Firm Size and Compensation Dynamics with Risk Aversion and Persistent Private Information

Gerard Maideu-Morera

Toulouse School of Economics

August 25, 2023

Introduction

- Financing constraints slow down firms' growth over their lifecycle
- Literature: Dynamic contracts (cash flow diversion model) to provide microfoundations

Introduction

- Financing constraints slow down firms' growth over their lifecycle
- Literature: Dynamic contracts (cash flow diversion model) to provide microfoundations
- But:
  1. **Entrepreneur risk neutral**: Backload compensation at little (or no) cost
  2. **Productivity shocks i.i.d**: Limits gains misreporting and difference in preferences for future contract arrangements

Introduction

- Financing constraints slow down firms' growth over their lifecycle
- Literature: Dynamic contracts (cash flow diversion model) to provide microfoundations
- But:
  1. **Entrepreneur risk neutral**: Backload compensation at little (or no) cost
  2. **Productivity shocks i.i.d**: Limits gains misreporting and difference in preferences for future contract arrangements

This paper: Firm size and compensation dynamics when entrepreneur is risk averse and has persistent private information?

1. Risk aversion decouples firm size and compensation dynamics
2. Interaction risk aversion + persistence on firm size dynamics

Preview results

- **Optimal contract:**

	Risk neutral (Clementi and Hopenhayn (2006))	**Risk averse + persistent**
Firm size (drift)	$\nearrow$	$\searrow$ (constant if i.i.d)
Compensation	once reach FB	smoothed & variance $\nearrow$

Preview results

- **Optimal contract:**

	Risk neutral (Clementi and Hopenhayn (2006))	**Risk averse + persistent**
Firm size (drift)	$\nearrow$	$\searrow$ (constant if i.i.d)
Compensation	once reach FB	smoothed & variance $\nearrow$

- (Quasi-) **Implementation:** wealth + equity share; pledge shares

	Risk neutral (Clementi and Hopenhayn (2006))	**Risk averse + persistent**
Promised utility	Equity value	Private Wealth
Equity share (drift)	$\nearrow$	$\searrow$ (constant if i.i.d)

Literature

- **Dynamic financial contracting:** Clementi and Hopenhayn (2006), Albuquerque and Hopenhayn (2004), Biais et al. (2007), Biais et al. (2010), DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007a), DeMarzo and Fishman (2007b), DeMarzo et al. (2012) and Clementi et al. (2010)
  - **Risk aversion:** He (2012) and Di Tella and Sannikov (2021) (Hidden savings problems)
  - **Persistence:** DeMarzo and Sannikov (2016), Fu and Krishna (2019) and Krasikov and Lamba (2021)

    $\rightarrow$ RN + persistence: firm size still converges to FB. Need both RA + persistence
- **Dynamic Mirrlees:** Kapička (2013), Farhi and Werning (2013), Golosov et al. (2016a), Makris Pavan (2020) and Hellwig (2021)

  $\rightarrow$ Use first-order approach and incentive-adjusted measures. Connection labor wedge dynamics

Outline

1. Model and set up lender's problem
2. Optimal allocation
   - Firm size dynamics
   - Consumption dynamics (Generalized Inverse Euler Equation)
3. Numerical simulations
4. Implementation

Cash flow diversion model

- Risk-averse entrepreneur (agent) needs funds k_t (firm size) from a lender (principal) to operate a project

Cash flow diversion model

- Risk-averse entrepreneur (agent) needs funds k_t (firm size) from a lender (principal) to operate a project
- Entrepreneur's expected continuation utility

$$w_t = \mathbb{E}\Big[\sum_{\tau=0}^{\infty} \beta^{\tau} u(c_{t+\tau})\Big]$$

Cash flow diversion model

- Risk-averse entrepreneur (agent) needs funds k_t (firm size) from a lender (principal) to operate a project
- Entrepreneur's expected continuation utility

$$w_t = \mathbb{E}\Big[\sum_{\tau=0}^{\infty} \beta^\tau u(c_{t+\tau})\Big]$$

- Every period, cash flow $f(k_t, \theta_t)$, where $\theta_t \in [\underline{\theta}, \overline{\theta}]$ is the entrepreneur's productivity
  - θ_t is private information, denote $\theta^t = \{\theta_1, ..., \theta_t\}$
  - Follows a persistent Markov process, conditional density $\varphi_t(\theta_t|\theta_{t-1})$ (and cdf $\Phi(\theta_t|\theta_{t-1})$)

$$\rho_t(\theta^t) \equiv \frac{\frac{\partial}{\partial \theta_{t-1}}(1 - \Phi(\theta_t|\theta_{t-1}))}{\varphi_t(\theta_t|\theta_{t-1})} \geq 0$$

  - Assume $f_\theta > 0$ and $f_{\theta k} > 0$

Cash flow diversion model

- Lender does not observe returns, entrepreneur can misreport and divert a fraction of the cash flow
  - After report $f(k_t, \widetilde{\theta}_t)$, ask repayment $b_t(\widetilde{\theta}_t)$ and advance funds $k_{t+1}(\widetilde{\theta}_t)$

Cash flow diversion model

- Lender does not observe returns, entrepreneur can misreport and divert a fraction of the cash flow
  - After report $f(k_t, \widetilde{\theta}_t)$, ask repayment $b_t(\widetilde{\theta}_t)$ and advance funds $k_{t+1}(\widetilde{\theta}_t)$
- Deadweight loss $(1-\iota) \in [0,1)$ on diverted funds

$$c_t = f(k_t, \theta_t) - (1-\iota)\left(f(k_t, \theta_t) - f(k_t, \widetilde{\theta}_t)\right) - b_t(\widetilde{\theta}_t)$$

If don't misreport $c_t = f(k_t, \theta_t) - b_t(\theta_t)$. Cannot overreport $\widetilde{\theta}_t \leq \theta_t$

Cash flow diversion model

- Lender does not observe returns, entrepreneur can misreport and divert a fraction of the cash flow
  - After report $f(k_t, \widetilde{\theta}_t)$, ask repayment $b_t(\widetilde{\theta}_t)$ and advance funds $k_{t+1}(\widetilde{\theta}_t)$
- Deadweight loss $(1-\iota) \in [0,1)$ on diverted funds

$$c_t = f(k_t, \theta_t) - (1-\iota)\left(f(k_t, \theta_t) - f(k_t, \widetilde{\theta}_t)\right) - b_t(\widetilde{\theta}_t)$$

If don't misreport $c_t = f(k_t, \theta_t) - b_t(\theta_t)$. Cannot overreport $\widetilde{\theta}_t \leq \theta_t$

- Every period capital fully depreciates and entrepreneur cannot save by himself

Principal's problem

- Direct mechanism: report $\sigma^t \to$ allocations $\left(k_t(\sigma^{t-1}), b_t(\sigma^t)\right)$
- Risk-neutral lender problem

$$K(v) = \min_{\{k_t(\theta^{t-1}), b_t(\theta^t)\}} \mathbb{E}_0 \left[\sum_{t=1}^{\infty} q^t \left(k_{t+1} \left(\theta^t\right) - b_t \left(\theta^t\right) \right) \right]$$

$$\text{s.t} \quad \mathbb{E}_0 \left[w_1 \left(\theta^1\right) \right] \geq v \;\; (PK)$$

$$w_t(\theta^t) \geq w_t^\sigma(\theta^t) \;\; \forall \sigma, \theta^t \;\; (IC)$$

Principal's problem

- Direct mechanism: report $\sigma^t \to$ allocations $\left(k_t(\sigma^{t-1}), b_t(\sigma^t)\right)$
- Risk-neutral lender problem

$$K(v) = \min_{\{k_t(\theta^{t-1}), b_t(\theta^t)\}} \mathbb{E}_0 \left[\sum_{t=1}^{\infty} q^t \left(k_{t+1}\left(\theta^t\right) - b_t\left(\theta^t\right) \right) \right]$$

$$\text{s.t} \quad \mathbb{E}_0 \left[w_1\left(\theta^1\right) \right] \geq v \quad (PK)$$

$$w_t(\theta^t) \geq w_t^{\sigma}(\theta^t) \ \ \forall \sigma, \theta^t \quad (IC)$$

- **First Order Approach** (Kapicka (2013), Farhi Werning (2013), Pavan Segal Toikka (2014))
  - Local IC constraint (mimick type right below)
  - Check global IC numerically

full details

Optimal Allocation

Firm size dynamics

In the FB (no private info)

$$\frac{1}{q} = \mathbb{E}\left[f_k(k_{t+1}(\theta^t), \theta_{t+1})|\theta_t\right]$$

Firm size dynamics

$$\frac{1}{q} = \mathbb{E}\left[f_k(k_{t+1}(\theta^t), \theta_{t+1})(1 - \tau^k(\theta^{t+1}))|\theta_t\right]$$

Firm size dynamics

$$\frac{1}{q} = \mathbb{E}\left[f_k(k_{t+1}(\theta^t), \theta_{t+1})(1 - \tau^k(\theta^{t+1}))|\theta_t\right]$$

Proposition

$$\tau^k(\theta^{t+1}) = \iota \underbrace{\frac{\theta_{t+1} f_{\theta k}(\theta^{t+1})}{f_k(\theta^{t+1})}}_{\substack{>0 \\ \textit{Elasticity } f_k \textit{ w.r.t } \theta_{t+1}}} \times \underbrace{\widetilde{\mu}_{t+1}(\theta^{t+1})}_{\textit{"Normalized shadow cost insurance"}} \times \underbrace{\frac{1 - \Phi(\theta_{t+1}|\theta_t)}{\theta_{t+1}\varphi(\theta_{t+1}|\theta_t)}}_{\textit{Pareto tail}} \geq 0$$

Firm size dynamics

$$\frac{1}{q} = \mathbb{E}\left[f_k(k_{t+1}(\theta^t), \theta_{t+1})(1 - \tau^k(\theta^{t+1}))|\theta_t\right]$$

Proposition

$$\tau^k(\theta^{t+1}) = \iota \underbrace{\frac{\theta_{t+1} f_{\theta k}(\theta^{t+1})}{f_k(\theta^{t+1})}}_{\substack{>0 \\ \textit{Elasticity } f_k \textit{ w.r.t } \theta_{t+1}}} \times \underbrace{\widetilde{\mu}_{t+1}(\theta^{t+1})}_{\textit{"Normalized shadow cost insurance"}} \times \underbrace{\frac{1 - \Phi(\theta_{t+1}|\theta_t)}{\theta_{t+1}\varphi(\theta_{t+1}|\theta_t)}}_{\textit{Pareto tail}} \geq 0$$

- With **persistent private information:** $\widetilde{\mu}_{t+1}(\theta^{t+1})$ (and so $\tau^k(\theta^{t+1})$) tend to increase over time
  - If promise more insurance $t+1$ lower cost screening at t
  - Implies firm size tends to **decrease** over time!
  - As labor tax Dynamic Mirrlees (Farhi Werning (2013), Makris Pavan (2020))

Compensation Dynamics

Proposition

The consumption process satisfies a Generalized Inverse Euler Equation (Hellwig (2021))

$$\frac{q}{\beta}\hat{\mathbb{E}}\left[\frac{1}{u'(\theta^{t+1})}|\theta^t\right] = \frac{1}{u'(\theta^t)}(1 + s(\theta^t))$$

where

$$s(\theta^t) = \left(\frac{f_\theta(\theta^t)u''(\theta^t)}{u'(\theta^t)} - \hat{\mathbb{E}}\left[\rho_{t+1}(\theta^{t+1})\frac{\iota f_\theta(\theta^{t+1})u''(\theta^{t+1})}{u'(\theta^{t+1})}|\theta^t\right]\right)\frac{f_k(\theta^t)}{f_{\theta k}(\theta^t)}\tau^k(\theta^t)$$

- LHS: cost transferring compensation to $t+1$ in IC way. Compensation smoothed over time
- $s(\theta^t)$ accounts for changes information rents. Higher $\rho \rightarrow$ "savings" less discouraged

Numerical Simulations

Parametrization

- Utility:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

- Production function:

$$f(k,\theta) = z\theta k^{\alpha}$$

- Productivity:

$$\theta_t = \theta_{t-1}^{\rho}\varepsilon_t$$

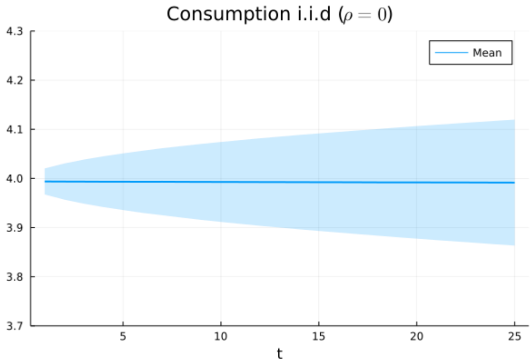
  with $\log(\varepsilon_t) \sim N(\mu, \sigma_\varepsilon^2)$

- Calibration: $\beta = q = 0.95$, $\alpha = 3/4$, $\sigma = 2$, $\iota = 0.95$, $\mu = 1$ and $\sigma_\varepsilon^2 = 0.01$. Compare $\rho = 0$ and $\rho = 0.7$

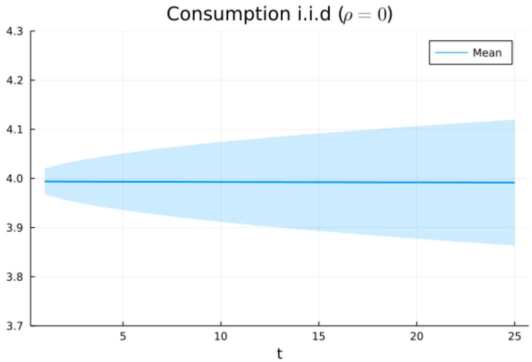
- Dynamic programming (squared) with optimal control problem at every point in state space $(\lambda_-, \gamma_-, k, \theta_-)$

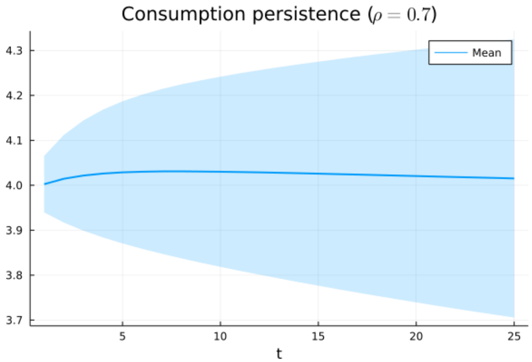
- Monte Carlo simulation: 10^6 draws over 25 periods

Compensation dynamics



Compensation dynamics



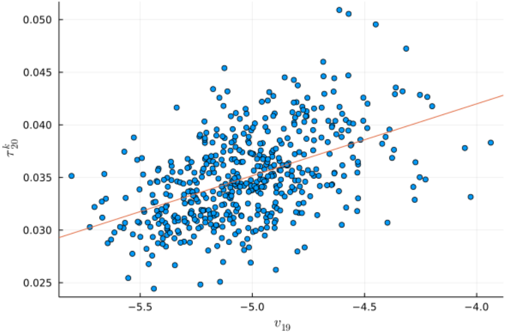


l/r

Separation compensation and firm size

- Promised utility $v_{t-1} = \mathbb{E}[w(\theta^t)|\theta_{t-1}]$

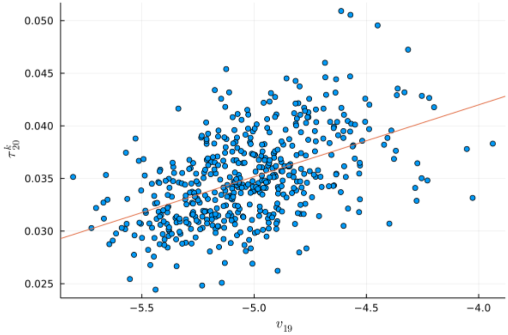
Figure: Wedges and promised utility at age 20 ($\rho = 0.7$)



Separation compensation and firm size

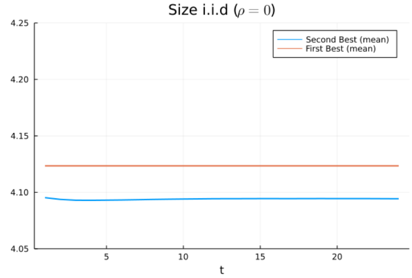
- Promised utility $v_{t-1} = \mathbb{E}[w(\theta^t)|\theta_{t-1}]$

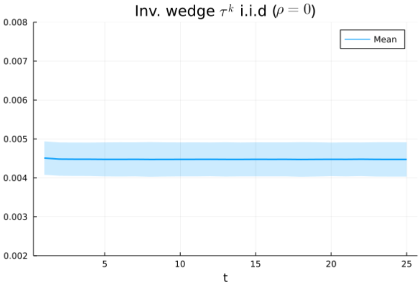
Figure: Wedges and promised utility at age 20 ($\rho = 0.7$)



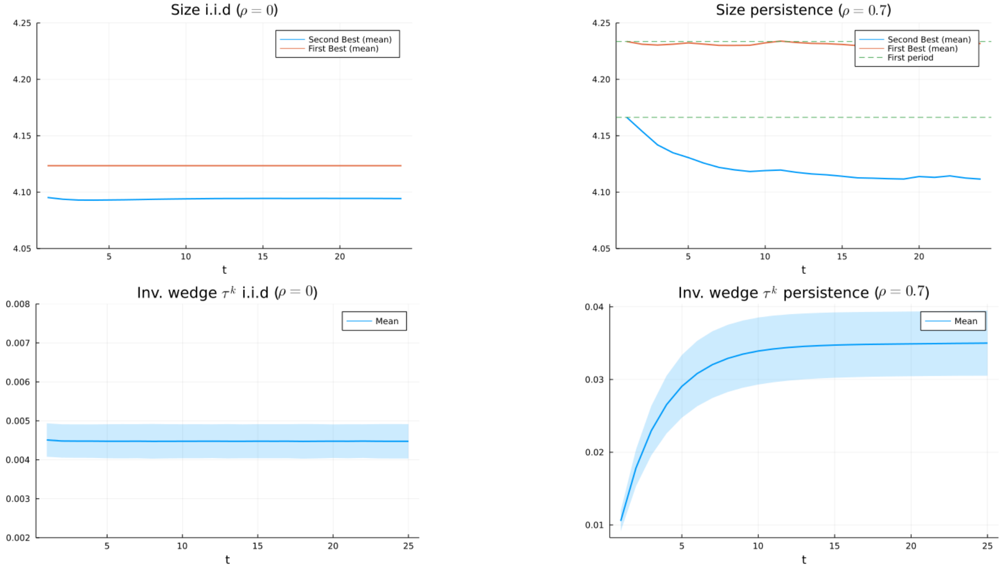
- With risk-neutrality, one-to-one relation firm size and promised utility (Clementi and Hopenhayn (2006))
- Promised insurance $\Delta_{t-1} = \mathbb{E}\left[\rho(\theta^t)\frac{\partial w(\theta^t)}{\partial \theta_t}|\theta_{t-1}\right]$ drives firm size dynamics

Firm size dynamics





Firm size dynamics



Implementation

Implementation

- **Risk neutral:** promised utility $(v_t) \iff$ value equity (Clementi and Hopenhayn (2006))
- **Risk averse:** promised utility $(v_t) \iff$ private wealth
  - v_t linked to consumption but unrelated to sensitivity consumption-returns

Implementation

- **Risk neutral:** promised utility $(v_t) \iff$ value equity (Clementi and Hopenhayn (2006))
- **Risk averse:** promised utility $(v_t) \iff$ private wealth
  - v_t linked to consumption but unrelated to sensitivity consumption-returns

- Implementation i.i.d: Constant equity share & pledge shares to borrow (Fabisik (2019)) given the implied wealth regs problem results

Implementation

- **Risk neutral:** promised utility $(v_t) \iff$ value equity (Clementi and Hopenhayn (2006))
- **Risk averse:** promised utility $(v_t) \iff$ private wealth
  - v_t linked to consumption but unrelated to sensitivity consumption-returns
- Implementation i.i.d: Constant equity share & pledge shares to borrow (Fabisik (2019)) given the implied wealth regs problem results
- Implementation persistence: Extra state variable, promised insurance:

$$\Delta_{t-1} = \mathbb{E}\left[\rho(\theta^t)\frac{\partial w(\theta^t)}{\partial \theta_t}|\theta_{t-1}\right] \iff \text{Time-varying equity share}$$

Intuition: If buy equity at $t+1$ to type θ_t', less attractive offer for types $\theta_t'' > \theta_t'$ (because have higher expected returns) $\implies$ helps screen types

  - But constant equity share still performs well (connection capital gains & dynamic info rents)

Implications

- Both models get positive relation between equity share and firm size, but
  - Risk neutral equity share drifts upwards (Clementi and Hopenhayn (2006))
  - Risk averse + persistent equity share drifts downwards
  - With risk neutral (and i.i.d) firm size converges to FB because the entrepreneur becomes the only owner of the firm (debt and outside equity $\to 0$)
  - E.g. venture capital

Implications

- Both models get positive relation between equity share and firm size, but
  - Risk neutral equity share drifts upwards (Clementi and Hopenhayn (2006))
  - Risk averse + persistent equity share drifts downwards
  - With risk neutral (and i.i.d) firm size converges to FB because the entrepreneur becomes the only owner of the firm (debt and outside equity $\rightarrow 0$)
  - E.g. venture capital

Conclusions

- Use first order-approach and change of measure to study a cash flow diversion model with persistent private information and risk aversion

Findings:

- Firm size tends to decrease over time
- Compensation smoothed + variance $\nearrow$ (Generalized IEE)
- (Quasi-) Implementation:
  - promised utility $\Longleftrightarrow$ wealth (not equity as risk neutral)
  - promised insurance $\Longleftrightarrow$ equity share
- Challenging to get both firm size and equity share dynamics right

Regressions iid types

	$Consumption_t$			
	(1)	(2)	(3)	(4)
$returns_t$	0.0479***	0.0479***	0.0484***	0.0626***
	(16781.39)	(411.86)	(141.92)	(1130.54)
v_{t-1}	0.199***		0.199***	0.199***
	(31258.41)		(2798.00)	(31290.36)
$returns_{t-5}$		0.0474***		
		(407.27)		
$returns_t * v_{t-1}$			-0.0000173	
			(-1.40)	
$returns_t^2$				-0.00127***
				(-265.37)
N	4900000	4400000	4900000	4900000
R^2	0.999	0.071	0.999	0.999

- (2) and consumption follows RW: compensation perfectly smoothed across periods
- (3) effect returns on compensation doesn't depend promised utility
- (4) compensation close to linear in returns

Back

Immiseration in the very long run

Figure: Marginal utility with $\rho = 0$ (i.i.d)

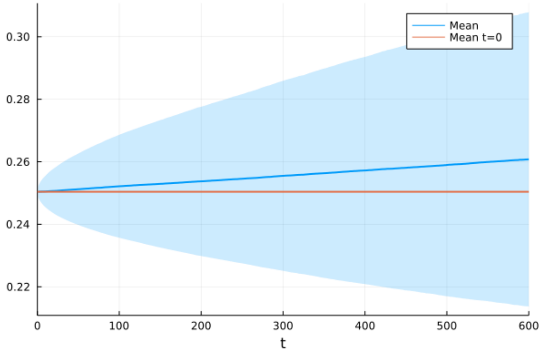
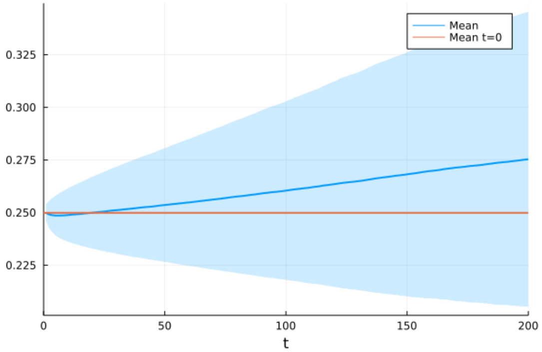


Figure: Marginal utility with $\rho = 0.7$ (i.i.d)



Back

Methodology from Dynamic Public Finance

1. Persistent Private Info → **First Order Approach** (Kapicka (2013), Farhi Werning (2013), Pavan Segal Toikka (2014), Golosov Troshkin Tsyvinski (2016))

$$\frac{\partial}{\partial \theta_t} w_t\left(\theta^t\right) = u'\left(c\left(\theta^t\right)\right)\phi f_\theta\left(k_t\left(\theta^{t-1}\right), \theta_t\right) + \beta \underbrace{\mathbb{E}[\rho(\theta^{t+1})\frac{\partial w_{t+1}\left(\theta^{t+1}\right)}{\partial \theta_{t+1}}|\theta_t]}_{\equiv \Delta_t(\theta^t) \text{ "Promised insurance"}}$$

Check global IC numerically

2. Risk aversion → **Incentive-adjusted probability measures** (Hellwig (2021))

$$\hat{\varphi}(\theta_t|\theta_{t-1}) = \frac{\varphi(\theta_t|\theta_{t-1})m(\theta^t)}{\mathbb{E}[\varphi(\theta_t|\theta_{t-1})m(\theta^t)|\theta_{t-1}]} \;\; with \;\; \frac{m'(\theta^t)}{m(\theta^t)} = \frac{u''(\theta^t)\phi f_\theta(\theta^t)}{u'(\theta^t)} < 0$$

- To characterize $\mu_t(\theta^t)$ (and Euler eq.): Higher if want to provide more insurance around θ^t

Back

Consumption dynamics

Proposition

The consumption process satisfies a Generalized Inverse Euler Equation

$$\frac{q}{\beta}\hat{\mathbb{E}}\left[\frac{1}{u'(\theta^{t+1})}|\theta^t\right] = \frac{1}{u'(\theta^t)}(1 + s(\theta^t))$$

where

$$s(\theta^t) = \left(\frac{f_\theta(\theta^t)u''(\theta^t)}{u'(\theta^t)} - \hat{\mathbb{E}}\left[\rho_{t+1}(\theta^{t+1})\frac{\iota f_\theta(\theta^{t+1})u''(\theta^{t+1})}{u'(\theta^{t+1})}|\theta^t\right]\right)\frac{f_k(\theta^t)}{f_{\theta k}(\theta^t)}\tau^k(\theta^t)$$

Consumption dynamics

Proposition

The consumption process satisfies a Generalized Inverse Euler Equation

$$\frac{q}{\beta}\hat{\mathbb{E}}\left[\frac{1}{u'(\theta^{t+1})}|\theta^t\right] = \frac{1}{u'(\theta^t)}(1+s(\theta^t))$$

where

$$s(\theta^t) = \left(\frac{f_\theta(\theta^t)u''(\theta^t)}{u'(\theta^t)} - \hat{\mathbb{E}}\left[\rho_{t+1}(\theta^{t+1})\frac{\iota f_\theta(\theta^{t+1})u''(\theta^{t+1})}{u'(\theta^{t+1})}|\theta^t\right]\right)\frac{f_k(\theta^t)}{f_{\theta k}(\theta^t)}\tau^k(\theta^t)$$

- If persistence not too high $s(\theta^t) < 0 \rightarrow$ variance consumption and expected marginal utility permanently increasing (*immiseration*)
- Varying $k_{t+1}(\theta^t)$ extra force for immiseration

Back

Parametrization

- Utility:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

- Production function:

$$f(k, \theta) = z\theta k^{\alpha}$$

- Productivity:

$$\theta_t = \theta_{t-1}^{\rho} \varepsilon_t$$

  with $\log(\varepsilon_t) \sim N(\mu, \sigma_{\varepsilon}^2)$

- Calibration: $\beta = q = 0.95$, $\alpha = 3/4$, $\sigma = 2$, $\iota = 0.95$, $\mu = 1$ and $\sigma_{\varepsilon}^2 = 0.01$. Compare $\rho = 0$ and $\rho = 0.7$
- Dynamic programming (squared) with optimal control problem at every point in state space $(\lambda_-, \gamma_-, k, \theta_-)$
- Monte Carlo simulation: 10^6 draws over 25 periods

Back

Recursive planning problem

$$K_t(v_{t-1}, \Delta_{t-1}, \theta^{t-1}, k_t) =$$

$$\min \int \left[k_{t+1}(\theta^t) - b_t(\theta^t) + qK_{t+1}(v_t(\theta^t), \Delta_t(\theta^t), \theta^t, k_{t+1}(\theta^t))\right]\varphi(\theta_t|\theta^{t-1})d\theta_t$$

$$\textbf{(PK)}\ w_t(\theta^t) = u(c(\theta^t)) + \beta v_t(\theta^t)$$

$$v_{t-1} = \int w_t(\theta^t)\varphi(\theta_t|\theta^{t-1})d\theta_t$$

$$\textbf{(IC)}\ \dot{w}(\theta) = u'(c(\theta^t))\phi f_\theta(k_t, \theta_t) + \beta\Delta_t(\theta^t) \quad [\mu(\theta^t)]$$

$$\Delta_{t-1} = \int w_t(\theta^t)\frac{\partial\varphi(\theta_t|\theta^{t-1})}{\partial\theta^{t-1}}d\theta_t$$

$$\textbf{(Feasibility)}\quad c(\theta^t) = f(k_t, \theta_t) - b_t(\theta^t)$$

Back

Entrepreneur's problem implementation i.i.d

$$W_1 = W_0 + \frac{\chi \overline{f}(k_{SB})}{1-q}$$

To first order $\frac{dc_t}{df(k_{SB},\theta_t)} \approx (1-q)\chi$. From the regressions: $\hat{\chi} = \frac{\beta_{returns}}{(1-q)} = 0.958 \approx \phi$

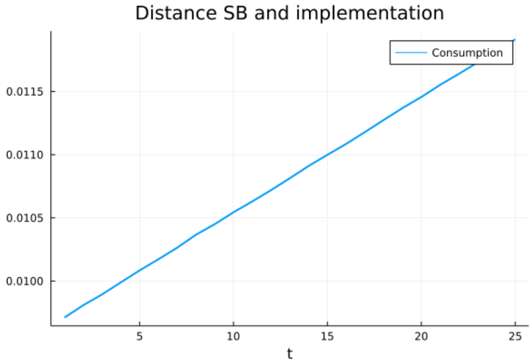
$$\begin{aligned} \mathcal{W}\left(W_t, \theta_t\right) &= \max_{\widetilde{\theta}} u\left(\widetilde{c}_t\right) + \beta \mathbb{E}\left[\mathcal{W}(W_{t+1}, \theta_{t+1})\right] \\ \text{s.t } W_{t+1} &= qC\left(W_t, \widetilde{\theta}_t\right) \\ c_t &= (1-q)C\left(W_t, \widetilde{\theta}_t\right) \\ \widetilde{c}_t &= c_t + \phi\left(f\left(k_{SB}, \theta\right) - f\left(k_{SB}, \widetilde{\theta}\right)\right) \end{aligned} \tag{1}$$

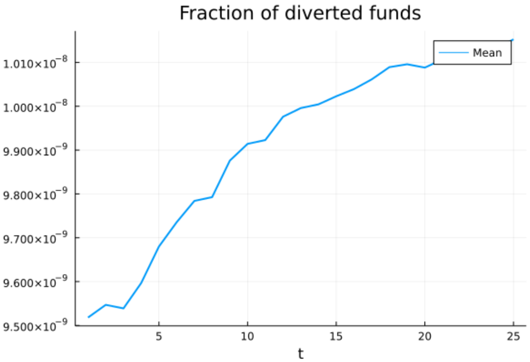
where

$$C(W_t, \theta_t) = \frac{1}{q} W_t + \hat{\chi}\left(f(k_{SB}, \widetilde{\theta}_t) - \overline{f}(k_{SB})\right)$$

Back

Plots implementation iid





Back