Optimal Energy Policy and the Inequality of Climate Change

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Abstract

What is the optimal policy to fight climate change? Taxation of carbon and fossil fuels has strong redistributive effects across countries: (i) curbing energy demand is costly for developing economies, which are the most affected by climate change in the first place, (ii) taxation has strong general equilibrium effects through energy markets and trade reallocation. Through the lens of an Integrated Assessment Model (IAM) with heterogeneous countries, I show that optimal carbon policy depends crucially on the availability of redistribution instruments. After characterizing the Social Cost of Carbon (SCC), I derive formulas for second-best fossil fuel taxes in the presence of inequalities in climate damages and incomes, redistributive effects through energy and good trade, and participation constraints if countries can exit climate agreements. I show that a uniform carbon should be reduced twofold in the presence of inequality. If country-specific carbon taxes are available, the distribution of carbon prices is proportionally related to the level of income: poor and hot countries should pay lower energy taxes than rich and cold countries. These qualitative results are general and I propose extensions with international trade, uncertainty, or participation constraints when countries can leave climate agreements.

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1 Introduction

The climate is warming due to greenhouse gas emissions generated by economic activity. More than 500 gigatons of carbon have been emitted through the burning of fossil fuels, and global atmospheric temperatures have increased by more than $1.1°C$ since the industrial revolution. The sources of these emissions are unequally distributed: developed economies account for over 65% of cumulative greenhouse gas (GHG) emissions – $\sim$ 25% each for the European Union countries and the United States, while some developing countries have barely emitted anything compared to their population level. In Figure 1, we see how much individuals in each country have exceeded their carbon budget – a fixed number of gigatons of CO_2 per inhabitant: countries in red have emitted cumulated emissions per capita much higher than their allocated budget. This measure of emissions highly correlates with local development and GDP in each region as seen in Figure 2.

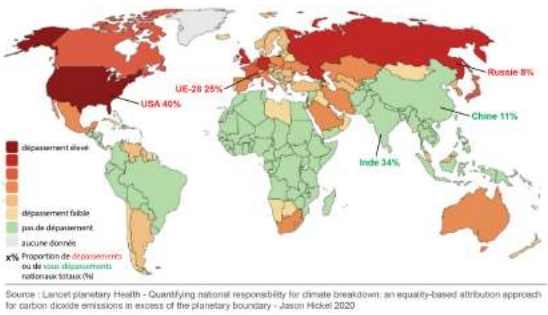

Figure 1: Excess of carbon budget

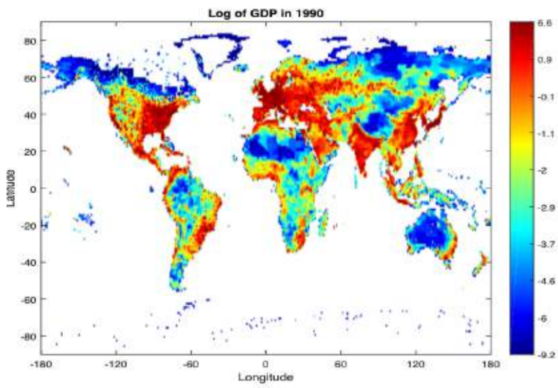

Figure 2: Local GDP

However, the consequences of global warming are also unequal: the increase in temperatures will disproportionately affect developing countries where the climate is already warm. Most emerging and low-income economies lie geographically closer to the tropics and the equator and tend to be most vulnerable to global warming. Figure 3 displays an adaptation index that compiles different measures of likelihood and vulnerability of the region to extreme events, loss in biodiversity, drought and heatwaves, or sea level rising among other factors. We observe that this correlates very closely with local temperatures as seen in Figure 4. Moreover, these indices covary negatively with the region's GDP as seen above.

These two layers of inequalities raise the question: which countries will be affected the most by climate change? Do these different dimensions of heterogeneity matter when measuring the future costs of global warming and optimal carbon taxation? In that context, we need to understand how to design climate policy in the presence of externalities and inequality. Indeed, carbon taxation has strong redistributive effects across countries. Should inequality – development level, temperature, and the ownership of fossil fuel reserves, etc. – be taken into account when implementing climate policy?

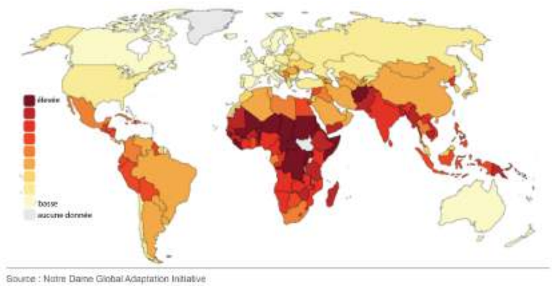

Figure 3: Adaptation Index

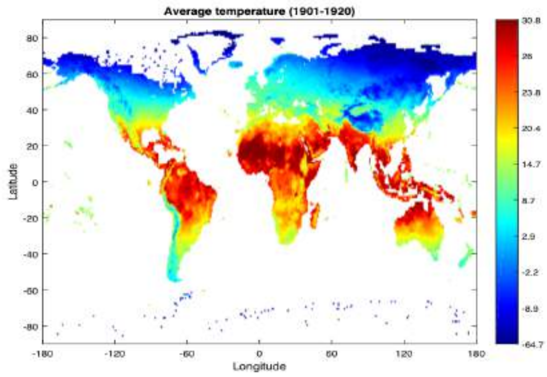

Figure 4: Local temperature

To answer these classical questions in climate economics, I develop a simple yet flexible model of climate economics. This extends the standard Neoclassical Growth – Integrated Assessment model to include heterogeneous regions. These regions – or individual countries – are (i) heterogeneous in income and level of development and several other dimensions, (ii) could be affected differently by the global climate and (iii) are interacting with each other through good and energy markets as well as the climate system through emissions and temperature. This theoretical framework is of the same family as heterogeneous agent models, or Mean-Field Games.

Since the quantitative framework is very general, I first provide an extremely simple toy model to provide intuitions. Keeping the same notion of externality and interactions in energy markets, the design of optimal policy and the characterization of the Social Cost of Carbon (SCC) carry through. The main result is the following. In the presence of inequality and climate externality, a world social planner would solve both issues at once. First, they would impose a carbon tax that accounts for the climate externality, i.e. the Social Cost of Carbon. in a Pigouvian fashion. Second, they would redistribute across countries using lump-sum transfers, for example taxing lump-sum European and American countries and transferring to South Asian and African Countries. It is well-known that in addition to the tragedy of the commons, there are strong policy constraints that prevent perfect redistribution even in the case of optimal taxation policy

As a result, I consider a Second-Best Ramsey policy and a larger set of suboptimal policies that would search for an alternative way to fight climate change. Despite being unable to redistribute freely across countries due to limitations on lump-sum transfers, a planner adapts its tax policy. I show that optimal taxation changes in three ways compared to the standard Pigouvian result in Representative Agents models: (i) the level of the SCC accounts for inequality and the correlation between poverty and vulnerability to climate change, (ii) the taxation of energy also account for redistributive effects of the energy price – due to change in terms-of-trade between exporters and importers and (iii) the distribution of carbon tax is correlated with the level of development: richer/advanced economies should be imposed a higher tax simply because they have lower marginal utilities of consumption and can hence "afford" to pay higher taxes without being excessively affected. These findings are very general and I develop a quantitative model to provide policy recommendations.

Using a quantitative macroeconomics model with several market forces and frictions, I compute the optimal policy in that context. First, I evaluate the heterogeneous welfare costs of global warming in a panel of 40 countries. In this framework, countries are heterogeneous in many dimensions – population, productivity, temperature, etc. – and in each of them a household consumes, borrows subject to credit constraints, invests in physical capital, produces homogenous goods using capital and energy, and chooses between carbon-intensive fossil fuels and carbon-neutral clean energy. Moreover, different countries are interacting on the world market for fossil energy where a representative competitive supplier extracts fossil fuels and redistribute energy rent to individual countries. The different countries are also interacting through the global climate: both atmospheric and local temperatures rise when the cumulative stock of emissions rises. However, climate damage is an externality and there are no incentives to curb emissions. This model is very general and is flexible enough to add numerous extensions. Simulating the model sequentially in continuous time amounts to solving differential equations, and I develop a new methodology to handle the solution of this infinite-dimensional system.

Second, in this framework, I design the optimal Ramsey policy. Using advances in public finance and optimal taxation in heterogeneous agents modeling, I show how to design the planner's problem and decentralize the optimal taxes with heterogeneous regions. I show how optimal Pigouvian taxes should be adapted to account for (i) redistribution effects of fossil fuel taxations, (ii) the social cost of carbon due to climate externalities, (iii) the effect of these taxes on energy markets and on the redistribution of the fossil energy rent and (iv) the distortion of energy choice both in level and in composition between different sources. As a result, the world optimal carbon policy may not be as simple as *Carbon tax* $=$ *Social Cost of Carbon*, and the taxation should be adapted to the specific situation of each country.

Third, using this theoretical model, I derive several closed-form results to inform the various mechanisms at hand in this environment. First, inequality affects the Social Cost of Carbon as the world SCC is a weighted average of local marginal damages, with weights representing the distributional effects: with the actual distribution of temperatures and outputs, the SCC is higher in this heterogeneous agent world than in a representative agent one. Moreover, in standard Integrated Assessment Models, which factors determine the Social Cost of Carbon? I derive a simple yet general formula and show that the price of carbon is linear in GDP/level of development of the country and in the temperature gap from optimal climate, where proportionality constants depend on climate and damage parameters. These results contrast with the recent developments of this literature which rely on computational models that tend to be opaque and sometimes theoretically untractable.

Related literature

This paper stands at the intersection of several subfields of macroeconomics, climate economics and computational and mathematical economics.

First, since this project considers an Integrated Assessment model (IAM) with heterogeneous countries, this is naturally related to the classical approach of IAM by Nordhaus. I use a similar neoclassical model with a climate block and damage of higher temperatures, as in the DICE model, Nordhaus (1993) Nordhaus (2017). Regarding policy design, few papers actually build optimal taxation policies to tackle climate change. Golosov, Hassler, Krusell and Tsyvinski (2014) is the major exception and develops the first-best policy in a representative agent model and optimal tax as a function of the SCC and a closed-form formula of the climate parameters. Moreover, Hillebrand and Hillebrand (2019) develop several transfer policies that are Pareto improvement to the competitive.

Second, I also relate to the scientific literature that has reexamined the empirical performance of IAMs, review the calibration, and derive analytical formula as in Dietz, van der Ploeg, Rezai and Venmans (2021), Dietz and Venmans (2019), Ricke and Caldeira (2014) or Folini et al. (2021). Adopting the best practice from this literature, I consider a macroeconomic model where I derive closed-form expressions for the social cost of carbon and the asymptotics of this general class of model.

Third, and importantly, handling country heterogeneity, I also relate to a booming literature on computational climate economy models, such as Hassler, Krusell, Olovsson and Reiter (2020), Krusell and Smith (2022) and Kotlikoff, Kubler, Polbin and Scheidegger (2021*b*). In a model that is extremely related, I adopt a different methodology – using the sequential formulation – and I study the optimal policy when heterogeneity and externality matter for the price of carbon.

Fourth, in a related field, the spatial-economic geography literature has done important advances in studying the heterogeneous impact of climate change. In this field, important frictions and adaptation mechanisms have been studied, such as migration, international trade or sector reallocation, such as Cruz and Rossi-Hansberg (2021), Cruz Álvarez and Rossi-Hansberg (2022), Rudik et al. (2021) or Bilal and Rossi-Hansberg (2023). In comparison, I assume away strategic complementarities such as migration or trade, as it would not be tractable in this sequential formulation. However, I do consider forward-looking heterogeneous agents and design optimal policy in this context.

Fifth, I also consider specific details to the energy markets that borrow from a literature that studies market frictions such as exhaustible resources and market power, such as Hotelling (1931), Heal and Schlenker (2019) and Bornstein, Krusell and Rebelo (2023). I keep the energy market simple, but I show that the path of emissions and hence the Social Cost of Carbon depends greatly on the details of the pricing of fossil energy.

Sixth, I develop a framework that is flexible enough to handle aggregate uncertainty, such as climate risk and business cycle fluctuation. The Stochastic DICE model of Cai and Lontzek (2019) and Lontzek, Cai, Judd and Lenton (2015) or the general approach to study model uncertainty and ambiguity aversion applied to climate change in Barnett, Brock and Hansen (2020), Barnett, Brock and Hansen (2022) are particularly related. If the inclusion of aggregate risk is preliminary in the present paper, I provide intuitions in the toy model and will integrate this in forthcoming works.

Seventh, I also relate to a thriving literature that studies optimal policy design in Heterogeneous Agents models. Solving Ramsey policy, Le Grand et al. (2021), Bhandari et al. (2021*a*), Dávila and Schaab (2023) or McKay and Wolf (2022) propose different approach to conduct monetary and fiscal policy in HANK models. In my framework, I solve the Ramsey policy sequentially and solve climate externalities and Pigouvian taxation in the presence of heterogeneity rather than managing business cycle fluctuations.

Eighth and lastly, I also integrate advances from the mathematical literature on the Probabilistic Formulation of Mean Field Games. Classical references such as the Lasry-Lions approach of the PDE system, Cardaliaguet (2013/2018) or even Pham and Wei (2017) all rely on Dynamic Programming principle. Recently, the solution of the master equation has been very fruitful as in Cardaliaguet et al. (2015) or Bilal (2021). However, a probabilistic approach has realized that the Pontryagin maximum principle extends to the stochastic case, as in Yong and Zhou (1999) or the Mean-Field / McKean Vlasov infinite dimensional case, as in Carmona et al. (2015), Carmona and Delarue (2018) or Carmona and Laurière (2022). Using this approach in the deterministic case with shooting algorithms in large dimensions, I solve the model, compute the social cost of carbon and design optimal policy. For the case with aggregate risk, I borrow intuitions from Carmona et al. (2016), Bourany (2018) and Carmona and Delarue (2018) to solve the Stochastic FBSDE system.

2 Toy model

In this section, we develop the simplest version of the quantitative model covered in the next section. The goal is to provide intuitions on the effects of heterogeneity across countries, the source of climate externality related to energy markets, and the implementation of optimal policy.

The model is static and all the decisions are taken in one period. Consider two countries $i = N, S$, for *North* and *South* that are heterogeneous in three dimensions that will be detailed below. A unique household in each country consumes the good c_i that is produced by the representative firm with energy e_i.[1] In each of these countries, an energy producer extracts energy and sells this input at a price q^e on international markets. It earns profits and is owned by the household. Moreover, the countries are subject to climate damage represented by the productivity term $\mathcal{D}_i(\mathcal{S})$ as in Nordhaus DICE models. We will describe each agent's problem in turn. Finally, a government, whose objective will be specified in the next section, imposes an energy tax t_i^e and distributes lump-sum transfers t_i^{ls} in each country.

First, the Household is passive and consumes their labor income w_i, the profit of the energy firm of it's country π_i^e and the lump-sum transfers given by the government.

$$\begin{aligned} &\mathcal{V}_i = U(c_i) \\ &c_i = w_i + \pi_i^e + \mathrm{t}_i^{ls} \qquad\qquad [\lambda_i] \end{aligned}$$

Second, the representative firm produces a homogeneous good[2] using energy e_i and household labor with constant return to scale technologies. We normalize the fixed labor supply to 1, such that e_i represents energy use per capita. The production function $F(e_i)$ is thus increasing and concave in e_i, i.e. $F'(e) > 0$ and $F''(e) < 0$ and features Inada conditions. This firm maximize profits:

$$\max_{e_i} \ \mathcal{D}_i(\mathcal{S}) z_i F(e_i) - (q^e + \mathrm{t}_i^e) e_i - w_i$$

where t_i^e is an energy tax paid per unit of energy.

Note that since the good firm's technology is Constant Return to Scale (CRS), labor income is simply the residual of firms' revenue, and hence we can aggregate firms and household budget into a single constraint:

$$c_i + (q^e + \mathbf{t}_i^e) e_i = \mathcal{D}_i(\mathcal{S}) z_i F(e_i) + q_i^e e_i - c_i(e_i) + \mathrm{t}_i^{ls}$$

Both countries are subject to climate damages $\mathcal{D}_i(\mathcal{S})$ caused by climate externalities related

[1] Generalization of this model, with differing $\mathcal{P}_i$, endowments of inputs in the production function (e.g. capital k_i or labor ℓ_i), do not change the qualitative implication of this framework, ass we will show in the quantitative model.

[2] This good can be traded costlessly across countries and its price is the numeraire, and hence normalized to 1.

to the world energy consumption:

$$\mathcal{S} = \mathcal{S}_0 + \overbrace{e_S + e_N}^{=\text{GHG emissions}}$$

where energy consumption and emissions are measured in Tons of Carbon or CO_2. This depends on the energy mix between fossil fuels used for energy and renewables. However, this is taken as given in the short run in our static equilibrium. The quantitative model introduces this endogenous channel of energy choice.

The global carbon emission stock is not internalized by households in their energy consumption decision leading to damage $\mathcal{D}_i(\mathcal{S})$ that affects country i's effective productivity, as in standard Integrated Assessment models, e.g. Nordhaus DICE models.

In each country, an energy producer in extracts energy e_i^x – for example oil, gas or coal – maximizing its profit, subject to convex cost $c(E)$, i.e. $c'(E) > 0$ and $c''(E) > 0$ that is paid in the homogenous good.

$$\pi_i^e = \max_{e_i^x} q^e e_i^x - c_i(e_i^x)$$
$$\Rightarrow \qquad q^e = c_i'(E) \qquad \& \qquad \pi_i^e := c_i'(e_i^x)e_i^x - c_i(e_i^x)$$

subject the energy price q^e. Since energy is traded without friction on international markets, this price is set to clear the supply and demand:

$$e_N + e_S = e_N^x + e_S^x$$

Heterogeneity *North* and *South* are symmetric in all regards, except for differences in three parameters. First, the South and the North are different in terms of productivity z_i: $z_S < z_N$. Here, we consider a wide definition of z_i as productivity residuals that can account for technology, efficiency, market frictions, and institutions. This results in the North producing more, and being richer, leading to inequality in consumption.[3] Second, we furthermore inequality in energy reserves, and assume that $c_N'(e) > c_S'(e)$. This implies that northern countries have larger production and energy rent. Third, we consider that the Southern country is subject to stronger damages of climate, $\mathcal{D}_S(\mathcal{S}) < \mathcal{D}_N(\mathcal{S})$ for all $\mathcal{S}$ the stock of carbon emissions. In this sense, the damage parameter $\gamma_i = -\frac{\mathcal{D}_i'(\mathcal{S})}{\mathcal{S}\mathcal{D}_i(\mathcal{S})}$ is higher in the South such that $\gamma_S > \gamma_N$. All these differences yield heterogeneity in consumption in the competitive equilibrium and motives for redistribution.

The Competitive Equilibrium is a system of price q^e and allocation $\{c_i, e_i, e_i^x\}_i$ such that (i)

[3] Indeed, assuming $F(e)$ is Cobb Douglas $F(e) = \bar{\ell}^{1-\alpha}e^\alpha$, with $\bar{\ell} = cst$, we obtain $\alpha\mathcal{D}_i(\mathcal{S})z_i e_i^{\alpha-1} = q^e$ leading to

$$e_i = \big(\alpha\mathcal{D}_i(\mathcal{S})z_i/q^e\big)^{1/(1-\alpha)} \qquad y_i - q^e e_i = \big(\mathcal{D}_i(\mathcal{S})z_i\big)^{1/(1-\alpha)}(q^e)^{-\alpha/(1-\alpha)}\big[\alpha^{\alpha/(1-\alpha)} - \alpha^{1/(1-\alpha)}\big]$$

which is increasing in z_i and $\mathcal{D}_i(\mathcal{S})$.

the good firm maximizes profits, and (ii) the energy producers choose production e_i^x to maximize profit, and market clear for both goods and energy:

$$\sum_{i=N,S} c_i + c_i(e_i^x) = \sum_{i=N,S} \mathcal{D}_i(\mathcal{S}) z_i F(e_i) \qquad\qquad e_N + e_S = e_N^x + e_S^x$$

The competitive equilibrium results in the following optimality conditions. First, for consumption, the multiplier λ_i represents the marginal value of wealth, i.e. marginal utility of consumption.

$$\lambda_i = U'(c_i) \qquad \textit{with} \qquad c_i = \mathcal{D}_i(\mathcal{S}) z_i F(e_i) + q^e(e_i^x - e_i) + c_i(e_i^x) + \mathrm{t}_i^{ls}$$

where consumption depends on production, energy cost, and net energy export.

The second optimality for energy use for production writes as follow:

$$MPe_i = q^e + \mathrm{t}_i^e \qquad \textit{with} \qquad MPe_i := \mathcal{D}_i(\mathcal{S}) z_i F'(e_i)$$

This corresponds to the standard tradeoff Marginal Product of Energy = Energy Price.

This competitive equilibrium is inefficient: Indeed, the climate damages $\mathcal{D}_i(\mathcal{S})$ are not internalized, and energy consumption might be too high depending on the economic cost of global warming $\mathcal{D}_i(\mathcal{S})$.

Moreover, economic inequality results from the heterogeneity in productivity and climate damage since $c_N > c_S$ we have $\lambda_S > \lambda_N$. Redistribution from the North to the South could be desirable from a utilitarian point of view. This inequality in consumption and damages arises despite trade openness[4].

We explore how the Ramsey planner would allocate consumption and energy in such an environment.

2.1 Social planner allocation with full transfers

Consider a Social Planner who could make the agent's decisions, subject to the resource constraints in goods and energy as well as the climate externality.

$$\max_{\{c_i, e_i\}_{i=N,S}} \sum_{i=N,S} \omega_i U(c_i)$$

$$\sum_{i=N,S} c_i + c_i(e_i^x) = \sum_{i=N,S} \mathcal{D}_i(\mathcal{S}) z_i F(e_i) \qquad [\lambda]$$

$$e_N + e_S = e_N^x + e_S^x \qquad [\mu^e]$$

$$\mathcal{S} := \mathcal{S}_0 + e_S + e_N$$

[4] One could also consider trade and financial autarky and lack of redistribution across countries: production in one country can not be exported or transferred to another country. That would strengthen that heterogeneity

where λ is the shadow value of the good market clearing and μ^e the one of the energy market clearing. We consider a welfare function, where the countries are weighted with Pareto weights ω_i. In the following, we denote the social planner allocation $\{\hat{c}_i, \hat{e}_i\}_{i=N,S}$ to distinguish it from the competitive equilibrium.

Choosing the consumption on behalf of the agents yields a redistribution motive:

$$[c_i] \qquad \lambda = \omega_i U'(c_i) \qquad \Rightarrow \qquad \omega_N U'(\hat{c}_N) = \omega_S U'(\hat{c}_S)$$

Depending on the Pareto weights there is a motive for transferring consumption across countries.

Regarding the choice of energy inputs:

$$[e_i] \;\&\; [e_i^x] \qquad c'(e_i^x) = \frac{\mu^e}{\lambda} = \mathcal{D}_i(\mathcal{S}) z_i F'(e_i) + \underbrace{\sum_{j=N,S} \mathcal{D}_j'(\mathcal{S}) z_i F(e_i)}_{=\overline{SCC}}$$

we see an additional term that represents the cost of emitting one ton of carbon in terms of forgone production. This term is the social cost of carbon (SCC) in the social planner allocation and represents the marginal global damage of climate change.

We turn now to how to decentralize such allocation. We consider a planner who has access to all instruments $\{\mathrm{t}_i^e, \mathrm{t}_i^{ls}\}_i$, and in particular lump-sum transfers t_i^{ls} across countries. The energy optimality rewrites:

$$MPe_i := \mathcal{D}_i(\mathcal{S}) z_i F'(\hat{e}_i) = q^e + \mathrm{t}^e$$

$$\text{with} \qquad q^e = c'(e_i^x) \qquad \mathrm{t}^e = \overline{SCC} = \sum_{j=N,S} \mathcal{D}_j'(\mathcal{S}) z_i F(\hat{e}_j)$$

Importantly, the carbon tax $\mathrm{t}_i^e = \mathrm{t}^e$ is equal to the social cost of carbon. We see that this result relies on the existence of lump-sum transfers. Indeed, the budget constraint in this equilibrium allocation writes

$$\hat{c}_i = \mathcal{D}_i(\mathcal{S}) z_i F(\hat{e}_i) - (q^e + \mathrm{t}^e)\hat{e}_i + q^e \hat{e}_i^x + c_i(\hat{e}_i^x) + \mathrm{t}_i^{ls}$$

where the transfers t_i^{ls} are such that $\omega_N U'(\hat{c}_N) = \omega_S U'(\hat{c}_S)$. In particular, summing the two budget constraints[5] yields:

$$\mathrm{t}_N^{ls} + \mathrm{t}_S^{ls} = \mathrm{t}^e \sum_i \hat{e}_i \qquad \mathrm{t}_S^{ls} > 0 \qquad \mathrm{t}_N^{ls} < 0$$

implying there is potentially lump-sum redistri[[[bution from North to South as we assumed[6] $z_S < z_N$ and $\theta_S < \theta_N$, under reasonable parametrization for the Pareto weight[7]. There exists a set of

[5] with $E = \hat{e}_N + \hat{e}_S$

$$\sum_i \hat{c}_i + q^e E + \mathrm{t}^e \sum_i \hat{e}_i = \sum_i \mathcal{D}_i(\mathcal{S}) z_i F(\hat{e}_i) + q^e E - \sum_i c(e_i^x) + \sum_i \mathrm{t}_i^{ls}$$

[6] Given that $\mathrm{t}_i^{ls} = u'^{-1}(\frac{\lambda}{\omega_i}) - \mathcal{D}_i(\mathcal{S}) z_i F(\hat{e}_i) - \pi_i^e(e_{it}^x) - (q^e + \mathbf{t}^e)\hat{e}_i$

[7] In particular, if the Pareto weight are large enough, i.e. $\omega_S \geq \lambda / u'(c_S)$ i.e. more than the weight imposed by

Pareto weights $\omega_i = 1/U'(c_i)$ – the so-called Negishi weights – such that this motive disappears $\mathrm{t}_S^{ls} = \mathrm{t}_N^{ls}$.

In the following, we forbid this assumption of lump-sum transfers: indeed if development aid exists, in practice full redistribution to cover the difference in technology, market frictions and institutions with lump-sum transfers and taxes is politically unfeasible.

2.2 Ramsey Problem with uniform carbon tax & limited transfers

Consider now a Social Planner that takes into account the constraints that prevent the full lump-sum redistribution. Subject to competitive equilibrium optimality conditions, and the same market frictions – climate externality and the absence of financial instruments for transfers across countries, the planner takes the decisions of consumption and energy to maximize the welfare function with weights ω_i for each country. We denote the Ramsey allocation $\{\tilde{c}_i, \tilde{e}_i\}_i$ to distinguish it from the competitive equilibrium $\{c_i, e_i\}$ and the First-best planner allocation $\{\hat{c}_i, \hat{e}_i\}$.

$$\mathbb{W} = \max_{\{\tilde{c}_i, \tilde{e}_i\}_i} \sum_{i=N,S} \omega_i U(c_i)$$

The consumption and energy allocation are subject to the budget constraint, where the household is imposed a uniform energy tax t^e. As in Weitzmann (2014), I consider a uniform carbon tax for all countries, as a social-planner policy resulting from every country agreement. In the next section, I will consider different tax rates for each country. In both cases, I assume away cross-country transfers, as the revenue of the tax is redistributed lump-sum $\tilde{t}_i^{ls} = \mathrm{t}^e e_i$. The household-firm constraint writes:

$$\tilde{c}_i + (q^e + \mathrm{t}^e)\tilde{e}_i = \mathcal{D}_i(\mathcal{S}) z_i F(\tilde{e}_i) + (q^e \tilde{e}_i^x - c_i(\tilde{e}_i^x)) + \mathrm{t}_i^{ls}$$

A particularity of the Second-Best policy is that agents are still acting optimally. Therefore, energy is still priced at competitive price by energy firms, and household/firm still consume energy optimally:

$$q^e = c'(e_i^x) \qquad\qquad \pi_i^e(e_i^x) = c'(e_i^x)e_i^x - c_i(e_i^x)$$

$$q^e + \mathrm{t}^e = MPe_i$$

As a result, using the Primal Approach in public finance, the Ramsey maximization problem states

$$\mathbb{W} = \max_{\{\tilde{c}_i, \tilde{e}_i,\}_i} \sum_{i=N,S} \omega_i \, U(c_i)$$

$$s.t \qquad \tilde{c}_i + MPe_i \tilde{e}_i = \mathcal{D}_i(\mathcal{S}) z_i F(\tilde{e}_i) + c'(\tilde{e}_i^x)\tilde{e}_i^x - c_i(\tilde{e}_i^x) + \mathrm{t}^e \tilde{e}_i \qquad [\phi_i] \qquad \forall\, i = N, S$$

$$\mathcal{S} := \mathcal{S}_0 + e_N + e_S \qquad\qquad e_N + e_S = e_N^x + e_S^x \qquad [\mu^e]$$

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the shadow value of good discounted by marginal utility of the South consumption

Moreover, the Lagrange Multipliers ϕ_i represent the Social Value of relaxing the budget constraint. The consumption allocation yield simply:

$$\omega_i U'(c_i) = \phi_i$$

Let us define an inequality factor that will be important in all the following tax formulas:

$$\widehat{\phi}_i = \frac{\phi_i}{\overline{\phi}} = \frac{\omega_i U'(c_i)}{\frac{1}{2}(\omega_N U'(c_N) + \omega_S U'(c_S))} \lesseqgtr 1$$

where $\overline{\phi} = \frac{1}{2}(\omega_N U'(c_N) + \omega_S U'(c_S))$ the average marginal utility, which will be the "money-welfare" conversion factor for the social planner in the context where there is no full redistribution. This factor $\widehat{\phi}_i$ will be high for relatively poorer countries – or countries with a high Pareto weight ω_i.

Now, we derive the choice of energy, that will integrate all of the distortions of that model. The combination of optimality conditions for demand e_i and supply e_i^x gives:

$$\phi_i \mathrm{t}^e = \underbrace{\phi_i \, \mathcal{D}_i(\mathcal{S}) z_i F''(\tilde{e}_i)\tilde{e}_i}_{=\text{demand distortion}} \;-\; \underbrace{\phi_i \, c_i''(\tilde{e}_i^x)\tilde{e}_i^x}_{=\text{supply distortion}} \; \underbrace{- \textstyle\sum_j \phi_j \mathcal{D}_j'(\mathcal{S}) z_j F(e_j)}_{\propto \text{ SCC}}$$

Before, providing a general formula for the tax, note that we need to aggregate the different countries since we consider a single tax instrument for the world. We see that the energy choice of the planner can into account several forms of redistribution.

First, climate change affects countries differently according to their marginal damages $\mathcal{D}_j(\mathcal{S})$, but this damage is now scales by the marginal utility/inequality factor $\widehat{\phi}_i \propto \omega_i U'(c_i)$ since the planner doesn't implement full redistribution. Rescaled in monetary unit, with the conversion factor $\overline{\phi}_i$, the SCC writes:

$$SCC := -\frac{\partial \mathbb{W}/\partial \mathcal{S}}{\partial \mathbb{W}/\partial c} = -\frac{1}{\overline{\phi}} \sum_j \phi_j \mathcal{D}_j'(\mathcal{S}) z_j F(e_j) = -\sum_j \widehat{\phi}_j \mathcal{D}_j'(\mathcal{S}) z_j F(e_j)$$

In particular, in this heterogeneous countries model with limited redistribution, the Social Cost of Carbon integrates the distribution of consumption/income under the factor $\widehat{\phi}_i$,

$$\begin{aligned} SCC &:= -\sum_j \widehat{\phi}_j \mathcal{D}_j'(\mathcal{S}) z_j F(e_j) = -2\mathbb{E}_j \left(\widehat{\phi}_j \mathcal{D}_j'(\mathcal{S}) z_j F(e_j) \right) \\ &= 2\,\mathbb{E}_j[-\mathcal{D}_j'(\mathcal{S}) z_j F(e_j)] + 2\,\mathbb{C}\mathrm{ov}_j \Big(\frac{\omega_j U'(c_j)}{\frac{1}{2}\sum_j \omega_j U'(c_j)}, -\mathcal{D}_j'(\mathcal{S}) z_j F(e_j) \Big) \\ &\lesseqgtr 2\,\mathbb{E}_j[-\mathcal{D}_j'(\mathcal{S}) z_j F(e_j)] = \overline{SCC} \end{aligned}$$

where $\overline{SCC} = 2\mathbb{E}_j[-\mathcal{D}_j'(\mathcal{S}) z_j F(e_j)]$ is the Social Cost of Carbon in the model where full redistribution is available, or equivalently a representative agent model where redistributive concerns are absent. Note that since $\mathbb{E}_j(\cdot)$ is a mean over countries j, we need to multiply by the number of countries (2 here) to obtain the sum of local damages.

Is the SCC higher in the model with inequality compared to the one-agent setting? First, we have that low-income countries have a lower consumption and hence higher marginal utility of consumption, $c_S < c_N$ and $\widehat{\phi}_S > \widehat{\phi}_N$. Second, we assumed stronger damages $\mathcal{D}'_S(\mathcal{S}) > \mathcal{D}'_N(\mathcal{S})$. However, third, productivity and income are higher in high-income countries, and $z_N > z_S$ implies $F(e_N) > F(e_S)$. Therefore, the covariance between $\widehat{\phi}_i$, $\bar{y}_i = z_i F(e_i)$ and $\mathcal{D}_i(\mathcal{S})$ is ambiguous. Quantitatively, in most Integrated Assessment models, the local cost of climate change $\mathcal{D}'_i(\mathcal{S}) y_i$ is strongly correlated with income y_i, as there larger production loss of climate change in richer countries.

Second, one can define similarly the social value of rent (exporters) that accounts for the redistributive effect of the energy supply on price and profit $\pi'_j(\tilde{e}^x_i) = c''_i(\tilde{e}^x_i)$. We derive the Social Value of Rent (SVR) as:

$$
\begin{aligned}
SVR &= \frac{1}{2}\frac{1}{\bar{\phi}} \sum_j \phi_j \pi'_j(e^x_j) = \mathbb{E}_j\Big(\widehat{\phi}_j \pi'_j(e^x_j)\Big) \\
&= \mathbb{E}_j\big(\pi'_j(e^x_j)\big) + \mathbb{C}\mathrm{ov}_j\Big(\tfrac{\omega_j U'(c_j)}{\frac{1}{2}\sum_j \omega_j U'(c_j)}, \pi'_j(e^x_j)\Big) < \mathbb{E}_j\big(\pi'_j(e^x_j)\big)
\end{aligned}
$$

where the last inequality comes from the assumption that North has a larger endowment in energy resources and hence energy rent. Since we assume perfect competition, the rent distortion ultimately depends on the supply elasticity

$$
\pi'_j(e^x_j) = c''_i(e^x_j) e^x_j = q^e \nu^e_i
$$

with ν^e_i the inverse supply elasticity, which is constant in the iso-elastic case $q^e = c'_i(e) = \bar{\nu}_i e^{\nu^e_i}$. As a result, the Social Value of Rent is positive when the energy supply is inelastic – price and profit vary a lot for small changes in quantity produced – and it is null when the energy production is Constant Return to Scale (CRS) when $\nu = 0$.

Third, changing the energy price and quantity redistributes across energy users/importers through the change in price along the demand curve. We derive the Social Cost of Energy (SCE) as:

$$
\begin{aligned}
SCE &= \frac{1}{2}\frac{1}{\bar{\phi}} \sum_j \phi_j \mathcal{D}_j(\mathcal{S}) z_j F''(e_j) e_j = \mathbb{E}_j\Big(\widehat{\phi}_j\, \mathcal{D}_j(\mathcal{S}) z_j F''(e_j) e_j\Big) \\
&= \mathbb{E}_j\big(\mathcal{D}_j(\mathcal{S}) z_j F''(e_j) e_j\big) + \mathbb{C}\mathrm{ov}_j\Big(\tfrac{\omega_j U'(c_j)}{\frac{1}{2}\sum_j \omega_j U'(c_j)}, \mathcal{D}_j(\mathcal{S}) z_j F''(e_j) e_j\Big) \\
&> \mathbb{E}_j\big(\mathcal{D}_j(\mathcal{S}) z_j F''(e_j) e_j\big)
\end{aligned}
$$

where the last inequality comes from the fact that lower-income economies are more sensitive to energy price distortion. This comes from the fact that $z_i F''(e_i) e_i$ is related to the energy share and demand elasticity:

$$
\mathcal{D}_i(\mathcal{S}) z_i F''(e_i) e_i = \frac{q^e}{\sigma^e_i}(s^e_i - 1) < 0
$$

where $s_i^e = \frac{e_i q^e}{y_i} < 1$ is the energy share in production and σ_i^e is energy demand elasticity. One can derive – in the CES case[8] – that $s_i^e \propto (z_i/q^e)^{\sigma-1}$. In the case where energy is a low-substitution input, such that $\sigma_i^e < 1$, we have that $z_N > z_S$ implies that $s_N^e < s_S^e$. Moreover, empirical evidence has shown that emerging economies rely strongly on fossil-fuel supply, making their production function more inelastic to changes in energy prices. Note that again that term is null if the energy demand/production function is constant return to scale in energy such that $s_i^e = 1$.

As a result, the *level* of the optimal energy taxation policy account for these three distributional motives (i) climate damage in SCC, (ii) distortion in energy supply and rent in SVR and (iii) energy demand in SCE. For (ii) and (iii), taxation is isomorphic to a terms-of-trade manipulation between the exporters and the importers in trade theory.

$$
\begin{aligned}
MPe_i &= c'(E) + \mathbf{t}^e \\
\mathbf{t}_i^e &= SCC - SVR + SCE \\
&= -\sum_j \widehat{\phi}_j \mathcal{D}_j'(\mathcal{S}) z_j F(e_j) \;-\; \frac{1}{2}\sum_j \widehat{\phi}_j \pi_j'(e_j^x) \;+\; \frac{1}{2}\sum_j \widehat{\phi}_j \mathcal{D}_j(\mathcal{S}) z_j F''(e_j) e_j \\
&= 2\mathbb{E}_j\Big(\widehat{\phi}_j y_j \gamma_j \mathcal{S}\Big) - q^e \mathbb{E}_j\Big(\widehat{\phi}_j \nu_i^e\Big) - q^e \mathbb{E}_j\Big(\widehat{\phi}_j \frac{1-s_i^e}{\sigma_i^e}\Big)
\end{aligned}
$$

where $\gamma_i = -\frac{\mathcal{D}_i'(\mathcal{S})}{\mathcal{D}_i(\mathcal{S})\mathcal{S}}$ is the marginal damage of climate change[9], $y_i = \mathcal{D}_i(\mathcal{S}) z_i F(e_i)$ is total production, ν_i^e the inverse energy supply elasticity, s_i^e the energy cost shares, and σ_i^e the energy demand elasticity. We see these three motives matter with a single tax and lump-sum rebate. Compared to the economy with full redistribution, the tax can be smaller if (i) the cost of climate $\gamma_j y_j$ is concentrated in richer countries, with low $\widehat{\phi}_i$, (ii) the supply elasticity is high – low ν_i^e – in poorer countries, high $\widehat{\phi}_i$, (iii) the demand elasticity σ_i^e is small in poorer, high $\widehat{\phi}_i$, countries.

However, if the planner has access to a distribution of carbon tax rates (or carbon price), with the presence of inequality, the *distribution* of the tax changes as we will see in the next section.

2.3 Ramsey Problem with heterogeneous carbon tax & limited transfers

We consider a case where the Social Planner would implement a policy with a distribution of country-specific carbon tax. I again assume away cross-country transfers, and the revenue of the carbon tax is rebated lump-sum $\tilde{t}_i^{ls} = \mathrm{t}_i^e e_i$. The welfare objective is the same, and the budget constraints become:

$$
\tilde{c}_i + (q^e + \mathrm{t}_i^e)\tilde{e}_i = \mathcal{D}_i(\mathcal{S}) z_i F(\tilde{e}_i) + (q^e \tilde{e}_i^x - c_i(\tilde{e}_i^x)) + \mathrm{t}_i^{ls}
$$

[8] With CRS production $F(e,\ell) = z\Big((1-\epsilon)^{\frac{1}{\sigma}} \ell^{\frac{\sigma-1}{\sigma}} + \epsilon^{\frac{1}{\sigma}} e^{\frac{\sigma-1}{\sigma}}\Big)$ we obtain that $s_i^e = \frac{eq}{y} = \epsilon(z_i/q^e)^{\sigma-1}$

[9] In particular, this is a constant parameter in the Damage function used in DICE model $\mathcal{D}_i(\mathcal{S}) = e^{-\gamma_i \mathcal{S}^2}$.

All the optimality conditions, for energy demand and supply are internalized by the planner and remain identical:

$$q^e = c'(e_i^x) \qquad \pi_i^e(e_i^x) = c'(e_i^x)e_i^x - c_i(e_i^x)$$
$$q^e + \mathrm{t}^e = MPe_i$$
$$e_N + e_S = e_N^x + e_S^x$$

The planner keeps the same motive for redistribution given the inequality factor coming from the shadow value ϕ_i of the budget constraint and the Household consumption decisions:

$$\omega_i U'(c_i) = \phi_i \qquad \widehat{\phi}_i = \frac{\phi_i}{\overline{\phi}} = \frac{\omega_i U'(c_i)}{\frac{1}{2}(\omega_N U'(c_N) + \omega_S U'(c_S))} \lesseqgtr 1$$

The optimality condition for energy demand e_i and supply e_i^x become:

$$\mathrm{t}_i^e = \frac{1}{\widehat{\phi}_i} \underbrace{\sum_j \widehat{\phi}_j\big(- \mathcal{D}_j'(\mathcal{S}) z_j F(\tilde{e}_j)\big)}_{\propto \text{ SCC}} - \frac{1}{\widehat{\phi}_i} \underbrace{\sum_j \widehat{\phi}_j c_j''(\tilde{e}_j^x)\tilde{e}_j^x}_{=\text{supply distortion}} + \underbrace{\mathcal{D}_i(\mathcal{S}) z_i F''(\tilde{e}_i)\tilde{e}_i}_{=\text{demand distortion}}$$

$$\mathrm{t}_i^e = \frac{1}{\widehat{\phi}_i}\Big(SCC - SVR\Big) - SCE_i$$

$$\mathrm{t}_i^e = \frac{1}{\widehat{\phi}_i}\, 2\,\mathbb{E}_j\Big(\widehat{\phi}_j \gamma_j \mathcal{S} y_j\Big) - \frac{1}{\widehat{\phi}_i}\, q^e\, \mathbb{E}_j\Big(\widehat{\phi}_j \nu_j^e\Big) - \frac{1 - s_i^e}{\sigma_i^e}$$

where $\gamma_i = -\frac{\mathcal{D}_i'(\mathcal{S})}{\mathcal{D}_i(\mathcal{S})\mathcal{S}}$ is the marginal damage of climate change, y_i is total output, ν_i^e the inverse energy supply elasticity, s_i^e the energy cost shares, and σ_i^e the energy demand elasticity.

We see that the planner would accommodate country-specific levels of inequality for the distribution of carbon prices. Indeed, for a given – potentially arbitrary – distribution of Pareto weights ω_i, the optimal carbon tax is relatively lower for poorer countries for several reasons:
(i) the Pigouvian Social Cost of Carbon is *discounted* by the country level of inequality $\widehat{\phi}_i$: the planner understand that energy is used in production and would not reduce consumption even further than it already is. The global climate damage leads to a high carbon tax for rich countries that have low marginal utility of consumption $\widehat{\phi}_i \propto \omega_i U'(c_i) \approx 0$ and can in some sort "afford" the distortion brought by the carbon tax.

Similarly, (ii) the General Equilibrium effect on the energy supply affects every country's energy profit, as represented by the Social *SVR*, and this tax motive is also discounted by the level of income $\widehat{\phi}_i$, for the same reason as for the SCC. Lastly, (iii) the energy demand is affected by this country-specific tax, and the planner would adjust this local distortion by modulating the distribution of carbon price to account for local demand effects, as represented by the energy cost share s_i^e and elasticity σ_i^e.

These main findings – that the *level* and the *distribution* of carbon taxes change with inequality – are general and hold in a dynamic quantitative model that I develop in the next sections.

3 Quantitative model

We develop a framework with neoclassical foundations and rich heterogeneity across regions. The time is continuous $t \in [t_0, \infty)$, where[10] $t_0 = 2000$. The countries/regions are indexed by $i \in \mathbb{I}$. They can be heterogenous in an arbitrary number of dimensions[11] s.

In each country, we consider 4 representative agents: (i) a household doing consumption/saving decisions, (ii) a homogeneous good producer using capital, labor and energy, (iii) an energy firm that extracts fossil-fuels and (iv) a renewable energy producer.

As of now, this model includes several individual states $s_i = \{z_i, \mathcal{P}_i, \bar{\nu}_i, \gamma_i, \Delta_i, \xi_i, w_i, \tau_i, \mathcal{R}_i\}$, respectively productivity z, population $\mathcal{P}_i$, marginal cost of producing fossil fuels $\bar{\nu}$, climate vulnerability γ_i, geographic factors for temperature scaling Δ, and carbon intensity of the fossil energy mix ξ, which are six dimensions of heterogeneity that are time-invariant. In addition, country wealth w, local temperature τ, and local reserve of fossil fuel energy sources $\mathcal{R}$ change over time. Moreover, the world is subject to global states which can also time-varying $S = \{\mathcal{T}, \mathcal{S}\}$ which are respectively world atmospheric temperature $\mathcal{T}$, world atmospheric carbon concentration $\mathcal{S}$. All these variables will be explained in turn below.

Countries interact with the rest of the world through several channels: (i) Each country can trade financial assets b_i in world markets, with $b_{it} > 0$ for saving and $b_{it} < 0$ for borrowing. (ii) The consumption of fossil-fuel energy is traded in a world energy market at price q_t^f and (iii) Fossil consumption releases carbon emissions in the atmosphere $\mathcal{S}_t$ which increase world temperatures $\mathcal{T}_t$ and local temperature τ_{it}. Moreover, in a later extension, we will consider bilateral trade in goods between countries. We will present the four different agents in turn.

3.1 Country Household

At each instant t, each region $i \in \mathbb{I}$ is populated by a representative household of population size $\mathcal{P}_{it}$. This population is increasing at a growth rate exogenously determined n, and $\dot{\mathcal{P}}_{it} = n\mathcal{P}_{it}$. As a result, the population is given as $\mathcal{P}_{it} = \mathcal{P}_{i0}e^{nt}$.

This representative household owns the representative firm that is producing output with total factor productivity z_{it}. This total factor productivity also grows with a deterministic growth rate $\bar{g}$, giving a TFP level of $z_{it} = z_{i0}e^{\bar{g}t}$. In the tradition of the Neoclassical model, we normalize all the economic variables of the model by the rate of effective population $z_t\mathcal{P}_t = e^{(n+\bar{g})t}$, leaving only the relative difference between countries' population $\mathcal{P}_i \equiv \mathcal{P}_{i0}$ and productivity $z_i \equiv z_{i0}$. In the following, each country's agent solves an independent dynamic control problem and is subject to global variables that we shall denote with capital letters – for example, $\mathcal{T}_t$ for global temperature

[10] In the application we will consider an interval $t \in [t_0, t_T]$ with $t_0 = 2000$ and $t_T = 2100$.

[11] More precisely, state variables of heterogeneity can be split in two, $s = \{\underline{s}, \overline{s}\}$, where ex-ante heterogeneity is constant over time or relate to initial conditions and is denoted $\underline{s}$, while ex-post heterogeneity $\overline{s}$ changes over time depending on the fluctuations of the regions variables. In practice, with the method used, $\underline{s}$ can be arbitrarily large, but the size of ex-post heterogeneity $\overline{s}$ needs to be controlled, as we will explained in the computational section below.

or $\mathcal{E}_t$ for global emissions explained below.

The household in the country $i \in \mathbb{I}$ consumes the homogeneous final good $c_t \equiv c_{it}$ and is subject to the region's temperature $\tau_t \equiv \tau_{it}$. They can save and borrow in a liquid financial asset b_{it} at a world interest rate $r_t^\star$. Moreover, they can invest and hold that wealth in capital k_{it} to be rented to the homogeneous good producer at rate r_{it}^k.

Household supply their inelastic labor $\bar{\ell}_i = \mathcal{P}_i$ to the final good firms, receiving the wage income v_{it}. Moreover, the household receives the profit that the fossil sector generates $\pi_i^f = \pi_i^f(q_t^f, e_{it}^x, \mathcal{R}_{it})$, that will be detailed below. They maximize the present discounted utility, with the discount rate ρ, and solves the following intertemporal problem.

$$\mathcal{V}_{it_0} = \max_{\{c_{it}, b_{it}, k_{it}\}} \int_{t_0}^{\infty} e^{-(\rho-n)t} u_i(c_{it}, \tau_{it})\, dt$$

The utility that households receive from consumption is also scaled by a damage function, which represents the direct impact of temperature.

$$u_i(c_{it}, \tau_{it}) = u\Big(\mathcal{D}_i^u(\tau_{it}) c_{it}\Big) \qquad\qquad u(\mathcal{D}\, c) = \frac{(\mathcal{D}c)^{1-\eta}}{1-\eta}$$

We aggregate the bond and capital of the individual country as a single wealth variable $w_{it} = k_{it} + b_{it}$, and rescale labor income and wealth per effective unit of labor v_{it}, accounting for TFP and population growth $\bar{g} + n$, it yields the dynamics:

$$\dot{w}_{it} = \big(r_t^\star - (n + \bar{g})\big) w_{it} + v_{it} + \mathrm{t}_{it}^{ls}$$

on $t \in [t_0, t_T]$ where the dynamics of wealth starts from initial condition $w_{t_0} = k_0 + b_0$. The return on capital is $r_{it}^k = MPk_{it} - \delta$ which is equalized to the bond return $r_{it}^k = r_t^\star$ in the absence of other financial market frictions. Capital is thus a control variable. Finally, the household receives lump-sum transfers t_i^{ls} that are now arbitrary. We will go at length on the various policy designs in later sections. This wealth level constitutes the first dimension of ex-post heterogeneity.

3.2 Final good firms

In each country $i \in \mathbb{I}$, a representative firm is producing the homogeneous final good using different inputs: labor, capital, and energy[12], coming for fossil or renewable sources. The firm

[12] The original – unnormalized – production function:

$$Y_t = F(K_t, E_t, L_t) = \mathcal{D}(\tau_t) z_t \Big[(1-\varepsilon)^{\frac{1}{\sigma}} \big(K_t^\alpha L_t^{1-\alpha}\big)^{\frac{\sigma-1}{\sigma}} + \varepsilon^{\frac{1}{\sigma}} \big(z_t^e E_t\big)^{\frac{\sigma-1}{\sigma}}\Big]^{\frac{\sigma}{\sigma-1}}$$

We divide the output level Y_t by the growth trend in population and TFP $e^{(n+\bar{g})t}$ and by initial population $\mathcal{P}_0 \equiv L_t$ to obtain output per effective capita.

maximizing profit, i.e. output per capita $y = \mathcal{D}^y(\tau) z f(\cdot)$, net of input costs:

$$\max_{k_{it}, e_{it}} \mathcal{D}_i^y(\tau_{it}) z_i f(k_{it}, e_{it}) - v_{it} - q^e_{it} e_{it} - (r^\star_t + \delta) k_{it}$$

where temperature τ, relative productivity z, capital stock per effective capita k and energy input per effective capita e all affect production. The temperature τ_{it} affects the productivity through damages $\mathcal{D}_y(\tau_{it})$. This is the source of climate externality as will detailed below. The gross production function is a CES aggregate between the capital-labor bundle k and energy e:

$$f(k_{it}, e_{it}) = \left[(1-\varepsilon)^{\frac{1}{\sigma}} k_{it}^{\alpha \frac{\sigma-1}{\sigma}} + \varepsilon^{\frac{1}{\sigma}} \left(z^e_t e_{it} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

with $\sigma < 1$, such as energy is complementary in production[13] and where directed technical change z^e_t is exogenous and deterministic. This directed – energy augmenting – technical change allows an increase in output for a given energy consumption mix. An upward trend in such technology is sometimes argued to be behind the "relative decoupling" of developed economies: an increase in production and value-added simultaneous to a decline in energy consumption. For now, this trend is taken exogenously increasing at rate $z^e_t = \bar{z}^e e^{g_e t}$, but in an extension of the model, we consider an endogenous directed technical change. Moreover, energy used in production comes from two sources: either fossil e^f_{it} and renewable e^r_{it} for every country i, as detailed below.

Energy demand

Given the demand for energy inputs e_t in each country, the firm has the choice among two sources of energy: one fossil-fuel source in finite supply e^f_t and one renewable source e^r_t. We consider that these two sources are substitutable, and total energy inputs quantity e_t is given by the CES aggregator, where σ_e represents the elasticity of substitution.

$$e_t = \left(\omega_f^{\frac{1}{\sigma_e}} (e^f_t)^{\frac{\sigma_e - 1}{\sigma_e}} + (1-\omega_f)^{\frac{1}{\sigma_e}} (e^r_t)^{\frac{\sigma_e-1}{\sigma_e}} \right)^{\frac{\sigma_e}{\sigma_e - 1}} \qquad \text{if} \qquad \sigma_e \in (1, \infty)$$

$$e_t = e^f_t + e^r_t \qquad \text{if} \qquad \sigma_e \to \infty$$

subject to the budget for energy expenditures:

$$q^e_t e_t = e^f_t \left(q^f_t + \mathrm{t}^f_{it} \right) + e^r_t q^r_t$$

[13] If $\sigma = 1$ we have the Cobb Douglas : $f(k_t, e_t) = \bar{\varepsilon} z^{e\ \varepsilon}_t k^\alpha_t e^\varepsilon_t$

As a result, demand curves for both fossil and renewable energies are given by usual CES demands:

$$\frac{e_t^f}{e_t} = \omega^f \left(\frac{q_t^f}{q^e}\right)^{-\sigma_e} \qquad \& \qquad \frac{e_t^r}{e_t} = (1-\omega^f)\left(\frac{q_t^r}{q_t^e}\right)^{-\sigma_e}$$

$$q_t^e = \left(\omega_f (q_t^r)^{1-\sigma_e} + (1-\omega_f)(e_t^r)^{1-\sigma_e}\right)^{\frac{1}{1-\sigma_e}} \qquad \text{if} \qquad \sigma_e \in (1,\infty)$$

$$q_t^e = \min\{q_t^f, q_t^r\} \qquad \text{if} \qquad \sigma_e \to \infty$$

where the price of the energy bundle q_t is some weighted sum of the energy price of fossil fuel q_t^f and renewable $q_t^{e,r}$.

Climate damage and externality

Change in temperatures τ_{it} in each country $i \in \mathbb{I}$ – given in degree Celsius, $^\circ C$ – affects the productivity with a Damage function $\mathcal{D}_y(\tau_t)$. This scaler increase with $\tau < \tau_i^\star$ and decreases when $\tau < \tau_i^\star$, where the "optimal temperature" $\tau_i^\star$ such that $\mathcal{D}_y(\tau_i^\star) = 1$. We consider the "optimal" temperature as:

$$\tau_i^\star = \alpha^\tau \tau_{it_0} + (1-\alpha^\tau)\tau^\star$$

where τ_{it_0} is the initial temperature in country i and $\tau^\star = 15.5^\circ C$ is an optimal level of yearly temperature for temperate climates, as used in Kotlikoff et al. (2021*b*). This flexible formulation allows for differing degrees of adaptability depending on the value of α^τ. Hot temperatures do not affect countries with long histories of cold vs. hot climates in the same way, due to the presence of adaptation structures – i.e. air conditioning vs. heating infrastructures.

Productivity decays to zero when temperatures are extremely cold or hot $\lim_{\tau\to-\infty} \mathcal{D}_y(\tau) = \lim_{\tau\to\infty} \mathcal{D}_y(\tau) = 0$. We follow Nordhaus formalism and use a quadratic function for the damage function:

$$\mathcal{D}_y(\tau) = \begin{cases} e^{-\gamma_y^\oplus \frac{1}{2}(\tau-\tau_i^\star)^2} & \text{if } \tau > \tau_i^\star \\ e^{-\gamma_y^\ominus \frac{1}{2}(\tau-\tau_i^\star)^2} & \text{if } \tau < \tau_i^\star \end{cases}$$

where $\gamma_y^\oplus$ and $\gamma_y^\ominus$ represent damage parameters on output respectively for hot v.s. cold temperatures – and they are different to allow for asymmetry on climate impact.

The utility that households receive from consumption is also scaled by a similar damage function, which represents the direct impact on population likelihood of mortality – for example, due to heatwaves or extreme weather events – as a direct scaler of consumption.

$$\mathcal{D}_u(\tau) = \begin{cases} e^{-\gamma_u^\oplus \frac{1}{2}(\tau-\tau_i^\star)^2} & \text{if } \tau > \tau_i^\star \\ e^{-\gamma_u^\ominus \frac{1}{2}(\tau-\tau_i^\star)^2} & \text{if } \tau < \tau_i^\star \end{cases}$$

where $\gamma_u^\oplus$ and $\gamma_u^\ominus$ represent also the damage parameters, but on the direct impact on utility and mortality, respectively, for hot v.s. cold temperatures.

In the previous graph, we present an example of such damage function for two countries, USA and India, with the distribution of temperature (approximated by a normal distribution),

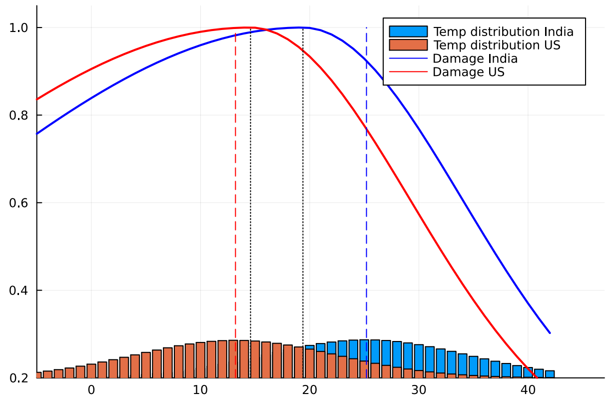


Figure 5: Damage function for two example countries, US and India

their average yearly temperature (respectively $13.5°C$ and $25°C$) in dashed lines and their optimal temperature in dotted black lines (respectively $15°C$ and $20°C$)

3.3 Energy firms

Fossil fuel extraction and exploration

Fossil energy is produced and sold in a centralized market at the world level. A continuum of competitive producers is extracting the fuel quantity e^x_{it} from their respective pool of resources $\mathcal{R}_{it}$, with production cost $\nu_i(e^x_{it}, \mathcal{R}_{it})$.

Fossil energy can be shipped costlessly around the world, where the global market in energy clears:

$$\sum_{\mathbb{I}} e^x_{it} = \sum_{\mathbb{I}} e^{(n+\bar{g})t}\, \mathcal{P}_i\, e^f_{it}$$

where the demand comes from the aggregation of individual energy per capita inputs in each country $i \in \mathbb{I}$ and energy input is rescaled by the population and technology exponential trends $e^{(n+\bar{g})t}$.

Moreover, the fossil-fuel reserves $\mathcal{R}_{it}$ are depleted with extraction e^x_{it}, but can be regenerated by exploration, which require investment ι^x_t to obtain $\delta^R \iota^x_t$ additional reserves for an exploration cost $\mu(\iota^x_t, \mathcal{R}_t)$

$$\dot{\mathcal{R}}_{it} = -e^x_{it} + \delta^R \iota^x_{it}$$

The parameter δ^R can be interpreted in two ways: first, it can represent the probability intensity $\delta^R \iota^x_t$ of finding developable reserves among possible reserves ι^x_{it} in a continuum of fossil fuel fields and mines. Second, it can also represent the fraction of individual producers discovering developable reserves, aggregating up a representative producer. This stylized model is a simplified version of the rich framework developed in Bornstein et al. (2023).

Moreover, the fossil-fuel producer hence faces a modified Hotelling finite-resources problem – c.f. Heal and Schlenker – allowing for exploration of additional reserves. As a result, its dynamic problem is given by :

$$v^e(\mathcal{R}_{it_0}) = \max_{\{e_t^x, \iota_t^x\}_{t \geq t_0}} \int_{t_0}^{\infty} e^{-\rho t} \pi_i^f \left(q_t^f, \mathcal{R}_{it}, e_{it}^x, \iota_{it}^x\right) dt$$

$$\text{with} \qquad \pi_i(q_t^f, \mathcal{R}_{it}, e_{it}^x, \iota_{it}^x) = q_t^f e_{it}^x - \nu_i(e_{it}^x, \iota_{it}^x) - \mu_i(\iota_{it}^x, \mathcal{R}_{it})$$

$$s.t. \qquad \dot{\mathcal{R}}_t = -e_{it}^x + \delta^R \iota_{it}^x \qquad\qquad \sum_{\mathbb{I}} e_{it}^x = \sum_{\mathbb{I}} \mathcal{P}_{i0} e^{(n+\bar{g})t} e_{it}^f$$

This can be solved using the Pontryagin maximum principle, where we denote λ_t^R the Hotelling rent, which is the costate of the resource depletion dynamics. The price of the fossil energy supplied and the optimal exploration are given by optimality conditions:

$$[e_{it}^x] \qquad q_t^f = \nu_{e^x}(e_{it}^{x\,\star}, \mathcal{R}_{it}) + \lambda_{it}^R$$

$$[\iota_{it}^x] \qquad \delta^R \lambda_{it}^R = \mu_\iota(\iota_{it}^{x\,\star}, \mathcal{R}_{it})$$

Price is hence the sum of marginal cost, plus an additional rent meant to price the finiteness of the resource. Moreover, the dynamics of that Hotelling rent are given by the equation:

$$\dot{\lambda}_{it}^R = \rho \lambda_{it}^R + \nu_R(e_t^x, \mathcal{R}_{it}) + \mu_R(\iota_{it}^x, \mathcal{R}_{it})$$

In standard Hotelling models without stock effects – i.e. where $\nu_R(e^x, \mathcal{R}) = 0$ and no exploration $\mu(\iota^x, \mathcal{R}) = 0$ – we have the standard expression for the finite resource rent $\dot{\lambda}_t^R = \rho \lambda_t^R$ and $\lambda_t^R = e^{\rho t} \lambda_{t_0}^R$, and $R_t \to 0$ as $t \to \infty$. In our context, the rent grows less fast because (i) the producer anticipate that the depletion of reserves will increase marginal cost in the future $\nu_R(E^\star, \mathcal{R}) < 0$ and (ii) it can invest in exploration, increasing future reserves which can lower even further the future cost of exploring $\mu_R(\iota^\star, \mathcal{R}) < 0$.

As a result, with functional forms that yield isoelastic supply curves for fossil energy extraction and exploration, we can solve the dynamics of the rent price.[14]

$$\nu_i(e_{it}^x, \mathcal{R}_{it}) = \frac{\bar{\nu}_i}{1+\nu} \left(\frac{e_{it}^x}{\mathcal{R}_{it}}\right)^{1+\nu} \mathcal{R}_{it} \qquad\qquad \mu_i(\iota_{it}^x, \mathcal{R}_{it}) = \frac{\bar{\mu}_i}{1+\mu} \left(\frac{\iota_{it}^x}{\mathcal{R}_{it}}\right)^{1+\mu} \mathcal{R}_{it}$$

Note that this market for fossil fuels is in equilibrium: an aggregate supply curve (q_t^f, E_t^f) determined by the aggregation of fossil-fuel producers $E_t^f = \sum_i e_{it}^x$ meets the demand coming from the aggregation of all individual countries (q_t^f, e_{it}^f). Moreover, fossil fuels emit CO_2 and other GHG emissions, as we will see in the next section.

Renewable energy production

Renewable energy is not subject to the finiteness of the stock of reserves and is produced

[14] Details of the fossil energy producers can be found in appendix .

with capital k_t^e.

$$e_t^r = z_t^r f(k_t^r)$$

Furthermore, carbon emissions associated with renewable energy are null, minimizing the externality on the climate when the energy transition is complete. We assume that capital k_t^r is fungible with the capital k_t that produces the homogeneous good and is hence subject to the same interest $r_t^\star$ on the common world capital market

$$q_t^r z_t^r f'(k_t^r) = r_t$$

where q_t^r is the price of that renewable energy demanded. We make these stylized assumptions to keep the model tractable.

For now, renewable energy production is assumed constant return to scale, i.e. $f^r(k_t^r) = k_t^r$. As a result, the price of non-fossil energy q_t^r is given exogenously by:

$$q_t^r = \frac{r_t^\star}{z_t^r}$$

Moreover, if the two sources of energy are perfectly substitutable, i.e. $\sigma_e \to \infty$, then we obtain that renewables act as a perfect "backstop" technology to fossil fuel. If q_t^f grows up to then all the energy is produced using renewable $e_t = e_t^r$ and the emissions collapse to zeros. This example is analyzed in Heal and Schlenker (2019) in a simpler model.

3.4 Climate system, emissions and externality

Economic activity are emitting carbon and other greenhouse gas emissions, which change the climate and increase the temperature of the atmosphere. Due to these activities coming from the energy sector, each country is emitting CO_2 per effective capita:

$$\epsilon_{it} = \xi_i^f \mathcal{P}_i e_{it}^f$$

where ξ^f denote the carbon content of fossil fuels[15]. As a result, since the energy use is normalized by growth of TFP and population, the absolute amount of global emissions aggregates to:

$$\mathcal{E}_t = \sum_{i \in \mathbb{I}} e^{(n+\bar{g})t} \epsilon_{it} = e^{(n+\bar{g})t} \sum_{\mathbb{I}} \xi_i^f \mathcal{P}_i e_{it}^f di$$

[15]We can consider an alternative, like in Nordhaus' DICE model, with

$$\epsilon_{it} = \xi^f (1 - \vartheta_{it}) e_{it}^f \mathcal{P}_i \qquad \& \qquad \mathcal{E}_t = e^{(n+\bar{g})t} \sum_{\mathbb{I}} \xi^f (1 - \vartheta_{it}) e_{it}^f \mathcal{P}_i$$

where ϑ_t represents the abatement policy taken in country i. It represents all the policies that allow reducing the emissions for a given choice of the energy mix – for example, additional environmental regulations or investment in carbon capture technology – with a convex cost $c(\vartheta_{it}) e_{it}^f$. Its optimal choice can be determined as solution of the FOC $c'(\vartheta_i) e_{it}^f = 0 \;\Rightarrow\; \vartheta_i = 0$ (business as usual) or $c'(\vartheta_i) = -\xi_i \mathbf{t}_{it}^f$ (second best with carbon tax).

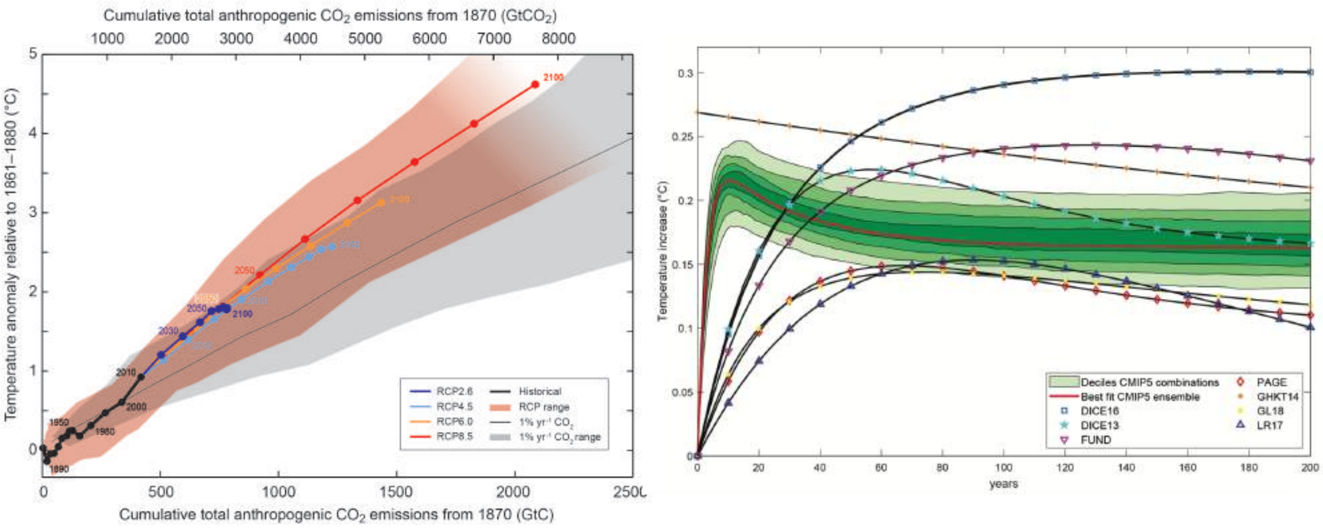
These emissions are released in the atmosphere, adding up to the cumulative stock of greenhouse gas $\mathcal{S}_t$.

$$\dot{\mathcal{S}}_t = \mathcal{E}_t - \delta_s \mathcal{S}_t$$

However, a part of these emissions exit the atmosphere and can be stored in oceans or the biosphere, discounting the current stocks by an amount δ_s. Moreover, these cumulative emissions push the global atmospheric temperature $\mathcal{T}_t$ upward linearly with parameter χ with some inertia and delay represented by parameter ζ

$$\dot{\mathcal{T}}_t = \zeta\big(\chi \mathcal{S}_t - (\mathcal{T}_t - \bar{\mathcal{T}}_{t_0})\big)$$

This simple two-equations climate system is a good approximation of large-scale climate models[16] with a small set of parameters $\xi^f, \delta_s, \zeta, \chi$.



Linear temperature model - IPCC report / Dietz, van der Ploeg, Rezai, Venmans (2021)

More particularly, ζ is the inverse of persistence, and modern calibrations set $\zeta \approx 0.1$ is such that the pick of emissions happens after 10 years. Dietz et al (2021) show that classical IAM models such at Nordhaus' DICE tend to set ζ too low, generating a too large inertia of the climate system, as shown in the figure below. Moreover, if $\zeta \to \infty$, temperature reacts immediately and we obtain a linear model – which is a good long-run approximation:

$$\mathcal{T}_t = \bar{\mathcal{T}}_{t_0} + \chi \mathcal{S}_t = \bar{\mathcal{T}}_{t_0} + \chi \int_{t_0}^{t} \sum_{\mathbb{I}} e^{(n+\bar{g})t} \epsilon_{it} \, ds \Big|_{GtC}$$

[16]These climate models have typically much more complex climate block, adding 3 to 4 more state variables, with $\mathbf{J}$ the vector of carbon "boxes": layers of the atmosphere and sinks such as layers of oceans:

$$\dot{\mathbf{J}}_t = \Phi^J \mathbf{J}_t + \rho^e \sum_{\mathbb{I}} \xi^f \mathcal{P}_i e_i^f$$
$$F_t = \mathcal{F}(\mathbf{J}_t) \qquad \dot{\mathcal{T}}_t = \Phi^T \mathcal{T} + \eta F_t$$

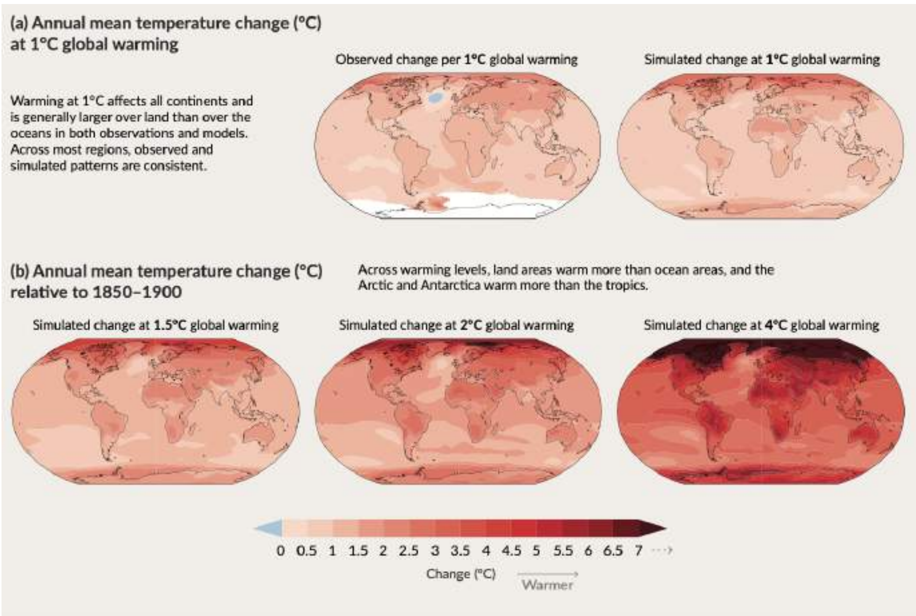
with F_t Carbon forcing and ρ^e, vector of parameters, Φ^J and Φ^T Markovian transition matrices and $\mathcal{F}(\cdot)$ a non-linear function.

As we see, the global externality depends on the path of individual policies $\epsilon_{it} \propto e^f_{it}$ as of function of the endogenous states of the country $\{w_i, \tau_i\}$, as well as the growth rates $\bar{g}+n$ of the economy, i.e. TFP and population.

The temperature in country i is affected by global warming of the atmosphere $\mathcal{T}_t$ with sensitivity Δ_i

$$\dot{\tau}_{it} = \Delta_i \, \dot{\mathcal{T}}_t$$

Atmospheric temperature $\mathcal{T}_t$ translates into local temperature τ_{it} according to a pattern scaler Δ_i that depends on the geographic properties of country i – like temperature, latitude, longitude, elevation, distance from coasts and water bodies, vegetation, and albedo (sunlight reflexivity due to ice, vegetation and soil properties)[17]. Evidence of this temperature scaling is displayed in the following map from the IPCC report.



[17]This pattern scaling could be simplified with a simple linear equation as a first-order approximation $\Delta_i = 1.537 - 0.0288 \times \tau_{it_0}$. Moreover, this scaling could be made more realistic and time-varying using a non-linear function of temperature $\Delta_i \equiv \Delta(\tau_{it})$.

4 Competitive equilibrium and Business as usual

4.1 Household / Firm

First, since the Household owns the three firms – final good, fossil, and renewable energy – we can aggregate profits and household budget constraint, which gives:

$$\dot{w}_{it} = \big(r^\star_t - (n+\bar{g})\big)w_{it} + \pi^f_{it} + \mathcal{D}^y(\tau_{it})z_{it}f(k_{it}, e^f_{it}, e^r_{it}) - (r^\star + \delta)k_{it} - \big(q^f_t + \mathrm{t}^f_{it}\big)e^f_{it} - q^r_{it}e^r_{it} - c_{it} + \mathrm{t}^{ls}_i$$

which yields a single optimal control problem. However, the consumption/saving that relates to the path of wealth w_{it} and the firms decisions in energy e_{it} and capital k_{it} are still separated, which provide different optimality conditions.

To solve for the competitive equilibrium and the optimal decision of the Household, we use the Pontryagin Maximum Principle. The Hamiltonian of the individual country with individual states $\mathbf{s} = \{s_i\}_i = \{z_i, \mathcal{P}_i, \bar{\nu}_i, \gamma_i, \Delta_i, \xi_i, w_i, \tau_i\}_i$, individual controls $\mathbf{c} = \{c, b, k, e^f, e^r\}_i$ and costates/Lagrange multipliers, $\lambda = \{\lambda^w, \lambda^\tau, \lambda^S\}$ writes as follow:

$$\mathcal{H}(\mathbf{s}, \mathbf{c}, \lambda) = u(c, \tau) + \lambda^w \dot{w} + \lambda^\tau \dot{\tau} + \lambda^S \dot{S}$$

The equilibrium relations for the household consumption/saving problem boil down to the standard neoclassical model dynamics and for each country $i \in \mathbb{I}$, we obtain a system of coupled ODEs.

$$\begin{cases} \dot{\lambda}^w_{it} &= \lambda^w_{it}(\rho + \eta\bar{g} - r^\star_t) \\ \lambda^w_{it} &= u_c(c_{it}, \tau_{it}) \end{cases}$$

where λ^w_{it} is the costate for the wealth w_{it} of country i, i.e. the marginal value of an additional unit of wealth optimal should be increasing if the world interest rate exceeds the discount factor ρ. Using the law of motion and the definition of the marginal value of wealth, we obtain the Euler equation:

$$\frac{\dot{c}_{it}}{c_{it}} = \frac{1}{\eta}\left(r^\star_t + \eta\bar{g} - \rho\right) + \gamma_i(\tau_{it} - \tau^\star_i)\dot{\tau}_{it}$$

The dynamics of local temperature appear in the Euler equation. Indeed, because the marginal utility of consumption is affected directly by changes in temperature, an increase in temperature in the future triggers substitution from present to future consumption through saving.

Moreover, the capital and energy choices simply result from static optimization between price/cost and marginal return of those inputs in the production.

$$\begin{cases} q^f_t + \mathrm{t}^f_{it} &= MPe^f_{it} \\ r^\star_{it} &= MPk_{it} - \delta \end{cases} \qquad\qquad q^r_{it} = MPe^r_{it}$$

where $MPx = \partial_x[\mathcal{D}^y(\tau)zf(k, e^f, e^r)]$ for $x \in \{k, e^f, e^r\}$. Moreover, the bonds are in zero net supply, and hence the aggregate wealth should equal the aggregate capital stock

$$\sum_{i\in\mathbb{I}} \mathcal{P}_i b_{it} = 0 \qquad \Rightarrow \qquad \sum_{i\in\mathbb{I}} \mathcal{P}_i w_{it} = \sum_{i\in\mathbb{I}} \mathcal{P}_i k_{it}$$

4.2 Fossil energy market

The dynamics of Hoteling rents λ^R_{it} for the fossil energy price q^f_t are described above and listed here for completeness:

$$\begin{cases} q^f_t &= \nu_{e^x}(e^x_{it}, \mathcal{R}_{it}) + \lambda^R_{it} \qquad\qquad \delta^R \lambda^R_{it} = \mu_\iota(\iota^x_{it}, \mathcal{R}_{it}) \\ \dot{\lambda}^R_{it} &= \rho\lambda^R_{it} + \nu_R(e^x_{it}, \mathcal{R}_{it}) + \mu_R(\iota^x_t, \mathcal{R}_{it}) \\ \dot{\mathcal{R}}_{it} &= -e^x_{it} + \delta^R \iota^x_{it} \end{cases}$$

where the optimal extraction and exploration depend on the dynamic Hoteling rent λ^R_{it} that varies with stock effects due to depleting reserves $\mathcal{R}_{it}$.

Moreover, the energy market clears between the demand of individual countries and supply from the fossil energy firm:

$$E^f_t = \sum_{i\in\mathbb{I}} e^x_{it} = \sum_{i\in\mathbb{I}} \mathcal{P}_i e^{(n+\bar{g})t} e^f_{it}$$

4.3 Local cost of carbon and Climate system

In addition, the climate block for carbon stock $\mathcal{S}_t$ and temperature τ_{it} are valued with the costates λ^S_{it} and λ^τ_{it}, representing respectively the marginal value of adding an additional unit of carbon in the atmosphere $\mathcal{S}_t$ and the marginal value of increasing local temperature by an additional degree. Recalling the dynamics of the climate system,

$$\begin{cases} \mathcal{E}_t &= \sum_{i\in\mathbb{I}} \epsilon_{it} = \sum_{\mathbb{I}} e^{(n+\bar{g})t} \xi_i \mathcal{P}_i e^f_{it} \\ \dot{\mathcal{S}}_t &= \mathcal{E}_t - \delta^s \mathcal{S}_t \\ \dot{\tau}_{it} &= \zeta\big(\Delta_i \chi \mathcal{S}_t - (\tau_{it} - \tau_{it_0})\big) \end{cases}$$

we can use the Pontryagin principle to pin down the dynamics of the local cost of carbon. First, the shadow value of increasing temperatures is affected by the cost of climate on both the productivity effect $\mathcal{D}^y(\tau) z f(k,e)$ and the utility effect $u(\mathcal{D}^u(\tau)c)$.

$$\dot{\lambda}^\tau_{it} = \lambda^\tau_{it}(\rho+\zeta) + \underbrace{\gamma^y_i(\tau_{it} - \tau^\star_i)\mathcal{D}^y(\tau_{it})}_{-\partial_\tau \mathcal{D}^y} f(k_{it}, e_{it})\lambda^w_{it} + \underbrace{\gamma^u_i(\tau_{it} - \tau^\star_i)\mathcal{D}^u(\tau_{it})}_{-\partial_\tau \mathcal{D}^u} u'(\mathcal{D}^u(\tau)c_{it})c_{it}$$

Indeed, this shadow value increases with marginal damages, scaled by both marginal utility of wealth λ^w_{it} and consumption $u'(\mathcal{D}^u(\tau)c_{it})$. This change in the marginal value of temperature affects directly the shadow value of adding carbon in the atmosphere according to the dynamics of λ^S_{it}:

$$\dot{\lambda}^S_{it} = \lambda^S_{it}(\rho + \delta^s) - \zeta\,\chi\,\Delta_i\,\lambda^\tau_{it}$$

We hence see why adding an extra unit of carbon in the atmosphere has a differential impact of different regions due to heterogeneous costs of temperature and vulnerability to climate synthesized by the pattern scaling parameters Δ_i and marginal damages γ^y_i and γ^u_i.

The Local Cost of carbon is a common measure used by climate scientists and climate economists to summarize the marginal welfare cost of carbon in monetary terms. The Cost of Carbon is an equilibrium concept, in the sense that it depends on the trajectories of temperatures but also on production and consumption. In the competitive equilibrium, the climate externality of fossil fuel use is not internalized and households do not take climate damage into account for choosing consumption, production, and energy decisions. A typical microfoundation of such an assumption is to consider infinitesimal agents and regions, such that $\partial_{e^f_i}\mathcal{E}_t = 0$. However, one doesn't need such an assumption to analyze the cost of externality, especially when looking at large countries with the United States, China or India that have large carbon footprints.

Moreover, with or without infinitesimal agents, it doesn't prevent the households to be rational and to anticipate perfectly the evolution of climate in the region. The Local Cost of Carbon (LCC) represents such a welfare measure that is normalized into monetary units according to the marginal utility of wealth/consumption in the region, as indeed the monetary value of one unit of welfare is different across regions due to inequality in consumption $\frac{\partial \mathcal{V}_{it}}{\partial c_{it}} = \lambda^w_{it} = u_c(c_{it}) \neq u_c(c_{jt}) = \lambda^w_{jt}$.

In continuous time, and using our framework of the Pontryagin Maximum Principle, this local cost of carbon rewrite easily as the ratio of the two costates:

$$LCC_{it} := \frac{\frac{\partial \mathcal{V}_{it}}{\partial S_t}}{\frac{\partial \mathcal{V}_{it}}{\partial c_{it}}} = -\frac{\lambda^S_{it}}{\lambda^w_{it}}$$

In the competitive equilibrium, this measure integrates the cost of climate on locality i even in any suboptimal policy. Note that is *not* the social cost of carbon (SCC) as the SCC would integrate spillovers of each country on the rest of the world and a potentially optimal path of consumption. However, this notion is exactly analogous to the Local Cost of Carbon concept developed in Cruz Álvarez and Rossi-Hansberg (2022).

As a result, following the dynamics of the LCC amounts to solve for the dynamics of both costates λ^w_{it} and λ^S_{it}.

4.4 General Equilibrium

A complete description of the system can be found in appendix B. In this framework, there are types of interaction mechanisms between the different countries $i \in \mathbb{I}$.

First, the emissions from each country affect the global climate and local temperatures, creating these heterogeneous impacts and costs of climate change λ. Second, fossil energy markets clear such that the energy demand from all the individual countries impact the fossil fuel price q_t^f and has redistributive effects on the fossil energy rent $\pi_t(E^f, \mathcal{R})$. Third, the bonds market also clears as assets are in zero net supply, and individual savings and consumptions depend on the path of world interest rate as well as collateral constraints. However, there are no bilateral flows between individual countries, such as migration or bilateral trade and capital flow.

This makes this system of ordinary differential equations (ODEs) the specificity of being strongly coupled. Despite the infinite dimensionality of this system, this problem is well-posed, as it is the solution of Forward Backward McKean Vlasov system of ordinary differential equations. Despite the possibility many global interactions, i.e. each country interacts with global variables affected by the entire distribution of agents – atmospheric temperature $\mathcal{T}_t$, fossil energy price q_t^f, world interest rate $r_t^\star$ – one can not add bilateral flow. Allowing bilateral/local interaction may make the problem ill-posed, as explained in Boucekkine, Camacho and Zou (2009) and in the sense that there is no existence of solutions to the problem. We hence assume solely global interactions in the scope of this paper. The definition of competitive equilibrium is as follows:

Definition 4.1. *Given, ex-ante heterogeneity* $\{z_i, \mathcal{P}_i, \bar{\nu}_i, \gamma_i, \Delta_i, \xi_i\}$ *and initial conditions* $\{w_{it_0}, \tau_{it_0}, \mathcal{R}_{it_0}\}$ *and* $\{\mathcal{S}_{t_0}, \mathcal{T}_{t_0}\}$ *a competitive equilibrium is a continuum of sequences of states* $\{w_{it}, \tau_{it}, \mathcal{R}_{it}\}_{it}$ *and* $\{\mathcal{S}_t, \mathcal{T}_t\}_t$*, policies* $\{c_{it}, b_{it}, k_{it}, e_{it}^f, e_{it}^r, e_{it}^x, \iota_{it}^x\}_{it}$*, and price sequences* $\{q_t^f, q_t^r, r_t^\star\}$ *such that:*

- *Households choose policies* $\{c_{it}, b_{it}\}_{it}$ *to maximize their utility subject to budget constraint*
- *Final good firms choose policies* $\{k_{it}, e_{it}^f, e_{it}^r\}$ *to maximize profit.*
- *Renewable energy firm produce* $\{e_{it}^r\}$ *to maximize static profits*
- *The fossil fuels firms extract and explores* $\{e_{it}^x, \iota_{it}^x\}$ *to maximize profit*
- *Emissions* $\mathcal{E}_t$ *affect climate* $\{\mathcal{S}_t, \mathcal{T}_t\}_t$*, &* $\{\tau_{it}\}_{it}$ *following the climate system dynamics.*
- *Prices* $\{q_t^f, q_{it}^r, r_t^\star\}$ *adjust to clear the markets for fossil and renewable energy and bonds,*

$$E_t^f = \sum_{\mathbb{I}} e_{it}^x = \sum_{\mathbb{I}} e^{(\bar{g}+n)t} \mathcal{P}_i e_{it}^f \qquad e_{it}^r = z_i^r f(k_{it}^r) \qquad \sum_{i \in \mathbb{I}} b_{it} = 0$$

while the last good market clears by Walras law

$$\sum_{\mathbb{I}} \mathcal{P}_i c_{it} + \sum_{\mathbb{I}} \mathcal{P}_i [\nu_i(e_{it}^x, \mathcal{R}_{it}) + \mu_i(\iota_{it}^x, \mathcal{R}_{it})] + \sum_{\mathbb{I}} \mathcal{P}_i [\dot{k}_{it} + (n + \bar{g} + \delta) k_{it}] = \sum_{\mathbb{I}} \mathcal{P}_i \mathcal{D}_i^y(\tau_{it}) z_i f(k_{it}, e_{it})$$

This Business-as-usual scenario features unrestricted use of fossil energy until its price increases when resources are depleted. In particular, temperatures increase to high levels, and climate damages are large. We will analyze the result in the quantitative section below. We now turn to the optimal policy to take into account the climate externalities.

5 Optimal climate policy – First-Best

We consider the optimal policy of a social planner that maximizes the weighted sum of the Household utility, where the Pareto weights ω_i are arbitrary[18], and subject to the resource constraints of the economy.

By choosing all the agent decisions, consumption c_{it}, bonds and capital b_{it} and k_{it}, energy e^f_{it} and e^r_{it}, it would internalize the climate externality due to emissions $\mathcal{E}_t$ and increase in temperature τ_{it}. We denote by $\mathcal{V}_t$ the aggregate welfare in this social planner equilibrium.

$$\mathcal{W}_{t_0} = \max_{\{c,b,k,e^f,e^r,e^x,\iota,\mathcal{E}\}} \int_{t_0}^{\infty} \sum_{\mathbb{I}} e^{-(\rho+n)t}\, \omega_i\, \mathcal{P}_i\, u\big(\mathcal{D}^u(\tau_{it})\, c_{it}\big)\; dt$$

subject to the resource constraints of the economy and the energy and climate system:

$$\sum_{\mathbb{I}} \mathcal{P}_i c_{it} + \sum_{i\in\mathbb{I}} \nu(e^x_{it}, \mathcal{R}_{it}) + \mu(\iota^x_{it}, \mathcal{R}_{it}) + \sum_{\mathbb{I}} \mathcal{P}_i[\dot{k}_{it} + (n+\bar{g}+\delta)k_{it}] = \sum_{\mathbb{I}} \mathcal{P}_i \mathcal{D}^y_i(\tau_{it}) z_i f(k_{it}, e_{it}) \qquad [\widehat{\lambda}_t]$$

$$E^f_t = \sum_{\mathbb{I}} e^x_{it} = \sum_{\mathbb{I}} e^{(\bar{g}+n)t} \mathcal{P}_i e^f_{it} \qquad e^r_{it} = z^r_i f(k^r_{it}) \qquad [\widehat{\lambda}^r_{it}]$$

$$\mathcal{E}_t = \sum_{\mathbb{I}} e^{(n+\bar{g})t} \xi_i \mathcal{P}_i e^f_{it} \qquad \dot{\mathcal{S}}_t = \mathcal{E}_t - \delta^s \mathcal{S}_t \qquad [\widehat{\lambda}^S_t]$$

$$\dot{\tau}_{it} = \zeta\big(\Delta_i \chi \mathcal{S}_t - (\tau_{it} - \tau_{it_0})\big) \qquad [\widehat{\lambda}^\tau_{it}]$$

$$\dot{\mathcal{R}}_{it} = -e^x_t + \delta^R \iota^x_t \qquad [\widehat{\lambda}^R_t]$$

The Social planner choses, consumption/saving c_{it}, energy mix e^f_{it}, extraction e^x_t and exploration ι^x_t, as well as the trajectories of dynamic states $\{w, \tau, \mathcal{R}\}_{it}$. Note that the planner has discount factor $\widetilde{\rho}$ which might be different than the agent discount parameter ρ, and notably smaller, if we believe the planner could be more patient. Moreover, we denote the Lagrange multiplier of the Social Planner allocation by $\widehat{\lambda}$'s. We observe now that the market clearing for goods has a common shadow value $\widehat{\lambda}_t$ for all locations $i \in \mathbb{I}$ at the difference to the competitive equilibrium.

The result is analogous to the toy model example. The choice of consumption solves for redistribution motive, as the planner searches for equalizing marginal utility, subject to the Pareto weights:

$$\omega_i u_c(c_{it}, \tau_{it}) = \widehat{\lambda}_t = \omega_j u_c(c_{jt}, \tau_{jt})$$

with marginal utility $u_c(c_{it}, \tau_{it}) = \mathcal{D}^u(\tau_{it}) u'(\mathcal{D}^u(\tau_{it})c) = \mathcal{D}^u(\tau_{it})^{1-\eta} c_{it}^{-\eta}$, with the CRRA functional form. Despite the possibility, in the competitive equilibrium, to trade in goods, bonds, and energy, strong inequality exists due to differences in productivity, energy rents or climate damage. As a result, the social planner, would like to redistribute consumption and this would be done using lump-cum transfers in the decentralized equilibrium.

The fossil energy choice is similar to the toy model since the marginal utility of consumption

[18] The only constraint we impose is that they integrate to one $\sum_{\mathbb{I}} \omega_i = 1$

are equalized to $\widehat{\lambda}$ across countries.

$$MPe^{f}_{it}\,\widehat{\lambda}_t = \nu_{e^x}(e^x_{it}, \mathcal{R}_{it})\,\widehat{\lambda}_t \;+\; \widehat{\lambda}^R_t \;-\xi_i\,\widehat{\lambda}^S_t$$

We see that the planner equalizes the marginal product of fossil energy $MPe^f = \mathcal{D}^y(\tau)zf_{e^f}(k, e^f, e^r)$ to its shadow cost. This marginal cost is the sum of different channels: first, the marginal extraction cost $\nu(\cdot)$, second, the social Hoteling rent $\widehat{\lambda}^R_t/\widehat{\lambda}_t$ and third integrates the climate damage $\widehat{\lambda}^S_t/\widehat{\lambda}_t$ as we will see in the next section.

The conditions for the choice of renewable energy and capital are standard in the neoclassical model:

$$\widehat{\lambda}^r_{it}/\widehat{\lambda}_t = MPe^r_{it}$$

$$\dot{\widehat{\lambda}}_t = (\widehat{r}_t + \eta\bar{g} - \widetilde{\rho})\,\widehat{\lambda}_t \qquad\qquad \widehat{r}_t = MPk_{it}$$

where $\widehat{r}_t$ is the shadow price of capital which is equalized across countries. Moreover, the capital choice is not constrained by borrowing limits, because goods can be allocated and transferred freely between regions and time periods.

5.1 Social Cost of Carbon

In this optimal allocation, the marginal cost of adding one unit of carbon in the atmosphere $\mathcal{S}_t$ can be summarized by the Social Cost of Carbon:

$$\overline{SCC}_t := -\frac{\frac{\partial \mathcal{V}_t}{\partial \mathcal{S}_t}}{\frac{\partial \mathcal{V}_t}{\partial \hat{c}_{it}}} = -\frac{\widehat{\lambda}^S_t}{\widehat{\lambda}_t}$$

We see that since the marginal utility of consumption/ marginal value of wealth is equalized across countries $\partial\mathcal{V}_t/\partial\hat{c}_{it} = \omega_i u_c(c_{it}, \tau_{it}) = \widehat{\lambda}_t$, the normalization of the welfare cost $\widehat{\lambda}^S_t$ into monetary unit is not ambiguous, and doesn't depend on the country one chooses. The welfare cost of carbon evolve again with the marginal damage of temperature:

$$\dot{\widehat{\lambda}}^{\tau}_{it} = \widehat{\lambda}^{\tau}_{it}(\rho + \zeta) + \underbrace{\gamma^y_i(\tau_{it} - \tau^\star_i)\mathcal{D}^y(\tau_{it})}_{-\partial_\tau \mathcal{D}^y} f(k_{it}, e_{it})\lambda_t p_i + \underbrace{\gamma^u_i(\tau_{it} - \tau^\star_i)\mathcal{D}^u(\tau_{it})}_{-\partial_\tau \mathcal{D}^u} u'(\mathcal{D}^u(\tau)c_{it})c_{it}p_i$$

Again, the shadow value increases with marginal damages, scaled by the common marginal value of wealth $\widehat{\lambda}_{it}$ and consumption $u'(\mathcal{D}^u(\tau)c_{it})$. The marginal value of temperature again affects the shadow value of carbon, but this time in an aggregate fashion, where all of the costs for all countries $\lambda^\tau_{it}, \forall i \in \mathbb{I}$:

$$\dot{\widehat{\lambda}}^{S}_{it} = \widehat{\lambda}^S_{it}(\rho + \delta^s) - \zeta\,\chi\,\sum_{\mathbb{I}} \Delta_i\,\widehat{\lambda}^\tau_{it}$$

Given the dynamics of this welfare cost of carbon, the fossil energy choice boils down

$$MPe^f_{it} = \partial_E \nu(E^f_t, \mathcal{R}_t) + \frac{\widehat{\lambda}^R_t}{\widehat{\lambda}_t} \underbrace{-\xi_i \frac{\widehat{\lambda}^S_t}{\widehat{\lambda}_t}}_{=\xi_i SCC_t}$$

with the conversion parameter "energy to carbon" ξ_i for each country.

From this optimality condition, we recover the standard Representative agent's result that Pigouvian Taxation should equal the marginal damage from the externality, exactly as in the result of Golosov et al. (2014). In particular, the carbon tax is equal across countries. To see that taxation result, let us analyze the decentralization of such allocation in the competitive equilibrium.

5.2 Decentralization, taxation, and transfers

We recall the budget constraint of each agent and augment it with two tax instruments that will be necessary for the planner to decentralize the optimal allocation: first, a fossil fuel tax t^f_{it} is used to account for the climate externality, second, lump-sum transfers are used to tax or transfers lump-sum to each country.

$$\dot{w}_{it} = r^\star_t w_{it} + \mathcal{D}^y(\tau_{it}) z_{it} f(k_{it}, e^f_{it}, e^r_{it}) + \theta^R_i \pi^R_t - (n + \bar{g} + \delta) k_{it} - (q^f_t + \mathrm{t}^f_{it}) e^f_{it} - q^r_{it} e^r_{it} - c_{it} + \mathrm{t}^{ls}_{it}$$

First, turning to the energy tax, we see how the planner's first-order condition can be decentralized:

$$MPe^f_{it} = \underbrace{\nu_{e^x}(e^x_{it}, \mathcal{R}_{it}) + \frac{\widehat{\lambda}^R_t}{\widehat{\lambda}_t}}_{=\text{price } q^f_t} - \xi_i \underbrace{\frac{\widehat{\lambda}^S_t}{\widehat{\lambda}_t}}_{=-\overline{SCC}_t}$$

$$MPe^f_{it} = q^f_t + \xi_i \mathbf{t}^S_t \qquad\qquad \mathbf{t}^S_t = \overline{SCC}_t$$

In particular, the carbon tax is equal across countries, thanks to the adjacent equalization of marginal utility of consumption / marginal value of wealth. To achieve such equalization in the decentralization, the planner needs to use lump-sum transfers:

$$\omega_i \partial_c u(c_{it}, \tau_{it}) = \widehat{\lambda}_t = \omega_j \partial_c u(c_{jt}, \tau_{jt}) \Rightarrow \qquad\qquad c_{it} = u_c^{-1}(\widehat{\lambda}_t | \tau_{it})$$

and, using the budget constraint above, one obtains such consumption levels using lump-sum transfers:

$$c_{it} = (r^\star_t - n - \bar{g}) w_{it} + \mathcal{D}^y(\tau_{it}) z_{it} f(k_{it}, e^f_{it}, e^r_{it}) + \pi^f_{it} - \delta k_{it} - (q^f_t + \xi_i \mathrm{t}^S_t) e^f_{it} - q^r_{it} e^r_{it} - \dot{w}_{it} + \mathrm{t}^{ls}_{it}$$

In particular, lump-sum transfers (per efficient unit of population) allow redistributing across countries and across time:

$$\int_{t_0}^{\infty} e^{(n+\bar{g})t} \sum_{\mathbb{I}} \mathcal{P}_i \mathrm{t}_{it}^{ls}\, dt = 0$$

In particular, in situations where the technology difference z_i, energy comparative advantage $\bar{\nu}_i$, vulnerability to climate γ_i or Pareto weights ω_i are very heterogeneous such that consumption differentials in the equilibrium without policy intervention are large, one can show that some countries would receive positive lump-sum transfers $\exists j, s.t.\ \mathrm{t}_j^{ls} > 0$ and some would have to pay lump-sum taxes $\exists j', s.t.\ \mathrm{t}_{j'}^{ls} < 0$. This implies that such decentralized allocation features direct lump-sum transfers across countries.

The question is whether such lump-sum transfers are feasible politically. Would a world central planner be able to solve world inequality by imposing lump-sum transfers, for example taxing North America and Europe and rebating it lump-sum to Africa or South Asia? The representative agent framework such as Golosov et al. (2014) or heterogeneous agent models with unrestricted redistribution such as Hillebrand and Hillebrand (2019) all assume the availability of such lump-sum transfers.

In the next section, we will analyze the policies where this family of policies is not feasible for political, governance, or economic reasons. Imposing such constraints prevents redistribution and equalization of marginal utilities across countries, and requires to solve for different kinds of optimal policy problems.

6 Ramsey problem and optimal energy policy

We again consider the optimal policy of a social planner that maximizes the weighted sum of the Household utility, now subject to the optimality conditions of the agents. In this context, it would not only internalize all the dynamics of economic variables, the climate, and energy markets but also the decisions that households and firms take.

The Ramsey planner chooses consumption/saving c_{it}, energy mix e_{it}^f and e_{it}^r, the extraction and exploration E_t^f and $\mathcal{I}_t$ as well as the trajectories of dynamic states $(\tau, \mathcal{S}, \mathcal{R})$ indirectly:

$$\mathcal{W}_{t_0} = \max_{\{c,b,k,e^f,e^r,e^x,\iota^x\}} \int_{t_0}^{\infty} \sum_{\mathbb{I}} e^{-(\rho+n)t}\ \omega_i\ \mathcal{P}_i\ u\big(\mathcal{D}(\tau_{it})c_{it}\big)\ dt$$

subject to (i) the optimality conditions of households, for c_i, b_i, k_i, e_i^f, e_i^r, (ii) the optimality conditions of the Fossil fuel producers for e_i^x, ι and $\mathcal{R}$ and (iii) the Climate and temperature dynamics τ_i and $\mathcal{S}$. We apply the Pontryagin Maximum Principle in $\mathbb{I}$-dimension[19] – the details of the entire system can be found in appendix appendix C. The Lagrange multipliers corresponding

[19]Note that a previous version of this work studied a continuum of countries, resulting in a Mean-Field Game for the competitive equilibrium and a system of McKean Vlasov differential equations for the Ramsey policy

to states dynamics equations are denoted ψ's and the ones corresponding to market clearing are named with μ's. Note that the social planner has full commitment, in the sense that decisions taken in the initial period t_0 are binding until the end of times and there is no time inconsistency.

We provide some intuitions of the most important results and those that connect with the rest of the literature.

First, the optimality for consumption yields the marginal value of wealth ψ^w_{it}. This multiplier informs on the value of consumption in country i and measures directly the extent of inequality across countries. This is directly related to the marginal utility of consumption and the distortion of the saving decisions:

$$[c_{it}] \qquad \omega_i \psi^w_{it} = \underbrace{\omega_i \partial_c u(c_i, \tau_{it})}_{\text{=direct effect}} + \underbrace{\omega_i \psi^c_{it} \partial_{cc} u(c_i, \tau_i)}_{\text{=indirect effect on savings}}$$

This expression for the social shadow value of wealth is analogous to the "marginal value of liquidity" in heterogenous agents analysis like Le Grand, Martin-Baillon and Ragot (2021) and Dávila and Schaab (2023). Unlike the previous analysis in section 5, there is inequality in consumption, and the planner can not equalize marginal utilities:

$$\omega_i \partial_c u(c_{it}, \tau_{it}) \neq \omega_j \partial_c u(c_{jt}, \tau_{jt})$$
$$\omega_i \psi^w_{it} \neq \omega_j \psi^w_{jt}$$

Moreover, appendix C shows under what specific conditions the consumption/saving choice of the planner and the household coincide. In such cases, we obtain that $\psi^c_{it} = 0$ and there is no time-varying difference between household's marginal value of wealth $\lambda^w_{it} = u_c(c, \tau)$ and social planner marginal value of wealth $\psi^w_{it} = \omega_i \mathcal{P}_i u_c(c, \tau)$, except for the population and Pareto weight. In that context, the agent and the planner would make the same consumption/saving decisions along the transition path.

That shadow value for wealth ψ^w_{it} allows us to build a measure of inequality, by comparing the individual value with the average value:

$$\widehat{\psi}^w_{it} = \frac{\omega_i \mathcal{P}_i \psi^w_{it}}{\overline{\psi}^w_t} \lesseqgtr 1 \qquad \text{with} \qquad \overline{\psi}^w_t = \frac{1}{\mathcal{P}} \sum_{\mathbb{I}} \omega_i \mathcal{P}_i \psi^w_{it}$$

If the ratio is higher than 1, we can argue that the country is relatively poorer, with a lower welfare than the average household.

Given this inequality factor, we can derive the Social Cost of Carbon, as an aggregation of the Local Cost of Carbon in the different locations $i \in \mathbb{I}$. This relies on the costate for the Carbon Stock in the atmosphere ψ^S_t, which matters for the Ramsey energy policy.

6.1 Second Best – Social cost(s) of carbon

In this model, the social cost of carbon is written simply:

$$SCC_t := -\frac{\frac{\partial \mathcal{W}_t}{\partial \mathcal{S}_t}}{\frac{\partial \mathcal{W}_t}{\partial c_t}} = -\frac{\psi_t^S}{\overline{\psi}_t^w}$$

The costate for the stock of carbon $\mathcal{S}_t$ measures the social shadow value of an additional ton of GHG emitted in the atmosphere. To convert this welfare measure into monetary units, one should renormalize it using the marginal value of wealth or capital $\partial\mathcal{W}_t/\partial c_t \equiv \partial\mathcal{W}_t/\partial w_t$. As the cost of climate is a global measure, the standard naive intuition from the "representative agent" framework is to use the average marginal value $\overline{\psi}_t^w$. This allows us to consider an average SCC, but we will see that redistribution terms need to be accounted for in the optimal taxation results.

To measure the welfare cost of climate damage, one can follow the dynamics of ψ_t^S along the trajectories of climate and aggregate temperatures. Applying the Pontryagin Max Principle in this Ramsey problem – or using integration by part as in the proof of the PMP – we can follow this shadow value for carbon $\mathcal{S}$ that depends on the costate for local temperatures.

$$\begin{aligned}\dot{\psi}_{it}^{\tau} = \psi_{it}^{\tau}(\rho+\zeta) &+ \underbrace{\gamma_i^y(\tau_{it}-\tau_i^\star)\mathcal{D}^y(\tau_{it})}_{-\partial_\tau \mathcal{D}^y} f(k_{it},e_{it})\psi_{it}^w + \underbrace{\gamma_i^u(\tau_{it}-\tau_i^\star)\mathcal{D}^u(\tau_{it})}_{-\partial_\tau \mathcal{D}^u} u'(\mathcal{D}^u c_{it})c_{it} \\ &+ \gamma_i^y(\tau_{it}-\tau_i^\star)\mathcal{D}^y(\tau_{it})z_i\big[k_{it}\partial_k f(k_{it},e_{it}) + e_{it}\partial_e f(k_{it},e_{it})\big] \\ &+ \gamma_i^u(\tau_{it}-\tau_i^\star)\mathcal{D}^u(\tau_{it})u'(\mathcal{D}^u c)(1-\gamma)\psi_{it}^c\end{aligned}$$

The marginal value for country i of being subject to an increase in local temperature is measured by ψ_t^τ. It increases with different terms: the temperature gap $\tau_{it}-\tau_i^\star$, due to the convexity of the damage function, the damage sensitivity to temperature for TFP γ_i^y and utility/mortality γ_i^u. Moreover, in contrast to the costate in the competitive equilibrium and the first-best allocations, the Ramsey planner needs to take into account the optimal decisions of agents, and how temperature changes distort the first-order conditions of the optimizing agents. These terms depend on how the decisions on consumption $u_c(c,\tau)$ and capital and energy choices – related to MPe_i and MPk_i – are changed with temperature τ_{it}. If the Ramsey planner would make the same decisions as the agents – for example if they have the same preference and the climate externality doesn't distort those choices – then we would recover the same Social Cost of Carbon exposed in the First-Best allocation. In Ramsey plans, this is in general not true, and therefore the SCC and the Pigouvian tax would account for these distortions.

Furthermore, as before, the marginal cost for country i of releasing carbon in atmosphere ψ_t^S is directly and globally affected by the marginal value of temperatures:

$$\dot{\psi}_t^S = \psi_t^S(\widetilde{\rho}+\delta^s) - \zeta\,\chi\sum_{\mathbb{I}}\Delta_i\,\omega_i p_i \psi_{it}^\tau$$

through the climate parameters: ζ the climate inverse persistence (e.g. lags), χ the climate sensitivity and Δ_i the catching up effect" of temperature in cold locations.

Moreover, the marginal damage affects all the countries locally and symmetrically through a value ψ^{τ}_{it}. These gain/costs are cumulated additively, as we see from the previous ODE for ψ^S_t, and we can perform this (exact) decomposition:

$$\psi^S_t = \sum_{\mathbb{I}} \omega_i \mathcal{P}_i \psi^S_{it} \qquad\qquad \dot{\psi}^S_{it} = \psi^S_{it}(\widetilde{\rho} + \delta^s) - \zeta\, \chi\, \Delta_i\, \psi^{\tau}_{it}$$

where the local costate ψ^S_{it} follow an analogous ODE for each location $i \in \mathbb{I}$.

More particularly, the Social Cost of Carbon can hence be expressed as a weighted sum of this local measure that we denote Local Social Cost of carbon LCC_{it}. This cost is local as it takes into account the individual damages in location $i \in \mathbb{I}$, and it is normalized in monetary unit ψ^w_{it}, which is the marginal value of wealth/income in location i. However, it is also social because the Ramsey planner is choosing the optimal energy, emissions, and temperature paths internalizing the global damages across countries.

$$LCC_{it} = -\frac{\psi^S_{it}}{\psi^w_{it}}$$

$$SCC_t = -\frac{\psi^S_t}{\overline{\psi}^w_t} = -\sum_{\mathbb{I}} \overbrace{\frac{\omega_i \mathcal{P}_i \psi^w_{it}}{\overline{\psi}^w_t}}^{=\text{inequality measure}} \overbrace{\frac{\psi^S_{it}}{\psi^w_{it}}}^{=-LCC_{it}}$$

$$SCC_t = -\sum_{\mathbb{I}} \widehat{\psi}^w_{it} \;\; LCC_{it}$$

As a result, we can express the Social Cost of Carbon as:

$$\begin{aligned} SCC_t &= -\sum_{\mathbb{I}} \widehat{\psi}^w_{it} \;\; LCC_{it} \\ &= \mathcal{P}\mathbb{E}^{\mathbb{I}}\big[LCC_{it}\big] + \mathcal{P}\mathbb{C}\text{ov}^{\mathbb{I}}\Big(\widehat{\psi}^w_{it}, LCC_{it}\Big) \qquad \lesseqgtr \mathbb{E}^{\mathbb{I}}\big[LCC_{it}\big] =: \overline{SCC}_t \end{aligned}$$

where the last inequality depends on whether the marginal damage – i.e. high local temperature τ_{it} – tends to be correlated with development levels y_i, i.e. lower production, consumption and hence a higher marginal utility of consumption $\widehat{\psi}^w_{it}$. Note, as in the First Section, the total SCC is a sum over location – and not an average as suggested by the mean $\mathbb{E}^{\mathbb{I}}[\cdot]$ – and one needs to rescale it by world population $\mathcal{P}$.

To conclude, the presence of heterogeneity and the correlation between local damage and poverty increases the Social Cost of Carbon from the Social Planner's perspective. In the following section, we summarize the different concepts of the social cost of carbon and we solve closed-form for the SCC in the long-run in appendix D.

6.2 Redistributive and distortive effect of energy taxes

As in the Toy model presented in the first section, energy taxes have strong redistributive through energy markets.

First, let us turn toward the social value of fossil energy supply, called SVF and it represents the global value of changing marginally the global fossil market, by manipulating supply, profits, and reserves:

$$SVF_t = \underbrace{-\frac{1}{\mathcal{P}} \sum_{\mathbb{I}} \widehat{\psi}_{it}\, \nu_{ee}(e^x_{it}, \mathcal{R}_{it}) e^x_{it}}_{=\text{supply distortion}} \quad + \quad \underbrace{\frac{1}{\mathcal{P}} \sum_{\mathbb{I}} \widehat{\psi}_{it} \big[\psi^R_{it} - \lambda^R_{it}\big]}_{=\text{difference planner/agents reserve valuation}}$$

This Social Cost of Fossil (SVF) accounts for the two different redistributive effects of lowering the energy demand / relaxing the energy market clearing: (i) it implies moving down the supply curve, which distorts the supply of each country $\nu_{ee}(e^x_{it}, \mathcal{R}_{it}) e^x_{it}$. In particular, it hurts the large producers, or countries with inelastic supply, i.e. high inverse elasticity ν_i. This sum is weighted with our measure of inequality related to marginal value of wealth $\widehat{\psi}^w_{it}$. It is larger if poorer countries experience large swings in supply.Moreover, with isoelastic functional form, it yields:

$$\nu_{ee}(e^x_{it}, \mathcal{R}_{it}) e^x_{it} = \nu \nu_e(e^x_{it}, \mathcal{R}_{it}) = (q^f_t - \lambda^R_{it})\nu$$

(ii) decreasing demand and extraction also imply "leaving fossil fuels in the group", which is proportional to the valuation of reserves. However, since fossil fuels are already valued by the energy firms, this term only accounts for the additional value of the social planner that accounts for the extraction e^x_{it} and exploration ι^x_{it} distortions. Moreover, this difference is related to the curvature of the exploration cost $\mu_{\iota\iota}(\iota^x_{it}, \mathcal{R}_{it}) e^x_{it} / \delta^R$, such that

$$\psi^R_{it} - \lambda^R_{it} = -\frac{\mu_{\iota^x \iota^x}(\iota^x_{it}, \mathcal{R}_{it}) e^x_{it}}{\delta^R \mu_{\iota^x}(\iota^x_{it}, \mathcal{R}_{it})} = -\mu \frac{e^x_{it}}{\delta^R \iota^x_{it}}$$

where the last equality relies on the isoelastic exploration cost functional form. The planner would value less the location that extracts more than they find new reserves $e^x_{it} > \delta^R \iota^x_{it}$, or that have an elastic cost of exploration, with high inverse elasticity μ. Again the planner would weight the different agents according to their inequality weights $\widehat{\psi}^w_{it}$.

The Social Value of Fossil is thus :

$$SVF_t = -\mathbb{E}^{\mathbb{I}}\big[(q^f_t - \lambda^R_{jt})\nu\big] - \mathrm{Cov}^{\mathbb{I}}\big(\widehat{\psi}^w_{jt}, (q^f_t - \lambda^R_{jt})\nu\big) - \mathbb{E}^{\mathbb{I}}\big[\mu \frac{e^x_{jt}}{\delta^R \iota^x_{jt}}\big] - \mathrm{Cov}^{\mathbb{I}}\big(\widehat{\psi}^w_{jt}, \mu \frac{e^x_{jt}}{\delta^R \iota^x_{jt}}\big) < 0$$

Lastly, we see how a change in energy price affect demand for fossil fuels. Denoting it the Social Cost of Energy, like in the toy model section, it rewrites the same way, for a more general production function.

$$SCR_t = \frac{1}{\mathcal{P}}\sum_{\mathbb{I}} \omega_i \mathcal{P}_i \widehat{\psi}^w_{it} MMPe^f_{it} e^f_{it} = \frac{1}{\mathcal{P}}\sum_{\mathbb{I}} \omega_i \mathcal{P}_i \widehat{\psi}^w_{it} \mathcal{D}^y(\tau_{it}) z_i e^f_{it} f_{e^f}(k_{it}, e^f_{it}, e^r_{it})$$

With CES functional form we have

$$\begin{aligned} MPe^f_{it} &= MPe_i\big(\frac{e^f_t}{\omega e_t}\big)^{-\frac{1}{\sigma_e}} \\ \Rightarrow \qquad MMPe^f_{it} e^f_{it} &= -q^f_t \frac{1-s^f_{it}}{\sigma^e} \end{aligned}$$

where $s^f_{it} = \frac{q^f_t e^f_{it}}{q^e_{it} e_{it}}$ is the fossil energy share in the energy mix.

6.3 Optimal energy policy and decentralization

In this section, we uncover our main result that derives the optimal policy for energy. We derive the optimality conditions for the Ramsey planner, in particular for fossil energy e^f_{it} and the other equilibrium relations are detailed in appendix C. We will see that it integrates the different redistribution motives that we detailed above. However, the Ramsey planner, by internalizing these externalities, would like to distort agents' optimality conditions, which include the curvature of demand and cost functions.

Despite these numerous notations, a sufficient optimal policy for satisfying this condition is the following:

$$\mathrm{t}^f_t = SCC_t + SVF_t + SCF_t$$

where

$$\begin{aligned} SCC_t &= \mathbb{E}^{\mathbb{I}}\big[LCC_{it}\big] + \mathbb{C}\mathrm{ov}^{\mathbb{I}}\Big(\widehat{\psi}^w_{it}, LCC_{it}\Big) \\ SVF_t &= \underbrace{-\frac{1}{\mathcal{P}}\sum_{\mathbb{I}} \widehat{\psi}_{it}\, \nu_{ee}(e^x_{it}, \mathcal{R}_{it}) e^x_{it}}_{=\text{supply distortion}} \quad + \quad \underbrace{\frac{1}{\mathcal{P}}\sum_{\mathbb{I}} \widehat{\psi}_{it}\big[\psi^R_{it} - \lambda^R_{it}\big]}_{=\text{difference planner/agents reserve valuation}} \\ &= -\frac{1}{\mathcal{P}}\sum_{\mathbb{I}} \widehat{\psi}_{it}(q^f_t - \lambda^R_{it})\nu - \frac{1}{\mathcal{P}}\sum_{\mathbb{I}} \widehat{\psi}_{it} \frac{e^x_{it}}{\delta^R \iota^x_{it}} \mu \\ SCR_t &= \sum_{\mathbb{I}} \omega_i \mathcal{P}_i \widehat{\psi}^w_{it} MMPe^f_{it} e^f_{it} \\ &= -q^f_t \frac{1}{\mathcal{P}} \sum_{\mathbb{I}} \omega_i \mathcal{P}_i \widehat{\psi}^w_{it} \frac{1-s^f_{it}}{\sigma^e} \end{aligned}$$

Where the SCC the social value of carbon, and LCC_{it} the local social cost of carbon studied in section 6.1 and SVF the social value of fossil fuel supply are detailed in the section above. As in the Toy model of section 1, these terms account for both externality and redistribution effects.

In that formula, similarly to the Toy model of section 2, we see that the *level* of the carbon tax is common for all countries and accounts for (i) the climate externality, (ii) the fossil cost redistribution for producers and (iii) the cost distortion for importer and firms relying on fossil fuels. As a result, even without climate externality $SCC_t = 0$, the fossil fuel tax t_t^f would not be zero. It accounts for the manipulation of terms of trade because of the wealthy exporters and the relatively poorer importers. Such a result holds as long as the fossil production is traded internationally in a market where agents have different marginal utilities of consumption, and if there are redistributive effects of rents and moving along the curvature of the supply curve. These motives would be absent in models like Golosov et al. (2014) where the production function is constant return to scale.

Case with country-specific carbon tax

As in the Toy Model section, consider an experiment with country-specific taxes would allow to correct some of these redistributive concerns. In that case, not only the *level* but also the *distribution* of the fossil fuel/carbon tax is affected by redistribution motives. The optimal tax would be:

$$\mathrm{t}_t^f = \frac{1}{\widehat{\psi}_{it}^w}\Big[SCC_t + SVF_t\Big] + SCF_{it}$$

with $SCF_{it} \propto MMPe_{it}^f e_{it}^f$ the local demand distortion.

Indeed the ratio $1/\widehat{\psi}_{it}^w$ is the inverse of our inequality measure developed at the beginning of this section. It implies that richer/colder countries, which have higher consumption and lower marginal utilities will be charged a higher carbon tax, and conversely for poorer countries that will be charged a lower tax:

$$\text{low}\quad c_{it}\quad \text{high}\quad \tau_{it} \qquad \Rightarrow \qquad \text{high}\quad \widehat{\psi}_{it}^w \;\propto\; \partial_c u(c_{it}, \tau_{it}) \qquad \Rightarrow \qquad \text{low}\quad \mathrm{t}_{it}^f$$

everything else being constant, in particular SCC_t and SVF.

7 Unilateral climate policy

We consider a distribution of unilateral social planners, one for each country $i \in \mathbb{I}$, and would take decisions for consumption, capital, and energy. The problem is very similar to the competitive equilibrium of section 4. The main difference is the internalization of the climate externality in each country: the planner would take into of its own country's energy impact on the world climate. The main assumption for this reasoning to hold is that the mass of each country j is not zero: $\int_{\mathbb{I}} \mathbb{1}\{i = j\} di > 0$, i.e. if the regions are not infinitesimal. As a result, in each country, the planner would solve the following problem, with individual welfare $\mathcal{W}_{it}$

$$\mathcal{W}_{it_0} = \max_{c_i, b_i, k_i, e_i^f, e_i^r} \int_{t_0}^{\infty} e^{-(\widetilde{\rho}+n)t}\; u\big(\mathcal{D}^u(\tau_{it})\, c_{it}\big) p_i \, dt$$

Note that this problem corresponds to the planner of section section 6, with Pareto weight distribution to be degenerate at $\omega_i = 1$ and $\omega_j = 0\ \forall j \neq i$.

In the rest of this section, which is forthcoming, we consider an optimal allocation full commitment, where no planner has any incentive to deviate from the plan set up in period t_0. Assuming away full-commitment implies strategic interactions between countries, and since the mass of agents is not zero, this $N-$player differential game can prove hard to solve both mathematically and computationally.

8 Long-run analysis

In this section, we provide analytical results of the Competitive equilibrium, First-Best and Ramsey allocations on the cost of carbon, the path of emissions, and temperature in the asymptotic stationary equilibrium.

8.1 The Social Cost of Carbon

Given the path for the costate that informs on the social value of carbon emission, we canÂ find a balance-growth path that keeps the SCC stationary. We consider the long-run equilibrium where the terminal time horizon $T \to \infty$. In this context, only a stable temperature makes the system stationary, such that the emissions entering the atmosphere $\mathcal{E}_t$ are exactly offset by the one rejected outside the climate system δ_i

$$\mathcal{E}_t = \delta_s \mathcal{S}_t \qquad \text{and} \qquad \tau_t \to \tau_\infty$$

Depending on the trajectory of emissions between t_0 and t_T – when $\mathcal{E}_t \approx \delta_s \mathcal{S}_t$ – there are different cumulative emission/atmospheric carbon level $\mathcal{S}_t$ possible and hence different distribution of temperature $\mathcal{T}_T$ and $\{\tau_i\}_i$.

In particular, it is not difficult to guess the ordering between Competitive equilibrium (CE), Unilateral policy (UP), Ramsey allocation (RA), and First Best Allocation (FB) :

$$\mathcal{T}_T^{FB} < \mathcal{T}_T^{RA} < \mathcal{T}_T^{UP} < \mathcal{T}_T^{CE}$$

Solving the stationary differential equations at the limit $t \to T \to \infty$, we find an analytical characterization for the Social Cost of Carbon.

Proposition:
In the stationary competitive equilibrium, the Ramsey or the First Best allocations, the Social Cost of Carbon can be expressed as:

$$SCC_t \equiv \frac{1}{\overline{\psi}_t^w} \, \frac{\chi}{\widetilde{\rho} + \delta^s} \int_{\mathbb{I}} \Delta_i(\tau_{i,\infty} - \tau_i^\star) \Big(\gamma_i^y \mathcal{D}^y(\tau_{i,\infty}) y_{i,\infty} \psi_{i,\infty}^k + \gamma_i^u \mathcal{D}^u(\tau_{i,\infty}) \, \omega_i \, u'(\mathcal{D}^u c_{i,\infty}) \, c_i \Big) di$$

This formula is analogous to the Social Cost of Carbon expressed in Golosov et al. (2014). Considering a linear instead of quadratic damage function – and only applied to TFP, without direct effects on mortality, would yield an exactly identical expression. We rely on a different set of assumptions – stationarity and continuous time – while the analysis in Golosov et al. (2014) relies on a representative agent, full depreciation every discrete period, and log-utility assumptions such that income and substitution forces in consumption/saving offset each other to yield such formula.

In particular, the noticeable feature is the proportionality of the SCC with $y_{i,\infty}$ and $c_{i\infty}$ and the temperature gap $(\tau_{i,\infty} - \tau_i^\star)$. If countries are richer, and more developed, the marginal damage has a larger economic impact. Moreover, due to the convexity of the damage function, the cost of carbon increases with temperature: hotter countries have more to lose from an additional increase in temperature. The extent of this proportionality depends on the exact calibration of the damage parameters $\gamma_i^y = \gamma_i^\oplus$ or $\gamma_i^\ominus$ for productivity impact and $\gamma_i^u = \gamma_{u,i}^\oplus$ or $\gamma_{u,i}^\ominus$ for mortality effects. More work is needed to make these damage parameters empirically grounded, as studied in Carleton et al. (2022)

Moreover, the SCC is proportional to the extent that the country is warming faster than the world's atmosphere due to geographical factors Δ_i.

Finally, these different effects are scaled with the effective discount factor – the rate of the social planner and including the depreciating of carbon due to the exit of the greenhouse gas from the atmosphere. This highlight in a very clear fashion how the discount factor affects the Social Cost of Carbon, as raised in the debate Stern and Stern (2007) and Nordhaus (2007).

Moreover, the ratio $1/\overline{\psi}_t^w$ and ψ_{it}^w in the expression of the Social Cost of Carbon highlight the importance of inequality for the computation of carbon price.

To study this, one could also consider the *"Local cost of carbon"* as the marginal damage for the region $i \in \mathbb{I}$:

$$LCC_{it} = \frac{\chi}{\widetilde{\rho} + \delta^s} \Delta_i(\tau_{i,\infty} - \tau_i^\star)\Big(\gamma_i^y y_{i,\infty} + \gamma_i^u c_{i,\infty}\Big)$$

with output $y_{i,\infty} = \mathcal{D}_i^y(\tau_{it}) z_i f(k_{it}, e_{it})$. Again, considering a single country, this formula boils down to the SCC for a representative country. Taking heterogeneous countries and following the same logic as above, we observe that:

$$\begin{aligned} SCC_t &= \int_{\mathbb{I}} \widehat{\psi}_{it}^w \, LCC_{it} \, di \\ &= \mathbb{E}^{\mathbb{I}}\big[LCC_{it}\big] + \mathbb{C}\text{ov}^{\mathbb{I}}\Big(\widehat{\psi}_{it}^w, LCC_{it}\Big) > \mathbb{E}^{\mathbb{I}}\big[LCC_{it}\big] =: \overline{SCC}_t \end{aligned}$$

This covariance between $\widehat{\psi}_{it}^w = \psi_{it}^w / \overline{\psi}_t^w$ and the LCC_i that is proportional to y_i and $\tau_{i,\infty} - \tau_i^\star$ is clearly positive as we will explore in our quantitative experiments. This is obviously identical to the theoretical result we showed above in the non-stationary path. In this long-run context, the covariance is easier to compute as it relies on less assumptions on preferences and technology as it can be directly measured from the data on τ_{it}, y_{it} and c_{it}.

8.2 Green Growth and decoupling from energy

Empirically, energy use has correlated strongly with GDP levels and industrial production in the last century, as seen in figures in **??**. However, lowering GHG emissions tend to go hand in hand with reducing energy consumption. This asks the question of the possibility of decoupling between economic growth and energy supply, and fossils in particular.

To examine this in our framework, let us study the optimality conditions for energy and express the energy share in the final output.

$$\begin{cases} MPe_i = {z_i}^{1-\frac{1}{\sigma}} {y_{it}}^{\frac{1}{\sigma}} \, \varepsilon^{\frac{1}{\sigma}} \, (z^e_{it})^{1-\frac{1}{\sigma}} {e_{it}}^{-\frac{1}{\sigma}} & = q^e_t \\ MPe_i \big(\frac{e^f_t}{\omega e_t}\big)^{-\frac{1}{\sigma_e}} & = q^{e,f}_t \\ MPe_i \big(\frac{e^r_t}{(1-\omega) e_t}\big)^{-\frac{1}{\sigma_e}} & = q^{e,r}_t \end{cases}$$

As a result, the total energy share writes:

$$s_{e,t} := \frac{e_{it} q^e_t}{y_{it}} = (q^e_t)^{1-\sigma} z_i^{\sigma-1} (z^e_t)^{\sigma-1} \varepsilon$$

Since all the variable are already expressed in efficient unit per capita, accounting for the trend in population n and TFP growth $\bar{g}$, we have z_i constant and all the variables growth in absolute value. However, all the other variables can feature additional long-run trends, such as energy price $\dot{q}^e_t / q^e_t = g_q$ or directed technical change $\dot{z}^e_t / z^e_t = g_e$.

We consider two case: (i) the cost share of energy stays stable in output and (ii) this share falls to zeros.

$$\begin{aligned} &(i) \qquad s_{e,t} \rightarrow_{t\rightarrow\infty} \bar{s}_e \qquad && \Leftrightarrow \qquad && g_q(1-\sigma) + g_e(\sigma - 1) = 0 \\ &(ii) \qquad s_{e,t} \rightarrow_{t\rightarrow\infty} 0 \qquad && \Leftrightarrow \qquad && g_q - g_e < 0 \end{aligned}$$

In our quantitative exercise, following empirical evidence that energy share $s_{e,t}$ tends to comove strongly with energy price q^e_t, we assume that $\sigma < 1$ and energy is a complementary factor in production. As result, $g_e = g_q$ for (i) and $g_e > g_q$ for (ii). For the energy share to stay stable or decline, directed technical change should at least compensate for the increase in price.

To determine the path of price in our context, recall the supply side of the energy market, we have:

$$\frac{\dot{q}^e_t}{q^e_t} = s_{ef,t} \frac{\dot{q}^{e,f}_t}{q^{e,f}_t} + s_{er,t} \frac{\dot{q}^{e,r}_t}{q^{e,r}_t}$$

where $s_{ef,t} = \frac{e^f_t q^{e,f}_t}{e_t q_t}$ is the expenditure share in fossil and $s_{er,t} = 1 - s_{ef,t}$ the share in renewable. Recall that in our context,

$$q^f_t = \Big(\frac{E^f_t}{\mathcal{R}_t}\Big)^{\nu} + \lambda^R_t \qquad \Rightarrow \qquad \frac{\dot{q}^f_t}{q^f_t} = s_{\mathcal{C}} \nu \Big(\frac{\dot{E}^f_t}{E^f_t} - \frac{\dot{\mathcal{R}}_t}{\mathcal{R}_t}\Big) + (1 - s_{\mathcal{C}}) \frac{\dot{\lambda}^R_t}{\lambda^R_t}$$

where $s_{\mathcal{C}} = \frac{\mathcal{C}_E(\cdot)}{q_t^f}$ is the share of marginal in the fossil price, and $\frac{\dot{\lambda}_t^R}{\lambda_t^R}$ is the growth of the Hotelling rent, which is ρ at the first order. Obviously if extraction rate is faster than exploration of new reserves, the price will grow to infinity. Moreover, the rent of the monopolist will at least grow at the speed ρ in the first order,

Similarly, to get decoupling from fossils in the energy mix, we must have $g_r = \frac{\dot{q}_t^{e,r}}{q_t^{e,r}} < \frac{\dot{q}_t^{e,f}}{q_t^{e,f}} = g_f$. In this case, $g_q \to g_r$.

To conclude, to obtain a balance green growth equilibrium in our context, we need: (i) fossil prices to grow sufficiently fast due to extraction or rise in Hotelling rents, (ii) the price of renewables to grow less fast than fossils and (iii) that the directed technical change grows at a rate at least faster than the growth in the relative price of the resulting energy.

8.3 Path of emissions and temperature

We saw that the cost of carbon depends mostly on the resulting final temperatures once the economy and climate reach a stationary path where temperatures stay constant. This level matters and varies enormously as it depends linearly on the path of emissions:

$$\tau_{it} - \tau_{it_0} = \Delta_i \, \chi \int_{t_0}^{T} e^{-\delta_s(T-t)} \mathcal{E}_t dt$$

As a result, replacing the aggregate emissions, we obtain:

$$\tau_{it} - \tau_{it_0} = \Delta_i \, \chi \, \xi \, \omega \int_{t_0}^{T} e^{(n+\bar{g})t-\delta_s(T-t)} q_t^{f \ -\sigma_e} \int_{j\in\mathbb{I}} \big(z_j z_{j,t}^e \mathcal{D}(\tau_{j,t})\big)^{\sigma-1} y_{j,t} \, q_{j,t}^{\sigma_e-\sigma} dj \ dt$$

where the path of world emissions $\{\epsilon_j\}_j$ has been expressed by fossil energy demand $e_j^f(q_t^f, z_j, z_{j,t}^e)$. In the long-run, the local temperature will uniquely be affected by the externality of the world economy, along with geographical factors determining warming Δ_i, the climate sensitivity parameter χ and the carbon exit from atmosphere δ_s,

We observe that the path of emissions depends positively on the growth of population n and aggregate productivity $\bar{g}$, the deviation of output from trend y_j & relative TFP z_j, the directed technical change z_t^e. Fossil demand is also shaped by the elasticity of energy in output σ, the Fossil energy price $q^{e,f}$ and its long run growth rate g^{q^f}, as expressed above. Finally, the change in energy mix, renewable share ω and price q_t^r & elasticity of the energy source σ_e are factors that would help reduce these paths of emissions.

To analyze this asymptotic behavior, we perform an approximation of this resulting temperature at terminal time. T.

$$\frac{\dot{\tau}_T}{\tau_T} \propto \ n \ + \ \bar{g}^y - (1-\sigma)\big(g_e - \widetilde{\gamma}\big) + (\sigma_e - \sigma)(1-\omega)g^{q^r} - (\sigma_e(1-\omega) + \sigma\omega)g^{q^f}$$

This decomposition is reminiscent of a Generalized Kaya (or $I = PAT$) identity, where

Emission growth can be decomposed as

$$\varepsilon_{it} = \frac{\epsilon_{it}}{e_{it}} \frac{e_{it}}{y_{it}} \frac{y_{it}}{p_{it}} p_{it}$$

where y_{it} is already the output per capita. Taking the growth rate of this decomposition, we obtain the formula above. This show how important the path of energy prices g^{q^f} and g^{q^r} and technology g_e matter for future path of emissions and climate.

9 Calibration

The calibration of this model is preliminary, and will be updated to match (i) empirical moments on output growth, production, population demographic and energy markets (ii) reasonable estimates of the SCC. In particular, parameters denoted by $\star$ are subject to future changes. As of now, this calibration is aimed at simulate a first version of the model to provide intuitions of economic and climate mechanisms. Many of the parameters are taken or inspired by the rest of the literature.

Table 1: Baseline calibration

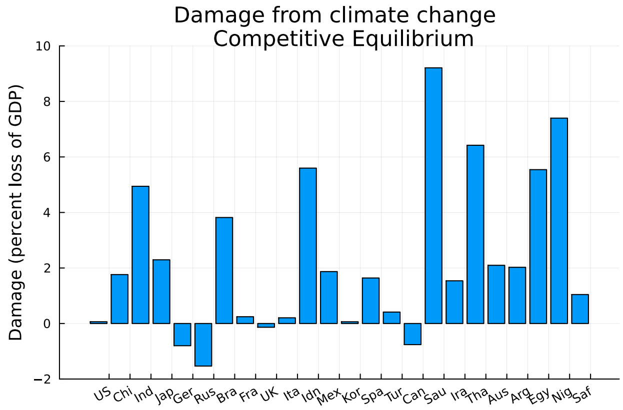
Technology & Energy markets			
α	0.35	Capital share in $f(\cdot)$	Capital/Output ratio
ϵ	0.12	Energy share in $f(\cdot)$	Energy cost share (8.5%)
σ	0.3	Elasticity capital-labor vs. energy	Complementarity in production (c.f. Bourany 2020)
ω	0.8	Fossil energy share in $e(\cdot)$	Fossil/Energy ratio
σ_e	2.0	Elasticity fossil-renewable	Slight substitutability & Study by Stern
δ	0.06	Depreciation rate	Investment/Output ratio
$\bar{g}$	$0.01^\star$	Long run TFP growth	Conservative estimate for growth
g_e	$0.01^\star$	Long run energy directed technical change	Conservative / Acemoglu et al (2012)
g_r	$0.01^\star$	Long run renewable price increase	Conservative / Match price fall in renewable
ν	$2^\star$	Extraction elasticity of fossil energy	Conservative extraction / Krusell et al (2022)
μ	$2^\star$	Exploration elasticity of fossil energy	Cubic exploration cost / Krusell et al (2022)
δ^R	$0.45^\star$	Probability of new reserves discovery	Conservative / Krusell et al (2022)
Preferences & Time horizon			
ρ	0.03	HH Discount factor	Long term interest rate & usual calib. in IAMs
$\widetilde{\rho}$	$0.03^\star$	SP Discount factor	Planner as patient as Households
η	2.5	Risk aversion	Positive utility in steady state
n	$0.01^\star$	Long run population growth	Conservative estimate for growth
ω_i	1	Pareto weights	Uniforms / Utilitarian Social Planner
T	90	Time horizon	Horizon 2100 years since 2011
Climate parameters			
ξ	0.81	Emission factor	Conversion 1 $MTOE \Rightarrow 1\ MT\ Carbon$
ζ	0.3	Inverse climate persistence / inertia	Sluggishness of temperature $\sim 10-15$ years
χ	$2.1/1e6$	Climate sensitivity	Pulse experiment: $100\, GtC \equiv 0.21°C$ medium-term warming
δ_s	0.0014	Carbon exit from atmosphere	Pulse experiment: $100\, GtC \equiv 0.16°C$ long-term warming
$\gamma^\oplus$	$0.00234^\star$	Damage sensitivity	Conservative estimate: Nordhaus' DICE
$\gamma^\ominus$	$0.2 \times \gamma^{\oplus\ \star}$	Damage sensitivity	Conservative estimate: Nordhaus' DICE
α^τ	$0.2^\star$	Weight historical climate for optimal temp.	Marginal damage decorrelated with initial temp.
$\tau^\star$	15.5	Optimal yearly temperature	Average spring temperature / Developed economies
Parameters calibrated to match data			
p_i		Population	Data – World Bank 2011
z_i		TFP	To match GDP Data – World Bank 2011
τ_i		Local Temperature	To match temperature of largest city
$\mathcal{R}_i$		Local Fossil reserves	To match data from BP Energy review

10 Quantitative Experiment

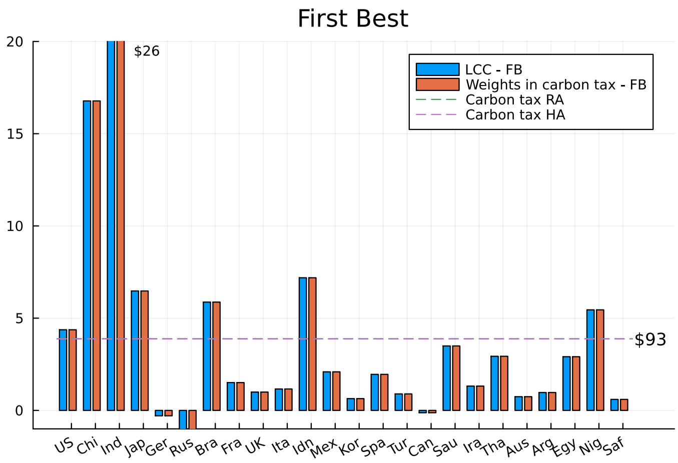
We collect data on 24 countries, selected as the union of the 15 largest in terms of population and the 15 largest in terms of total GDP. As a result, it includes both small but rich countries as well as large but lower-income economies.

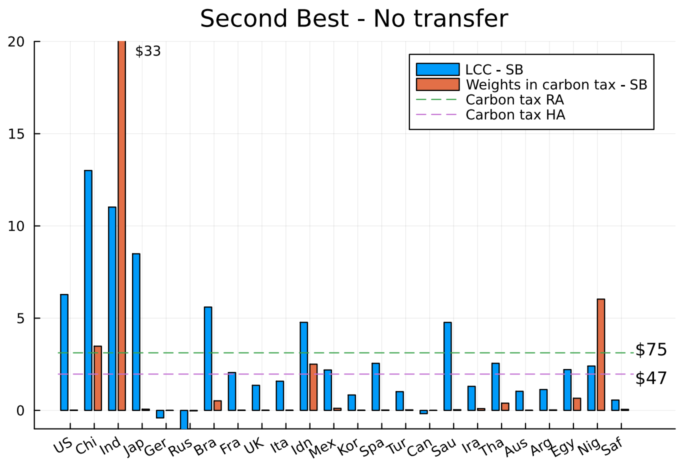
We use the local temperatures of the largest city as well as GDP, energy use, CO_2 emissions, population from international data from the World Bank. In particular, I calibrate productivity residual z to match the distribution of output per capita at the steady state, assumed to be the mean over the years 2000-2011.

More work is needed to match the data and to make the model empirically grounded.



In the following two graphs, I plot the difference, in the long run stationary competitive equilibrium, between the distribution of local cost of carbon $LCC_i = \lambda^{\mathcal{S}}_{it}/\lambda^{w}_{it}$ and the local cost of carbon reweighted by our measure of inequality $\widehat{\lambda}^{w}_{it} = \lambda^{w}_{it}/\overline{\lambda}^{w}_{it}$, and hence $LWCC_{it} = \widehat{\lambda}^{w}_{it}\, LCC_{it} = \lambda^{\mathcal{S}}_{it}/\bar{\lambda}^{k}_{t}$



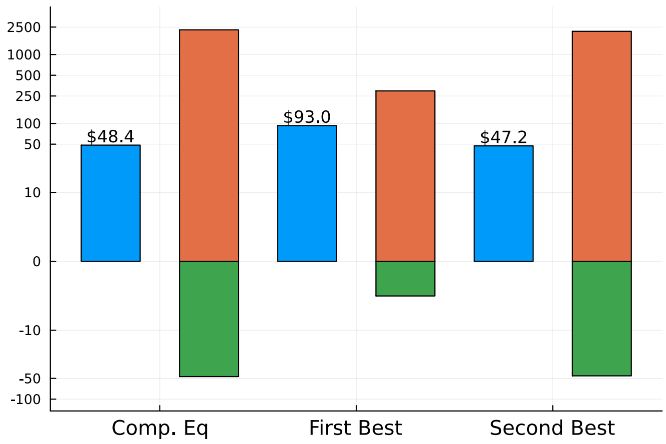


In the following graph, I plot a decomposition, in the long run stationary competitive equilibrium, of the Social Cost of Carbon between what is driven by the marginal utility of consumption $\bar{\lambda}_t$ and the welfare impact of climate change λ^{S}_{t}.

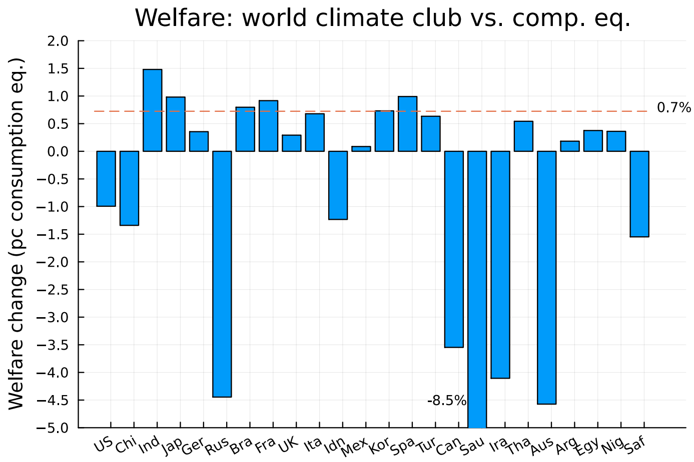
$$SCC_t := -\frac{\partial \mathcal{W}_t/\partial \mathcal{S}_t}{\partial \mathcal{W}_t/\partial c_t} = -\frac{\lambda^S_t}{\lambda^w_t} = -\frac{\sum_{\mathbb{I}} \lambda^S_{it} di}{\frac{1}{I}\sum_{\mathbb{I}} \lambda_{it}}$$

with the decomposition:

$$\log(SCC_t) = \log(-\lambda^S_t) - \log(\lambda^w_t)$$



In the next graph, I display the difference between the second-best climate policy $\mathcal{W}_i$ and the competitive equilibrium $\mathcal{V}_i$ in consumption equivalent welfare units.



The rest of this section is forthcoming

11 Conclusion

In this paper, I show how to design optimal climate policy in presence of inequalities and constraints on redistribution instruments. Indeed, if the optimal policy can not transfer across countries as in the First-Best allocation, then second-best policy would account for the redistributive effects of taxation and lower the burden for developing countries while increasing taxes for richer countries. Additional constraints, such as countries' political incentives to participate in a climate agreement are analyzed in subsequent work to provide policy recommendations for climate policy.

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A Energy producers – fossil fuel company

We consider the simplest functional forms, yielding isoelastic supply curves for fossil energy extraction and exploration:

$$\nu(E,\mathcal{R}) = \frac{\bar{\nu}}{1+\nu}\Big(\frac{E}{\mathcal{R}}\Big)^{1+\nu}\mathcal{R} \qquad\qquad \mu(\mathcal{I}^e,\mathcal{R}) = \frac{\bar{\mu}}{1+\mu}\Big(\frac{\mathcal{I}^e}{\mathcal{R}}\Big)^{1+\mu}\mathcal{R}$$

Setting up the Hamiltonian,

$$\mathcal{H}(\mathcal{R}_t,\lambda_t^R,E_t,\mathcal{I}_t^f) = \pi_t(\mathcal{R}_t,E_t,\mathcal{I}_t^f) + \lambda_t^R(\delta^R\mathcal{I}_t^f - E_t)$$

The optimal decisions are given by:

$$[E_t] \qquad q_t^{e,f} = \nu_E(E,R) + \lambda_t^R = \bar{\nu}\Big(\frac{E_t}{\mathcal{R}_t}\Big)^{\nu} + \lambda_t^R$$

$$[\mathcal{I}_t] \qquad \lambda_t^R\delta^R = \mu_I(\mathcal{I}_t,\mathcal{R}_t) = \bar{\mu}\Big(\frac{\mathcal{I}_t}{\mathcal{R}_t}\Big)^{\mu} \qquad\qquad \mathcal{I}_t = \mathcal{R}_t\big(\frac{\lambda_t^R\delta}{\bar{\mu}}\big)^{1/\mu}$$

The Pontryagin Maximum Principle yields the dynamics of the costate :

$$\begin{aligned}
-\dot{\lambda}_t^R + \rho\lambda_t^R &= \partial_R\mathcal{H}(R,E^\star,I^\star) \\
\dot{\lambda}_t^R &= \rho\lambda_t^R + \partial_R\nu(E_t^\star,\mathcal{R}_t) + \partial_R\mu(\mathcal{I}_t^\star,\mathcal{R}_t) \\
\dot{\lambda}_t^R &= \rho\lambda_t^R - \frac{\bar{\nu}\nu}{1+\nu}\Big(\frac{E_t^\star}{\mathcal{R}_t}\Big)^{1+\nu} - \frac{\bar{\mu}\mu}{1+\mu}\Big(\frac{I_t^\star}{\mathcal{R}_t}\Big)^{1+\mu} \\
\dot{\lambda}_t^R &= \rho\lambda_t^R - \frac{\bar{\nu}\nu}{1+\nu}\Big(\frac{E_t^\star}{\mathcal{R}_t}\Big)^{1+\nu} - \frac{\bar{\mu}\mu}{1+\mu}\Big(\frac{I_t^\star}{\mathcal{R}_t}\Big)^{1+\mu}
\end{aligned}$$

Replacing it with the optimal decisions, we obtain a non-linear equation for the Hotelling rent:

$$\dot{\lambda}_t^R = \rho\lambda_t^R - \frac{\bar{\nu}^{-1/\nu}\nu}{1+\nu}\big(q^{e,f} - \lambda_t^R\big)^{1+1/\nu} - \frac{\bar{\mu}^{-1/\mu}\mu}{1+\mu}\big(\delta^R\lambda_t^R\big)^{1+1/\mu}$$

Moreover, we should add the transversality conditions

$$\lim_{t\to\infty} e^{-\rho t}\lambda_t^R\mathcal{R}_t = 0$$

and since we know that λ_t^R grows less fast than $e^{\rho t}$, we have the transversality respected even if $\mathcal{R}_t \not\to 0$ when $t \to \infty$.

This implies a (highly!) non-linear ODE for the Hotelling rent λ_t^R, where λ_0^R is chosen such that $\mathcal{R}_t = 0$ by terminal time $t = \bar{t}$. We can "simplify" the ODE, in the case where the cost are quadratic $\mu = \nu = 1$ and

$$\dot{\lambda}_t^R = \rho\lambda_t^R + \frac{1}{2\bar{\nu}}\big(q_t^{e,f} - \lambda_t^R\big)^2 + \frac{1}{2\bar{\mu}}\big(\delta^R\lambda_t^R\big)^2$$

We see that the Hotelling rent account for the extraction cost (scaled by $\bar{\nu}$) and the exploration cost (scaling in $\bar{\mu}$) and depend on the price/inverse demand for determining the quantity produced in equilibrium.

A stationary solution can be found in the case where $\dot{\lambda}_t^R = 0$

$$\rho\lambda_t^R + \frac{1}{2\bar{\nu}}\big(q_t^{e,f} - \lambda_t^R\big)^2 + \frac{1}{2\bar{\mu}}\big(\delta^R\lambda_t^R\big)^2 = 0$$
$$\rho\lambda_t^R - \frac{1}{\bar{\nu}}q_t^{e,f}\lambda_t^R + \frac{1}{2\bar{\nu}}(\lambda_t^R)^2 + \frac{1}{2\bar{\nu}}(q_t^{e,f})^2 + \frac{1}{2\bar{\mu}}(\delta^R)^2(\lambda_t^R)^2 = 0$$
$$\lambda_\infty^R = \frac{\frac{q_t^{e,f}}{\bar{\nu}} - \rho \pm \sqrt{(\frac{q_t^{e,f}}{\bar{\nu}} - \rho)^2 - (\frac{1}{\bar{\nu}} + \frac{\delta^2}{\bar{\mu}})\frac{1}{\bar{\nu}}(q_t^{e,f})^2}}{\frac{1}{\bar{\nu}} + \frac{\delta^2}{\bar{\mu}}}$$

We obtain two stationary positive solutions: for a given energy price (demanded) $q^{e,f}$, in one equilibrium, the rent is very high, incentiving a lot of exploration as a share of reserve ($\mathcal{I}/\mathcal{R}$ is high) but the production is relatively low ($q^{e,f} - \lambda^R$ is low and so is the marginal cost and quantity $E/\mathcal{R}$). In a second stationary equilibrium, the rent is lower and the marginal cost is higher since the extraction is larger as a share of reserves. Note, that this stationary equilibrium is not consistent with state $\mathcal{R}_t$ dynamics since the reserves are depleting at different rates: only the first case is consistent with a sustainable level of extraction and exploration.

B Competitive equilibrium

Dynamics of the individual state variables $s_{it} = (k_{it}, \tau_{it}, z_i, p_i, \theta_i, \gamma_i, \Delta_i, \xi_i)$ and aggregate ones $(\mathcal{S}_t, \mathcal{T}_t, \mathcal{R}_t)$:

$$\dot{w}_t = r_t^\star w_{it} + \mathcal{D}(\tau_t)f(k_t, e_t) - (n + \bar{g} + \delta + r_t^\star)k_t + \theta\pi_t^f - c_t - q_t^e e_t - c(\vartheta_t)e_t^f$$
$$\mathcal{E}_t = e^{(n+\bar{g})t}\int_{\mathbb{S}}\xi(1-\vartheta_{it})e_{it}^f p_{it}ds$$
$$\dot{\tau}_{it} = \zeta(\Delta_i\,\chi\mathcal{S}_t - (\tau_{it} - \tau_{it_0})) \qquad \dot{\mathcal{S}}_t = \mathcal{E}_t - \delta_s\mathcal{S}_t$$
$$\dot{\mathcal{R}}_t = -E_t^f + \delta_R\mathcal{I}_t \qquad q_t^{e,f} = \bar{\nu}(E_t^f/\mathcal{R}_t)^\nu$$

Household problem: Pontryagin Maximum Principle

$$\mathcal{H}^{hh}(s, \{c, k, e^f, e^r\}, \{\lambda\}) = u(c_i, \tau_i) + \lambda_{it}^w\Big(r_t^\star w_{it} + \mathcal{D}(\tau_{it})f(k_t, e_t) - (n + \bar{g} + \delta + r_t^\star)k_t + \theta\pi_t^f - q_t^f e_{it}^f - q_{it}^r e_{it}^r - c_t\Big)$$

$$[c_t] \qquad u'(c_{it}) = \lambda_{it}^w$$
$$[k_t] \qquad MPk_{it} = r_t^\star[e_t^f] \qquad MPe_{it}^f$$
$$[e_t^r] \qquad MPe_{it}^r = \mathcal{D}(\tau_{it})z\;\partial_e f(k_{it}, e_{it})\Big(\frac{e_{it}^r}{(1-\omega)e_{it}}\Big)^{-\frac{1}{\sigma_e}} = q_{it}^r$$
$$[k_t] \qquad \dot{\lambda}_t^w = \lambda_t^w\big(\rho - r_t^\star\big)$$

Fossil Energy Monopoly problem:

$$\mathcal{H}^m(\mathcal{R}_t, \lambda_t^R, E_t, \mathcal{I}_t^f) = \pi_t(E_t^f, \mathcal{I}_t^f, \mathcal{R}_t) + \lambda_t^R(\delta^R \mathcal{I}_t^f - E_t)$$

$$[\mathcal{R}_t] \qquad \dot{\lambda}_t^R = \rho\lambda_t^R + \frac{\bar{\nu}\nu}{1+\nu}\Big(\frac{E_t^\star}{R_t}\Big)^{1+\nu} + \frac{\bar{\mu}\mu}{1+\mu}\Big(\frac{I_t^\star}{R_t}\Big)^{1+\mu}$$

$$[E_t^f] \qquad q_t^{e,f} = \nu_E(E, \mathcal{R}) + \lambda_t^R = \bar{\nu}\Big(\frac{E_t}{\mathcal{R}_t}\Big)^{\nu} + \lambda_t^R$$

$$[\mathcal{I}_t] \qquad \lambda_t^R \delta^R = \mu_I(\mathcal{I}_t^f, R_t) = \bar{\mu}\Big(\frac{\mathcal{I}_t^f}{\mathcal{R}_t}\Big)^{\mu} \qquad \mathcal{I}_t^f = R_t\big(\frac{\lambda_t^R \delta}{\bar{\mu}}\big)^{1/\mu}$$

C Optimal policy and Ramsey problem

Welfare criterion:

$$\mathcal{W}_{t_0} = \max_{\{c,b,k,e^f,e^r,e^x,\iota^x\}} \int_{t_0}^{\infty}\int_{\mathbb{I}} e^{-(\rho+n)t}\ \omega_i\ \mathcal{P}_i\ u\big(\mathcal{D}(\tau_{it})c_{it}\big)\ dt$$

Household:

$$\dot{w}_{it} = \big(r_t^\star - (n+\bar{g})\big)w_{it} + v_{it} + \mathrm{t}_{it}^{ls}$$
$$v_{it} = \mathcal{D}_i^y(\tau_{it})z_i f(k_{it}, e_{it}) - q_{it}^e e_{it} - (r_t^\star + \delta)k_{it}$$

Combine:

$$\dot{w}_{it} = \big(r_t^\star - (n+\bar{g})\big)w_{it} + \pi_{it}^f + \mathcal{D}^y(\tau_{it})z_{it} f(k_{it}, e_{it}^f, e_{it}^r) - (r^\star + \delta)k_{it} - \big(q_t^f + \mathrm{t}_{it}^f\big)e_{it}^f - q_{it}^r e_{it}^r - c_{it} + \mathrm{t}_i^{ls}$$

Optimality conditions of the Household:

$$\begin{cases} \dot{\lambda}_{it}^w &= \lambda_{it}^w(\rho + \eta\bar{g} - r_t^\star) \\ \lambda_{it}^w &= u_c(c_{it}, \tau_{it}) \end{cases}$$

Climate system:

$$\begin{cases} \dot{\mathcal{S}}_t &= \sum_{\mathbb{I}} e^{(n+\bar{g})t}\xi_i \mathcal{P}_i e_{it}^f - \delta^s \mathcal{S}_t \\ \dot{\tau}_{it} &= \zeta\big(\Delta_i \chi \mathcal{S}_t - (\tau_{it} - \tau_{it_0})\big) \end{cases}$$

Firms inputs optimality conditions, capital and energy demand

$$\begin{cases} q_t^f + \mathrm{t}_{it}^f &= MPe_{it}^f \\ r_{it}^\star &= MPk_{it} - \delta \end{cases} \qquad q_{it}^r = MPe_{it}^r$$

Energy firms dynamic decisions:

$$\begin{cases} q_t^f &= \nu_E(e_{it}^x, \mathcal{R}_{it}) + \lambda_{it}^R \\ \dot{\lambda}_{it}^R &= \rho\lambda_{it}^R + \nu_R(e_{it}^x, \mathcal{R}_{it}) + \mu_R(\iota_t^x, \mathcal{R}_{it}) \\ \dot{\mathcal{R}}_{it} &= -e_{it}^x + \delta^R \iota_{it}^x \end{cases} \qquad \delta^R \lambda_{it}^R = \mu_\iota(\iota_{it}^x, \mathcal{R}_{it})$$

Energy market clears

$$E_t^f = \sum_{i\in\mathbb{I}} e_{it}^x = \sum_{i\in\mathbb{I}} \mathcal{P}_i e^{(n+\bar{g})t} e_{it}^f$$

Bond market clears:

$$\sum_{i\in\mathbb{I}} \mathcal{P}_i b_{it} = 0 \qquad \Rightarrow \qquad \sum_{i\in\mathbb{I}} \mathcal{P}_i w_{it} = \sum_{i\in\mathbb{I}} \mathcal{P}_i k_{it}$$

Reformulation of Energy firm profit.

$$\begin{aligned}\mathcal{P}_i \pi_{it}^f(q_t^f, e_{it}^x, \iota_{it}, \mathcal{R}_{it}) &= q_t^f\, e_{it}^x - \nu(e_{it}^x, \mathcal{R}_{it}) - \mu(\iota_t^x, \mathcal{R}_{it}) \\ &= \Big(\nu(e_{it}^x, \mathcal{R}_{it}) - \mu_\iota(\iota_{it}^x, \mathcal{R}_{it})/\delta\Big) e_{it}^x - \nu(e_{it}^x, \mathcal{R}_{it}) - \mu(\iota_t^x, \mathcal{R}_{it})\end{aligned}$$

Ramsey problem, Hamiltonian:

$$\begin{aligned}\mathcal{H}(\mathbf{s}, \mathbf{c}, \psi) = \sum_{\mathbb{I}} \omega_i \mathcal{P}_i u(c_{it}, \tau_{it}) &+ \sum_{\mathbb{I}} \omega_i \mathcal{P}_i \psi_{it}^w \Big(\big(r_t^\star - (n+\bar{g})\big) w_{it} \\ &+ \frac{1}{\mathcal{P}_i} \Big[\Big(\nu_{e^x}(e_{it}^x, \mathcal{R}_{it}) + \mu_\iota(\iota_{it}^x, \mathcal{R}_{it})/\delta \Big) e_{it}^x - \nu(e_{it}^x, \mathcal{R}_{it}) + \mu(\iota_t^x, \mathcal{R}_{it}) \Big] \\ &+ \mathcal{D}^y(\tau_{it}) z_{it} f(k_{it}, e_{it}^f, e_{it}^r) - MPk_{it}k_{it} - MPe_{it}^f e_{it}^f - MPe_{it}^r e_{it}^r - c_{it} + \mathrm{t}_i^{ls} \Big) \\ + \underbrace{\sum_{\mathbb{I}} \omega_i \mathcal{P}_i \psi_{it}^S}_{=\psi_t^S} &\Big(\sum_{\mathbb{I}} e^{(n+\bar{g})t} \xi_i \mathcal{P}_i e_{it}^f - \delta^s \mathcal{S}_t \Big) \\ + \sum_{\mathbb{I}} \omega_i \mathcal{P}_i \psi_{it}^\tau &\Big(\zeta \big(\Delta_i \chi \mathcal{S}_t - (\tau_{it} - \tau_{it_0}) \big) \Big) \\ + \sum_{\mathbb{I}} \omega_i \psi_{it}^R &\Big(- e_{it}^x + \delta^R \iota_{it}^x \Big) \\ + \sum_{\mathbb{I}} \omega_i \mathcal{P}_i \psi_{it}^\lambda &\Big(\lambda_{it}(\rho + \eta\bar{g} - r_t^\star) \Big) + \sum_{\mathbb{I}} \omega_i \mathcal{P}_i \phi_{it}^c \big(u_c(c_{it}, \tau_{it}) - \lambda_{it}^w \big) \\ + \mu_t^b \sum_{i\in\mathbb{I}} \mathcal{P}_i &\big(w_{it} - k_{it} \big) \;+\; \mu_t^e \sum_{i\in\mathbb{I}} \big(e_{it}^x - \mathcal{P}_i e_{it}^f \big)\end{aligned}$$

Static optimality conditions:

- Consumption: $[c_{it}]$

$$\omega_i \mathcal{P}_i \psi_{it}^w = \omega_i \mathcal{P}_i u_c(c_i, \tau_{it}) + \omega_i \mathcal{P}_i \psi_{it}^c u_{cc}(c_{it}, \tau_{it})$$

- Capital $[k_{it}]$

$$-\mu_t^b \mathcal{P}_i - \omega_i \mathcal{P}_i \psi_{it}^w MMPk_{it}k_{it} = 0$$

$$\mu_t^b = -\frac{1}{\mathcal{P}} \sum_{\mathbb{I}} \omega_i \mathcal{P}_i \psi_{it}^w MMPk_{it}k_{it}$$

- Energy extraction $[e^x_{it}]$

$$\mu^e_t + \omega_i \psi^w_{it} \nu_{ee}(e^x, \mathcal{R}) e^x_{it} + \omega_i \psi^w_{it} \underbrace{\mu_\iota(\iota^x_{it}, \mathcal{R}_{it})/\delta}_{=\lambda^R_{it}} - \omega_i \psi^R_{it} = 0$$

$$\mu^e_t = \underbrace{-\frac{1}{\mathcal{P}} \sum_{\mathbb{I}} \omega_i \mathcal{P}_i \psi^w_{it} \nu_{ee}(e^x_{it}, \mathcal{R}_{it}) e^x_{it} +}_{=\text{supply distortion}} \underbrace{\frac{1}{\mathcal{P}} \sum_{\mathbb{I}} \omega_i \mathcal{P}_i \psi^R_{it}}_{=\text{planner reserve valuation}} - \underbrace{\frac{1}{\mathcal{P}} \sum_{\mathbb{I}} \omega_i \mathcal{P}_i \psi^w_{it} \lambda^R_{it}}_{=\text{agents reserve valuation}}$$

- Energy exploration $[\iota^x_{it}]$

$$\omega_i \psi^w_{it} \Big(\mu_{\iota\iota}(\iota^x_{it}, \mathcal{R}_{it}) e^x_{it}/\delta - \mu_\iota(\iota^x_{it}, \mathcal{R}_{it}) \Big) + \omega_i \psi^R_{it} \delta^R = 0$$

$$\psi^w_{it} \lambda^R_{it} \delta^R \big(\frac{e^x_{it}}{\delta^R \iota^x_{it}} \mu - 1 \big) + \psi^R_{it} \delta^R = 0$$

where the last line is taking the iso-elastic functional form. Note that is the marginal cost of extraction is constant, then $\psi^R_{it} = \lambda^R_{it} \psi^w_{it}$ and planner and agents' reserve valuation are aligned.

- Fossil energy consumption $[e^f_{it}]$

$$\omega_i \mathcal{P}_i \psi^w_{it} [\mathrm{t}^f_t - MMPe^f_{it}] + \psi^S_t e^{(n+\bar{g})t} \xi_i \mathcal{P}_i - \mathcal{P}_i \mu^e_t = 0$$

$$\underbrace{\Big(\frac{1}{\mathcal{P}} \sum_{\mathbb{I}} \omega_i \mathcal{P}_i \psi^w_{it} \Big)}_{=\overline{\psi}^w_t} \mathrm{t}^f_t = \underbrace{-\psi^S_t}_{\propto SCC_t} e^{(n+\bar{g})t} \overline{\xi} + \underbrace{\sum_{\mathbb{I}} \omega_i \mathcal{P}_i \psi^w_{it} MMPe^f_{it}}_{=\text{demand distortion}} + \underbrace{\mu^e_t}_{=\text{supply distortion}}$$

with $\overline{\xi} = \frac{1}{\mathcal{P}} \sum_i \mathcal{P}_i \xi_i$

- Interest rate $r^\star_t$

$$\sum_{\mathbb{I}} \omega_i \mathcal{P}_i \psi^w_{it} w_{it} = \sum_{\mathbb{I}} \omega_i \mathcal{P}_i \psi^\lambda_{it} \lambda_{it}$$

Pontryagin Principle: Optimality wrt dynamic variables

- Wealth $[w_{it}]$ – effectively bonds $[b_{it}]$

$$\mathcal{H}_w(\cdot) = \omega_i \mathcal{P}_i \psi^w_{it} (r^\star_t - (n + \bar{g})) - \mathcal{P}_i \mu^b_t$$

- Temperature $[\tau_{it}]$:

$$\begin{aligned} \mathcal{H}_\tau(\cdot) = \omega_i \mathcal{P}_i u_\tau(c_{it}, \tau_{it}) &+ \omega_i \mathcal{P}_i \psi^c_{it} u_{c\tau}(c_{it}, \tau_{it}) \\ &+ \omega_i \mathcal{P}_i \psi^w_{it} \Big[\mathcal{D}^y_\tau(\tau_{it}) z_{it} f(k_{it}, e^f_{it}, e^r_{it}) - \tfrac{\partial}{\partial \tau} \big(MPk_{it} k_{it} + MPe^f_{it} e^f_{it} + MPe^r_{it} e^r_{it} \big) \Big] \\ &- \zeta \omega_i \mathcal{P}_i \psi^\tau_{it} \end{aligned}$$

- Carbon atmospheric stock: $[\mathcal{S}_t]$:

$$
\begin{aligned}
\mathcal{H}_{\mathcal{S}_i}(\cdot) &= \zeta\omega_i\mathcal{P}_i\psi^{\tau}_{it}\Delta_i\chi - \delta^s\omega_i\mathcal{P}_i\psi^S_{it} \\
\mathcal{H}_{\mathcal{S}}(\cdot) &= \zeta\chi\sum_{\mathbb{I}}\omega_i\mathcal{P}_i\psi^{\tau}_{it}\Delta_i - \delta^s\sum_{\mathbb{I}}\omega_i\mathcal{P}_i\psi^S_{it} \\
\mathcal{H}_{\mathcal{S}}(\cdot) &= \zeta\psi^{\tau}_t - \delta^s\psi^S_t \\
\zeta\to 1 \qquad \Rightarrow \qquad \psi^S_t &= \sum_{\mathbb{I}}\omega_i\mathcal{P}_i\psi^S_{it} \approx \sum_{\mathbb{I}}\omega_i\mathcal{P}_i\psi^{\tau}_{it}\,\Delta_i\chi
\end{aligned}
$$

- Marginal value of wealth $[\lambda_{it}]$

$$
\mathcal{H}_{\lambda}(\cdot) = \omega_i\mathcal{P}_i\psi^{\lambda}_{it}(\rho + \eta\bar{g} - r^{\star}_t) - \omega_i\mathcal{P}_i\psi^c_{it}
$$

D Closed form solution for the Social Cost of Carbon

Solving for the shadow cost of carbon and temperature $\Leftrightarrow$ solving ODE

$$
\begin{aligned}
\dot{\psi}^{\tau}_{it} &= \psi^{\tau}_t(\widetilde{\rho} + \Delta\zeta) + \gamma^y_j(\tau - \tau^{\star})\mathcal{D}^y(\tau)f(k,e)\psi^w_t + \gamma^u_j(\tau - \tau^{\star})\mathcal{D}^u(\tau)\underbrace{u'(\mathcal{D}^u(\tau)c)}_{=\psi^w_{it}}c \\
\dot{\psi}^S_t &= {\psi^S}_t(\widetilde{\rho} + \delta^s) - \int_{\mathbb{I}}\Delta_i\zeta\chi\psi^{\tau}_{it}
\end{aligned}
$$

We need to solve for ψ^{τ}_t and ψ^{S}_t. In stationary equilibrium $\dot{\psi}^S_t = \dot{\psi}^{\tau}_t = 0$. As a result, we obtain:

$$
\begin{aligned}
\psi^{\tau}_{it} &= -\int_t^{\infty} e^{-(\widetilde{\rho}+\zeta)u}(\tau_u - \tau^{\star})\Big(\gamma^y_j\mathcal{D}^y(\tau_u)y_{\tau}\psi^w_u + \gamma^u_j\mathcal{D}^u(\tau_u)u'(\mathcal{D}^u(\tau_u)c_u)c_u\Big)du \\
\psi^{\tau}_{it} &= -\frac{1}{\widetilde{\rho} + \Delta\zeta}(\tau_{\infty} - \tau^{\star})\Big(\gamma^y_j\mathcal{D}^y(\tau_{\infty})y_{\infty} + \gamma^u_j\mathcal{D}^u(\tau_{\infty})c_{\infty}\Big)\psi^w_{\infty} \\
\psi^S_t &= -\int_t^{\infty} e^{-(\widetilde{\rho}+\delta^s)u}\zeta\chi\int_{\mathbb{I}}\Delta_j\psi^{\tau}_{j,u}dj\ du \\
&= \frac{1}{\widetilde{\rho} + \delta^s}\zeta\chi\int_{\mathbb{I}}\Delta_j\psi^{\tau}_{j,\infty}dj \\
&= -\frac{\chi}{\widetilde{\rho} + \delta^s}\frac{\zeta}{\widetilde{\rho} + \zeta}\int_{\mathbb{I}}\Delta_j(\tau_{j,\infty} - \tau^{\star})\Big(\gamma^y_j\mathcal{D}^y(\tau_{j,\infty})y_{\infty} + \gamma^u_j\mathcal{D}^u(\tau_{j,\infty})c_{j,\infty}\Big)\psi^w_{j,\infty}dj \\
\psi^S_t &\xrightarrow[\zeta\to\infty]{} -\frac{\chi}{\widetilde{\rho} + \delta^s}\int_{\mathbb{I}}\Delta_j(\tau_{j,\infty} - \tau^{\star})\Big(\gamma^y_j\mathcal{D}^y(\tau_{j,\infty})y_{j,\infty}\ +\ \gamma^u_j\mathcal{D}^u(\tau_{j,\infty})c_{j,\infty}\Big)\psi^w_{j,\infty}dj
\end{aligned}
$$

which proves the analytical formula in the main text.

Moreover, observing that we obtained an expression for the Social Cost, we can rewrite it as the integral of Local Cost, invoking Fubini's theorem:

$$\begin{aligned}
\psi_t^S &= -\int_t^{\infty} e^{-(\widetilde{\rho}+\delta^s)u}\zeta\chi\Delta_j\psi_{j,u}^{\tau}dj\ du \\
&= -\int_{\mathbb{I}}\int_t^{\infty} e^{-(\widetilde{\rho}+\delta^s)u}\zeta\chi\Delta_j\psi_{j,u}^{\tau}\,du\,dj \\
&= \int_{\mathbb{I}}\psi_{j,t}^S dj \\
\text{with}\qquad \psi_{j,t}^S &= \int_t^{\infty} e^{-(\widetilde{\rho}+\delta^s)u}\zeta\chi\Delta_j\psi_{j,u}^{\tau}\,du \\
&\xrightarrow[\zeta\to\infty]{} -\frac{\chi}{\widetilde{\rho}+\delta^s}\Delta_j(\tau_{j,\infty}-\tau^{\star})\Big(\gamma_j^y\mathcal{D}^y(\tau_{j,\infty})y_{j,\infty}+\gamma_j^u\mathcal{D}^u(\tau_{j,\infty})c_{j,\infty}\Big)\psi_{j,\infty}^w
\end{aligned}$$