas normalization requires that the representing $\bar{V}$ is the certainty equivalent for the preference.[2]

Definition 2. *Fix a partition* Π *and* $V:\Omega\times\mathbb{R}^{\Omega}\to\mathbb{R}$*. We say that* (V,Π) *is an* interim (generalized) expectation *if and only if for each* $\omega\in\Omega$ *the function* $V(\omega,\cdot):\mathbb{R}^{\Omega}\to\mathbb{R}$ *is normalized, monotone, and continuous and the function* $V(\cdot,f):\Omega\to\mathbb{R}$ *is* Π*-measurable and*

$$V\left(\omega,f1_{\Pi(\omega)}+h1_{\Pi(\omega)^{c}}\right)=V(\omega,f)\quad\forall\omega\in\Omega,\forall f,h\in\mathbb{R}^{\Omega}. \tag{2}$$

A conditional expectation is a pair formed by an *ex-ante* (generalized) expectation and an *interim* (generalized) expectation that satisfies some consistency properties.

Definition 3. *Let* (V,Π) *be an interim expectation.*

1. *We say that* $(V_{\circ},V,\Pi)$ *is a* lower conditional expectation *if and only if* $V_{\circ}$ *is an ex-ante expectation such that*

$$V_{\circ}(f)\leq V_{\circ}(V(\cdot,f))\qquad\forall f\in\mathbb{R}^{\Omega}. \tag{3}$$

2. *We say that* (V°,V,Π) *is an* upper conditional expectation *if and only if* V° *is an ex-ante expectation such that*

$$V^{\circ}(f)\geq V^{\circ}(V(\cdot,f))\qquad\forall f\in\mathbb{R}^{\Omega}. \tag{4}$$

3. *We say that* $\left(\bar{V},V,\Pi\right)$ *is a* nonlinear conditional expectation *if and only if it is both a lower and an upper conditional expectation.*

Compared to standard conditional expectations, a nonlinear conditional expectation only relaxes the assumption of linearity from both $\bar{V}$ and V, as point 3 implies that

$$\bar{V}(f)=\bar{V}(V(\cdot,f))\qquad\forall f\in\mathbb{R}^{\Omega}. \tag{5}$$

[2] We are implicitly assuming that the utility index $u:\mathbb{R}\to\mathbb{R}$ coincides with the identity function. In the multi-agent setting of Section 3, this assumption is without loss of generality as long as the risk preferences of all agents are homogeneous since we can always interpret each $f\in\mathbb{R}^{\Omega}$ as a utility act.

This is tantamount to weakening the assumption of independence, retaining consequentialism and dynamic consistency. Consequentialism takes care of (2), while dynamic consistency is the main axiom behind the law of iterated expectations in (5). However, it is well known that full-fledged dynamic consistency is restrictive outside the realm of subjective expected utility, especially with uncertainty-averse preferences (see, for example, Ghirardato [15] and Siniscalchi [46]). Therefore, in points 1 and 2, we consider ex-ante expectations that are consistent with the interim expectation yet possibly exhibit a reduction/increase in the uncertainty aversion as the agent receives information.[3]

Whenever the agent has a lower conditional expectation $V_\circ$, her interim preference can be rationalized by $V_\circ$ provided that receiving the interim information reduces the uncertainty aversion of the agent. Indeed, evaluating the uncertain act f with $V_\circ$ induces a lower evaluation than first evaluating the uncertainty within the cells of Π with V and then the residual uncertainty with $V_\circ$. Such condition is satisfied by existing updating rules for preferences under uncertainty, as we show next.

Example 1 (Choquet expected utility with proxy updating)**.** We analyze the class of preferences and updating rule recently proposed by Gul and Pesendorfer [19]. Consider a totally monotone capacity $\nu : 2^\Omega \to [0,1]$ and a partition Π.[4] In the ex-ante stage, the agent evaluates every act $f \in \mathbb{R}^\Omega$ with the Choquet integral of f with respect to ν, denoted as $V_\circ(f)$. Recall that the core of ν is defined as

$$core(\nu) = \left\{\mu \in \Delta(\Omega) : \forall E \in 2^\Omega, \mu(E) \geq \nu(E)\right\}$$

and that $V_\circ(f) = \min_{\mu \in core(\nu)} \mathbb{E}_\mu(f)$. We let $\mu_\nu \in \Delta(\Omega)$ denote the Shapley value corresponding to ν. The interim expectations at state ω are:

$$V(\omega, f) = \min_{\mu \in core_\circ(\nu)} \mathbb{E}_{p_\mu(\omega,\cdot)}(f) \qquad \forall f \in \mathbb{R}^\Omega$$

[3] In Online Appendix E, we show that this somewhat informal use of the terms "uncertainty aversion reduction/increase" is backed by formal axioms linking the ex-ante and interim preferences of the agents.

[4] A capacity ν is totally monotone if and only if, for all $k \geq 2$ and all $E_1, ..., E_k \in 2^\Omega$, $\nu\left(\cup_{i=1}^k E_i\right) \geq \sum_{\{J:\emptyset \neq J \subseteq \{1,...,k\}\}} (-1)^{|J|+1} \nu\left(\cap_{j\in J} E_j\right)$.

where $p_\mu(\omega,\cdot)$ is the conditional probability of μ given Π and

$$core_\circ(\nu) = \{\mu \in core(\nu) : \forall E \in \Pi, \mu(E) = \mu_\nu(E)\}. \tag{6}$$

In words, each agent updates her preferences with full Bayesian updating but starting from the restricted set $core_\circ(\nu)$. In this case, the results in [19, Axiom C.4 and Theorem 1] imply that $(V_\circ, V, \Pi)$ is a lower conditional expectation. ▲

Instead, an upper conditional expectation rationalizes the interim expectation of the agent provided that it features less uncertainty aversion than the interim preferences. Indeed, evaluating the uncertain act f with V° induces a higher evaluation than first evaluating the uncertainty within the cells of Π with V and then the residual uncertainty with V°. The next example shows that with maxmin expected utility (see Gilboa and Schmeidler [16]) and full Bayesian updating, we obtain an upper conditional expectation.

Example 2 (Maxmin expected utility with full Bayesian updating)**.** Let C be a compact and convex set of probabilities over Ω and let Π be a partition. Define

$$V_C^\circ(f) = \min_{\mu\in C} \mathbb{E}_\mu(f) \qquad \forall f \in \mathbb{R}^\Omega \tag{7}$$

and

$$V_C(\omega, f) = \min_{p\in C_\omega} \mathbb{E}_p(f) \qquad \forall \omega \in \Omega, \forall f \in \mathbb{R}^\Omega, \tag{8}$$

where, for all $\omega \in \Omega$, $C_\omega = \{p_\mu(\omega,\cdot) : \mu \in C\}$. Then (V_C°, V_C, Π) is an upper conditional expectation. Moreover, it is well known that if C is rectangular for Π (see Epstein and Schneider [12]), then (V_C°, V_C, Π) is a nonlinear conditional expectation.[5] ▲

The observation that Bayesian updating induces an upper common ex-ante expectation also holds for the class of divergence preferences introduced in Maccheroni et al. [32]. We illustrate this class in Example 5 below, where we consider multiplier preferences a la Hansen and Sargent [23].

Another class of preferences that induces an upper common ex-ante expectation is the one that evaluates a random variable with its value at risk (VaR) paired with Bayesian updating.

[5] C is rectangular if and only if $C = \left\{\sum_{l=1}^L p_{\mu_l}(E_l,\cdot)\,\mu(E_l) : \mu, \mu_1, ..., \mu_L \in C\right\}$, where $\Pi = \{E_1, ..., E_L\}$ and we denote the update on the E_l cell by $p_\mu(E_l,\cdot)$.

Example 3 (Quantile Maximization)**.** Consider a partition Π and a probability $\mu \in \Delta(\Omega)$. In the ex-ante stage, the agent evaluates every act $f \in \mathbb{R}^{\Omega}$ with the τ-quantile (Rostek [41]), with $\tau \in (0,1)$. That is,

$$V^{\circ}(f) = \inf \{z \in \mathbb{R} : \mu(\{\omega \in \Omega : f(\omega) \leq z\}) \geq \tau\}.$$

Analogously, the interim expectation V of the agent in state ω evaluates the act with respect to the τ-quantile but using $p_{\mu}(\omega, \cdot)$ in place of μ. Then, for a sufficiently low quantile, i.e., for τ sufficiently close to 0, the pair (V°, V, Π) is an upper conditional expectation.[6] Importantly, this means that the widely used VaR, which involves portfolio evaluation by looking at the left tail quantiles, generally features an increase of uncertainty aversion as information is received. Similarly, one can link high quantiles with lower conditional expectations. ▲

We close this section with a notion of full support for rational preferences. Indeed, as Samet [42], we mostly focus on the case of full support.[7] Given states $\bar{\omega}, \omega \in \Omega$, we say that $\bar{\omega}$ is $V(\omega, \cdot)$-essential if and only if there exists an $\varepsilon > 0$ such that for each $f \in \mathbb{R}^{\Omega}$ and for each $\delta \geq 0$

$$V\left(\omega, f + \delta 1_{\{\bar{\omega}\}}\right) - V(\omega, f) \geq \varepsilon\delta. \tag{9}$$

In the linear case, we clearly have that $\bar{\omega}$ belongs to the support of $p_{\mu}(\omega, \cdot)$ if and only if $\bar{\omega}$ is $V(\omega, \cdot)$-essential. For the general case, we say that an interim expectation (V, Π) has full support if and only if for all $\omega \in \Omega$ each $\bar{\omega} \in \Pi(\omega)$ is $V(\omega, \cdot)$-essential.

3 (Un-)common ex-ante preferences

We now consider a finite set of agents $I = \{1, ..., n\}$, each endowed with an interim expectation (V_i, Π_i). Given the collection of partitions $\{\Pi_i\}_{i \in I}$ for the agents, that is, an *information structure*, we denote by $\Pi_{\sup}$ and $\Pi_{\inf}$ the *meet* and the *join* of the parti-

[6] In particular, any $\tau < \min_{\omega \in \Omega} p_{\mu}(\omega, \{\omega\})$ would work.

[7] Corollary 1 does not rely on the full-support assumption per se but rather on a regularity condition of the sequences of higher-order beliefs (cf. Definition 4). Our full-support condition, paired with the absence of non-trivial public information, implies that the regularity condition holds. However, this can be verified directly and independently of the full-support assumption (cf. Example 5).

tions.[8] They respectively correspond to the public information among the agents and the aggregate information collectively held by the agents.

In a multi-agent setting, it might be convenient to view V_i as an operator from $\mathbb{R}^\Omega$ to $\mathbb{R}^\Omega$. In this case, the ω-th component of this operator is $V_i(\omega, f)$ for all $f \in \mathbb{R}^\Omega$. This rewriting turns out to be useful in order to formally discuss higher-order expectations. For instance, given two agents $i, j \in I$ and an act $f \in \mathbb{R}^\Omega$, the expectation of agent i at state ω about the evaluation of act f by agent j is $V_i(\omega, V_j(f))$. Moreover, if we do not fix a state $\omega \in \Omega$, we obtain the second-order evaluation (of i through j) $V_i \circ V_j : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$. We next illustrate the relevance of this concept in a stylized asset-pricing model.

Example 4 (Forecasting the forecaster)**.** Consider a state-contingent asset $f \in \mathbb{R}^\Omega$ in a discrete-time economy with $t \in \mathbb{N}$ periods. Each index $i \in I$ represents a continuum of speculative traders with the same interim expectations (V_i, Π_i). Let $(i_1, ..., i_t) \in I^t$, with $t \in \mathbb{N}$, be a finite sequence of agents' classes. In period 0, an external agent is endowed with the asset. In period 1, she has to sell the asset to one of the agents in class i_1. The price is determined by Bertrand competition among the potential buyers. In period 2, the agent of class i_1 holding the asset has to sell it to an agent in class i_2 according to the same procedure as above and then leaves the economy. This scheme proceeds until period t when the agent of class i_t holding the asset is paid its realized value.[9]

We can easily solve for the unique equilibrium by backward induction. In period t, the willingness to pay for the asset of an agent in class i_t, and therefore the (state-contingent) equilibrium price, is exactly $V_{i_t}(f)$. Given Bertrand competition among potential buyers, for an agent in class i_{t-1}, the (state-contingent) value of the asset is then $V_{i_{t-1}} \circ V_{i_t}(f)$. Iterating this backward reasoning up to period 1, the initial (state-contingent) price of the asset is $V_{i_1} \circ V_{i_2} \circ ... \circ V_{i_{t-1}} \circ V_{i_t}(f) \in \mathbb{R}^\Omega$. This highlights the importance of higher-order expectations in market interactions.[10] ▲

Following Samet [42], we call a sequence $(i_t)_{t \in \mathbb{N}}$ in I an *I-sequence* if and only if for each individual $i \in I$, $i = i_t$ for infinitely many t indexes.

[8] That is, $\Pi_{\sup}$ is the finest among all partitions that are coarser than each Π_i, and $\Pi_{\inf}$ is the coarsest among all partitions that are finer than each Π_i.

[9] This model is a variation of classical models of sequential speculative trading such as Harrison and Kreps [24] and Morris [37], where we also allow for non-SEU preferences of the traders.

[10] Toward pointing out the direct role of higher-order expectations, we assumed that the agents know the class of the potential buyers (and hence their interim expectations). In Section 4, we characterize the equilibrium of the related beauty-contest game where the relevant class of buyers is uncertain.

Definition 4. *We say that a collection* $\{(V_i, \Pi_i)\}_{i\in I}$ *of interim expectations exhibits* convergence to a deterministic limit *if and only if for all I-sequences* $\iota = (i_t)_{t\in\mathbb{N}}$ *and for all* $f \in \mathbb{R}^\Omega$*, there exists* $k_{f,\iota} \in \mathbb{R}$ *such that*

$$\lim_{t\to\infty} V_{i_t} \circ V_{i_{t-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(f) = k_{f,\iota} 1_\Omega.$$

In this case, for each I-sequence $\iota = (i_t)_{t\in\mathbb{N}} \in I^{\mathbb{N}}$ *define* $\bar{V}_\iota : \mathbb{R}^\Omega \to \mathbb{R}$ *by* $\bar{V}_\iota(f) = k_{f,\iota}$.

If there is convergence to a deterministic limit, then the sequences of higher-order expectations of the agents converge to a limit whose value, being a constant function of the state, is trivially common knowledge. Our first result shows that there is convergence to a deterministic limit, provided that all the interim expectations of the agents have full support and there is no non-trivial public event. Moreover, the rate of convergence is *quasi-exponential*; that is, it is exponential in the number of times that all the agents have been repeated in the sequence.

Theorem 1. *If* $\{(V_i, \Pi_i)\}_{i\in I}$ *is a collection of full support interim expectations such that* $\Pi_{\sup} = \{\Omega\}$*, then* $\{(V_i, \Pi_i)\}_{i\in I}$ *exhibits convergence to a deterministic limit. Moreover, there exist* $\varepsilon \in (0,1)$ *and* $C \in \mathbb{R}_+$ *such that for each I-sequence* $(i_m)_{m\in\mathbb{N}}$ *and for each* $\tau, t \in \mathbb{N}$*, if every* $i \in I$ *appears at least* τ *times in* $(i_1, ..., i_t)$*, then*

$$\left|\left|\bar{V}_\iota(f) 1_\Omega - V_{i_t} \circ ... \circ V_{i_1}(f)\right|\right|_\infty \leq C\varepsilon^\tau \,||f||_\infty \qquad \forall f \in \mathbb{R}^\Omega.$$

Quasi-exponential convergence provides a bound on the approximation error for computing the limit higher-order expectation of f given ι using the t-th order expectation. In particular, the bound improves in t only if additional expectations of *all* the agents are involved. This theorem generalizes Proposition 2 of Samet [42] in several dimensions. Most importantly, it allows for nonlinear expectations. This generality makes it impossible to represent the operator as stochastic matrices, but Proposition 5 shows that, for the class of rational preferences we consider, it is still possible to extract a useful network of connections between agent-state pairs from the nonlinear operator. Loosely speaking, pair (i, ω) is connected to (j, ω') if agent i believes that while in state ω agent j nonlinear interim expectation is responsive to the payoff in state ω'. We then show that the full support of the interim expectations paired with the absence of non-trivial public events implies that this network is strongly connected and that iterations of operators with such underlying strongly connected structures induce convergence to a deterministic limit. In doing so, we also improve Samet's original result by adding a rate to this convergence.

We next illustrate the meaning of quasi-exponential convergence to a deterministic limit in the asset-pricing example.

Example (Forecasting the forecaster continued)**.** Suppose that the assumptions of Theorem 1 are satisfied. Then, rather than looking at a fixed-length sequence, we consider an infinite sequence of classes $(i_t)_{t\in\mathbb{N}}$. We can focus on I-sequences as, if the identity of classes are iid draws with full support on I, then with probability 1, an I-sequence is realized. With this, Theorem 1 guarantees that, for a truncation $(i_1, ..., i_{\bar{t}})$ of $(i_t)_{t\in\mathbb{N}}$ such that each agent appears sufficiently many times, the dependence of the initial equilibrium price on the realized state of the world is arbitrarily (and exponentially) small. Intuitively, the willingness to pay of an agent in class i_1 does not significantly depend on the state as she knows that the selling value depends on a large number of subsequent transactions. This and the assumption $\Pi_{\text{sup}} = \{\Omega\}$ imply that many of the subsequent buyers will care about the value of the asset also in states that are ruled out by the information of i_1. ▲

3.1 Common ex-ante expectations

A natural question that emerges in this setting is whether the interim preferences of the agents are consistent with a common ex-ante expectation.

Definition 5. *We say that* $V_\circ$ *(resp.* V°*) is a* lower *(resp.* upper*)* common ex-ante expectation *for* $\{(V_i, \Pi_i)\}_{i\in I}$ *if and only if* $(V_\circ, V_i, \Pi_i)$ *(resp.* (V°, V_i, Π_i)*) is a lower (resp. upper) conditional expectation for all* $i \in I$*. When* $\bar{V}$ *is both a lower and an upper common ex-ante expectation for* $\{(V_i, \Pi_i)\}_{i\in I}$*, we say that* $\bar{V}$ *is a* nonlinear common ex-ante expectation *for* $\{(V_i, \Pi_i)\}_{i\in I}$*.*

We let $\mathcal{V}_\circ$ and $\mathcal{V}^\circ$ denote the sets of lower and upper common ex-ante expectations for $\{(V_i, \Pi_i)\}_{i\in I}$ respectively. Clearly, their intersection is the set of nonlinear common ex-ante expectations for $\{(V_i, \Pi_i)\}_{i\in I}$. It is plain that in the case each $V_i(\omega, \cdot)$ is SEU, the nonemptiness of this intersection amounts to the existence of a *common prior.*

The sets $\mathcal{V}_\circ$ and $\mathcal{V}^\circ$ might contain multiple elements. However, we focus on two selections: (i) the more optimistic ex-ante expectation that is more uncertainty averse than the interim expectations and (ii) the more pessimistic ex-ante expectation that is less uncertainty averse than the interim expectations. Let

$$V_*(f) = \sup_{V_\circ \in \mathcal{V}_\circ} V_\circ(f) \quad \text{and} \quad V^*(f) = \inf_{V^\circ \in \mathcal{V}^\circ} V^\circ(f) \qquad \forall f \in \mathbb{R}^\Omega.$$

denote the maximal and minimal elements of $\mathcal{V}_\circ$ and $\mathcal{V}^\circ$.

We now show that both V_* and V^* are always well-defined and provide a characterization of them in terms of the higher-order expectations of the agents.

Proposition 1. *Let $\{(V_i, \Pi_i)\}_{i\in I}$ be a collection of interim expectations. Both V_* and V^* are well-defined and, respectively, a lower and an upper common ex-ante expectation for $\{(V_i, \Pi_i)\}_{i\in I}$. Moreover, if $\{(V_i, \Pi_i)\}_{i\in I}$ exhibits convergence to a deterministic limit, then, for every $f \in \mathbb{R}^\Omega$,*

$$V_*(f) = \inf_{\iota\in I^{\mathbb{N}}:\iota \textit{ is an I-sequence}} \bar{V}_\iota(f) \quad \textit{and} \quad V^*(f) = \sup_{\iota\in I^{\mathbb{N}}:\iota \textit{ is an I-sequence}} \bar{V}_\iota(f).$$

The interpretation is that by looking at the lowest (resp. highest) limit of the iterated expectations, we exactly identify the minimal changes in uncertainty aversion between the ex-ante and interim stages needed to jointly rationalize the interim preferences of the agents. In turn, this implies that $V_*(f) \leq V^*(f)$ for all $f \in \mathbb{R}^\Omega$, that is, the ex-ante preferences V_* and V^* are ranked in terms of their uncertainty aversion.

Example (Forecasting the forecaster continued)**.** In the setting of Example 4, fix an I-sequence $\iota = (i_n)_{n\in\mathbb{N}}$ and recall that the initial equilibrium price of asset f, for the game with length t, is equal to the random variable $V_{i_t} \circ V_{i_{t-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(f)$. In this case, by Theorem 1, as we let t go to infinity, the limit price is deterministic and equal to $\bar{V}_\iota(f)$. Moreover, by Proposition 1, the limit initial price satisfies

$$V_\circ(f) \leq \bar{V}_\iota(f) \leq V^\circ(f) \tag{10}$$

for all upper and lower common ex-ante expectations $V_\circ \in V_\circ$ and $V^\circ \in V^\circ$, and, more accurately, $\bar{V}_\iota(f) \in [V_*(f), V^*(f)]$. For example, equation (10) implies that if the traders are maxmin agents and share the same set of ex-ante probabilistic models $C \subseteq \Delta(\Omega)$, then, under full Bayesian updating, the initial limit price with private information $\bar{V}_\iota(f)$

is smaller than the common ex-ante evaluation $V^{\circ}(f) = \min_{p \in C} \int f dp$. Indeed, the initial equilibrium price is the result of a compounded pessimistic evaluation due to full Bayesian updating and iterated minimization across all the updated probabilistic models. ▲

Combining our previous results, we get a characterization for the existence of a nonlinear common ex-ante expectation $\bar{V}$ for $\{(V_i, \Pi_i)\}_{i \in I}$: under convergence to a deterministic limit, there exists a common ex-ante expectation if and only if the deterministic limit of all the I-sequences of higher-order expectations is the same. This generalizes the main result of Samet [42] to the class of rational preferences.

Corollary 1. *Let $\{(V_i, \Pi_i)\}_{i \in I}$ be a collection of interim expectations that exhibits convergence to a deterministic limit. The following statements are equivalent:*

(i) There exists a nonlinear common ex-ante expectation $\bar{V}$ for $\{(V_i, \Pi_i)\}_{i \in I}$;

(ii) For each $f \in \mathbb{R}^{\Omega}$ there exists $k_f \in \mathbb{R}$ such that for each I-sequence $(i_t)_{t \in \mathbb{N}}$

$$\lim_{t \to \infty} V_{i_t} \circ V_{i_{t-1}} \circ \ldots \circ V_{i_2} \circ V_{i_1}(f) = k_f 1_{\Omega}.$$

(iii) We have $V_* = V^*$.

In this case, for each $f \in \mathbb{R}^{\Omega}$, we have $V_*(f) = V^*(f) = \bar{V}(f) = k_f$.

As an immediate consequence of Theorem 1 and Corollary 1, we get that our characterization of the nonlinear common ex-ante expectation holds provided that agents' interim preferences have full support and there is no public information. Next, we illustrate the (asset-pricing) equilibrium implications of the existence of a nonlinear common ex-ante expectation.

Example (Forecasting the forecaster continued)**.** Assume that the agents have a nonlinear common ex-ante expectation $\bar{V}$. For a sufficiently long truncation of $(i_t)_{t \in \mathbb{N}}$, the initial equilibrium price is approximately state-independent and equal to the common ex-ante evaluation $\bar{V}(f)$ of the asset. In words, under a nonlinear common ex-ante expectation, the order of trades does not affect the initial price. Conversely, for any two arbitrary I-sequences truncated at $\bar{t} \in \mathbb{N}$, we can falsify the existence of a nonlinear common ex-ante expectation by checking whether the corresponding equilibrium prices are sufficiently different. ▲

On the one hand, Corollary 1 provides sufficient conditions for the existence of a nonlinear common ex-ante expectation, as well as a way to compute it. On the other hand, mutual dynamic consistency with respect to all the information structures of the agents is very restrictive under ambiguity-averse preferences, as pointed out by Ellis [11] and Gumen and Savochkin [20]. Therefore, our corollary implies that the order of trades in Example 4 is generally relevant for the equilibrium price and more so with ambiguity aversion. This creates scope for the manipulation of the frequency with which agents with a given information structure can trade to affect the equilibrium price. In other words, without SEU, for example, under ambiguity-averse preferences, there is scope for the optimal design of the sequential trading protocol.

Clearly, whenever the ex-ante preferences of the agents are of the form of Example 1 (resp. 2) and all the agents update with the proxy updating rule (resp. full Bayesian updating) with respect to their information partitions $(\Pi_i)_{i=1}^n$, there is a lower (resp. upper) common ex-ante expectation. We next provide a last concrete example where there exists a nonlinear common ex-ante expectation.

Example 5 (Multiplier expectations and misspecification aversion)**.** Here, we consider multiplier preferences (see Hansen and Sargent [23], axiomatized in Strzalecki [48]). Let $\mu \in \Delta(\Omega)$ have full support and let Π be a partition. Define

$$\bar{V}_{\lambda,\mu}(f) = \min_{\mu' \in \Delta(\Omega)} \{\mathbb{E}_{\mu'}(f) + \lambda R(\mu' || \mu)\} \qquad \forall f \in \mathbb{R}^\Omega \tag{11}$$

and

$$V_{\lambda,\mu}(\omega, f) = \min_{p \in \Delta(\Omega)} \{\mathbb{E}_p(f) + \lambda R(p || p_\mu(\omega, \cdot))\} \qquad \forall \omega \in \Omega, \forall f \in \mathbb{R}^\Omega \tag{12}$$

where $\lambda > 0$ and $R(\cdot||\cdot)$ is the relative entropy. The agent has a probability model of reference μ, but she does not fully trust it. She is willing to consider other models μ', nevertheless the farther they are in terms of relative entropy from μ (resp., its update), the less plausible they are, and the smaller role they play in the minimization (11) (resp., (12)). Here, λ is a parameter that captures the agent's aversion to the potential misspecification of μ: the lower λ, the more the decision maker considers other probability models p. It is well known that $\left(\bar{V}_{\lambda,\mu}, V_{\lambda,\mu}, \Pi\right)$ is a nonlinear conditional expectation (see Maccheroni et al. [33, Section 5.2]). Next, assume that each agent has an information partition Π_i and her conditional interim expectations $(V_{i,\lambda,\mu}, \Pi_i)$ are computed according to (12) with respect to Π_i. In this case, the common prior μ uniquely defines the nonlinear common ex-ante expectation $\bar{V}_{\lambda,\mu}$.

It is possible to generalize multiplier expectation so as to take into account ambiguity aversion as in Cerreia-Vioglio et al. [10]. Formally, rather than a single model, let us fix a set $\mathcal{M} \subseteq \Delta(\Omega)$ of probabilities with full support over Ω and a partition Π. In particular, assume that $\mu_{|\Pi} = \mu'_{|\Pi}$ for all $\mu, \mu' \in \mathcal{M}$, that is, there is no model uncertainty with respect to the events that are Π-measurable. Next, define $\bar{V}_{\lambda,\mathcal{M}}(f) = \min_{\mu\in\mathcal{M}} \bar{V}_{\lambda,\mu}(f)$ for all $f \in \mathbb{R}^{\Omega}$. Similarly as before, assume that each agent has an information partition Π_i and her conditional interim expectation (V_i, Π_i) is $V_{i,\lambda,\mathcal{M}}(\omega, f) = \min_{\mu\in\mathcal{M}} V_{i,\lambda,\mu}(\omega, f)$. For every $i \in I$, if Π_i is coarser than Π, then $\left(\bar{V}_{\lambda,\mathcal{M}}, V_{i,\lambda,\mathcal{M}}, \Pi_i\right)$ is a nonlinear conditional expectation. The interpretation is that the agents are uncertain about the probabilistic model beyond their aggregate information $\Pi_{\inf}$. Moreover, the agents are averse to misspecification about the model restricted on $\Pi_{\inf}$. ▲

3.2 Characterization and computation of extreme ex-ante expectations

In this section, we explain how extreme ex-ante expectations are characterized axiomatically, and we operationalize them by providing an algorithm to compute them. Readers chiefly interested in the application to beauty contests can skip this subsection.

3.2.1 Axiomatic characterization

We develop a full-fledged formal axiomatization in Online Appendix E. Here, we sketch our findings. We start by considering a collection of preference relation $\left\{\left(\succsim, \{\succsim_{\omega,\Pi_i}\}_{\omega\in\Omega}\right)\right\}_{i=1}^{n}$ where $\succsim$ is interpreted as the ex-ante preference and each $\succsim_{\omega,\Pi_i}$ corresponds to the interim preferences of agent i in state ω.

Under standard monotonicity, continuity, and weak order axioms, it is easy to show that these preference relations are represented, respectively, by functions $\bar{V}$ and $V_i(\omega, \cdot)$ that are normalized and monotone. We next impose the axioms corresponding to the information interpretation of these preferences. We first require that for every $i \in \{1, ..., n\}$ and every $\omega, \omega' \in \Omega$, $\Pi_i(\omega) = \Pi_i(\omega')$ implies $\succsim_{\omega,\Pi_i} = \succsim_{\omega',\Pi_i}$, consistently with the interpretation that interim preferences should not reflect more information than the one available to the agent.

We then require that states outside $\Pi_i(\omega)$ are null for preference $\succsim_{\omega,\Pi_i}$, an axiomatic form of consequentialism.

With this, the key task becomes to provide axioms that link the ex-ante and interim preferences so that the former is a lower ex-ante expectation for the former. The key axiom turns out to be dynamic subconsistency (also called weak dynamic consistency in Gul and Pesendorfer, [19, Axiom C4]): For each $f \in \mathbb{R}^{\Omega}$ and for each g measurable with respect to the partition Π_i,

$$g \succsim_{\omega,\Pi_i} f, \qquad \forall \omega \in \Omega \implies g \succsim f.$$

It captures a form of higher uncertainty aversion for the ex-ante preference (for the uncertainty revealed by Π_i). Indeed, it requires that if an act g that is certain conditional on Π_i is interim preferred to an arbitrary act f, then this preference is maintained at the ex-ante stage. We show that if and only this axiom is satisfied, jointly with the ones mentioned above, the representing pair $\bar{V}$ and $\{V_i(\omega,\cdot)\}_{\omega\in\Omega, i\in\{1,\dots,n\}}$ is a lower common ex-ante expectation.

Finally, we show that to single out the extreme lower ex-ante expectation, the pair must be required to have "as much dynamic consistency as possible" in a well-defined sense. Indeed, we show V_* represents the unique preference $\succsim$ that satisfies the aforementioned axioms together with $\{\succsim_{\omega,\Pi_i}\}_{\omega\in\Omega, i\in\{1,\dots,n\}}$ and is such that (i) for every act $f \in \mathbb{R}^{\Omega}$ there is at least one agent that is dynamically consistent in the evaluation of that act and (ii) that all the agents are dynamically consistent with respect to the acts measurable with respect to any of the partitions Π_i. Formally, (i) for each $f \in \mathbb{R}^{\Omega}$ there exists $i \in \{1,\dots,n\}$ such that $\bar{V}(f) = \bar{V}(V_i(\cdot,f))$ and (ii) for $i,j \in \{1,\dots,n\}$ if h is Π_i-measurable, $\bar{V}(h) = \bar{V}(V_j(\cdot,h))$.[11]

[11] The last result that singles out V_* is proved for the case of two players. It is known that extending the results about iterated expectations from the two-player to arbitrary many cases involves additional difficulties; see Hellman [25]. The reason here is the same as there: with multiple players, I-sequences are not equivalent to permutation sequences in which players always show up in the same order. Still, the direction of the result in which axiom implies that the ex-ante expectation is the extreme lower ex-ante expectation holds for arbitrary players. More generally, a similar analysis can be conducted for (extreme and non) upper ex-ante expectations.

3.2.2 Algorithm to construct extreme common ex-ante expectations

In this section, we provide an operational algorithm to compute V_* starting from an arbitrary collection of interim expectations $\{(V_i, \Pi_i)\}_{i\in I}$. Differently from the characterization of V_* in Proposition 1, this algorithm does not involve a minimization over the infinite set of I-sequences. Let

$$\hat{V}(f) = \max_{\omega\in\Omega} f(\omega) \qquad \forall f \in \mathbb{R}^{\Omega}.$$

Clearly, $\hat{V}$ is an ex-ante expectation. Next, define recursively the sequence $\left\{\hat{V}^{\tau}\right\}_{\tau\in\mathbb{N}}$ of real-valued functions over $\mathbb{R}^{\Omega}$ by $\hat{V}^1 = \hat{V}$ and

$$\hat{V}^{\tau+1}(f) = \min_{i\in I} \hat{V}^{\tau}(V_i(f)) \qquad \forall f \in \mathbb{R}^{\Omega}, \forall \tau \in \mathbb{N}.$$

For example, $\hat{V}^2(f) = \min_{i\in I} \max_{\omega\in\Omega} V_i(\omega, f)$ assigns to each act f its worst most favorable interim expectation across all agents. In general, $\hat{V}^{\tau+1}$ assigns to f its worst most favorable τ-th order interim expectation across all sequences of τ agents.

Proposition 2. *We have that* $\lim_{\tau} \hat{V}^{\tau}$ *exists and it is equal to* V_*.

4 Equilibrium and (un-)common ex-ante preferences

In this section, we consider the equilibrium implications of common ex-ante expectations for a class of coordination games. In each of the following applications, the equilibrium $\sigma^{\beta} = \left(\sigma_i^{\beta}\right)_{i\in I} \in \left(\mathbb{R}^{\Omega}\right)^n$ is described by the following fixed-point condition:

$$\sigma_i^{\beta}(\omega) = V_i\left(\omega, (1-\beta)\hat{f} + \beta \sum_{j\in I} w_{ij}\sigma_j^{\beta}\right) \qquad \forall \omega \in \Omega, \forall i \in I. \tag{13}$$

Here, $\hat{f} \in \mathbb{R}^{\Omega}$ is a payoff-relevant fundamental, $\beta \in (0,1)$ parametrizes the relative importance of *coordination* with other agents over *adaptation* to the fundamental, and $W = \{w_{ij}\}_{i,j\in I} \in \mathbb{R}^{n\times n}$ is a stochastic matrix where each w_{ij} captures the relative importance of agent j for i.[12]

[12] A matrix $W \in \mathbb{R}^{n\times n}$ is stochastic if and only if $w_{ij} \geq 0$ for all $i, j \in I$ and $\sum_{j\in I} w_{ij} = 1$ for all $i \in I$.

The interpretation is that the equilibrium outcome for agent i coincides with her (generalized) expectation of a combination of the fundamental and the equilibrium outcomes of the other players. This kind of fixed-point condition is ubiquitous in models of asset pricing with beauty contests (cf. Morris and Shin [38]), networks of financial institutions (cf. Jackson and Pernoud [30]), and price competition (cf. Angeletos and Pavan [2]) as we show in Section 4.4. In the SEU case, the high-coordination limit ($\beta \to 1$) of the equilibrium strategies is used to select an equilibrium of the pure-coordination games and can be related to the common prior expectation of the asset (cf. Shin and Williamson [45] and Golub and Morris [18]). Analogously, the characterization of this limit and its relation to the ex-ante preferences we have defined will be the main focus of our analysis.

4.1 Beauty contests: coordination and equilibrium

As a leading application, we consider a beauty-contest model with random matching and private information (as in Golub and Morris [18]) that generalizes the forecasting the forecaster example of Section 3. Each $i \in I$ represents a continuum of agents sharing the same information partition Π_i. Time is discrete, and there is a random variable $\hat{f} \in \mathbb{R}^\Omega$ denoting the only asset in this economy that is sequentially traded with random matching. Let $\beta \in (0,1)$. At every period $t \in \mathbb{N}$, if an agent in class i holds the asset, with probability $(1-\beta)$, she has to liquidate the asset and obtain its fundamental (uncertain) value $\hat{f}$. With complementary probability β, she privately has to sell the asset to an agent from a randomly selected class and then leaves the game. The matching probabilities, conditional on not liquidating the asset, are described by a stochastic matrix W, where w_{ij} is the probability with which an agent in class i is matched to class j. In particular, the random matching and liquidation are independent of the state. After the realization of the matched class j, the agents in j compete a la Bertrand, offering a price to the asset holder in i, who decides to whom to sell the asset. This mechanism implies that in equilibrium, the offered price is equal to the (common) willingness to pay for the asset of the agents in class j. If an agent in class j acquires the asset, then the game continues to period $t+1$.[13]

[13] Observe that there is no relevant learning over time since the past owners of the asset have left the game. Moreover, conditional on non-liquidation, even if the asset holder would learn something about the state $\omega \in \Omega$ from the offers of the agents in j, accepting the highest offer is still dominant.

We study the equilibria of this game for variational preferences (cf. Maccheroni et al. [32]). A collection of interim expectations $\{(V_i, \Pi_i)\}_{i\in I}$ is *variational* if and only if for every $i \in I$ and $\omega \in \Omega$, there exists a lower semicontinuous, grounded, and convex cost function $c_{i,\omega} : \Delta(\Omega) \to [0, \infty]$ such that

$$V_i(\omega, f) = \min_{p\in\Delta(\Omega)} \{\mathbb{E}_p(f) + c_{i,\omega}(p)\} \tag{14}$$

for all $f \in \mathbb{R}^\Omega$.[14] Variational interim expectations exhibit violations of subjective expected utility due to aversion to ambiguity, a widely documented trait. The interpretation is that each agent considers the evaluation of the act under many probabilistic models, and $c_{i,\omega}$ penalizes more the models (subjectively) deemed less plausible. In particular, the probabilistic models p for which $c_{i,\omega}(p) = 0$ represent the "benchmark" models that i trusts the most in state ω. All the examples of preferences we have introduced, except quantile maximization, are variational.

Assumption 1 The collection of interim expectations $\{(V_i, \Pi_i)\}_{i\in I}$ has full support, is such that $\Pi_{\sup} = \{\Omega\}$, and is variational and W is strongly connected.

A (Markov) *strategy* for an agent in class $i \in I$ is a random variable $\sigma_i \in \mathbb{R}^\Omega$ that is measurable with respect to the information structure Π_i. In particular, from the point of view of agents in i, the strategies $\sigma_j \in \mathbb{R}^\Omega$ of agents in any class j are state-dependent offers that can be evaluated through their interim preferences. Let Σ denote the set of profiles of strategies for the n classes of agents, respectively. For every $\beta \in (0, 1]$, if we fix a profile of strategies $\sigma = (\sigma_j)_{j\in I} \in \Sigma$, then the corresponding (state-dependent) willingness to pay for asset $\hat{f}$ of any agent in class $i \in I$ is:

$$S_{\beta,i}(\sigma) = V_i\left((1-\beta)\hat{f} + \beta \sum_{j\in I} w_{ij}\sigma_j\right) \qquad \forall \omega \in \Omega. \tag{15}$$

The *equilibria* of this game correspond to the fixed points of the map $S_\beta(\cdot) : \Sigma \to \Sigma$, that is, $\sigma^\beta \in \Sigma$ is an equilibrium if and only if it satisfies equation (15).

Proposition 3. *For every $\beta \in (0, 1)$, there exists a unique equilibrium $\sigma^\beta \in \Sigma$ of the*

[14] A cost function c is grounded if and only if $\min_{p\in\Delta(\Omega)} c(p) = 0$.

game. Moreover, there exists $C \in \mathbb{R}_+$ such that, for every $\beta \in (0,1)$,

$$\max_{i,j \in I, \omega, \omega' \in \Omega} \left| \sigma_i^\beta (\omega) - \sigma_j^\beta (\omega') \right| \leq (1-\beta) C \max_{\omega, \omega' \in \Omega} \left| \hat{f}(\omega) - \hat{f}(\omega') \right|. \tag{16}$$

The inequality in equation (16) gives a bound on the maximum level of disagreement among the equilibrium asset evaluations. The RHS is monotonically decreasing in β and *linearly* vanishes as we let coordination become more important, that is, $\beta \to 1$. This implies that the price of the asset becomes constant across states and agents in the limit.

4.2 Beauty contests: coordination and misspecification neutrality

In this section, we characterize the unique equilibrium σ^β as coordination becomes more and more important, i.e., $\beta \to 1$. Define the set of interim benchmark beliefs

$$Q = \left\{ q \in (\Delta(\Omega))^{I \times \Omega} : \forall (i,\omega) \in I \times \Omega, \forall \omega' \in \Pi_i(\omega), q_{i,\omega'} = q_{i,\omega}, c_{i,\omega}(q_{i,\omega}) = 0 \right\}.$$

Each $q \in Q$ is a collection of interim beliefs for all the agents and states that are (i) measurable with respect to the information of the corresponding agents and (ii) most trusted in the given state. This can be combined with the network structure W to obtain an *interaction structure* $W^q \in \mathbb{R}_+^{(I \times \Omega) \times (I \times \Omega)}$ among agent-state pairs capturing both the interim beliefs of the agents as well as the strength of their links. Formally, we let

$$w^q_{(i,\omega)(j,\omega')} = w_{ij} q_{i,\omega}(\omega') \qquad \forall i,j \in I, \forall \omega, \omega' \in \Omega. \tag{17}$$

Under SEU interim preferences, there is a unique interaction structure (introduced by Golub and Morris [18]) pinned down by the network W and the posterior beliefs of the agents. In the present setting, model uncertainty translates into a multiplicity of relevant interim beliefs, hence into a multiplicity of interaction structures. However, this multiplicity is disciplined by both the information and the interim preferences of the agents.

Lemma 1. *For each $q \in Q$, there exists a unique (row vector) $\gamma^q \in \Delta(I \times \Omega)$ such that $\gamma^q = \gamma^q W^q$.*

This is a consequence of the connectedness properties of each W^q implied by $\Pi_{\text{inf}} =$

$\{\Omega\}$, full support of $\{V_i, \Pi_i\}_{i\in I}$, and that W is strongly connected. We are now ready to state the main result of this section.

Theorem 2. *For all $i \in I$ and $\omega \in \Omega$,*

$$V_*\left(\hat{f}\right) \leq \lim_{\beta\to 1} \sigma_i^\beta(\omega) = \min_{q\in Q} \sum_{(j,\omega')\in I\times\Omega} \gamma^q_{j,\omega'} \mathbb{E}_{q_{j,\omega'}}\left(\hat{f}\right). \tag{18}$$

Therefore, if there exists a nonlinear common ex-ante expectation $\bar{V}$ for $\{(V_i, \Pi_i)\}_{i\in I}$, then, for all $i \in I$ and $\omega \in \Omega$,

$$\bar{V}\left(\hat{f}\right) \leq \lim_{\beta\to 1} \sigma_i^\beta(\omega).$$

First, we observe that, in the limit where the coordination motive prevails, the equilibrium price is independent of the realized state and the agent's identity. In particular, the limit selects an equilibrium of the pure coordination game where the asset is payoff irrelevant. This generalizes a well-known fact under subjective expected utility (cf. Golub and Morris [18]).

Second, the constant limit price equals the most cautious average of the benchmark evaluations of $\hat{f}$ that are consistent with the network structure. Notably, the cautious selection of the benchmark models q from Q induced by the market interaction has two roles. While selecting beliefs that evaluate the asset in a cautious way (i.e., to keep the first-order evaluations $\mathbb{E}_{q_{i,\omega}}\left(\hat{f}\right)$ low), it also determines how the heterogeneous evaluations are aggregated through the eigenvector centrality γ^q of the interaction structure.

Third, our formula points out that the strong coordination motives in the market attenuate the ambiguity concern exhibited by the equilibrium evaluation. Intuitively, the asymmetric information of the traders combined with their coordination motive implies that the equilibrium prices are less variable across states than the fundamental itself. Indeed, when evaluating the asset conditional on private information $\Pi_i(\omega)$, the agent does not only need to take into consideration what f pays in the states in $\Pi_i(\omega)$. Indeed, facing the possibility of trading the asset to those j such that $w_{ij} > 0$, they have to take into account their evaluation, which will also depend on the asset payment in the states in $\cup_{\hat{\omega}\in\Pi_i(\omega)}\Pi_j(\hat{\omega})$, i.e., the states that j believes are possible in one of the states i believes to be possible. It is this dependence on more states that dampens the variability of the asset price compared to the underlying fundamental, and, of course, this dampening is only amplified by the fact that the evaluation of j itself depends on the ones of their neighbors k such that $w_{jk} > 0$. Therefore, the uncertainty-averse traders evaluate owning the asset more favorably than the fundamental.

More formally, we have

$$\lim_{\beta \to 1} \sigma_i^\beta (\omega) \geq V_i \left(\omega, \hat{f}\right) \qquad \forall i \in I, \forall \omega \in \Omega,$$

since each collection of beliefs $q \in Q$ satisfy $c_{i,\omega} (q_{i,\omega}) = 0$ for all $i \in I$ and $\omega \in \Omega$. In turn, this immediately yields the lower bound in equation (18) and, when there exists a common ex-ante evaluation, we actually have $V_* \left(\hat{f}\right) = \bar{V} \left(\hat{f}\right)$, implying that the equilibrium price is higher than the *shared* ex-ante evaluation. This is a sharp difference with respect to the case of SEU interim preferences where, under a common prior, the limit equilibrium price coincides with the prior expectation.

We next show that in several important cases, this ambiguity reduction completely mutes the agent's concern.

Corollary 2. *Assume that, for all* $i \in I$ *and* $\omega \in \Omega$, *it holds* $\arg\min_{p \in \Delta(\Omega)} c_{i,\omega} (p) = \left\{q_{i,\omega}^*\right\}$. *For all* $i \in I$ *and* $\omega \in \Omega$,

$$\lim_{\beta \to 1} \sigma_i^\beta (\omega) = \sum_{(i,\omega) \in I \times \Omega} \gamma_{i,\omega}^{q^*} \mathbb{E}_{q_{i,\omega}^*} \left(\hat{f}\right).$$

Moreover, if the collection $\left\{q_{i,\omega}^*\right\}_{i \in I, \omega \in \Omega}$ *admits a common prior* μ^* *we have*

$$\lim_{\beta \to 1} \sigma_i^\beta (\omega) = \mathbb{E}_{\mu^*} \left(\hat{f}\right).$$

This result characterizes an extreme form of ambiguity-aversion reduction. Indeed, whenever each interim preference has a unique most trusted benchmark model, the limit equilibrium price is equal to an ex-ante SEU evaluation of the asset, implying that only the interim benchmark models matter as the importance of coordination grows. This reduction is particularly stark when the agents' benchmark models are consistent with a common prior μ^*. In this case, the ex-ante evaluation of the asset according to this probabilistic model is the limit price equilibrium, and this limit is the same regardless of the ambiguity attitudes and the network structure. Therefore, whenever μ^* is highly misspecified with respect to the "objective" probability model ν^*, there is a divergence between the limit market price $\lim_{\beta\to 1}\sigma_i^\beta(\omega)$ and the rational-expectations value $\mathbb{E}_{\nu^*}\left(\hat{f}\right)$ pair to the full gap $\mathbb{E}_{\mu^*}\left(\hat{f}\right)-\mathbb{E}_{\nu^*}\left(\hat{f}\right)$, with no attenuation whatsoever despite the shared concern for misspecification.

In the next example, we illustrate this phenomenon within the class of multiplier preferences with Bayesian updating from a common prior.

Example 6. Suppose that, in the ex-ante stage, the agents share the same unique benchmark model $\mu^* \in \Delta(\Omega)$, but they are averse to misspecification with possibly different attitudes: each $i \in I$ evaluates $\hat{f}$ as

$$\min_{p\in\Delta(\Omega)}\left\{\mathbb{E}_p\left(\hat{f}\right)+\lambda_i R\left(p||\mu^*\right)\right\}$$

where $(\lambda_i)_{i\in I} \in \mathbb{R}^n_{++}$ is a profile of misspecification concern indexes. After having observed their own private information, the agents update the benchmark model to $p_{\mu^*,i}(\omega,\cdot)$. Therefore, the interim evaluation of i at ω is

$$V_i(\omega, f)=\min_{p\in\Delta(\Omega)}\left\{\mathbb{E}_p(f)+\lambda_i R\left(p||p_{\mu^*,i}(\omega,\cdot)\right)\right\} \qquad \forall f\in\mathbb{R}^\Omega.$$

In this case, Corollary 2 implies that

$$\lim_{\beta\to 1}\sigma_i^\beta(\omega)=\mathbb{E}_{\mu^*}\left(\hat{f}\right) \qquad \forall i\in I, \forall\omega\in\Omega.$$

That is, the ambiguity is completely washed out, and the price converges to the expected evaluation of the asset, independently of the attitudes towards misspecification. If these attitudes are homogeneous, i.e., $\lambda_i=\lambda$ for all $i \in I$, then there exists a common ex-ante expectation

$$\bar{V}(f)=\min_{p\in\Delta(\Omega)}\left\{\mathbb{E}_p(f)+\lambda R\left(p||\mu^*\right)\right\} \qquad \forall f\in\mathbb{R}^\Omega$$

and a wedge between $\bar{V}\left(\hat{f}\right)$ and $\lim_{\beta\to 1}\sigma_i^\beta(\omega)$ arises whenever the asset pays a different amount in each state. More generally, this wedge remains present between V_* and $\lim_{\beta\to 1}\sigma_i^\beta(\omega)$ even when the misspecification attitudes are heterogeneous. ▲

4.3 Beauty contests: unambiguous information structure

Here, we consider an important particular case: the agents are unambiguous with respect to the information structure while still possibly perceiving ambiguity about the fundamental $\hat{f}$, i.e., there is no *strategic ambiguity.* In this case, the first-order expectations of the agents exhibit perceived ambiguity and ambiguity aversion, whereas the higher-order expectations do not; that is, they are SEU. Formally, we say that the *information structure is unambiguous* if and only if for every $i \in I$, V_i is Π_{inf}-affine, that is

$$V_i\left(\omega, (1-\alpha)\, h + \alpha g\right) = (1-\alpha)\, V_i\left(\omega, h\right) + \alpha V_i\left(\omega, g\right)$$

for all $\alpha \in (0,1)$, for all $\omega \in \Omega$, and for all $g, h \in \mathbb{R}^{\Omega}$ where g is Π_{inf}-measurable. This implies that V_i is linear over the vector space of elements $g \in \mathbb{R}^{\Omega}$ that are Π_{inf}-measurable. This restriction is reasonable, for instance, in games where the agents repeatedly interact and can observe the actions of the coplayers after each interaction. In this case, if the agents are correctly specified, then their beliefs will converge to the true distribution on Π_{inf}.

Proposition 4. *If the information structure is unambiguous, then for all $i \in I$ and $\omega \in \Omega$,*

$$\lim_{\beta \to 1} \sigma_i^{\beta}\left(\omega\right) \in \left[V_*\left(\hat{f}\right), V^*\left(\hat{f}\right)\right].$$

Moreover, if there exists a nonlinear common ex-ante expectation $\bar{V}$ for $\{(V_i, \Pi_i)\}_{i\in I}$, then, for all $i \in I$ and $\omega \in \Omega$,

$$\lim_{\beta \to 1} \sigma_i^{\beta}\left(\omega\right) = \bar{V}\left(\hat{f}\right).$$

Whenever the traders are not ambiguous regarding events in their information structures, the extreme ex-ante preferences give both an upper and lower bound for any possible equilibrium selection. Next, observe that whenever a nonlinear common ex-ante expectation exists, the identity $\bar{V} = V_* = V^*$ implies that the limit equilibrium $\lim_{\beta \to 1} \sigma^{\beta}$ is well defined and equal to the ex-ante evaluation. This is an implication of the common prior assumption under SEU (cf. Golub and Morris [18]) that we extend to the unambiguous-information case. Finally, comparing the second parts of Theorem 2 and of Proposition 4, we observe that the only ambiguity the market interaction can tame is the one about the information structures of the agents.

4.4 Additional application: price competition

As mentioned above, the previous analysis only depends on the equilibrium equation (13) regardless of the specifics of the underlying games. Here, we provide an alternative foundation of (13) based on a price-competition model. Concretely, n firms are competing on prices. We fix a random variable $\hat{f} \in \mathbb{R}^{\Omega}$ representing the state of the economy, and we let y denote its realization. The interpretation is that there is aggregate uncertainty about y. Each firm i chooses the price $x_i \in \mathbb{R}$ for its good, has 0 production costs, and its payoff function $u_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ depends on the state y as well as the entire profile of prices $x \in \mathbb{R}^n$: $u_i(x, y) = D_i(x, y) x_i$ where $D_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the demand function faced by firm i and is defined as $D_i(x, y) = \beta \sum_{j \in I} w_{ij} x_j + (1-\beta) y - x_i/2$ for some $\beta \in (0, 1)$ and a stochastic and strongly connected matrix W with $w_{jj} = 0$ for all $j \in I$. The demand faced by firm i negatively depends on its own price and positively depends on the state of the economy and on the prices of the other firms, respectively, with coefficients $(1-\beta)$ and β. As usual, the interpretation is that the firms compete on the same market with partially differentiated products, and w_{ij} captures the similarity of products i and j. Suppose also that $\{(V_i, \Pi_i)\}_{i \in I}$ is a collection of maxmin (cf. Example 2) interim preferences. In particular, let $C_{i,\omega} \subseteq \Delta(\Omega)$ denote the set of interim probabilistic models of agent i at state ω.

As before, a strategy $\sigma_i \in \mathbb{R}^{\Omega}$ of agent i is measurable with respect to Π_i. Given a strategy profile σ_{-i} for the coplayers of i, the problem faced by i given state $\omega \in \Omega$ is $\max_{x_i \in \mathbb{R}} \min_{p \in C_{i,\omega}} \mathbb{E}_p \left(\left((1-\beta) \hat{f} + \beta \sum_{j \in I} w_{ij} \sigma_j \right) x_i - \frac{x_i^2}{2} \right)$. With this, the first-order condition characterizing the equilibrium σ^{β} for every $\beta \in (0, 1)$ is

$$\sigma_i^{\beta}(\omega) = \min_{p \in C_{i,\omega}} \mathbb{E}_p \left((1-\beta) \hat{f} + \beta \sum_{j \in I} w_{ij} \sigma_j^{\beta} \right) \qquad \forall \omega \in \Omega, \forall i \in I, \tag{19}$$

which is just a particular case of equation (13), so that, mutatis mutandis, all the previous analysis applies to this competition game as well.

5 Related literature

Our work lies at the intersection of several strands of literature, including decision theory, game theory, and information economics. Our Theorem 1 and Corollary 1 generalize to rational preferences the common-prior characterization of Samet [42]. In the case of SEU, the latter has been previously extended to compact spaces of uncertainty in Hellman [25] and to more general payoff-relevant spaces in Golub and Morris [17].

More recently, the existence of a nonlinear common ex-ante expectation for non-ambiguity-neutral preferences under both dynamic consistency and consequentialism has been studied by Ellis [11]. This paper shows that if the agents' information has a product structure in addition to the previous properties, then their interim preferences cannot exhibit violations of Savage's sure-thing principle for acts that are measurable with respect to the aggregate information. However, the following facts limit the implications of this critical result for our analysis: (i) We also consider and characterize weaker versions of common dynamic consistency, which allow for violations of Savage's sure-thing principle (ii) We never impose a product structure for the information of the agents which in turn would rule out hard evidence about the interim types of the opponents (e.g., E-mail-game like information structures have such hard evidence) (iii) For the class of games that we consider in Section 4, even the residual ambiguity about the fundamental state is relevant for the equilibrium outcomes.

Our applications generalize the standard beauty-contest settings in Shin and Williamson [45], Allen et al. [1], or Golub and Morris [18] by allowing for ambiguity aversion and obtaining notable equilibrium implications. In general, our work proposes a viable theory for games under incomplete information without SEU. In this regard, Epstein and Wang [13] introduce a universal type space for a class of preferences very similar to the rational one analyzed in the current paper. We improve on this work by characterizing the collections of finite type spaces that admit some degree of ex-ante mutual agreement within this universal type space. On the more applied side, we contribute to a recent growing literature that studies the joint effect of ambiguity aversion and differential information on equilibrium prices (see Huo et al. [26] and the citations therein).

Relatedly, we improve on the analysis of incomplete-information games under uncertainty of Kajii and Ui [28] by considering variational preferences and deriving equilibrium properties for a specific class of coordination games. Moreover, we focus here on simultaneous-move games rather than analyzing the effect of ambiguity aversion in multi-stage games such as Battigalli et al. [3]-[4], and Hanany et al. [22], which in turn provide a very different set of results.

Our results are complementary to the extended literature on no-trade results without SEU. On the one hand, Billot et al. [5], Rigotti et al. [40], and Strzalecki and Werner [49] study efficient allocations under ambiguity with public information, as opposed to the private-information setting of the current paper. On the other hand, Kajii and Ui [29] and Martins-da-Rocha [34] provide no-trade characterizations of the existence of common ex-ante benchmark beliefs.

Finally, our work is related to the extended literature on updating non-SEU preferences under (relaxations of) consequentialism and dynamic consistency as in Ghirardato [15], Epstein and Schneider [12], Maccheroni et al. [33], Hanany and Klibanoff [21], and Gumen and Savochkin [20]. However, we take an interim approach rather than deriving or studying a given updating rule, as in the works above. We derive the ex-ante preferences that are consistent with the given interim ones. This allows us to connect our results to existing updating rules by comparing the prescribed ex-ante preferences with the ones we obtain from the interim preferences and derive new insights into their implications in strategic interactions.

6 Conclusion

The results of this paper can also be used as a stepping stone for further analysis of games beyond SEU. Here, we highlight some open questions and future research avenues.

First, as already stressed, despite our analysis following an interim approach, our results can be used in games of incomplete information with general preferences under uncertainty and a *given* set of updating rules. Indeed, the disagreement bound in Proposition 3 and the limit characterization in Theorem 2 did not put any intertemporal restriction on the agents' preferences. So, for example, if all the agents are maxmin, share the same ex-ante set of probability models, and update their beliefs with full Bayesian updating, then our results give tools to study how the equilibrium outcomes change with respect to the agents' private information. Therefore, our results can be a stepping stone toward an information design model in beauty contests under non-SEU preferences.

Second, our framework enables us to revisit some classical results for SEU agents on incomplete information games to understand whether they carry on with more general preferences. An example is a result established in Nielsen et al. [39] that if a stochastically monotone function (often interpreted as the price of an asset) of the beliefs is common knowledge across the players, their beliefs coincide. The result extends if the information structure is unambiguous but may fail more generally.

Third, our framework and results are the first steps toward a general analysis of approximate common knowledge under model uncertainty. The standard analysis based on p-belief operators of Monderer and Samet [35] can be extended to a setting with multiple interim beliefs, for example, by requiring that all the interim probability models assign probability p to an event. In particular, within this richer framework, we can also ask about the strategic implications of approximate common unambiguity of an event, that is, for each agent, all the interim belief of that agent assigns the same probability to that event. Our examples suggest that this might well have significant strategic consequences such as contagion or taming of ambiguity aversion among agents.

Relatedly, approximate common knowledge has been recently studied in a learning setting under SEU by Frick et al. [14]. Their common-learning results complement Samet's convergence result on higher-order expectations by showing that their KL divergence relative to the prior distribution decreases monotonically along any sequence. On the one hand, our Theorem 1 offers an alternative distance between higher-order and prior expectations. On the other hand, our setting can be used to analyze common learning in the presence of ambiguity, for example, by resorting to the learning rules studied in Lanzani [31].

Finally, our analysis is a stepping stone to obtaining sharper equilibrium refinements in complete information games. Indeed, in the SEU world, Kajii and Morris [27] pioneered a robust approach that selects only the subset of equilibria that are limit points of *every* sequence of incomplete information games, approximating the original complete information game. An even sharper refinement would only select equilibria that are limit points, including elaborations under incomplete information and non-SEU preferences.

A Appendix: Mathematical preliminaries

Since Ω is finite, with a small abuse of notation, we equivalently view Ω as the set $J = \{1, ..., \bar{n}\}$. We also denote the elements of the canonical basis of $\mathbb{R}^{\bar{n}}$ by e^j for all $j \in J$. The composition of normalized, monotone, and continuous operators is an operator which shares the same properties. A normalized and monotone operator $T : \mathbb{R}^{\bar{n}} \to \mathbb{R}^{\bar{n}}$ is linear if and only if there exists a stochastic $\bar{n} \times \bar{n}$ matrix M such that $T(f) = Mf$ for all $f \in \mathbb{R}^{\bar{n}}$. All products of $\bar{n} \times \bar{n}$ matrices are to be intended backward/left, that is, $\Pi_{l=1}^{k+1} M_l = M_{k+1} \Pi_{l=1}^{k} M_l = M_{k+1} ... M_1$ for all $k \in \mathbb{N}$. Define $I_{\bar{n}}$ to be the $\bar{n} \times \bar{n}$ identity matrix. Given $j, j' \in J$ we say that j *is strongly monotone with respect to* j' (under T) if and only if there exists $\varepsilon_{jj'} \in (0, 1)$ such that for each $f \in \mathbb{R}^{\Omega}$ and for each $\delta \geq 0$

$$T_j \left(f + \delta e^{j'} \right) - T_j (f) \geq \varepsilon_{jj'} \delta. \tag{20}$$

We say that j *is constant with respect to* j' if and only if

$$T_j \left(f + \delta e^{j'} \right) - T_j (f) = 0 \qquad \forall f \in \mathbb{R}^{\Omega}, \forall \delta \geq 0. \tag{21}$$

We say that T is *dichotomic* if and only if for each $j, j' \in J$, j is either strongly monotone with respect to j' or constant.

Definition 6. *Let T be a monotone operator. We say that $A(T)$ is the* indicator matrix *of T if and only if its jj'-th entry is such that*

$$a_{jj'} = \begin{cases} 1 & j \text{ is strongly monotone wrt } j' \\ 0 & \text{otherwise} \end{cases} \qquad \forall j, j' \in J.$$

The indicator matrix $A(M)$ of an $\bar{n} \times \bar{n}$ nonnegative matrix M is defined to be such that $a_{jj'} = 1$ if and only if $m_{jj'} > 0$ and $a_{jj'} = 0$ if and only if $m_{jj'} = 0$. We say that $A(T)$ is *nontrivial* if and only if for each $j \in J$ there exists $j' \in J$ such that $a_{jj'} = 1$. The indicator matrix $A(T)$ of a monotone operator T induces a natural partition of J. Recall that given a nonnegative $\bar{n} \times \bar{n}$ matrix A with *nonnull rows*, we can partition the set $J = \{1, ..., \bar{n}\}$ with the partition $\{J_l(A)\}_{l=1}^{m_A+1}$ of essential and inessential indexes of A. The first m_A sets consist of the essential classes while $J_{m_A+1}(A)$ consists of all inessential indexes and it might be empty. This is the case if A is symmetric, that is, $a_{jj'} = a_{j'j}$ for all

$j, j' \in J$. Instead, there always exists at least a nonempty class of essential indexes $J_1(A)$, see, e.g., Seneta [44]. We call $\Pi(A) = \{J_l(A)\}_{l=1}^{m_A+1}$ the partition of A. When $A = A(T)$ where T is normalized, monotone, and continuous and $A(T)$ is nontrivial, we denote by $\Pi(T)$ the partition $\Pi(A(T))$.

Lemma 2. *Let (V, Π) be an interim expectation with full support. The following statements are equivalent:*

(i) $a_{jj'} = 1$*;*

(ii) $\Pi(\omega_j) = \Pi(\omega_{j'})$*.*

In particular, $A(V)$ is symmetric, $a_{jj} = 1$ for all $j \in J$, $\Pi(V) = \Pi$, and V is dichotomic.

Given a stochastic matrix M, we denote by $\delta(M) = \min_{j,j' \in J: m_{jj'} > 0} m_{jj'}$ and $d(M) = \min_{j \in J} m_{jj}$. The next result builds on [6, Proposition 8].

Proposition 5. *If $T : \mathbb{R}^{\bar{n}} \to \mathbb{R}^{\bar{n}}$ is normalized, monotone, continuous, and such that $A(T)$ is nontrivial, then there exists a compact and convex set $\mathcal{M}(T)$ of $\bar{n} \times \bar{n}$ stochastic matrices such that $A(M) \geq A(T)$ for all $M \in \mathcal{M}(T)$ and for each $f \in \mathbb{R}^{\bar{n}}$ there exists $M(f) \in \mathcal{M}(T)$ such that $T(f) = M(f) f$. Moreover, if T is dichotomic, then $\mathcal{M}(T)$ can be chosen to be such that $A(M) = A(T)$ for all $M \in \mathcal{M}(T)$.*

Theorem 3. *Let $\{T_i\}_{i \in I}$ be a finite collection of normalized, monotone, and continuous dichotomic operators. If 1) $A(T_i)$ is symmetric for all $i \in I$, 2) $a_{i,jj} = 1$ for all $i \in I$ and for all $j \in J$, 3) the meet of the partitions $\{\Pi(T_i)\}_{i \in I}$ is $\{\Omega\}$, then for each I-sequence $(i_m)_{m \in \mathbb{N}}$ and for each $f \in \mathbb{R}^{\bar{n}}$ we have that $\lim_{m \to \infty} T_{i_m} \circ \ldots \circ T_{i_1}(f)$ exists and is a constant vector. Moreover, for each I-sequence $(i_m)_{m \in \mathbb{N}}$ and for each $\tau, t \in \mathbb{N}$, if i appears at least τ times in $(i_1, ..., i_t)$ for all $i \in I$, then*

$$\left|\left| \lim_{m \to \infty} T_{i_m} \circ \ldots \circ T_{i_1}(f) - T_{i_t} \circ \ldots \circ T_{i_1}(f) \right|\right|_\infty \leq \left(1 - \delta^{2^{\bar{n}^2} \bar{n}^2}\right)^{\tau 2^{-\bar{n}^2} - 1} ||f||_\infty ,$$

where $\delta = \inf_{i \in I, M \in \mathcal{M}(T_i)} \delta(M) > 0$.

B Appendix: Section 3

Proof of Theorem 1. By Lemma 2 and since $\{(V_i, \Pi_i)\}_{i\in I}$ is a finite set of full support interim expectations, we have that $A(V_i)$ is symmetric, $\Pi(V_i) = \Pi_i$, and V_i is dichotomic for all $i \in I$. Moreover, we have that $a_{i,jj} = 1$ for all $j \in J$ and for all $i \in I$. By Theorem 3 and since the meet of $\{\Pi(V_i)\}_{i\in I}$ is $\{\Omega\}$, we can conclude that for each I-sequence $\iota = (i_t)_{t\in\mathbb{N}}$ and for each $f \in \mathbb{R}^\Omega$ we have that $\lim_{m\to\infty} V_{i_m} \circ ... \circ V_{i_1}(f) = k_{\iota,f} 1_\Omega$ for some $k_{\iota,f} \in \mathbb{R}$. Moreover, there exist $\hat{\delta} = \left(\inf_{i\in I, M\in\mathcal{M}(V_i)} \delta(M)\right)^{2^{\bar{n}^2}\bar{n}^2} \in (0,1)$ and $\hat{t} = 2^{\bar{n}^2} \in \mathbb{N}$ such that for each I-sequence $(i_m)_{m\in\mathbb{N}}$ and for each $\tau, t \in \mathbb{N}$, if i appears at least τ times in $(i_1, ..., i_t)$ for all $i \in I$, then $||k_{f,\iota} 1_\Omega - V_{i_t} \circ ... \circ V_{i_1}(f)||_\infty \leq \left(1-\hat{\delta}\right)^{\frac{\tau}{\hat{t}}-1} ||f||_\infty$. Finally, the last part of the statement follows setting $C = \frac{1}{1-\hat{\delta}}$ and $\varepsilon = \left(1-\hat{\delta}\right)^{\frac{1}{\hat{t}}}$. ■

Whenever $\{(V_i, \Pi_i)\}_{i\in I}$ exhibits convergence to a deterministic limit, pick an arbitrary $\omega \in \Omega$ and define $V_\star : \mathbb{R}^\Omega \to \mathbb{R}$ and $V^\star : \mathbb{R}^\Omega \to \mathbb{R}$ by $V_\star(f) = \inf_{\iota\in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} \bar{V}_\iota(f)(\omega)$ and $V^\star(f) = \sup_{\iota\in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} \bar{V}_\iota(f)(\omega)$ for all $f \in \mathbb{R}^\Omega$. Clearly, we have that $V_\star \leq V^\star$.

Proof of Proposition 1. The first part of the statement immediately follows by Lemma 8. Since $V_\star$ (resp. $V^\star$) is a pointwise infimum (resp. supremum) of normalized and monotone functionals, so is $V_\star$ (resp. $V^\star$). Fix $f \in \mathbb{R}^\Omega$ and $i \in I$. Consider also an I-sequence ι'. Since $\{(V_i, \Pi_i)\}_{i\in I}$ exhibits convergence to a deterministic limit, we have $k_{V_i(f),\iota'} 1_\Omega = \lim_{t\to\infty} V_{i'_t} \circ V_{i'_{t-1}} \circ ... \circ V_{i'_2} \circ V_{i'_1}(V_i(f)) = \lim_{t\to\infty} V_{i''_t} \circ V_{i''_{t-1}} \circ ... \circ V_{i''_2} \circ V_{i''_1}(f) = k_{f,\iota''} 1_\Omega$ where ι'' is the I-sequence such that $\iota''_1 = i$ and $\iota''_t = \iota'_{t-1}$ for all $t \in \mathbb{N}\setminus\{1\}$. This implies that $k_{V_i(f),\iota'} = k_{f,\iota''} \geq \inf_{\iota\in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} k_{f,\iota} = \inf_{\iota\in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} \bar{V}_\iota(f) = V_\star(f)$. Since ι' was arbitrarily chosen, this implies that $V_\star(V_i(f)) = \inf_{\iota\in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} \bar{V}_\iota(V_i(f)) = \inf_{\iota\in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} k_{V_i(f),\iota} \geq V_\star(f)$, proving that $V_\star \in \mathcal{V}_\circ$. Next, consider $V' \in \mathcal{V}_\circ$ and suppose by contradiction that $V'(g) > V_\star(g)$ for some $g \in \mathbb{R}^\Omega$. Since $V'(g) > V_\star(g)$, there exists an I sequence ι such that $V'(g) 1_\Omega > \lim_{t\to\infty} V_{i_t} \circ V_{i_{t-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(g) = k_{g,\iota} 1_\Omega$. Since V' is normalized and continuous at $k_{g,\iota} 1_\Omega$ by Lemma 7, $V'(g) > V'(k_{g,\iota} 1_\Omega) = V'\left(\lim_{t\to\infty} V_{i_t} \circ V_{i_{t-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(g)\right) \geq V'(g)$, a contradiction. This proves that $V_\star = V_*$. A symmetric argument shows that $V^\star = V^*$. ■

Denote by P the set of permutations of agents, that is, bijections $\rho : \{1, ..., n\} \to \{1, ..., n\}$. Given $\rho \in P$, we denote by $V_\rho : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ the operator defined by

$$V_\rho = V_{\rho(1)} \circ V_{\rho(2)} \circ ... \circ V_{\rho(n)}. \tag{22}$$

As usual, we also denote by V_ρ^t the composition $\underbrace{V_\rho \circ ... \circ V_\rho}_{t\text{-times}}$ for all $t \in \mathbb{N}$ and for all $\rho \in P$.

Proof of Corollary 1. We prove the equivalence between (i) and (ii). The equivalence between (ii) and (iii) immediately follows by Proposition 1.

(i) implies (ii). By assumption, for each I-sequence $\iota = (i_t)_{t\in\mathbb{N}}$ and for each $f \in \mathbb{R}^\Omega$ we have that $\lim_{m\to\infty} V_{i_m} \circ ... \circ V_{i_1}(f) = k_{\iota,f} 1_\Omega$ for some $k_{\iota,f} \in \mathbb{R}$. By Lemma 7 and since $\bar{V}$ is an ex-ante expectation and $\left(\bar{V}, V_i, \Pi_i\right)$ is a nonlinear conditional expectation, we have that $k_{\iota,f} = \bar{V}(k_{\iota,f} 1_\Omega) = \bar{V}(\lim_{m\to\infty} V_{i_m} \circ ... \circ V_{i_1}(f)) = \lim_{m\to\infty} \bar{V}(V_{i_m} \circ ... \circ V_{i_1}(f)) = ... = \bar{V}(f)$, proving the implication.

(ii) implies (i). Fix a permutation $\bar{\rho} \in P$. Define the I-sequence $(i_k)_{k\in\mathbb{N}}$ by $i_k = \bar{\rho}(k \bmod n)$ for all $k \in \mathbb{N}$ such that $k \bmod n \neq 0$ and $i_k = \bar{\rho}(n)$ for all $k \in \mathbb{N}$ such that $k \bmod n = 0$. Define $\hat{V} : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ by $\hat{V}(f) = \lim_{\tau\to\infty} V_{\bar{\rho}}^\tau(f)$ for all $f \in \mathbb{R}^\Omega$. By assumption, we have that $\hat{V}$ is well defined and $\hat{V}(f)$ is a constant function for all $f \in \mathbb{R}^\Omega$. Since $V_{\bar{\rho}}$ is the composition of normalized, monotone, and continuous operators, so is $V_{\bar{\rho}}^\tau$ for all $\tau \in \mathbb{N}$ and, by passing to the limit, $\hat{V}$ is normalized and monotone. By assumption, we also have that $\hat{V}(f) = \lim_{\tau\to\infty} V_\rho^\tau(f)$ for all $f \in \mathbb{R}^\Omega$ and for all $\rho \in P$. Since $\hat{V}$ is normalized and monotone and $\hat{V}(f)$ is a constant function for all $f \in \mathbb{R}^\Omega$, we also have that $\hat{V}\left(\hat{V}(f)\right) = \hat{V}(f)$ for all $f \in \mathbb{R}^\Omega$, that is, $\hat{V} \circ \hat{V} = \hat{V}$. Define also $\bar{V} : \mathbb{R}^\Omega \to \mathbb{R}$ by $\bar{V}(f) = \hat{V}_1(f)$ for all $f \in \mathbb{R}^\Omega$. Since $\hat{V} \circ \hat{V} = \hat{V}$, it is immediate to see that $\bar{V}$ is an ex-ante expectation such that $\bar{V} \circ \hat{V} = \bar{V}$. This implies that for each $f \in \mathbb{R}^\Omega$ and for each $\rho \in P$

$$\bar{V}(V_\rho(f)) = \bar{V}\left(\hat{V}(V_\rho(f))\right) = \bar{V}\left(\lim_{\tau\to\infty} V_\rho^\tau(V_\rho(f))\right) = \bar{V}\left(\lim_{\tau\to\infty} V_\rho^{\tau+1}(f)\right) = \bar{V}(f). \quad (23)$$

Consider $i \in I$. Consider any permutation such that $\tilde{\rho}(1) = i$. By (23), we have that $\bar{V} \circ V_{\tilde{\rho}} \circ V_i = \bar{V} \circ V_i$. Consider the permutation $\hat{\rho}$ such that $\hat{\rho}(i') = \tilde{\rho}(i'+1)$ for all $i' \in \{1, ..., n-1\}$ and $\hat{\rho}(n) = i$. Define also $\tilde{V} = \bar{V} \circ V_i$. It follows that $\tilde{V}$ is an ex-ante expectation. Since $\bar{V} \circ V_{\tilde{\rho}} \circ V_i = \bar{V} \circ V_i$, we can conclude that $\tilde{V} \circ V_{\hat{\rho}} = \bar{V} \circ V_i \circ V_{\hat{\rho}} = \bar{V} \circ V_{\tilde{\rho}} \circ V_i = \bar{V} \circ V_i = \tilde{V}$. By induction, this implies that $\tilde{V} \circ V_{\hat{\rho}}^\tau = \bar{V} \circ V_i = \tilde{V}$ for all $\tau \in \mathbb{N}$. By (23) and Lemma 7 and since $\tilde{V}$ is an ex-ante expectation, $\bar{V} \circ \hat{V} = \bar{V}$, and $\tilde{V} \circ V_{\hat{\rho}}^\tau = \bar{V} \circ V_i = \tilde{V}$ for all $\tau \in \mathbb{N}$, we can conclude that $\bar{V}(f) = \bar{V}\left(\hat{V}(f)\right) = \bar{V}\left(V_i\left(\hat{V}(f)\right)\right) = \tilde{V}\left(\lim_{\tau\to\infty} V_{\hat{\rho}}^\tau(f)\right) = \bar{V}(V_i(f))$ for all $f \in \mathbb{R}^\Omega$, yielding that $\bar{V} \circ V_i = \bar{V}$. Since i was arbitrarily chosen, the statement follows. ■

Proof of Proposition 2. By induction, we have that each $\hat{V}^\tau$ is an ex-ante expectation. Fix $f \in \mathbb{R}^\Omega$. Since each V_i is an interim expectation, if $\tau \geq 2$, then we have that $\hat{V}^{\tau+1}(f) = \min_{i\in I} \hat{V}^\tau (V_i(f)) = \min_{i\in I} \min_{i'\in I} \hat{V}^{\tau-1}(V_{i'}(V_i(f))) \leq \min_{i\in I} \hat{V}^{\tau-1}(V_i(f)) = \hat{V}^\tau(f)$. Since f was arbitrarily chosen, this implies that $\hat{V}^{\tau+1} \leq \hat{V}^\tau$ for all $\tau \in \mathbb{N}\backslash\{1\}$. Define $\hat{V}^\infty : \mathbb{R}^\Omega \to \mathbb{R}$ by $\hat{V}(f) = \lim_\tau \hat{V}^\tau(f)$ for all $f \in \mathbb{R}^\Omega$. Since $\left\{\hat{V}^\tau(f)\right\}_{\tau\in\mathbb{N}}$ is an eventually decreasing sequence bounded from below by $\min_{\omega\in\Omega} f(\omega)$, $\hat{V}^\infty$ is a well defined ex-ante expectation. By construction, we have that $\hat{V}^{\tau+1}(f) \leq \hat{V}^\tau(V_i(f))$ for all $f \in \mathbb{R}^\Omega$ and $i \in I$. By passing to the limit, we obtain that $\hat{V}^\infty(f) \leq \hat{V}^\infty(V_i(f))$ for all $f \in \mathbb{R}^\Omega$ and for all $i \in I$, which in turn yields that $\hat{V}^\infty \leq V_*$ by definition of V_*. By induction assume that $\hat{V}^\tau \geq V_*$. It follows that $\hat{V}^{\tau+1}(f) = \min_{i\in I} \hat{V}^\tau(V_i(f)) \geq \min_{i\in I} V_*(V_i(f)) \geq V_*(f)$ for all $f \in \mathbb{R}^\Omega$, proving the inductive step. This yields that $\hat{V}^\infty \geq V_*$ and, in particular, $\hat{V}^\infty = V_*$. ■

C Appendix: Section 4

The elements of $\left(\mathbb{R}^\Omega\right)^n$ are vectors of n components, $\mathbf{f}$, where each component i, f_i, is an element of $\mathbb{R}^\Omega$. We endow $\left(\mathbb{R}^\Omega\right)^n$ with the norm $\|\ \|_* : \left(\mathbb{R}^\Omega\right)^n \to [0,\infty)$ defined by $\|\mathbf{f}\|_* = \sup_{i\in I} \|f_i\|_\infty$ for all $\mathbf{f} \in \left(\mathbb{R}^\Omega\right)^n$. Define $\hat{\mathbf{f}} \in \left(\mathbb{R}^\Omega\right)^n$ as $\hat{f}_i = \hat{f}$ for all $i \in I$. For every monotone operator $R : \left(\mathbb{R}^\Omega\right)^n \to \left(\mathbb{R}^\Omega\right)^n$ define the adjacency matrix $\bar{A}(R) \in \{0,1\}^{(n\times\bar{n})\times(n\times\bar{n})}$ as follows. For every $i,j \in I$ we set $\bar{a}_{(i,\omega)(j,\omega')}(R) = 1$ if and only if there exist $\mathbf{f} \in \left(\mathbb{R}^\Omega\right)^n$ and $\delta \geq 0$ such that $R_{i,\omega}\left(\mathbf{f} + \delta e^{j,\omega'}\right) - R_{i,\omega}(\mathbf{f}) > 0$. Moreover, we say that a class of indices Z, $\emptyset \neq Z \subseteq I \times \Omega$, is closed and strongly connected with respect to a matrix $A \in \{0,1\}^{(n\times\bar{n})\times(n\times\bar{n})}$ if and only if (i) for each $z, z' \in Z$ there exists a path $\{z_l\}_{l=1}^K \subseteq Z$ such that $a_{z_l z_{l+1}} = 1$ for all $l \in \{1,...,K-1\}$, $z_1 = z$ and $z_K = z'$; (ii) for each $z \in Z$, $a_{zz'} = 1$ implies $z' \in Z$.

Proof of Proposition 3. By Lemma 14, it follows that, for every $\beta \in (0,1)$, S_β is a contraction with respect to the supnorm and it admits a unique fixed point $\sigma^\beta \in \Sigma$. With this, the result follows by Lemma 12 and applying [7, Theorem 2] with $T = S_1$. ■

Next, let $\mathcal{W} \subseteq \mathbb{R}_+^{(n\times\bar{n})\times(n\times\bar{n})}$ denote the set of stochastic matrices over $I \times \Omega$ and define $\partial S_1(0) = \left\{\hat{W} \in \mathcal{W} : \forall (i,\omega) \in I \times \Omega, w_{i,\omega} \in \partial S_{1,i,\omega}(0)\right\}$, where $\partial S_{1,i,\omega}(0) \subseteq \Delta(I \times \Omega)$ is the superdifferential of the concave functional $S_{1,i,\omega}$ at 0. Let $s \in int(\Delta(I))$ denote the unique probability vector that satisfies $s = sW$, where uniqueness and strict positivity follow from the fact that W is strongly connected.

Proof of Lemma 1. By Lemma 12, there exists a unique class of indices Z, $\emptyset \neq Z \subseteq I \times \Omega$, that is closed and strongly connected with respect to $A(S_1)$ and, in addition, every row of $A(S_1)$ is not null. Given that S_1 is concave, it follows easily from the definition of $\partial S_1(0)$ that, for each $\hat{W} \in \partial S_1(0)$, Z is the unique closed and strongly connected class of indices with respect to $A\left(\hat{W}\right)$. Fix $q \in Q$. By Lemma 13, $W^q \in \partial S_1(0)$, so that Z is the unique closed and strongly connected class of indices with respect to $A(W^q)$. Next, observe that, for each $\gamma \in \Delta(I \times \Omega)$, we have $\gamma = \frac{1}{2}\gamma I + \frac{1}{2}\gamma W^q = \gamma\left(\frac{I+W^q}{2}\right)$ if and only if $\gamma = \gamma W^q$. In addition, given that $A\left(\frac{I+W^q}{2}\right) \geq A(W^q)$, it follows by [6, Corollaries 8.1 and 8.2] and [47, Theorem 2.2.5] that there exists a unique $\gamma^q \in \Delta(I \times \Omega)$ such that $\gamma^q = \gamma^q\left(\frac{I+W^q}{2}\right)$. By the previous claim, γ^q is also the unique probability vector such that $\gamma^q = \gamma^q W^q$. Given that $q \in Q$ was arbitrarily chosen, the statement follows. ■

Proof of Theorem 2. First, recall that S_1 is normalized, monotone, translation invariant, concave and, by Lemma 10, $S_1(\mathbf{f}) = \mathbf{f}$ if and only if there exists $m \in \mathbb{R}$ such that $f_i = f_{i'} = m1_\Omega$ for all $i, i' \in I$. With this, for all $(i, \tilde{\omega}) \in I \times \Omega$, $\lim_{\beta\to 1}\sigma_i^\beta(\tilde{\omega}) = \min_{\{\eta\in\Delta(I\times\Omega):\exists q\in Q,\eta=\eta W^q\}}\sum_{(i,\omega)\in I\times\Omega}\eta_{i,\omega}\hat{f}(\omega) = \min_{q\in Q}\sum_{(i,\omega)\in I\times\Omega}\gamma_{i,\omega}^q\hat{f}(\omega)$, where the first equality follows by [7, Lemma 4 and Proposition 14] and the second equality follows by Lemma 1. Next, fix $q \in Q$ and observe that

$$\sum_{(i,\omega)\in I\times\Omega}\gamma_{i,\omega}^q\mathbb{E}_{q_{i,\omega}}\left(\hat{f}\right) = \sum_{(i,\omega)\in I\times\Omega}\gamma_{i,\omega}^q\left[\sum_{(j,\omega')\in I\times\Omega}q_{i,\omega}(\omega')\,w_{ij}\hat{f}(\omega')\right] = \sum_{(i,\omega)\in I\times\Omega}\gamma_{i,\omega}^q\hat{f}(\omega).$$

This proves the equality in (18).

We now prove the left inequality in (18). Fix $\bar{\imath} \in I$. By the previous part, we know that there exists $m \in \mathbb{R}$ such that $\lim_{\beta\to 1}\sigma_i^\beta(\omega) = m$ for all $(i,\omega) \in I \times \Omega$. By contradiction, assume that $V_*\left(\hat{f}\right) > m$. By Lemmas 7 and 15, we can conclude that $m = V_*(m1_\Omega) = \lim_{\beta\to 1}V_*\left(\sigma_i^\beta\right) \geq V_*\left(\hat{f}\right) > m$ yielding a contradiction.

The second part of the statement directly follows by the first part and by Theorem 1 and Corollary 1. ■

Proof of Corollary 2. The first part of the statement follows from Theorem 2 and from the fact that, by assumption, $Q = \{q^*\}$. By Lemma 1, there exists a unique probability vector $\gamma^{q^*} \in \Delta(I \times \Omega)$ such that $\gamma^{q^*} = \gamma^{q^*}W^{q^*}$. Now, for each $(i,\omega) \in I \times \Omega$, define

$\gamma^{\mu^*} \in \Delta(I \times \Omega)$ as $\gamma^{\mu^*}_{i,\omega} = s_i \mu^*(\omega)$ and observe that

$$\sum_{(j,\omega')\in I\times\Omega} \gamma^{\mu^*}_{i,\omega} w^{q^*}_{(j,\omega')(i,\omega)} = \sum_{(j,\omega')\in I\times\Omega} s_j \mu^*(\omega') w_{ji} q^*_{j,\omega'}(\omega) = \sum_{j\in I} s_j w_{ji} \sum_{\omega'\in\Omega} \mu^*(\omega') p_{\mu^*,j}(\omega',\omega)$$
$$= \mu^*(\omega) \sum_{j\in I} s_j w_{ji} = \mu^*(\omega) s_i = \gamma^{\mu^*}_{i,\omega}.$$

This show that $\gamma^{\mu^*} = \gamma^{\mu^*} W^{q^*}$, proving that $\gamma^{q^*} = \gamma^{\mu^*}$. Finally, we have $\sum_{(i,\omega)\in I\times\Omega} \gamma^{q^*}_{i,\omega} \mathbb{E}_{q^*_{i,\omega}}\left(\hat{f}\right) = \sum_{(i,\omega)\in I\times\Omega} \gamma^{\mu^*}_{i,\omega} \mathbb{E}_{q^*_{i,\omega}}\left(\hat{f}\right) = \sum_{(i,\omega)\in I\times\Omega} s_i \mu^*(\omega) \mathbb{E}_{p_{\mu^*,i}(\omega,\cdot)}\left(\hat{f}\right) = \mathbb{E}_{\mu^*}\left(\hat{f}\right)$, proving the second part of the statement. ■

Proof of Proposition 4. Fix $\beta \in (0,1)$. By Lemma 14, we have that $\sigma^\beta_i = S_{\beta,i}\left(\sigma^\beta\right) = V_i\left((1-\beta)\hat{f} + \beta\sum_{l=1}^n w_{il}\sigma^\beta_l\right)$ for all $i \in I$. This implies that σ^β_i is Π_i-measurable and, in particular, Π_{inf}-measurable for all $i \in I$. Since V_i is Π_{inf}-affine, this implies that

$$\sigma^\beta_i = V_i\left((1-\beta)\hat{f} + \beta\sum_{l=1}^n w_{il}\sigma^\beta_l\right) = (1-\beta)V_i\left(\hat{f}\right) + \beta\sum_{l=1}^n w_{il} V_i\left(\sigma^\beta_l\right) \quad \forall i \in I. \tag{24}$$

By Lemma 9, since V_i is Π_{inf}-affine for every $i \in I$, we have that V_* is such that

$$V_*((1-\alpha)h + \alpha g) \geq (1-\alpha)V_*(h) + \alpha V_*(g) \tag{25}$$

and V^* is such that

$$V^*((1-\alpha)h + \alpha g) \leq (1-\alpha)V^*(h) + \alpha V^*(g) \tag{26}$$

for all $\alpha \in (0,1)$ and for all $g, h \in \mathbb{R}^\Omega$ where g is Π_{inf}-measurable. By (24), (25), (26) and since each $V_i\left(\hat{f}\right)$ is Π_i-measurable, hence Π_{inf}-measurable, we have that, for each $i \in I$, $V_*\left(\sigma^\beta_i\right) = V_*\left((1-\beta)V_i\left(\hat{f}\right) + \beta\sum_{l=1}^n w_{il}V_i\left(\sigma^\beta_l\right)\right) \geq (1-\beta)V_*\left(\hat{f}\right) + \beta\sum_{l=1}^n w_{il}V_*\left(\sigma^\beta_l\right)$, and $V^*\left(\sigma^\beta_i\right) = V^*\left((1-\beta)V_i\left(\hat{f}\right) + \beta\sum_{l=1}^n w_{il}V_i\left(\sigma^\beta_l\right)\right) \leq (1-\beta)V^*\left(\hat{f}\right) + \beta\sum_{l=1}^n w_{il}V^*\left(\sigma^\beta_l\right)$. Define $x_* \in \mathbb{R}^n$ to be such that $x_{*i} = V_*\left(\sigma^\beta_i\right) - V_*\left(\hat{f}\right)$ for all $i \in I$. We can conclude that $x_* \geq \beta W x_*$. Assume by contradiction that $x_{*i'} = \min_{i\in I} x_{*i} < 0$. Since W is a stochastic matrix, we have $x_{*i'} \leq (Wx_*)_{i'}$. Since $\beta \in (0,1)$ was arbitrarily chosen, it follows that $x_{*i'} < \beta(Wx_*)_{i'}$, yielding the contradiction $x_{*i'} < \beta(Wx_*)_{i'} \leq x_{*i'}$. Therefore, we must have $V_*\left(\sigma_i^\beta\right) \geq V_*\left(\hat{f}\right)$ for all $i \in I$ and for all $\beta \in (0,1)$. By taking the limit for $\beta \to 1$ in the previous inequality and by Lemma 7 and Theorem 2, we get $\lim_{\beta\to 1} \sigma_i^\beta(\omega) \geq V_*\left(\hat{f}\right)$ for all $\omega \in \Omega$ and for all $i \in I$. Analogous steps yield that $\lim_{\beta\to 1} \sigma_i^\beta(\omega) \leq V^*\left(\hat{f}\right)$ for all $\omega \in \Omega$ and for all $i \in I$. The second part of the statement follows from the first part, Theorem 1, and Corollary 1. ■

References

[1] F. Allen, S. Morris, and H.S. Shin, Beauty contests and iterated expectations in asset markets, The Review of Financial Studies, 19, 719-752, 2006.

[2] G.M. Angeletos, and A. Pavan, Efficient use of information and social value of information. Econometrica, 75, 1103-1142, 2007.

[3] P. Battigalli, E. Catonini, G. Lanzani, and M. Marinacci, "Ambiguity attitudes and self-confirming equilibrium in sequential games", Games and Economic Behavior, 115, 1-29, 2019.

[4] P. Battigalli, A. Francetich, G. Lanzani, and M. Marinacci, "Learning and self-confirming long-run biases", Journal of Economic Theory, 183, 740-785, 2019.

[5] A. Billot, A. Chateauneuf, I. Gilboa, and J.M. Tallon, "Sharing beliefs: Between agreeing and disagreeing," Econometrica, 68, 685-694, 2000.

[6] S. Cerreia-Vioglio, R. Corrao, and G. Lanzani, Dynamic Opinion Aggregation: Long-run Stability and Disagreement, The Review of Economic Studies, forthcoming, 2024.

[7] S. Cerreia-Vioglio, R. Corrao, and G. Lanzani, Nonlinear fixed points and Stationarity: Economic Applications, mimeo, 2023.

[8] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Uncertainty averse preferences, Journal of Economic Theory, 146, 1275–1330, 2011.

[9] S. Cerreia-Vioglio, P. Ghirardato, F. Maccheroni, M. Marinacci, and M. Siniscalchi, Rational preferences under ambiguity, Economic Theory, 48, 341–375, 2011.

[10] S. Cerreia-Vioglio, L.P. Hansen, F. Maccheroni, and M. Marinacci, Making decisions under model misspecification, University of Chicago, Becker Friedman Institute for Economics Working Paper, (2020-103), 2020.

[11] A. Ellis, On dynamic consistency in ambiguous games, Games and Economic Behavior, 111, 241-249, 2018.

[12] L. G. Epstein and M. Schneider, Recursive multiple-priors, Journal of Economic Theory, 113, 1–31, 2003.

[13] L.G. Epstein and T. Wang, "Beliefs about beliefs" without probabilities, Econometrica, 64, 1343-1373, 1996.

[14] M. Frick, R. Iijima, and Y. Ishii, Learning Efficiency of Multi-Agent Information Structures, Journal of Political Economy, forthcoming, 2023.

[15] P. Ghirardato, Revisiting Savage in a conditional world, Economic Theory, 20, 83-92, 2002.

[16] I. Gilboa and D. Schmeidler, Maxmin expected utility with non-unique prior, Journal of Mathematical Economics, 18, 141–153, 1989.

[17] B. Golub and S. Morris, Higher-Order Expectations, SSRN 2979089, 2017.

[18] B. Golub and S. Morris, Expectations, Networks, and Conventions, mimeo, 2018.

[19] F. Gul and W. Pesendorfer, Evaluating ambiguous random variables from Choquet to maxmin expected utility, Journal of Economic Theory, 192, 105129, 2021.

[20] A. Gumen, and A. Savochkin, Dynamically stable preferences, Journal of Economic Theory, 148, 1487-1508, 2013.

[21] E. Hanany, and P. Klibanoff, Updating Ambiguity Averse Preferences, The B.E. Journal of Theoretical Economics, 9, 2009.

[22] E. Hanany, P. Klibanoff, and S. Mukerji, Incomplete information games with ambiguity averse players, American Economic Journal: Microeconomics, 12, 135-87, 2020.

[23] L. P. Hansen and T. J. Sargent, Robust control and model uncertainty, American Economic Review, 91, 60–66, 2001.

[24] J.M. Harrison and D.M. Kreps, Speculative investor behavior in a stock market with heterogeneous expectations, The Quarterly Journal of Economics, 92, 323-336, 1978.

[25] Z. Hellman, Iterated expectations, compact spaces, and common priors, Games and Economic Behavior, 72, 163-171, 2011.

[26] Z. Huo, M. Pedroni, and G. Pei, Bias and Sensitivity Under Ambiguity, mimeo, 2023.

[27] A. Kajii and S. Morris, The robustness of equilibria to incomplete information, Econometrica, 65, 1283-1309, 1997.

[28] A. Kajii and T. Ui, Incomplete information games with multiple priors, The Japanese Economic Review, 56, 332-351, 2005.

[29] A. Kajii and T. Ui, Interim efficient allocations under uncertainty, Journal of Economic Theory,144, 337-353, 2009.

[30] M.O. Jackson, and A. Pernoud, Systemic risk in financial networks: A survey, Annual Review of Economics, 13, 171-202, 2021.

[31] G. Lanzani, Dynamic concern for misspecification, mimeo, 2022.

[32] F. Maccheroni, M. Marinacci, and A. Rustichini, Ambiguity aversion, robustness, and the variational representation of preferences, Econometrica, 74, 1447–1498, 2006.

[33] F. Maccheroni, M. Marinacci, and A. Rustichini, Dynamic variational preferences, Journal of Economic Theory, 128, 4–44, 2006.

[34] V.F. Martins-da-Rocha, Interim efficiency with MEU-preferences, Journal of Economic Theory, 145, 1987-2017, 2010.

[35] D. Monderer, and D. Samet, Approximating common knowledge with common beliefs, Games and Economic Behavior, 1, 170-190, 1989.

[36] S. Morris, The common prior assumption in economic theory, Economics & Philosophy, 11, 227-253, 1995.

[37] S. Morris, Speculative investor behavior and learning, The Quarterly Journal of Economics, 111, 1111-1133, 1996.

[38] S. Morris, and H.S. Shin, Social value of public information, American Economic Review, 92, 1521-1534, 2002.

[39] L.T. Nielsen, A. Brandenburger, J. Geanakoplos, R. McKelvey, and T. Page, Common knowledge of an aggregate of expectations, Econometrica, 58, 1235-1239, 1990.

[40] L. Rigotti, C. Shannon, and T. Strzalecki, Subjective beliefs and ex ante trade, Econometrica, 76, 1167-1190, 2008.

[41] M. Rostek, Quantile maximization in decision theory, The Review of Economic Studies, 77, 339-371, 2010.

[42] D. Samet, Iterated expectations and common priors, Games and Economic Behavior, 24, 131–141, 1998.

[43] D. Schmeidler, Subjective probability and expected utility without additivity, Econometrica, 57, 571–587, 1989.

[44] E. Seneta, Non-negative Matrices and Markov Chains, 2nd ed., Springer, New York, 1981.

[45] H.S. Shin, T. Williamson, How much common belief is necessary for a convention?, Games and Economic Behavior, 13, 252-268, 1996.

[46] M. Siniscalchi, Dynamic choice under ambiguity, Theoretical Economics, 6, 379-421, 2011.

[47] D. W. Stroock, An Introduction to Markov Processes, 2nd ed., Springer Verlag, Berlin, 2014.

[48] T. Strzalecki, Axiomatic foundations of multiplier preferences, Econometrica, 79, 47-73, 2011.

[49] T. Strzalecki, and J. Werner, Efficient allocations under ambiguity, Journal of Economic Theory, 146, 1173-1194, 2011.

D Online appendix: Omitted proofs

Lemma 3. *Let $S, T : \mathbb{R}^{\bar{n}} \to \mathbb{R}^{\bar{n}}$ be monotone and define $\hat{A} = A(T \circ S)$, $\tilde{A} = A(S)$, and $A = A(T)$. If there exists $k \in J$ such that $a_{jk} > 0$ and $\tilde{a}_{kj'} > 0$, then $\hat{a}_{jj'} > 0$. In particular, we have that:*

1. *If $\{T_h\}_{h \in \{1,...,H\}}$ is a collection of monotone operators from $\mathbb{R}^{\bar{n}}$ to $\mathbb{R}^{\bar{n}}$ and the jj'-th entry of $\Pi_{h=1}^{H} A(T_h)$ is strictly positive, then the jj'-th of $A(T_H \circ ... \circ T_1)$ is strictly positive.*

2. *If $t \in \mathbb{N}$ and the jj'-th entry of $A(T)^t$ is strictly positive, then the jj'-th of $A(T^t)$ is strictly positive.*

Proof. By assumption, there exists $k \in \{1, ..., \bar{n}\}$ such that $a_{jk}, \tilde{a}_{kj'} > 0$, that is, there exist $\varepsilon_{jk}, \varepsilon_{kj'} \in (0,1)$ such that for each $f \in \mathbb{R}^{\bar{n}}$ and for each $\delta \geq 0$, $S_k\left(f + \delta e^{j'}\right) - S_k(f) \geq \varepsilon_{kj'}\delta$ and $T_j\left(f + \delta e^k\right) - T_j(f) \geq \varepsilon_{jk}\delta$. Since S is monotone, this implies that $S\left(f + \delta e^{j'}\right) \geq S(f) + \varepsilon_{kj'}\delta e^k$ for all $f \in \mathbb{R}^{\bar{n}}$ and for all $\delta \geq 0$. Since T is monotone, this yields that for each $f \in \mathbb{R}^{\bar{n}}$ and for each $\delta \geq 0$, $T_j\left(S\left(f + \delta e^{j'}\right)\right) \geq T_j\left(S(f) + \varepsilon_{kj'}\delta e^k\right) \geq T_j(S(f)) + \varepsilon_{jk}\varepsilon_{kj'}\delta$. Since $\varepsilon_{jk}\varepsilon_{kj'} \in (0,1)$, this proves that, under $T \circ S$, j is strongly monotone with respect to j', proving that $\hat{a}_{jj'} > 0$ and the main part of the statement.

1. Consider a collection of H monotone operators from $\mathbb{R}^{\bar{n}}$ to $\mathbb{R}^{\bar{n}}$: $\{T_h\}_{h \in \{1,...,H\}}$. We prove by finite induction the statement that, for each $l \in \{1, ..., H\}$, if the jj'-th entry of $\Pi_{h=1}^{l} A(T_h)$ is strictly positive, then the jj'-th of $A(T_l \circ ... \circ T_1)$ is strictly positive.
Initial step. Assume $l = 1$. In this case, we trivially have that $A(T_1) = \Pi_{h=1}^{l} A(T_h)$.
Inductive step. Assume the statement is true for l. We prove it is true for $l + 1$. Define $S = T_l \circ ... \circ T_1$ and $T = T_{l+1}$. As before, set $\tilde{A} = A(S)$, $A = A(T)$, and $\hat{A} = A(T \circ S) = A(T_{l+1} \circ ... \circ T_1)$. Finally, define by $a_{jj'}^{(l)}$ (resp., $a_{jj'}^{(1)}$ and $a_{jj'}^{(l+1)}$) the generic jj'-th entry of $\Pi_{h=1}^{l} A(T_h)$ (resp., $A(T_{l+1})$ and $\Pi_{h=1}^{l+1} A(T_h)$). Observe that $a_{jj'}^{(l+1)} = \sum_{k=1}^{\bar{n}} a_{jk}^{(1)} a_{kj'}^{(l)}$. If the jj'-th entry of $\Pi_{h=1}^{l+1} A(T_h)$ is strictly positive, then $a_{jj'}^{(l+1)} > 0$, yielding that $a_{jk}^{(1)}, a_{kj'}^{(l)} > 0$ for some $k \in J$. By inductive hypothesis, we have that $a_{kj'}^{(l)} > 0$ implies that $\tilde{a}_{kj'} > 0$ as well as $a_{jk} > 0$. By the main part of the statement, we can conclude that $\hat{a}_{jj'} > 0$, proving the inductive step.

The statement follows by finite induction.

2. By point 1, the statement trivially follows. ■

Lemma 4. *Let $\{B_k\}_{k\in\{1,...,K\}}$ be a finite collection of $\bar{n}\times\bar{n}$ nonnegative matrices such that $b_{k,jj}>0$ for all $k\in\{1,...,K\}$ and for all $j\in J$. If $A(B_k)$ is symmetric for all $k\in\{1,...,K\}$, then $A(B_K...B_1)\geq A(B_k)$ for all $k\in\{1,...,K\}$ and $\Pi(A(B_K...B_1))$ is coarser than $\Pi(B_k)$ for all $k\in\{1,...,K\}$.*

Proof. Define $B=\Pi_{k=1}^K B_k$. By induction, we prove that $A(\Pi_{k=1}^m B_k)\geq A(B_k)\geq I_{\bar{n}}$ for all $k\in\{1,...,m\}$ and for all $m\in\{1,...,K\}$. By definition and since $b_{1,jj}>0$ for all $j\in J$, if $m=1$, then $A(\Pi_{k=1}^1 B_k)=A(B_1)\geq I_{\bar{n}}$. By point 1 of Lemma 3 and inductive hypothesis and since $b_{k,jj}>0$ for all $k\in\{1,...,K\}$ and for all $j\in J$, if $m,m+1\in\{1,...,K\}$, then $A(B_{m+1})A(\Pi_{k=1}^m B_k)\geq I_{\bar{n}}A(B_k)$ and $A\left(\Pi_{k=1}^{m+1}B_k\right)\geq A(A(B_{m+1})A(\Pi_{k=1}^m B_k))\geq A(I_{\bar{n}}A(B_k))=A(B_k)\geq I_{\bar{n}}$ for all $k\in\{1,...,m\}$. By point 1 of Lemma 3 and inductive hypothesis, we also have that $A(B_{m+1})A(\Pi_{k=1}^m B_k)\geq A(B_{m+1})I_{\bar{n}}$ and $A\left(\Pi_{k=1}^{m+1}B_k\right)\geq A(A(B_{m+1})A(\Pi_{k=1}^m B_k))\geq A(A(B_{m+1})I_{\bar{n}})=A(B_{m+1})\geq I_{\bar{n}}$. The statement follows by finite induction. In particular, this yields that $A(B_K...B_1)\geq A(B_k)\geq I_{\bar{n}}$ for all $k\in\{1,...,K\}$. Consider $k\in\{1,...,K\}$. Since $A(B_k)$ is symmetric, any index $j\in J$ is essential under B_k. Let $l\in\{1,...,m_{B_k}\}$ and $j\in J_l(B_k)$. We have two cases:

1. $j\in J_{l'}(A(B))$ for some $l'\in\left\{1,...,m_{A(B)}\right\}$. Consider $j'\in J_l(B_k)$. It follows that $j\overset{B_k}{\longleftrightarrow}j'$. Since $A(B)\geq A(B_k)$, we have that $j\overset{A(B)}{\longleftrightarrow}j'$, yielding that $j'\in J_{l'}(A(B))$. This implies that $J_l(B_k)\subseteq J_{l'}(A(B))$.

2. $j\in J_{m_B+1}(A(B))$. Consider $j'\in J_l(B_k)$. It follows that $j\overset{B_k}{\longleftrightarrow}j'$. Since $A(B)\geq A(B_k)$, we have that $j\overset{A(B)}{\longleftrightarrow}j'$, yielding that $j'\in J_{m_{B+1}}(A(B))$. Otherwise, since $j\overset{A(B)}{\longleftrightarrow}j'$, if $j'\notin J_{m_{B+1}}(B)$, then j' would be essential under $A(B)$ and so would be j, a contradiction. This implies that $J_l(B_k)\subseteq J_{m_B+1}(A(B))$. ■

Lemma 5. *If $T:\mathbb{R}^{\bar{n}}\to\mathbb{R}^{\bar{n}}$ is normalized, monotone, and continuous, then there exists a compact and convex set $\mathcal{M}(T)$ of $\bar{n}\times\bar{n}$ stochastic matrices such that for each $f\in\mathbb{R}^{\bar{n}}$ there exists $M(f)\in\mathcal{M}(T)$ such that $T(f)=M(f)f$. Moreover, if j is constant with respect to j', then $m_{jj'}=0$ for all $M\in\mathcal{M}(T)$.*

Proof. Let $j\in J$. Define the binary relation $\succsim_j^*$ on $\mathbb{R}^\Omega$ by $f\succsim_j^* g$ if and only if $T_j(\lambda f+(1-\lambda)h)\geq T_j(\lambda g+(1-\lambda)h)$ for all $\lambda\in(0,1]$ and $h\in\mathbb{R}^{\bar{n}}$. By [2] and since T_j is normalized, monotone, and continuous, we have that there exists a compact and convex set C_j of $\Delta(\{1,...,\bar{n}\})$ such that

$$f \succsim_j^* g \iff \langle f,p\rangle \geq \langle g,p\rangle \qquad \forall p \in C_j \tag{27}$$

and

$$T_j(f) = \alpha_j(f) \min_{p\in C_j} \langle f,p\rangle + (1-\alpha_j(f)) \max_{p\in C_j} \langle f,p\rangle \quad \forall f \in \mathbb{R}^{\bar{n}} \tag{28}$$

where $\alpha_j : \mathbb{R}^{\bar{n}} \to [0,1]$. Observe also that if j is constant with respect to j', then $e^{j'} \sim_j^* 0$. By (27), it follows that

$$p_{j'} = 0 \qquad \forall p \in C_j. \tag{29}$$

Since C_j is compact, for each $f \in \mathbb{R}^{\bar{n}}$ define $p_{\min,f}, p_{\max,f} \in C_j$ such that $\langle f, p_{\min,f}\rangle = \min_{p\in C_j}\langle f,p\rangle$ and $\langle f, p_{\max,f}\rangle = \max_{p\in C_j}\langle f,p\rangle$. By (28) and since C_j is convex, it follows that $p_{j,f} = \alpha_j(f)\,p_{\min,f} + (1-\alpha_j(f))\,p_{\max,f} \in C_j$ such that $T_j(f) = \langle f, p_{j,f}\rangle$ for all $f \in \mathbb{R}^{\bar{n}}$. Fix $f \in \mathbb{R}^{\bar{n}}$. Since j was arbitrarily chosen, define $M(f)$ to be the matrix whose j-th row entries correspond to the entries of $p_{j,f}$. It follows that $T(f) = M(f)\,f$. Moreover, $M(f)$ belongs to the set $\mathcal{M}(T)$ of matrices M whose j-th row belongs to C_j. Since each of these sets is compact and convex, so is $\mathcal{M}(T)$. Since f was arbitrarily chosen, the statement follows. By construction of $\mathcal{M}(T)$ and (29), it follows that if j is constant with respect to j', then $m_{jj'} = 0$ for all $M \in \mathcal{M}(T)$. ■

Proof of Proposition 5. For each $j, j' \in J$ if j is strongly monotone with respect to j', consider $\varepsilon_{jj'} \in (0,1)$ as in (20) otherwise let $\varepsilon_{jj'} = 1/2$. Define $\tilde{M}$ to be such that $\tilde{m}_{jj'} = a_{jj'}\varepsilon_{jj'}$ for all $j,j' \in J$ where $a_{jj'}$ is the jj'-th entry of $A(T)$. Since each row of $A(T)$ is not null, for each $j \in J$ there exists $j' \in J$ such that $a_{jj'} = 1$ and, in particular, $\tilde{m}_{jj'} > 0$. This implies that $\sum_{l=1}^{\bar{n}} \tilde{m}_{jl} > 0$ for all $j \in J$. Define also $\varepsilon = \min\left\{\min_{j\in J}\sum_{l=1}^{\bar{n}} \tilde{m}_{jl}, 1/2\right\} \in (0,1)$. Define the stochastic matrix $\bar{M}$ to be such that $\bar{m}_{jj'} = \tilde{m}_{jj'}/\sum_{l=1}^{\bar{n}} \tilde{m}_{jl}$ for all $j,j' \in J$. Clearly, we have that for each $j,j' \in J$, $\bar{m}_{jj'} > 0$ if and only if $\tilde{m}_{jj'} > 0$ if and only if $a_{jj'} = 1$. This yields that $A(\bar{M}) = A(T)$. Next, consider $f,g \in \mathbb{R}^{\bar{n}}$ such that $f \geq g$. Define $g^0 = g$. For each $j' \in \{1,...,\bar{n}-1\}$ define $g^{j'} \in \mathbb{R}^{\bar{n}}$ to be such that $g_j^{j'} = f_j$ for all $j \leq j'$ and $g_j^{j'} = g_j$ for all $j \geq j'+1$. Define

$g^{\bar{n}} = f$. Note that $f = g^{\bar{n}} \geq ... \geq g^1 \geq g^0 = g$. It follows that

$$T_j(f) - T_j(g) = \sum_{j'=1}^{\bar{n}} \left[T_j\left(g^{j'}\right) - T_j\left(g^{j'-1}\right) \right] \geq \sum_{j'=1}^{\bar{n}} a_{jj'} \varepsilon_{jj'} \left(g_{j'}^{j'} - g_{j'}^{j'-1} \right) = \sum_{j'=1}^{\bar{n}} \tilde{m}_{jj'} \left(f_{j'} - g_{j'} \right)$$
$$= \left(\sum_{l=1}^{\bar{n}} \tilde{m}_{jl} \right) \left(\sum_{j'=1}^{\bar{n}} \bar{m}_{jj'} \left(f_{j'} - g_{j'} \right) \right) \geq \varepsilon \sum_{j'=1}^{\bar{n}} \bar{m}_{jj'} \left(f_{j'} - g_{j'} \right) \qquad \forall j \in J.$$

This implies that

$$f \geq g \implies T(f) - T(g) \geq \varepsilon \bar{M} (f - g) = \varepsilon \left(\bar{M} f - \bar{M} g \right). \tag{30}$$

Define $S : \mathbb{R}^{\bar{n}} \to \mathbb{R}^{\bar{n}}$ by $S(f) = \frac{T(f) - \varepsilon \bar{M} f}{1 - \varepsilon}$ for all $f \in \mathbb{R}^{\bar{n}}$. By definition of S and (30) and since $\bar{M}$ is a stochastic matrix and T is normalized, monotone, and continuous, it is immediate to see that S is normalized, monotone, and continuous. We can rewrite T to be such that

$$T(f) = \varepsilon \bar{M} f + (1 - \varepsilon) S(f) \qquad \forall f \in \mathbb{R}^{\bar{n}}. \tag{31}$$

Consider the set $\mathcal{M}(S)$ of Lemma 5. Define $\mathcal{M}(T) = \varepsilon \bar{M} + (1 - \varepsilon) \mathcal{M}(S)$. Since $\mathcal{M}(S)$ is compact and convex, $A(T) = A\left(\bar{M}\right)$, and $\varepsilon \in (0, 1)$, it follows that $\mathcal{M}(T)$ is compact and convex and $A(M) \geq A\left(\bar{M}\right) = A(T)$ for all $M \in \mathcal{M}(T)$. By (31) and since for each $f \in \mathbb{R}^{\bar{n}}$ there exists $\hat{M}(f) \in \mathcal{M}(S)$ such that $S(f) = \hat{M}(f) f$, for each $f \in \mathbb{R}^{\bar{n}}$ we have that $T(f) = M(f) f$ where $M(f) = \varepsilon \bar{M} + (1 - \varepsilon) \hat{M}(f) \in \mathcal{M}(T)$.

Finally, consider $j, j' \in J$. Since $A(M) \geq A(T)$, if the jj'-entry of $A(T)$ is 1 so is the one of $A(M)$ for all $M \in \mathcal{M}(T)$. Assume that the jj'-entry of $A(T)$ is 0. Since $A(T) = A\left(\bar{M}\right)$, the jj'-entry of $A\left(\bar{M}\right)$ is 0 too. Since T is dichotomic, it follows that for each $f \in \mathbb{R}^{\bar{n}}$ and for each $\delta \geq 0$

$$\varepsilon \sum_{l=1}^{\bar{n}} \bar{m}_{jl} f_l + (1 - \varepsilon) S_j\left(f + \delta e^{j'}\right) = \varepsilon \sum_{l=1}^{\bar{n}} \bar{m}_{jl} \left(f_l + \delta e_l^{j'} \right) + (1 - \varepsilon) S_j\left(f + \delta e^{j'}\right)$$
$$= T_j\left(f + \delta e^{j'}\right) = T_j(f) = \varepsilon \sum_{l=1}^{\bar{n}} \bar{m}_{jl} f_l + (1 - \varepsilon) S_j(f).$$

Since $\varepsilon \in (0, 1)$, we can conclude that $S_j\left(f + \delta e^{j'}\right) = S_j(f)$ for all $f \in \mathbb{R}^{\bar{n}}$ and for all $\delta \geq 0$, that is, j is constant with respect to j' under S. By Lemma 5, we have that $m_{jj'} = 0$

for all $M \in \mathcal{M}(S)$. Since $\mathcal{M}(T) = \varepsilon \bar{M} + (1-\varepsilon)\mathcal{M}(S)$ and $\bar{m}_{jj'} = 0$, we can conclude that the jj'-entry of $A(M)$ is 0 for all $M \in \mathcal{M}(T)$. Since j and j' were arbitrarily chosen, we can conclude that $A(M) = A(T)$ for all $M \in \mathcal{M}(T)$. ■

Lemma 6. *Let M and $\bar{M}$ be two $\bar{n} \times \bar{n}$ stochastic matrices. If $A(\bar{M})$ is symmetric and $0 < d(\bar{M})$, then we have that $A(\bar{M}M) \geq A(M)$ and*

1. *$\delta(\bar{M}M) \geq \delta(M)$, provided $A(\bar{M}M) = A(M)$.*

2. *$\delta(\bar{M}M) \geq \delta(M)\delta(\bar{M})$, provided $A(\bar{M}M) > A(M)$.*

Moreover, if $\{M_k\}_{k=1}^{\infty}$ is a sequence of $\bar{n} \times \bar{n}$ stochastic matrices such that $A(M_k)$ is symmetric, $\delta(M_k) \geq \delta > 0$, and $d(M_k) > 0$ for all $k \in \mathbb{N}$, then

$$\delta\left(\prod_{k=1}^{m} M_k\right) \geq \delta^{\bar{n}^2} \qquad \forall m \in \mathbb{N}. \tag{32}$$

Proof. Since $d(\bar{M}) > 0$, it follows that $\bar{m}_{jj} > 0$ for all $j \in J$. This implies that the jj-th entry of $A(\bar{M})$ is 1 for all $j \in J$, and, in particular, if the jj'-th entry of $A(M)$ is strictly positive, so is the one of $A(\bar{M})A(M)$. By point 1 of Lemma 3, we can conclude that $A(\bar{M}M) \geq A(M)$. We have two cases:

1. $A(\bar{M}M) = A(M)$. Set $\hat{M} = \bar{M}M$ and consider $\hat{m}_{jj'} > 0$. We next prove that for each $l \in \{1, ..., \bar{n}\}$
$$m_{lj'} = 0 \implies \bar{m}_{jl} = 0. \tag{33}$$
By contradiction, assume that there exists $\bar{l} \in \{1, ..., \bar{n}\}$ such that $m_{\bar{l}j'} = 0$ and $\bar{m}_{j\bar{l}} > 0$. Since $A(\hat{M}) = A(\bar{M}M) = A(M)$ and $\hat{m}_{jj'} > 0$ and $m_{\bar{l}j'} = 0$, we would have that $m_{jj'} > 0$ and $\hat{m}_{\bar{l}j'} = 0$. Since $A(\bar{M})$ is symmetric, we would also have that $\bar{m}_{\bar{l}j} > 0$, yielding that $\hat{m}_{\bar{l}j'} \geq \bar{m}_{\bar{l}j} m_{jj'} > 0$, a contradiction with $\hat{m}_{\bar{l}j'} = 0$. By (33), we can conclude that $\hat{m}_{jj'} = \sum_{l=1}^{\bar{n}} \bar{m}_{jl} m_{lj'} \geq \sum_{l=1}^{\bar{n}} \bar{m}_{jl}\delta(M) = \delta(M)$, proving the statement.

2. $A(\bar{M}M) > A(M)$. Set $\hat{M} = \bar{M}M$. In this case, if $\hat{m}_{jj'} > 0$, then $\bar{m}_{j\bar{l}} m_{\bar{l}j'} > 0$ for some $\bar{l} \in \{1, ..., \bar{n}\}$ and, in particular, $\bar{m}_{j\bar{l}}, m_{\bar{l}j'} > 0$. It follows that $\hat{m}_{jj'} = \sum_{l=1}^{\bar{n}} \bar{m}_{jl} m_{lj'} \geq \bar{m}_{j\bar{l}} m_{\bar{l}j'} \geq \delta(\bar{M})\delta(M)$, proving the statement.

Consider a sequence $\{M_k\}_{k=1}^{\infty}$ of $\bar{n} \times \bar{n}$ stochastic matrices such that $A(M_k)$ is symmetric, $\delta(M_k) \geq \delta > 0$, and $d(M_k) > 0$ for all $k \in \mathbb{N}$. By induction and the previous part, we have that $A\left(\prod_{k=1}^{m+1} M_k\right) = A\left(M_{m+1}\prod_{k=1}^{m} M_k\right) \geq A\left(\prod_{k=1}^{m} M_k\right)$ for all $m \in \mathbb{N}$. Define $f : \mathbb{N} \to \{0,1\}$ by $f(1) = 1$ and

$$f(m+1) = \begin{cases} 1 & \text{if } A\left(\prod_{k=1}^{m+1} M_k\right) > A\left(\prod_{k=1}^{m} M_k\right) \\ 0 & \text{if } A\left(\prod_{k=1}^{m+1} M_k\right) = A\left(\prod_{k=1}^{m} M_k\right) \end{cases} \qquad \forall m \in \mathbb{N}.$$

By induction, we prove that

$$\delta\left(\prod_{k=1}^{m} M_k\right) \geq \delta^{\sum_{k=1}^{m} f(k)} \qquad \forall m \in \mathbb{N}. \tag{34}$$

Initial step. Assume $m = 1$. Since $f(1) = 1$, $\delta\left(\prod_{k=1}^{m} M_k\right) = \delta(M_1) \geq \delta = \delta^{\sum_{k=1}^{m} f(k)}$.

Inductive step. Assume the statement is true for $m \in \mathbb{N}$. We prove it is true for $m+1$. Since $A\left(\prod_{k=1}^{m+1} M_k\right) \geq A\left(\prod_{k=1}^{m} M_k\right)$, we have two cases:

1. $A\left(\prod_{k=1}^{m+1} M_k\right) > A\left(\prod_{k=1}^{m} M_k\right)$. In this case, we have that $f(m+1) = 1$. By the first part of the statement and inductive hypothesis, we have that

$$\delta\left(\prod_{k=1}^{m+1} M_k\right) = \delta\left(M_{m+1}\prod_{k=1}^{m} M_k\right) \geq \delta(M_{m+1})\,\delta\left(\prod_{k=1}^{m} M_k\right) \geq \delta\delta^{\sum_{k=1}^{m} f(k)} = \delta^{\sum_{k=1}^{m+1} f(k)}.$$

2. $A\left(\prod_{k=1}^{m+1} M_k\right) = A\left(\prod_{k=1}^{m} M_k\right)$. In this case, we have that $f(m+1) = 0$. By the first part of the statement and inductive hypothesis, we have that

$$\delta\left(\prod_{k=1}^{m+1} M_k\right) = \delta\left(M_{m+1}\prod_{k=1}^{m} M_k\right) \geq \delta\left(\prod_{k=1}^{m} M_k\right) \geq \delta^{\sum_{k=1}^{m} f(k)} = \delta^{\sum_{k=1}^{m+1} f(k)}.$$

Thus, (34) follows by induction. Since $\left\{A\left(\prod_{k=1}^{m} M_k\right)\right\}_{m\in\mathbb{N}}$ is an increasing sequence with upper bound the $\bar{n}\times\bar{n}$ square matrix whose entries are all 1s, we observe that $f(k)=1$ for at most $\bar{n}^2$ indices, yielding that $\sum_{k=1}^{m} f(k)\le \bar{n}^2$ for all $m\in\mathbb{N}$, proving (32). ■

Lemma 7. *If $\bar{V}:\mathbb{R}^{\Omega}\to\mathbb{R}$ is an ex-ante expectation, then it is continuous at constant functions.*

Proof. Consider $k\in\mathbb{R}$ and a sequence of functions $\{f_m\}_{m\in\mathbb{N}}\subseteq\mathbb{R}^{\Omega}$ such that $f_m\to k1_{\Omega}$. Since $f_m\to k1_{\Omega}$ and Ω is finite, we have that $\lim_{m\to\infty}\min_{\omega\in\Omega} f_m(\omega)=k=\lim_{m\to\infty}\max_{\omega\in\Omega} f_m(\omega)$. Since $\bar{V}$ is normalized and monotone, we also have that $\min_{\omega\in\Omega} f_m(\omega)\le\bar{V}(f_m)\le\max_{\omega\in\Omega} f_m(\omega)$ for all $m\in\mathbb{N}$. By passing to the limit and since $\bar{V}$ is normalized, we have that $\lim_{m\to\infty}\bar{V}(f_m)=k=\bar{V}(k1_{\Omega})$, proving continuity at $k1_{\Omega}$. ■

Proof of Lemma 2. (i) implies (ii). Let $j,j'\in J$. Since $a_{jj'}=1$, we have that j is strongly monotone with respect to j'. By contradiction, assume that $\Pi(\omega_j)\neq\Pi(\omega_{j'})$. Since Π is a partition, it follows that $\Pi(\omega_j)\cap\Pi(\omega_{j'})=\emptyset$. Since (V,Π) is an interim expectation and j is strongly monotone with respect to j', we thus have that there exists $\varepsilon_{jj'}\in(0,1)$ such that $0=V\left(\omega_j,01_{\Pi(\omega_j)}+1_{\{\omega_{j'}\}}1_{\Pi(\omega_j)^c}\right)-V(\omega_j,0)=V\left(\omega_j,1_{\{\omega_{j'}\}}\right)-V(\omega_j,0)\ge\varepsilon_{jj'}>0$, a contradiction.

(ii) implies (i). Note that $\Pi(\omega_j)=\Pi(\omega_{j'})$ only if $\omega_{j'}\in\Pi(\omega_j)$. Since (V,Π) is an interim expectation with full support, we have that each $\bar{\omega}\in\Pi(\omega_j)$ is $V(\omega_j,\cdot)$-essential and, in particular, so is $\omega_{j'}$, yielding that $a_{jj'}=1$.

By the previous part of the proof and since $\Pi(\omega_j)=\Pi(\omega_j)$ for all $j\in J$ and $A(V)$ is $\{0,1\}$-valued, we thus have that both $a_{jj'}=1$ and $a_{j'j}=1$ hold if and only if $\Pi(\omega_j)=\Pi(\omega_{j'})$, proving that $A(V)$ is symmetric, $a_{jj}=1$ for all $j\in J$, and $\Pi(V)=\Pi$. Finally, for all $j,j'\in J$, if j is not strongly monotone with respect to j', we can conclude that $a_{jj'}=0$ and $\omega_{j'}\notin\Pi(\omega_j)$. Since $V\left(\omega,f1_{\Pi(\omega)}+h1_{\Pi(\omega)^c}\right)=V(\omega,f)$ for all $\omega\in\Omega$ and for all $f,h\in\mathbb{R}^{\Omega}$, this implies that

$$V\left(\omega_j,f+\delta 1_{\{\omega_{j'}\}}\right)=V\left(\omega_j,f1_{\Pi(\omega_j)}+\delta 1_{\{\omega_{j'}\}}1_{\Pi(\omega_j)}+01_{\Pi(\omega_j)^c}\right)=V(\omega_j,f)$$

for all $f\in\mathbb{R}^{\Omega}$ and for all $\delta\ge 0$, yielding that j is constant with respect to j'. This implies that V is dichotomic. ■

Proof of Theorem 3. Define

$$\hat{t} := 2^{\bar{n}^2}.$$

By Proposition 5, we have that $I_{\bar{n}} \leq A(T_i) = A(M)$ for all $M \in \mathcal{M}(T_i)$ and for all $i \in I$. Since $\mathcal{M}(T_i)$ is compact for all $i \in I$ and I is finite, this implies that

$$\delta := \inf_{i \in I, M \in \mathcal{M}(T_i)} \delta(M) > 0.$$

Define also

$$\hat{\delta} := \delta^{\hat{t}\bar{n}^2} > 0.$$

Consider $f \in \mathbb{R}^{\bar{n}}$ and an I-sequence $(i_t)_{t\in\mathbb{N}}$. Define $f_t = T_{i_t} \circ ... \circ T_{i_1}(f) \in \mathbb{R}^{\bar{n}}$ for all $t \in \mathbb{N}$ and set $f_0 = f$. By Proposition 5, there exists a sequence $\{M_t\}_{t\in\mathbb{N}}$ of $\bar{n} \times \bar{n}$ stochastic matrices such that $M_t \in \mathcal{M}(T_{i_t})$ and $T_{i_t}(f_{t-1}) = M_t f_{t-1}$ for all $t \in \mathbb{N}$. Set $t_0 = 0$. Define recursively the following subsequence $t_{h+1} = \min\{m > t_h : \{i_{t_h+1}, ..., i_m\} \supseteq I\}$, for all $h \geq 0$. We next proceed by steps.

Step 1: $A\left(\Pi_{t=t_h+1}^{t_{h+1}} M_t\right) \geq I_{\bar{n}}$ and $\Pi\left(A\left(\Pi_{t=t_h+1}^{t_{h+1}} M_t\right)\right) = \{\Omega\}$ for all $h \in \mathbb{N}_0$.

Proof of the Step. Fix $h \in \mathbb{N}_0$. Since $I_{\bar{n}} \leq A(T_{i_t}) = A(M_t)$ for all $t \in \{t_h+1, ..., t_{h+1}\}$, we have that $A(M_t)$ has a strictly positive diagonal and it is symmetric for all $t \in \{t_h+1, ..., t_{h+1}\}$. By Lemma 4 and since $\{t_h+1, ..., t_{h+1}\} \supseteq I$ and the meet of the partitions $\{\Pi(T_i)\}_{i\in I}$ is $\{\Omega\}$, so is the meet of the partitions $\{\Pi(M_t)\}_{t=t_h+1}^{t_h+1}$, yielding that $\Pi\left(A\left(M_{t_{h+1}} ... M_{t_h+1}\right)\right) = \{\Omega\}$. By Lemma 4, we also have that $A\left(\Pi_{t=t_h+1}^{t_{h+1}} M_t\right) \geq A(M_t) \geq I_{\bar{n}}$ for all $t \in \{t_h+1, ..., t_{h+1}\}$. □

Step 2: $\delta\left(\Pi_{t=t_h+1}^{t_{h+1}} M_t\right) \geq \delta^{\bar{n}^2}$ for all $h \in \mathbb{N}_0$.

Proof of the Step. Fix $h \in \mathbb{N}_0$. By Lemma 6 and since $A(M_t) = A(T_{i_t})$ is symmetric, $\delta(M_t) \geq \delta > 0$, and $d(M_t) > 0$ for all $t \in \mathbb{N}$, the statement follows. □

Define $\bar{M}_h = \Pi_{t=t_h+1}^{t_{h+1}} M_t$ for all $h \in \mathbb{N}_0$. By Steps 1 and 2 and [7, Lemma 4.8 and Theorem 4.19], we have that $\Pi_{h=0}^{m} \bar{M}_h$ converges to a stochastic matrix M whose rows coincide to each other and, in particular, that $\left|\left|M - \Pi_{h=0}^{\tau-1} \bar{M}_h\right|\right|_\infty \leq \left(1 - \hat{\delta}\right)^{\frac{\tau}{\hat{t}}-1}$ for all $\tau \in \mathbb{N}$. This implies that $\Pi_{l=1}^{m} M_l \to M$ and, in particular, that for each $\tau, t \in \mathbb{N}$, if i appears at least τ times in $(i_1, ..., i_t)$ for all $i \in I$, then $||M - \Pi_{l=1}^{t} M_t||_\infty \leq \left|\left|M - \Pi_{h=0}^{\tau-1} \bar{M}_h\right|\right|_\infty \leq \left(1 - \hat{\delta}\right)^{\frac{\tau}{\hat{t}}-1}$. Finally, it follows that $\lim_{m\to\infty} T_{i_m} \circ ... \circ T_{i_1}(f) = \lim_{m\to\infty} \Pi_{l=1}^{m} M_l f = Mf$, and, in particular, that for each $\tau, t \in \mathbb{N}$, if i appears at least τ times in $(i_1, ..., i_t)$ for all $i \in I$, then $||\lim_{m\to\infty} T_{i_m} \circ ... \circ T_{i_1}(f) - T_{i_t} \circ ... \circ T_{i_1}(f)||_\infty = ||Mf - (\Pi_{l=1}^t M_t) f||_\infty \leq \left(1 - \delta^{2^{\bar{n}^2}\bar{n}^2}\right)^{\tau 2^{-\bar{n}^2}-1} ||f||_\infty$ proving the statement. ■

Lemma 8. *The sets $\mathcal{V}_\circ$ and $\mathcal{V}^\circ$ are nonempty and V_* and V^* are well defined and respectively a lower and an upper common ex-ante expectation for $\{(V_i, \Pi_i)\}_{i\in I}$.*

Proof of Lemma 8. Define $V_\circ(f) = \min_{\omega\in\Omega} f(\omega)$ and $V^\circ(f) = \max_{\omega\in\Omega} f(\omega)$ for all $f \in \mathbb{R}^\Omega$. It is immediate to see that both $V_\circ$ and V° are ex-ante expectations. Next, fix $f \in \mathbb{R}^\Omega$, and observe that since

$$V_i(\omega, f) \in \left[\min_{\omega'\in\Omega} f(\omega'), \max_{\omega'\in\Omega} f(\omega')\right] \qquad \forall \omega \in \Omega, \forall i \in I,$$

we have that $V_\circ(V_i(f)) = \min_{\omega\in\Omega} V_i(\omega, f) \geq \min_{\omega'\in\Omega} f(\omega') = V_\circ(f)$ and $V^\circ(V_i(f)) = \max_{\omega\in\Omega} V_i(\omega, f) \leq \max_{\omega'\in\Omega} f(\omega') = V^\circ(f)$ for all $i \in I$. This proves that $V_\circ$ and V° are respectively lower and upper common ex-ante expectations for $\{(V_i, \Pi_i)\}_{i\in I}$, hence that $V_\circ$ and V° are nonempty. We next show that V_* and V^* are well defined lower and upper common ex-ante expectations for $\{(V_i, \Pi_i)\}_{i\in I}$. First, observe that

$$V_*(k1_\Omega) = \sup_{V_\circ\in\mathcal{V}_\circ} V_\circ(k1_\Omega) = \sup_{V_\circ\in\mathcal{V}_\circ} k = k \qquad \forall k \in \mathbb{R}$$

and that, for all $f, g \in \mathbb{R}^\Omega$ with $f \geq g$, we have $V_*(f) = \sup_{V_\circ\in\mathcal{V}_\circ} V_\circ(f) \geq \sup_{V_\circ\in\mathcal{V}_\circ} V_\circ(g) = V_*(g)$, where the inequality follows from monotonicity of each $V_\circ \in V_\circ$. With this, V_* is an ex-ante expectation. Next, fix $f \in \mathbb{R}^\Omega$ and $V_\circ \in V_\circ$. For each $i \in I$, we have $V_\circ(f) \leq V_\circ(V_i(f)) \leq \sup_{V'_\circ\in\mathcal{V}_\circ} V'_\circ(V_i(f)) = V_*(V_i(f))$. Given that $V_\circ \in V_\circ$ was arbitrarily chosen, it follows that $V_*(f) = \sup_{V_\circ\in\mathcal{V}_\circ} V_\circ(f) \leq V_*(V_i(f))$ proving that V_* is a lower common ex-ante expectation. With exactly the same steps we can show that V^* is an upper common ex-ante expectation. ■

Lemma 9. *Let $\{(V_i, \Pi_i)\}_{i\in I}$ be a collection of interim expectations that exhibits convergence to a deterministic limit. The following facts are true*

1. *If V_i is concave for all $i \in I$, then V_* is concave. If in addition V_i is positive homogeneous (resp. translation invariant) for all $i \in I$, then V_* is positive homogeneous (resp. translation invariant).*

2. *If V_i is $\Pi_{\inf}$-affine for all $i \in I$, then $V_*((1-\lambda) f + \lambda g) = (1-\lambda) V_*(f) + \lambda V_*(g)$ for all $\lambda \in (0,1)$ and for all $f, g \in \mathbb{R}^\Omega$ where g is $\Pi_{\inf}$-measurable.*

Proof. 1. Consider an I-sequence $\iota = (i_k)_{k\in\mathbb{N}} \in I^{\mathbb{N}}$. Consider $f, g \in \mathbb{R}^\Omega$ and $\lambda \in (0,1)$. Since each V_{i_1} is concave, we have that $V_{i_1}(\lambda f + (1-\lambda) g) \geq \lambda V_{i_1}(f) + (1-\lambda) V_{i_1}(g)$. By induction, assume that $V_{i_k} \circ V_{i_{k-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(\lambda f + (1-\lambda) g) \geq \lambda V_{i_k} \circ V_{i_{k-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(f) + (1-\lambda) V_{i_k} \circ V_{i_{k-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(g)$. Since $V_{i_{k+1}}$ is a concave interim expectation, we have that

$$
\begin{aligned}
& V_{i_{k+1}} \circ V_{i_k} \circ V_{i_{k-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(\lambda f + (1-\lambda) g) \\
& \geq \lambda V_{i_{k+1}} \circ V_{i_k} \circ V_{i_{k-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(f) + (1-\lambda) V_{i_{k+1}} \circ V_{i_k} \circ V_{i_{k-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(g).
\end{aligned}
$$

By passing to the limit, we obtain that $\bar{V}_\iota(\lambda f + (1-\lambda) g) 1_\Omega \geq \lambda \bar{V}_\iota(f) 1_\Omega + (1-\lambda) \bar{V}_\iota(g) 1_\Omega$, proving that $\bar{V}_\iota$ is concave. Since ι was arbitrarily chosen, we have that $\bar{V}_\iota$ is concave for every I-sequence ι. Finally, given that, by Proposition 1, we have $V_*(f) = \inf_{\iota\in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} \bar{V}_\iota(f)$ for all $f \in \mathbb{R}^\Omega$, it follows that V_* is concave. With similar steps we can prove the second part of the first item.

2. Consider an I-sequence $\iota = (i_k)_{k\in\mathbb{N}} \in I^{\mathbb{N}}$. Consider $f, g \in \mathbb{R}^\Omega$ where g is $\Pi_{\inf}$-measurable, and $\lambda \in (0,1)$. Since each V_i is $\Pi_{\inf}$-affine, we have that $V_{i_1}(\lambda f + (1-\lambda) g) = \lambda V_{i_1}(f) + (1-\lambda) V_{i_1}(g)$. By induction, assume that $V_{i_k} \circ V_{i_{k-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(\lambda f + (1-\lambda) g) = \lambda V_{i_k} \circ V_{i_{k-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(f) + (1-\lambda) V_{i_k} \circ V_{i_{k-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(g)$. Since $V_{i_{k+1}}$ is $\Pi_{\inf}$-affine and $V_{i_k} \circ V_{i_{k-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(g)$ is $\Pi_{\inf}$-measurable, we have that

$$
\begin{aligned}
& V_{i_{k+1}} \circ V_{i_k} \circ V_{i_{k-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(\lambda f + (1-\lambda) g) = V_{i_{k+1}}\left(V_{i_k} \circ V_{i_{k-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(\lambda f + (1-\lambda) g)\right) \\
& = \lambda V_{i_{k+1}} \circ V_{i_k} \circ V_{i_{k-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(f) + (1-\lambda) V_{i_{k+1}} \circ V_{i_k} \circ V_{i_{k-1}} \circ ... \circ V_{i_2} \circ V_{i_1}(g).
\end{aligned}
$$

By Theorem 1 we can pass to the limit and obtain that $\bar{V}_\iota(\lambda f + (1-\lambda) g) 1_\Omega = \lambda \bar{V}_\iota(f) 1_\Omega + (1-\lambda) \bar{V}_\iota(g) 1_\Omega$, proving that $\bar{V}_\iota$ is $\Pi_{\inf}$-affine. Since ι was arbitrarily chosen, we have that $\bar{V}_\iota$ is $\Pi_{\inf}$-affine for every I-sequence ι. Finally, given that, by Proposition 1, we have $V_*(f) = \inf_{\iota\in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} \bar{V}_\iota(f)$ for all $f \in \mathbb{R}^\Omega$, it follows that

$$
\begin{aligned}
V_*((1-\lambda) f + \lambda g) &\geq \lambda \inf_{\iota\in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} \bar{V}_\iota(f) + (1-\lambda) \inf_{\iota\in I^{\mathbb{N}}:\iota \text{ is an } I\text{-sequence}} \bar{V}_\iota(g) \\
&= (1-\lambda) V_*(f) + \lambda V_*(g)
\end{aligned}
$$

for all $\lambda \in (0,1)$ and for all $f, h \in \mathbb{R}^\Omega$ where g is $\Pi_{\inf}$-measurable. The statement for V^* follows from completely symmetric steps. ■

Lemma 10. *Let $\mathbf{f} \in \left(\mathbb{R}^\Omega\right)^n$. The following statements are equivalent:*

(i) $S_1(\mathbf{f}) = \mathbf{f}$;

(ii) There exists $m \in \mathbb{R}$ such that $f_i = f_{i'} = m1_\Omega$ for all $i, i' \in I$.

Proof. (i) implies (ii). By assumption, we have that $f_i = V_i\left(\sum_{l=1}^n w_{il} f_l\right)$ for all $i \in I$. By Proposition 5 and Lemma 2, for each $i \in I$ there exists an $\bar{n} \times \bar{n}$ stochastic matrix M_i whose diagonal is strictly positive and it is such that: 1) $A(V_i) = A(M_i)$ is symmetric, 2) $\Pi(M_i) = \Pi_i$, and 3) $V_i\left(\sum_{l=1}^n w_{il} f_l\right) = M_i\left(\sum_{l=1}^n w_{il} f_l\right) = \sum_{l=1}^n w_{il} M_i f_l$. It follows that $\mathbf{f}$ is also a fixed point of the operator $\tilde{S} : \left(\mathbb{R}^\Omega\right)^n \to \left(\mathbb{R}^\Omega\right)^n$ where $\tilde{S}_i(\mathbf{g}) = \sum_{l=1}^n w_{il} M_i g_l$ for all $i \in I$. We next show that $\tilde{S}(\mathbf{f}) = \mathbf{f}$ only if there exists $m \in \mathbb{R}$ such that $f_i = f_{i'} = m1_\Omega$ for all $i, i' \in I$. By contradiction, assume that there exist $\bar{\imath}, \bar{\imath}' \in I$ and $\omega_{\bar{j}}, \omega_{\bar{j}'} \in \Omega$ such that $f_{\bar{\imath}}\left(\omega_{\bar{j}}\right) = \max_{i \in I} \max_{j \in J} f_i(\omega_j) > \min_{i \in I} \min_{j \in J} f_i(\omega) = f_{\bar{\imath}'}\left(\omega_{\bar{j}'}\right)$. By induction, note that for each $t \in \mathbb{N}$

$$\tilde{S}_i^t(\mathbf{g}) = \sum_{\mathbf{i} \in I^{t+1}: i_1 = i} w_{i_1 i_2} \ldots w_{i_t i_{t+1}} M_{i_1} \ldots M_{i_t} g_{i_{t+1}} \qquad \forall \mathbf{g} \in \left(\mathbb{R}^\Omega\right)^n$$

and $\sum_{\mathbf{i} \in I^{t+1}: i_1 = i} w_{i_1 i_2} \ldots w_{i_t i_{t+1}} = 1$. Since W is strongly connected, there exists a sequence of agents $(\bar{\imath}_1, \ldots, \bar{\imath}_{\bar{t}+1})$ such that $\bar{t} \in \mathbb{N}$, $\{\bar{\imath}_1, \ldots, \bar{\imath}_{\bar{t}}\} \supseteq I$, and $\bar{\imath}_1 = \bar{\imath}_{\bar{t}+1} = \bar{\imath}$ with $w_{\bar{\imath}_l \bar{\imath}_{l+1}} > 0$ for all $l \in \{1, \ldots, \bar{t}\}$. By Lemma 4 and since $\{\bar{\imath}_1, \ldots, \bar{\imath}_{\bar{t}}\} \supseteq I$, we have that $\Pi\left(A\left(M_{\bar{\imath}_1} \ldots M_{\bar{\imath}_{\bar{t}}}\right)\right)$ is coarser than $\Pi(M_i) = \Pi_i$ for all $i \in I$. Since $\Pi_{\sup} = \{\Omega\}$, we can conclude that $\Pi\left(A\left(M_{\bar{\imath}_1} \ldots M_{\bar{\imath}_{\bar{t}}}\right)\right) = \{\Omega\}$, yielding that $M_{\bar{\imath}_1} \ldots M_{\bar{\imath}_{\bar{t}}}$ is strongly connected. By Lemma 4 and since the diagonal of each $M_{\bar{\imath}_l}$ is strictly positive, we also have that $M_{\bar{\imath}_1} \ldots M_{\bar{\imath}_{\bar{t}}}$ has a strictly positive diagonal. This implies that $M_{\bar{\imath}_1} \ldots M_{\bar{\imath}_{\bar{t}}}$ is primitive, that is, there exists $\tau \in \mathbb{N}$ such that each entry of $\left(M_{\bar{\imath}_1} \ldots M_{\bar{\imath}_{\bar{t}}}\right)^\tau$ is strictly positive. Since W is strongly connected there exists a sequence of agents $\left(\hat{\imath}_1, \ldots, \hat{\imath}_{\hat{t}+1}\right)$ such that $\hat{t} \in \mathbb{N}$, $\hat{\imath}_1 = \bar{\imath}$, and $\hat{\imath}_{\hat{t}+1} = \bar{\imath}'$ with $w_{\hat{\imath}_l \hat{\imath}_{l+1}} > 0$ for all $l \in \left\{1, \ldots, \hat{t}\right\}$. Next, recall that by Euclid's algorithm for each $l \in \{1, \ldots., \tau\bar{t} + 1\}$ there exists unique $q_l \in \mathbb{N}_0$ and $r'_l \in \{0, \ldots., \bar{t} - 1\}$ such that $l = q_l \bar{t} + r'_l$. We define $r_l = r'_l$ if $r'_l \in \{1, \ldots., \bar{t} - 1\}$ and $r_l = \bar{t}$ if $r'_l = 0$. Consider the sequence of agents $\left(\tilde{\imath}_1, \ldots, \tilde{\imath}_{\tau\bar{t}+\hat{t}+1}\right)$ where $\tilde{\imath}_l = \bar{\imath}_{r_l}$ for all $l \in \{1, \ldots., \tau\bar{t} + 1\}$ and

$\tilde{\imath}_l = \hat{\imath}_{l-\tau\bar{t}}$ for all $l \in \left\{\tau\bar{t}+1, ..., \tau\bar{t}+1+\hat{t}\right\}$. By construction, we have that $w_{\tilde{\imath}_l\tilde{\imath}_{l+1}} > 0$ for all $l \in \left\{1, ..., \tau\bar{t}+1+\hat{t}\right\}$. Since $\mathbf{f}$ is a fixed point of $\tilde{S}$, note that $\tilde{S}^{\tau}(\mathbf{f}) = \mathbf{f}$ for all $\tau \in \mathbb{N}$ and $f_{\bar{\imath}} = \tilde{S}_{\bar{\imath}}^{\tau\bar{t}+\hat{t}}(\mathbf{f}) = \sum_{\mathbf{i}\in I^{\tau\bar{t}+\hat{t}+1}:i_1=\bar{\imath}} w_{i_1i_2}...w_{i_{\tau\bar{t}+\hat{t}}i_{\tau\bar{t}+\hat{t}+1}} M_{i_1}...M_{i_{\tau\bar{t}+\hat{t}}} f_{i_{\tau\bar{t}+\hat{t}+1}}$. Define $f^{\mathbf{i}} = M_{i_1}...M_{i_{\tau\bar{t}+\hat{t}}} f_{i_{\tau\bar{t}+\hat{t}+1}}$ for all $\mathbf{i} \in I^{\tau\bar{t}+\hat{t}+1}$ such that $i_1 = \bar{\imath}$. We have that

$$f_{\bar{\imath}} = \sum_{\mathbf{i}\in I^{\tau\bar{t}+\hat{t}+1}:i_1=\bar{\imath}} w_{i_1i_2}...w_{i_{\tau\bar{t}+\hat{t}}i_{\tau\bar{t}+\hat{t}+1}} f^{\mathbf{i}}. \tag{35}$$

Since each M_i is an $\bar{n}\times\bar{n}$ stochastic matrix and $\max_{j\in J} f_i(\omega_j) \leq f_{\bar{\imath}}\left(\omega_{\bar{j}}\right)$ for all $i \in I$, we have that $\max_{j\in J} f^{\mathbf{i}}(\omega_j) \leq f_{\bar{\imath}}\left(\omega_{\bar{j}}\right)$ for all $\mathbf{i} \in I^{\tau\bar{t}+\hat{t}+1}$ such that $i_1 = \bar{\imath}$. We focus on the summand $w_{\tilde{\imath}_1\tilde{\imath}_2}...w_{\tilde{\imath}_{\tau\bar{t}+\hat{t}}\tilde{\imath}_{\tau\bar{t}+\hat{t}+1}} M_{\tilde{\imath}_1}...M_{\tilde{\imath}_{\tau\bar{t}+\hat{t}}} f_{\tilde{\imath}_{\tau\bar{t}+\hat{t}+1}} = w_{\tilde{\imath}_1\tilde{\imath}_2}...w_{\tilde{\imath}_{\tau\bar{t}+\hat{t}}\tilde{\imath}_{\tau\bar{t}+\hat{t}+1}} f^{\tilde{\imath}}$. By construction, we have that $w_{\tilde{\imath}_1\tilde{\imath}_2}...w_{\tilde{\imath}_{\tau\bar{t}+\hat{t}}\tilde{\imath}_{\tau\bar{t}+\hat{t}+1}} > 0$ and $M_{\tilde{\imath}_1}...M_{\tilde{\imath}_{\tau\bar{t}+\hat{t}}} f_{\tilde{\imath}_{\tau\bar{t}+\hat{t}+1}} = \left(M_{\bar{\imath}_1}...M_{\bar{\imath}_{\bar{t}}}\right)^{\tau} M_{\hat{\imath}_1}...M_{\hat{\imath}_{\hat{t}}} f_{\hat{\imath}_{\hat{t}+1}}$. Set $g = M_{\hat{\imath}_1}...M_{\hat{\imath}_{\hat{t}}} f_{\hat{\imath}_{\hat{t}+1}} = M_{\hat{\imath}_1}...M_{\hat{\imath}_{\hat{t}}} f_{\bar{\imath}'}$. Since each $M_{\hat{\imath}_l}$ is an $\bar{n}\times\bar{n}$ stochastic matrix with strictly positive diagonal, so is $M_{\hat{\imath}_1}...M_{\hat{\imath}_{\hat{t}}}$. Since $\max_{j\in J} f_{\bar{\imath}'}(\omega_j) \leq f_{\bar{\imath}}\left(\omega_{\bar{j}}\right)$ and $f_{\bar{\imath}'}\left(\omega_{\bar{j}'}\right) < f_{\bar{\imath}}\left(\omega_{\bar{j}}\right)$, this implies that $\min_{\omega\in\Omega} g(\omega) \leq g\left(\omega_{\bar{j}'}\right) < f_{\bar{\imath}}\left(\omega_{\bar{j}}\right)$ and $\max_{\omega\in\Omega} g(\omega) \leq f_{\bar{\imath}}\left(\omega_{\bar{j}}\right)$. Since each entry of $\left(M_{\bar{\imath}_1}...M_{\bar{\imath}_{\bar{t}}}\right)^{\tau}$ is strictly positive and $f^{\tilde{\imath}} = \left(M_{\bar{\imath}_1}...M_{\bar{\imath}_{\bar{t}}}\right)^{\tau} g$, we can conclude that $f^{\tilde{\imath}}(\omega) < f_{\bar{\imath}}\left(\omega_{\bar{j}}\right)$ for all $\omega \in \Omega$. By (35) and since $w_{\tilde{\imath}_1\tilde{\imath}_2}...w_{\tilde{\imath}_{\tau\bar{t}+\hat{t}}\tilde{\imath}_{\tau\bar{t}+\hat{t}+1}} > 0$ and $\max_{j\in J} f^{\mathbf{i}}(\omega_j) \leq f_{\bar{\imath}}\left(\omega_{\bar{j}}\right)$ for all $\mathbf{i} \in I^{\tau\bar{t}+\hat{t}+1}$, this implies that

$$0 = \sum_{\mathbf{i}\in I^{\tau\bar{t}+\hat{t}+1}:i_1=\bar{\imath}} w_{i_1i_2}...w_{i_{\tau\bar{t}+\hat{t}}i_{\tau\bar{t}+\hat{t}+1}} \left[f^{\mathbf{i}}\left(\omega_{\bar{j}}\right) - f_{\bar{\imath}}\left(\omega_{\bar{j}}\right)\right] \leq w_{\tilde{\imath}_1\tilde{\imath}_2}...w_{\tilde{\imath}_{\tau\bar{t}+\hat{t}}\tilde{\imath}_{\tau\bar{t}+\hat{t}+1}} \left[f^{\tilde{\imath}}\left(\omega_{\bar{j}}\right) - f_{\bar{\imath}}\left(\omega_{\bar{j}}\right)\right] < 0,$$

a contradiction.

(ii) implies (i). Since each V_i is normalized and W is a stochastic matrix, the statement is trivial. ■

Lemma 11. *Fix $i, j \in I$ and $\omega, \omega' \in \Omega$. The following are equivalent:*

(i) *$w_{ij} > 0$ and $\omega' \in \Pi_i(\omega)$;*

(ii) *$\underline{a}_{(i,\omega)(j,\omega')}(S_1) = 1$;*

(iii) *$\bar{a}_{(i,\omega)(j,\omega')}(S_1) = 1$.*

Proof. (i) implies (ii). By Lemma 2, there exists $\varepsilon > 0$ such that

$$V_i\left(\omega, f + \delta e^{\omega'}\right) - V_i(\omega, f) \geq \varepsilon\delta \qquad \forall f \in \mathbb{R}^{\Omega}, \forall \delta \geq 0.$$

Next, fix $\mathbf{f} = (f_l)_{l=1}^n \in \left(\mathbb{R}^\Omega\right)^n$ and $\delta \geq 0$, and observe that

$$S_{1,i,\omega}\left(\mathbf{f} + \delta e^{j,\omega'}\right) - S_{1,i,\omega}(\mathbf{f}) = V_i\left(\omega, \sum_{l=1}^n w_{il} f_l + w_{ij}\delta e^{\omega'}\right) - V_i\left(\omega, \sum_{l=1}^n w_{il} f_l\right) \geq \varepsilon w_{ij}\delta$$

proving the statement by setting $\varepsilon_{(i,\omega)(j,\omega')} = \varepsilon w_{ij}$.

(ii) implies (iii). Immediate.

(iii) implies (i). We prove the statement by contradiction. Fix $\mathbf{f} = (f_l)_{l=1}^n \in \left(\mathbb{R}^\Omega\right)^n$ and $\delta \geq 0$ and observe that $S_{1,i,\omega}\left(\mathbf{f} + \delta e^{j,\omega'}\right) - S_{1,i,\omega}(\mathbf{f}) = V_i\left(\omega, \sum_{l=1}^n w_{il} f_l + w_{ij}\delta e^{\omega'}\right) - V_i\left(\omega, \sum_{l=1}^n w_{il} f_l\right)$. Therefore, if either $w_{ij} = 0$ or $\omega' \notin \Pi_i(\omega)$, then $S_{1,i,\omega}\left(\mathbf{f} + \delta e^{j,\omega'}\right) = S_{1,i,\omega}(\mathbf{f})$. Given that $\mathbf{f}$ and δ were arbitrarily chosen, we obtain a contradiction. ■

Lemma 12. *There exists a unique class of indices Z, $\emptyset \neq Z \subseteq I \times \Omega$, that is closed and strongly connected with respect to $A(S_1)$ and, in addition, every row of $A(S_1)$ is not null.*

Proof. We have that $S_\beta(\mathbf{f}) = S_1\left((1-\beta)\hat{\mathbf{f}} + \beta\mathbf{f}\right)$ for all $\beta \in (0,1)$. Fix $\lambda \in (0,1)$ and define $S_1^\lambda = \lambda I + (1-\lambda) S_1$. Clearly, we have that, for each $\mathbf{f} \in \left(\mathbb{R}^\Omega\right)^n$, $S_1^\lambda(\mathbf{f}) = \mathbf{f}$ if and only if $S_1(\mathbf{f}) = \mathbf{f}$. Therefore, by Lemma 10, $S_1^\lambda(\mathbf{f}) = \mathbf{f}$ if and only if there exists $m \in \mathbb{R}$ such that $f_i = f_{i'} = m 1_\Omega$ for all $i, i' \in I$. By [1, Corollary 1 and part 2 of Proposition 1], it follows that there exists a unique class of indices Z', $\emptyset \neq Z' \subseteq I \times \Omega$, that is closed and strongly connected with respect to $\bar{A}\left(S_1^\lambda\right)$. It is easy to see that every row of $\bar{A}(S_1)$ is not null and that Z' is also closed and strongly connected with respect to $\bar{A}(S_1)$. In addition, by Lemma 11, every row of $A(S_1)$ is not null and Z' is closed and strongly connected with respect to $A(S_1)$. Finally, the statement follows by setting $Z = Z'$. ■

Lemma 13. *We have $\{W^q \in \mathcal{W} : q \in Q\} \subseteq \partial S_1(0)$.*

Proof. For every $(i,\omega) \in I \times \Omega$, by [5, Theorem 2.3.9], we have that $\partial S_{1,i,\omega}(0) = \{\rho \in \Delta(I \times \Omega) : \exists \tilde{q}_{i,\omega} \in \partial V_i(\omega, 0), \rho(j,\omega') = w_{ij}\tilde{q}_{i,\omega}(\omega')\}$ where $\partial V_i(\omega, 0)$ denotes the superdifferential of the concave functional $V_i(\omega, \cdot)$ evaluated at $0 \in \mathbb{R}^\Omega$. With this, the statement follows by the definition of $\partial S_1(0)$, the definition of each W^q in equation (17), and by [32, Theorem 18]. ■

Lemma 14. *If $\beta \in (0,1]$, then S_β is a β-contraction. In particular, there exists a unique $\sigma^\beta \in \left(\mathbb{R}^\Omega\right)^n$ such that $S_\beta^\tau\left(\hat{\mathbf{f}}\right) \overset{\|\,\|_*}{\to} \sigma^\beta$, $S_\beta\left(\sigma^\beta\right) = \sigma^\beta$, and $\left\|\sigma^\beta\right\|_* \leq \left\|\hat{f}\right\|_\infty$.*

Proof. Given that $\{(V_i, \Pi_i)\}_{i\in I}$ is a variational collection of interim expectations, it follows that $V_i(\omega, \cdot)$ is concave and translation invariant for all $i \in I$ and for all $\omega \in \Omega$. Therefore, by [4, p. 346], we have that

$$\begin{aligned}
\|S_{\beta,i}(\mathbf{f}) - S_{\beta,i}(\mathbf{g})\|_\infty &= \left\| V_i\left((1-\beta)\hat{f} + \beta \sum_{l=1}^n w_{il} f_l\right) - V_i\left((1-\beta)\hat{f} + \beta \sum_{l=1}^n w_{il} g_l\right)\right\|_\infty \\
&\leq \left\|(1-\beta)\hat{f} + \beta \sum_{l=1}^n w_{il} f_l - (1-\beta)\hat{f} - \beta \sum_{l=1}^n w_{il} g_l\right\|_\infty = \left\|\beta \sum_{l=1}^n w_{il}(f_l - g_l)\right\|_\infty \\
&\leq \beta \sum_{l=1}^n w_{il} \|f_l - g_l\|_\infty \leq \beta \|\mathbf{f} - \mathbf{g}\|_* \qquad \forall i \in I, \forall \mathbf{f}, \mathbf{g} \in \left(\mathbb{R}^\Omega\right)^n,
\end{aligned}$$

proving that $\|S_\beta(\mathbf{f}) - S_\beta(\mathbf{g})\|_* = \sup_{i\in I} \|S_{\beta,i}(\mathbf{f}) - S_{\beta,i}(\mathbf{g})\|_\infty \leq \beta \|\mathbf{f} - \mathbf{g}\|_*$ for all $\mathbf{f}, \mathbf{g} \in \left(\mathbb{R}^\Omega\right)^n$. By the Banach contraction principle, for each $\beta \in (0,1)$ we have that $S_\beta^\tau\left(\hat{\mathbf{f}}\right) \overset{\|\,\|_*}{\to} \sigma^\beta$ as well as $S_\beta\left(\sigma^\beta\right) = \sigma^\beta$ where σ^β is the unique fixed point of S_β for all $\beta \in (0,1)$. Finally, by [4, p. 346] and since V_i is normalized, we have that

$$\|S_{\beta,i}(\mathbf{f})\|_\infty = \left\| V_i\left((1-\beta)\hat{f} + \beta \sum_{l=1}^n w_{il} f_l\right)\right\|_\infty \leq (1-\beta)\left\|\hat{f}\right\|_\infty + \beta \sum_{l=1}^n w_{il} \|f_l\|_\infty \quad \forall i \in I, \forall \mathbf{f} \in \left(\mathbb{R}^\Omega\right)^n.$$

By induction, this implies that $\left\|S_\beta^\tau\left(\hat{\mathbf{f}}\right)\right\|_* \leq \left\|\hat{f}\right\|_\infty$ for all $\tau \in \mathbb{N}$. By passing to the limit, the statement follows. ■

Lemma 15. *We have $V_*\left(S_{\beta,i}^\tau\left(\hat{\mathbf{f}}\right)\right) \geq V_*\left(\hat{f}\right)$ for all $i \in I$, for all $\beta \in (0,1)$, and for all $\tau \in \mathbb{N}$ where $\hat{\mathbf{f}} \in \left(\mathbb{R}^\Omega\right)^n$ is such that $\hat{f}_i = \hat{f}$ for all $i \in I$. Moreover, we have $V_*\left(\sigma_i^\beta\right) \geq V_*\left(\hat{f}\right)$ for all $i \in I$ and for all $\beta \in (0,1)$.*

Proof. Fix $\beta \in (0,1)$. By Theorem 1, $\{(V_i, \Pi_i)\}_{i\in I}$ exhibits convergence to a deterministic limit, hence, by Lemma 9, V_* is concave. This implies that for each $i \in I$, and for each $\mathbf{f} \in \left(\mathbb{R}^\Omega\right)^n$

$$V_*(S_{\beta,i}(\mathbf{f})) = V_*\left(V_i\left((1-\beta)\hat{f} + \beta \sum_{l=1}^n w_{il} f_l\right)\right) \geq (1-\beta)V_*\left(\hat{f}\right) + \beta \sum_{l=1}^n w_{il} V_*(f_l)$$

We now prove the statement for $\tau = 1$. We have that for each $i \in I$, $V_*\left(S_{\beta,i}^1\left(\hat{\mathbf{f}}\right)\right) =$

$V_*\left(S_{\beta,i}\left(\hat{\mathbf{f}}\right)\right) \geq (1-\beta) V_*\left(\hat{f}\right) + \beta \sum_{l=1}^{n} w_{il} V_*\left(\hat{f}_l\right) = V_*\left(\hat{f}\right)$. Assume that the statement is true for $\tau \in \mathbb{N}$. Observe that for each $i \in I$, $V_*\left(S_{\beta,i}^{\tau+1}\left(\hat{\mathbf{f}}\right)\right) = V_*\left(S_{\beta,i}\left(S_{\beta}^{\tau}\left(\hat{\mathbf{f}}\right)\right)\right) \geq (1-\beta) V_*\left(\hat{f}\right) + \beta \sum_{l=1}^{n} w_{il} V_*\left(S_{\beta,l}^{\tau}\left(\hat{\mathbf{f}}\right)\right) \geq V_*\left(\hat{f}\right)$. The statement follows by induction. By Lemma 14, the previous part of the proof, and since by Lemma 9 V_* is a continuous ex-ante expectation, we have that $V_*\left(\sigma_i^{\beta}\right) = V_*\left(\lim_{\tau} S_{\beta,i}^{\tau}\left(\hat{\mathbf{f}}\right)\right) = \lim_{\tau} V_*\left(S_{\beta,i}^{\tau}\left(\hat{\mathbf{f}}\right)\right) \geq V_*\left(\hat{f}\right)$ for all $i \in I$ and $\beta \in (0,1)$ proving the statement. ■

E Online appendix: An axiomatic foundation

In this section, we consider a single decision maker with preferences over monetary acts or utility profiles, that is, $\mathbb{R}^{\Omega}$. We model the decision maker preferences via a binary relation $\succsim$ on $\mathbb{R}^{\Omega}$. We next list four important properties:

A 1 (Weak order)**.** *The binary relation* $\succsim$ *is complete and transitive.*

A 2 (Certainty equivalent)**.** *For each* $f \in \mathbb{R}^{\Omega}$ *there exists* $k \in \mathbb{R}$ *such that* $f \sim k1_{\Omega}$*.*

A 3 (Continuity)**.** *For each* $f, g, h \in \mathbb{R}^{\Omega}$ *the sets* $\{\lambda \in [0,1] : \lambda f + (1-\lambda) g \succsim h\}$ *and* $\{\lambda \in [0,1] : h \succsim \lambda f + (1-\lambda) g\}$ *are closed.*

A 4 (Monotonicity)**.** *For each* $f, g \in \mathbb{R}^{\Omega}$ *and for each* $h, k \in \mathbb{R}$

$$f \geq g \implies f \succsim g \text{ and } h > k \implies h1_{\Omega} \succ k1_{\Omega}.$$

On the one hand, transitivity and monotonicity are common assumptions of rationality while completeness reflects the burden of choice the decision maker faces. On the other hand, continuity is a technical assumption which will allow us to represent preferences through a continuous utility function. The assumption of certainty equivalent shares both features. It allows us to show that preferences admit a utility function, possibly not continuous, yet it takes a clear behavioral interpretation: the decision maker for each random variable admits an equivalent amount which received with certainty makes her indifferent to the random prospect. The above axioms define the following two nested class of preferences.

Definition 7. *Let $\succsim$ be a binary relation on $\mathbb{R}^{\Omega}$. We say that $\succsim$ is a* rational preference *if and only if it satisfies weak order, certainty equivalent, and monotonicity. We say that $\succsim$ is a* continuous rational preference *if and only if it satisfies weak order, continuity, and monotonicity.*

It is easy to show that continuous rational preferences are rational preferences. Continuous rational preferences were studied by Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi [2]. The next result is a version of their Proposition 1.

Proposition 6. *Let $\succsim$ be a binary relation on $\mathbb{R}^{\Omega}$. The following statements are equivalent:*

(i) $\succsim$ is a rational preference;

(ii) There exists a normalized and monotone functional $\tilde{V} : \mathbb{R}^{\Omega} \to \mathbb{R}$ such that

$$f \succsim g \iff \tilde{V}(f) \geq \tilde{V}(g). \tag{36}$$

Moreover, we have that:

1. *The functional $\tilde{V}$ is continuous if and only if $\succsim$ is a continuous rational preference.*
2. *The functional $\tilde{V}$ is the unique normalized functional satisfying (36).*

Proof. (ii) implies (i). It is routine.

(i) implies (ii). Since $\succsim$ satisfies certainty equivalent, for each $f \in \mathbb{R}^{\Omega}$ define k_f to be such that $k_f 1_{\Omega} \sim f$. Since $\succsim$ satisfies weak order and monotonicity, we have that k_f is unique. Define $\tilde{V} : \mathbb{R}^{\Omega} \to \mathbb{R}$ by $\tilde{V}(f) = k_f$ for all $f \in \mathbb{R}^{\Omega}$. Since $\succsim$ satisfies weak order and monotonicity, we have that

$$f \succsim g \iff k_f 1_{\Omega} \succsim k_g 1_{\Omega} \iff k_f \geq k_g \iff \tilde{V}(f) \geq \tilde{V}(g),$$

proving (36). Clearly, if $f = k1_{\Omega}$ for some $k \in \mathbb{R}$, we have that $\tilde{V}(k1_{\Omega}) = \tilde{V}(f) = k_f = k$, proving that $\tilde{V}$ is normalized. Finally, since $\succsim$ satisfies monotonicity, if $f \geq g$, then $f \succsim g$ and $\tilde{V}(f) \geq \tilde{V}(g)$, proving that $\tilde{V}$ is monotone.

1. The "Only if" is routine. "If". Since $\succsim$ satisfies weak order, continuity, and monotonicity, we have that $\succsim$ satisfies certainty equivalent. It follows that $\tilde{V}$ as defined above represents $\succsim$. Since $\succsim$ satisfies continuity, it follows that for each $f, g \in \mathbb{R}^{\Omega}$ and for each $c \in \mathbb{R}$

$$\left\{\lambda \in [0,1] : \tilde{V}\left(\lambda f + (1-\lambda) g\right) \leq c\right\} = \left\{\lambda \in [0,1] : c1_{\Omega} \succsim \lambda f + (1-\lambda) g\right\}$$

where the latter set is closed. By [3, Lemma 42], we have that $\tilde{V}$ is lower semicontinuous. By [3, Appendix A.3], upper semicontinuity follows similarly.

2. Assume that $\hat{V}$ is normalized and satisfies (36). We have that for each $f \in \mathbb{R}^{\Omega}$

$$\hat{V}(f) = \hat{V}\left(\hat{V}(f) 1_{\Omega}\right) \implies f \sim \hat{V}(f) 1_{\Omega} \implies \tilde{V}(f) = \tilde{V}\left(\hat{V}(f) 1_{\Omega}\right) = \hat{V}(f),$$

proving that $\hat{V} = \tilde{V}$. ■

We can now discuss conditional preferences. We assume that there are two periods 0 and 1. At 0, the decision maker has no information and has also preferences over $\mathbb{R}^{\Omega}$. At time 1, the decision maker observes an event E from a partition Π of Ω and updates her preferences. We model this by a pair $\left(\succsim, \{\succsim_{\omega}\}_{\omega \in \Omega}\right)$.

A 5 (Rationality)**.** *The binary relation $\succsim$ is a rational preference and $\succsim_{\omega}$ is a continuous rational preference for all $\omega \in \Omega$.*

A 6 (Conditional preferences)**.** *For each $\omega, \omega' \in \Omega$*

$$\Pi(\omega) = \Pi(\omega') \implies \succsim_{\omega} = \succsim_{\omega'}.$$

We thus assume that original and updated preferences are rational, where the latter are also assumed to be continuous. At the same time, we assume that if two states belong to the same event, then the corresponding updated preferences must be the same, incorporating exactly nothing more than the information embedded in Π.

For each partition Π we define by $B(\Pi)$ the subset of elements of $\mathbb{R}^{\Omega}$ which are Π-measurable.

A 7 (Consequentialism)**.** *For each $f, h \in \mathbb{R}^{\Omega}$ and for each $\omega \in \Omega$, $f1_{\Pi(\omega)} + h1_{\Pi(\omega)^c} \sim_{\omega} f$.*

A 8 (Dynamic subconsistency)**.** *For each $f \in \mathbb{R}^{\Omega}$ and for each $g \in B(\Pi)$, $g \succsim_{\omega} f$ for all $\omega \in \Omega$ implies $g \succsim f$.*

On the one hand, consequentialism imposes that updated preferences are only influenced by the states that are still relevant/possible. On the other hand, dynamic subconsistency is a form of monotonicity and it states that if interim f is weakly worse than a Π-measurable act g, no matter which event realized in Π, then g is weakly better than f also at time 0. By switching the order of f and g, we can define symmetrically dynamic superconsistency. The usual assumption of dynamic consistency is equivalent to assume dynamic sub and superconsistency.

Definition 8. *Let $\left(\succsim, \{\succsim_{\omega}\}_{\omega\in\Omega}\right)$ be a collection of binary relations on $\mathbb{R}^{\Omega}$. We say that $\left(\succsim, \{\succsim_{\omega}\}_{\omega\in\Omega}\right)$ is a* dynamic subconsistent rational preference *if and only if it satisfies the properties of rationality, conditional preferences, consequentialism, and dynamic subconsistency.*

The next result provides a behavioral foundation for nonlinear conditional expectations.

Proposition 7. *Let $\left(\succsim, \{\succsim_{\omega}\}_{\omega\in\Omega}\right)$ be a collection of binary relations on $\mathbb{R}^{\Omega}$. The following statements are equivalent:*

(i) $\left(\succsim, \{\succsim_{\omega}\}_{\omega\in\Omega}\right)$ *is a dynamic subconsistent rational preference;*

(ii) There exist two functions $\bar{V} : \mathbb{R}^{\Omega} \to \mathbb{R}$ and $V : \Omega \times \mathbb{R}^{\Omega} \to \mathbb{R}$ such that $\left(\bar{V}, V, \Pi\right)$ is a lower conditional expectation and for each $\omega \in \Omega$

$$f \succsim_{\omega} g \iff V(\omega, f) \geq V(\omega, g) \text{ and } f \succsim g \iff \bar{V}(f) \geq \bar{V}(g).$$

Proof. (ii) implies (i). It is routine.

(i) implies (ii). By Proposition 6 and since $\left(\succsim, \{\succsim_{\omega}\}_{\omega\in\Omega}\right)$ satisfies rationality, we have that there exists a normalized and monotone function $\bar{V} : \mathbb{R}^{\Omega} \to \mathbb{R}$ and a collection of normalized, monotone, and continuous functions $\{V_{\omega}\}_{\omega\in\Omega}$ from $\mathbb{R}^{\Omega}$ to $\mathbb{R}$ such that $\bar{V}$ represents $\succsim$ and V_{ω} represents $\succsim_{\omega}$ for all $\omega \in \Omega$. Define $V : \Omega \times \mathbb{R}^{\Omega} \to \mathbb{R}$ by $V(\omega, f) =$

$V_\omega(f)$ for all $(\omega, f) \in \Omega \times \mathbb{R}^\Omega$. By point 2 of Proposition 6 and since $\left(\succsim, \{\succsim_\omega\}_{\omega\in\Omega}\right)$ satisfies conditional preferences, we have that for each $\omega, \omega' \in \Omega$

$$\Pi(\omega) = \Pi(\omega') \implies \quad \succsim_\omega = \succsim_{\omega'} \implies V(\omega, \cdot) = V(\omega', \cdot),$$

proving that $V(\cdot, f)$ is Π-measurable for all $f \in \mathbb{R}^\Omega$. Since $\left(\succsim, \{\succsim_\omega\}_{\omega\in\Omega}\right)$ satisfies consequentialism, we have that for each $\omega \in \Omega$ and for each $f, h \in \mathbb{R}^\Omega$

$$f 1_{\Pi(\omega)} + h 1_{\Pi(\omega)^c} \sim_\omega f \implies V\left(\omega, f 1_{\Pi(\omega)} + h 1_{\Pi(\omega)^c}\right) = V(\omega, f).$$

Finally, for each $f \in \mathbb{R}^\Omega$ define $g \in \mathbb{R}^\Omega$ by $g(\omega) = V(\omega, f)$ for all $\omega \in \Omega$. It follows that $g \sim_\omega g 1_{\Pi(\omega)} \sim_\omega f$ for all $\omega \in \Omega$ and for all $f \in \mathbb{R}^\Omega$. Since $\left(\succsim, \{\succsim_\omega\}_{\omega\in\Omega}\right)$ satisfies dynamic subconsistency, we can conclude that $g \succsim f$ and, in particular, $\bar{V}(f) \leq \bar{V}(g) = \bar{V}(V(\cdot, f))$ for all $f \in \mathbb{R}^\Omega$. ■

If $\left(\succsim, \{\succsim_\omega\}_{\omega\in\Omega}\right)$ were to satisfy dynamic superconsistency in place of subconsistency, the result above would yield a foundation for upper conditional expectations. Finally, by assuming both, we would obtain a foundation for nonlinear conditional expectations. Clearly, in Proposition 7, linear conditional expectations are obtained by requiring in (i) $\succsim$ and each $\succsim_\omega$ to satisfy the axiom of independence. Similarly, maxmin conditional expectations, as in Example 2, are obtained by imposing c-independence.

E.1 Different information structures

We now consider different information structures, that is, partitions $\{\Pi_i\}_{i=1}^n$. Consequently, we consider the collection $\left\{\left(\succsim, \{\succsim_{\omega,\Pi_i}\}_{\omega\in\Omega}\right)\right\}_{i=1}^n$.

Corollary 3. *Let $\left\{\left(\succsim, \{\succsim_{\omega,\Pi_i}\}_{\omega\in\Omega}\right)\right\}_{i=1}^n$ be a collection of binary relations on $\mathbb{R}^\Omega$. The following statements are equivalent:*

(i) $\left(\succsim, \{\succsim_{\omega,\Pi_i}\}_{\omega\in\Omega}\right)$ is a dynamic subconsistent rational preference for all $i \in \{1, ..., n\}$;

(ii) There exist $n+1$ functions $\bar{V} : \mathbb{R}^\Omega \to \mathbb{R}$ and $V_i : \Omega \times \mathbb{R}^\Omega \to \mathbb{R}$ such that $\left(\bar{V}, V_i, \Pi_i\right)$ is a lower conditional expectation for all $i \in \{1, ..., n\}$, for each $\omega \in \Omega$ and for each

$i \in \{1, ..., n\}$

$$f \succsim_{\omega,\Pi_i} g \iff V_i(\omega, f) \geq V_i(\omega, g) \text{ and } f \succsim g \iff \bar{V}(f) \geq \bar{V}(g).$$

Proof. By Proposition 7 and the uniqueness part of Proposition 6, the statement immediately follows. ■

Proposition 8. *Let $\left\{\left(\succsim, \{\succsim_{\omega,\Pi_i}\}_{\omega\in\Omega}\right)\right\}_{i=1}^{n}$ be a collection of dynamic subconsistent rational preferences on $\mathbb{R}^{\Omega}$. The following statements are equivalent:*

(i) For each $i \in \{1, ..., n\}$ and for each $h \in B(\Pi_i)$ if $g \in B(\Pi_j)$ for some $j \in \{1, ..., n\}$ is such that $g \sim_{\omega,\Pi_j} h$ for all $\omega \in \Omega$, then $g \sim h$;

(ii) $\bar{V}(h) = \bar{V}(V_j(\cdot, h))$ for all $h \in B(\Pi_i)$ and for all $i, j \in \{1, ..., n\}$.

Proof. (ii) implies (i). It is routine.

(i) implies (ii). Consider $i \in \{1, ..., n\}$ and $h \in B(\Pi_i)$. Define $g \in \mathbb{R}^{\Omega}$ by $g(\omega) = V_j(\omega, f)$ for all $\omega \in \Omega$. Clearly, $g \in B(\Pi_j)$. Moreover, it follows that $g \sim_{\omega,\Pi_j} g1_{\Pi(\omega)} \sim_{\omega,\Pi_j} h$ for all $\omega \in \Omega$. We can conclude that $g \sim h$ and, in particular, $\bar{V}(h) = \bar{V}(g) = \bar{V}(V_j(\cdot, h))$. Since h as well as i and j were arbitrarily chosen, the statement follows. ■

We call the assumption of point (i) above: dynamic consistency for Π-measurable acts.

Proposition 9. *Let $\left\{\left(\succsim, \{\succsim_{\omega,\Pi_i}\}_{\omega\in\Omega}\right)\right\}_{i=1}^{n}$ be a collection of dynamic subconsistent rational preferences on $\mathbb{R}^{\Omega}$. The following statements are equivalent:*

(i) For each $f \in \mathbb{R}^{\Omega}$ there exists $i \in \{1, ..., n\}$ such that if $g \in B(\Pi_i)$ and $g \sim_{\omega,\Pi_i} f$ for all $\omega \in \Omega$, then $g \sim f$;

(ii) For each $f \in \mathbb{R}^{\Omega}$ there exists $i \in \{1, ..., n\}$ such that $\bar{V}(f) = \bar{V}(V_i(\cdot, f))$.

Proof. (ii) implies (i). It is routine.

(i) implies (ii). Consider $f \in \mathbb{R}^{\Omega}$ and i as in point (i). Define $g \in \mathbb{R}^{\Omega}$ by $g(\omega) = V_i(\omega, f)$ for all $\omega \in \Omega$. Clearly, $g \in B(\Pi_i)$ and $g \sim_{\omega,\Pi_i} f$ for all $\omega \in \Omega$, yielding that $g \sim f$ and, in particular, $\bar{V}(f) = \bar{V}(g) = \bar{V}(V_i(\cdot, f))$. Since f was arbitrarily chosen, the statement follows. ■

We call the assumption of point (i) above: "always some dynamic consistent".

Proposition 10. *Let $\left\{\left(\succsim, \{\succsim_{\omega,\Pi_i}\}_{\omega\in\Omega}\right)\right\}_{i=1}^{2}$ be a collection of dynamic subconsistent rational preferences on $\mathbb{R}^\Omega$ with $\Pi_{\sup} = \{\Omega\}$ and full-support interim expectations. The following statements are equivalent:*

(i) The collection satisfies dynamic consistency for Π-measurable acts and always some dynamic consistent;

(ii) $\bar{V} = V_*$.

Proof. With the usual notation, recall that $(V_1 \circ V_2)^t$ and $(V_2 \circ V_1)^t$ converge to a deterministic limit. Denote the corresponding limiting functionals by $\bar{V}_{12}$ and $\bar{V}_{21}$. Moreover, $V_*(f) = \min\left\{\bar{V}_{12}(f), \bar{V}_{21}(f)\right\}$ for all $f \in \mathbb{R}^\Omega$ and $\bar{V} \leq V_*$.

(ii) implies (i). By the previous two propositions, it is routine.

(i) implies (ii). By the first proposition, we have that

$$\bar{V}(V_1(h)) = \bar{V}(V_2(V_1(h))) \text{ and } \bar{V}(V_2(h)) = \bar{V}(V_1(V_2(h))) \quad \forall h \in \mathbb{R}^\Omega. \tag{37}$$

Consider $f \in \mathbb{R}^\Omega$. By the second proposition, we have that either $\bar{V}(f) = \bar{V}(V_1(f))$ or $\bar{V}(f) = \bar{V}(V_2(f))$. In the first case, we are going to show that

$$\bar{V}(f) = \bar{V}\left((V_2 \circ V_1)^t(f)\right).$$

By (37) and since $\bar{V}(f) = \bar{V}(V_1(f))$, the statement follows for $t = 1$. If the statement holds for t, since $(V_2 \circ V_1)^t(f) \in B(\Pi_2)$, we have that

$$\bar{V}\left((V_2 \circ V_1)^t(f)\right) = V\left(V_1(V_2 \circ V_1)^t(f)\right) = V\left(V_2\left(V_1(V_2 \circ V_1)^t\right)(f)\right) = V\left((V_2 \circ V_1)^{t+1}(f)\right),$$

proving the inductive step. The statement holds by induction, by passing to the limit, we obtain that $\bar{V}(f) = \bar{V}_{21}(f) \geq V_*(f)$ and $\bar{V}(f) = V_*(f)$. In the case $\bar{V}(f) = \bar{V}(V_2(f))$, a similar argument yields that $\bar{V}(f) = V_*(f)$. ■

References

[1] S. Cerreia-Vioglio, R. Corrao, and G. Lanzani, Dynamic Opinion Aggregation: Long-run Stability and Disagreement, The Review of Economic Studies, forthcoming, 2023.

[2] S. Cerreia-Vioglio, P. Ghirardato, F. Maccheroni, M. Marinacci, and M. Siniscalchi, Rational preferences under ambiguity, Economic Theory, 48, 341–375, 2011.

[3] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Uncertainty averse preferences, Journal of Economic Theory, 146, 1275–1330, 2011.

[4] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and A. Rustichini, Niveloids and their extensions: Risk measures on small domains, Journal of Mathematical Analysis and Applications, 413, 343–360, 2014.

[5] F. H. Clarke, Optimization and Nonsmooth Analysis, SIAM, Philadelphia, 1990.

[6] M. O. Jackson, Social and Economic Networks, Princeton University Press, Princeton, 2008.

[7] E. Seneta, Non-negative Matrices and Markov Chains, 2nd ed., Springer, New York, 1981.