Three reasons to price carbon under uncertainty: Accuracy of simple rules

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Abstract: An easy-to-interpret rule for the optimal risk-adjusted social cost of carbon is derived using perturbation analysis. This rule internalises the adverse effects of global warming on the risk of recurring climate-related disasters and the risk of irreversible climate tipping points as well as the usual adverse effect on total factor productivity. It approximates the true numerical optimum very well, especially if the small parameters (i.e., the share of damages in GDP, the sensitivity of the risk of disasters to temperature and the risk of climate tipping) are small enough and the discount rates corrected for growth and risk is not too small. The rule is also accurate with ongoing technical progress in fossil-fuel production and multiple economic sectors even though the rule is derived for a one-sector model without such technical progress.

Keywords: carbon pricing; damages; recurring macroeconomic and climate-related disasters; climate tipping point; perturbation analysis.

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1 Introduction

One of the first decisions President Biden took upon taking office was to ask economists to come up with the best evaluation of the social cost of carbon: the expected presented discounted value of all present and future damages from emitting one ton of carbon today (SCC). In response to the President's request, Rennert et al. (2022) have used improved probabilistic socioeconomic projections, climate models, damage functions, and discounting methods to consistently value risk. This substantially increases previous estimates of the SCC, leading to a preferred mean estimate of \$185/tCO_2 based on a near-term risk-free discount rate of 2%/year. This is 3.6 times higher than the US government's current value of \$51/tCO_2.[1] Adding all the economic, climatic and damage uncertainties thus substantially increases the SCC. This is crucial for policy formulation, since cost-benefit analysis then implies that the implied higher benefits of mitigation call for more stringent climate policies across the board.

To take account of the effects of climate tipping points and feedback loops on the derivation of the optimal risk-adjusted SCC as in, for example, Lemoine and Traeger (2016), Cai and Lontzek (2019), and Hambel et al. (2021a) is a complicated and challenging numerical exercise. This requires solving Hamilton–Jacobi–Bellman equations, which are notoriously difficult to solve, especially when they involve many state variables. Monte-Carlo methods for obtaining the optimal SCC are much cheaper but give misleading results as has been shown by Jensen and Traeger (2014). To properly allow for skewed distributions of shocks with gradual arrival of impacts to the climate system and damages and of temperature-dependent disasters and irreversible climate tipping points is still mostly uncharted territory. More importantly, it is difficult to obtain intuitive insights from numerical optimisation exercises. For that reason, analytical results are of critical importance. This is not a trivial undertaking, since the SCC has to be evaluated taking account of a wide range of uncertainties regarding the evolution of the economy, temperature, and global warming damages.

Integrated assessment studies make very different assumptions and yield, not surprisingly, very different estimates of the optimal SCC. Although much of the debate has centred on the choice of the utility discount rate (or pure rate of time preference), differences persist in assumptions about the economic growth rate, its distribution, and about the stochastic properties of the climate system and of global warming damages. The consequences of these assumptions remain relatively unexplored. Given the "black-box" nature of many integrated assessment models, it is not always clear where these differences come from. For example, they may arise from different rates of time impatience, different degrees of risk aversion, different elasticities of intertemporal substitution, and/or different damage coefficients. Substantial differences also result depending on whether stochastic shocks to the economy and the climate system, including the temperature-dependent risks of recurring climate disasters and irreversible climate tipping points, are taken account of or not.

[1] The estimate of \$185/tCO_2 incorporates updated scientific understanding throughout all components of the SCC in the open-source Greenhouse Gas Impact Value Estimator (GIVE) model, and responds to the near-term recommendations by the National Academies of Sciences, Engineering, and Medicine.

To shed light on the various drivers of the SCC, we derive an analytical and intuitive rule for the SCC that highlights the various determinants of the SCC under various types of uncertainty. This rule for the SCC also means that the computational task of solving a complex stochastic dynamic programming problem numerically is avoided. Furthermore, such a rule makes it easier to communicate with policy makers how preferences, attitudes to risk, and the various types of uncertainty affect the SCC. The rule shows how the SCC is affected by the adverse impact of global warming on aggregate output, the risk of recurring climate-related disasters, and the risk of irreversible climate tipping points.

We have three key three objectives. First, we want to obtain a tractable and intuitive but approximate expression for the optimal risk-adjustded SCC that internalises (1) the global warming externalities resulting from the effect of global warming on total factor productivity, (2) the risk of recurring climate-related disasters, and (3) the risk of irreversible climate tipping leading to an irreversible regime shift. To solve the Hamilton–Jacobi–Bellman equations that are needed to calculate the optimal SCC, we use perturbation methods. These methods have been used before in van den Bremer and van der Ploeg (2021) to obtain a rule for the optimal SCC that internalises the adverse effect of global warming on total factor productivity. Our first contribution is thus to extend this earlier rule to allow for recurring climate-related disasters and irreversible climate tipping points.

Second, we seek to evaluate the accuracy of this rule for the optimal risk-adjusted SCC by comparing it with the numerical optimum obtained by a finite-difference method under a wide range of circumstances. It turns out that the rule approximates the numerical optimum well, thus eliminating the computational burden of finding optimal climate policy numerically and giving analytical insights into the drivers of the optimal carbon price. The rule not only performs well when only internalising the effects of global warming on total factor productivity, but also performs well when the risks of recurring climate-related disasters and irreversible climate tipping points are internalised too. Our estimate of the optimal risk-adjusted SCC is better (compared to the numerical optimum) if damages are a small fraction of GDP, the risk of climate-related disasters does not react too strongly to temperature, the risk of climate tipping is small enough, and discount rates are not too low.

Third, we examine the robustness of our rule for the optimal risk-adjusted SCC by performing various tests of its accuracy *outside* the model for which it has been derived, namely by allowing for a falling cost of fossil fuel and having a two-sector model of economic growth. For both these cases, the rule for the optimal SCC, even though designed for a simpler model with constant cost of fossil fuel and one-sector growth, approximates the numerical optimum surprisingly well.

To make headway, we simplify the DSGE model of global warming and the economy that was analysed in van den Bremer and van der Ploeg (2021) by replacing the box models of the carbon stock and temperature dynamics by a model in which temperature rises linearly in cumulative emissions.[2] The cumulative emissions model is a good approximation to the results from complex climate models (Allen et al., 2009; Matthews et al., 2009; Dietz and Venmans, 2019). It is also in line with policy analysis of the Intergovernmental Panel on Climate

[2] Appendix F.2 discusses how more general carbon cycle and temperature dynamics affect our rule for the SCC.

Change (IPCC) and avoids the problem of excessive inertia in the temperature response that most well-known integrated assessment models of climate change and the economy suffer from (Dietz et al., 2021b).

We extend our earlier approach to macroeconomic uncertainty by including risks of rare macroeconomic disasters (Barro, 2006, 2009; Barro and Jin, 2011) as well as normally distributed shocks to economic growth. Together with Epstein–Zin preferences, this helps to deal with the equity premium and risk-free rate puzzles by better matching the observed risk-free rate and equity risk premium. We also extend our earlier approach to not only allow global warming to negatively impact total factor productivity (Nordhaus, 2017) but also to increase the risks of rare, recurring climate-related disasters and irreversible climate tipping points. The latter allows for the probability of an irreversible regime shift, in which the climate system shifts to one with a higher transient climate response to cumulative emissions at an unknown future date.

Our rule for the optimal SCC in absence of climate tipping risk is given in Result 1. It consists of a term to internalise the adverse effect of global warming on total factor productivity and a risk mitigation term that corresponds to the effect of global warming on the expected loss of climate-related disasters. These terms and thus the optimal SCC are proportional to the transient climate response to cumulative emissions and aggregate economic activity and inversely proportional to the risk- and growth adjusted discount rate. The optimal SCC in the presence of irreversible climate tipping is given in Result 2 and consists of three terms. The first term internalises the adverse effects of global warming on total factor productivity and the risk of climate-related disasters. The second term is a repricing term to take account of the probability that at some uncertain future time the transient climate response to cumulative emissions jumps up. The third term mitigates risk by internalising the adverse effect of global warming on the risk of climate tipping.

Our analysis builds on simple rules for the optimal SCC from *deterministic* growth models of climate and the economy. For example, Nordhaus (1991) derives an *approximate* rule for the optimal SCC under certainty, which is proportional to aggregate economic activity. Golosov et al. (2014) obtain a similar rule and give conditions under which the rule yields the *exact* welfare-maximising outcome in a DGSE framework.[3] Barrage (2014), van den Bijgaart et al. (2016), Rezai and van der Ploeg (2016), and Hambel et al. (2021b) perform a detailed numerical analysis of the performance of such approximate rules for the SCC under certainty and show that these are generally good approximations.

We follow Lemoine (2021), van den Bremer and van der Ploeg (2021) and Traeger (2023), and derive approximate,intuitive rules for the optimal SCC under uncertainty using *stochastic* integrated assessment, general equilibrium models of the economy and the climate. Our main innovation is that we allow for the effects of recurring temperature-dependent risks of recurring disasters and of irreversible climate tipping points. We also allow for standard forms of macroeconomic uncertainty (exogenous risk of rare macroeconomic disasters as well as

[3]These are logarithmic utility, Cobb–Douglas production, 100% depreciation of capital each period, a linear two-box model for the dynamics of atmospheric carbon, and damages to the logarithm of total factor productivity rising linearly in atmospheric carbon. This *exact* expression for the deterministic rule is extended to more general settings with a negative linear effect of atmospheric carbon on utility, mean reversion in the effects of damages on total factor productivity, less than 100% logarithmic depreciation, and policy makers who are more patient than private agents by van der Ploeg and Rezai (2022).

Brownian shocks). Our rule for the SCC thus internalises the three externalities resulting from emissions curbing economic production, increasing the frequency of climate-related disasters, and bringing forward the expected date of a climate tipping point.

Section 2 presents our stochastic integrated assessment model. Section 3 derives solutions to the model (in terms of the value function) using perturbation methods. These solutions are used in Section 4 to derive approximate rules for the optimal risk-adjusted SCC that take account of the three global warming externalities, considering the scenarios without (Section 4.1) and with climate tipping (Section 4.2) in turn. Section 5 discusses our calibration. Section 6 evaluates the numerical accuracy of our rules for the SCC by comparing to the SCC obtained from full numerical optimisation. Section 7 road-tests the numerical accuracy of our rules for the optimal risk-adjusted SCC in more general models than have been used to derive the rule. Section 8 concludes.

2 Stochastic Integrated Assessment Model

We specify a macroeconomic DSGE model with endogenous growth and add fossil fuel as a production factor whose combustion gives rise to global warming. The pure rate of time preference (or utility discount rate) is denoted by $\rho \geq 0$. The coefficient of relative risk aversion is denoted by $\gamma \geq 0$, the coefficient of intergenerational inequality aversion by $\eta \geq 0$, and the elasticity of intertemporal substitution by $1/\eta$. We use Epstein–Zin preferences to distinguish relative risk aversion, γ, from the inverse of the elasticity of intertemporal substitution, $1/\eta$ (Epstein and Zin, 1989). Empirical evidence suggests $\gamma > \eta$, which reflects a preference for early resolution of uncertainty (Vissing-Jørgensen and Attanasio, 2003). Much of the macro-finance literature sets $1/\eta < 1$, so that macroeconomic volatility depresses share prices. Other empirical evidence suggests $1/\eta > 1$ (Hall, 1988; Campbell, 1999; Vissing-Jørgensen, 2002), which is also common in climate economics (e.g., Nordhaus (2007); Gollier (2018)).

We use continuous-time recursive preferences (Duffie and Epstein, 1992):

$$J_t = \mathbb{E}_t \left[\int_t^{\infty} f(C_s, J_s) \mathrm{d}s \right] \text{ with } f(C_t, J_t) = \frac{1}{1-\eta} \frac{C_t^{1-\eta} - \rho[(1-\gamma)J_t]^{\frac{1-\eta}{1-\gamma}}}{[(1-\gamma)J_t]^{\frac{1-\eta}{1-\gamma}-1}}, \tag{1}$$

where the recursive aggregator $f(C_t, J_t)$ depends on aggregate consumption C_t and the value function J_t. If aversion to risk is the same as that to intertemporal fluctuations, $\gamma = \eta$, the aggregator function becomes $f(C_t, J_t) = \frac{C_t^{1-\gamma}}{1-\gamma} - \rho J_t$, in which case we have CRRA utility.

The dynamics of the aggregate capital stock, denoted by K_t, is given by

$$\mathrm{d}K_t = \left[I_t - \delta K_t - \frac{1}{2}\varphi \frac{I_t^2}{K_t} \right] \mathrm{d}t + K_t \sigma \mathrm{d}W_{tk} - K_{t-}\ell_e \mathrm{d}N_{te} - K_{t-}\ell_c \mathrm{d}N_{ct}, \tag{2}$$

where I_t denotes aggregate investment, $\delta \geq 0$ the depreciation rate of capital, φ the parameter for investment adjustment costs, W_{tk} a Wiener process modeling economic shocks, $\sigma \geq 0$ the relative volatility of capital, and K_0

is the given initial capital stock. N_{te} is a Poisson point process capturing the risk of macroeconomic disasters that are independent of climate change. N_{tc} is another Poisson point process that models climate-related disaster shocks. The parameters $\ell_e \geq 0$ and $\ell_c \geq 0$ indicate the relative sizes of the jumps, which are stochastic but independent of N_{te}, N_{tc}, and W_{tk}. The macroeconomic recovery rate $Z_i \equiv 1 - \ell_i$ follows a power distribution with parameter β_i, so its probability density function is $f_i(Z_i) = \beta_i Z_i^{\beta_i - 1}$ with $Z_i \in (0,1)$, where $\mathbb{E}[Z_i] = \frac{\beta_i}{\beta_i + 1}$ for i is e or c. The jump intensity of N_{te} is constant and denoted by λ_e. The expected time until the next disaster is $1/\lambda_e$.

The temperature anomaly is the difference in temperature since pre-industial times and is referred to as temperature T_t. It is driven by cumulative emissions since initial time zero, denoted by E_t, so that

$$T_t = T_0 + \chi E_t \tag{3}$$

with $\chi > 0$ the transient climate response to cumulative emissions (TCRE) and T_0 initial temperature relative to pre-industrial times. Since temperature responds immediately to cumulative emissions, we do not have excessive inertia in the temperature response to marginal emissions, for which the main integrated assessment models used by economists have been criticised (Dietz et al., 2021b).

The risk of climate-related disasters increases linearly in temperature, that is, $\lambda_{c,t} = \lambda_{0T}^c + \lambda_{1T}^c T_t$ with $\lambda_{1T}^c \geq 0$. Using the temperature equation (3) and with $\lambda_0^c \equiv \lambda_{0T}^c + \lambda_{1T}^c T_0$ and $\lambda_1^c \equiv \chi \lambda_{1T}^c$, the jump intensity for the point process N_c can be written as

$$\lambda_c(\lambda_1^c E_t) = \lambda_0^c + \lambda_1^c E_t. \tag{4}$$

Here, $\lambda_i \mathrm{d}t$ is the probability of a jump to occur in the infinitesimally small time interval $\mathrm{d}t$. We thus allow global warming to increase the intensity (or arrival rate) of climate-related disasters (but not the distribution of the size of these disasters).

Aggregate investment is $I_t = Y_t - C_t - bF_t$, where Y_t is aggregate production, F_t fossil-fuel use, and b the fixed production cost per unit of fossil fuel. The final goods production function is $Y_t = AK_t^{\alpha} F_t^{1-\alpha}$, where $0 < \alpha < 1$ and total factor productivity A is assumed to be constant.[4] Global warming induces proportional losses in total factor productivity, $A_t \equiv A^*(1 - D_t)$, where the damage ratio, $D_t = D_{0T} + D_{1T} T_t$ with $D_{1T} \geq 0$, rises linearly in temperature. The *expected* marginal effect of temperature on the damage ratio is thus D_{1T}. With $D_0 \equiv D_{0T} +$

[4]This production function stems from $Y = AK^{\beta} F^{1-\alpha} (K_a L)^{\alpha - \beta}$, where L denotes labour employed (without loss of generality set to 1), K_a denotes the economy-wide capital stock, and $0 < \beta < 1$. The economy-wide capital stock thus boosts the efficiency of labour, leading to a growth externality. In equilibrium, all firms are the same, so that the individual capital stock equals the economy-wide capital stock, $K = K_a$, and we have $Y = AK^{\alpha} F^{1-\alpha}$. The exponents of K and F add up to one, so there is endogenous growth. A production subsidy internalises the growth externality. It is easy to allow for renewable energy as a factor input. We assume that this is already optimised out and captured in A. We allow for technical progress in Section 7. We also abstract from labour-augmenting technical progress and population growth. Neither extension affects the derivation of our estimate of the optimal SCC presented in Results 1 and 2.

$D_{1T}T_0$ and $D_1 \equiv \chi D_{1T}$, the damage ratio and total factor productivity become

$$D(D_1E_t) = D_0 + D_1E_t, \qquad A(D_1E_t) = A^*\left[1-(D_0+D_1E_t)\right]. \tag{5}$$

The damage ratio thus rises and total factor productivity falls linearly in cumulative emissions.

The emissions rate is $\varpi_t F_t$, where ϖ_t is the emissions intensity, which falls due to technical progress at the rate of economic growth: $\varpi_t = \exp(-\int_0^t g_s \mathrm{d}s)$. Cumulative emissions follow from

$$\mathrm{d}E_t = \varpi F_t \mathrm{d}t, \tag{6}$$

where $E_0 = 0$ (as cumulative emissions are measured from $t = 0$).

To allow for the risk of an irreversible climate tipping point, there is a probability of a regime shift at an unknown date in the future, when the transient climate response to cumulative emissions jumps from χ to $\overline{\chi} > \chi$. The hazard rate of this tipping point increases with temperature as $h = h_{0T} + h_{0T}T_t$ or, defining $h_0 \equiv h_{0T} + h_{1T}T_0$ and $h_1 \equiv \chi h_{1T}$, with cumulative emissions as

$$h(h_1E_t) = h_0 + h_1E_t. \tag{7}$$

The probability that this tipping point occurs in an infinitesimally small period between time t and time $t+\mathrm{d}t$ equals $h(t)\mathrm{d}t$. Global warming thus makes a climate tipping point more likely to arrive soon. The probability that the tipping point has not occurred in the period up to t is $\exp\left(-\int_0^t h(s)\mathrm{d}s\right)$.

Finally, we assume that the parameters in the equations for $\lambda_c(\lambda_1^c E_t)$ (4), $D(D_1E_t)$ (5), and $h(h_1E_t)$ (7), including the pre- and post-tip values of the TCRE, χ and $\overline{\chi}$, are known and not stochastic, but we will consider stochastic shocks to the damage ratio and the TCRE in Appendix C. A detailed discussion of the effects of these stochastic shocks can be found in van den Bremer and van der Ploeg (2021).

3 Value Function Solutions using Perturbation Methods

Here we apply perturbation methods to derive an approximate solution for the value function that will be used to derive an estimate for the optimal SCC in Section 4. We must distinguish the pre-tip Hamilton–Jacobi–Bellman (HJB) equation (denoted without an overbar and characterized by the lower climate sensitivity χ) and the post-tip HJB equation (denoted with an overbar and characterized by the higher climate sensitivity $\overline{\chi}$, with $\overline{\chi} > \chi$), which we will consider in reverse order.

3.1 The Hamilton–Jacobi–Bellman equations

3.1.1 Post-tip HJB equation

The post-tip value function $\overline{J} = J(K,E)$ depends on the capital stock K and the stock of cumulative emissions E. The optimal solution must satisfy the HJB equation,

$$\max_{C,F}\Big[f(C,\overline{J}) + \frac{1}{\mathrm{d}t}\mathbb{E}_t\big[\mathrm{d}\overline{J}(K,E)\big]\Big] = 0, \tag{8}$$

where $\frac{1}{\mathrm{d}t}\mathbb{E}_t[\mathrm{d}\overline{J}]$ is Ito's differential operator applied to $\overline{J}$. Using Ito's lemma, the HJB equation becomes

$$\begin{aligned}\max_{C,F}\Big[f\big(C,\overline{J}\big) + \overline{J}_K\Big[A(\overline{D}_1E)K^{\alpha}F^{1-\alpha} - C - bF - \delta K - \frac{1}{2}\varphi\frac{I^2}{K}\Big] + \frac{1}{2}\overline{J}_{KK}K^2\sigma^2 + \overline{J}_E\varpi F \\ + \lambda_e\mathbb{E}\big[\overline{J}\big((1-\ell_e)K,E\big) - \overline{J}(K,E)\big] + \lambda_c(\overline{\lambda}_1^cE)\mathbb{E}\big[\overline{J}\big((1-\ell_c)K,E\big) - \overline{J}(K,E)\big]\Big] = 0,\end{aligned} \tag{9}$$

where $\overline{D}_1 = \overline{\chi}D_{1T}$ and $\overline{\lambda}_1^c = \overline{\chi}\lambda_{1T}^c$ have both been evaluated using the post-tip value of the TCRE, $\overline{\chi}$.

3.1.2 Pre-tip HJB equation

The pre-tip problem requires us to solve the HJB equation

$$\begin{aligned}\max_{C,F}\Big[f\big(C,J\big) + J_K\Big[A(D_1E)K^{\alpha}F^{1-\alpha} - C - bF - \delta K - \frac{1}{2}\varphi\frac{I^2}{K}\Big] + \frac{1}{2}J_{KK}K^2\sigma^2 + J_E\varpi F \\ + \lambda_e\mathbb{E}\big[J\big((1-\ell_e)K,E\big) - J(K,E)\big] + \lambda_c(\lambda_1^cE)\mathbb{E}\big[J\big((1-\ell_c)K,E\big) - J(K,E)\big] \\ + h(h_1E)\Big(\overline{J}(K,E) - J(K,E)\Big)\Big] = 0,\end{aligned} \tag{10}$$

where the absence of overbars on $D_1 = \chi D_{1T}$, $\lambda_1^c = \chi\lambda_{1T}^c$, and $h_1 = \chi h_{1T}$ now denotes evaluation using the pre-tip value of the TCRE, χ. Compared with the post-tip HJB equation (9), the additional final term in (10) represents the expected loss from a climatic tipping point.

3.2 Optimality conditions

The pre- and post-tip first-order conditions are of equivalent form, and we present the results in this section, which are valid for both circumstances, without overbars for convenience. Optimality requires equalising the marginal values of consumption and investment,

$$f_C(C,J) = C^{-\eta}\big((1-\gamma)J\big)^{\frac{\eta-\gamma}{1-\gamma}} = J_K\Big(1+\varphi\frac{I}{K}\Big), \tag{11}$$

and setting the marginal product of fossil fuel equal to its production cost, b, plus the SCC

$$(1-\alpha)\frac{Y}{F} = b + \varpi P, \tag{12}$$

where the optimal SCC, denoted by P, is the marginal disvalue of emitting an additional ton of carbon divided by the marginal value of consumption, $P \equiv -J_E/f_C > 0$. One way to implement the resulting first-best outcome in a decentralised market economy is to set the carbon price to the SCC and rebate the revenue to the private sector as lump-sum payments.

Since optimal use of fossil fuel is proportional to Y, we have, in equilibrium, an "AK" model of endogenous growth with the reduced-form aggregate production function

$$Y = B(D_1E, P)K, \tag{13}$$

where $B(D_1E,P) \equiv \left[A(D_1E)\left(\frac{1-\alpha}{b+\varpi P}\right)^{1-\alpha}\right]^{1/\alpha}$. Aggregate output (and fossil fuel use) decrease in global warming, captured by E, the carbon price, P, the sensitivity of (expected) damages to temperature, D_{1T}, and the transient climate response to cumulative emissions, χ (the latter two via $D_1 = \chi D_{1T}$). The carbon price P depends in turn on the state variables K and E.

3.3 Perturbation analysis

As in van den Bremer and van der Ploeg (2021), we assume that the damage coefficient D_1 is a small parameter. Since we want to extend the analysis with two additional global warming externalities, we now also assume that the sensitivity of the intensity of climate-related disasters with respect to cumulative emissions $\mathrm{d}\lambda_{c,t}/\mathrm{d}E_t = \lambda_1^c$ and the hazard rate of the tipping point h are small parameters. We assume that all three parameters are equally small[5]:

$$D_1 = \mathcal{O}(\epsilon), \quad \lambda_1^c = \mathcal{O}(\epsilon) \quad \text{and} \quad h = \mathcal{O}(\epsilon). \tag{14}$$

[5] Formally, small parameters must be non-dimensional. We thus require the (non-dimensional) damage ratio D to be small for all relevant values of E. The increase in jump intensity of climate disasters due to increases in the stock of cumulative emissions, $\Delta\lambda_c = \lambda_1^c \Delta E$ (%/year) and the hazard rate of climate tipping h (%/year) have be small relative to other rates (%/year) in the model, such as the growth rate of GDP $g^{(0)}$, for all relevant values of E. To make an assessment of whether the assumptions in (14) are reasonable, we use a temperature anomaly of $T = 1.5°$C and the parameter values in our calibration (see Section 5, using pre-tip values). We obtain $D = D_{1T}T = 1.4\%$, $\Delta\lambda_c \mathbb{E}[1-Z^c]/g^{(0)} = \lambda_{1T}^c \Delta T \mathbb{E}[1-Z^c]/g^{(0)} = 2.9\%$, where we have included the mean size of expected climate disasters to make a realistic assessment, and $h/g^{(0)} = h_{1T}\Delta T/g^{(0)} = 12\%$ (with $\Delta T = T - T_0 = 0.4°$C). These are clearly small numbers, which justifies the assumptions in (14). In practise, discounting of the future reduces the quantitative error introduced by larger values of the small parameters that arise in the future when temperatures are larger. The real test will come from the comparison to full numerical optimisation in Section 6.

We assume that solutions for the value function, J, and the policy variables, C and F, take the form (see also van den Bremer and van der Ploeg (2021)):

$$\begin{aligned} J(K,E) &= J^{(0)}(K,E;\epsilon E) + \epsilon J^{(1)}(K,E) + \mathcal{O}\left(\epsilon^2\right), \\ C(K,E) &= C^{(0)}(K,E;\epsilon E) + \epsilon C^{(1)}(K,E) + \mathcal{O}\left(\epsilon^2\right), \\ F(K,E) &= F^{(0)}(K,E;\epsilon E) + \epsilon F^{(1)}(K,E) + \mathcal{O}\left(\epsilon^2\right), \end{aligned} \tag{15}$$

where the structure of the solution is based on the underlying HJB equation. Similar equations can be written down with overbars for the post-tip problem. The first terms in the brackets (before the semicolon) on the right-hand side of the equations denote the functional dependence, whereas the second terms (after the semicolon) denote the order of the functional dependence if this order is higher than zero. We refer to $J^{(0)}(K,E;\epsilon E)$ as having a 'slow' functional dependence on E, as differentiating $J^{(0)}$ with respect to E increases the order in ϵ by 1 (unlike differentiating with respect to K, which leaves the order unchanged).[6] In equation (15), $J^{(0)}$ and $J^{(1)}$ denote, respectively, the zeroth- and first-order terms in the value function perturbation series, and similarly for consumption and fossil-fuel use. We note that the zeroth-order solution does take into account the effects of cumulative emissions on damages and on the risk of climate disasters, but not the effects of variations in this stock and of climate tipping, which arise at the next order. We will now consider the zeroth- and first-order solutions for the value function in turn.

3.3.1 Zeroth-order solution

The zeroth-order expressions for the policy variables and aggregate output follow from the first-order optimality conditions (11) and (12):

$$C^{(0)} = \left(J_K^{(0)}\right)^{-\frac{1}{\eta}} \left((1-\gamma)J^{(0)}\right)^{\frac{1}{\eta}\frac{\eta-\gamma}{1-\gamma}}, \qquad F^{(0)} = \left(\frac{(1-\alpha)A(\epsilon E)}{b}\right)^{\frac{1}{\alpha}} K, \qquad Y^{(0)} = B^{(0)}K, \tag{16}$$

where $B^{(0)} \equiv B(D_1E,0)$ is total factor productivity in the absence of a carbon price.

We conjecture that the zeroth-order solution is of the form $J^{(0)} = \psi_0(\epsilon E)K^{1-\gamma}$, where $\psi_0(\epsilon E)$ needs to be determined. Treating the effects of $D(\epsilon E)$ on total factor productivity and of $\lambda_c(\epsilon E)$ on the risk of rare macroeconomic disasters as small and constant for the purpose of deriving the zeroth-order solution, we substitute this conjecture and the first-order optimality conditions (16) into the HJB equation (9) to obtain the following implicit, nonlinear equation in $\psi_0^* \equiv (1-\gamma)\psi_0$:

$$\frac{1-\gamma}{1-\eta}\left(\left(\psi_0^*\right)^{\frac{\eta-1}{\eta(1-\gamma)}} - \rho\right) + (1-\gamma)g^{(0)}(\psi_0^*) - \frac{1}{2}\gamma(1-\gamma)\sigma^2 + \lambda_e\mathbb{E}\left[Z_e^{1-\gamma}-1\right] + \lambda_c(\epsilon E)\mathbb{E}\left[Z_c^{1-\gamma}-1\right] = 0, \tag{17}$$

[6] Appendix A.1 of van den Bremer and van der Ploeg (2021) explains how the perturbation method we use differs from a Taylor-series expansion.

where $g^{(0)}(\psi_0^*) = -\delta + \alpha\left(\frac{b}{1-\alpha}\right)^{1-\frac{1}{\alpha}} A(\epsilon E)^{\frac{1}{\alpha}} - \left(\psi_0^*\right)^{\frac{1-1/\eta}{1-\gamma}} - \frac{1}{2}\varphi\left(\alpha\left(\frac{b}{1-\alpha}\right)^{1-\frac{1}{\alpha}} A(\epsilon E)^{\frac{1}{\alpha}} - \left(\psi_0^*\right)^{\frac{1-1/\eta}{1-\gamma}}\right)^2$. We can solve this equation numerically for ψ_0^* and thus for $\psi_0 = \psi_0^*/(1-\gamma)$. Note that the expression for $g^{(0)}$ is not equal to the expected growth rate (e.g., of the capital stock) owing to the presence of non-zero mean disasters, but equals the expected growth rate in normal times when no disasters occur. The solution for ψ_0 depends on the stock of cumulative emissions through $A(\epsilon E)$ and $\lambda_c(\epsilon E)$. It is convenient to express the solution to (17) as

$$\psi_0 = \frac{1}{1-\gamma}\frac{\left(r^\star\right)^{\frac{-\eta(1-\gamma)}{1-\eta}}}{\left(1-\varphi i^{(0)}\right)^{1-\gamma}}, \tag{18}$$

where the (zeroth-order accurate) discount rate $r^\star$ is given by

$$r^\star = \rho + (\eta-1)\Big[g^{(0)} - \frac{1}{2}\gamma\sigma^2 - \frac{\lambda_e}{1-\gamma}\mathbb{E}\big[1-Z_e^{1-\gamma}\big] - \frac{\lambda_c(\epsilon E)}{1-\gamma}\mathbb{E}\big[1-Z_c^{1-\gamma}\big]\Big], \tag{19}$$

and the investment rate $i^{(0)} = I_t^{(0)}/K_t$, which is also needed to evaluate $g^{(0)} = i^{(0)} - \delta - \varphi\left(i^{(0)}\right)^2/2$, follows from the nonlinear implicit equation

$$-i^{(0)} + \alpha\left(\frac{b}{1+\alpha}\right)^{1-\frac{1}{\alpha}} A(\epsilon E)^{1/\alpha} - \frac{r^\star}{1-\varphi i^{(0)}} = 0, \tag{20}$$

which follows from substituting the solutions for the policy variables (16) and the zeroth-order solution for the value function (18) into the budget constraint, $Y - bF - I - C = 0$. In equations (16)-(20), we have made explicit the origin of the slow functional dependence of the solutions (i.e. ψ_0^*, $g^{(0)}$, $r^\star$, and $i^{(0)}$) on E, that is, through $A(\epsilon E)$ and $\lambda_c(\epsilon E)$.

Finally, we note that the zeroth-order solution is valid for both the pre- and post-tipping problems, as the effect of tipping arises at higher order, and the solution is shown here without overbars for convenience only.

3.3.2 First-order solution

In line with the ordering assumption (14), three effects can potentially contribute to the first-order solution: the effect of cumulative emissions on damages through $D(\epsilon E)$ and on the risk of macroeconomic disasters through $\lambda_c(\epsilon E)$ and the effect of climate tipping through $h(E) = \mathcal{O}(\epsilon)$. Since the marginal effects of cumulative emissions on the ratio of damages to output ($\mathrm{d}D/\mathrm{d}E = D_1\chi$) and on the risk of macroeconomic disasters ($\mathrm{d}\lambda_c/\mathrm{d}E = \lambda_1\chi$) are constant in our model, a first-order solution only arises in the pre-tip problem to account for climate tipping ($h(E)$). If damages are a nonlinear (convex) function of temperature, as in the DICE integrated assessment model (e.g., Nordhaus, 2017), and this degree of convexity exceeds the concavity of the dependence of temperature on the atmospheric carbon stock, a first-order solution must also take account of non-constant marginal damages, as is shown in van den Bremer and van der Ploeg (2021).[7] Here, a first-order solution would also arise if temperature-

[7] In the notation of van den Bremer and van der Ploeg (2021), whose carbon stock is non-cumulative, the present model corresponds to the case in which $\theta_{ET} = 0$ (and $\nu_\chi = \nu_\lambda = 0$) in van den Bremer and van der Ploeg (2021).

related risk of recurring climate-related disasters were a nonlinear instead of a linear function of the stock of cumulative emissions.

By substituting the optimality conditions (16) into the HJB equation (9) and retaining only first-order terms in ϵ, we obtain an equation that is solved by a value function of the form $\epsilon J^{(1)} = \epsilon\psi_1(E)K^{1-\gamma}$, where the coefficient $\psi_1(E)$ is given by (N.B., $J = J^{(0)} + \epsilon J^{(1)}$):

$$\epsilon\psi_1(E) = \frac{h(E)\left(\overline{\psi}_0(\epsilon E) - \psi_0(\epsilon E)\right)}{r^\star(\epsilon E)}, \tag{21}$$

where $\overline{\psi}_0$ and ψ_0 are the post- and pre-tipping values of the zeroth-order value function coefficient given by equation (18) and $r^\star$ is the (zeroth-order accurate) discount rate given by (19) (see Appendix A for a derivation). The first-order post-tipping value function is zero ($\overline{\psi}_1 = 0$), as climate tipping can only occur once in our model.

4 The Optimal Risk-Adjusted SCC

We will now proceed by using the zeroth- and first-order value function solutions derived in Section 3 to obtain simple rules that provide a leading-order estimate of the SCC. We will distinguish two cases: without climate tipping (Section 4.1) and with climate tipping (Section 4.2).

4.1 A rule for the optimal SCC without climate tipping

Without climate tipping, an estimate of the risk-adjusted SCC can be obtained based on the zeroth-order solution alone. We refer to this as our leading-order approximation of the SCC. For ease of notation, we will drop the time subscripts t throughout this section.

Result 1. *Without the risk of climate tipping, the leading-order approximation to the SCC is*

$$P_{\text{R1}} = \left[D_{1T} + \lambda_{1T}^c \frac{\mathbb{E}\left[1 - Z_c^{1-\gamma}\right]}{1-\gamma}\frac{q^{(0)}}{B^{(0)}}\right]\frac{\chi Y^{(0)}}{r^\star}, \tag{22}$$

where the discount rate adjusted for risk and growth is given by

$$r^\star = \rho + (\eta - 1)\left[g^{(0)} - \frac{1}{2}\gamma\sigma^2 - \frac{\lambda_e}{1-\gamma}\mathbb{E}\left[1 - Z_e^{1-\gamma}\right] - \frac{\lambda_c}{1-\gamma}\mathbb{E}\left[1 - Z_c^{1-\gamma}\right]\right]. \tag{23}$$

Here, $q^{(0)} = \frac{1}{1-\varphi i^{(0)}}$ denotes Tobin's Q with $i^{(0)}$ the investment rate for the zeroth-order solution, and $g^{(0)}$ denotes the expected growth rate net of macroeconomic and climate-related disasters.

Proof: This result follows from evaluating $P_{\text{R1}} = -\frac{J_E^{(0)}}{J_K^{(0)}} q^{(0)}$.

If the disaster sizes follow power distributions with parameters β_e and β_c, respectively (we require $\beta_e, \beta_c > \gamma - 1$), the expression for the optimal SCC (22) becomes $P_{\text{R1}} = \left[D_{1T} + \frac{\lambda_{1T}^c}{\beta_c + 1 - \gamma} \frac{q^{(0)}}{B^{(0)}} \right] \frac{\chi Y^{(0)}}{r^\star}$, where the discount rate adjusted for risk and growth (23) becomes $r^\star = \rho + (\eta - 1)\left(g^{(0)} - \frac{1}{2}\gamma\sigma^2 - \frac{\lambda_c}{1+\beta_c-\gamma} - \frac{\lambda_e}{1+\beta_e-\gamma} \right)$.

To interpret Result 1, note that the two terms in the square brackets in equation (22) correspond to the expected marginal damage per unit of aggregate output from having one degree higher temperature. The first term D_{1T} is the (expected) marginal effect of one degree higher temperature on the damage ratio D, and indicates the adverse effect of global warming on total factor productivity. The second term in the square brackets in (22) is the expected marginal loss of one degree higher temperature due to a higher risk of a rare recurring climate-related disaster. This term is proportional to the marginal increase in risk of such a disaster due to a higher temperature, i.e., $\lambda_1^c = \mathrm{d}\lambda_c(T)/\mathrm{d}T$, the TCRE (i.e., χ), and to the risk-adjusted expected loss (as a share of the capital stock) due to such a disaster, $\frac{\mathbb{E}[1-Z_c^{1-\gamma}]}{1-\gamma}$. This risk-adjusted expected loss increases in the coefficient of relative risk aversion, γ. Since a temperature-related disaster hits the capital stock, the risk-adjusted expected loss is in terms of units of lost capital. To get the expected loss in units of lost output of final goods, it must be multiplied by $q^{(0)}/B^{(0)} = q^{(0)}K/Y^{(0)}$ with $q^{(0)}K$ representing the replacement costs to rebuild the capital stock.

We also observe from equation (22) that our estimate of the optimal SCC is proportional to the transient climate response to cumulative emissions, χ, and to world aggregate output of final goods, $Y_t^{(0)}$.[8] Furthermore, our estimate of the SCC is inversely proportional to the risk- and growth-adjusted discount rate $r^\star$ given in equation (23). This rate is used to discount all future marginal damages of emitting one ton of carbon today, taking account of normal macroeconomic uncertainties and the risk of recurring climate-related disasters and of marginal damages and losses due to climate disasters that grow in line with the economy.

Hence, our expression for the optimal SCC (22) generalises the expressions derived in van den Bremer and van der Ploeg (2021), which ignored temperature-related risks of rare macroeconomics disasters and in which case the SCC reduces to $P_{\text{R1}} = D_{1T} \frac{\chi Y^{(0)}}{r^\star}$. The novelty of Result 1 is that it contains an additional term to correct for the adverse effects of global warming on the risk of recurring macroeconomic disasters. Due to the "AK" feature of our macroeconomic model, the SCC is proportional to the aggregate capital stock or GDP, which in turn are proportional to each other.

Both components of the SCC given in (22) decrease in the risk-adjusted discount rate (23). Hence, for a constant growth rate $g^{(0)}$, a lower pure rate of time preference (ρ) implies a lower discount rate and a higher SCC.[9] If the coefficient of intergenerational inequality aversion (η) exceeds one or equivalently the elasticity of intertemporal substitution ($1/\eta$) is less than one, equation (22) and (23) indicate that lower economic growth and

[8] In fact, the SCC is proportional to aggregate output that prevails in the absence of carbon pricing. This is a direct result of the approximations we have made.

[9] In contrast to exogenous Ramsey growth models such as Golosov et al. (2014) and Nordhaus (2017), our rate of economic growth $g^{(0)}$ is endogenous. Hence, there are indirect effects on the optimal SCC via the growth rate $g^{(0)}$. For example, the direct effect of a higher rate of pure time preference ρ is to lower the SCC and the indirect effect is to raise the SCC as economic growth is lowered (for $\eta > 1$). Together, the effect of a higher rate of pure time preference on the discount rate is always positive and thus always negative on the SCC). See also van den Bremer and van der Ploeg (2021).

higher macroeconomic uncertainty whether stemming from geometric Brownian motion or from negative climate-related disaster shocks lower the discount rate and increase the SCC.

It is instructive to decompose the discount rate (23) when disasters follow a power distribution:

$$r^\star = \underbrace{\rho}_{\substack{\text{pure time}\\ \text{preference}}} \quad \underbrace{+\eta g^{(0)}_{\text{total}}}_{\substack{\text{affluence}\\ \text{effect}}} \quad \underbrace{-g^{(0)}_{\text{total}}}_{\substack{\text{growing}\\ \text{damages}}} \quad \underbrace{-\frac{1}{2}(1+\eta)\gamma\sigma^2_{\text{total}}}_{\substack{\text{prudence}\\ \text{effect}}} \quad \underbrace{+\gamma\sigma^2_{\text{total}}}_{\substack{\text{insurance}\\ \text{effect}}}, \tag{24}$$

where $g^{(0)}_{\text{total}} \equiv g^{(0)} - \lambda_e/(1+\beta_e) - \lambda_c/(1+\beta_c)$ is the total expected growth rate, corrected for the expected effect of macroeconomic and climate-related disasters, and $\sigma^2_{\text{total}} \equiv \sigma^2 + 2\lambda_e/((1+\beta_e)(1+\beta_e-\gamma)) + 2\lambda_c/((1+\beta_c)(1+\beta_c-\gamma))$ can be thought of as a measure of total uncertainty, combining normally distributed shocks with the effects of macroeconomic and climate-related disasters.[10]

The first two terms in (24) correspond to the deterministic Keynes–Ramsey rule: the rate of pure time preference ρ plus the affluence effect $\eta g^{(0)}_{\text{total}}$. The affluence effect captures that, if growth and aversion to intergenerational inequality aversion are high, policy makers prefer to postpone climate action to periods if people are richer.[11] The third term, $-g^{(0)}_{\text{total}}$ corrects the discount rate for the growth rate of the economy and growing damages (as damages are proportional to output), so that a higher growth rate increases the SCC. Together, the second and the third terms act to increase the discount rate and reduce the SCC if intergenerational inequality is high enough (i.e, $\eta > 1$).

The fourth term $-\frac{1}{2}(1+\eta)\gamma\sigma^2_{\text{total}}$ is the prudence term, which indicates that regular macroeconomic uncertainty as well as the risk of macroeconomic and climate-related disasters necessitate precautionary saving, which depresses the interest rate in equilibrium. As a result, the SCC and the carbon price are higher. This effect is larger for higher degrees of relative risk aversion (γ), relative prudence ($1+\eta$) and macroeconomic volatility ($\sigma > 0$, $\lambda_c > 0$ and $\lambda_e > 0$).

The fifth term $\gamma\sigma^2_{\text{total}}$ is the insurance effect, stemming from the perfect correlation between damages and losses due to macroeconomic and climate-related disasters on the one hand and output on the other hand: in future states of nature, damages and losses are high when output and consumption are high and the marginal utility of consumption is low. Policy makers thus employ a higher discount rate and undertake less climate action. This effect is larger for higher degrees of relative risk aversion (γ) and macroeconomic volatility ($\sigma > 0$, $\lambda_c > 0$ and $\lambda_e > 0$).

[10]Note that the coefficient relative risk aversion γ appears in this measure of total uncertainty because of the non-normal nature of these shocks. As β_e and β_c are generally much larger than γ (see Section 5, the effect of γ is quantitatively unimportant, and can conceptually be ignored.

[11]The increased desire to consume and save less requires a higher interest rate for household-investors to willingly hold the safe asset, which is in fixed supply.

4.1.1 The equity premium and the risk-free rate

Without risk of climate tipping, the leading-order estimate of the equity premium is

$$r_p = \gamma\sigma^2 + \sum_{i=e,c} \lambda_i \mathbb{E}\left[(1-Z_i)(Z_i^{-\gamma}-1)\right] \tag{25}$$

and of the risk-free rate is

$$r_f = \rho + \eta g^{(0)} - \frac{1}{2}\gamma(1+\eta)\sigma^2 - \sum_{i=e,c} \lambda_i \mathbb{E}\left[(Z_i^{-\gamma}-1) + \frac{\eta-\gamma}{1-\gamma}(1-Z_i^{1-\gamma})\right]. \tag{26}$$

This result extends well-known expressions in the macro-finance literature to allow for temperature-dependent risk of climate-related disasters. The discount rate for evaluating the SCC, equation (23), can be written as

$$r^\star = r_f + r_p - g^{(0)}_{\text{total}}, \quad \text{where} \quad g^{(0)}_{\text{total}} \equiv g^{(0)} - \sum_{i=e,c} \lambda_i \mathbb{E}[1-Z_i] \tag{27}$$

The discount rate $r^\star$ is the return on risky assets $r_f + r_p$ minus the growth rate that takes account of of expected damages from disasters, $g^{(0)}_{\text{total}}$. While preference parameters are typically unobservable, the risk-free interest rate, the equity premium, and economic growth are observable. Equation (27) offers a market-based calibration strategy for preference parameters and other not directly observable parameters by matching asset-pricing moments and the economic growth rate.

4.2 A rule for the Optimal SCC with climate tipping

To allow for the risk of an irreversible climate tipping point, we will also make use of the first-order value function derived in Section 3.3.2.

4.2.1 Post-tip SCC

The optimal post-tip SCC can simply be obtained from Result 1 and equation (22), by ensuring that post-tip values of the TCRE are used:

$$\overline{P}_{\text{R1}} = \left[D_{1T} + \lambda^c_{1T}\frac{\mathbb{E}\left[1-Z_c^{1-\gamma}\right]}{1-\gamma}\frac{\overline{q}^{(0)}}{\overline{B}^{(0)}}\right]\frac{\overline{\chi}\overline{Y}^{(0)}}{\overline{r}^\star}, \tag{28}$$

where Tobin's Q $\overline{q}^{(0)}$, the capital-output ratio $\overline{B}^{(0)}$, output $\overline{Y}^{(0)}$ and the optimal discount rate $\overline{r}^\star$ must all be evaluated in a post-tipping world (i.e., with $\chi = \overline{\chi}$).

4.2.2 Pre-tip SCC

Result 2. *With temperature-dependent risks of recurring climate-related disasters and of irreversible climate tipping, the pre-tip SCC can be estimated by*

$$P_{\text{R2}} = P_{\text{R1}}\frac{\psi_0}{\psi} + \frac{h'(E)}{r^\star}\frac{Y^{(0)}q^{(0)}}{B^{(0)}}\frac{1}{1-\gamma}\left(\frac{\psi_0-\overline{\psi}_0}{\psi}\right) + \frac{h(E)}{r^\star}\left(\overline{P}_{\text{R1}}\frac{\overline{\psi}_0}{\psi} - P_{\text{R1}}\frac{\psi_0}{\psi}\right), \tag{29}$$

where $h'(E) = h_1$. The post-tip SCC $\overline{P}_{\text{R1}}$ and the pre-tip SCC in the absence of tipping P_{R1} can be obtained from Result 1 and are given by

$$P_{\text{R1}} = \left[D_{1T} + \lambda_{1T}^c\frac{\mathbb{E}\left[1-Z_c^{1-\gamma}\right]}{1-\gamma}\frac{q^{(0)}}{B^{(0)}}\right]\frac{\chi Y^{(0)}}{r^\star}, \qquad \overline{P}_{\text{R1}} = \left[D_{1T} + \lambda_{1T}^c\frac{\mathbb{E}\left[1-Z_c^{1-\gamma}\right]}{1-\gamma}\frac{\overline{q}^{(0)}}{\overline{B}^{(0)}}\right]\frac{\chi\overline{Y}^{(0)}}{\overline{r}^\star}. \tag{30}$$

Here, $q^{(0)} = \frac{1}{1-\varphi i^{(0)}}$ and $\overline{q}^{(0)} = \frac{1}{1-\varphi\overline{i}^{(0)}}$ denote Tobin's Q before and after tipping, respectively.

Proof: Using perturbation analysis, it can be shown that $\epsilon\psi_1 = h(E)\frac{\overline{\psi}_0(E)-\psi_0(E)}{r^*}$ (see Section 3.3.2) and thus $\psi(E) = \psi_0(E) + h(E)\frac{\overline{\psi}_0(E)-\psi_0(E)}{r^\star}$, where $\psi(E)$ is defined by $J \equiv \psi(E)K^{1-\gamma}$. The optimal SCC follows from evaluating $P = -\frac{J_E}{J_K}q = -\frac{\psi'(E)}{1-\gamma}qK$. We obtain[12]

$$P_{\text{R2}} = -\left[\psi_0'(E) + h'(E)\frac{\overline{\psi}_0(E)-\psi_0(E)}{r^\star} + h(E)\frac{\overline{\psi}_0'(E)-\psi_0'(E)}{r^\star}\right]\frac{q^{(0)}K}{1-\gamma}\frac{1}{\psi}. \tag{31}$$

Note that we have additionally assumed that the risk- and growth-adjusted discounted rate, $r^\star$, and Tobin's Q, $q^{(0)}$, are unaffected by marginal changes in the stock of cumulative emissions, E, which is approximately the case in all our numerical simulations. We can rewrite (31) as (29), using $P_{\text{R1}} = -\frac{\psi_0'(E)}{\psi_0(E)(1-\gamma)}q^{(0)}K$ and $\overline{P}_{\text{R1}} = -\frac{\overline{\psi}'(E)}{\overline{\psi}(E)(1-\gamma)}\overline{q}K$ without risk of climate tipping. Result 1 then establishes that the latter two prices are given by equation (30). Further details are given in Appendix A. □

If there is a risk of climate tipping ($h(E) > 0$), the anticipated future damages caused by a future upward jump in the transient climate response to cumulative emissions (TCRE) and the resulting higher temperatures need to be internalised.[13] To be precise, the carbon price needs to be adjusted to mitigate the risk of a climate tipping point (second term in the square brackets in equation (29)), and carbon needs to be repriced to allow for the anticipated effect that the TCRE jumps up in the future (third term in the square brackets in equation (29).

[12]Strictly speaking, the second term in the square brackets in (31) is $\mathcal{O}(\epsilon^2)$, whereas the first and third terms are $\mathcal{O}(\epsilon^1)$; the latter corresponds to the order to which our result is valid formally. Similarly, $1/\psi$ terms should formally be expanded in powers of ϵ with terms of $\mathcal{O}(\epsilon^2)$ and above neglected. We nevertheless retain both these types of higher-order terms in Result 2, as they are readily available, make Result 2 easier to interpret, and we do not foresee other additional terms arising at $\mathcal{O}(\epsilon^2)$ from Result 1. Furthermore, our full numerical optimisation in Section 6 confirm their inclusion improves the accuracy of our rule.

[13]If the arrival rate of climate tipping is zero, $h(E) = 0$, equation (29) reduces back to Result 1 because then ψ_1 and $\psi = \psi_0$.

The second term $\frac{h'(E)}{r^\star}\frac{Y^{(0)}q^{(0)}}{B^{(0)}}\frac{1}{1-\gamma}\left(\frac{\psi_0-\overline{\psi}_0}{\psi}\right)$ is positive if the hazard rate of tipping increases with temperature. Since the economy is worse off after the irreversible climate tipping point, we have $\overline{\psi}_0 < \psi_0$.[14] The term captures the expected marginal loss from having one degree higher temperature. This term implies that a larger dependence of the risk of irreversible climate tipping on temperature h_{1T}, a higher TCRE χ, combining as $h'(E) = h_{1T}\chi$, and a larger risk-adjusted loss to welfare $\frac{1}{1-\gamma}\left(\frac{\psi_0-\overline{\psi}_0}{\psi}\right) > 0$ resulting from the tip increase the optimal SCC. Just as for the risk of recurring climate-related disaster risks, this effect is more pronounced if the economy is more risk averse (as captured by the $\frac{1}{1-\gamma}\left(\frac{\psi_0-\overline{\psi}_0}{\psi}\right)$ term).

The third term $\frac{h(E)}{r^\star}\left(\overline{P}_{R1}\frac{\overline{\psi}_0}{\psi} - P_{R1}\frac{\psi_0}{\psi}\right)$ allows for the risk of climate tipping itself. It is equal to the tip arrival rate multiplied by the gap between the post-tip and the pre-tip SCC weighted by pre- and post-tip welfare, respectively, discounted at the pre-tip risk-adjusted discount rate. This term prices in that at some uncertain date in the future, the TCRE (χ) and thus damages jump up. This pushes up the optimal SCC (independent of whether the hazard rate of the climate tipping point is dependent on temperature and thus endogenous or not).

5 Market-Based Calibration

Our calibration strategy builds upon Pindyck and Wang (2013), van den Bremer and van der Ploeg (2021), and Hambel et al. (2022). In the past, the influence of climate change on asset markets was negligible and the historical impact of climate change on the economy was, if anything, moderate, at least in developed countries. Thus, we first calibrate the economic part of our model by disregarding climate damages (business as usual, BAU). We then discuss the climate parts. The resulting market-based calibration is summarised in Table 1. Section 6.3 considers ethics-based calibrations with a lower discount rate to capture that policy makers are more patient than the market.

5.1 Macroeconomic uncertainties

For the normal macroeconomic uncertainty we assume an annual volatility of 2% ($\sigma = 2\%/\text{year}^{1/2}$), matching the historical volatility of consumption (cf. Wachter, 2013). For the recurring macroeconomic and climate-related disasters, we assume that the recovery rates, $Z_i = 1-\ell_i$, $i \in \{e, c\}$, have power distributions over $(0,1)$ with parameters $\beta_i > 0$. The jump size distribution is thus determined by the density function $\zeta_i(Z_i) = \beta_i Z_i^{\beta_i - 1}$, $Z_i \in (0,1)$ (cf. Pindyck and Wang 2013), so that the n^{th} moment of the recovery rate is $\mathbb{E}[Z_i^n] = \frac{\beta_i}{\beta_i+n}$. We follow Barro and Jin (2011) and define a disaster as an event that destroys more than $\ell_e^\star = 10\%$ of the capital stock, GDP and aggregate consumption. With this cut-off, their historical consumption data suggest an annualdisaster probability of 3.8% and average loss of 20% when a disaster strikes: $\mathbb{E}[\ell_e|\ell_e > \ell_e^\star] = 0.2$ and $\lambda_e \int_0^{1-\ell_e^\star} \zeta_e(Z_e)\mathrm{d}Z_e = 3.8\%/\text{year}$. These values give coefficients $\beta_e = 8$ and $\lambda_e = 8.8\%/\text{year}$.

[14]Note that $\psi^\star \equiv (1-\gamma)\psi > 0$, and $\psi_0 > \overline{\psi}_0$ as welfare after climate tipping is lower, so that this term is indeed positive ($1-\gamma < 0$, $(\psi_0 - \overline{\psi}_0)/\psi < 0$).

For the temperature-related risk of rare climate-related disasters, we assume that the intensity of disasters rises linearly in temperature with $\lambda_0^c = 0.3\%$/year and $\lambda_{1T}^c = 9.6\%$/year/°C and that the expected loss is 1.5% and thus $\mathbb{E}[1 - Z_c] = 0.015$ (cf. Karydas and Xepapadeas, 2022 and Hambel et al., 2022 for more details). Fitting a power distribution, we obtain $\beta_c = 65.7$. To put those numbers into perspective, consider an initial temperature anomaly of $T_0 = 1.1$°C. This implies that the initial one-year probability of a climate disaster with an average damage of 1.5% of the capital stock ($17.25 trillion) is $1 - \exp(-(0.003 + 0.096\,T_0) \times 1.0) \approx 10.3\%$, so that on average every 9.7 years a climate disaster hits the economy and the expected annual damage to the capital stock is $1.78 trillion per year.[15] At 2 degrees Celsius, the one-year probability of a climate disaster increases to 17.7%, and a climate-related disaster strikes every 5.64 years. Note that climate-related disasters occur more frequently than non-climate-related macroeconomics disasters,[16] but tend to have a smaller negative impact on the economy when they strike.

5.2 Preferences and production

We follow van den Bremer and van der Ploeg (2021) and use an energy share of $1 - \alpha = 4.3\%$ and an energy cost parameter of $b = \$540$ per ton of carbon. Given an initial world GDP of $Y^{(0)} = \$115$ trillion/year, this corresponds in a business-as-usual scenario with $P = 0$ to an initial fossil use of $F_0 = 115 \times 0.043/0.54 = 9.16$ GtC/year (giga tons of carbon per year). We set the coefficient of intergenerational inequality aversion $\eta = 1.5$ corresponding to an elasticity of intertemporal substitution of 2/3, which is close to the value in the DICE model. There is an ongoing debate in the asset pricing literature on this parameter. E.g., Vissing-Jørgensen and Attanasio (2003), combine equity and consumption data and estimate an EIS ($1/\gamma$) of 1.5. On the other hand, Hall (1988) and estimate an EIS well below one.

Given these parameter choices, we calibrate the remaining parameters to match expected GDP growth $g^{(0)} = 2\%$ per year in normal times, so that without rare macroeconomic disasters we have an average consumption rate of $\frac{C^{(0)}}{B^{(0)}K} = 73\%$ of GDP, a risk-free interest rate of $r_f = 0.8\%$/year, an equity premium of 6.5%/year, a return on risky assets of 7.3%/year, and a Tobin's Q of $q^{(0)} = 1.38$ (cf. Pindyck and Wang, 2013; Hambel et al., 2021a). They imply that the growth- and risk-adjusted discount rate needed to calculate the SCC is $r^{\star} = 5.3\%$/year. Without climate change (as used for the zeroth-order approximation), we can obtain closed-form expressions for these quantities (see Appendix B) to pin down the remaining preference and production parameters: $\rho = 5.08\%$/year, $\eta = 5.347$, $A^* = 0.1231$, and $\varphi = 12.5$ (see Table 1).

[15] UN (2020) reports that extreme major weather events have roughly doubled from 1980-1999 to 2000-2019 (from 4,212 to 7,348 events) with a global economic cost of $1.63 and $2.97 trillion, respectively. So, our calibration is within this range. For further and more detailed discussion on the evidence of global warming on the frequency of climate-related events (river floods, tropical cyclones, crop failure, wildfires, droughts, and heatwaves), see Lange et al. (2020).

[16] For example, at 2 degrees Celsius $\lambda_c = \lambda_{T0}^c + \lambda_{T1}^c T = 19.5\%/\text{year} > \lambda_e = 8.8\%/\text{year}$. This is true for all $T > 0.89$°C.

Preferences	Coefficient of relative risk aversion: $\gamma = 5.347$ Coefficient of intergenerational inequality aversion: $\eta = 1.5$ Elasticity of intertemporal substitution: $1/\eta = 0.6667$ Pure rate of time preference: $\rho = 5.08\%$/year
Economy	Initial world GDP (2021): $Y_0^{(0)} = \$115$ trillion/year Initial world capital stock (2021): $K_0 = \$1150$ trillion Total factor productivity: $A^* = 0.1231$ and $B(0,0) = 0.1$ Adjustment cost parameter: $\varphi = 12.5$
Macroeconomic uncertainties	*Capital stock:* Growth in normal times: $\overline{g}^{(0)} = 2\%$/year Annual Volatility: $\sigma = 2\%/\text{year}^{1/2}$ *Macroeconomic disasters (not climate related):* Arrival rate of disasters: $\lambda^e = 8.8\%$/year Mean size of disasters: $\mathbb{E}[\ell_e] = 20\%$ Shape parameter of power distribution: $\beta^e = 8$ *Macroeconomic disasters (climate related):* Mean size of disasters: $\mathbb{E}[1 - Z^c] = 1.5\%$ Shape parameter of power distribution: $\beta^c = 65.7$ Arrival rate of disasters: $\lambda_{0T}^c = 0.3\%$/year and $\lambda_{1T}^c = 9.6\%$/year/°C
Fossil fuel	Initial global emissions in BAU scenario (2021): $F_0 = 9.16$ GtC/year Share of fossil fuel in value added: $1 - \alpha = 4.3\%$ Cost of fossil fuel: $b = 540$ \$/tC
Temperature	Initial temperature (2021): $T_0 = 1.1$°C Transient climate response to cumulative emissions (pre-tip): $\chi = 1.8$°C/TtC Initial cumulative emissions (2021): $E_0 = 0$ GtC Marginal effect of temperature: $D_{1T} = 0.9\%$/°C
Climate tipping point	Arrival rate of tipping point: $h_{0T} = 0$ and $h_{1T} = 0.6\%$/°C/year Transient climate response to cumulative emissions (post-tip): $\overline{\chi} = 2.5$°C/TtC

Table 1: Market-Based Calibration.

5.3 Climate system and global warming damages

For the climate tipping point, we assume that the TRCE jumps from $\chi = 1.8$°C/TtC before to $\overline{\chi} = 2.5$°C/TtC after the tip.[17] The instantaneous probability of tipping rises linearly in temperature with $h_0 = 0$ and $h_{1T} = 0.6\%$/°C/year. Initial temperature $T_0 = 1.1$°C in 2021. Cumulative emissions are measured relative to the total cumulative emission stock of 611.1 GtC in 2021 emitted since the beginning of the industrial revolution (consistent with $\chi = 1.8$°C/TtC), so that $E_0 = 0$.

Nordhaus and Moffat (2017) perform a meta study of global warming damages and come up with a preferred estimate of 0.18% of output per temperature squared (i.e., $D = 0.0018T^2$), to which they add a 25% for unmeasured damages. Hence, their damage function $D_{NM17} = 0.00225T^2$, and the losses due to global warming at 1.5, 2, and 2.5 degrees Celsius equal $D(T = 1.5°C) = 0.51\%$, $D(T = 2°C) = 0.9\%$, and $D(T = 2.5°C) = 1.41\%$, respectively. We

[17] These values are well in line with the findings of Allen et al. (2009) and Matthews et al. (2009).

calibrate our damage function so that it fits the slope of the damage function of Nordhaus and Moffat (2017) (i.e., $\mathrm{d}D_{NM17}/\mathrm{d}T = 0.0045T$) at a temperature increase of 2°C, setting $D_0 = 0$ and $D_{1T} = 0.9\%/°C$.

6 How Accurate is the Rule for the Optimal SCC?

To test the accuracy of our approximate rules for the optimal SCC given by Result 1 and Result 2, we use the market-based calibration in Table 1 and our rules to calculate an estimate of the optimal SCC. Section 6.1 compares this with a SCC obtained from full numerical optimisation of the model based on a grid-based finite-difference approach (cf. Munk and Sørensen (2010) and see Appendix E).[18] Section 6.2 shows how sensitive the accuracy of our rules for the optimal SCC is with respect to our small-parameter assumptions, relating to the strength of each of the three global warming externalities. Section 6.3 acknowledges that the market-based calibration leads to a very low SCC and investigates the size of the optimal SCC and the accuracy of our rules if policy makers are more patient than private-sector agents. Such an ethics-based calibration also offers a more challenging test of our simple rules.

6.1 Accuracy of rule for optimal SCC with market-based calibration

6.1.1 The SCC in 2021

Table 2 summarises the results for six different settings using the market-based calibration in Table 1. We find that errors of the simple rule compared to the full numerical optimisation are small. We note that values of the SCC are relatively small, which is due to the relatively high discount rates that follow from the market-based calibration (cf. values of $r^\star$ in Table 2).

Table 2, panel (a) indicates that the accuracy of the simple rule in Result 1 is excellent in models without a climate tipping point; the deviation between our estimate of the optimal SCC from our simple rule and the numerical solution is always less than 0.7%. The error is higher if the rule also takes account of the temperature-related risk of recurring macroeconomic disasters, but still negligible for practical purposes. Table 2, panel (b) shows the analogous results when there is a temperature-related risk of an irreversible climate tipping point. The risk of climate tipping calls for a higher price of carbon, which is in line with the detailed analysis of Dietz et al. (2021a), who find that, collectively, climate tipping points increase the SCC by about a quarter (our effect is smaller in magnitude). With climate tipping, our estimate of the optimal SCC based on the simple rule in Result 2 is somewhat less accurate than without tipping, although the deviation between the simple rule and the fully nonlinear numerical solution is still very small. Our estimate of the optimal SCC when all the three global

[18] In Appendix E, we show that by eliminating K, the problem can be reduced to a numerical optimisation problem in a single state variable, E, and time. We solve this using 4 time steps per year and 100 nodes for E. More time steps or a finer grid do not increase the accuracy of the numerical results, and we can conclude the numerical solutions have converged.

(a) Without a climate tipping point

Method of calculation	Rule ($/tCO$_2$)	Numerical ($/tCO$_2$)	Error	$r^\star$ (%/year)
TFP damages only	9.60	9.60	−0.04%	5.30%
Temperature-related recurring disasters only	23.53	23.73	−0.85%	5.23%
both TFP damages and temperature-related disasters	33.17	33.40	−0.69%	5.23%

(b) With a climate tipping point

Method of calculation	Rule ($/tCO$_2$)	Numerical ($/tCO$_2$)	Error	$r^\star$ (%/year)
TFP damages only	10.33	10.62	−2.72%	5.29%
Temperature-related recurring disasters only	26.41	26.35	−0.23%	5.22%
both TFP damages and temperature-related disasters	36.67	37.12	−1.21%	5.22%

Table 2: Calculation of the Optimal SCC ($/tCO$_2$) using the Market-Based Calibration. Results are based on the market-based calibration summarised in Table 1 and compare the performance of the rule to the numerical solution. The table also shows how the three climate externalities affect the growth- and risk-adjusted discount rate $r^\star$, compared to the value of 5.30%/year in the absence of climate change.

warming externalities are simultaneously present leads to an error in our estimate of the optimal SCC of only −1.21%.[19]

The final columns of Table 2 indicate that the growth- and risk-adjusted discount rate, $r^\star$, falls, especially as a result of recurring climate-related disasters. This is due to precautionary savings, and it further boosts the increase in the SCC.

6.1.2 The SCC as a function of temperature

Figure 1 shows how higher temperatures than our starting temperature of 1.1°C affect the accuracy of the rules for the optimal SCC. Grey lines (——) show the results obtained by the simple rules while black lines (——) show the results obtained from the full numerical optimisation. Panel (a) shows the results when global warming only affect total factor productivity through the damage function. Panel (b) shows the results when temperature-related recurring climate-related disasters also occur. Panel (c) shows the results when all three global warming externalities are present.

The optimal SCC increases with temperature in each of the three panels even though the marginal impact of temperature on output D_{1T} and the marginal effect of temperature on the risk of recurring climate-related disasters λ^c_{1T} and on the risk of a climate tipping point h_{1T} are constant. The reason for this is that the discount rate $r^\star$ given in equation (23) reacts negatively to higher temperatures due to the higher risks of climate-related disasters, $\lambda_c(T(E))$, provided that the elasticity of intertemporal substitution is less than one (and the coefficient of intergenerational inequality aversion $\eta > 1$, as is the case in the calibration in Table 1). Hence, the effect of

[19]We note that all our estimates of the SCC slightly under-estimate the SCC compared to the full numerical optimisation. If needed, higher-order terms in the perturbation series could be derived to reduce the error margins.

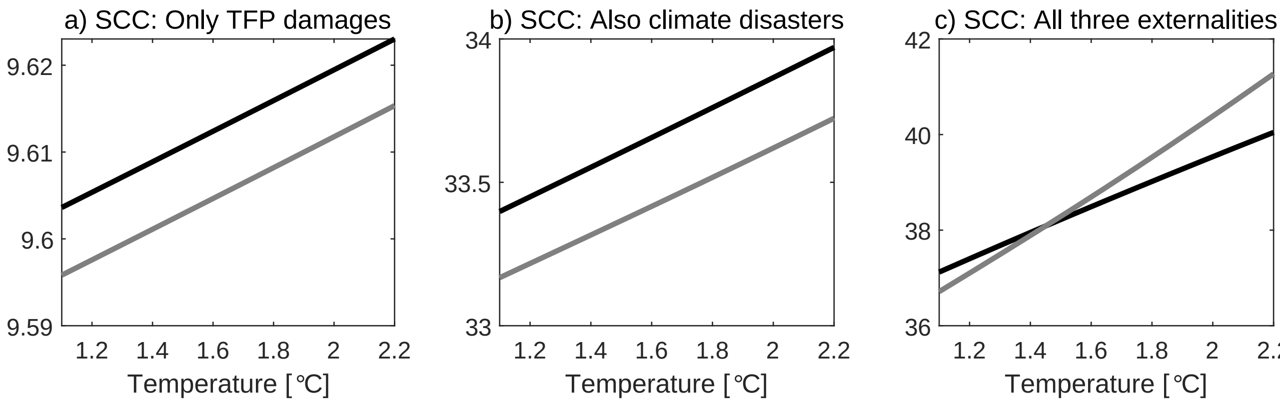

Figure 1: Temperature Sensitivity of Optimal SCC (\$/tCO_2) with Market-Based Calibration. The graphs depict the sensitivity of the optimal SCC with respect to temperature. Grey lines (——) show the results obtained by the simple rules while black lines (——) show the results obtained from full numerical optimisation. Panel (a) shows the results when global warming only affects total factor productivity. Panel (b) shows the results when there is also a temperature-related risk of recurring climate-related disasters. Panel (c) shows the results when all three global warming externalities are present.

higher temperatures on the optimal SCC must be positive, as is confirmed in each of the panels of Figure 1. Notice that the optimal SCC and temperature would be negatively related in models with an elasticity of intertemporal substitution greater than one (i.e., $\eta < 1$). Table 2 and Figure 1 also indicate that the temperature-related risk of recurring climate-related disasters significantly boosts the SCC (roughly by a factor 2.5), much more so than the temperature-related risk of an irreversible climate tipping point.

6.2 Testing the small-parameter assumptions

Our perturbation analysis gives an accurate estimate of the optimal SCC when our three small parameters are 'small'. How small is 'small'? To test whether the small parameters are indeed small enough, we examine how the accuracy of the rule varies with the size of the three small parameters. Figure 2 does this for the first two small parameters, i.e., the marginal effect of temperature on the damage ratio D_{1T} and the marginal effect of temperature on the intensity of climate-related macroeconomic disasters λ^c_{1T}, by showing how the optimal SCC, both evaluated with our simple rule and with numerical optimisation, varies with these two parameters (without risk of a climate tipping point, $h = 0$). Given an initial temperature anomaly of $T_0 = 1.1°$C, the benchmark value $\lambda^c_{1T} = 9.6\%$/year/°C means that a climate disaster happens every 9.7 years, while a value of $\lambda^c_{1T} = 19.2\%$/year/°C (double) means that such an event happens every 5.2 years on average. The left-hand panel confirms that a higher adverse effects of temperature on total factor productivity D_{1T} and on the risk of recurring disasters λ^c_{1T} increase the SCC according to our rule. The right-hand panel shows that the relative error compared with full numerical optimisation varies from close to zero to at most -1.5%. The rule thus performs well for a realistic range of the size of the first two small parameters with the market-based calibration.

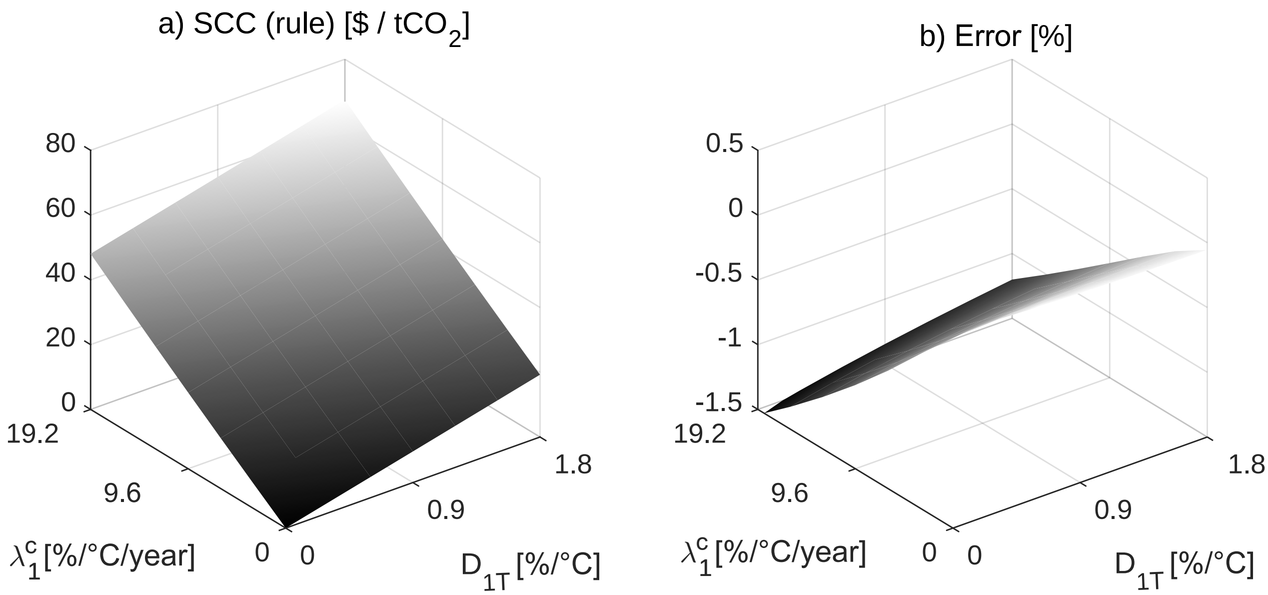

Figure 2: Sensitivity of the SCC and Accuracy without Climate Tipping Risk ($h = 0$)**.** Panel (a) illustrates how the marginal effect of temperature on the damage ratio D_{1T} and the marginal effect of temperature on the intensity of climate-related macroeconomic disasters λ_1^c affect the optimal SCC if temperature is fixed at $T_0 = 1.1$°C. Note that the benchmark calibration values are $D_{1T} = 0.9\%/°\text{C}$ and $\lambda_1^c = 0.3 + 9.6 \times 1.1 = 10.9\%$/year. Panel (b) illustrates the relative deviation between the simple rule and the numerical optimum as a function of these parameters.

We also consider the accuracy of our rule when there is a risk of a climate tipping point. Figure 3 is the same as Figure 2 but with $h_{1T} = 0.6\%/°\text{C/year}$. We confirm again that the optimal SCC is higher, the more sensitive the adverse effects of global warming on total factor productivity and on the risk of recurring climate-related disasters are to temperature (D_{1T} and λ_{1T}^c, respectively). The main difference is that the error compared to full numerical optimisation is a little higher, up to -2% (instead of the -1.21% reported in Table 6.1).

Now we test the accuracy of our rule with respect to the third small parameter: the hazard rate of climate tipping h. To do so, we vary the marginal effect of temperature on this hazard rate, h_{1T}. Figure 4, panel (a) shows how varying the size of this parameter impacts the optimal SCC. For this purpose, we consider the benchmark model with all three externalities and vary h_{1T}. The figure shows that a higher intensity of climate tipping points increases the SCC almost linearly according to our rule as shown in the grey line (——), where the apparently quadratic error reflects the validity of our estimate for the SCC up first order in the small parameter, resulting in second-order (quadratic) errors. The figure also suggests that the true SCC obtained numerically (——) reacts in a more concave manner to the tipping probability even if this probability is linear in temperature. Panel (b) shows that the relative error compared with numerical optimisation varies from less than 1% if tipping is switched off to about 2% if h_{1T} is double that in the benchmark calibration. The rule thus performs very well.

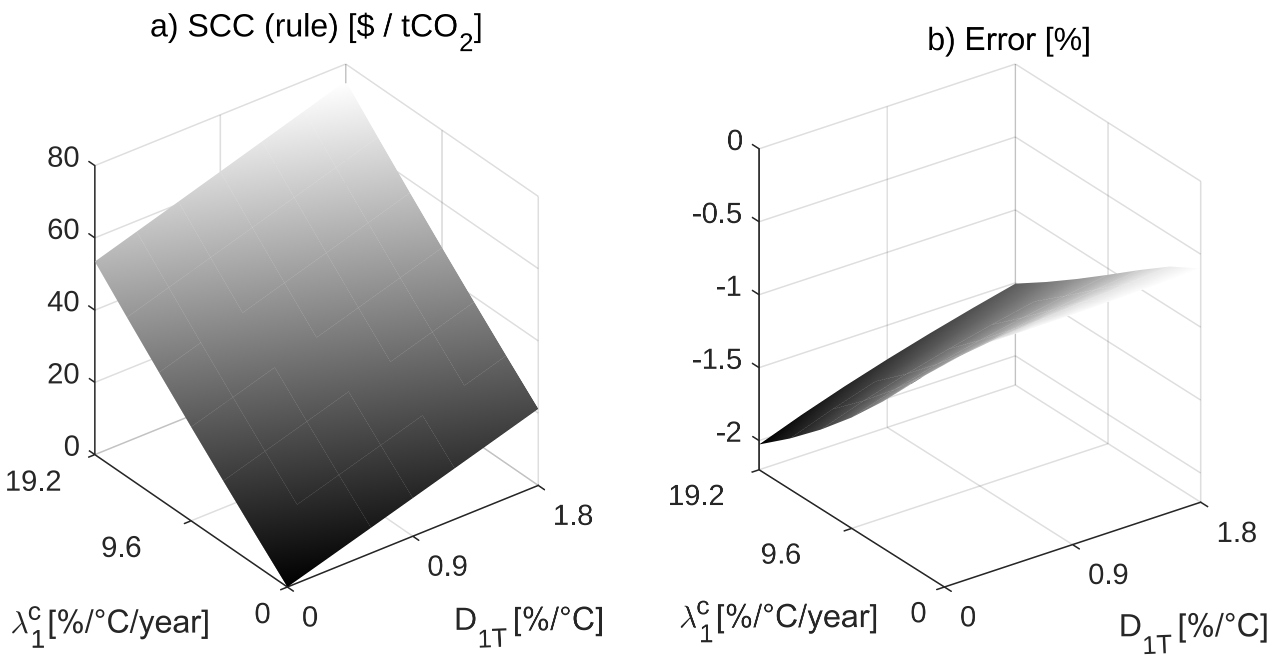

Figure 3: Sensitivity of the SCC and Accuracy with Climate Tipping Risk (h_{1T} = 0.6%/°C/year). Panel (a) illustrates how the marginal effect of temperature on the damage ratio D_{1T} and the marginal effect of temperature on the intensity of climate-related macroeconomic disasters λ_{1T}^c affect the optimal SCC (for $T_0 = 1.1$°C). Panel (b) illustrates the relative deviation between the simple rule and the numerical optimum as a function of the same parameters.

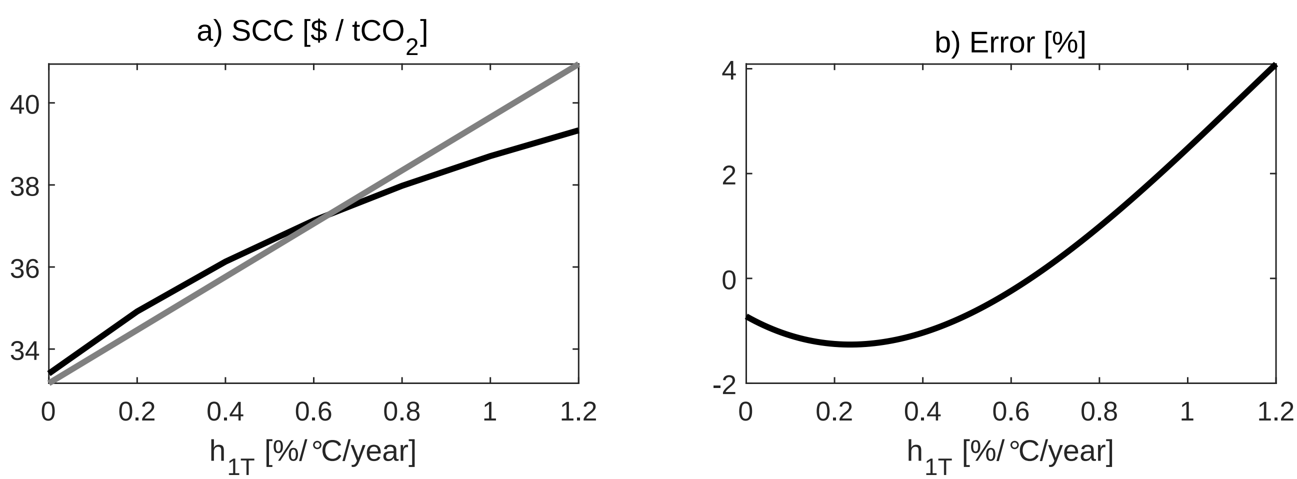

Figure 4: Tipping Risk Sensitivity of the Optimal SCC (D_{1T} = 0.9%/°C, λ_{1T}^c = 9.6%/°C/year). Panel (a) illustrates how the marginal effect of temperature on the intensity of climate tipping h_{1T} affects the optimal social cost of carbon (for $T_0 = 1.1$°C). The benchmark value is h_{1T} = 0.6%/°C/year. The black line (——) show the numerical solution and the grey line (——) the SCC determined by Result 2. Panel (b) illustrates the relative deviation between the simple rule and the numerical optimum as a function of h_{1T}.

6.3 Ethics-based calibration with lower discount rates

So far, we have studied a market-based calibration of our integrated assessment model that matches the equity premium r_p in equation (25), the return on safe assets or risk-free rate r_f in equation (26), the return on risk assets $r_f + r_p$, and the rate of economic growth. This leads to the growth- and risk-adjusted discount rate (23), to be used for calculating our estimate of the optimal SCC from equation (22) (Result 1) or (42) (Result 2).

However, a market-based calibration may not coincide with the preferences of a government, which are more likely to be based on ethical or, alternatively, on political-economy considerations. For example, our calibration has a rather high pure rate of time preference of $\rho = 5.08\%$/year. In contrast, Stern et al. (2006) follows Frank Ramsey and argue that discounting the utility of future generations is unethical and, therefore, an almost zero pure rate of time preference should be used. Here, we analyse what happens to the optimal SCC if we follow a more ethical approach with a lower utility discount rate of $\rho = 2.27\%$/year, corresponding to a growth- and risk-adjusted discount rate of $r^\star = 3\%$/year, and an even lower utility discount rate of $\rho = 1.06\%$/year, corresponding to $r^\star = 2\%$/year, instead of the market-based $\rho = 5.08\%$/year given in Table 1 and the corresponding $r^\star = 5.3\%$/year. These ethics-based choices of the utility discount rate assume that policy makers are more patient than the private sector.[20]

Table 3 reports our estimates of the optimal SCC for these two lower discount rates. Our conclusions are as follows. First, our simple rule for the SCC, if we allow for all three types of global warming externalities, gives a SCC of \$91/tCO_2 if $r^\star = 3\%$/year and \$182/tCO_2 if $r^\star = 2\%$/year. Second, with the lower discount rates, Table 3 indicates that the effect of recurring climate disasters is even more substantial compared to the effect of the climate tipping point. Third, unsurprisingly, the numerical errors increase for lower discount rates, although errors are still at most -3.3% compared to the numerical optimum.

To analyse the effects of the three types of global warming externalities and the risk-adjusted discount rate $r^\star$ on the optimal SCC over time, Figure 5 plots the evolution of the optimal SCC in nine scenarios. The three columns refer to the three different values of $r^\star$, and the three rows to the three types of climate externalities. In the first row, there are only damages to TFP, in the second row there is also the risk of recurring climate-related disasters, and the last row shows the SCC with all three types of externalities. The black lines (——) depict the median evolution of the SCC in the respective scenario, the dashed lines (- - -) the 5% and 95% quantiles, and the dotted lines (⋯⋯) show a particular sample path.[21] Comparing the first and the second column, it is apparent that recurring climate disasters give the SCC a boost. In contrast, the tipping point provides a relatively small additional contribution to the median evolution of the SCC. However, if the tipping risk materialises in a particular path, as shown in (⋯⋯), then the SCC experiences a significant jump upward of about 25%. The result holds regardless of the discount rate, but the SCC decreases sharply in $r^\star$. It is also shown that for lower

[20] Barrage (2018) shows that the government then needs two instruments to achieve the first-best outcome: a carbon tax to correct for the global warming externalities and a capital subsidy to correct for private-agent savings being sub-optimally low. See also van der Ploeg and Rezai (2022).

[21] All paths are simulated using the same random numbers in all nine scenarios.

(a) Benchmark calibration with market annual discount rate of $r^\star = 5.3\%$/year

Method of calculation	Rule ($/t$CO_2$)	Numerical ($/t$CO_2$)	Error
TFP damages only	9.60	9.60	−0.04%
TFP damages and recurring climate disasters	33.17	33.40	−0.69%
TFP damages, climate disasters, and climate tipping	36.67	37.12	−1.21%

(b) With lower annual discount rate of $r^\star = 3\%$/year

Method of calculation	Rule ($/t$CO_2$)	Numerical ($/t$CO_2$)	Error
TFP damages only	17.01	17.06	−0.28%
TFP damages and recurring climate disasters	75.78	77.26	−1.91%
TFP damages, climate disasters, and climate tipping	90.67	91.62	−1.04%

(c) With even lower annual discount rate of $r^\star = 2\%$/year

Method of calculation	Rule ($/t$CO_2$)	Numerical ($/t$CO_2$)	Error
TFP damages only	25.47	25.63	−0.63%
TFP damages and recurring climate disasters	139.19	143.88	−3.26%
TFP damages, climate disasters, and climate tipping	181.87	179.50	1.32%

Table 3: Ethics-Based Calculation of Optimal SCC (US $/t$CO_2$) with Lower Discount Rates. Apart from the lower choice of utility discount rates ρ in panels (b) and (c) (i.e., $\rho = 2.27\%$ and 1.06%, respectively, instead of 5.08% per year), to ensure that $r^\star$ equals 3%/year and 2%/year, respectively, results are based on the calibration summarised in Table 1.

discount rates the 5% quantile of SCC increases faster than in the market-based calibration with the higher utility discount rate.

7 Road-Testing the SCC Rule in Different Models

Here we conduct two more challenging tests of the robustness of our simple rule by checking how well it performs compared to the true numerical optimum in models that are different from what we have used to derive our rule. We investigate the robustness of our rule when there is technical progress in the production of fossil fuel and when the economy has two sectors, a green and a carbon-intensive one. Neither of these alterations of our economic model affects our rule for the optimal SCC except through its effects on aggregate economic activity (see Appendix D). We also extend our rule to allow for shocks to the damage ratio and show that this gives accurate predictions of the optimal SCC.

7.1 Technical progress in fossil fuel extraction

The model we have used in the previous sections to derive the optimal SCC assumes a constant cost of fossil fuel in terms of units of final goods, b. Here, we assume that this cost b drops by 2% per year due to technical

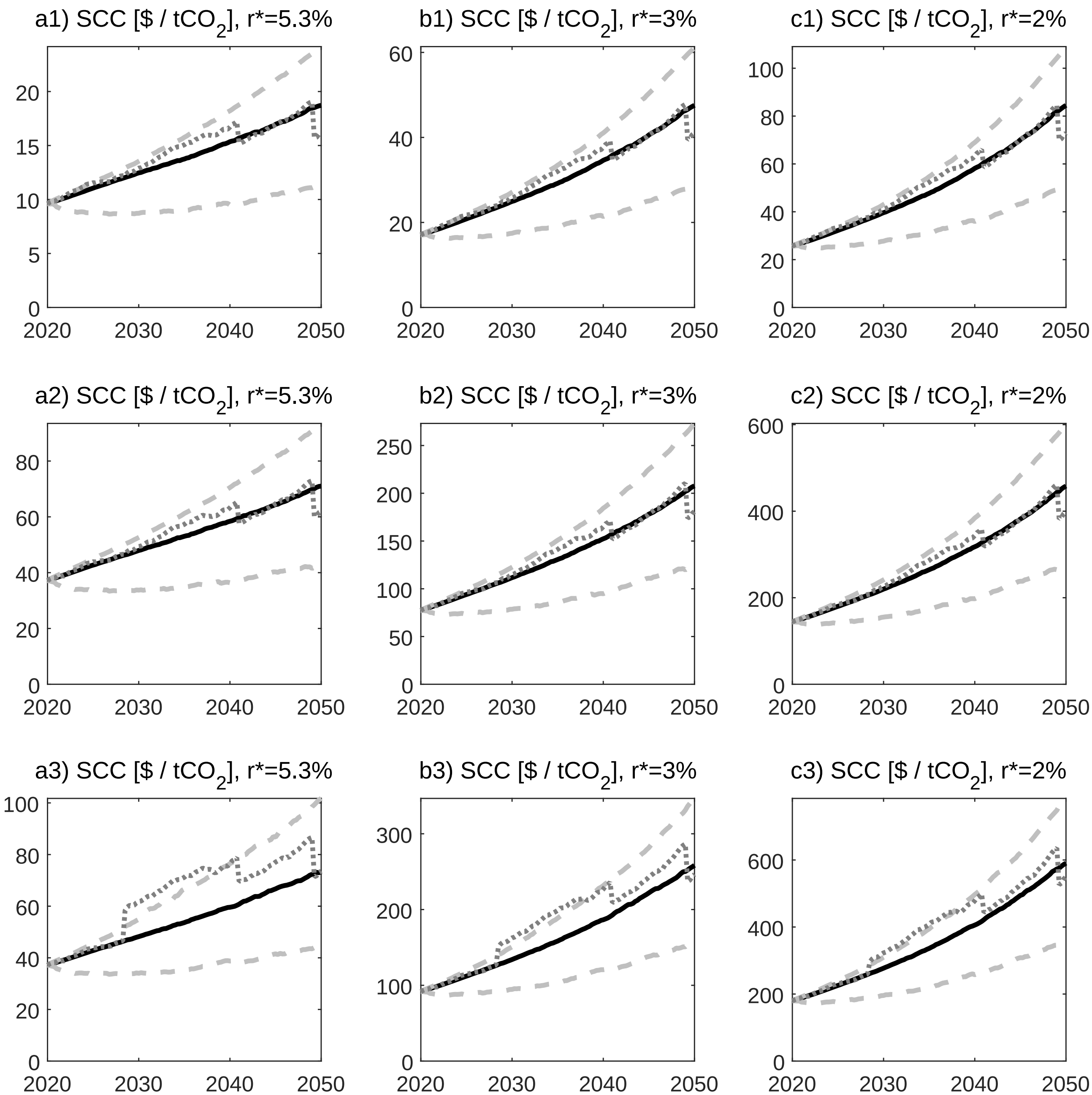

Figure 5: Simulation of the Optimal SCC. Column a) refers to the market-based calibration. Columns b) and c) refer to the ethics-based choices of utility discount rates. Row 1 considers the case with TFP damages only, row 2 considers TFP damages and recurring climate disasters, and row 3 considers all three externalities. The black lines (——) depict the median evolution of the optimal SCC, the dashed lines (- - -) give 5% and 95% quantiles, and the dotted lines (⋯⋯) give a sample path.

progress and check whether our rule for the optimal SCC (which depends on b through $Y^{(0)}$, cf. (13)) performs well compared with the numerical optimum. Table 5 in Appendix D indicates that the simple rule for the optimal SCC is a good estimate of the true numerical optimum for each of the three discount rates, but less so if there is a risk of a climate tipping point and the discount rate is low. The rule thus performs reasonably well even if allowance is made for technological progress in fossil fuel production. As can be seen by comparing Table 5 in Appendix D with Table 3, the reason is that the SCC itself is not much affected by technological progress in fossil fuel production.

7.2 Two-sector model of the economy

We also investigate the accuracy of our rule by road-testing it in a model with a carbon-intensive and a green production sector. We use the model of Hambel et al. (2022) which we refer to for technical details. The green sector uses carbon-free energy as input; the carbon-emitting sector deploys fossil fuel, leading to emissions, warming and damages to aggregate output as well as higher risks of climate-related disasters and a climate tipping point. Both energy sources are available at a cost, and the consumption goods produced in the two sectors are perfect substitutes. The economy can reallocate capital from the dirty to the green capital stock which is also costly. From an computational point of view such a model is significantly more complex than the benchmark framework as it is burdened by an additional state variable, which leads to a multiplication of the run time of the solution algorithm. We emphasise that our simple rule is not affected by this issue.

Table 6 in Appendix D indicates that our simple rule works very well in the two-sector economy in most specifications even though it has been derived for the one-sector economy. In case of climate tipping and low discount rates, the errors are a bit larger than in our benchmark specification.We have also tested the rule for the optimal SCC in settings with imperfect substitutes between clean and carbon-intensive final goods (with the consumption goods produced in both sectors aggregated in a CES-consumption bundle). The results indicate that this has an almost negligible influence on the optimal SCC.

7.3 Stochastic shocks to the damage ratio and the TCRE

So far, we have assumed a deterministic value for the marginal effect of temperature on the damage ratio, D_{1T}. If we allow for stochastic shocks to the damage ratio where the distribution of these shocks is skewed and displays mean reversion, we slightly adjust our simple rules along the lines of van den Bremer and van der Ploeg (2021). This new rule is given in Result 3 in C.2 and indicates that stochastic shocks to the damage ratio increase the optimal SCC provided these shocks have a skewed distribution ($\theta > 0$). We find numerically that shocks to the damage ratio (i) indeed increase the optimal SCC, and (ii) do not affect the accuracy of our rule. Moreover, since only the expectation of the TCRE matters, stochastic shocks to the TCRE do not affect the optimal SCC if they are normally distributed.

8 Concluding Remarks

We have presented an integrated assessment model of climate and the economy with a wide range of uncertainties, ranging from regular (normal) macroeconomic shocks and risks of recurring macroeconomic disasters to recurring climate-related macroeconomic disasters and an irreversible climate tipping point whose arrival rates increase with global warming. We have applied perturbation analysis to obtain a tractable and intuitive rule for the optimal SCC that deals with all these uncertainties. This rule provides an estimate of the optimal SCC that is formally exact when three small parameters are infinitesimally small and provides a very accurate estimate of the risk-adjusted SCC compared to a full numerical optimisation for realistic values of these small parameters. The rule internalises three externalities arising from the negative effects of emissions on temperature and production damages and the positive effects of emissions and temperature on the frequency of temperature-dependent recurring climate disasters and the risk of a climate tipping point.

We have compared this rule for the optimal SCC, which can be evaluated on the back of an envelope, with the value we obtain from full numerical optimisation of our integrated assessment model using the method of finite differences for solving the Hamilton–Jacobi–Bellman equation. This comparison makes clear that our rule for the optimal SCC is very accurate for our market-based calibration. Although this rule performs less well with a more ethics-based calibration with lower discount rates, it still provides close estimates of the optimal SCC. A decomposition analysis of the optimal SCC suggests that the temperature-dependent risks of recurring climate-related disasters and of a climate tipping point significantly increase the SCC by a factor of 4 to 6 compared to when only the adverse effects of global warming on productivity are taken account of. Climate tipping and especially recurring climate-related disasters require a much higher price of carbon and a much more ambitious climate policy. We have also shown that the present rule can readily be adapted to allow for the presence of skewed and persistent shocks to the damage ratio. These shocks also require a higher carbon price.

We have also challenged the robustness of our rule for the optimal SCC by road-testing it in more general models for which the rule was not designed. If there is ongoing technological progress in the production of fossil fuel instead of a constant cost of fossil fuel or if there is two-sector instead of one-sector growth with either imperfect or perfect substitution between the carbon-intensive and green final goods, the rule for the optimal SCC still performs remarkably well.

Both the accuracy and robustness of our estimates of the SCC are thus good, especially if the parameters that we assume to be small —- the sensitivities of the damage ratio and the risk of climate-related disasters with respect to temperature and the risk of climate tipping — are indeed small enough, as they are in our calibration, and the discount rate is not too small. If we allowed for higher-order terms in the derivation of the optimal SCC, the accuracy of our rule would improve further.

In future research, we will examine how rules for the optimal SCC perform if extended in various directions.[22] One can extend rules for the optimal SCC given in Results 1 and 2 to allow for *multiple* recurring climate-related disasters, varying from floods, droughts, hurricanes to bush fires. Each of these disasters may have different intensities and distributions. Following Cai et al. (2016) and Lemoine and Traeger (2016) one can also allow for multiple climate tipping points.[23] Further extensions of the rules for the SCC may deal with more general carbon cycle and temperature dynamics, uncertainty in and convexity of the damage ratio, uncertainty in the temperature response, and correlations between the shocks to the climate, damages, and the economic system. These extensions may also deal with nonlinear positive feedback effects in the climate system and with tail risks arising from skewed distributions.

We foresee that AI-based techniques will have an increasing role to play in the derivation of such simple rules, as we have derived in this paper. For example, Friedl et al. (2023) have developed deep-learning based algorithms to derive optimal climate policies in the face of wide-ranging types of economic, climatic and damage uncertainties including tipping points and parametric uncertainty quantification. A novelty is that a Gaussian-based surrogate model is estimated that is used to analyse the social cost of carbon with respect to uncertain model parameters. Our perturbation analysis offers an extension of generalised asset-pricing formulae for the social cost of carbon. In future work, symbolic regression may be used to obtain a rule for the optimal SCC, which is both accurate and can be interpreted along the lines of existing asset-pricing theories of the SCC.[24] Such developments have the potential to provide a step change in our understanding of carbon pricing in uncertain environments.

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[22] See Appendix F and van den Bremer and van der Ploeg (2021) for some such extensions previously considered.

[23] Lenton et al. (2008) and Lenton (2021) have identified nine of these; e.g., melting and disintegration of the Antarctic or Greenland Ice Sheet, melting of the permafrost, or breakdown of the Atlantic meridional overturning circulation.

[24] In a different field of application but using techniques we will foresee will be applied to derive rules for the SCC, Haefner et al. (2023) have recently obtained perturbations of existing wave models to explain rogue waves using large-scale machine learning. They use causal analysis, deep learning, parsimony-guided model selection, and symbolic regression to obtain a symbolic model (i.e., an equation or 'rule') for oceanic rogue waves consistent with the underlying physical processes, both confirming and extending existing theories.

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APPENDICES

A Derivation of the first-order value function

To derive the first-order value function $J^{(1)}$, we begin by noting from the pre-tip HJB equation (10) that the first-order conditions (11) and (12) are unaffected by the climate-tipping term in the HJB equation (the rightmost term in (10)), as this term does not depend directly on the policy variables C and F.

Furthermore, we note that, as a result of the linearity of the dependence of the risk of climate-related recurring disasters λ_c and the damage function D on E, their marginal effects $\mathrm{d}\lambda_c/\mathrm{d}E = \lambda_1^c$ and $\mathrm{d}D/\mathrm{d}E = D_1$ are constant and not a function of E, resulting in $J_E^{(0)}$ (as obtained from (18)) being only a 'slow'[25] function of E. As a result, the first-order 'forcing' term $J_E^{(0)}\omega F^{(0)}$ does not give rise to a first-order value function that is a 'fast' function of E and thus does not yield an additional contribution to the SCC ($P \propto J_E$). We therefore ignore this term.

Inspection of the HJB equation (10) shows that a first-order solution of the form $\epsilon J^{(1)} = \epsilon\psi_1 K^{1-\gamma}$ can be found. Substitution of the series expansion (15) into the HJB equation (10), in which the optimal solutions for the policy variables have already been substituted in from the first-order conditions (11) and (12), and retaining only first-order terms in $h = \mathscr{O}(\epsilon)$ gives:

$$
\begin{aligned}
K^{1-\gamma}(\overline{\psi}_0-\psi_0)+\frac{1}{2}K^{1-\gamma}(-1+\gamma)\gamma\sigma^2\psi_1+K^{1-\gamma}\Delta\lambda\psi_1 K^{1-\gamma}(1-\gamma)\left((i^{(1)}-\omega i^{(0)}i^{(1)})\psi_0+\left(-\delta+i^{(0)}-\frac{1}{2}\omega\left(i^{(0)}\right)^2\right)\psi_1\right)\\
+\frac{1}{1-\eta}\Bigg[-K^{1-\gamma}(\gamma-\eta)\left(-K^{1-\gamma}(-1+\gamma)\psi_0\right)^{-\frac{1}{1-\gamma}+\frac{\eta}{1-\gamma}}\Bigg[-\rho\left(-K^{1-\gamma}(-1+\gamma)\psi_0\right)^{\frac{1-\eta}{1-\gamma}}\\
+\left((-1)^{-1/\eta}K(1-\gamma)^{\frac{1-\eta}{(-1+\gamma)\eta}}(-1+\omega i^{(0)})^{-1/\eta}\psi_0^{\frac{1-\eta}{(-1+\gamma)\eta}}\right)^{1-\eta}\Bigg]\psi_1\\
+\left(-K^{1-\gamma}(-1+\gamma)\psi_0\right)^{1-\frac{1-\eta}{1-\gamma}}\Bigg[-\frac{(-1+\eta)\rho\left(-K^{1-\gamma}(-1+\gamma)\psi_0\right)^{\frac{1}{1-\gamma}-\frac{\eta}{1-\gamma}}\psi_1}{(-1+\gamma)\psi_0}\\
+\frac{1}{(-1+\gamma)\eta}(-1)^{1-\frac{1}{\eta}}K(1-\gamma)^{-\frac{1}{-1+\gamma}+\frac{1}{(-1+\gamma)\eta}}(1-\eta)(-1+\omega i^{(0)})^{-1-\frac{1}{\eta}}\psi_0^{-1-\frac{1}{-1+\gamma}+\frac{1}{(-1+\gamma)\eta}}\times\\
\left((-1)^{-1/\eta}K(1-\gamma)^{\frac{1-\eta}{(-1+\gamma)\eta}}(-1+\omega i^{(0)})^{-1/\eta}\psi_0^{\frac{1-\eta}{(-1+\gamma)\eta}}\right)^{-\eta}\left(-\omega i^{(1)}\psi_0+\gamma\omega i^{(1)}\psi_0+\psi_1-\eta\psi_1-\omega i^{(0)}\psi_1+\eta\omega i^{(0)}\psi_1\right)\Bigg]\Bigg]=0,
\end{aligned}
$$

where $\Delta\lambda$ is a shorthand for $\lambda_e\mathbb{E}[Z_e^{1-\gamma}-1]+\lambda_c\mathbb{E}[Z_c^{1-\gamma}-1]$, and we have divided by h. This equation still has two unknowns, ψ_1 and $i^{(1)}$, and must be solved together with the budget constraint, $Y-bF-I-C=0$, at the same order of approximation. Again using the optimal solutions for the policy variables from the first-order conditions

[25] The terms 'fast' and 'slow' in perturbation analysis are used to denote the functional dependence on E. A 'fast' dependence leaves the order of the term unchanged upon differentiation. A 'slow' dependence increases the order.

(11) and (12) and retaining only the first-order terms in $h = \mathcal{O}(\epsilon)$ yields a second equation relating ψ_1 and $i^{(1)}$:

$$-i^{(1)} + \frac{(-1)^{-1/\eta}(1-\gamma)^{\frac{1}{(\gamma-1)\eta}-\frac{1}{\gamma-1}}\psi_0^{\frac{1}{(\gamma-1)\eta}-\frac{1}{\gamma-1}-1}(\omega i^{(0)}-1)^{-\frac{1}{\eta}-1}(-\eta\psi_1+\eta\omega i^{(0)}\psi_1-\omega i^{(0)}\psi_1+\gamma\omega i^{(1)}\psi_0-\omega i^{(1)}\psi_0+\psi_1)}{(\gamma-1)\eta} = 0, \tag{32}$$

which can be solved for $i^{(1)}$ in terms of ψ_1:

$$i^{(1)} = \frac{(-1)^{-1/\eta}(1-\gamma)^{\frac{1}{(\gamma-1)\eta}-\frac{1}{\gamma-1}}\psi_0^{\frac{1}{(\gamma-1)\eta}-\frac{1}{\gamma-1}-1}(\omega i^{(0)}-1)^{-\frac{1}{\eta}-1}(-\eta\psi_1+\eta\omega i^{(0)}\psi_1-\omega i^{(0)}\psi_1+\psi_1)}{(\gamma-1)\eta\left(\frac{(-1)^{-1/\eta}\omega(1-\gamma)^{\frac{1}{(\gamma-1)\eta}-\frac{1}{\gamma-1}+1}\psi_0^{\frac{1}{(\gamma-1)\eta}-\frac{1}{\gamma-1}}(\omega i^{(0)}-1)^{-\frac{1}{\eta}-1}}{(\gamma-1)\eta}+1\right)}. \tag{33}$$

Substituting $i^{(1)}$ from (33) and the zeroth-order solution for ψ_0 from (18), we obtain after considerable but trivial manipulation:

$$\epsilon\psi_1 = \frac{h\left(\overline{\psi}_0 - \psi_0\right)}{r^\star}, \tag{34}$$

where $r^\star$ was obtained as part of the zeroth-order solution and is given by (19).

B Details of the market-based calibration

Assuming no negative impact of climate change on the economy, we can derive closed-form expressions for key economic variables in our model. Given the parameter values for economic uncertainty and the share of fossil-fuel use, we calibrate the remaining parameters under that assumption to match an expected GDP growth rate of $g^{(0)} = 2\%$ in normal times, i.e., in the absence of rare macroeconomic disasters, an average consumption rate of $\chi^{(0)} \equiv \frac{C^{(0)}}{B^{(0)}K} = 73\%$ of GDP, a risk-free interest rate of $r_f = 0.8\%$/year, an equity risk premium of $r_p = 6.5\%$/year, a return on risky assets of 7.3%/year, and a Tobin's Q of $q^{(0)} = 1.38$. The following equations constitute a non-linear system that relates ρ, γ, $B^{(0)}$, δ, and φ to those quantities.

$$\chi^{(0)} = \frac{q^{(0)}}{B^{(0)}}\left[\rho + (\eta-1)\left(\overline{g}^{(0)} - 0.5\gamma\sigma^2 - \frac{\lambda_e}{\beta_e-\gamma+1}\right)\right], \tag{35}$$

$$\overline{g}^{(0)} = -\delta + B^{(0)}\left(1-\chi^{(0)}-\alpha\right) - \frac{1}{2}\varphi\left(B^{(0)}\right)^2\left(1-\chi^{(0)}-\alpha\right)^2 - \frac{\lambda_e}{\beta_e+1}, \tag{36}$$

$$r_f = \rho + \eta\overline{g}^{(0)} - \frac{1}{2}\gamma(1+\eta)\sigma_c^2 - \lambda_e\frac{(\eta-\gamma)(\alpha_e-\gamma)+\gamma(\beta_e-\gamma+1)}{(\alpha_e-\gamma)(\alpha_e-\gamma+1)}, \tag{37}$$

$$r_p = \gamma\sigma^2 + \lambda_e\gamma\left[\frac{1}{\beta_e-\gamma} - \frac{\beta_e}{(\beta_e+1)(\beta_e-\gamma+1)}\right], \tag{38}$$

$$q^{(0)} = \frac{1}{1-\varphi i^{(0)}}. \tag{39}$$

For the derivation of these equations and for further details, we refer to Pindyck and Wang (2013) and Hambel et al. (2022).

C Stochastic shocks to the damage ratio and the TCRE

Damages from global warming are notoriously uncertain (see Nordhaus and Moffat (2017)). We can allow for stochastic shocks to the damage ratio in the same manner as in van den Bremer and van der Ploeg (2021). Let the damage ratio be given by

$$D(T,\mu) = 1 - D_{1T}\max(\mu,0)^{1+\theta}T, \tag{40}$$

where μ is a persistent normally-distributed shock to the damage function, and $\theta \geq 0$.[26] Here, θ controls the skewness of the damage distribution. The stochastic process for μ is independent of the process driving economic activity and is described by the Ornstein–Uhlenbeck process,[27]

$$\mathrm{d}\mu = \nu(\overline{\mu} - \mu) + \sigma_\mu \mathrm{d}W_\mu, \tag{41}$$

where W_μ is a Wiener process, σ_μ is the volatility, ν is the mean-reversion coefficient, and $\overline{\mu}$ is the steady-state value of μ. With uncertainty, the volatility and skewness of the damage ratio increase with time.[28] We assume that shocks to the TCRE are distributed with means χ (pre-tipping) and $\overline{\chi}$ (post-tipping). This suggests the following extended rule for the optimal SCC.

Result 3. *With all three global warming externalities and stochastic shocks to the damage ratio and normally distributed shocks to the TCRE, the leading-order estimate of the optimal SCC is*

$$P_{\mathrm{R3}} = P_{\mathrm{R1}}\frac{\psi_0}{\psi} + \frac{h(E_t)}{r^\star}\left(\overline{P}_{\mathrm{R1}}\frac{\overline{\psi}}{\psi} - P_{\mathrm{R1}}\frac{\psi_0}{\psi}\right) + h'(E_t)\frac{Y_t^{(0)}q_t^{(0)}}{B^{(0)}r^\star}\frac{\overline{\psi}-\psi_0}{(\gamma-1)\psi}, \tag{42}$$

where $h'(E_t) = h_1\chi$. The post-tip SCC $\overline{P}_{\mathrm{R1}}$ and the pre-tip SCC in the absence of tipping P_{R1} can be obtained from Result 1 and are given by

$$P_{\mathrm{R1}} = \left[D_{1T}\Delta + \lambda_{1T}^c\frac{\mathbb{E}\left[1-Z_c^{1-\gamma}\right]}{1-\gamma}\frac{q_t^{(0)}}{B^{(0)}}\right]\frac{\chi Y_t^{(0)}}{r^\star}, \qquad \overline{P}_{\mathrm{R1}} = \left[D_{1T}\Delta + \lambda_{1T}^c\frac{\mathbb{E}\left[1-Z_c^{1-\gamma}\right]}{1-\gamma}\frac{\overline{q}_t^{(0)}}{\overline{B}^{(0)}}\right]\frac{\overline{\chi}Y_t^{(0)}}{\overline{r}^\star}.$$

Tobin's Q $q^{(0)}$, and the discount rate $r^\star$ are as in Result 2, and the correction for skewed damage shocks with mean reversion is

$$\Delta \equiv \overline{\mu}^{1+\theta}\left(1 + \frac{1}{2}\theta(1+\theta)\frac{(\sigma_\mu/\overline{\mu})^2}{r^\star + 2v}\right). \tag{43}$$

[26]For simplicity, we abstract from damage ratios that are a convex function of temperature.

[27]In equation (40), we truncate this process, so as to formally exclude negative values of μ, which typically have negligibly small probability. An alternative is the Cox–Ingersol–Ross process, which avoids negative outcomes.

[28]Note that $\mu(t)$ is normally distributed with mean $\mu(0)e^{-\nu t} + \overline{\nu}(1-e^{-\nu t})$ and variance $\frac{\sigma_\mu^2(1-e^{-2\nu t)})}{2\nu}$.

(a) Without a climate tipping point

Method of calculation	**Rule** ($/t$CO_2$)	**Numerical** ($/t$CO_2$)	**Error**
TFP damages	11.72	12.31	-4.86%
TFP damages and temperature-related disasters	35.32	36.21	-2.45%

(b) With a climate tipping point

Method of calculation	**Rule** ($/t$CO_2$)	**Numerical** ($/t$CO_2$)	**Error**
TFP damages	12.77	13.81	-7.53%
TFP damages and temperature-related disasters	38.95	40.46	-3.73%

Table 4: Calculation of the Optimal SCC (US $/t$CO_2$) with Shocks to the Damage Ratio and Market-Based Calibration. The table summarises the SCC in four different settings with the calibration summarised in Table 1 and compares the performance of the rule to the numerically optimised value of the SCC.

Proof: Along similar lines to the proof in van den Bremer and van der Ploeg (2021).

We see from equation (43) that if shocks to the damage ratio are normally distributed ($\theta = 0$), the optimal SCC is unaffected by damage ratio uncertainty because then $\Delta = \overline{\mu}$ is a constant. However, equation 43 indicates that, if shocks to the damage ratio have a right-skewed distribution ($\theta > 0$), damage ratio uncertainty pushes up the optimal SCC. The effect of damage uncertainty on the SCC is larger if skew (θ) and normalised volatility ($\sigma_\mu/\overline{\mu}$) are high and mean reversion (ν) and the risk- and growth-adjusted interest rate are low.

Since shocks to the transient climate response to cumulative emissions are normally distributed, they do not affect this rule for the optimal SCC.

C.1 Calibration of shocks to the damage ratio

For the stochastic shocks to the damage ratio (see section C), we follow van den Bremer and van der Ploeg (2021) and set the steady-state value of these shocks to $\overline{\mu} = 0.28$. This value ensures that the long-run damage ratio equals our calibrated value of $D_{1T} = \overline{\mu}^{3.7} = 0.9\%$ reported in Table 1. The annual volatility, skew parameter and mean reversion coefficient for the distribution of these shocks are set to $\sigma_\mu = 2.3\%$, $\theta = 2.7$, and $\nu = 0.05$ per year, respectively. The mean reversion coefficient implies that damage shocks persist on average for 20 years.

C.2 Accuracy of the rule with stochastic shocks to the damage ratio

The numerical analysis in Table 4 indicates that the rule for the optimal SCC in Result 3 generally produces fairly accurate SCCs. However, with recurring climate-related disasters, the accuracy of the simple rule is a bit higher because they dominate the TFP damage and its uncertainty. It is not surprising that the additional state variable μ reduces the accuracy of the simple rule somewhat.

(a) Benchmark calibration with market-based discount rate of $r^\star = 5.3\%$/year

Method of calculation	Rule ($/t$CO_2$)	Numerical ($/t$CO_2$)	Error
TFP damages	9.60	9.57	0.31%
TFP damages and recurring climate disasters	33.12	32.62	1.53%
TFP damages, climate disasters, and climate tipping	36.67	37.03	−0.97%

(b) With lower discount rate of $r^\star = 3\%$/year

Method of calculation	Rule ($/t$CO_2$)	Numerical ($/t$CO_2$)	Error
TFP damages	17.06	16.67	2.29%
TFP damages and recurring climate disasters	75.78	76.33	−0.73%
TFP damages, climate disasters, and climate tipping	90.67	91.46	−0.86%

(c) With even lower discount rate of $r^\star = 2\%$/year

Method of calculation	Rule ($/t$CO_2$)	Numerical ($/t$CO_2$)	Error
TFP damages	25.47	24.35	4.40%
TFP damages and recurring climate disasters	139.19	142.72	−2.54%
TFP damages, climate disasters, and climate tipping	181.87	179.64	1.24%

Table 5: Road-Testing the Rule for the Optimal SCC (US $/t$CO_2$) with Technical Progress in Fossil Fuel Production. The rule is tested outside the realm for which it was derived, namely when there is steady technical progress in fossil fuel extraction instead of a constant cost of fossil fuel.

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D Numerical results for two extensions in Section 7

Here we present Tables 5 and 6, which give the numerical results for two extensions of our core integrated assessment models by allowing for (1) the cost of green energy to fall due to learning by doing; and (2) a green and a brown economic sector (see Section 7).

E Numerical solution approach

In order to solve the model numerically, we first decompose the value function into two parts and reduce the number of state variables by one. This leads to a simplified HJB equation, which can then be solved numerically.

E.1 Auxiliary calculations

Aggregate consumption is given by

$$C \equiv A[1 - D(E,\chi,\mu)]K^\alpha F^{1-\alpha} - I - bF.$$

(a) Benchmark calibration with market-based discount rate of $r^\star = 5.3\%$/year

Method of calculation	**Rule** ($/tCO$_2$)	**Numerical** ($/tCO$_2$)	**Error**
TFP damages	9.60	9.58	0.21%
TFP damages and recurring climate disasters	33.12	32.93	0.57%
TFP damages, climate disasters, and climate tipping	36.67	37.28	−1.63%

(b) With lower discount rate of $r^\star = 3\%$/year

Method of calculation	**Rule** ($/tCO$_2$)	**Numerical** ($/tCO$_2$)	**Error**
TFP damages	17.06	16.90	0.94%
TFP damages and recurring climate disasters	75.78	75.11	0.88%
TFP damages, climate disasters, and climate tipping	90.67	88.37	2.60%

(c) With even lower discount rate of $r^\star = 2\%$/year

Method of calculation	**Rule** ($/tCO$_2$)	**Numerical** ($/tCO$_2$)	**Error**
TFP damages	25.47	25.00	1.85%
TFP damages and recurring climate disasters	139.19	133.34	4.20%
TFP damages, climate disasters, and climate tipping	181.87	168.75	7.77%

Table 6: Road-testing the Optimal SCC (US \$/tCO$_2$) in a Two-Sector Model of the Economy. The rule is tested outside the realm for which it was derived, namely when there is a carbon-intensive and a green sector instead of one sector of the economy.

The HJB equation for the general case with each of the three global warming externalities and damage uncertainty is

$$0 = \max_{F,I} \Big[\frac{1}{1-\eta} \frac{C^{1-\eta} - \rho[(1-\gamma)J]^{\frac{1-\eta}{1-\gamma}}}{[(1-\gamma)J]^{\frac{1-\eta}{1-\gamma}-1}} + J_K\big(-\delta K + I - \tfrac{1}{2}\varphi(I/K)^2 K\big)$$
$$+ J_E \varpi F + \tfrac{1}{2} J_{KK} K^2 \sigma^2 + J_\mu \nu(\overline{\mu} - \mu) + \tfrac{1}{2} J_{\mu\mu} \sigma_\mu^2$$
$$+ \sum_{i=e,c} \lambda_i (T_0 + \chi E)\big(\mathbb{E}[J(K(1-\ell_i), E, \chi, \mu)] - J\big) + H(T_0 + \chi E)\big(J(K, E, \overline{\chi}, \mu) - J\big) 1_{\{\chi = \chi_0\}} \Big].$$

Define $f = F/K$, $i = I/K$, and $c = C/K$. Conjecture a value function of the form

$$J(K, E, \mu, \chi) = \frac{1}{1-\gamma} K^{1-\gamma} V(E, \mu, \chi).$$

The Epstein–Zin aggregator is then given by

$$\frac{1}{1-\eta} \frac{C^{1-\eta} - \rho[(1-\gamma)J]^{\frac{1-\eta}{1-\gamma}}}{[(1-\gamma)J]^{\frac{1-\eta}{1-\gamma}-1}} = \rho\theta c^{1-\eta} V^{1-1/\theta} u(K) - \rho\theta u(K) V,$$

where $u(K) = \frac{1}{1-\gamma}K^{1-\gamma}$ and $\theta = \frac{1-\gamma}{1-\eta}$. Substituting this conjecture and its partial derivatives into the HJB equation and dividing by $u(K)$ yields the following HJB equation for $V = V(E,\mu,\chi)$:

$$0 = \max_{F,I}\Big[\rho\theta V^{1-1/\theta}c^{1-\eta} - \rho\theta V + V(1-\gamma)\big(-\delta + i - \tfrac{1}{2}\varphi i^2\big) + V_E f K_0 - \tfrac{1}{2}\gamma(1-\gamma)\sigma^2 V + V_\mu \nu(\overline{\mu} - \mu)$$
$$+ \tfrac{1}{2}V_{\mu\mu}\sigma_\mu^2 + V\sum_{i=e,c}\lambda_i(T_0 + \chi E)\mathbb{E}[(1-\ell_i)^{1-\gamma} - 1] + H(T_0 + \chi E)\big(V(E,\overline{\chi},\mu) - V\big)1_{\{\chi=\chi_0\}}\Big]. \tag{44}$$

where $c = A(E,\chi,\mu)f^{1-\alpha} - i - bf$.

E.2 Numerical algorithm

Basic idea We face a time-homogeneous problem with an infinite time horizon. Since the boundary conditions on V are unknown, we transform the problem into a similar one with a finite time horizon denoted by $t_{\max}$. In our implementation, we consider a model with a finite time horizon and choose a conjecture for $V(t_{\max}, E, \mu)$, which mimics the shape of the true yet unknown value function.[29] Starting with this terminal condition, we work backwards through the time grid until the differences between the value function in $t+1$ and t become negligibly small and the solution converges to that of an infinite time horizon.

Definition of the grid We use a grid-based solution approach to solve the non-linear PDE. We discretize the (t,E,μ)-space using an equally-spaced lattice. Its grid points are defined by

$$\big\{(t_n, E_i, \mu_j) \,|\, n = 0,\cdots,N_t,\ i = 0,\cdots,N_E,\ j = 0,\cdots,N_\mu\big\},$$

where $t_n = n\Delta_t$, $E_i = i\Delta_E$, and $\mu_j = j\Delta_\mu$ for some fixed grid size parameters Δ_t, Δ_E, and Δ_μ that denote the distances between two grid points. The numerical results are based on a choice of $N_E = 100$, $N_\mu = 50$ and 4 time steps per year. Our results hardly change if we use a finer grid or more time steps per year. In the sequel, $V_{n,i,j}$ denotes the approximated value function at the grid point (t_n, E_i, μ_j) and $\pi_{n,i,j}$ refers to the corresponding set of optimal controls. We apply an implicit finite-difference scheme.

Finite-differences approach In this paragraph, we describe the numerical solution approach in more detail. We adapt the numerical solution approach used by Munk and Sørensen (2010). The numerical procedure works as follows. At any point in time, we make a conjecture for the optimal strategy $\pi^*_{n,i,j}$. A good guess is the value at the previous grid point since the abatement strategy varies only slightly over a small time interval, i.e., we set $\pi_{n-1,i,j} = \pi^*_{n,i,j}$. Substituting this guess into the HJB equation yields a semi-linear PDE:

$$0 = \rho\theta V^{1-1/\theta}c^{1-\eta} + M_1 V + M_2 V_E + M_3 V_{EE} + M_4 V_\mu + M_5 V_{\mu\mu} \tag{45}$$

[29] We have tested several conjectures, and find that the concrete choice of $V(t_{\max}, E, \mu)$ does only affect how long it takes until the algorithm converges, but does not affect the limit itself. An obvious conjecture is the leading-order approximation for the value function, i.e., $V = (1-\gamma)\psi_0$.

with state-dependent coefficients $M_i = M_i(t, E, \mu)$, see equation (44). Due to the implicit approach, we approximate the time derivative by forward finite differences. In the approximation, we use the so-called 'up-wind' scheme that stabilizes the finite-difference approach. Therefore, the relevant finite differences at the grid point (n, i, j) are given by

$$
\begin{aligned}
&\mathrm{D}_E^+ V_{n,i,j} = \frac{V_{n,i+1,j} - V_{n,i,j}}{\Delta_E}, \qquad \mathrm{D}_E^- V_{n,i,j} = \frac{V_{n,i,j} - V_{n,i-1,j}}{\Delta_E},\\
&\mathrm{D}_\mu^+ V_{n,i,j} = \frac{V_{n,i,j+1} - V_{n,i,j}}{\Delta_\mu}, \qquad \mathrm{D}_\mu^- V_{n,i,j} = \frac{G_{n,i,j} - G_{n,i,j-1}}{\Delta_\mu},\\
&\mathrm{D}_{EE}^2 V_{n,i,j} = \frac{V_{n,i+1,j} - 2V_{n,i,j} + V_{n,i-1,j}}{\Delta_E^2}, \qquad \mathrm{D}_{\mu\mu}^2 V_{n,i,j} = \frac{V_{n,i,j+1} - 2V_{n,i,j} + V_{n,i,j-1}}{\Delta_\mu^2}\\
&\mathrm{D}_t^+ V_{n,i,j} = \frac{V_{n+1,i,j} - V_{n,i,j}}{\Delta_t}.
\end{aligned}
$$

Substituting these expressions into the PDE above yields the following semi-linear equation for the grid point (t_n, m_i, τ_j)

$$
\begin{aligned}
V_{n+1,i,j}\frac{1}{\Delta_t} = V_{n,i,j}\Big[&-M_1 + \frac{1}{\Delta_t} + \mathrm{abs}\Big(\frac{M_2}{\Delta_E}\Big) + \mathrm{abs}\Big(\frac{M_4}{\Delta_\mu}\Big) + 2\frac{M_3}{\Delta_E^2} + 2\frac{M_5}{\Delta_\mu^2}\Big]\\
&+ \mu_{n,i-1,j}\Big[\frac{M_2^-}{\Delta_E} - \frac{M_3}{\Delta_E^2}\Big] + V_{n,i+1,j}\Big[-\frac{M_2^+}{\Delta_E} - \frac{M_3}{\Delta_E^2}\Big]\\
&+ V_{n,i,j-1}\Big[\frac{M_4^-}{\Delta_\mu} - \frac{M_5}{\Delta_\mu^2}\Big] + V_{n,i,j+1}\Big[-\frac{M_4^+}{\Delta_\mu} - \frac{M_5}{\Delta_\mu^2}\Big]\\
&+ \delta\theta V_{n,i,j}^{1-1/\theta} c_{n,i,j}^{1-1/\psi}
\end{aligned}
$$

Therefore, for a fixed point in time each grid point is determined by a non-linear equation. This results in a non-linear system of $(N_\mu + 1)(N_E + 1)$ equations that can be solved for the vector

$$
V_n = (V_{n,1,1}, \cdots, V_{n,1,N_S}, V_{n,2,1}, \cdots, V_{n,2,N_S}, \cdots, V_{n,N_T,1}, \cdots, V_{n,N_E,N_\mu}).
$$

Using this solution we update our conjecture for the optimal controls at the current point in the time dimension. We apply the above-mentioned first-order conditions and finite-difference approximations of the corresponding derivatives. Finally, we calculate the optimal SCC using those partial derivatives.

Post-tip problem To determine the optimal SCC with climate tipping, we solve the HJB equation (9) for the post-tip problem numerically as described above. We then solve the HJB equation (10) for the pre-tip problem in a similar fashion. Given the solution $\overline{V}$ of the post-tip problem on the whole grid, the semi-linear PDE of the pre-tip problem read

$$0 = \rho\theta V^{1-1/\theta} c^{1-\eta} + M_1 V + M_2 V_E + M_3 V_{EE} + M_4 V_\mu + M_5 V_{\mu\mu} + H(T_0 + \chi E)[\overline{V} - V]. \tag{46}$$

This semi-linear PDE can now be solved along the lines of the post-tip problem.

F More general rules for the optimal SCC

In this section we briefly discuss various extension of our estimate of the optimal SCC given in Results 1, 2, and 3. In particular, we allow for multiple recurring climate-related disasters and climate tipping points, more general carbon cycle and temperature dynamics, and uncertainty in and convexity of the damage ratio.

F.1 Multiple disasters or tipping points

We will indicate the different climate-related disasters by the subscript i and the different climate tipping points by the subscript j to allow for to allow for different risks and impacts. Following the notation in Results 2 and 3, the post-tip SCC $\overline{P}_j$ for tipping element j and the pre-tip SCC in the absence of tipping $P^{(0)}$ are

$$P^{(0)}_{\text{R1}} = \left[D_{1T} + \sum_i \lambda^c_{1,i} \frac{\mathbb{E}\left[1 - Z^{1-\gamma}_{c,i}\right]}{1-\gamma} \frac{q^{(0)}}{B^{(0)}} \right] \frac{\chi Y^{(0)}}{r^\star}, \qquad \overline{P}_{j,\text{R1}} = \left[D_{1T} + \sum_i \lambda^c_{1,i} \frac{\mathbb{E}\left[1 - Z^{1-\gamma}_{c,i}\right]}{1-\gamma} \frac{\overline{q}^{(0)}}{\overline{B}^{(0)}} \right] \frac{\overline{\chi} Y^{(0)}}{\overline{r}^\star}.$$

Provided the risks of recurring climate-related disasters and climate tipping points are independent, it is straightforward to extend our rule for the optimal SCC for multiple disasters and tipping points by setting

$$P_{\text{R2}} = \sum_j \left[P^{(0)} \frac{\psi_0}{\psi_j} + \frac{h_j(E)}{r^\star} \left(\overline{P}_j \frac{\overline{\psi}_j}{\psi_j} - P^{(0)} \frac{\psi_0}{\psi_j} \right) + h'_j(E) \frac{Y^{(0)} q^{(0)}}{B^{(0)} r^\star} \frac{\psi_0 - \overline{\psi}_j}{(1-\gamma)\psi_j} \right],$$

and equation (23) becomes $r^\star_{\text{R1}} = \rho + (\eta - 1)\Big(g^{(0)} - \frac{1}{2}\gamma\sigma^2 + \sum_i \left(\frac{\lambda_{ci}(T)}{1-\gamma}\mathbb{E}[1 - Z^{1-\gamma}_{ci}]\right) + \frac{\lambda_e}{1-\gamma}\mathbb{E}[1 - Z^{1-\gamma}_e]\Big)$. It is thus straightforward to extend Result 2 and allow for multiple, different temperature-related risks of climate-related disasters or climate tipping points.

F.2 More general carbon cycle and temperature dynamics

Following Matthews et al. (2009), Allen et al. (2009), and policy analysis of the IPCC, we have modelled temperature as a function of cumulative emissions. However, most integrated assessment models have used linear dynamic models of carbon cycle and temperature dynamics to mimick the big climate science models. These linear models typically specify the dynamics of the I stocks of carbon by

$$\mathrm{d}S_i = (\varrho_{1i} F e^{-gt} - \varrho_{2i} S_i)\mathrm{d}t, \qquad i = 1, \dots, I, \tag{47}$$

and the stock of atmospheric carbon by $S_A = \sum_{i=1}^{I} \varrho_{3i} S_i$. Here the parameters $0 \leq \varrho_{1i} \leq 1, i = 1, \dots, I$ satisfy $\sum_{i=1}^{I} \varrho_{1i} = 1$, and regulate the share of emissions that goes to each carbon box. The parameters $\varrho_{2i} \geq 0, i = 1, \dots, I$, indicate the decay rate of carbon in each box. The different carbon stocks may refer to the bottom and lower parts of the oceans and the bottom and lower parts of the atmosphere or may be the result of a mathematical fitting exercise to mimick the responses of the detailed climate science models. Note that there is always a part of atmospheric carbon that stays up in the atmosphere forever for all practical intents and purposes, which is captured by setting $\varrho_{21} = 0$. The parameters $0 \leq \varrho_{3i} \leq 1$, satisfy $\sum_{i=1}^{I} \varrho_{3i} = 1$ and define the stock of atmospheric carbon. For example, Golosov et al. (2014) uses a 2-box model with $I = 2$, Nordhaus (2017) uses a 3-box model with $I = 3$. A popular model among climate scientists is the 4-box model with $I = 4$ put forward by Joos et al. (2013).

Following Golosov et al. (2014), we suppose that the convexity of the function relating the damage ratio to temperature more or less outweighs the concavity of the logarithmic Arrhenius' law relating temperature to the stock of atmospheric carbon. For example, in Nordhaus (2017) the damage ratio is a quadratic function of temperature. The reduced-form damage ratio is then approximately a *linear* function of the stock of atmospheric carbon: $D = D_{0A} + D_{1A} S_A$ instead of our formulation with the stock of cumulative emissions, $D(D_1 E) = D_0 + \chi D_{1T} E$ in equation (5). We also assume that the intensities of recurring climate disasters and of irreversible climate tipping points are slightly convex functions of temperature to offset the concave (logarithmic) function relating temperature to the atmospheric stock, so that these intensities are also roughly linear functions of the stock of atmospheric carbon. Hence, the marginal intensities of recurring disasters and of tipping points with respect to the stock of atmospheric carbon are approximately constant and given by λ_A and h_A, respectively.

Armed with these assumptions, it follows that $1/r_\star$ in the rule for the optimal SCC in Results 2 and 3 become $\frac{\varrho_{1,1}}{r^\star} + \frac{\varrho_{1,2}}{r^\star + \varrho_{2,2}} + .. + \frac{\varrho_{1,I}}{r^\star + \varrho_{2,I}}$ So, the shorter the lifetime of carbon in a particular box, the less the emissions that enter this box contribute the SCC. It is easy to extend this result to allow for stochastic shocks to the damage ratio (cf. Result 2).