Cross-Sectional Dynamics Under Network Structure: Theory & Marcoeconomic Applications

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Motivation

- Common in economics: cross-section of units/agents, linked by network ties
- Theory and empirics: network amplifies unit-level shocks, implies comovement of cross-sectional variables
- How does network-induced comovement play out over time?
- **Literature:** Two restrictive cases:
  - **innovations transmit contemporaneously**
    e.g. Acemoglu et al. 2012, 2016, Elliott et al. 2014

    → static model, links of all order play out simultaneously
  - **innovations transmit one link per period**
    e.g. Long & Plosser 1983, Golub & Jackson 2010, Carvalho & Reischer 2021

    → tenable in theory, less so in empirics

Contribution

- Econometric framework that can speak to dynamics implied by networks
  - VAR parameterized s.t. innovations transmit cross-sectionally via bilateral links
  - Can accommodate general patterns on how innovations travel through network over time
- Applicable in two distinct lines of empirical work with cross-sectional time series
  - estimate dynamic network (peer) effects,
    with network given or estimated (+ shrink to observed links)
  - dimensionality-reduction technique for modeling (c.s.) time series

→ Two applications

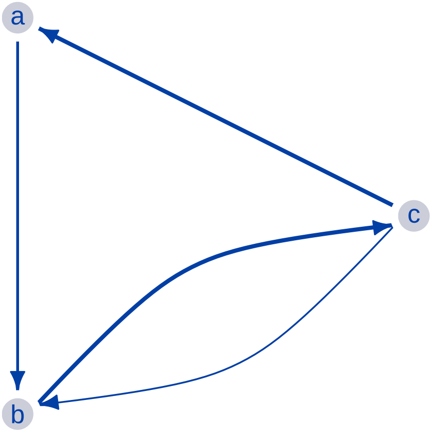
Related Literature: Model

Networks in econometrics

- Spatial Autoregressive Models:
  - identify network effects in static framework
    Manski 1993, Lee 2007, Bramouillé et al. 2009, de Paula et al. 2020, ...
    → I look at dynamic, contagion-like network effects
  - some work on lagged/dynamic network effects
    Knight et al. 2016, Zhu et al. 2017, Yang & Lee 2019, ...
    → I relate TS properties to network and timing of network effects, generalize latter, & show how to conduct inference on both
- Networks in time series (TS) econometrics:
  - represent TS model output as network
    Diebold & Yilmaz 2009, 2014, Barigozzi & Brownlees 2018, ...
    → I use network to obtain a TS model
  - restrict TS models using networks
    Pesaran et al. 2004, Chudik & Pesaran 2011, Caporin et al. 2022, ...
    → I focus on simpler/clearer case & assume tranmission via links → analytical results

Bilateral Connections in Networks

details



$$A = \begin{bmatrix} 0 & 0 & .8 \\ .7 & 0 & .6 \\ 0 & .8 & 0 \end{bmatrix}$$

shows direct links

$$A^2 = \begin{bmatrix} 0 & .64 & 0 \\ 0 & .48 & .56 \\ .56 & 0 & .48 \end{bmatrix}$$

shows 2nd order connections

...

Lagged Innovation Transmission via Bilateral Links

VAR(1):

$$y_t = A y_{t-1} + u_t \ ,$$

$$\rightarrow y_{it} = \sum_{j=1}^{n} a_{ij} y_{j,t-1} + u_{it}$$

- Interpret A as network: innovations travel one link per period

→ Granger Causality at horizon $h = 1, 2, ...$ given by hth order network connections:

illustration

$$\frac{\partial y_{i,t+h}}{\partial y_{j,t}} \Big| \mathcal{F}_t = (A^h)_{ij} \ .$$

- Used in theory:
    - Long & Plosser (1983): sectoral output under one period delay in I-O conversion
    - Golub & Jackson (2010): study of societal opinion formation through friendship ties

Lagged Innovation Transmission via Bilateral Links

NVAR(p, 1): (particular version of NAR(p) in Zhu et al. 2017)

$$x_\tau = \alpha_1 A x_{\tau-1} + ... + \alpha_p A x_{\tau-p} + v_\tau \ , \quad \alpha = (\alpha_1, ..., \alpha_p) \in \mathbb{R}^p \ .$$

- Assuming $\alpha_l \neq 0 \, \forall \, l$, x_j Granger-causes x_i at horizon h iff there exists a connection from i to j of at least one order $k \in \{\underline{k}, \underline{k}+1, ..., h\}$, where $\underline{k} = ceil(h/p)$.

$$\rightarrow \frac{\partial x_{i,\tau+h}}{\partial v_{j,\tau}} \Big| \mathcal{F}^x_\tau = c^h_{\underline{k}}(\alpha) \left[A^{\underline{k}} \right]_{ij} + ... + c^h_h(\alpha) \left[A^h \right]_{ij} \ .$$

- i.e. x_τ driven by lagged network effects, with transmission spread out over p periods
- α shows time profile of transmission

Lagged Innovation Transmission via Bilateral Links

$$x_\tau = \alpha_1 A x_{\tau-1} + ... + \alpha_p A x_{\tau-p} + v_\tau \ , \quad \alpha = (\alpha_1, ..., \alpha_p) \in \mathbb{R}^p \ .$$

If x_τ observed every $q > 1$ periods – $\{y_t\}_{t=1}^T = \{x_{tq}\}_{t=1}^T$ –, then

- for GC at horizon h, links of order $k \in \{\underline{k}, \underline{k}+1, ..., hq\}$ matter, $\underline{k} = ceil(hq/p)$
- holds for $q \in \mathbb{Q}_{++}$, and also for flow variables (for $q \in \mathbb{N}$ or $q^{-1} \in \mathbb{N}$)

→ " **NVAR**(p, q) " stationarity relation to contemp. transmision inference

Application 1: Motivation

Macro literature on production networks:

- assuming contemporaneous input-output-conversion, shows:
  Horvath 2000, Acemoglu et al. 2012, 2016, Bouakez et al. 2014, ...
  - supply chain network amplifies sectoral shocks
  - strength of effect on aggregates depends on sector's position in network
- exception: one period-lagged I-O-conversion → NVAR(1,1)
  Long & Plosser (1983), Carvalho & Reischer (2021)
  - generates endogenous BCs (persistence in aggregates)
  - model-persistence matches empirics,
    calibrated model gives improved forecasts of agg. IP relative to statistical models

This application:

- How does amplification materialize over time?
- Does network-position shape timing of effect?
- Estimate roles of exogenous shock persistence vs. lagged IO conversion Foerster et al. (2011)

Estimation/Setup

- Generalized version of LP: firms use inputs produced in last p periods
  $\rightarrow$ at some model-frequency, sectoral prices $\sim$ NVAR$(p, 1)$:

$$x_\tau \approx \alpha_1 A x_{\tau-1} + ... + \alpha_p A x_{\tau-p} + v_\tau \ ,$$

  with $\alpha_l \geq 0 \; \forall \; l$ and $\sum_{l=1}^{p} \alpha_l = 1$. theory

- Observation freq. potentially $\neq$ network interaction freq.: $\{y_t\}_{t=1}^T = \{x_{qt}\}_{t=1}^T$
  $\rightarrow$ I consider $q \in \{1/3, \ 1/2, \ 1, \ 2, \ 4\}$,
  i.e. quarterly, bi-monthly, monthly, bi-weekly and weekly network interactions

- 51 sectors, Jan 2005 - Aug 2022, I-O-matrix from 2010 data

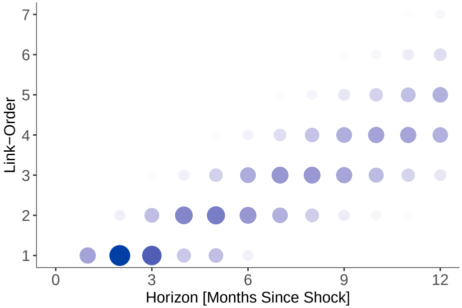
- For now, let $v_{i\tau} \overset{iid}{\sim} N(0, \sigma_i^2)$, get $(\hat{\alpha}, \hat{\sigma})_{MLE}$ for different (p, q) & select model via IC

- Work in progress: $v_{i\tau} = \lambda_i f_\tau + \varepsilon_{i\tau}$, $f_\tau, \varepsilon_{i\tau} \sim$ AR(1)
  $\rightarrow$ Determine roles of exogenous shock persistence vs. lagged I-O-conversion

Results: Impulse Responses & Their Composition

more results

Relevance of Link-Orders Across Horizons



Input-Output Links to Utilities Sector

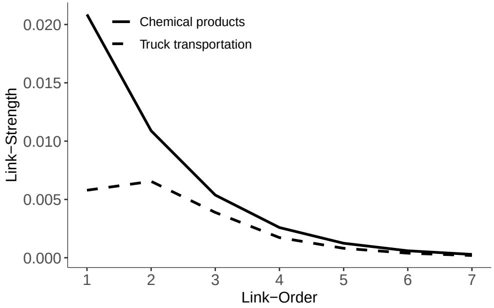


Figure: Transmission of Price Shocks via Supply-Chain Links (1)

Recall: $$\frac{\partial y_{i,t+h}}{\partial u_{j,t}} = c^h_{\underline{k}}(\alpha)\left(A^{\underline{k}}\right)_{ij} + \ldots + c^h_h(\alpha)\left(A^h\right)_{ij} \ , \quad \underline{k} = ceil(h/p) \ .$$

Results: Impulse Responses & Their Composition

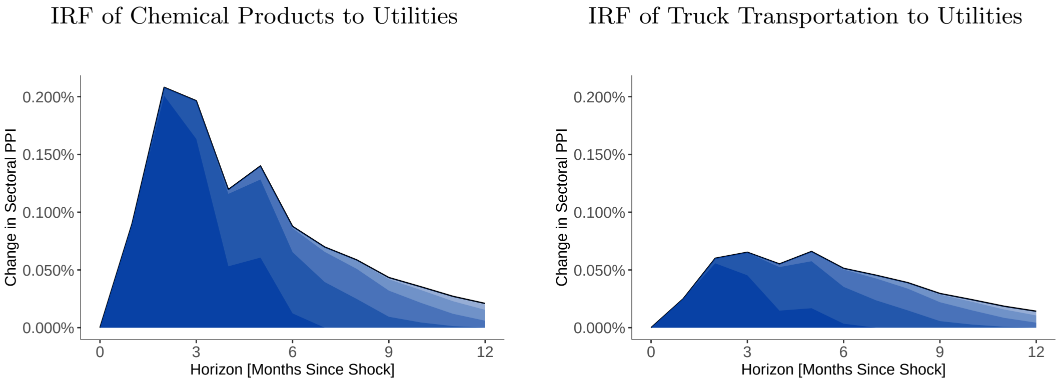


Figure: Transmission of Price Shocks via Supply-Chain Links (2)

Recall: $$\frac{\partial y_{i,t+h}}{\partial u_{j,t}} = c^h_{\underline{k}}(\alpha)\left(A^{\underline{k}}\right)_{ij} + ... + c^h_h(\alpha)\left(A^h\right)_{ij} \ , \quad \underline{k} = ceil(h/p) \ .$$

Results: Agg. Response to Sectoral Price Shocks

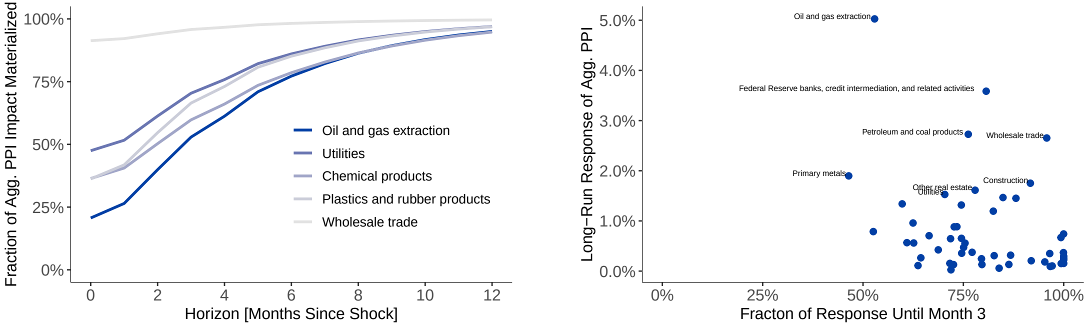


Figure: Size & Timing of Aggregate PPI Response to Sectoral Shocks

Notes: Left panel shows time profile of effect of sectoral price shocks on aggregate PPI for few selected sectors. Right panel relates strength of effect on aggregate PPI to its timing. Shock sizes: one standard deviation.

Application 2: Motivation

How to model industrial production dynamics across 44 countries?

- Even for this moderate size of cross-section, unrestricted VAR not feasible
- NVAR(p, q): well-performing, simple-to-estimate and interpretable alternative details

→ Estimate (α, A), A sparse !

- Assumption: a few bilateral links drive dynamics of whole cross-section

Relation to Alternative Dimensionality-Reduction Techniques

- Combines insights from factor models / RR regression (Velu et al. 1986, Stock & Watson 2002, ...) and sparse / shrinkage methods (Tibshirani 1996, ...)

$$\text{Recall NVAR}(p,1): \quad y_t = X_t(A)\alpha + u_t \ , \quad \text{with } X_t = A[y_{t-1}, ..., y_{t-p}] \ .$$

- Equivalence betw. factor model & NVAR$(p,1)$, with # factors = rank(A): details
  - $y_t \sim \text{NVAR}(p,1) \quad \Rightarrow \quad y_t \sim \text{FM}$
  - $y_t \sim \text{FM} + f_t \sim \text{NVAR}(p,1) \quad \Rightarrow \quad y_t \sim \text{NVAR}(p,1)$, for n large
- Expect: Network-VAR preferred when dynamics driven by many micro links rather than few influential units (see Boivin & Ng, 2006)

Data & Forecasting Setup

Data:

- Use IMF & OECD data on monthly IP series
- Compute growth rate relative to same month previous year, subtract mean
- January 2001 - January 2020, 44 countries

Forecasting Exercise:

- Use sample end dates from December 2017 to December 2019
- Consider forecasts of up to 24 months ahead (COVID-19 excluded)
- For $p = 1:6$, compare
  - NVAR$(p, 1)$ + Lasso-shrinking of a_{ij} to zero, select λ based on BIC (Zou, Hastie & Tibshirani 2007) details
  - PC-FM: select # of factors based on Bai & Ng (2002), fit VAR(p) for factors

Results: Forecasting more results

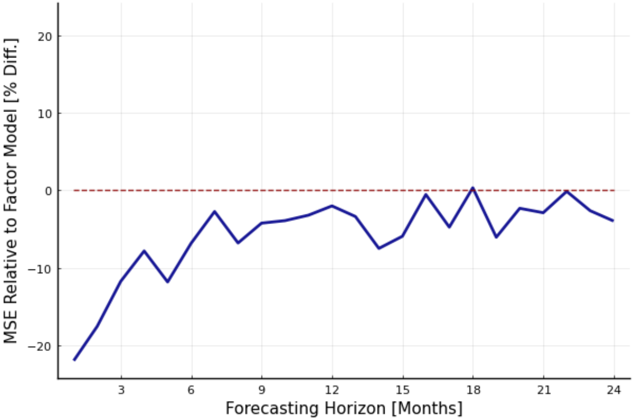


Figure: Out-Of-Sample Forecasting Performance: NVAR$(4, 1)$ vs. Factor Model

Notes: Plot depicts percentage difference between out-of-sample Mean Squared Errors generated by NVAR$(4, 1)$ to those generated by Principal Components Factor Model.

Conclusion

- I propose econometric framework for cross-sectional time series exploiting network structure
- I apply it to estimate how supply shocks propagate through US supply chain network and affect dynamics of sectoral prices
- I apply it to forecast cross-country IP dynamics, assuming & estimating network

Bilateral Connections in Networks back

- Network: $n \times n$ adjacency matrix A with elements a_{ij}
- Directed and weighted: $a_{ij} \in [0,1]$ shows strength of (direct) link from i to j
- Walk: product of direct links a_{ij} that lead from i to j over some intermediary units

$$\text{e.g.} \quad a_{i,k_1} a_{k_1,k_2} a_{k_2,j} \quad : \quad \text{walk from } i \text{ to } j \text{ of length } 3$$

- $(A^K)_{ij}$: sum of all walks from i to j of length K ("Kth order connection from i to j")

$$\text{e.g.} \quad A = \begin{bmatrix} 0 & 0 & .8 \\ .7 & 0 & .6 \\ 0 & .8 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0 & .64 & 0 \\ 0 & .48 & .56 \\ .56 & 0 & .48 \end{bmatrix}, \quad A^3 = \begin{bmatrix} .448 & 0 & .384 \\ .336 & .448 & .288 \\ 0 & .384 & .448 \end{bmatrix}.$$

Lagged Innovation Transmission via Bilateral Links

back

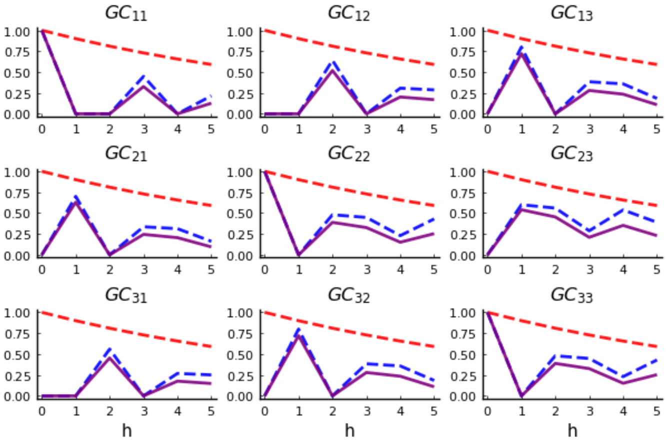


Figure: Example Generalized Impulse Responses For NVAR$(1,1)$

Notes: Panel (i,j) shows $(A^h)_{ij}$ in blue, α^h in red and GC^h_{ij}, their product, in purple.

Time Aggregation of Lagged Transmission Patterns back

- Let $x_\tau = \alpha_1 A x_{\tau-1} + \alpha_2 A x_{\tau-2} + \alpha_3 A x_{\tau-3} + v_\tau$, and $\{y_t\}_{t=1}^T = \{x_{2t}\}_{t=1}^T$.
- We get

$$x_\tau = \left[\alpha_2 A + \alpha_1^2 A^2\right] x_{\tau-2} + \left[(\alpha_1\alpha_2 + 2\alpha_1\alpha_3)A^2\right] x_{\tau-4}$$
$$+ v_\tau + \alpha_1 A v_{\tau-1} + (\alpha_3 A + \alpha_1\alpha_2 A^2) v_{\tau-3} + \text{terms in } x_{\tau-6}, x_{\tau-7} \ .$$

$$\to y_t \approx \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \Theta_0 u_t + \Theta_1 u_{t-1} \ ,$$

with $\Phi_1 = \alpha_2 A + \alpha_1^2 A^2$, $\quad \Phi_2 = (\alpha_1\alpha_2 + 2\alpha_1\alpha_3)A^2$

$u_t = [v_\tau', v_{\tau-1}']'$, $\quad u_{t-1} = [v_{\tau-2}', v_{\tau-3}']'$, ,

$\Theta_0 = [I_n, \alpha_1 A]$, $\quad \Theta_1 = \left[0_n, \alpha_3 A + \alpha_1\alpha_2 A^2\right]$.

Contemporaneous Innovation Transmission via Bilateral Links back

- Under contemporaneous network interactions,

$$\tilde{y} = A\tilde{y} + \varepsilon = (A + A^2 + A^3 + ...)\varepsilon \; .$$

→ Acemoglu et al. (2012): network A amplifies granular shocks ε_j, implies cross-sectional comovement in $\{\tilde{y}_i\}_{i=1}^n$

- Result: for NVAR$(p, 1)$, $y_t = \alpha_1 A y_{t-1} + ... + \alpha_p A y_{t-p} + u_t$, we have that

$$lim_{h\to\infty} \sum_{j=0}^{h} \frac{\partial y_{t+h}}{\partial u_{t+j}} = \frac{\partial \tilde{y}}{\partial \varepsilon} = (I - A)^{-1} \; , \qquad (\text{for} \sum_{l=1}^{p} \alpha_l = 1)$$

→ Taking stance on timing of network effects, y_t can speak to (transition) dynamics

Stationarity of NVAR$(p, 1)$ back

Let x_τ follow an NVAR$(p, 1)$

$$x_\tau = \alpha_1 A x_{\tau-1} + ... + \alpha_p A x_{\tau-p} + v_\tau \ ,$$

where $v_\tau \sim WN$, and assume $\alpha_l \neq 0$ for at least one l. Let $a = \sum_{l=1}^{p} |\alpha_l|$.

1a x_τ is WS if for all Eigenvalues λ_i of A it holds that $|\lambda_i| < 1/a$.

1b If in addition $\alpha_1, ..., \alpha_p \geq 0$, this condition is both necessary and sufficient.

2 x_τ is WS iff the univariate AR(p)

$$\check{x}_\tau = \lambda_i \alpha_1 \check{x}_{\tau-1} + ... + \lambda_i \alpha_p \check{x}_{\tau-p} + \check{v}_\tau$$

is WS for all Eigenvalues λ_i of A.

Inference: NVAR$(p, 1)$ back

$$y_t \quad = \alpha_1 A y_{t-1} + ... + \alpha_p A y_{t-p} + u_t \quad = X_t(A)\alpha + u_t \quad = A z_t(\alpha) + u_t \ .$$

Given A: OLS estimator for α:

$$\hat{\alpha}_{OLS}|A = \left[\sum_{t=1}^{T} X_t' X_t\right]^{-1} \left[\sum_{t=1}^{T} X_t' y_t\right] \ , \quad X_t = [Ay_{t-1}, ..., Ay_{t-p}] \ .$$

- consistent and asymp. Normal for n, T & $(n, T) \longrightarrow \infty$

Joint estimation, (α, A)**:** OLS-Ridge estimator for A, shrinking to B:

$$\hat{A}_{OLS}|\alpha = [Y'Z + \lambda B] \left[Z'Z + \lambda\Sigma\right]^{-1}$$

- get $(\hat{\alpha}, \hat{A})_{OLS}$ by iterating until convergence (Meng & Rubin 1993)
- normalize $||\alpha||_1 = 1$ (e.g.)
- consistent and asymp. Normal for $T \longrightarrow \infty$

Inference: NVAR(p, q), $q > 1$ back

- In principle, could apply EM algorithm (data augmentation):

$$x_\tau = \alpha_1 A x_{\tau-1} + ... + \alpha_p A x_{\tau-p} + v_\tau \ , \quad \tau = 1 : T_\tau \ ,$$
$$y_{\tau/q} = x_\tau \quad \text{if } \tau/q \in \mathbb{N} \ ,$$

- However, point ID not guaranteed;

  e.g. for $q = 2, p = 1$, can identify α_1 up to sign: $y_t = \alpha_1^2 A^2 y_{t-1} + \eta_t$.

- Akin to AR(p) observed every $q > 1$ periods (Palm & Nijman 1984), and to estimating continuous time models from discrete time data (Phillips 1973)

- Mapping between α and $\gamma(\alpha)$ in VARMA approx. for y_t not bijective

$$y_t \approx \sum_{l=1}^{p^*} \mathbf{\Phi_l} y_{t-l} + u_t \ , \quad u_t = \sum_{l=0}^{p^*-1} \Theta_l \eta_{t-l} \ , \quad \mathbf{\Phi_l} = \sum_{g=1}^{q^*} \gamma_{lg}(\alpha) A^g \ .$$

$\to$ use prior info or structural model

Asymptotics: $\hat{\alpha}_{OLS}|A$ in NVAR$(p,1)$

$T \longrightarrow \infty$

- Model correct: $y_t = X_t\alpha + u_t$
- $\mathbb{E}_{t-1}[u_t] = 0$, $\quad \mathbb{E}_{t-1}[u_t u_t'] = \Sigma$
- y_t ergodic and strictly stationary

$n \longrightarrow \infty$

- Model correct: $y_{it} = x_{it}'\alpha + u_{it}$
- $\mathbb{E}_{t-1}[u_t] = 0$, $\quad \mathbb{E}_{t-1}[u_{it}u_{is}] = \sigma^2$ if $t = s$ and zero otherwise
- A_n converges to some limit s.t.
  - $\frac{1}{n}\sum_{i=1}^{n}\left(A_{n,i\cdot}y_{t-l}\right)'\left(A_{n,i\cdot}y_{t-k}\right) \longrightarrow \mathbb{E}\left[\left(A_{i\cdot}y_{t-l}\right)'\left(A_{i\cdot}y_{t-k}\right)\right]$
  - $\frac{1}{n}\sum_{i=1}^{n}\left(A_{n,i\cdot}y_{t-l}\right)' u_{it} \longrightarrow \mathbb{E}\left[\left(A_{i\cdot}y_{t-l}\right)' u_{it}\right]$
  - $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(A_{n,i\cdot}y_{t-l}\right)' u_{it} \Rightarrow N\left(\mathbb{E}\left[\left(A_{i\cdot}y_{t-l}\right)' u_{it}\right], \mathbb{V}\left[\left(A_{i\cdot}y_{t-l}\right)' u_{it}\right]\right)$

Theory back

Assume n sectors, rep. firm produces variety i by using labor and inputs $j = 1 : n$:

$$y_{i\tau} = z_{i\tau} l_{i\tau}^{b_i} \prod_{j=1}^{n} x_{ij\tau}^{a_{ij}} , \quad b_i > 0 , \;\; a_{ij} \geq 0 , \quad b_i + \sum_{j=1}^{n} a_{ij} = 1 .$$

- If $x_{ij\tau}$ is variety j bought at τ: $p_\tau = Ap_\tau + \varepsilon_\tau$, $\varepsilon_\tau = -log(z_\tau)$ (e.g. Acemoglu et al., 2012)
- If $x_{ij\tau}$ is variety j bought at $\tau - 1$: $p_\tau = Ap_{\tau-1} + \varepsilon_\tau$ (Long & Plosser 1983, Carvalho & Reischer 2021)

→ If $x_{ij\tau}$ is CES-aggregate of variety j bought at $\{\tau - p, ..., \tau - 1\}$:

$p_\tau \approx \alpha_1 Ap_{\tau-1} + ... + \alpha_p Ap_{\tau-p} + \varepsilon_\tau$, for some $\alpha_l \geq 0$, $l = 1 : p$, and $\sum_{l=1}^{p} \alpha_l = 1$

Data back

Input-Output Matrix from Bureau of Economic Analysis (BEA)

- 64 mostly 3- and 4-digit sectors (due to PPI availability)
- I take data for 2010
- Following Acemoglu et al. (2016), links defined as $a_{ij} \equiv \frac{sales_{j \to i}}{sales_i}$ (valid for general p as $\beta \to 1$)

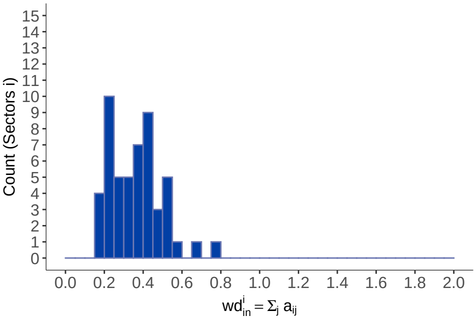
Monthly sector-level PPI data from Bureau of Labor Statistics (BLS)

- 51 BEA-sectors, January 2005 - August 2022
- I take logs and subtract sector-specific linear time trend and seasonality (since the assumed process is stationary)

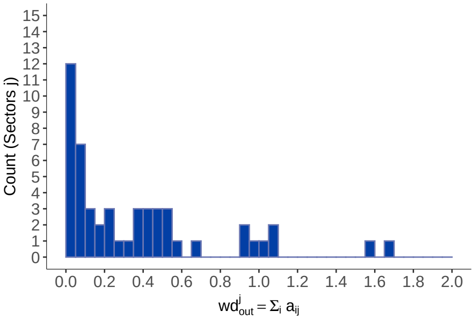
Data: Input-Output Network

- Density: 16.88 %
- Average shortest path: 2.41, longest shortest path: 7

(a) Weighted In-Degrees



(b) Weighted Out-Degrees



Notes: Left panel plots weighted in-degrees (column-wise sums of A), shows sectors' differing reliance on intermediate inputs. Right panel plots weighted out-degrees (row-wise sums of A), shows sectors' differing importance as suppliers to other sectors.

Data: Input-Output Network

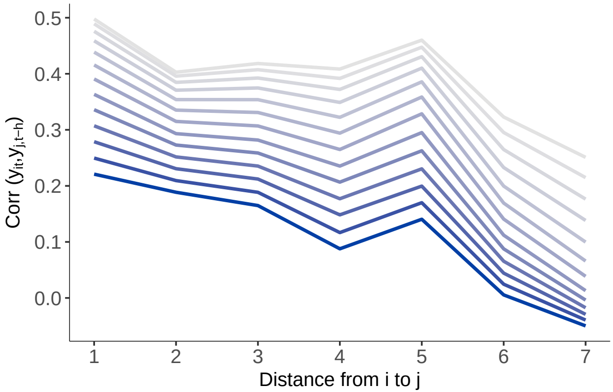


Figure: Network Distance And The Correlation of Sectoral Inflation

Notes: Figure plots average correlation of sectoral prices for different distances between them. Lightest blue line refers to contemporaneous correlations. Darker lines show average correlation of sector i with lagged values of sector j as function of distance from i to j. Lags from 1 to 12 months. Series are de-trended and de-seasonalized log PPIs.

Data: PPI back

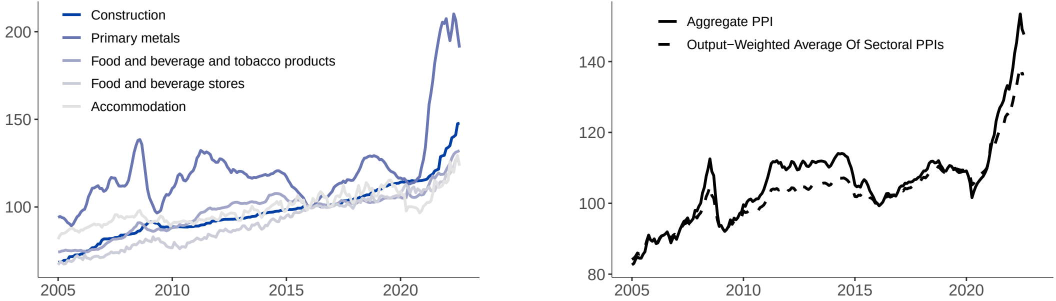


Figure: Aggregate & Sectoral PPIs

Notes: Left panel shows raw PPI series for few selected sectors. Right panel compares aggregate PPI (FRED Database) and output-weighted average of PPIs of studied sectors.

Estimation Results: Model Selection back

Table: Model Selection: Log MDD

		p					
		$1q$	$2q$	$3q$	$4q$	$5q$	$6q$
	1/3			19079			19044
	1/2		19384		18768		18690
q	1	20153	20056	19675	19879	18899	20218
	2	17546	19570	19248	20142	18662	19636
	4	18517	19808	19754	19655	18904	19301

Notes: Table shows log Marginal Data Density (MDD) across model specifications. Values for q (from top to bottom) refer to quarterly, bi-monthly, monthly, bi-weekly and weekly network interactions, while $p = mq$ implies last m months matter for dynamics.

Estimation Results back

Table: Estimation Results: α

	MLE	Mean	Low	High
α_1	0.1550	0.1557	0.1370	0.1745
α_2	0.3460	0.3382	0.3168	0.3605
α_3	0.2816	0.2865	0.2644	0.3129
α_4	0.0915	0.0991	0.0785	0.1174
α_5	0.1045	0.0975	0.0837	0.1135

Notes: First column shows Maximum Likelihood or Maximum A-Posteriori (MAP) Estimator, second refers to posterior mean, and Low and High report the bounds of the 95% Bayesian HPD credible sets.

Application 2: Motivation back

NVAR(p,q): sparse, flexible and interpretable alternative for modeling (high-dimensional) cross-sectional time series:

$$y_t = \sum_{l=1}^{p} \Phi_l(\alpha, A) y_{t-l} + u_t \ , \quad \Phi_l = \sum_{g=1}^{q} \alpha_{lg} A^g \ , \quad \alpha_{lg} \in \mathbb{R} \ , \quad a_{ij} \in [0,1] \ ,$$

- Sparsity:
  - $y_{it} = x_{it}'\alpha + u_{it}$ with $X_t = \left[\tilde{y}_{t-1}^1, \tilde{y}_{t-1}^2, ..., \tilde{y}_{t-1}^q, \tilde{y}_{t-2}^1, ..., \tilde{y}_{t-p}^q\right]_{(n \times qp)}$ and $\tilde{y}_{t-l}^g \equiv A^g y_{t-l}$

  $\rightarrow$ use A to reduce np covariates in $[y_{t-1}', ..., y_{t-p}']'$ to qp covariates in X_t
  - A can be sparse: higher-order network effects through A^2, A^3, ...
- Flexibility:
  - estimated network + general time dimension of network effects
  - like functional approximation using A as basis
- Interpretability:
  - bilateral connections drive dynamics
  - estimate network & whole set of spillover and spillback effects

Relation to Factor Model back

NVAR → FM

- $y_t = A[\alpha_1 y_{t-1} + \alpha_2 y_{t-2}] + u_t$ with A of rank $r \in 1:n$
- Write $A = B_{n\times r} C_{r\times n}$

$\rightarrow$ $y_t = \Lambda f_t + u_t,\ f_{kt} = \alpha_1 C_{k\cdot} y_{t-1} + \alpha_2 C_{k\cdot} y_{t-2}$ for $k = 1:r$

- (not unique: $A = BC = BQQ^{-1}C = \tilde{B}\tilde{C}$ for any $r \times r$ full-rank matrix Q)

Relation to Factor Model back

FM → NVAR

- $y_t = \Lambda f_t + \xi_t$, $\quad f_t = \Phi_1 f_{t-1} + \Phi_2 f_{t-2} + \eta_t$, with $f_t \in \mathbb{R}^r$
- Take r distinct vectors of weights $w^k = (w_1^k, ..., w_n^k)$, $k = 1 : r$, and consider $\sum_{i=1}^n w_i^k y_{it} = \sum_{i=1}^n w_i^k \Lambda_{i\cdot} f_t + \sum_{i=1}^n w_i^k \xi_{it}$
- If n large enough, $\bar{\xi}_t^k \equiv \sum_{i=1}^n w_i^k \xi_{it} \sim O_p(n^{-1/2})$ is negligible $\rightarrow Wy_t = W\Lambda f_t$

$$
\begin{aligned}
y_t &= \Lambda \left(\Phi_1 f_{t-1} + \Phi_2 f_{t-2} + \eta_t\right) + \xi_t \\
&= \Lambda\Phi_1 (W\Lambda)^{-1} W y_{t-1} + \Lambda\Phi_2 (W\Lambda)^{-1} W y_{t-2} + u_t \ ,
\end{aligned}
$$

- If $\Phi_l = \phi_l \Phi$ for $l = 1, 2$ (i.e. $f_t \sim$ NVAR(2,1)), then

$$
y_t = \Lambda\Phi(W\Lambda)^{-1} W[\phi_1 y_{t-1} + \phi_2 y_{t-2}] + u_t
$$

- Let $A = \Lambda\Phi(W\Lambda)^{-1}W$, $\alpha_l = \phi_l$

Estimation back

$$y_t = \sum_{l=1}^{p} \alpha_l A y_{t-l} + u_t \ , \quad \alpha \equiv (\alpha_1, ..., \alpha_p) \in \mathbb{R}^p \ , \quad a_{ij} \in [0,1] \ ,$$

- To identify (α, A), normalize $||\alpha||_1 = 1$ and change domain of a_{ij} to $\mathbb{R}_+$
- Consider OLS with Lasso penalty (λ) on a_{ij}
- Get $(\hat{\alpha}, \hat{A})$ by iterating on

$$\hat{\alpha}_{LS}|A = \left[\sum_{t=1}^{T} X_t' X_t\right]^{-1} \left[\sum_{t=1}^{T} X_t' y_t\right] \ ,$$

$$\hat{a}_{ij,LS}|(\alpha, A_{i,-j}) = max\{0, \check{a}_{ij}\} \ , \quad \check{a}_{ij} = \frac{\sum_{t=1}^{T}(y_{it} - A_{i,-j} z_{-j,t}) z_{jt} - \lambda}{\sum_{t=1}^{T} z_{jt}^2} \ .$$

Results: Estimated Network back

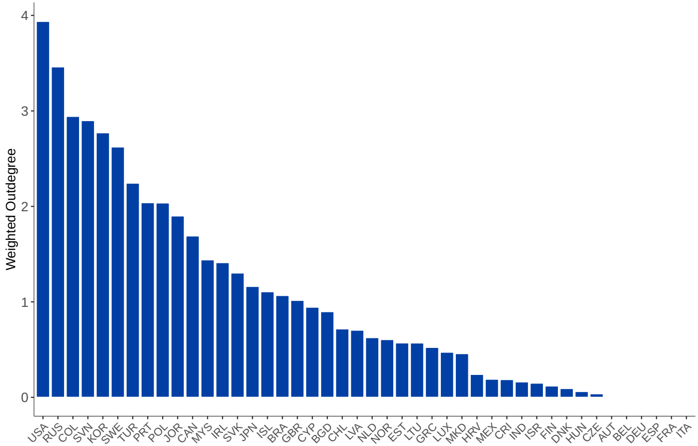


Figure: Weighted Outdegrees In The Estimated Network

Notes: Plot shows weighted outdegrees in estimated network as relevant for cross-country monthly IP dynamics.

Results: Impulse Responses & Their Composition back

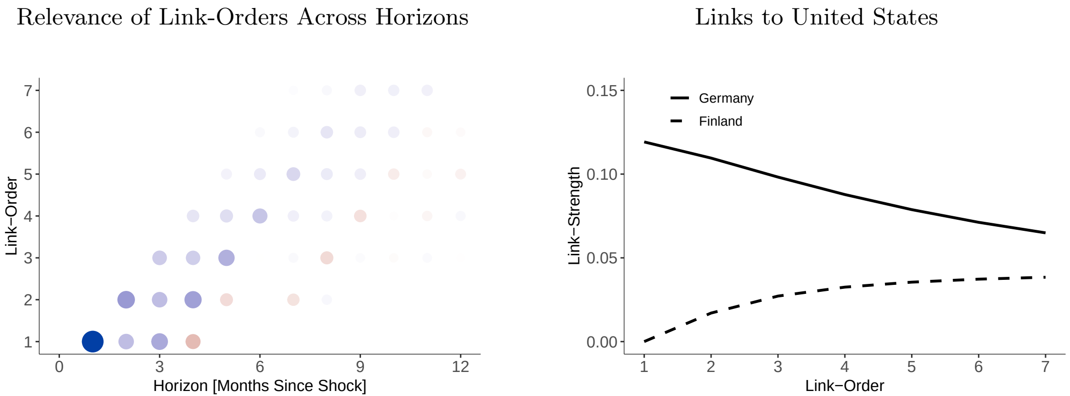


Figure: Network-Induced Transmission of Inudstrial Production Innovations (1)

Notes: Left panel shows importance of different connection-orders for transmission as function of time elapsed since shock took place. Right panel shows connections of different order from Germany and Finland to United States.

Results: Impulse Responses & Their Composition

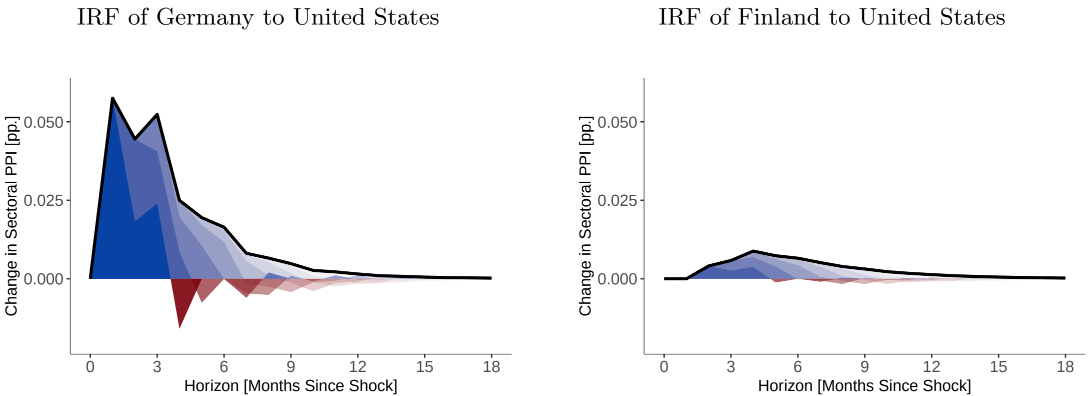


Figure: Network-Induced Transmission of Inudstrial Production Innovations (2)

Notes: The two panels show the Impulse-Response Functions (IRFs) of German and Finnish IP growth, respectively, to a one standard deviation increase in US IP growth.