Procurement with rationing of capacity constrained suppliers

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Overview

1 Introduction

2 Bayes Nash Equilibrium with 2 players

3 Extensions

4 Literature Review

5 Conclusion and policy implications

Introduction

- Procurement auctions are ubiquitous.
- Procurement targets might be larger than any individual firm's capacity.
- Cumulative capacity of suppliers does not exactly equate target. Allocation and market clearing problem.
- This study: an auction with multiple winners and a rationing rule, employed in renewable energy auctions of India.
- Contributes to literature on procurement with multiple suppliers (Anton **and** Yao, 1989), capacity constrained suppliers (Chaturvedi, 2015).

Introduction: Relevant institutional details

- Auctions conducted by Solar Energy Corporation of India (SECI) and National Thermal Power Corporation (NTPC).
- Awards the right to build a solar/wind power plant of certain capacity, and sell its production for 25 years.
- The capacity award and tariff on produced electricity determined during auction.
- Allocation is decided in 2 stage auctions.
- Here: analyse second stage, which is an open uniform price auction with rationing rule, and publicly known capacities.
- Relevant information from first stage: reserve bid, set of players in 2^{nd} stage and their corresponding capacity.

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  - As auction proceeds, the bid on the clock reduces.
  - Bidders exit the arena permanently at any displayed bid if they don't want the award at that or lower bid.
  - At every exit, auction continues if there is excess demand, else auction ends and
  - **Rationing rule**: last exiting bidder supplies the residual of target and capacities of bidders still in the auction.
  - The tariff for winning bidders is the bid at which auction ends.

Introduction: Example with target=300

Bidder	Capacity	Price	Award	Target
1	100	3.4	100	
2	60	3.4	60	
3	40	3.4	40	
4	150	3.4	100	**300**
5	100	3.8	0	
6	400	3.9	0	
7	200	4.2	0	

Table: Final allocations

Bidding strategies: preview of results

- Characterize BNE in pure strategies for 2 players with privately known cost, drawn independently from same distribution
- **Key theoretical result:** Player with highest capacity is less aggressive, and can exit at reserve bid with positive probability.
- There is a unique non-pooling equilibrium for 2 players, in addition to pooling equilibria.
- **Policy implication:** Rationing and capacity constraint $\implies$ inefficient selection.
- Paper contains extensions with 3 bidders, and asymmetric cost distributions.

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Notations and assumptions

- Auctioneer sets procurement target M and reserve bid b^R.
- Simultaneously, N risk-neutral bidders publicly reveal their respective capacities q_i.
- Bidder B_i discovers her marginal cost c_i. $c_i \in [\underline{c}, \bar{c}]$. Private information.
- $c_i \overset{i.i.d}{\sim} F(c)$, where F is atomless and public information.
  $f(c) = F'(c)$; $\sigma(c) = f(c)/F(c)$,
  Key-assumption: $\sigma'(c) < 0$ (monotone hazard rate).

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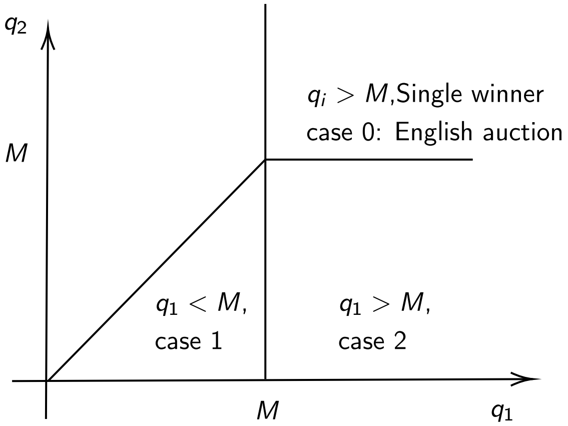
  she exits when clock shows b_i, if no other bidder in the arena has exited at bids weakly greater than b_i.
- $\beta_i(c_i)$: equilibrium bid function of B_i.
- Characterize bayes nash equilibria for auctions with 2 bidders.

Possible cases with 2 players ($q_1 > q_2$)

Assume $q_1 > q_2$ w.l.o.g



Equilibrium: Case 1 (2 small bidders)

- $q_i < M$, $q_1 + q_2 > M$.
- Ex-post payoffs:
  Winning: $\pi_i^W(b_i; c_i, \mathrm{q}, \mathrm{b}_{-i}) = q_i(p - c_i)$;
  Losing: $\pi_i^L(b_i; c_i, \mathrm{q}, \mathrm{b}_{-i}) = (M - q_{-i})(p - c_i)$;
  $p = Max\{b_1, b_2\}$ is the uniform price.
- Tie breaking rule: Residual award to B_1.

Equilibrium: Case 1 (2 small bidders)

- Expected payoff

$$\pi_i(b_i; c_i, b_{-i}) = Pr(b_{-i} < b_i)(M - q_{-i})(b_i - c_i) \\ + Pr(b_{-i} > b_i)q_i\mathbb{E}_F((b_{-i} - c_i)|b_{-i} > b_i)$$

- 2 pooling BNE- B_i exits when the clock starts ($b_i = b^R$), and B_{-i} never exits ($b_{-i} = -\infty$).

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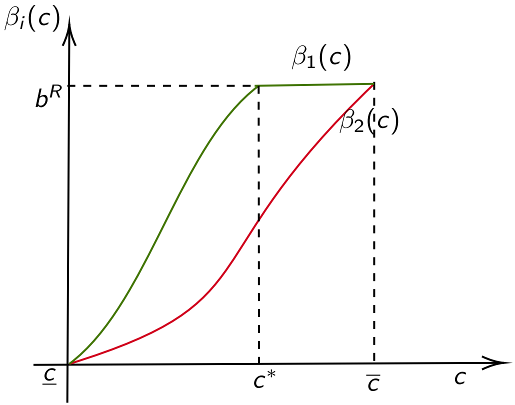
- 2 pooling BNE- B_i exits when the clock starts ($b_i = b^R$), and B_{-i} never exits ($b_{-i} = -\infty$).
  Requires a crazy type. Inefficient allocation.
- Any other equilibrium?

Equilibrium example, $q_1 > q_2$



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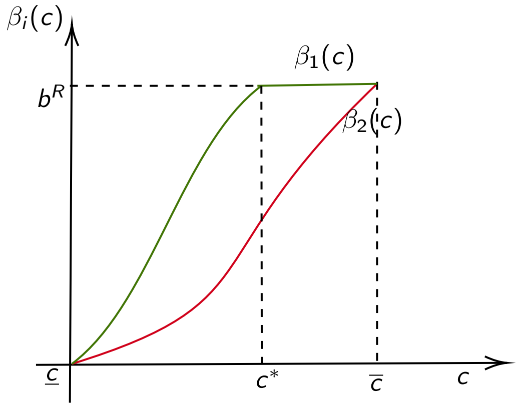

Bunching by larger player at b^R.

Equilibrium formal statement: Case 1

Lemma 1

For each B_i, $\beta_i(c)$ constitute a non-pooling Bayes Nash Equilibrium of the 2 player clock auction with rationing if and only if it satisfies following properties:

1. $\beta_i(c)$ *is non-decreasing in* c.
2. $\beta_i(c)$ *is continuous and atomless for* $b < b^R$ *for both* i.
3. $lim_{c \to \underline{c}^+} \beta_i(c) = \underline{c}$
4. *For each player* B_i, $\beta_i(c)$ *solves:*

$$\sigma(\beta_{-i}^{-1}(\beta_i(c)))\beta_{-i}^{-1'}(\beta_i(c))(\beta_i(c) - c)(q_1 + q_2 - M) = (M - q_{-i}) \tag{1}$$

5. $\beta_2(\bar{c}) = b^R$, *and* $\exists c^*$ *such that* $\beta_1(c) = b^R$ $, \forall c \in [c^*, \bar{c}]$.

▸▸ Proof

Equilibrium: Intuition behind B_1 bunching

- Define $\phi_i(b)$ as inverse of bid function, $\beta_i(c)$, wherever invertible.

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- Cost of reducing bid by db: $(M - q_{-i})db$.
- B_1 has higher residual quantity vis-a-vis B_2, which makes competition costly for her on the margin.

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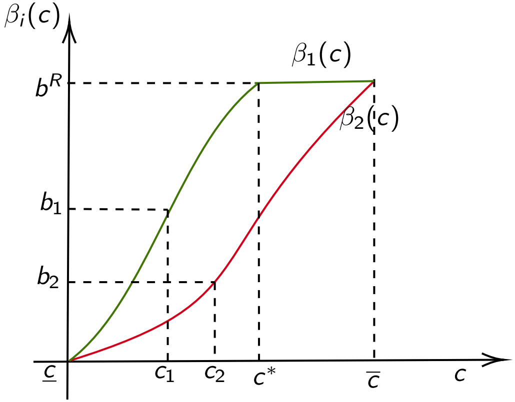
- Define $\phi_i(b)$ as inverse of bid function, $\beta_i(c)$, wherever invertible.
- Cost of reducing bid by db: $(M - q_{-i})db$.
- B_1 has higher residual quantity vis-a-vis B_2, which makes competition costly for her on the margin.
- Benefit of reducing bid by db:
  $\frac{f(\phi_{-i}(b))}{F(\phi_{-i}(b))}\phi'_{-i}(b)db(b - \phi_i(b))(q_1 + q_2 - M)$
- Higher bid gives her higher markup $(= b - \phi_1(b))$, which leads to a Marginal Benfit high enough to compensate for this cost.

Inefficient allocation



Equilibrum: Existence and Uniqueness

Theorem 1

Equilibrium described by Lemma 1 exists and is unique

Proof

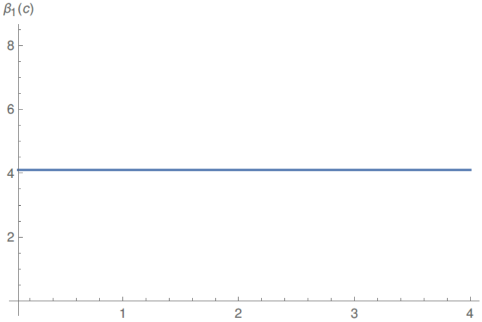
Equilibrium: Case 2 (1 large, 1 small bidder)

- $q_1 = M$, $q_2 < M$.
- $\pi_1^W = q_1(p - c_1)$, $\pi_1^L = (M - q_2)(p - c_i)$ where $p = Max\{b_1, b_2\}$.
- $\pi_2^W = q_2(p - c_1)$, $\pi_2^L = 0$
- B_2 bids her cost $(= c_2)$. B_1 maximises

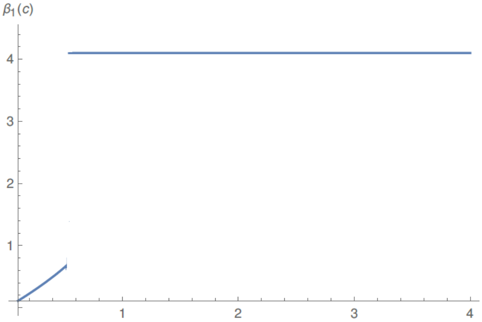
$$\pi_1(b_1; c_1, \beta_2(c)) = (M - q_2)(b_1 - c_1)F(b_1) + q_1 \int_{b_1}^{b^R} (x - c_1)dF(x)$$

- FOC for internal optima for B_1: $\sigma(\beta_1(c_1))(\beta_1(c_1) - c_1) = \frac{M - q_2}{q_2}$.

Case 2: examples



(a) $M = q_1 = 100,\ q_2 = 40$



(b) $M = q_1 = 100,\ q_2 = 80$

Figure: $\beta_1(c)$. $b^R = 4.1$, F is constrained Log-Normal. $\mu = 1$, $\sigma = 1$; $c_i \in [0, 4]$

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Asymmetric cost distributions:

If capacities same, B_i bunches if she thinks her cost is higher.

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1. Distributions can be ordered according to Reversed hazard rate, $\sigma_i(c) = \frac{f_i(c)}{F_i(c)}$.
   - B_2 is less competitive and bunches if $\sigma_2(c) > \sigma_1(c)\frac{M-q_2}{M-q_1}$.
   - Intuitively, **if B_2 is more likely to have high costs, she bunches**
2. Distributions have different suprema of supports, but same reverse hazard rates.
   - B_2 bunches only if $\bar{c}_2 > \bar{c}_1 + \Delta(M, q_1, q_2)$.
   - Intuitively, **if B_2 is likely to have higher costs, she bunches**

Formal result

Thus, efficiency can be restored if costs are drawn asymmetrically.

Extending to more players

- Add a very small bidder: Bidder B_3 with $q_3 < q_2$ while $q_1 + q_2 > M$; B_1 bunches.
- Semi-seperating equilibrium exists and has unique structure if $b^R = \bar{c}$. Pooling equilibria always exist. Formal results

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Literature

- Auctions:
  - Chaturvedi (2015) studied procurement with capacity constrained bidders, but only through simulations.
  - Krishna (2009) provides a good synthesis of ascending price auctions.
  - Split award auctions à la Anton **and** Yao (1989), Anton, Brusco **and** Lopomo (2010)
- Game of exit:
  - Levin (2004) provides a synthesis of results on symmetric war of attrition with single winner.
  - Nalebuff **and** Riley (1985) analyses asymmetric war of attrition, posits continuum of equilibria.
- Renewable energy auctions in India (Probst **andothers**, 2020; Ryan, 2021)

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Conclusion

- Analyse a novel auction mechanism being used in renewable energy auctions in India.
- With 2 players, there is a partially separating equilibrium, characterised by bunching at reserve by high quantity player
- With 3 players, partially separating equilibrium always exists if $b^R = \bar{c}$; pooling equilibrium always exists.
- While developed for procurement, the results can extend to a game of exit without sunk costs.
- If the costs are drawn from different distributions, inefficiency can be reduced if low quantity player is more likely to have higher cost.

Thanks!

Thank you!

Feedback or paper requests at manpreet.singh@psemail.eu

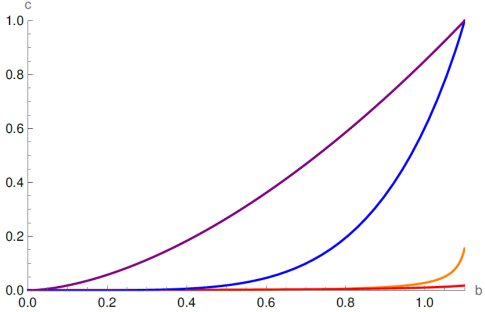
Overview

6 Appendix

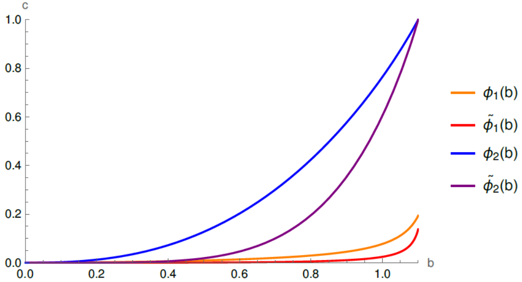
Comparative statics with respect to q_i

$q_1 \uparrow \implies B_1$ gains$\uparrow$ if win $\implies B_1$ more competitive

Also $\implies B_2$'s residual$\downarrow \implies B_2$ more competitive $\implies B_1$ response unclear



(a) ϕ: $q_1 = 60, q_2 = 50$
$\tilde{\phi}$: $q_1 = 80, q_2 = 50$



(b) ϕ: $q_1 = 60, q_2 = 50$
$\tilde{\phi}$: $q_1 = 60, q_2 = 45$

$$M = 100,\ c_i \overset{i.i.d}{\sim} U[0,1];\ b^R = 1.1.$$

Comparative statics in symmetric equilibrium

- Suppose $q_1 = q_2 = q < M$, $2q > M$, and $c_i \overset{i.i.d}{\sim} U(0,1)$.
- Equilibrium bid function is:

$$\beta(c) = \begin{cases} c^{\frac{2q-M}{M-q}}\left(b^R + \frac{2q-M}{2M-3q}(1 - c^{\frac{2M-3q}{M-q}})\right) & ; \; M \neq 1.5q \\ c.b^R - c.ln(c) & ; \; M = 1.5q \end{cases}$$

Corollary 1

Consider a symmetric clock auction with supplier rationing. Any increase in q (or decrease in M) makes players less competitive for all c as long as $M > 1.5q$, more competitive as long as $M < 1.5q$, and has no effect as long as $M = 1.5q$.

2P0F Extensions with different F

- Suppose $c_i \overset{i.i.d}{\sim} F_i(c)$, s.t $\sigma_1(c) < \sigma_2(c) \forall c$ or vice-versa.
- B_1 bunches if $\frac{\sigma_1(c)}{\sigma_2(c)} > \frac{M-q_1}{M-q_2} \forall c$.

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- B_1 bunches if $\frac{\sigma_1(c)}{\sigma_2(c)} > \frac{M-q_1}{M-q_2} \forall c$.
- Suppose $c_i \in [\underline{c}, \bar{c}_i]$ where $\bar{c}_1 < \bar{c}_2$ but F_is are such that $\sigma_1(c) = \sigma_2(c) \forall c < \bar{c}_1$.
- $\exists \Delta(M, q_1, q_2)$ such that B_2 bunches if $\bar{c}_2 > \bar{c}_1 + \Delta(M, q_1, q_2)$.

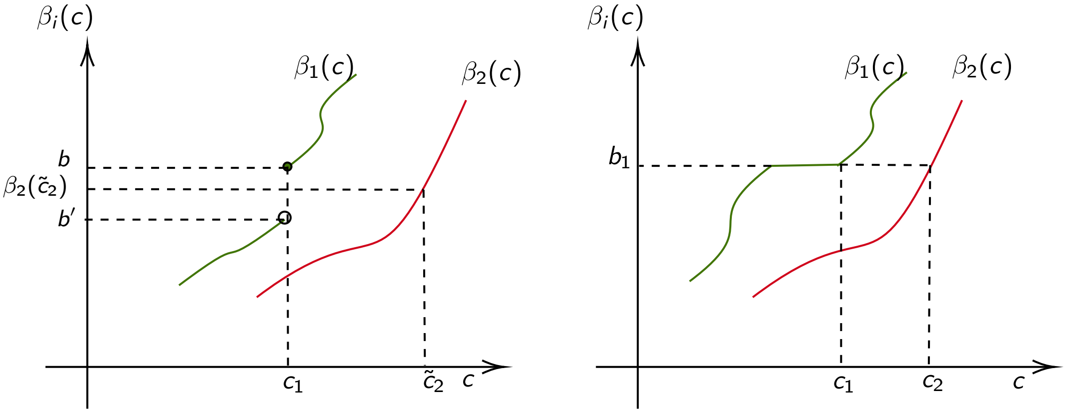
Lemma 1: Sketch of proof of property (i)

- Expected payoff of B_i follows SCP-IR, when B_{-i} plays non-decreasing strategy. Consider $b_1' > b_1$, $c_1' > c_1$.
- $A(b_1', b_1, c_1, b_2) \equiv \pi_1(b_1', c_1; b_2) - \pi_1(b_1, c_1; b_2) > 0$

$$
\begin{aligned}
&\pi_1(b_1', c_1'; b_2) - \pi_1(b_1, c_1'; b_2) \\
=&(M - q_2)[(b_1' - c_1')Pr(b_2 < b_1') - (b_1 - c_1')Pr(b_2 < b_1)] \\
&+ q_1[E(b_2 - c_1'|b_2 > b_1')Pr(b_2 > b_1') - E(b_2 - c_1'|b_2 > b_1)Pr(b_2 > b_1)] \\
=&(M - q_2)[(b_1' - c_1 + c_1 - c_1')Pr(b_2 < b_1') - (b_1 - c_1 + c_1 - c_1')Pr(b_2 < b_1)] \\
&+ q_1[E(b_2 - c_1 + c_1 - c_1'|b_2 > b_1')Pr(b_2 > b_1') \\
&- E(b_2 - c_1 + c_1 - c_1'|b_2 > b_1)Pr(b_2 > b_1)] \\
=&\underbrace{A(b_1', b_1, c_1, b_2)}_{>0} + \underbrace{(M - q_2 - q_1)}_{<0}\underbrace{(c_1 - c_1')}_{<0}\underbrace{[Pr(b_2 < b_1') - Pr(b_2 < b_1)]}_{>0}
\end{aligned}
$$

Proof of property (ii)

Show the deviations as depicted below:



(a) (b)

Proof of other properties

- For (iii), same argument as Bertrand
- (i) and (ii) imply $\beta_i(c)$ is invertible. Define inverse $\phi_i(b)$ as:

$$\phi_i(b) := \begin{cases} \beta_i^{-1}(b) & \text{for } b < b^R \\ Inf\{c : \beta_i(c) = b^R\} & \text{for } b = b^R \end{cases}$$

- (iv) is FOC for optimisation at interior point
- At any point of intersection (b, c), $\frac{\phi_2'(b)}{\phi_1'(b)} = \frac{M-q_2}{M-q_1} > 1$.
- Thus, at max 1 intersection as shown in the figure.

Back

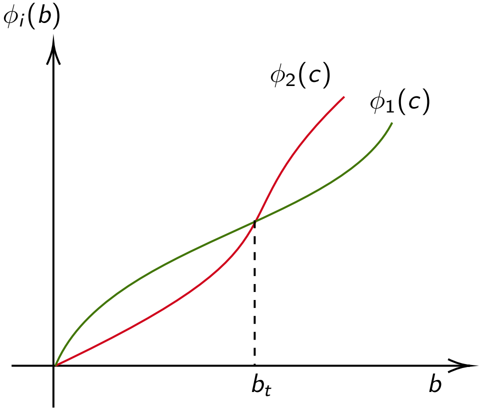


Figure: Possible intersection between $\phi_1(b)$ and $\phi_2(b)$

Back

Property (v)

- As $c \to \underline{c}^+$, $\beta_1(c) \to \underline{c}^+, \beta_2(c) \to \underline{c}^+$. Thus, $\lim_{b \to \underline{c}^+} \phi_i(b) = \underline{c}$
- Consider some $\delta \to 0^+$. Suppose $\phi_i(\underline{c} + \delta/n) = \underline{c} + \epsilon_i(\delta/n)$, $n \in N, n \geq 1$.
- $\phi_i(\underline{c} + \delta) - \phi_i(\underline{c} + \delta/n) \approx \frac{n-1}{n}\delta\phi_i'(\underline{c} + \delta)$ for each i

$$\frac{\phi_2'(\underline{c} + \delta)}{\phi_1'(\underline{c} + \delta)} \approx \frac{\phi_2(\underline{c} + \delta) - \phi_2(\underline{c} + \delta/n)}{\phi_1(\underline{c} + \delta) - \phi_1(\underline{c} + \delta/n)} = \frac{\epsilon_2(\delta) - \epsilon_2(\delta/n)}{\epsilon_1(\delta) - \epsilon_1(\delta/n)}$$

$$\frac{\phi_2'(\underline{c} + \delta)}{\phi_1'(\underline{c} + \delta)} = \frac{M - q_2}{M - q_1} \frac{\epsilon_1(\delta)}{\epsilon_2(\delta)} \frac{\delta - \epsilon_2(\delta)}{\delta - \epsilon_1(\delta)} = \frac{\epsilon_2(\delta) - \epsilon_2(\delta/n)}{\epsilon_1(\delta) - \epsilon_1(\delta/n)}$$

$$\implies \underbrace{\frac{M - q_2}{M - q_1}}_{>1} \approx \underbrace{\frac{\epsilon_2(\delta)(\delta - \epsilon_1(\delta))}{\epsilon_1(\delta)(\delta - \epsilon_2(\delta))}}_{>1, \text{ if } \epsilon_2(\delta) > \epsilon_1(\delta)} \quad \underbrace{\frac{\epsilon_2(\delta) - \epsilon_2(\delta/n)}{\epsilon_1(\delta) - \epsilon_1(\delta/n)}}_{\substack{>1, \text{ if } \epsilon_2(\delta) > \epsilon_1(\delta) \\ \text{because } \epsilon_i(\delta/n) \to 0 \text{ as } n \to \infty}}$$

Proof of Theorem 1

Define:

$$\phi_i(b) := \begin{cases} \beta_i^{-1}(b) & for \;\; b < b^R \\ Inf\{c : \beta_i(c) = b^R\} & for \;\; b = b^R \end{cases}$$

Sketch of proof (assuming $\underline{c} = 0$):

1. Consider a sequence $\{\frac{\delta}{2^n}\}_{n=1}^{\infty}$, where $\delta \in (0, \bar{c})$.

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Sketch of proof (assuming $\underline{c} = 0$):

1. Consider a sequence $\{\frac{\delta}{2^n}\}_{n=1}^{\infty}$, where $\delta \in (0, \bar{c})$.
2. For each n, show the uniqueness and existence of solution $(\phi_{1n}(b), \phi_{2n}(b))$, where $dom(\phi_{in}(b)) = [\frac{\delta}{2^n}, b^R]\ \forall i$, to this BVP:

$$\phi'_{2n}(b) = \frac{M - q_2}{q_1 + q_2 - M} \frac{1}{\sigma(\phi_{2n}(b))(b - \phi_{1n}(b))}$$
$$\phi'_{1n}(b) = \frac{M - q_1}{q_1 + q_2 - M} \frac{1}{\sigma(\phi_{1n}(b))(b - \phi_{2n}(b))} \quad (2)$$

$\phi_{2n}(b^R) = \bar{c}$, $\phi_{2n}(\frac{\delta}{2^n}) = \phi_{1n}(\frac{\delta}{2^n}) = \frac{\delta}{2^n}$.

Sketch of proof theorem 1

3. Define a function w_{in} over the domain $[0, b^R]$ as $w_{in}(b) = \phi_{in}(b)$ for $b \in [\frac{\delta}{2^n}, b^R]$ and $w_{in}(b) = \phi_{in}(\frac{\delta}{2^n})$ otherwise.

Sketch of proof theorem 1

3. Define a function w_{in} over the domain $[0, b^R]$ as $w_{in}(b) = \phi_{in}(b)$ for $b \in [\frac{\delta}{2^n}, b^R]$ and $w_{in}(b) = \phi_{in}(\frac{\delta}{2^n})$ otherwise.
4. Monotone convergence theorem implies w_{in} converges, and show that $\phi_{in}(\frac{\delta}{2^n}) \to 0$ as $n \to \infty$, which shows property (iii).

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Proof of Theorem 1

- Suppose 2 equilibria ϕ and $\hat{\phi}$ such that $\hat{\phi}_1(b^R) = \hat{c^*} > c^* = \phi_1(b^R)$.
- $\hat{\phi}_1(b)$ and $\phi_1(b)$ can't intersect. If they intersect at some (b^t, c^t), then there are 2 solutions to the boundary value problem defined by FOCs and $\phi_2(b^R) = \bar{c}$, $\phi_1(b^t) = c^t$.

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- Thus, $\hat{\phi}_1(b) > \phi_1(b) \forall b \leq b^R$.

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- Thus, $\hat{\phi}_1(b) > \phi_1(b) \forall b \leq b^R$.
- From F.O.Cs, $\sigma(\hat{\phi}_2(b))\hat{\phi}_2{}'(b) > \sigma(\phi_2(b))\phi_2{}'(b)$.
- If $\hat{\phi}_2(b) \geq \phi_2(b) \forall b$, $\hat{\phi}_2{}'(b) > \phi_2'(b) \forall b$. This must hold true at b^R. However, for $\phi_2(b^R) = \hat{\phi}_2(b^R)$, we need $\hat{\phi}_2{}'(b^R) < \phi_2'(b^R)$. Contradiction.
- If $\exists b_2^t$ where $\hat{\phi}_2$ and ϕ_2 intersect, $\hat{\phi}_2{}'(b_2^t) < \phi_2'(b_2^t)$. Then $\sigma(\hat{\phi}_2(b_2^t))\hat{\phi}_2{}'(b_2^t) < \sigma(\phi_2(b_2^t))\phi_2{}'(b_2^t)$ which $\implies$ $\hat{\phi}_1(b_2^t) < \phi_1(b_2^t)$. Contradiction.
- Thus, if $\hat{\phi}_1(b) > \phi_1(b)$, $\hat{\phi}_2(b) < \phi_2(b)$ $\forall b$. Implies point of intersection has monotonic relation with $\phi_1(b^R)$. Thus, $\exists$ only one c^* such that $\phi_1(\underline{c}) = \phi_2(\underline{c})$.

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Lemma for asymmetric support 2 players

Lemma 2

For each B_i, $\beta_i(c)$ constitutes a non-trivial BNE of the 2 player asymmetric clock auction with rationing if only if it satisfies following properties:

1. $\beta_i(c)$ *is non-decreasing in* c.
2. $\beta_i(c)$ *is continuous and atomless for* $b < b^R$ *for both* i.
3. $\lim_{c \to \underline{c}^+} \beta_i(c) = \underline{c}$, $\forall i$.
4. *For each player* B_i, $\beta_i(c)$ *solves:*

$$\sigma_{-i}(\beta_{-i}^{-1}(\beta_i(c)))\beta_{-i}^{-1'}(\beta_i(c))(\beta_i(c) - c)(q_1 + q_2 - M) = (M - q_{-i}) \tag{3}$$

5. $\exists \Delta$ *such that if* $\bar{c}_2 - \bar{c}_1 < \Delta$, $\exists c_1^*$ *such that* $\beta_1(c) = b^R$, $\forall c \in [c_1^*, \bar{c}_1]$ *and* $\beta_2(\bar{c}_2) = b^R$, *else,* $\exists c_2^*$ *such that* $\beta_2(c) = b^R$, $\forall c \in [c_2^*, \bar{c}_2]$ *and* $\beta_1(\bar{c}_1) = b^R$

Proof of lemma 2

Properties (i) to (iv) are same as before. Thus, $\phi_2'(b) > \phi_1'(b)$ at point of intersection. For property (v),

- $\phi_2(b) > \phi_1(b)$, in the same way as before.
- If $\bar{c}_1 > \bar{c}_2$, B_1 bunches.
- Suppose $\bar{c}_1 \leq \bar{c}_2$ and consider two pairs of supremum of support of (c_1, c_2), $(\bar{c}_1, \bar{c}_1)$ and $(\bar{c}_1, \hat{\bar{c}}_2)$ such that $\hat{\bar{c}}_2 > \bar{c}_1$.
- Denote the corresponding equilibrium inverse bid functions generated from these suprema as $\phi_i(b)$ and $\hat{\phi}_i(b)$ respectively.
- From Lemma 1, we know that $\phi_1(b^R) = c^* < \bar{c}_1$ and $\phi_2(b^R) = \bar{c}_1$ and that $\underset{b \to \underline{c}^+}{lim} \phi_i(c) = \underline{c}$ for both i.
- Either $\hat{\phi}_2(b^R) > \phi_2(b^R) = \bar{c}_1$ or $\hat{\phi}_2(b^R) = \hat{c}_2^* < \phi_2(b^R) = \bar{c}_1$.

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Contradictions

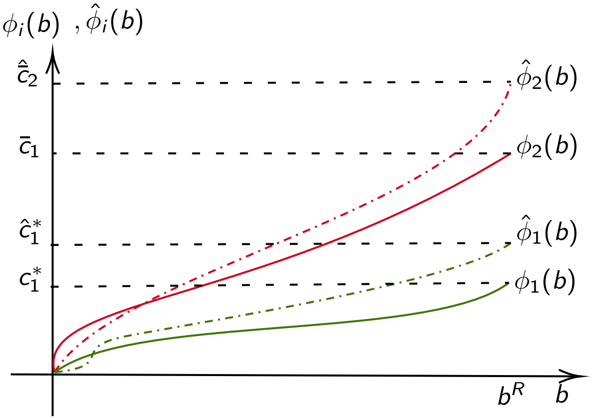

Figure: Intersecting solution curves

2 small 1 very small firm

- Framework same as before, except a player B_3 with quantity $q_3 < q_2 < q_1$, $q_i < M$, $q_1 + q_2 > M$, $q_i + q_3 < M$.
- Relevant concept is PBE, exit of B_3 starts a subgame.
- $\mathcal{P}(b)$ is the set of partially rationed bidders at any bid b
- $\mathcal{L}(b)$ is the set of fully rationed bidders.
- Here, $\mathcal{P}(b^R) = \{B_1, B_2\} = \mathcal{A}2$, $\mathcal{L}(b^R) = \{B_3\}$.
- Equilibrium bid function of B_i denoted by $\beta_{i,\mathcal{A}2,B_3}(c)$ in the subgame with all players, and $\beta_{i,\mathcal{A}2,\emptyset}(c)$ in the subgame started by B_3's exit

$$\begin{aligned}\pi_i(b_i; c_i, \mathbf{b}) =& (M - q_{-i} - q_3)(b_i - c_i)Pr(b_i = max_j\{b_j\}) \\ &+ q_i\mathbb{E}(b_{-i} - c_i | b_{-i} > b_3, b_{-i} > b_i)Pr(b_{-i} = max_j\{b_j\}) \\ &+ \mathbb{E}(\pi^*_{i,\mathcal{A}2,\emptyset}(b_3) | b_i < b_3, b_{-i} < b_3)Pr(b_3 = max_j\{b_j\})\end{aligned}$$

where $\pi^*_{i,\mathcal{A}2,\emptyset}(b_3)$ is the payoff for B_i in the subgame started by B_3's exit.

Equilibrium

Lemma 3

$\beta_{3,\mathcal{A}2,B_\mathbf{3}}(c) = c$. $\beta_{i,\mathcal{A}2,B_\mathbf{3}}(c)$ for $i \in \{1,2\}$, gives a PBE if and only if:

1. *$\beta_{i,\mathcal{A}2,B_\mathbf{3}}(c)$ is non-decreasing in c.*
2. *$\beta_{i,\mathcal{A}2,B_\mathbf{3}}(c)$ is continuous and atomless for $b < b^R$ for both i.*
3. *$\beta_{i,\mathcal{A}2,B_\mathbf{3}}(\underline{c}) = \underline{c}$, $\forall i$.*
4. *$\forall i$, $\beta_{i,\mathcal{A}2,B_\mathbf{3}}(c_i)$, solve following differential equations:*

$$\begin{aligned}&(\pi^*_{i,\mathcal{A}2,\emptyset}(b;c_i) - (M - q_{-i} - q_3)(\beta_{i,\mathcal{A}2,B_\mathbf{3}}(c_i) - c_i))\sigma(\beta_{i,\mathcal{A}2,B_\mathbf{3}}(c_i))1_{b\leq\bar{c}} \\ &+ (\beta_{i,\mathcal{A}2,B_\mathbf{3}}(c_i) - c_i)(\sum_j q_j - M)\sigma(\beta^{-1}_{-i,\mathcal{A}2,B_\mathbf{3}}(\beta_{i,\mathcal{A}2,B_\mathbf{3}}(c_i)))\beta^{-1\prime}_{-i,\mathcal{A}2,B_\mathbf{3}}(\beta_{i,\mathcal{A}2,B_\mathbf{3}}(c_i)) = M - q_{-i} - q_3\end{aligned} \tag{4}$$

   where $\pi^*_{i,\mathcal{A}2,\emptyset}(b;c_i)$ is B_i's continuation value if B_3 exits at b.

5. *BNE in the subgame started by B_3's exit at b is as per Lemma 2*
6. *$\exists c_1^* \leq \bar{c}$ such that $\beta_{1,\mathcal{A}2,B_\mathbf{3}}(c) = b^R, \forall c \in [c_1^*, \bar{c}]$. $\beta_{2,\mathcal{A}2,B_\mathbf{3}}(\bar{c}) = b^R$ if $b^R > \bar{c}$ and $\lim \beta \ldots (c) = b^R$ if $b^R = \bar{c}$*

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Proof

Proof of (i) proceeds as before, except for some adjustment for continuation value. I can write continuation value of B_1 when B_3 exits at some bid c_3 as:

$$\pi^*_{1,\mathcal{A}2,\emptyset}(c_3,c_1) = \underset{b_1''\leq c_3}{Max}\left[(M-q_2)(b_1''-c_1)\frac{F(\phi_2^{sg}(b_1''))}{a(c_3)} + q_1\int_{b_1''}^{c_3}(x-c_1)\frac{dF(\phi_2^{sg}(x))}{a(c_3)}\right]$$

$$\pi^*_{1,\mathcal{A}2,\emptyset}(c_3,c_1) = \underset{b_1''\leq c_3}{Max}\left[(M-q_2)(b_1''-c_1+c_1'-c_1')\frac{F(\phi_2^{sg}(b_1''))}{a(c_3)} + q_1\int_{b_1''}^{c_3}(x-c_1+c_1'-c_1')\frac{dF^{sg}(\phi_2(x))}{a(c_3)}\right]$$

$$\implies \pi^*_{1,\mathcal{A}2,\emptyset}(c_3,c_1) \leq \underset{b_1''\leq c_3}{Max}\left[(M-q_2)(x-c_1')\frac{F(\phi_2^{sg}(b_1''))}{a(c_3)} + q_1\int_{b_1''}^{c_3}(x-c_1')\frac{dF(\phi_2^{sg}(x))}{a(c_3)}\right]$$

$$+ \underset{b_1''\leq c_3}{Max}\left[(M-q_2)(c_1'-c_1)\frac{F(\phi_2^{sg}(b_1''))}{a(c_3)} + q_1\int_{b_1''}^{c_3}(c_1'-c_1)\frac{dF(\phi_2^{sg}(x))}{a(c_3)}\right]$$

$$\implies \pi_1(c_3,c_1') - \pi_1(c_3,c_1) \geq -\underset{b_1''\leq c_3}{Max}\left[(M-q_2)(c_1'-c_1)\frac{F(\phi_2^{sg}(b_1''))}{a(c_3)} + q_1\int_{b_1''}^{c_3}(c_1'-c_1)\frac{dF(\phi_2^{sg}(x))}{a(c_3)}\right] \geq q_1(c_1'-c_1) \tag{5}$$

Rest of the proof would proceed as in 2 small firms case.

Proof

Proof of (i) proceeds as before, except for some adjustment for continuation value. I can write continuation value of B_1 when B_3 exits at some bid c_3 as:

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$$\pi^*_{1,\mathcal{A}2,\emptyset}(c_3,c_1) = \underset{b_1''\le c_3}{Max}\left[(M-q_2)(b_1''-c_1+c_1'-c_1')\frac{F(\phi_2^{sg}(b_1''))}{a(c_3)} + q_1\int_{b_1''}^{c_3}(x-c_1+c_1'-c_1')\frac{dF^{sg}(\phi_2(x))}{a(c_3)}\right]$$

$$\implies \pi^*_{1,\mathcal{A}2,\emptyset}(c_3,c_1) \le \underset{b_1''\le c_3}{Max}\left[(M-q_2)(x-c_1')\frac{F(\phi_2^{sg}(b_1''))}{a(c_3)} + q_1\int_{b_1''}^{c_3}(x-c_1')\frac{dF(\phi_2^{sg}(x))}{a(c_3)}\right]$$

$$+ \underset{b_1''\le c_3}{Max}\left[(M-q_2)(c_1'-c_1)\frac{F(\phi_2^{sg}(b_1''))}{a(c_3)} + q_1\int_{b_1''}^{c_3}(c_1'-c_1)\frac{dF(\phi_2^{sg}(x))}{a(c_3)}\right]$$

$$\implies \pi_1(c_3,c_1') - \pi_1(c_3,c_1) \ge -\underset{b_1''\le c_3}{Max}\left[(M-q_2)(c_1'-c_1)\frac{F(\phi_2^{sg}(b_1''))}{a(c_3)} + q_1\int_{b_1''}^{c_3}(c_1'-c_1)\frac{dF(\phi_2^{sg}(x))}{a(c_3)}\right] \ge q_1(c_1'-c_1) \tag{5}$$

Rest of the proof would proceed as in 2 small firms case.

Proof of (ii),(iii),(iv) same as before. Back

Proving property (v)

At any point of intersection (b_t, c_t) of $\phi_{1,\mathcal{A}2,B_\mathbf{3}}(b)$ and $\phi_{2,\mathcal{A}2,B_\mathbf{3}}(b)$,

$$\frac{\phi'_{2,\mathcal{A}2,B_\mathbf{3}}(b_t)}{\phi'_{1,\mathcal{A}2,B_\mathbf{3}}(b_t)} = \frac{M - q_2 - q_3 - (\pi^*_{1,\mathcal{A}2,\emptyset}(b_t, c_t) - (M - q_2 - q_3)(b_t - c_t))\sigma(b_t)}{M - q_1 - q_3 - (\pi^*_{2,\mathcal{A}2,\emptyset}(b_t, c_t) - (M - q_1 - q_3)(b_t - c_t))\sigma(b_t)}$$
$$\implies \frac{\phi'_{2,\mathcal{A}2,B_\mathbf{3}}(b_t)}{\phi'_{1,\mathcal{A}2,B_\mathbf{3}}(b_t)} = \frac{(M - q_2 - q_3) - q_3(b_t - c_t)\sigma(b_t)}{(M - q_1 - q_3) - (\sum_{j=1}^{3} q_j - M)(b_t - c_t)\sigma(b_t)} > 1 \quad (6)$$

Proving property (v)

At any point of intersection (b_t, c_t) of $\phi_{1,\mathcal{A}2,B_3}(b)$ and $\phi_{2,\mathcal{A}2,B_3}(b)$,

$$
\begin{aligned}
\frac{\phi'_{2,\mathcal{A}2,B_3}(b_t)}{\phi'_{1,\mathcal{A}2,B_3}(b_t)} &= \frac{M - q_2 - q_3 - (\pi^*_{1,\mathcal{A}2,\emptyset}(b_t, c_t) - (M - q_2 - q_3)(b_t - c_t))\sigma(b_t)}{M - q_1 - q_3 - (\pi^*_{2,\mathcal{A}2,\emptyset}(b_t, c_t) - (M - q_1 - q_3)(b_t - c_t))\sigma(b_t)} \\
\implies \frac{\phi'_{2,\mathcal{A}2,B_3}(b_t)}{\phi'_{1,\mathcal{A}2,B_3}(b_t)} &= \frac{(M - q_2 - q_3) - q_3(b_t - c_t)\sigma(b_t)}{(M - q_1 - q_3) - (\sum_{j=1}^{3} q_j - M)(b_t - c_t)\sigma(b_t)} > 1
\end{aligned} \tag{6}
$$

Thus, at most one intersection, as in 2P0F.

Proving property (v)

At any point of intersection (b_t, c_t) of $\phi_{1,\mathcal{A}2,B_3}(b)$ and $\phi_{2,\mathcal{A}2,B_3}(b)$,

$$\begin{aligned}\frac{\phi'_{2,\mathcal{A}2,B_3}(b_t)}{\phi'_{1,\mathcal{A}2,B_3}(b_t)} &= \frac{M - q_2 - q_3 - (\pi^*_{1,\mathcal{A}2,\emptyset}(b_t, c_t) - (M - q_2 - q_3)(b_t - c_t))\sigma(b_t)}{M - q_1 - q_3 - (\pi^*_{2,\mathcal{A}2,\emptyset}(b_t, c_t) - (M - q_1 - q_3)(b_t - c_t))\sigma(b_t)} \\ \implies \frac{\phi'_{2,\mathcal{A}2,B_3}(b_t)}{\phi'_{1,\mathcal{A}2,B_3}(b_t)} &= \frac{(M - q_2 - q_3) - q_3(b_t - c_t)\sigma(b_t)}{(M - q_1 - q_3) - (\sum_{j=1}^{3} q_j - M)(b_t - c_t)\sigma(b_t)} > 1\end{aligned} \tag{6}$$

Thus, at most one intersection, as in 2P0F.In the immediate neighbourhood of $\underline{c}$, if B_1 bunches in the subgame,

$$\begin{aligned}\frac{\delta - \epsilon_1(\delta)}{\delta - \epsilon_2(\delta)} \frac{(q_3\delta + \epsilon_2(\delta)(\epsilon_2(\delta) - \epsilon_2(\delta/n))(q_1 + q_2 + q_3 - M))}{(q_1 + q_2 + q_3 - M)(\delta + \epsilon_1(\delta)(\epsilon_1(\delta) - \epsilon_1(\delta/n)))} &= \frac{M - q_2 - q_3}{M - q_1 - q_3} \\ \frac{\epsilon_2(\delta)(\epsilon_2(\delta) - \epsilon_2(\delta/n))}{\epsilon_1(\delta)(\epsilon_1(\delta) - \epsilon_1(\delta/n))} \approx \frac{q_3}{q_1 + q_2 + q_3 - M} &< 1\end{aligned} \tag{7}$$

proof contd.

Equation in (7) together imply

$$\frac{\delta - \epsilon_1(\delta)}{\delta - \epsilon_2(\delta)} \frac{\epsilon_2(\delta)(\epsilon_2(\delta) - \epsilon_2(\delta/n))}{\epsilon_1(\delta)(\epsilon_1(\delta) - \epsilon_1(\delta/n))} = \frac{M - q_2 - q_3}{M - q_1 - q_3}$$

which implies $\epsilon_2(\delta) > \epsilon_1(\delta)$.

proof contd.

Equation in (7) together imply

$$\frac{\delta - \epsilon_1(\delta)}{\delta - \epsilon_2(\delta)} \frac{\epsilon_2(\delta)(\epsilon_2(\delta) - \epsilon_2(\delta/n))}{\epsilon_1(\delta)(\epsilon_1(\delta) - \epsilon_1(\delta/n))} = \frac{M - q_2 - q_3}{M - q_1 - q_3}$$

which implies $\epsilon_2(\delta) > \epsilon_1(\delta)$.Thus, no bunching by B_1.

proof contd.

Equation in (7) together imply

$$\frac{\delta - \epsilon_1(\delta)}{\delta - \epsilon_2(\delta)} \frac{\epsilon_2(\delta)(\epsilon_2(\delta) - \epsilon_2(\delta/n))}{\epsilon_1(\delta)(\epsilon_1(\delta) - \epsilon_1(\delta/n))} = \frac{M - q_2 - q_3}{M - q_1 - q_3}$$

which implies $\epsilon_2(\delta) > \epsilon_1(\delta)$.Thus, no bunching by B_1.

No contradiction when bunching by B_2.

proof contd.

Equation in (7) together imply

$$\frac{\delta - \epsilon_1(\delta)}{\delta - \epsilon_2(\delta)} \frac{\epsilon_2(\delta)(\epsilon_2(\delta) - \epsilon_2(\delta/n))}{\epsilon_1(\delta)(\epsilon_1(\delta) - \epsilon_1(\delta/n))} = \frac{M - q_2 - q_3}{M - q_1 - q_3}$$

which implies $\epsilon_2(\delta) > \epsilon_1(\delta)$.Thus, no bunching by B_1.
No contradiction when bunching by B_2.

Necessary and sufficient conditions for B_2 bunching

If B_2 bunches in the subgame, $\exists\ \tilde{\epsilon}_2(\delta) < \epsilon_2(\delta)$ such that B_2 pools for costs between $\underline{c} + \tilde{\epsilon}_2(\delta)$ and $\underline{c} + \epsilon_2(\delta)$. Therefore,

$$\frac{\sigma(\underline{c} + \tilde{\epsilon}_2(\delta))}{\sigma(\underline{c} + \epsilon_1(\delta))} \frac{\phi'_{2,\mathcal{A}2,\emptyset}(\underline{c} + \delta)}{\phi'_{1,\mathcal{A}2,\emptyset}(\underline{c} + \delta)} \frac{\delta - \epsilon_1(\delta)}{\delta - \tilde{\epsilon}_2(\delta)} = \frac{M - q_2}{M - q_1}$$

which implies that $\frac{\delta - \epsilon_1(\delta)}{\delta - \tilde{\epsilon}_2(\delta)} \frac{\tilde{\epsilon}_2(\delta)(\tilde{\epsilon}_2(\delta) - \tilde{\epsilon}_2(\delta/n))}{\epsilon_1(\delta)(\epsilon_1(\delta) - \epsilon_1(\delta/n))} = \frac{M - q_2}{M - q_1}$, and ultimately I can infer that $\frac{\delta - \epsilon_1(\delta)}{\delta - \epsilon_2(\delta)} \frac{\epsilon_2(\delta)}{\epsilon_1(\delta)} \frac{(\epsilon_2(\delta) - \epsilon_2(\delta/n))}{(\epsilon_1(\delta) - \epsilon_1(\delta/n))} > \frac{M - q_2}{M - q_1}$.

Necessary and sufficient conditions for B_2 bunching

If B_2 bunches in the subgame, $\exists\ \tilde{\epsilon}_2(\delta) < \epsilon_2(\delta)$ such that B_2 pools for costs between $\underline{c} + \tilde{\epsilon}_2(\delta)$ and $\underline{c} + \epsilon_2(\delta)$. Therefore,

$$\frac{\sigma(\underline{c} + \tilde{\epsilon}_2(\delta))}{\sigma(\underline{c} + \epsilon_1(\delta))} \frac{\phi'_{2,\mathcal{A}2,\emptyset}(\underline{c} + \delta)}{\phi'_{1,\mathcal{A}2,\emptyset}(\underline{c} + \delta)} \frac{\delta - \epsilon_1(\delta)}{\delta - \tilde{\epsilon}_2(\delta)} = \frac{M - q_2}{M - q_1}$$

which implies that $\frac{\delta - \epsilon_1(\delta)}{\delta - \tilde{\epsilon}_2(\delta)} \frac{\tilde{\epsilon}_2(\delta)(\tilde{\epsilon}_2(\delta) - \tilde{\epsilon}_2(\delta/n))}{\epsilon_1(\delta)(\epsilon_1(\delta) - \epsilon_1(\delta/n))} = \frac{M - q_2}{M - q_1}$, and ultimately I can infer that $\frac{\delta - \epsilon_1(\delta)}{\delta - \epsilon_2(\delta)} \frac{\epsilon_2(\delta)}{\epsilon_1(\delta)} \frac{(\epsilon_2(\delta) - \epsilon_2(\delta/n))}{(\epsilon_1(\delta) - \epsilon_1(\delta/n))} > \frac{M - q_2}{M - q_1}$.

This condition is satisfied Back

Theorem for existence uniqueness

Theorem 2

If $b^R > \bar{c}$, equilibrium described by Lemma 3 may not always exist, but when it exists, it is unique. If $b^R = \bar{c}$, the equilibrium exists and is unique.

If B_2 bunching in the subgame started by B_3's exit at some bid, it's bunching in subgame started at any such bid. Thus, FOCs can be written as:

$$
\begin{aligned}
&(q_1 + q_2 + q_3 - M)(b - \phi_{1,\mathcal{A}2,B_3}(b))\sigma(b)1_{b\leq\bar{c}} \\
&+ (b - \phi_{1,\mathcal{A}2,B_3}(b))(q_1 + q_2 + q_3 - M)\sigma(\phi_{2,\mathcal{A}2,B_3}(b))\phi'_{2,\mathcal{A}2,B_3}(b) = M - q_2 - q_3 \\
&q_3(b - \phi_{2,\mathcal{A}2,B_3}(b))\sigma(b)1_{b\leq\bar{c}} \\
&+ (b - \phi_{2,\mathcal{A}2,B_3}(b))(q_1 + q_2 + q_3 - M)\sigma(\phi_{1,\mathcal{A}2,B_3}(b))\phi'_{1,\mathcal{A}2,B_3}(b) = M - q_1 - q_3
\end{aligned}
\tag{8}
$$

Back

proof contd

For any bids less than $\bar{c}$, the equations 8 can be rewritten as:

$$(b - \phi_{1,\mathcal{A}2,B_{\mathbf{3}}}(b)))(\sigma(b) + \sigma(\phi_{2,\mathcal{A}2,B_{\mathbf{3}}}(b))\phi'_{2,\mathcal{A}2,B_{\mathbf{3}}}(b)) = \frac{M - q_2 - q_3}{q_1 + q_2 + q_3 - M}$$

$$(b - \phi_{2,\mathcal{A}2,B_{\mathbf{3}}}(b))\left(\frac{q_3}{(q_1 + q_2 + q_3 - M)}\sigma(b) + \sigma(\phi_{1,\mathcal{A}2,B_{\mathbf{3}}}(b))\phi'_{1,\mathcal{A}2,B_{\mathbf{3}}}(b)\right) = \frac{M - q_1 - q_3}{q_1 + q_2 + q_3 - M} \tag{9}$$

As in theorem 1, it can be shown that solution to BVP defined by above, and boundary condition, $\phi_{2,\mathcal{A}2,B_{\mathbf{3}}}(\bar{c}) = c_2^*$ and $\lim_{b \to \underline{c}^+} \phi_{i,\mathcal{A}2,B_{\mathbf{3}}}(c) = \underline{c}$ for exactly one c_2^*.

proof contd

For any bids less than $\bar{c}$, the equations 8 can be rewritten as:

$$(b - \phi_{1,\mathcal{A}2,B_{\mathbf{3}}}(b)))(\sigma(b) + \sigma(\phi_{2,\mathcal{A}2,B_{\mathbf{3}}}(b))\phi'_{2,\mathcal{A}2,B_{\mathbf{3}}}(b)) = \frac{M - q_2 - q_3}{q_1 + q_2 + q_3 - M}$$

$$(b - \phi_{2,\mathcal{A}2,B_{\mathbf{3}}}(b))\left(\frac{q_3}{(q_1 + q_2 + q_3 - M)}\sigma(b) + \sigma(\phi_{1,\mathcal{A}2,B_{\mathbf{3}}}(b))\phi'_{1,\mathcal{A}2,B_{\mathbf{3}}}(b)\right) = \frac{M - q_1 - q_3}{q_1 + q_2 + q_3 - M} \tag{9}$$

As in theorem 1, it can be shown that solution to BVP defined by above, and boundary condition, $\phi_{2,\mathcal{A}2,B_{\mathbf{3}}}(\bar{c}) = c_2^*$ and $\lim_{b \to \underline{c}^+} \phi_{i,\mathcal{A}2,B_{\mathbf{3}}}(c) = \underline{c}$ for exactly one c_2^*.

Suppose that the solution gives $\phi_{i,\mathcal{A}2,B_{\mathbf{3}}}(\bar{c}) = c_i^*$.

proof contd

For $b > \bar{c}$, the IVP of concern is:

$$(b - \phi_{1,\mathcal{A}2,B_3}(b))(q_1 + q_2 + q_3 - M)\sigma(\phi_{2,\mathcal{A}2,B_3}(b))\phi'_{2,\mathcal{A}2,B_3}(b) = M - q_2 - q_3$$
$$(b - \phi_{2,\mathcal{A}2,B_3}(b))(q_1 + q_2 + q_3 - M)\sigma(\phi_{1,\mathcal{A}2,B_3}(b))\phi'_{1,\mathcal{A}2,B_3}(b) = M - q_1 - q_3$$

$\phi_{i,\mathcal{A}2,B_3}(\bar{c}) = c_i^*$, which will a solution such that $\phi_{2,\mathcal{A}2,B_3}(b^R) = \bar{c}$ for exactly one value of b^R, for a given set of M, q_1, q_2, q_3.

proof contd

For $b > \bar{c}$, the IVP of concern is:

$$(b - \phi_{1,\mathcal{A}2,B_3}(b))(q_1 + q_2 + q_3 - M)\sigma(\phi_{2,\mathcal{A}2,B_3}(b))\phi'_{2,\mathcal{A}2,B_3}(b) = M - q_2 - q_3$$
$$(b - \phi_{2,\mathcal{A}2,B_3}(b))(q_1 + q_2 + q_3 - M)\sigma(\phi_{1,\mathcal{A}2,B_3}(b))\phi'_{1,\mathcal{A}2,B_3}(b) = M - q_1 - q_3$$

$\phi_{i,\mathcal{A}2,B_3}(\bar{c}) = c_i^*$, which will a solution such that $\phi_{2,\mathcal{A}2,B_3}(b^R) = \bar{c}$ for exactly one value of b^R, for a given set of M, q_1, q_2, q_3.
Thus, equilibrium may not always exist.

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