Relaxing Instrument Exogeneity with Common Confounders

IV with Mismeasured Confounders

Christian Tien

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Outline

Setup

Linear Example

Identification

Returns to Education

Semiparametric Estimation

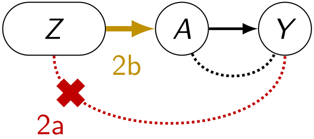
Conclusion

General Idea

Quantity of interest: *Causal effect* of treatment A on outcome Y.	$\theta_0 = \int Y(a)\pi(a)\,\mathrm{d}\mu_A(a)$
A is endogenous (simultaneity, unobserved confounders).	$Y(a) \not\perp\!\!\!\perp A$
We want to use relevant instruments Z for A.	$A(z) \neq A$ if $z \neq Z$
Instruments NOT unconditionally exogenous.	$Y(a) \not\perp\!\!\!\perp Z$
The unobserved *common confounders* U fully explain the association between Z and proxies W.	$Z \perp\!\!\!\perp W \mid U$
Instruments would be exogenous conditional on the common confounders U.	$Y(a) \perp\!\!\!\perp Z \mid U$

IV

Figure: DAG of an IV model

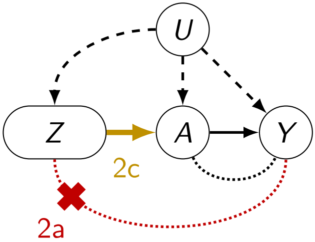


Assumption (IV Model)

1. *SUTVA:* $Y = Y(A, Z)$
2a. *Instrument Exogeneity:*
$Y(a, z) = Y(a) \perp\!\!\!\perp Z.$
2b. *Instrument Relevance:*
For any $g(A) \in L_2(A)$,
$\mathbb{E}[g(A)|Z] = 0$ *only if* $g(A) = 0$.

Unobservable Confounders U

Figure: DAG with unobserved confounders



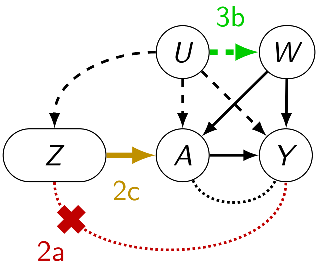
Assumption (Confounded IV)

2a. ***Cond.*** *Instrument Exogeneity:*
$Y(a, z) = Y(a) \perp\!\!\!\perp Z \mid \mathbf{U}$.

2c. ***Cond.*** *Instrument Relevance:*
For any $g(A, \mathbf{U}) \in L_2(A, \mathbf{U})$,
$\mathbb{E}\left[g(A, \mathbf{U}) | Z\right] = 0$ *only if* $g(A, \mathbf{U}) = 0$.

Proxies W for Unobservables U

Figure: Introducing proxies W



Assumption (Confounded IV with relevant proxies)

2a. *Cond. Instrument Exogeneity:*
$Y(a,z) = Y(a) \perp\!\!\!\perp Z \mid U.$

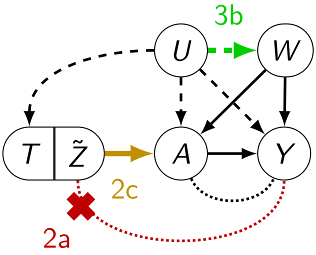
2c. *Cond. Instrument Relevance:*
For any $g(A,U) \in L_2(A,U)$,
$\mathbb{E}\left[g(A,\mathbf{U})|Z\right] = 0$ *only if* $g(A,\mathbf{U}) = 0$.

3a. *Proxy Exogeneity:*
$W(z) = W \perp\!\!\!\perp Z \mid U.$

3b. *Proxy Relevance:*
For any $g(U) \in L_2(U)$,
$\mathbb{E}\left[g(U)|W\right] = 0$ *only if* $g(U) = 0$.

Index Sufficiency

Figure: Focus on Z



Assumption (Confounded IV with rel. proxies and index sufficiency)

2a. *Cond. Instrument Exogeneity:* $Y(a,z) = Y(a) \perp\!\!\!\perp Z \mid U$.

2b. *Index sufficiency:* $U \perp\!\!\!\perp Z \mid T$ *for some* $T = \tau(Z)$.

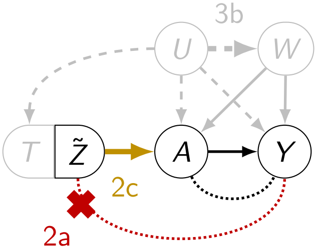
2c. *Cond. Instrument Relevance:* For any $g(A,T) \in L_2(A,T)$, $\mathbb{E}\left[g(A,T) | Z\right] = 0$ *only if* $g(A,T) = 0$.

3a. *Proxy Exogeneity:* $W(z) = W \perp\!\!\!\perp Z \mid U$.

3b. *Proxy Relevance:* For any $g(U) \in L_2(U)$, $\mathbb{E}\left[g(U) | W\right] = 0$ *only if* $g(U) = 0$.

Block Backdoor Path

Figure: Retrieving standard IV model



Assumption 1.1

2a. *Cond. Instrument Exogeneity*: $Y(a,z) = Y(a) \perp\!\!\!\perp Z \mid U$.

2b. *Index sufficiency*: $U \perp\!\!\!\perp Z \mid T$ *for some* $T = \tau(Z)$.

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3a. *Proxy Exogeneity*: $W(z) = W \perp\!\!\!\perp Z \mid U$.

3b. *Proxy Relevance*: *For any* $g(U) \in L_2(U)$, $\mathbb{E}\left[g(U)|W\right] = 0$ *only if* $g(U) = 0$..

Related literature

- Instrumental Variables
  - with linear separability of unobservables in the outcome model [Newey and Powell, 2003]
  - for average structural function identification with strict monotonicity in the first stage reduced form [Imbens and Newey, 2009]
- Proximal learning
  - General proximal learning [Deaner, 2018, Tchetgen Tchetgen et al., 2020, Cui et al., 2020]
  - Control function approach [Nagasawa, 2018]
- Index sufficiency assumption
  - on unobserved heterogeneity for average effect identification in panel data [Liu et al., 2021]
- Semiparametric estimation
  - with nested nuisances [Chernozhukov et al., 2022]
  - with nuisances as solutions to (possibly ill-posed) inverse problems [Bennett et al., 2023]

Linear Example

Equation	Exogeneity	Relevance
$Y = A\beta + W\upsilon_Y + U\gamma_Y + \varepsilon_Y,$	$\mathbb{E}\left[\varepsilon_Y Z\right] = \mathbf{0},$	
$A = Z\zeta + W\upsilon_A + U\gamma_W + \varepsilon_A,$		$\text{rank}\left(\mathbb{E}\left[A^\intercal Z \mid T\right]\right) = d_A$
$Z = U\gamma_Z + \varepsilon_Z,$		$\text{rank}(\gamma_Z) = d_U < d_Z,$
$W = U\gamma_W + \varepsilon_W,$	$\mathbb{E}\left[\varepsilon_W^\intercal \varepsilon_Z\right] = \mathbf{0},$	$\text{rank}(\gamma_W) = d_U \leq d_W$

3SLS procedure I

1. Reduced form (rank-restricted) regression of W on Z:
   1.1 d_U known: With appropriate normalisation of δ_{WU},

$$\underset{N\times d_W}{W} = \underset{N\times d_Z}{Z}\underset{d_Z\times d_U}{\delta_{ZU}}\underset{d_U\times d_W}{\delta_{WU}^{\intercal}} + \epsilon_W \implies \underset{N\times d_U}{T} = \underset{N\times d_Z}{Z}\underset{d_Z\times d_U}{\delta_{ZU}}$$

$$\left(\hat{\delta}_{ZU}, \hat{\delta}_{WU}\right) = \arg\min_{(\delta_{ZU},\delta_{WU})} \sum_{j=1}^{d_W}\sum_{i=1}^{N}\left(w_{i,j} - z_i^{\intercal}\delta_{ZU}\delta_{WU,j}^{\intercal}\right)^2$$

   If $d_W = d_U$, $\hat{\delta}_{ZU} = (Z^{\intercal}Z)^{-1} Z^{\intercal} W$ (OLS), $\delta_{WU} = I$.

   1.2 d_U unknown:
   U are the set of unobserved variables explaining all correlation between W and Z.
   - If $d_U \geq \min\{d_W, d_Z\}$, then rank $\left(\mathbb{E}\left[w_i z_i^{\intercal}\right]\right) = \min\{d_W, d_Z\}$.
   - If $d_U < \min\{d_W, d_Z\}$, then $d_U = \text{rank}\left(\mathbb{E}\left[w_i z_i^{\intercal}\right]\right)$.

3SLS procedure II

A test for a sufficient (not necessary) condition of W's relevance for U (and necessary condition for Z's relevance for A given T) [Chen and Fang, 2019] is

$$H_0 : \text{rank}\left(\mathbb{E}\left[w_i z_i^{\intercal}\right]\right) \leq r_0 < \min\{d_W, d_Z\} \text{ vs}$$
$$H_1 : \text{rank}\left(\mathbb{E}\left[w_i z_i^{\intercal}\right]\right) > r_0.$$

Reject H_0 for $r_0 = \min\{d_W, d_Z\} - 1$ (suggests $d_U \geq \min\{d_W, d_Z\}$):

- If $d_Z < d_W$: $d_U \geq d_Z$, so Z never relevant for A given T.
- If $d_W < d_Z$: $d_U \geq d_W$, so either W just-relevant for U ($d_W = d_U$), or W not relevant for U ($d_W < d_U$). Both are observationally equivalent.

If H_0 not rejected: Proceed with $d_U = r_0$ in step 1.1.

3SLS procedure III

2. Reduced form OLS regression of A on any subset of instruments Z_0 of dimension $N \times (d_Z - d_U)$ and T, W.

$$\underset{N\times d_A}{A} = \underset{N\times(d_Z-d_U)}{Z_0}\underset{(d_Z-d_U)\times d_A}{\delta_{Z_0A}} + \underset{N\times d_U}{T}\underset{d_U\times d_A}{\delta_{TA}} + \underset{N\times d_W}{W}\underset{d_W\times d_A}{\delta_{WA}} + \epsilon_A$$

Test of relevance of Z for A given T is an underidentification test [Windmeijer, 2021]:

$$H_0 : \text{rank}\left(\delta_{Z_0A}\right) < d_A \text{ vs } H_1 : \text{rank}\left(\delta_{Z_0A}\right) = d_A$$

Reject H_0: Z is relevant for A given T.

3SLS procedure IV

3. Outcome OLS regression of Y on exogenous variation in A $(Z_0\delta_{Z_0A})$, and T, W.

$$\underset{N\times 1}{Y} = \underset{N\times d_A}{(Z_0\delta_{Z_0A})}\underset{d_A\times d_1}{\delta_{AY}} + \underset{N\times d_U}{T}\underset{d_U\times 1}{\delta_{TY}} + \underset{N\times d_W}{W}\underset{d_W\times 1}{\delta_{WY}} + \epsilon_Y$$

Consistent estimator of β: $\hat{\delta}_{AY}$. Closed form expression:

$$\hat{\delta}_{AY} = \left(Z_0^\intercal M_{\hat{T}} Z_0\right)^{-1}\left(Z_0^\intercal M_{\hat{T}} Y\right)$$

$$M_{\hat{T}} = I - Z\left(Z^\intercal Z\right)^{-1}\left[Z^\intercal W\right]_{\text{rr}} \left(\left[W^\intercal Z\right]_{\text{rr}}\left(Z^\intercal Z\right)^{-1}\left[Z^\intercal W\right]_{\text{rr}}\right)^{-1}\left[W^\intercal Z\right]_{\text{rr}}\left(Z^\intercal Z\right)^{-1} Z^\intercal$$

$$\left[Z^\intercal W\right]_{\text{rr}} := Z^\intercal Z\hat{\delta}_{ZU} \overset{\text{if } \{d_W = d_U\}}{=} Z^\intercal W$$

Obtaining a valid control function T

Lemma 1

Assume $W \perp\!\!\!\perp Z \mid U$ (3a). Take any $\tau \in L_2(Z)$, where $T := \tau(Z)$, such that $U \perp\!\!\!\perp Z \mid T$. Then, also $W \perp\!\!\!\perp Z \mid T$.

Obtaining a valid control function T

Lemma 1

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Lemma 2

Assume $W \perp\!\!\!\perp Z \mid U$ *(3a), and for any* $g(U) \in L_2(U)$, $\mathbb{E}\left[g(U)|W\right] = 0$ *only when* $g(U) = 0$ *(3b). Take any* $\tau \in L_2(Z)$, *where* $T := \tau(Z)$, *such that* $W \perp\!\!\!\perp Z \mid T$. *Then, also* $U \perp\!\!\!\perp Z \mid T$.

Linearly separable outcome model

Assumption 3.1

There exists some function $k_0 \in L_2(A)$ such that

$$Y = Y(A) = k_0(A) + \varepsilon, \qquad \mathbb{E}\left[\varepsilon | Z, U\right] = \mathbb{E}\left[\varepsilon | U\right]. \tag{1}$$

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Theorem 1

Let assumptions (2b/2c/3a/3b) and 3.1 hold. Any $h \in \mathcal{L}_2(A, T)$ for which $\mathbb{E}\left[Y|Z\right] = \mathbb{E}\left[h(A,T)|Z\right]$, satisfies $h(A,T) = k_0(A) + \mathbb{E}\left[\varepsilon|T\right]$. Consequently, $\theta_0 := \int_{\mathcal{A}} Y(a)\pi(a)\,\mathrm{d}\mu_A(a) = \mathbb{E}_{\mathcal{T}}\left[\int_{\mathcal{A}} h(a,T)\pi(a)\,\mathrm{d}\mu_A(a)\right]$.

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A simple plug-in estimator would be the empirical equivalent $\hat{\theta} = \mathbb{E}_{\mathcal{T},n}\left[\int_{\mathcal{A}} \hat{h}(a,T)\pi(a)\,\mathrm{d}\mu_A(a)\right]$.

First stage monotonicity

Assumption 3.2 (Monotonicity)

$$A = h(Z, \eta) \tag{2}$$

1. *$h(Z, \eta)$ is strictly monotonic in η with probability 1.*
2. *η is a continuously distributed scalar with a strictly increasing conditional CDF $F_{\eta|U}$ on the conditional support of η.*
3. *$Z \perp\!\!\!\perp \eta \mid U$.*

Disturbance conditional on T

Lemma 3

$$F_{\eta|T} := \int_{\mathcal{U}} F_{A|Z,U}(A, Z, u) f_{U|T}(u, T) \, \mathrm{d}\mu_U(u)$$

is a strictly increasing CDF on the conditional support of η*, and* $Z \perp\!\!\!\perp \eta \mid T$*.*

Conditional unconfoundedness

Theorem 2

Let

$$V := F_{A|Z}(A, Z). \tag{3}$$

Under assumption 3.2, $V = F_{\eta|T}(\eta)$*, and*

$$A \perp\!\!\!\perp Y(a) \mid (V, T), \text{ for all } a \in \mathcal{A}. \tag{4}$$

Average structural function identification

Assumption 3.3 (Common Support)

For all $a \in \mathcal{A}$, the support of (V, T) equals the support of (V, T) conditional on A.

Theorem 3

Suppose (2a/2b/3a/3b) [relaxed IV model], 3.2 [monotonicity], and 3.3 [common support] hold. Then, $\theta_0 := \int_{\mathcal{A}} Y(a)\pi(a)\,\mathrm{d}\mu_A(a)$ is identified by

$$\theta = \mathbb{E}_{\mathcal{V},\mathcal{T}}\left[\int_{\mathcal{A}} \mathbb{E}\left[Y|A=a,(V,T)=(v,t)\right]\pi(a)\,\mathrm{d}\mu_A(a)\right].$$

NLS97 Data

- Y Household net worth at 35: continuous, in USD
- A BA degree: 1 if BA degree obtained, 0 otherwise

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- Other biases: Selection on unobservables into obtaining BA degree (at least partly result of individual optimisation)

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- X Covariates: sex, college GPA, parental education/net worth, siblings, region, etc

Assumption 4.1

1. *Linear model:* $Y = \alpha_Y + A\beta + U\gamma_Y + W\upsilon_Y + X\eta_Y + \varepsilon_Y.$

2a. *Cond. Instr. Exogeneity:* $\mathbb{E}\left[\varepsilon_Y^\intercal Z\right] = 0.$

2c. *Cond. Instr. Relevance:* $A = \alpha_A + Z\zeta + U\gamma_A + W\upsilon_A + X\eta_A + \varepsilon_A$

$$rank\left(\mathbb{E}\left[\left.(Z\zeta)\,A\right|\,T, X\right]\right) = d_A.$$

3a. *Proxy Exogeneity:* $W = \alpha_W + U\gamma_W + X\eta_W + \varepsilon_W,\ \mathbb{E}\left[\varepsilon_W^\intercal Z\right] = 0.$

3b. *Proxy Relevance:* $rank(\gamma_W) = d_U \leq d_W$

$$\Longrightarrow \beta = \frac{\mathbb{E}\left[(Z\zeta)\,Y \mid T, X\right]}{\mathbb{E}\left[(Z\zeta)\,A \mid T, X\right]}$$

1. Find T and test relevance of Z, W for U I

- Linear projection

$$\mathbb{E}\left[U|Z,X\right] = Z\delta_{ZU} + X\delta_{XU}$$
$$\mathbb{E}\left[W|Z,X\right] = \tilde{\alpha}_W + Z\delta_{ZW} + X\delta_{XW}$$

- By definition: $\text{rank}(\delta_{ZW}) = \min\left\{d_U, \min\{d_Z, d_W\}\right\}$.
- Hypothesis test: For some $r < \min\{d_Z, d_W\}$,

$$H_0 : \text{rank}(\delta_{ZW}) \leq r, \text{ vs } H_1 : \text{rank}(\delta_{ZW}) > r.$$

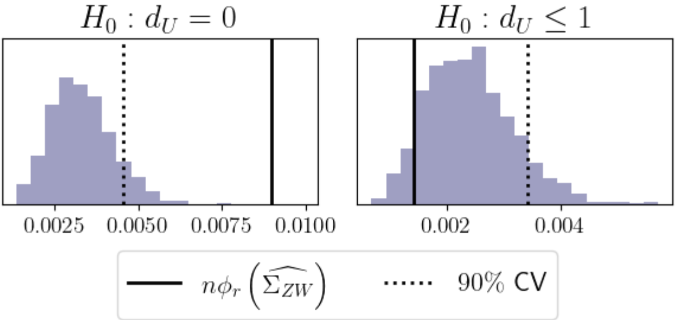
  Do not reject H_0: $d_U \leq r < \min\{d_Z, d_W\}$.
- Normalisation: $\delta_{ZW} := \underset{d_Z \times r}{\gamma_Z} \underset{r \times d_Z}{\gamma_W}$ with γ_W unitary (columns form orthonormal vectors as in SVD): $\hat{T} := Z\hat{\gamma}_Z$.

1. Find T and test relevance of Z, W for U II

- Equivalent test: For some $r < \min\{d_Z, d_W\}$,

$$H_0 : \phi_r\left(\gamma_Z \gamma_W\right) = 0, \text{ vs } H_1 : \phi_r\left(\gamma_Z \gamma_W\right) > 0,$$

where $\phi_r(A) := \sum_{j>r} \pi_j^2(A)$ (sum of $(r+1)$-th to smallest singular value squared).



1. Find T and test relevance of Z, W for U III

- $d_U = \text{rank}(\delta_{ZW}) \leq 1$: one-dimensional U explains correlation between Z, W given X.

2. Test relevance of Z for A given T

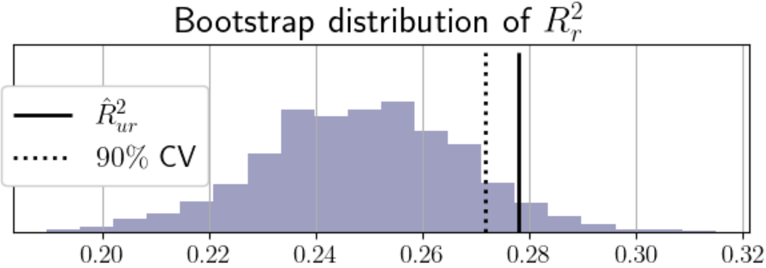
- Construct control function: $\hat{T} := Z\hat{\gamma}_Z$
- Compare restricted and unrestricted R^2:

$$A = \tilde{\alpha}_{A,ur} + Z\tilde{\zeta} + X\tilde{\eta}_{A,ur} + \tilde{\varepsilon}_{A,ur} \implies R^2_{ur} \quad (5)$$

$$A = \tilde{\alpha}_{A,r} + T\tilde{\gamma}_A + X\tilde{\eta}_{A,r} + \tilde{\varepsilon}_{A,r} \implies R^2_r \quad (6)$$

Hypothesis test:

$$H_0 : R^2_r = R^2_{ur}, \text{ vs } H_1 : R^2_r < R^2_{ur}.$$



3. Exogeneity of Z conditional on U I

$$Y = \alpha_Y + A\beta + U\gamma_Y + W\upsilon_Y + X\eta_Y + \varepsilon_Y.$$

2a. Cond. Instr. Exogeneity: $\mathbb{E}\left[\varepsilon_Y^\intercal Z\right] = 0.$

- Untestable in this just-identified model
- Does T reflect the hypothesised confounder ability?

Table: Construction of $\hat{T} = Z\hat{P}_{0,1}\hat{\Pi}_{0,1}$ (normalised to $\sigma(\hat{T}) = 1$)

	GPA				ASVAB
	English	Math	SocSci	LifeSci	percentile
$\hat{T}$	0.574	0.213	0.282	0.212	-0.278

- Weighted subject-GPA describes 94.4% of $\hat{T}$.
- Seems to reflect overall GPA and thus general ability.

3. Exogeneity of Z conditional on U II

Table: Effect of normalised $\hat{T}$ on W

	drink	smoke	try marijuana	run away	attack someone
$\hat{T}$	-0.071	-0.093	-0.101	-0.054	-0.093
Pr	0.653	0.468	0.296	0.106	0.189

		sell drugs	destroy property	steal $< 50\$$	steal $> 50\$$
$\hat{T}$		-0.052	-0.051	-0.051	-0.036
Pr		0.089	0.320	0.379	0.079

- One standard-deviation increase in $\hat{T}$ significantly reduces risky behaviour.
- Expected for confounder ability.

4. Estimation

$$\hat{\beta}_{\mathrm{OLS}} = \left(A^\intercal M_{W,X} A\right)^{-1} \left(A^\intercal M_{W,X} Y\right)$$

$$\hat{\beta}_{\mathrm{PL}} = \left(A^\intercal M_{\hat{T},X} A\right)^{-1} \left(A^\intercal M_{\hat{T},X} Y\right),$$

where $\hat{T}$ is constructed from the correlation of (A, Z) and W.

$$\hat{\beta}_{\mathrm{IV}} = \left(A^\intercal P_Z M_{W,X} A\right)^{-1} \left(A^\intercal P_Z M_{W,X} Y\right)$$

$$\hat{\beta}_{\mathrm{ICC}} = \left(A^\intercal P_Z M_{\hat{T},X} A\right)^{-1} \left(A^\intercal P_Z M_{\hat{T},X} Y\right),$$

where $\hat{T}$ is constructed from the correlation of Z and W.

Notation: Projection $P_X = X\left(X^\intercal X\right)^{-1} X^\intercal$ and annihilator $M_X = I - P_X$.

Results

Table: Estimates with different estimators (NLS97 data, $n = 1,890$)

	OLS	PL	IV	ICC
A	59,173	33,799	204,578	122,665
	(9,542)	(10,765)	(34,265)	(49,138)
$\hat{T}$		27,879		15,829
		(5,415)		(7,613)

- Positive ability bias (was ambiguous).
- Negative general selection bias (as expected).
- Standard error increases about 40% in ICC compared to IV.

Generalised setting

For all $k_0(\tau_0, g_0) \in \mathcal{K}_0(\tau_0, g_0)$, and $\tau_0(g_0) \in \mathcal{T}_{\text{valid}}(g_0)$,

$$\theta_0 = \mathbb{E}\left[m_0(O; k_0(\tau_0, g_0))\right], \tag{7}$$

$$\mathcal{K}_0(\tau, g) := \left\{k \in \mathcal{K} : \ \mathbb{E}\left[k(A; \tau, g)|Z\right] = g(Z) - \tau(Z)\right\} \tag{8}$$

$$\mathcal{T}_{\text{valid}}(g) := \left\{\tau \in \mathcal{T} : \mathbb{E}\left[g(Z)|W\right] = \mathbb{E}\left[\tau(Z)|W\right]\right\} \tag{9}$$

$$g_0(Z) := \mathbb{E}\left[Y|Z\right] \tag{10}$$

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$$g_0(Z) := \mathbb{E}\left[Y | Z\right] \tag{10}$$

Features of this setting:

- Identifiable θ_0 restricted by $\mathcal{K}_0(\tau, g)$ and thus $\mathcal{T}$: More complex τ can reduce the set of identifiable θ_0 (better choose valid τ of small complexity)
- Nested dependence of nuisances [Chernozhukov et al., 2022]
- Weak identification of nuisances: Non-uniqueness and ill-posedness [Bennett et al., 2023]

Continuous linear functionals

Assume that $k \mapsto \mathbb{E}\left[m(O;k)\right]$ for $k \in \mathcal{K}$ is a continuous linear functional over $\mathcal{K}$, such that by the Riesz representation theorem

$$\mathbb{E}\left[m_0(O;k)\right] = \mathbb{E}\left[\alpha_{k,0}(A)k(A)\right] \ \forall k \in \mathcal{K}. \tag{11}$$

Strong instrument relevance

Define the linear operator $P^{A,\mathcal{K}}_{\mathcal{L}_2(Z)}$ and its adjoint $P^{\mathcal{L}_2(Z)}_{A,\mathcal{K}}$ (where Π is the projection operator):

$$\left[P^{A,\mathcal{K}}_{\mathcal{L}_2(Z)}k\right](Z) := \mathbb{E}\left[k(A)|Z\right]$$

$$\left[P^{\mathcal{L}_2(Z)}_{A,\mathcal{K}}q_k\right](A) := \Pi_{\mathcal{K}}\,\mathbb{E}\left[q_k(Z)|A\right] = \Pi_{\mathcal{K}}\left[q_k(Z)|A\right]$$

Assumption 5.1 (Strong instrument relevance)

$\alpha_{k,0} \in \mathcal{N}^{\perp}\left(P^{\mathcal{L}_2(Z)}_{A,\mathcal{K}}P^{A,\mathcal{K}}_{\mathcal{L}_2(Z)}\right)$, *i.e.*

$$\Xi_{k,0} \neq \emptyset, \ \textit{where} \quad \Xi_{k,0} := \arg\min_{\xi_k \in \mathcal{K}}\left(\frac{1}{2}\mathbb{E}\left[\mathbb{E}\left[\xi_k(A)|Z\right]^2\right] - \mathbb{E}\left[m_0(O;\xi_k)\right]\right) \tag{12}$$

$$= \left\{\xi_k \in \mathcal{K} : P^{\mathcal{L}_2(Z)}_{A,\mathcal{K}}P^{A,\mathcal{K}}_{\mathcal{L}_2(Z)}\xi_k = \alpha_{k,0}\right\}. \tag{13}$$

Debiasing step 1

Debiased moment wrt k:

$$m_1(O; k, \tau, g, q_k) = m_0(O; k) + q_k(Z) \left(g(Z) - \tau(Z) - k(A; \tau, g)\right)$$

$$\mathcal{Q}_k := \left\{q_k \in \mathcal{L}_2(Z) : q_k(Z) = \mathbb{E}\left[\xi_k(A)|Z\right] \ \forall \xi_k \in \mathcal{K}\right\}$$

$$\mathcal{Q}_{k,0} := \left\{q_k \in \mathcal{L}_2(Z) : q_k(Z) = \mathbb{E}\left[\xi_{k,0}(A)|Z\right] \ \forall \xi_{k,0} \in \Xi_{k,0}\right\}$$

Debiasing step 2

Debiased moment wrt k and τ:

$$m_2(O;k,\tau,g,q_k,q_\tau) = m_1(O;k,\tau,g,q_k) + q_\tau(W;q_k)\left(\tau(Z) - g(Z)\right)$$

$$\mathcal{Q}_\tau(q_k) := \left\{ q_\tau \in \mathcal{L}_2(W) : q_\tau = \mathbb{E}\left[\xi_\tau(Z;q_k)|W\right],\ \forall \xi_\tau(q_k) \in \mathcal{T}, q_k \in \mathcal{Q}_k \right\}$$

$$\mathcal{Q}_{\tau,0}(q_k) := \left\{ q_\tau \in \mathcal{L}_2(W) : q_\tau = \mathbb{E}\left[\xi_{\tau,0}(Z;q_k)|W\right],\ \forall \xi_{\tau,0}(q_{k,0}) \in \Xi_{\tau,0}(q_k), q_k \in \mathcal{Q}_k \right\}$$

$$\Xi_{\tau,0}(q_k) := \arg\min_{\xi_\tau(q_k)\in\mathcal{T}} \left(\frac{1}{2}\mathbb{E}\left[\mathbb{E}\left[\xi_\tau(Z;q_k)|W\right]^2\right] - \mathbb{E}\left[q_k(Z)\xi_\tau(Z;q_k)\right]\right).$$

For this step, use that the continuous linear functional $\tau \mapsto \mathbb{E}\left[q_\tau(W;q_k)\tau(Z)\right]$ is strongly identified in this setting.

Intermediate τ-functional strongly identified

Theorem 4 (Strong identification of θ_0)
Suppose $\Pi_{\mathcal{T}}\left[q(W)|Z\right] = \Pi_{\mathcal{T}}\left[\mathbb{E}\left[q(W)|U\right]|Z\right]$ for any $q \in \mathcal{L}_2(W)$ for $\mathcal{T} \subseteq \mathcal{L}_2(Z)$ (relaxation of $W \perp\!\!\!\perp Z \mid U$), and assumption 3b (completeness of W for U) hold. Also, suppose the functional $\tau \mapsto \mathbb{E}\left[m(O;\tau)\right]$ is continuous and linear over $\mathcal{T}$. Then, θ_0 is strongly identified with the following holding true for the functional's Riesz representer:

$$\alpha \in \mathcal{N}^{\perp}(P_{Z,\mathcal{T}}^{\mathcal{L}_2(U)} P_{\mathcal{L}_2(U)}^{Z,\mathcal{T}}) = \mathcal{N}^{\perp}(P_{Z,\mathcal{T}}^{\mathcal{L}_2(W)} P_{\mathcal{L}_2(W)}^{Z,\mathcal{T}}). \tag{14}$$

Debiasing step 3

Debiased moment wrt k, τ, and g:

$$m_3(O; k, \tau, g, q_k, q_\tau, \alpha_g) = m_2(O; k, \tau, g, q_k, q_\tau) + \alpha_g(Z; q_k, q_\tau)\left(Y - g(Z)\right)$$

$$\Xi_{g,0}(q_k, q_\tau) := \arg\min_{\xi_g(q_k, q_\tau) \in \mathcal{L}_2(Z)} \left(\frac{1}{2}\,\mathbb{E}\left[\xi_g(Z; q_k, q_\tau)^2\right] - \mathbb{E}\left[\left(q_k(Z) - q_\tau(W; q_k)\right)\xi_g(Z; q_k, q_\tau)\right]\right).$$

Each debiased moment identifies θ

Lemma 4

Assume $\theta_0 = \mathbb{E}\left[m_0(O; k_0(\tau_0, g_0))\right]$ *and 5.1. Then,*

$$\begin{aligned}\theta_0 = \mathbb{E}\left[m_0(O; k_0)\right] &= \mathbb{E}\left[m_1(O; k_0, \tau_0, g_0, q_{k,0})\right] \\ &= \mathbb{E}\left[m_2(O; k_0, \tau_0, g_0, q_{k,0}, q_{\tau,0})\right] \\ &= \mathbb{E}\left[m_3(O; k_0, \tau_0, g_0, q_{k,0}, q_{\tau,0}, \alpha_{g,0})\right].\end{aligned}$$

Each of m_j for $j \in \{1, 2, 3\}$ identify θ_0 because in each consecutive debiasing step a conditionally mean-zero term is added to the previous moment.

Robustness of final debiased moment

Theorem 5 (Robust error decomposition)

Suppose $\theta_0 = \mathbb{E}\left[m_0(O; k_0(\tau_0, g_0))\right]$, *assumption 5.1, and the conditions for theorem 4 hold for* $q = q_\tau$. *Then,*

$$\begin{aligned}
&\mathbb{E}\left[m_3(O; k, \tau, g, q_k, q_\tau, \alpha_g)\right] - \theta_0 \\
&= \mathbb{E}\left[\Big(q_{k,0}(Z) - q_k(Z)\Big)\left(k(A; \tau, g) - k_0(A; \tau_0, g_0)\right)\right] \\
&\quad + \mathbb{E}\left[\left(q_{\tau,0}(W; q_k) - q_\tau(W; q_k)\right)\left(\tau_0(Z; g_0) - \tau(Z; g)\right)\right] \\
&\quad + \mathbb{E}\left[\left(\alpha_{g,0}(Z; q_k, q_\tau) - \alpha_g(Z; q_k, q_\tau)\right)\left(g(Z) - g_0(Z)\right)\right].
\end{aligned}$$

Double robustness, Neyman orthogonality, and an error decomposition in terms of projections on more favourable subspaces (circumventing ill-posedness problems) follow.

Conclusion

- Identification approach between IV and proximal learning
- Allows some endogeneity in instruments, as long as relevant proxies exist for the unobserved causes of instrument endogeneity
- Motivated by traditional economic identification problems with self-selection into treatment (returns to education)
- Semiparametric estimation with $\sqrt{n}$-rates possible under assumptions on nuisance convergence rates in terms of more favourable projected errors (circumvents potential ill-posedness of inverse problems)

Thank You

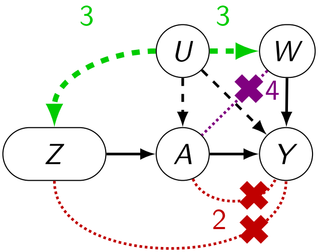
arXiv 2301.02052

ct493@cam.ac.uk

Appendix

Proximal Learning [Tchetgen Tchetgen et al., 2020]

Figure: Proximal learning

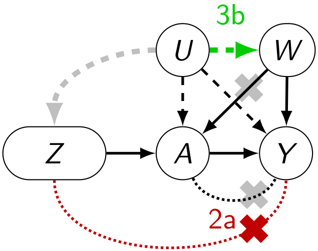


Assumption (Proximal learning)

2. *Cond. Exogeneity of action and its aligned proxy:*
   $Y(a,z) = Y(a) \perp\!\!\!\perp (A,Z) \mid U.$
3. *Relevance of both proxies:*
   For any $g(U) \in L_2(U)$,
   $\mathbb{E}\left[g(U)|Z\right] = 0$ *only if* $g(U) = 0$,
   $\mathbb{E}\left[g(U)|W\right] = 0$ *only if* $g(U) = 0$.
4. *Exogeneity of outcome-aligned proxies:*
   $W(a,z) = W \perp\!\!\!\perp (A,Z) \mid U.$

Relaxations compared to proximal learning

Figure: Proximal learning



Assumption

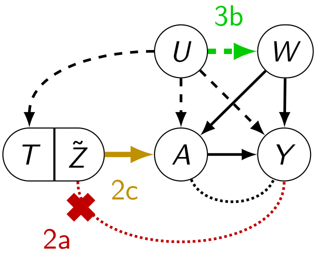
2a. *Cond. Exogeneity of action-aligned proxy:*
$Y(a, z) = Y(a) \perp\!\!\!\perp Z \mid U.$

3a. *Proxy Exogeneity:*
$W(z) = W \perp\!\!\!\perp Z \mid U.$

3b. *Relevance of outcome-aligned proxy:*
For any $g(U) \in L_2(U)$,
$\mathbb{E}\left[g(U)|W\right] = 0$ *only if* $g(U) = 0.$

Additional assumptions compared to proximal learning

Figure: Proximal learning



Assumption

2a. *Cond. Instrument Exogeneity:* $Y(a, z) = Y(a) \perp\!\!\!\perp Z \mid U$.

2b. *Index sufficiency:* $U \perp\!\!\!\perp Z \mid T$ *for some* $T = \tau(Z)$.

2c. *Cond. Instrument Relevance:* *For any* $g(A, T) \in L_2(A, T)$, $\mathbb{E}\left[g(A, T)|Z\right] = 0$ *only if* $g(A, T) = 0$.

3a. *Proxy Exogeneity:* $W(z) = W \perp\!\!\!\perp Z \mid U$.

3b. *Relevance of outcome-aligned proxy:* *For any* $g(U) \in L_2(U)$, $\mathbb{E}\left[g(U)|W\right] = 0$ *only if* $g(U) = 0$.

Intuition for relevance requirements

1. $\text{rank}(\gamma_W) = d_U \leq d_W$ ensures that $\mathbb{E}_{\text{L}}\left[U|Z\right]$ is proportional to $\mathbb{E}_{\text{L}}\left[W|Z\right]$. Keep $\mathbb{E}_{\text{L}}\left[U|Z\right]$ fixed by keeping $\mathbb{E}_{\text{L}}\left[W|Z\right]$ fixed (via T).

Intuition for relevance requirements

1. $\text{rank}(\gamma_W) = d_U \leq d_W$ ensures that $\mathbb{E}_{\text{L}}\left[U|Z\right]$ is proportional to $\mathbb{E}_{\text{L}}\left[W|Z\right]$. Keep $\mathbb{E}_{\text{L}}\left[U|Z\right]$ fixed by keeping $\mathbb{E}_{\text{L}}\left[W|Z\right]$ fixed (via T).
2. $\mathbb{E}\left[A^{\intercal}Z|\,T\right] = d_A$: Use remaining variation in Z to instrument for A while keeping $\mathbb{E}_{\text{L}}\left[U|Z\right]$ fixed (via T). Necessary for this is $(d_Z - d_U) \geq d_A$.

The obvious

Lemma 5

Assume $W \perp\!\!\!\perp Z \mid U$ (3a). Take any $\tau \in L_2(Z)$, where $T := \tau(Z)$, such that $U \perp\!\!\!\perp Z \mid T$. Then, also $W \perp\!\!\!\perp Z \mid T$.

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$$
\begin{aligned}
& f_{W,Z|T}(W,Z|T) \\
& \quad = \int_{\mathcal{U}} \underbrace{f_{W|Z,U,T}(W|Z,u,T)}_{=f_{W|U}(W|u)} \underbrace{f_{Z|U,T}(Z|u,T)}_{=f_{Z|T}(Z|T)} f_{U|T}(u,T)\,\mathrm{d}\mu_U(u) \\
& \quad = f_{Z|T}(Z|T) \underbrace{\int_{\mathcal{U}} f_{W|U}(W|u) f_{U|T}(u,T)\,\mathrm{d}\mu_U(u)}_{f_{W|T}(W|T)} \implies W \perp\!\!\!\perp Z \mid T
\end{aligned}
$$

The slightly less obvious I

Lemma 6

Assume $W \perp\!\!\!\perp Z \mid U$ (3a), and for any $g(U) \in L_2(U)$, $\mathbb{E}\left[g(U)|W\right] = 0$ only when $g(U) = 0$ (3b). Take any $\tau \in L_2(Z)$, where $T := \tau(Z)$, such that $W \perp\!\!\!\perp Z \mid T$. Then, also $U \perp\!\!\!\perp Z \mid T$.

The slightly less obvious II

Write $f_{W|Z}(W, Z)$ in two separate ways using T and relate them.

1. $f_{W|Z}(W, Z) = \int_{\mathcal{U}} f_{W|U}(W|u) f_{U|Z}(u|Z) \, \mathrm{d}\mu_U(u)$ by $W \perp\!\!\!\perp Z \mid U$
2. $f_{W|Z}(W, Z) = f_{W|T}(W, T) = \int_{\mathcal{U}} f_{W|U}(W, u) f_{U|T}(u, T) \, \mathrm{d}\mu_U(u)$ by construction of $T = \tau(Z)$ such that $W \perp\!\!\!\perp Z \mid T$.

$$\int_{\mathcal{U}} f_{W|U}(W, u) \left(f_{U|Z}(u|Z) - f_{U|T}(u|T) \right) \mathrm{d}\mu_U(u) = 0$$

$$\int_{\mathcal{U}} \left(f_{U|Z}(u|Z) - f_{U|T}(u|T) \right) \frac{f_W(W)}{f_U(u)} f_{U|W}(u, W) \, \mathrm{d}\mu_U(u) = 0$$

$$\mathbb{E}_U \left[\left. \frac{\left(f_{U|Z}(u|Z) - f_{U|T}(u|T) \right)}{f_U(u)} \right| W \right] f_W(W) = 0$$

The slightly less obvious III

Then, for any Z, let $g_Z(U) := \frac{\left(f_{U|Z}(u|Z) - f_{U|T}(u|T)\right)}{f_U(u)}$.

$$\mathbb{E}_U \left[g_Z(u) \middle| W \right] f_W(W) = 0$$

Completeness of W for U (3b) implies $g_Z(U) = 0$, and thus $f_{U|Z}(U|Z) = f_{U|T}(U|T)$, meaning $U \perp\!\!\!\perp Z \mid T$.

Valid control functions

Valid control functions $\tau \in L_2(Z)$ satisfy two conditions:

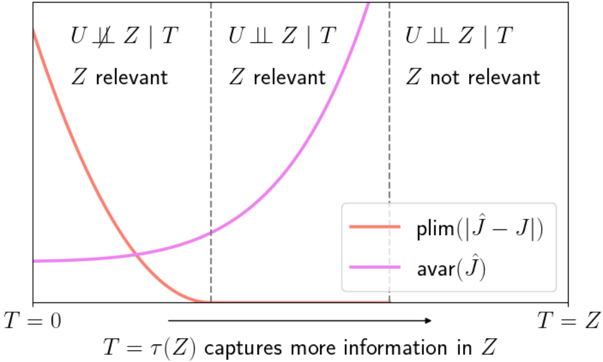
1. Conditional independence of W and Z:

$$W \perp\!\!\!\perp Z \mid \tau(Z).$$

2. Conditional relevance of Z for A:

$$\mathbb{E}\left[g(A, \tau(Z)|Z\right] = 0 \text{ only if } g(A, \tau(Z)) = 0.$$

Optimal control functions



- Optimal: minimum complexity $\tau(Z)$ subject to validity, e.g. $f(W|Z)$

Specification test

- Standard specification test for sufficient complexity of τ_1 vs τ_2.

$$H_0 : Z \perp\!\!\!\perp Y(a) \mid \tau_1(Z), \text{ vs } H_1 : Z \not\perp\!\!\!\perp Y(a) \mid \tau_1(Z).$$

- Test whether $\tau_1(.)$ valid, if $\tau_2(.)$ valid.
- Under H_0:
  - $\text{plim}\left(\hat{\theta}(\tau_1)\right) = \text{plim}\left(\hat{\theta}(\tau_2)\right)$,
  - $\text{avar}\left(\hat{\theta}(\tau_1)\right) < \text{avar}\left(\hat{\theta}(\tau_2)\right)$.
- Under H_1:
  - $\text{plim}\left(\hat{\theta}(\tau_1)\right) \neq \text{plim}\left(\hat{\theta}(\tau_2)\right)$.

Simple linear model

$$Y = \alpha_Y + A\beta + U \underset{d_U \times 1}{\gamma_Y} + W \underset{d_W \times 1}{\upsilon_Y} + X \underset{d_X \times 1}{\eta_Y} + \varepsilon_Y,$$

$U\gamma_Y$ Probably positive effect of ability U on net worth Y, by salary and non-salary mediation [Griliches, 1977].

ε_Y All variation in Y, which is jointly unexplained by (A, U, W, X). Individual-specific, heterogeneous characteristics.

Expected bias I

- i chooses whether to obtain a BA degree by maximising expected utility subject to information set $\mathcal{I}$:

$$A = \underset{a \in \{0,1\}}{\arg\max} \left(\mathbb{E}\left[u(Y(a)) - c(a) | A = a, \mathcal{I} \right] \right),$$

  $u : \mathcal{Y} \to \mathbb{R}$ is a diminishing returns utility function
  $c : \{0,1\} \to \mathbb{R}$ is a cost function for obtaining a BA degree.

- Optimal decision rule assuming full information:

$$\begin{aligned} A &= \underset{a \in \{0,1\}}{\arg\max} \left(u(Y(a)) - c(a) \right), \\ &= \mathbb{1}\left(u(Y(1)) - u(Y(0)) > c(1) - c(0) \right). \end{aligned}$$

Expected bias II

$U \Uparrow$ Ambiguous bias direction

$$\mathbb{1}\Big(\underbrace{u(Y(1)) - u(Y(0))}_{\Downarrow \text{ as } Y(a)\Uparrow \text{ and } \Delta(Y(1)-Y(0))=0} > \underbrace{c(1) - c(0)}_{\Downarrow \text{ when ability higher}} \Big)$$

$\varepsilon_Y \Uparrow$ Positive bias direction

$$\mathbb{1}\Big(\underbrace{u(Y(1)) - u(Y(0))}_{\Downarrow \text{ as } Y(a)\Uparrow \text{ and } \Delta(Y(1)-Y(0))=0} > c(1) - c(0) \Big)$$

- Byproduct: Separate ability bias from other biases.
- Theoretical models can produce differing bias directions.
- Empirical validation would be useful.

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